DIRICHLET PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS WITH COMPLEX VARIABLES

by

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To my mother, father and brothers...

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ABSTRACT

DIRICHLET PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS WITH COMPLEX VARIABLES

The theory of boundary-value problems for complex partial differential equations is investigated very intensively, because of its importance for physics and technology of everything in simply connected applied areas. It is far from being complete.

In this thesis the results of the Dirichlet boundary value problem are for homogeneous and inhomogeneous complex partial differential equations are collected and analyzed.

This study consists of four chapters. In the first and second chapters, we give historical background of the Dirichlet problem and some literature search about this problem. Some basic definitions and theorems from functional analysis and some technical preliminaries are presented. Moreover, we give some introduction in complex methods for partial differential equations and the necessary tools that serve to solve higher order boundary value problems with complex variables.

Chapter 3 is devoted to the investigation of the Dirichlet problem for the one dimensional partial differential equations with complex variable in the unit disc $\mathbb{D} := \{z : z < 1\}$ of the complex plane.

In the Chapter 4, we studied the Dirichlet problem for the two dimensional partial differential equations with complex variable in $\mathbb{D}^2 := \mathbb{D}_1 \times \mathbb{D}_2 = \{z = (z_1, z_2) : |z_k| < 1, k = 1, 2\}$.

ÖZET

KOMPLEKS DEĞERLİ KISMİ TÜREVLİ DİFERANSİYEL DENKLEMLER İÇİN DIRICHLET PROBLEMİ

Kompleks kısmi türevli diferansiyel denklemler için sınır değer problemlerinin teorisi, fizikte ve uygulamalı alanların teknolojisindeki öneminden dolayı yoğun bir şekilde araştırılmaktadır. Bu çalışmalar tamamlanabilmiş olmaktan uzaktır.

Bu tezde homojen ve homojen olmayan kompleks diferansiyel denklemler için Dirichlet sınır değer probleminin sonuçları toplanmış ve analiz edilmiştir.

Bu çalışma dört bölümden oluşmaktadır. İlk bölümde ve ikinci bölümünde, problemin tarihsel geçmişini ve literatür taramasını verdik. Fonksiyonel analizin bazı temel tanım ve teoremleri ve teknik ön bilgiler tanıtılmıştır. Buna ek olarak kısmi türevli diferansiyel denklemler için kompleks metodların birtakım giriş kısmı verilmiştir ve kompleks değerli yüksek mertebeden sınır değer problemlerinin çözümüne yardımcı olan gerekli araçlar da tanıtılmıştır.

Üçüncü bölüm bir boyutlu durumda, kompleks uzayın alt bölgesi olan $\mathbb{D} := \{z : z < 1\}$ de, kompleks değişkenli Dirichlet probleminin incelenmesine ayrılmıştır.

Dördüncü bölümde iki boyutlu durumda, $\mathbb{D}^2 := \mathbb{D}_1 \times \mathbb{D}_2 = \{z = (z_1, z_2) : |z_k| < 1, k = 1, 2\}$ 'de kompleks değişkenli Dirichlet problemi ele alınmıştır.

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1. INTRODUCTION

... "Mathematics is the science where everything is evident."
Kronecker

The theory of complex partial differential equations and its applications is an effective branch in mathematics which has grown extremely. Complex partial differential equations have been a major research area in magnetism, shell theory, medical imaging, electricity, etc. Boundary value problems for complex partial differential equations combines knowledges and methods from many fields of mathematics, for examples, complex analysis, functional analysis, partial differential equations, equations of mathematical physics, computational mathematics etc. The theory of complex boundary value problems started with the work of [15] and [16], after that developed by I. N. Vekua [7], F.D. Gakhov [3], W. Wendland [17].

Some results are achieved which very quickly enhance the development of generalized analytic functions so complex boundary value problems, reference to [18, 19, 3, 4, 7, 20, 21, 22, 23].

The most important aim of the theory of complex boundary value problems is to derive solutions in closed or analytic form. The investigation of boundary value problems for analytic functions has relation to many branches. Analytic functions are related with the Cauchy-Riemann operator $\partial_{\bar{z}}$. So one aspect is to investigate complex boundary value problems for different type of functions, e.g. functions with several complex variables, generalized analytic functions, functions satisfying the Cauchy- Riemann equation, functions satisfying the Poisson equation, functions satisfying the Beltrami equation, reference to [18, 7, 27].

Also some results in this direction i.e. to derive solutions are obtained for special kinds of equations, namely, for the Cauchy- Riemann equation, for the Beltrami equation, for elliptic equations, see [18], [24], [25], [3], [26], [4], [23], [18]. Additionally, great interest has arisen for polyharmonic and for polyanalytic equations, see [28] ,[29], [30], [31], [32].

Along with it, various kinds of conditions imposed on the boundary conditions lead to different complex boundary value problems, for instance the Riemann-Hilbert problem, the Schwarz problem, the Dirichlet problem see [33, 34, 35, 37].

Moreover, besides the study in the classical unit disc, they have been considered for some particular domains, for example, fan-shaped domain, a half unit disc, a triangle, the upper half plane, a quarter plane, a circular ring, etc. [36, 38, 39, 40, 41, 42, 45, 55, 47, 48, 49, 50, 51, 52, 53, 54].

On the other hand, boundary value problems have extended to higher dimensional spaces as a polydisc, a sphere, see [43], [44].

The complex boundary value problems of arbitrary order related with integral representation formulas. The relation was investigated by H. Begehr, firstly. Essential goal of the theory of complex boundary value problem is to find solutions to them under the appropriate conditions. The fundamental tools in this direction are the Gauss Theorem and Cauchy-Pompeiu representation formula. Also, the higher order Cauchy-Pompeiu operators $T_{m,n}$ introduced by H. Begehr and G. Hile, help to obtain solution for higher order complex partial differential equations.

Complex form Cauchy-Pompeiu representation formula is a generalization of the Cauchy formula for analytic functions. The area integral arising in that formula is called the Pompeiu operator which has an important role in treating complex partial differential equations. It is studied extensively by I.N. Vekua [7]. If $f \in L_p(D;\mathbb{C})$, $p > 1$, then $\partial_{\bar{z}}(Tf) = f$ and $\partial_z(Tf) =: \Pi f$.

The operator Πf is a strongly singular integral operator. From the Cauchy-Pompeiu representation formula it follows that for any $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$ can be attained by known values on the boundary and values of the first order derivative inside of the domain. Besides this, for $f \in L_p(D;\mathbb{C}), p > 1, \gamma \in C(\partial D;\mathbb{C}), w(z)$ can be constructed through

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}.$$

By the properties of the Pompeiu operator, w is derived as a solution to the differential equation $w_{\bar{z}} = f \quad \text{in} \quad D$. This idea is originated from I.N.Vekua and he proposed an idea to represent the solution of these type of problems in the form $w = \psi + Tf$, where ψ is an analytic function [7].

The model equations are basic inhomogeneous partial differential equations. The differential operator is obtained by the product of powers of $\partial_{\bar{z}}$ and some powers of its complex conjugate ∂_z. Some fundamental operators are the Laplace operator $\partial_{\bar{z}}\partial_z$, the Bitsadze operator $\partial_{\bar{z}}\partial_{\bar{z}}$, complex conjugate of the Bitsadze operator $\partial_z\partial_z$.

Since, the results for the model equations are valid for more general equations of arbitrary order with some boundary conditions, the main goal of the investigations on model equations is to extend them to arbitrary order.

In this study, we are mainly interested in the Dirichlet problem of arbitrary order for model and linear equations, with complex variables. The Dirichlet problem is reduced to a singular integral equation by the idea of I.N. Vekua [7] for generalized analytic functions. In this direction the Pompeiu operator Tf and its weak derivative Πf are used. By the integral operator $T_{m,n}$, [8] which is obtained by the iteration of the Pompeiu operator and its conjugate operator is the main tool for reducing the higher order complex partial differential equations to a singular integral equation.

The theory of boundary-value problems for complex partial differential equations is far from being complete. Besides the working group at Free University, Berlin (Germany) there are working groups in Istanbul and Ankara (Turkey), Astana and Almaty (Kazakhstan), Caracas (Venesolana), Delhi (India), Yerevan (Armenia), Minsk (Belarus), Dushanbe (Tajikistan), Tbilisi(Georgia) involved in the reseach.

In this thesis the results of the Dirichlet boundary value problem are for homogeneous and inhomogeneous complex partial differential equations are collected and analyzed.

This study consists of four chapters. In Chapter 1 and Chapter 2, we give historical background of the Dirichlet problem and some literature search about this problem. Some basic definitions and theorems from functional analysis and some technical preliminaries are presented. Moreover, we give some introduction in complex methods for partial differential equations and the necessary tools that serve to solve higher order boundary value problems with complex variables.

Chapter 3 is devoted to the investigation of the Dirichlet problem for the one dimensional partial differential equations with complex variable in the unit disc $\mathbb{D} := \{z : z < 1\}$ of the complex plane. Firstly, we start with the Dirichlet problem for first order complex model partial differential equations [2]. An alternative proof is given for the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation by reducing the problem to the one for analytic function. Then the Dirichlet problem is considered for a general complex partial differential equation. It is reduced into a singular integral equation. Then the Dirichlet problem for second order complex model partial differential equations are treated [2], [56].

The Dirichlet problem for the Laplace operator in $\mathbb{D}$,

$$w_{z\bar{z}} = 0 \quad \text{in} \quad \mathbb{D}, \quad w = 0 \quad \text{on} \quad \partial\mathbb{D},$$

has only trivial solution. On the other hand, according to Bitsadze [5], the Dirichlet problem for the Bitzadze operator

$$w_{\bar{z}\bar{z}} = 0 \quad \text{in} \quad \mathbb{D}, \quad w = 0 \quad \text{on} \quad \partial\mathbb{D},$$

has infinitely many linearly independent solutions. We showed that the Dirichlet problem for the complex conjugate of the Bitzadze operator has infinitely many linearly independent solutions, too [56].

Later, we consider the Dirichlet problem for higher order complex model partial differential equations [56]. It is seen that these type of problems have infinitely many linearly independent solutions. By taking additional boundary conditions we can make them well-posed problems, [56].

In Chapter 4, we carry these ideas in Chapter 3 to bidisc $\mathbb{D}^2 := \mathbb{D}_1 \times \mathbb{D}_2 = \{z = (z_1, z_2) : |z_k| < 1, k = 1, 2\}$. Firstly, we derive the solution of the Dirichlet problem of second order model partial differential equations by using the main results in [2]. Then, we extend the boundary value problem to a general linear differential equation. Under suitable solvability conditions, it is seen that the boundary value problem has a unique solution. To reach that conclusion, the problem is reduced into a singular integral equation. After that, solvability of the singular integral equation is investigated. For this investigation we mainly use the Fredholm theory.

Then we derive solution of the Dirichlet problem of fourth order model partial differential equations using the results that we have obtained for second order partial differential equations.

2. PRELIMINARIES

2.1. BASIC DEFINITIONS AND THEOREMS FROM FUNCTIONAL ANALYSIS

Definition 2.1.1. *[12] Let* $(E, \|.\|_E)$ *be a normed space. A function* $f : [a,b] \to E$ *is called Hölder continuous of order* $\alpha \in (0,1]$ *if there are nonnegative real constants* $c = c(f)$ *such that*

$$\|f(t) - f(s)\|_E \leq c \mid t - s \mid^{\alpha} \tag{2.1}$$

for all $s, t \in [a,b]$*. If* $\alpha = 1$*, then the function f satisfies a Lipschitz condition. Therefore, every Hölder continuous function of order* $\alpha = 1$ *is Lipschitz continuous.*

Theorem 2.1.2. *Every Lipschitz continuous function is uniformly continuous.*

Theorem 2.1.3. *[13] Let X be a Banach space and* $S \in B(X)$*. If* $\|S\| < 1$*, then* $I - S$ *is invertible. Moreover,* $\|(I - S)^{-1}\| \leq (1 - \|S\|)^{-1}$.

Definition 2.1.4. *[14] Let X, Y be Banach spaces over the same field. Let* $B(X,Y)$ *denote the set of all bounded operators from X to Y.* $S \in B(X,Y)$ *is said to be a Fredholm operator if* $kerS := S^{-1}(0)$ *and* $cokerS := Y \diagup imS$ *are finite-dimensional; the following quantities, called nullity and deficiency of S, are finite:*

$$\alpha(S) := nulS := dimkerS; \quad \beta(S) := defS := dimcokerS. \tag{2.2}$$

$indS := \alpha(S) - \beta(S) \in \mathbb{Z}$ *is Fredholm index of S.*

Theorem 2.1.5 (Fredholm Alternative)**.** *[14] For an index-zero Fredholm operator S either of the following mutually exclusive events takes place:*

(1) The equation $Sx = 0$ *has only the zero solution. The equation* $Sx = y$ *has a unique solution given an arbitrary right hand side.*

(2) *The equation* $Sx = 0$ *has a nonzero solution. The equation* $Sx = 0$ *has finitely many linearly independent solutions.*

Theorem 2.1.6 (Fredholm Theorem)**.** *[14] Let* $K \in K(X)$*, where* $K(X)$ *is the set of all compact operators on* X*. Then* $I - K$ *is an index-zero Fredholm operator.*

Definition 2.1.7. *[14] Let* $S \in B(X,Y)$*, an operator* $T \in B(Y,X)$ *is a left approximate inverse of* S *if* $TS - I \in K(X)$*.* $R \in B(Y,X)$*, is a right approximate inverse of* S *if* $AR - I \in K(Y)$*. An operator* $U \in B(Y,X)$ *is an approximate inverse of* S *if* U *is both right and left approximate inverse of* S*. An operator has an approximate inverse is called approximately invertible.*

Theorem 2.1.8 (Noether Criterion)**.** *[14] An operator is a Fredholm operator if and only if it is approximately invertible.*

Theorem 2.1.9 (Nikolskii Criterion)**.** *[14] An operator is index-zero Fredholm operator if and only if it is the sum of an invertible operator and a compact operator.*

2.2. NOTATIONS AND TECHNICAL PRELIMINARIES

Let $\mathbb{C}$ be the complex plane of the variable $z = x + iy$ and $x, y \in \mathbb{R}$, x is called the real part of z and is denoted by Rez, y is called the imaginary part of z and is denoted by Imz. The complex number $z = x - iy$ is called the conjugate of z. The extended complex plane is denoted by $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

A domain D in $\mathbb{C}$ is called regular if it is bounded and its boundary ∂D is being a smooth curve.

The complex partial differential operators of first order are defined by

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Let the complex-valued function $w(z,\bar{z}) = u + iv$ be defined in $D \in \mathbb{C}$ and let $u = u(x,y)$, $v = v(x,y)$ are real-valued functions. When u and v are differentiable and w is independent of $\bar{z}$ in an open set $\mathbb{C}$, the function w is called analytic in the set. The functions u, v then satisfy the Cauchy-Riemann system of two partial differential equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v, \tag{2.3}$$

which is equivalent to the complex homogeneous Cauchy-Riemann equation

$$\partial_{\bar{z}} w = 0.$$

Theorem 2.2.1. *[6] Let* w *be an analytic function in a simply connected domain* $D \subset \mathbb{C}$ *and let* Γ *be a simple closed curve,* $\Gamma \subset D$*. Then*

$$\int_{\Gamma} w(z)dz = 0.$$

The fundamental tools for solving boundary value problems for complex partial differential equations are the Gauss Theorem and the Cauchy-Pompeiu representation.

Theorem 2.2.2 (Gauss Theorem, complex form)**.** *[6] Let* $D \subset \mathbb{C}$ *be a regular domain, and* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C}), \quad z = x + iy$*, then*

$$\int_{D} w_{\bar{z}}(z)dxdy = \frac{1}{2i}\iint_{\partial D} w(z)dz$$

and

$$\int_{D} w_{z}(z)dxdy = -\frac{1}{2i}\iint_{\partial D} w(z)d\bar{z}.$$

From the Gauss Theorem the Cauchy-Pompeiu representation formulas can be obtained.

Theorem 2.2.3 (Cauchy-Pompeiu representations)**.** *[6] Let* $D \subset \mathbb{C}$ *be a regular domain,* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C}), \quad \zeta = \xi + i\eta$*, then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_{D} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\iint_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}$$

hold for all $z \in D$.

Let us define the integral operator, which is used to solve boundary value problems for the inhomogeneous Cauchy-Riemann equation.

Definition 2.2.4. *[7] For* $f \in L_1(D;\mathbb{C})$ *the integral operator*

$$Tf(z) = -\frac{1}{\pi}\iint_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C},$$

is called Pompeiu operator. By $\overline{T}$ *we denote*

$$\overline{T}f(z) = -\frac{1}{\pi}\iint_D f(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}, \quad z \in \mathbb{C}.$$

The Pompeiu operator is studied in detail by I.N.Vekua [7] plays a critical role in treating the boundary value problems for the Cauchy-Riemann equation. It has some important properties listed below.

Theorem 2.2.5. *[6] Let* $D \subset \mathbb{C}$ *be a bounded domain. If* $f \in L_1(D;\mathbb{C})$ *then* Tf *is analytic in* $\mathbb{C} \setminus \overline{D}$*, vanishing at infinity.*

Theorem 2.2.6. *[6] If* $f \in L_1(D;\mathbb{C})$ *then*

$$\iint_D Tf(z)\varphi_{\bar{z}}(z)dxdy + \iint_D f(z)\varphi(z)dxdy = 0,$$

where φ *is an arbitrary complex-valued function in* D *being continuously differentiable and having compact support in* D.

The Theorem 2.2.6 implies that for $f \in L_1(D)$, Tf is differentiable with respect to $\bar{z}$ in weak sense with

$$\partial_{\bar{z}}Tf = f \quad \text{in} \quad D. \tag{2.4}$$

Theorem 2.2.7. *[11] If* $f \in L_p(D), p > 1$, *then* Tf *has generalized first order derivative with respect to* z *equal to*

$$\Pi f(z) := -\frac{1}{\pi}\iint_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}, \quad z \in \mathbb{C}. \tag{2.5}$$

It is a singular integral operator being understood in the Cauchy principal sense,

$$\Pi f(z) := \lim_{\epsilon \to 0} -\frac{1}{\pi}\iint_{D_\epsilon} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}, \quad z \in \mathbb{C},$$

where D_ϵ is the domain $D \setminus \{\zeta : |\zeta - z| \leqslant \epsilon\}$. They can be analyzed with the theory of Calderon-Zygmund [9].

Pompeiu Integral Operators of Higher Order

By iteration of the Pompeiu operator T, and its conjugate $\overline{T}$, a hierarchy of kernel functions and higher order integral operators are constructed. Then, related to the integral operators, Cauchy-Pompeiu representation formulas are derived.

Definition 2.2.8. *[10] Let* $m, n \in \mathbb{Z}$, $0 \leq m + n$ *and* $0 < m^2 + n^2$,

$$K_{m,n}(z) = \begin{cases} \dfrac{(-m)!\,(-1)^m}{(n-1)!\,\pi} z^{m-1}\bar{z}^{n-1} & \text{if} \quad m \leq 0 \\[2ex] \dfrac{(-n)!\,(-1)^n}{(m-1)!\,\pi} z^{m-1}\bar{z}^{n-1} & \text{if} \quad n \leq 0 \\[2ex] \dfrac{z^{m-1}\bar{z}^{n-1}}{(m-1)!\,(n-1)!\,\pi}\left(\log|z|^2 - \displaystyle\sum_{r=1}^{m-1}\frac{1}{r} - \sum_{s=1}^{n-1}\frac{1}{s}\right) & \text{if} \quad m, n \geq 1. \end{cases}$$

For $0 \leq m^2 + n^2$, these kernel functions determine fundamental solutions to $\partial_z^m \partial_{\bar{z}}^n$. Kernel functions of the form in Definition 2.2.8 are studied in detail in [11]. Their elementary properties are

$$\partial_z K_{m,n}(z) = K_{m-1,n}(z),$$

$$\partial_{\bar{z}} K_{m,n}(z) = K_{m,n-1}(z),$$

$$K_{m,n} = \overline{K_{n,m}}.$$

Moreover, simple calculations confirm that, for radii R,

$$\iint\limits_{|z|\le R} \mid K_{m,n}(z) \mid dxdy < \infty, \quad \text{if} \quad 0 \le m+n.$$

Definition 2.2.9. *[10] For a domain $D \subset \mathbb{C}$, $f \in L_1(D;\mathbb{C})$ and $m,n \in \mathbb{Z}$ with $0 \le m+n$*

$$T_{m,n}f(z) = \iint\limits_{D} K_{m,n}(z-\zeta)f(\zeta)d\xi d\eta \quad \textit{if} \quad 1 \le m^2+n^2,$$

$$T_{0,0}f(z) = f(z).$$

For particular choices of m and n, the operators $T_{m,n}$ are

- $T_{0,1}f(z) = Tf(z)$,
- $T_{1,0}f(z) = \overline{T}f(z)$,
- $T_{-1,1}f(z) = \Pi f(z)$,
- $T_{1,-1}f(z) = \overline{\Pi}f(z)$.

We will give some essential theorems that we use in our study. The results, that are recalled here without proofs, can be found in [11].

Theorem 2.2.10. *Let D be a bounded domain, and suppose $m+n \ge 1$, and let w be a complex valued function in $L_1(D)$. Then the integral $T_{m,n,D}w(z)$ converges absolutely for almost all $z \in \mathbb{C}$. Moreover, if*

(i) $1 \le p < 2$, *when* $m+n=1$,

(ii) $1 \le p \le \infty$, *when* $m+n=2$, $mn \le 0$,

(iii) $1 \le p < \infty$, *when* $m+n=2$, $mn > 0$,

(iv) $1 \le p \le \infty$, *when* $m+n \ge 3$,

then for any bounded domain Ω, $T_{m,n,D}w(z) \in L_p(\Omega)$ with

$$\|T_{m,n,D}w(z)\|_{p,\Omega} \le M(m,n,p,D,\Omega)\|w(z)\|_{1,D}. \tag{2.6}$$

Theorem 2.2.11. *Let D be a bounded domain, and suppose* $m + n \geq 1$, *and let* $w \in L_p(D)$ *where*

(i) $2 < p \leq \infty$, *when* $m + n = 1$,

(ii) $1 \leq p \leq \infty$, *when* $m + n = 2$, $mn \leq 0$,

(iii) $1 < p < \infty$, *when* $m + n = 2$, $mn > 0$,

(iv) $1 \leq p \leq \infty$, *when* $m + n \geq 3$.

Then $T_{m,n,D}w$ *exists as a Lebesgue integral for all z in* $\mathbb{C}$, $T_{m,n,D}w$ *is continuous in* $\mathbb{C}$, *and for* $|z| \leq R$ *with* $R > 0$,

$$|T_{m,n,D}w(z)| \leq M\|w(z)\|_{p,D}, \tag{2.7}$$

where $M = M(m,n,p,D)$ *in cases (i) and (ii),* $M = M(m,n,p,D,R)$ *in (iii) and (iv).*

We will discuss Hölder continuity of the integral $T_{m,n,D}w$.

Theorem 2.2.12. *Suppose* $m + n \geq 1$ *and* $mn \leq 0$, *let D be a bounded domain in* $\mathbb{C}$, *and assume that w is a complex valued function in* $L_p(D)$; *suppose also that*

(i) $2 < p < \infty$, *if* $m + n = 1$,

(ii) $2 < p \leq \infty$, *if* $m + n = 2$,

(iii) $1 \leq p < \infty$, *if* $m + n = 3$,

(iv) $1 \leq p \leq \infty$, *if* $m + n \geq 4$.

For $z \in \mathbb{C}$, *set*

$$v(z) := T_{m,n,D}w(z) = \iint_D K_{m,n}(z-\zeta)w(\zeta)d\xi d\eta.$$

Then, for $z_1, z_2 \in \mathbb{C}$, *say with* $|z_1|, |z_2| < R$,

$$|v(z_1) - v(z_2)| \leq M\|w\|_{p,D} \begin{cases} |z_1 - z_2| & \text{if } \; m+n \geq 0, \\ \\ |z_1 - z_2|^{\frac{(p-2)}{p}} & \text{if } \; m+n = 1, \end{cases}$$

where $M = M(m,n,p)$ *in case (i),* $M = M(m,n,p,D)$ *in cases (ii) and (iii),and* $M = M(m,n,p,D,R)$ *in case (iv).*

The following theorem is about strongly singular integral operators.

Theorem 2.2.13. *Assume* $m + n = 0$, $(m,n) \neq (0,0)$, *and let* w *be a complex valued function in* $L_p(D)$, *and*

$$\|T_{m,n,D}w\|_{p,D} \leq M(p)\|w\|_{p,D}. \tag{2.8}$$

Lemma 2.2.14. *Let* ρ *be a complex valued function and for* $\rho \in C_0^\infty(\mathbb{C})$,

$$\overline{T_{m,n,}\rho} = T_{n,m}\overline{\rho}, \quad \textit{if} \quad m+n \geq 0, \tag{2.9}$$

$$T_{m,n,}\rho = T_{m+1,n}\rho_z = T_{m,n+1}\rho_{\bar{z}}, \quad \textit{if} \quad m+n \geq 0, \tag{2.10}$$

$$\partial_z(T_{m,n}\rho) = T_{m-1,n}\rho, \quad \textit{if} \quad m+n \geq 1, \tag{2.11}$$

$$\partial_{\bar{z}}(T_{m,n}\rho) = T_{m,n-1}\rho, \quad \textit{if} \quad m+n \geq 1, \tag{2.12}$$

$$\partial_z(T_{m,n}\rho) = T_{m,n}\rho_z, \quad \textit{if} \quad m+n \geq 0, \tag{2.13}$$

$$\partial_{\bar{z}}(T_{m,n}\rho) = T_{m,n}\rho_{\bar{z}}, \quad \textit{if} \quad m+n \geq 0. \tag{2.14}$$

Note that iterating (2.13) and (2.14), we obtain that $T_{m,n,}\rho \in C^\infty(\mathbb{C})$ and satisfying

$$\frac{\partial^{k+l}}{\partial_z^k \partial_{\bar{z}}^l} T_{m,n}\rho = T_{m,n}\left(\frac{\partial^{k+l}\rho}{\partial_z^k \partial_{\bar{z}}^l}\right). \tag{2.15}$$

3. DIRICHLET PROBLEM FOR THE ONE-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH COMPLEX VARIABLE

In the present chapter, we consider the Dirichlet problem for arbitrary order partial differential equations with complex variable in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane. In the case of the unit disc Dirichlet problem and also other boundary value problems can be solved certainly. Because of this, for the unit disc the Pompeiu operator is known explicitly [1]. For that reason, in our study this particular domain is considered.

3.1. DIRICHLET PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section, we investigate the Dirichlet problem for first order partial differential equations with complex variable. We begin with the following theorem of Begehr [2] about the analytic functions.

Dirichlet problem for analytic functions. Find an analytic function $w(z)$ in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying for given $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$

$$w(z) = \gamma(z) \text{ on } \partial\mathbb{D}.$$

Firstly, we present Plemelj-Sokhotzki formula which is useful for our study in the future.

Theorem 3.1.1 (Plemelj-Sokhotzki). *[3] Let* Γ *be a smooth contour (open or closed) and* $\varphi(\zeta)$ *a function of position on the contour, which satisfies the Hölder condition. Then the Cauchy type integral*

$$\Phi(z) := \frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) \frac{d\zeta}{\zeta - z}$$

has the limiting values

$$\Phi^{+}(\tau) := \lim_{z \to \tau, z \in D^{+}} \Phi(z), \Phi^{-}(\tau) := \lim_{z \to \tau, z \in D^{-}} \Phi(z)$$

at all points of the contour Γ, *where* D^+ *is the bounded domain with* $\partial D^+ = \Gamma$ *and* $D^- = \widehat{\mathbb{C}}\setminus (D^+ \cup \Gamma)$. *Moreover, for* $\tau \in \Gamma$

$$\Phi^+(\tau) = \frac{1}{2}\varphi(\zeta) + \Phi(\tau),$$

$$\Phi^-(\tau) = \frac{-1}{2}\varphi(\zeta) + \Phi(\tau),$$

the singular integral

$$\int_\Gamma \varphi(\zeta)\frac{d\zeta}{\zeta - z}$$

being understood in the sense of the principal value.

Theorem 3.1.2. *[2] The Dirichlet problem for analytic functions is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\bar{z}d\zeta}{1 - \bar{z}\zeta} = 0\cdot \tag{3.1}$$

The solution then is uniquely given by the Cauchy integral

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta - z}\cdot \tag{3.2}$$

Since the Cauchy integral (3.2) provides an analytic function in $\mathbb{D}$ and in $\widehat{\mathbb{C}}\setminus \mathbb{D}$, from Plemelj-Sokhotzki formula it follows

$$\lim_{z\to\zeta, z\in\mathbb{D}^+} w(z) - \lim_{z\to\zeta, z\in\mathbb{D}^-} w(z) = \gamma(\zeta), \quad \zeta \in \partial\mathbb{D}.$$

To obtain

$$\lim_{z\to\zeta, z\in\mathbb{D}^+} w(z) = \gamma(\zeta), \quad \zeta \in \partial\mathbb{D},$$

the condition

$$\lim_{z\to\zeta, z\in\mathbb{D}^-} w(z) = 0$$

is necessary and sufficient.

Note. *The Plemelj-Sokhotzki formula holds if* $\gamma(\zeta)$ *is Hölder continuous but from the topological properties of the unit disc Hölder continuity is not needed [4].*

Necessary part. Let the Dirichlet problem can be solvable and $w(z)$ is the solution of this problem. From the statement $w(z)$ is analytic and having continuous boundary values

$$\lim_{z\to\zeta} w(z) = \gamma(\zeta), \quad \zeta \in \partial\mathbb{D}.$$

For $|z| > 1$, we consider the function

$$w\left(\frac{1}{\bar{z}}\right) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta - \frac{1}{\bar{z}}} = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\bar{z}}{\bar{\zeta} - \bar{z}}\frac{d\zeta}{\zeta}\cdot \tag{3.3}$$

For $|z| > 1$, we have that

$$\lim_{z\to\zeta}\frac{1}{\bar{z}} = \frac{1}{\bar{\zeta}} = \zeta, \quad \zeta \in \partial\mathbb{D}.$$

Thus,

$$\lim_{z\to\zeta} w\left(\frac{1}{\bar{z}}\right)$$

exists, i.e.

$$\lim_{z\to\zeta} w(z)$$

exists for $|z| > 1$. Applying formula (3.3), we get

$$w(z) - w\left(\frac{1}{\bar{z}}\right) = \frac{\gamma(z)}{2\pi i}\int_{\partial\mathbb{D}}\left\{\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} - 1\right\}\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\{\gamma(\zeta) - \gamma(z)\}\left\{\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} - 1\right\}\frac{d\zeta}{\zeta}.$$

Since $\gamma(z)$ is continuous, we may take the limit

$$\lim_{z\to\zeta}\left\{\gamma(z) + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\{\gamma(\zeta) - \gamma(z)\}\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\{\gamma(\zeta) - \gamma(z)\}\frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}}\frac{d\zeta}{\zeta}\right\}$$

$$+\lim_{z\to\zeta}\left\{\frac{-1}{2\pi i}\int_{\partial\mathbb{D}}\{\gamma(\zeta) - \gamma(z)\}\frac{d\zeta}{\zeta}\right\} \equiv \gamma(\zeta)\cdot$$

This is equivalent to

$$\lim_{z\to\zeta, z\in\mathbb{D}^+}\left\{w(z) - w\left(\frac{1}{\bar{z}}\right)\right\} \equiv \gamma(\zeta)\cdot$$

Therefore,

$$\lim_{z\to\zeta, z\in\mathbb{D}^+} w(z) - \lim_{z\to\zeta, z\in\mathbb{D}^-} w(z) \equiv \gamma(\zeta).$$

Since

$$\lim_{z\to\zeta, z\in\mathbb{D}^+} w(z) \equiv \gamma(\zeta),$$

we obtain

$$\lim_{z\to\zeta, z\in\mathbb{D}^-} w(z) = \lim_{z\to\zeta, z\in\mathbb{D}^+} w\left(\frac{1}{\bar{z}}\right) = 0\cdot$$

As $w(\infty) = 0$, applying the maximum principle for analytic functions, we get $w(z) \equiv 0$ for $|z| > 1$. That means that

$$w\left(\frac{1}{\bar{z}}\right) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} = 0.$$

Sufficient part. Let formula (3.1) holds. Adding (3.1) to (3.2), we obtain

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\left\{\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta}-\bar{z}} - 1\right\}\frac{d\zeta}{\zeta}\cdot$$

In a similar manner in previous part, we can write

$$\lim_{z\to\zeta, z\in\mathbb{D}^+} w(z) = \gamma(\zeta)\cdot$$

It can be easily seen that $w_{\bar{z}} = 0$ in $\mathbb{D}$.

Therefore the Dirichlet problem

$$w_{\bar{z}} = 0 \quad \text{in } \mathbb{D}, \quad w = 0 \text{ on } \partial\mathbb{D}$$

for a first-order homogeneous partial differential equation has only trivial solution.

Theorem 3.1.3. *[2] The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \quad in \quad \mathbb{D}, \quad w = \gamma \quad on \quad \partial\mathbb{D}$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} = \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\bar{z}d\xi d\eta}{1-\bar{z}\zeta}\cdot \tag{3.4}$$

The solution then is uniquely given by

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}\cdot \tag{3.5}$$

Firstly, we give the proof of the Theorem from [2].

The representation (3.4) follows from Theorem 2.2.3 if problem is solvable. The uniqueness is a consequence of Theorem 3.1.2. The equation (3.5) is a solution under (3.4) follows by observing the properties of the Pompeiu operator and from

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\left\{\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} - 1\right\}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\left\{\frac{1}{\zeta - z} + \frac{\bar{z}}{1 - \bar{z}\zeta}\right\}d\xi d\eta = \gamma(z) \tag{3.6}$$

for $|z| = 1$.

The equation (3.4) is also necessary follows from Theorem 3.1.2. Applying (3.1) to the boundary value of the analytic function $w - Tf$ in $\mathbb{D}$, i.e. to $\gamma - Tf$ on $\partial\mathbb{D}$ gives (3.4) because of

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{1}{\pi}\iint_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{d\tilde{\xi}d\tilde{\eta}}{\tilde{\zeta} - \zeta}\frac{\bar{z}}{1 - \bar{z}\zeta}d\zeta$$

$$= -\frac{1}{\pi}\iint_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\bar{z}}{1 - \bar{z}\zeta}\frac{d\zeta}{\zeta - \tilde{\zeta}}d\tilde{\xi}d\tilde{\eta}$$

$$= -\frac{1}{\pi}\iint_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{\bar{z}}{1 - \bar{z}\tilde{\zeta}}d\tilde{\xi}d\tilde{\eta}$$

as is seen from the Cauchy formula.

Proof. Using the Pompeiu operator the problem will be reduced to the ones for analytic function. Then, this problem can be defined as

$$(w - Tf)_{\bar{z}} = f \quad \text{in} \quad \mathbb{D}, \quad (w - Tf)(z) = \gamma - Tf \quad \text{on} \quad \partial\mathbb{D}.$$

That problem is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma(\zeta) - Tf(\zeta))\frac{\bar{z}d\zeta}{1 - \bar{z}\zeta} = 0$$

and the solution is uniquely obtained as

$$(w - Tf)(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma(\zeta) - Tf(\zeta))\frac{d\zeta}{\zeta - z}. \tag{3.7}$$

From Gauss Theorem it follows

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} Tf(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} = \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\bar{z}d\xi d\eta}{1-\bar{z}\zeta} \tag{3.8}$$

and from the Cauchy Pompeiu representation it follows

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} Tf(\zeta)\frac{d\zeta}{\zeta - z} = -Tf(z) - \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{d\xi d\eta}{\zeta - z} = 0. \tag{3.9}$$

□

In a similar manner we can study the Dirichlet problem

$$w_z = 0 \text{ in } \mathbb{D}, \quad w = 0 \text{ on } \quad \partial\mathbb{D} \tag{3.10}$$

for a first order model complex partial differential equation.

Theorem 3.1.4. *[2] The Dirichlet problem for the inhomogeneous complex conjugate Cauchy-Riemann equation in the unit disc*

$$w_z = f \textit{ in } \mathbb{D}, \quad w = \gamma(z) \textit{ on } \partial\mathbb{D}$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z| < 1$

$$-\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{zd\bar{\zeta}}{1-z\bar{\zeta}} = \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{zd\xi d\eta}{1-z\bar{\zeta}}. \tag{3.11}$$

The solution then is uniquely given by the formula

$$w(z) = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}. \tag{3.12}$$

Note that (3.11) and (3.12) can be attained by the one for the Theorem 3.1.3 through complex conjugation and using the fact that

$$\overline{\left(\frac{\partial w}{\partial \bar{z}}\right)} = \frac{\partial \bar{w}}{\partial z}.$$

Hence the Dirichlet problem

$$w_z = 0 \text{ in } \mathbb{D}, \quad w = 0 \text{ on } \partial\mathbb{D} \tag{3.13}$$

for first-order homogeneous complex conjugate Cauchy Riemann equation in $\mathbb{D}$ has only trivial solution.

3.1.1 Dirichlet Problem For Linear Differential Equations

In this section we will extend the notion of solution of Dirichlet Problem for model equations to a linear differential equation. We will try to find a solution to first order linear differential equation

$$w_{\bar{z}} + q_1(z)w_z + q_2(z)\overline{w_z} + r_1(z)w + r_2(z)\bar{w} = f(z) \text{ in } \mathbb{D} \tag{3.14}$$

satisfying the Dirichlet boundary condition

$$w(z) = \gamma(z) \text{ on } \partial\mathbb{D} \tag{3.15}$$

where $r_1(z), r_2(z), f(z) \in L_p(\mathbb{D}), q_1(z), q_2(z), r_1(z), r_2(z)$ are measurable bounded functions and satisfying

$$| q_1(z) | + | q_2(z) | \leq q_0 < 1. \tag{3.16}$$

Lemma 3.1.5. *The Dirichlet problem (3.14) and (3.15) is equivalent to the singular integral equation*

$$(I + \widehat{\Pi} + \widehat{K})g = f(z) - q_1\psi_z - q_2(z)\overline{\psi_z} - r_1(z)\psi - r_2\overline{\psi} \tag{3.17}$$

where $w(z) = \psi(z) + T_{0,1}g(z)$,

$$\begin{aligned} \widehat{\Pi}g &= q_1 T_{-1,1}g(z) + q_2\overline{T_{-1,1}g(z)} \\ &= q_1\Pi g(z) + q_2\overline{\Pi g(z)} \end{aligned} \tag{3.18}$$

and

$$\widehat{K}g = r_1(z)T_{0,1}g(z) + r_2(z)\overline{T_{0,1}g(z)}. \tag{3.19}$$

Proof. We write (3.14) as

$$w_{\bar{z}} = f(z) - (q_1(z)w_z + q_2(z)\overline{w_z} + r_1(z)w + r_2(z)\bar{w}). \tag{3.20}$$

Then,

$$w_{\bar{z}} = \widetilde{f(z)} \quad \text{in} \quad \mathbb{D}, \tag{3.21}$$

$$w(z) = \gamma(z) \text{ on } \partial\mathbb{D}, \tag{3.22}$$

where $\widetilde{f(z)} = f(z) - (q_1(z)w_z + q_2(z)\overline{w_z} + r_1(z)w + r_2(z)\bar{w})$. By Theorem 3.1.3 solution of (3.14) and (3.15) is $w(z) = \psi(z) + T_{0,1}\widetilde{f(z)}$. Let $w_{\bar{z}} = g(z)$. Then,

$$g(z) = (\psi(z) + T_{0,1}\widetilde{f(z)})_{\bar{z}}. \tag{3.23}$$

$$g(z) + q_1(z)w_z + q_2(z)\overline{w_z} + r_1(z)w + r_2(z)\bar{w} = f(z) \text{ in } \mathbb{D}. \tag{3.24}$$

Since $w_{\bar{z}} = g(z)$ in $\mathbb{D}$,

$$w(z) = \psi(z) + T_{0,1}g(z), \bar{w} = \overline{\psi(z) + T_{0,1}g(z)}, w_z = \psi_z + T_{-1,1}g(z), \overline{w_z} = \overline{\psi_z + T_{-1,1}g(z)}.$$

After substituting these derivatives the result follows:

$$g(z) + q_1(z)T_{-1,1}g(z) + q_2(z)\overline{T_{-1,1}g(z)} + r_1(z)T_{0,1}g(z) + r_2(z)\overline{T_{0,1}g(z)}$$

$$= f(z) - q_1\psi_z - q_2(z)\overline{\psi_z} - r_1(z)\psi - r_2\overline{\psi} \tag{3.25}$$

Therefore, $g = (I + \widehat{\Pi} + \widehat{K})^{-1}(f(z) - q_1\psi_z - q_2(z)\overline{\psi_z} - r_1(z)\psi - r_2\overline{\psi})$ satisfies (3.17) if and only if $w(z) = \psi(z) + T_{0,1}g(z)$ satisfies (3.14) with the boundary condition (3.15). □

Solvability of The Singular Integral Equation

Lemma 3.1.6. *If*

$$q_0 \parallel \Pi \parallel_{L_p(\mathbb{D})} < 1 \tag{3.26}$$

for $p > 1$, *then the operator* $I + \widehat{\Pi}$ *is invertible.*

Proof. By the properties of norm, we obtain

$$\parallel \widehat{\Pi}g \parallel_{L_p(\mathbb{D})} = \parallel q_1\Pi g + q_2\overline{\Pi g(z)} \parallel_{L_p(\mathbb{D})}$$

$$\leq \parallel q_1\Pi g \parallel_{L_p(\mathbb{D})} + \parallel q_2\overline{\Pi g(z)} \parallel_{L_p(\mathbb{D})}$$

$$= | q_1(z) || \parallel \Pi g \parallel_{L_p(\mathbb{D})} + | q_2(z) || \parallel \overline{\Pi g(z)} \parallel_{L_p(\mathbb{D})}$$

$$= (| q_1(z) | + | q_2(z) |) \| \Pi g \|_{L_p(\mathbb{D})}$$

$$\leq q_0 \| \Pi g \|_{L_p(\mathbb{D})} < 1.$$

If condition $q_0 \| \Pi \|_{L_p(\mathbb{D})} < 1$ holds, then we get $\| \widehat{\Pi} \|_{L_p(\mathbb{D})} < 1$. By Theorem 2.1.3 the operator $I + \widehat{\Pi}$ is invertible. □

Lemma 3.1.7. *For bounded functions $r_1(z), r_2(z)$ and for $p > 2$, the operator $\widehat{K}$ is a compact operator.*

Proof. Firstly, we will consider the boundedness of $\widehat{K}$.

$$| \widehat{K} g | = | r_1(z) T_{0,1} g(z) + r_2(z) \overline{T_{0,1} g(z)} |$$

$$\leq | r_1(z) T_{0,1} g(z) | + | r_2(z) \overline{T_{0,1} g(z)} |$$

By Theorem 2.2.11 we get

$$| \widehat{K} g | \leq | r_1(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})} + |r_2(z)| M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})}$$

$$= (| r_1(z) | + |r_2(z)|) M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})} \leq C(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})},$$

where $M(p, \mathbb{D}), C(p, \mathbb{D})$ are always nonnegative constants, depending on the quantities in the parentheses.

By Theorem 2.2.12 we obtain

$$|T_{0,1} g(z_1) - T_{0,1} g(z_2)| \leq M(p) \| g \|_{L_p(\mathbb{D})} |z_1 - z_2|^{\alpha},$$

where $0 < \alpha < 1$, $p > 2$. It is Hölder continuous, in particular $T_{0,1} g(z)$ is uniformly continuous. So, by Arzela-Ascoli Theorem, the operators in $\widehat{K} g$ are compact operators. □

Now, we can apply the Fredholm alternative.

Theorem 3.1.8. *If $q_0 \| \Pi \|_{L_p(\mathbb{D})} < 1$ is satisfied, then (3.14) with the boundary condition (3.15) has a solution of the form $w(z) = \psi(z) + T_{0,1} g(z)$, where g is a solution of the singular integral equation (3.17) and $\psi(z)$ is an analytic function in $\mathbb{D}$.*

Proof. If (3.26) is satisfied then by Lemma 3.1.6 is $I + \widehat{\Pi}$ is invertible. In Lemma 3.1.7, we have showed that $\widehat{K}$ is compact. By Nikolskii Criterion, the operator $I + \widehat{\Pi} + \widehat{K}$ is Fredholm operator with index zero. Theorem 2.1.5 implies that the singular integral (3.17) has the Fredholm alternative, i.e. it has at least a solution. Hence, if g is a solution of (3.17), then $w(z) = \psi(z) + T_{0,1}g(z)$ is a solution to (3.14) and (3.15). □

Also, we can find the solution of (3.14) and (3.15) by denoting the solution as $w = w_1 + w_2$ where w_1 is the solution of the problem

$$w_{1\bar{z}} = 0 \quad \text{in } \mathbb{D},\ w_1 = \gamma(z) \text{ on } \partial\mathbb{D}, \tag{3.27}$$

and w_2 is a solution of the problem

$$w_{2\bar{z}} + q_1(z)w_{2z} + q_2(z)\overline{w_{2z}} + r_1(z)w_2 + r_2(z)\bar{w}_2 = f(z) - q_1(z)w_{1z} - q_2(z)\overline{w_{1z}} - r_1(z)w_1$$

$$- r_2(z)\bar{w}_1 \quad \text{in } \mathbb{D}, \tag{3.28}$$

$$w_2 = 0 \quad \text{on} \quad \partial\mathbb{D}. \tag{3.29}$$

By Theorem 3.1.2 the solution of (3.27) is

$$w_1(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} \gamma(\zeta) \frac{d\zeta}{\zeta - z} = \psi(z).$$

Then the problem (3.28) and (3.29) becomes

$$w_{2\bar{z}} = f - q_1(z)\psi_z - q_2(z)\overline{\psi_z} - r_1(z)\psi - r_2(z)\bar{\psi} - q_1(z)w_{2z} - q_2(z)\overline{w_{2z}} - r_1(z)w_2$$

$$- r_2(z)\bar{w}_2 \quad \text{in} \quad \mathbb{D} \tag{3.30}$$

$$w_2(z) = 0 \quad \text{on} \quad \partial\mathbb{D}. \tag{3.31}$$

Then,

$$w_{2\bar{z}} = \widetilde{f_2} \text{ in } \mathbb{D},\ w_2 = 0 \text{ on } \quad \partial\mathbb{D} \tag{3.32}$$

where $\widetilde{f_2} = f - q_1(z)\psi_z - q_2(z)\overline{\psi_z} - r_1(z)\psi - r_2(z)\bar{\psi} - q_1(z)w_{2z} - q_2(z)\overline{w_{2z}} - r_1(z)w_2 - r_2(z)\bar{w}_2$. By Theorem 3.1.3

$$w_2 = T_{0,1}\widetilde{f_2}.$$

By previous method we conclude that the problem (3.30) and (3.31) is equivalent to the singular integral equation

$$(I + \widehat{\Pi} + \widehat{K})g_2 = \widetilde{f_2} \tag{3.33}$$

where $w_2 = T_{0,1}g_2$.

Thus, if g_2 is a solution of the singular integral equation (3.33), then $T_{0,1}g_2$ is a solution of the problem (3.30) and (3.31). Hence $w = w_1 + w_2 = \psi(z) + T_{0,1}g_2$ is a solution of (3.14) and (3.15).

Example 3.1.9. *Let us consider the problem*

$$w_{\bar{z}} + 2\bar{z}^2 w_z + 2z\overline{w_z} + z^3 w + z\bar{z}\bar{w} = 4z^2\bar{z} + 8z\bar{z}^4 + 8z^3\bar{z} + 2z^5\bar{z}^2 + 2z^3\bar{z}^3 \text{ in } \mathbb{D} \tag{3.34}$$

$$w(z) = 2z^2\bar{z}^2 \text{ on } \partial\mathbb{D}. \tag{3.35}$$

Then, $w_1 = 2$ *is a solution of the problem*

$$w_{1\bar{z}} = 0 \text{ in } \mathbb{D} \tag{3.36}$$

$$w_1 = 2z^2\bar{z}^2 \text{ on } \partial\mathbb{D}. \tag{3.37}$$

and $w_2 = 2z^2\bar{z}^2 - 2$ *is a solution of the problem*

$$w_{2\bar{z}} + 2\bar{z}^2 w_{2z} + 2z\overline{w_{2z}} + z^3 w_2 + z\bar{z}\bar{w}_2 = 4z^2\bar{z} + 8z\bar{z}^4 + 8z^3\bar{z} + 2z^5\bar{z}^2 + 2z^3\bar{z}^3 - 2z^3$$

$$- 2z\bar{z} \text{ in } \mathbb{D}, \tag{3.38}$$

$$w_2 = 0 \text{ on } \partial\mathbb{D}. \tag{3.39}$$

So, solution of the problem (3.34), (3.35) is $w(z) = w_1 + w_2 = 2z^2\bar{z}^2$.

Now, we consider the linear differential equation

$$w_z + q_1(z)w_{\bar{z}} + q_2(z)\overline{w_{\bar{z}}} + r_1(z)w + r_2(z)\bar{w} = f(z) \text{ in } \mathbb{D} \tag{3.40}$$

satisfying the Dirichlet boundary condition

$$w(z) = \gamma(z) \quad \text{on } \partial\mathbb{D}, \tag{3.41}$$

where $r_1(z), r_2(z), f(z) \in L_p(\mathbb{D}), q_1(z), q_2(z), r_1(z), r_2(z)$ are measurable bounded functions and satisfying

$$| q_1(z) | + | q_2(z) | \leq q_0 < 1. \tag{3.42}$$

Lemma 3.1.10. *The Dirichlet problem (3.40) and (3.41) is equivalent to the singular integral equation*

$$(I + \widehat{\Pi} + \widehat{K})g = f(z) - q_1\psi_{\bar{z}} - q_2(z)\overline{\psi_{\bar{z}}} - r_1(z)\psi - r_2\overline{\psi} \tag{3.43}$$

where $w(z) = \overline{\psi(z)} + T_{1,0}g(z)$,

$$\widehat{\Pi}g = q_1 T_{1,-1}g(z) + q_2\overline{T_{1,-1}g(z)}$$

$$= q_1\overline{\Pi}g(z) + q_2\Pi\overline{g(z)} \tag{3.44}$$

and

$$\widehat{K}g = r_1(z)T_{1,0}g(z) + r_2(z)\overline{T_{1,0}g(z).} \tag{3.45}$$

Proof. We write (3.40) as

$$w_z = f(z) - (q_1(z)w_{\bar{z}} + q_2(z)\overline{w_{\bar{z}}} + r_1(z)w + r_2(z)\bar{w}). \tag{3.46}$$

Then,

$$w_z = \widetilde{f(z)} \text{ in } \mathbb{D}, \tag{3.47}$$

$$w(z) = \gamma(z) \text{ on } \partial\mathbb{D}, \tag{3.48}$$

where $\widetilde{f(z)} = f(z) - (q_1(z)w_{\bar{z}} + q_2(z)\overline{w_{\bar{z}}} + r_1(z)w + r_2(z)\bar{w})$. By Theorem 3.1.4 solution of (3.40) and (3.41) is $w(z) = \overline{\psi(z)} + T_{1,0}\widetilde{f(z)}$. Let $w_z = g(z)$. Then,

$$g(z) = (\overline{\psi(z)} + T_{1,0}\widetilde{f(z)})_z. \tag{3.49}$$

Since $w_z = g(z)$ in $\mathbb{D}$,

$$w(z) = \overline{\psi(z)} + T_{1,0}g(z), \bar{w} = \psi(z) + \overline{T_{1,0}g(z)}, w_{\bar{z}} = (\overline{\psi(z)})_{\bar{z}} + T_{1,-1}g(z), \overline{w_{\bar{z}}} = \psi(z)_z$$

$$+\overline{T_{1,-1}g(z)}.$$

After substituting these derivatives the result follows

$$g(z) + q_1(z)T_{1,-1}g(z) + q_2(z)\overline{T_{1,-1}g(z)} + r_1(z)T_{1,0}g(z) + r_2(z)\overline{T_{1,0}g(z)}$$

$$= f - q_1(z)(\overline{\psi(z)})_{\bar{z}} - q_2(z)\psi(z)_z - r_1(z)\overline{\psi(z)} - r_2(z)\psi(z). \qquad (3.50)$$

Therefore, $g = (I + \widehat{\Pi} + \widehat{K})^{-1}(f(z) - q_1\psi_{\bar{z}} - q_2(z)\overline{\psi_{\bar{z}}} - r_1(z)\psi - r_2\overline{\psi})$ satisfies (3.43) if and only if $w(z) = \overline{\psi(z)} + T_{1,0}g(z)$ satisfies (3.40) with the boundary condition (3.41). □

Solvability of The Singular Integral Equation

Lemma 3.1.11. *If*

$$q_0 \, || \, \Pi \, ||_{L_p(\mathbb{D})} < 1 \qquad (3.51)$$

for $p > 1$*, then the operator* $I + \widehat{\Pi}$ *is invertible.*

Proof. By the properties of norm we obtain

$$|| \, \widehat{\Pi}g \, ||_{L_p(\mathbb{D})} = || \, q_1\overline{\Pi}g + q_2\Pi\overline{g(z)} \, ||_{L_p(\mathbb{D})}$$

$$\leq || \, q_1\overline{\Pi}g \, ||_{L_p(\mathbb{D})} + || \, q_2\Pi\overline{g(z)} \, ||_{L_p(\mathbb{D})}$$

$$= | \, q_1(z) \, ||| \, \overline{\Pi}g \, ||_{L_p(\mathbb{D})} + | \, q_2(z) \, ||| \, \Pi\overline{g(z)} \, ||_{L_p(\mathbb{D})}$$

$$= (| \, q_1(z) \, | + | \, q_2(z) \, |) \, || \, \Pi g \, ||_{L_p(\mathbb{D})}$$

$$\leq q_0 \, || \, \Pi g \, ||_{L_p(\mathbb{D})} < 1.$$

If condition $q_0 \, || \, \Pi \, ||_{L_p(\mathbb{D})} < 1$ holds, then we get $|| \, \widehat{\Pi} \, ||_{L_p(\mathbb{D})} < 1$. By Theorem 2.1.3 the operator $I + \widehat{\Pi}$ is invertible. □

Lemma 3.1.12. *For bounded functions* $r_1(z), r_2(z)$ *and for* $p > 2$*, the operator* $\widehat{K}$ *is a compact operator.*

Proof. Firstly, we will consider the boundedness of $\widehat{K}$.

$$| \, \widehat{K}g \, | = | \, r_1(z)T_{1,0}g(z) + r_2(z)\overline{T_{1,0}g(z)} \, |$$

$$\leq | \, r_1(z)T_{1,0}g(z) \, | + | \, r_2(z)\overline{T_{1,0}g(z)} \, | \, .$$

By Theorem 2.2.11, we get

$$| \widehat{K}g | \leq | r_1(z) | M(p, \mathbb{D}) \| g\|_{L_p(\mathbb{D})} + |r_2(z)| M(p, \mathbb{D}) \|g\|_{L_p(\mathbb{D})}$$

$$= (| r_1(z) | + |r_2(z)|) M(p, \mathbb{D}) \| g\|_{L_p(\mathbb{D})} \leq C(p, \mathbb{D}) \| g\|_{L_p(\mathbb{D})},$$

where $M(p, \mathbb{D}), C(p, \mathbb{D})$ are always nonnegative constants, depending on the quantities in the parentheses.

By Theorem 2.2.12, we obtain

$$|T_{1,0}g(z_1) - T_{1,0}g(z_2)| \leq M(p) \| g\|_{L_p(\mathbb{D})} |z_1 - z_2|^{\alpha},$$

where $0 < \alpha < 1,\ p > 2$. It is Hölder continuous, in particular $T_{1,0}g(z)$ is uniformly continuous. So, by Arzela-Ascoli Theorem, the operators in $\widehat{K}g$ are compact operators. □

Now, we can apply the Fredholm alternative.

Theorem 3.1.13. *If* $q_0 \, || \, \Pi \, ||_{L_p(\mathbb{D})} < 1$ *is satisfied, then (3.40) with the boundary condition (3.41) has a solution of the form* $w(z) = \overline{\psi(z)} + T_{1,0}g(z)$*, where* g *is a solution of the singular integral equation (3.43) and* $\psi(z)$ *is an analytic function in* $\mathbb{D}$*.*

Proof. If (3.51) is satisfied then by Lemma 3.1.11 $I + \widehat{\Pi}$ is invertible. In Lemma 3.1.12, we have showed that $\widehat{K}$ is compact. By Nikolskii Criterion, the operator $I + \widehat{\Pi} + \widehat{K}$ is Fredholm operator with index zero. Theorem 2.1.5 implies that the singular integral equation (3.43) has the Fredholm alternative, i.e. it has at least a solution. Hence, if g is a solution of (3.43), then $w(z) = \overline{\psi(z)} + T_{1,0}g(z)$ is a solution to (3.40) and (3.41). □

Also we can investigate the boundary value problem by separating into the boundary value problems

$$w_{1z} = 0 \quad \text{in } \mathbb{D},$$

$$w_1 = \gamma(z) \quad \text{on} \quad \partial\mathbb{D}$$

and

$$w_{2z} + q_1(z) w_{2\bar{z}} + q_2(z) \overline{w_{2\bar{z}}} + r_1(z) w_2 + r_2(z) \bar{w}_2$$

$$= f(z) - q_1(z)w_{1\bar{z}} - q_2(z)\overline{w_{1\bar{z}}} - r_1(z)w_1 - r_2(z)\bar{w}_1 \quad \text{in } \mathbb{D},$$

$$w_{2z} = 0 \quad \text{on} \quad \partial\mathbb{D}.$$

3.2. DIRICHLET PROBLEM FOR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section, we study the Dirichlet problem for second-order partial differential equations with complex variable.
As is well-known, the basic second order differential operators are the Laplace operator $\partial_z\partial_{\bar{z}}$, the Bitsadze operator $\partial_{\bar{z}}^2$, and ∂_z^2 which is the complex conjugate of the Bitsadze operator. In the present section we will consider the Dirichlet problem for second-order homogeneous and inhomogeneous partial differential equations.

Lemma 3.2.1. *[2] The Dirichlet problem for the Laplace equation*

$$w_{z\bar{z}} = 0 \textit{ in } \mathbb{D},\ w = 0 \textit{ on } \partial\mathbb{D}$$

is only trivially solvable, i.e. $w \equiv 0$.

Therefore, the Dirichlet problem for a inhomogeneous partial differential equation

$$w_{z\bar{z}} = f \text{ in } \mathbb{D},\ \ w = \gamma \text{ on } \partial\mathbb{D}$$

has a unique solution in $\mathbb{D}$ and we have the following theorem.

Theorem 3.2.2. *[2] The Dirichlet problem for the Poisson equation in the unit disc*

$$w_{z\bar{z}} = f \textit{ in } \mathbb{D},\ w = \gamma \textit{ on } \partial\mathbb{D} \tag{3.52}$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ *is uniquely given by the formula*

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\right)d\zeta - \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)G_1(z,\zeta)d\xi d\eta,$$

where $G_1(z,\zeta) = \log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2, z,\zeta \in \mathbb{D}, \zeta \neq z.$

As Bitsadze has shown in [5], it is not take place for the Dirichlet problem for the Bitsadze equation

$$w_{\bar{z}\bar{z}} = 0 \text{ in } \mathbb{D},\ w = 0 \text{ on } \partial\mathbb{D}. \tag{3.53}$$

Lemma 3.2.3. *[2] The Dirichlet problem (3.53) has infinitely many linearly independent solutions.*

This means that the Dirichlet problem for the Bitsadze equation is ill-posed. We can make a unique solution in $\mathbb{D}$ taking additional condition $w_{\bar{z}}$ on $\partial\mathbb{D}$. We have the following theorem.

Theorem 3.2.4. *[2] The Dirichlet problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar{z}\bar{z}} = f(z) \text{ in } \mathbb{D},\ w = \gamma_0,\ w_{\bar{z}} = \gamma_1 \text{ on } \partial\mathbb{D} \tag{3.54}$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z| < 1$

$$\frac{\bar{z}}{2\pi i}\int_{\partial\mathbb{D}}\left(\frac{\gamma_0(\zeta)}{1-\bar{z}\zeta} - \frac{\gamma_1(\zeta)}{\zeta}\right)d\zeta + \frac{\bar{z}}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\overline{\zeta - z}}{1-\bar{z}\zeta}d\xi d\eta = 0$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{\bar{z}}{1-\bar{z}\zeta}d\zeta - \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\bar{z}}{1-\bar{z}\zeta}d\xi d\eta = 0.$$

The solution then is uniquely given by the formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta + \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta.$$

Finally, we consider the Dirichlet problem for the second order model partial differential equation

$$w_{zz} = 0 \text{ in } \mathbb{D},\ w = 0 \text{ on } \partial\mathbb{D}. \tag{3.55}$$

We have the following result.

Lemma 3.2.5. *The Dirichlet problem (3.55) has infinitely many linearly independent solutions.*

Proof. Integrating $w(z)$, we get $w(z) = \overline{\theta_1(z)} + z\overline{\theta_2(z)}$ where $\theta_1(z)$ and $\theta_2(z)$ are analytic functions in $\mathbb{D}$. Since $w \equiv 0$ on $\partial\mathbb{D}$, $\bar{z}\overline{\theta_1(z)} + \overline{\theta_2(z)} = 0$ on $\partial\mathbb{D}$.

$\partial_{\bar{z}}\theta_2(z) = \partial_z\overline{\theta_2(z)} = 0$ and $\overline{\theta_2(z)} = -\bar{z}\overline{\theta_1(z)}$ on $\partial\mathbb{D}$. From Theorem 3.1.4 it follows that $\overline{\theta_2(z)} = -\bar{z}\overline{\theta_1(z)}$ in $\mathbb{D}$, too. Hence $w(z) = (1-|z|^2)\overline{\theta_1(z)}$ for arbitrary analytic function $\theta_1(z)$. In particular, if we take $\overline{\theta_1(z)} = \bar{z}^k$ then $w_k(z) = (1-|z|^2)\bar{z}^k$ is a solution of the Dirichlet problem for any $k \in \mathbb{N}$. It is easily seen that these solutions are linearly independent over $\mathbb{C}$. $\square$

This means that the Dirichlet problem for the complex conjugate of the inhomogeneous Bitsadze equation is ill-posed. We can make it well-posed problem, taking additional condition w_z on $\partial\mathbb{D}$. We have the following theorem.

Theorem 3.2.6. *The Dirichlet problem for the complex conjugate of the inhomogeneous Bitsadze equation in* $\mathbb{D}$

$$w_{zz} = f(z) \ \ in\ \mathbb{D}, \ \ w = \gamma_0, w_z = \gamma_1 \ on \ \ \partial\mathbb{D} \tag{3.56}$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z| < 1$

$$-\frac{z}{2\pi i}\int_{\partial\mathbb{D}}\left(\frac{\gamma_0(\zeta)}{1-z\bar{\zeta}} - \frac{\gamma_1(\zeta)}{\bar{\zeta}}\right)d\bar{\zeta} + \frac{z}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\zeta - z}{1-z\bar{\zeta}}d\xi d\eta = 0$$

and

$$-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{z}{1-z\bar{\zeta}}d\bar{\zeta} - \frac{1}{\pi}\int_{\mathbb{D}} f(\zeta)\frac{z}{1-z\bar{\zeta}}d\xi d\eta = 0.$$

The solution then is uniquely given by the formula

$$w(z) = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{\overline{\zeta - z}}d\bar{\zeta} + \frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{\overline{\zeta - z}}{\overline{\zeta - z}}d\xi d\eta\cdot$$

Proof. The problem is equivalent to the system

$$w_z = \omega \text{ in } \mathbb{D},\ w(z) = \gamma_0 \text{ on } \partial\mathbb{D},$$

$$\omega_z = f(z) \text{ in } \mathbb{D},\ \omega(z) = \gamma_1 \text{ on } \partial\mathbb{D}.$$

These problems are uniquely solvable according to Theorem 3.1.4 if and only if

$$-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{zd\bar{\zeta}}{1-z\bar{\zeta}}=\frac{1}{\pi}\iint_{\mathbb{D}}\omega(\zeta)\frac{zd\xi d\eta}{1-z\bar{\zeta}}, \tag{3.57}$$

$$-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{zd\bar{\zeta}}{1-z\bar{\zeta}}=\frac{1}{\pi}\iint_{\mathbb{D}}f(\zeta)\frac{zd\xi d\eta}{1-z\bar{\zeta}}. \tag{3.58}$$

Then, the solutions are

$$w(z)=-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_0(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta-z}}-\frac{1}{\pi}\iint_{\mathbb{D}}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}}, \tag{3.59}$$

$$\omega(z)=-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_1(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta-z}}-\frac{1}{\pi}\iint_{\mathbb{D}}f(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}}. \tag{3.60}$$

respectively.

Inserting ω into (3.57) and (3.59), applying formulas

$$\frac{1}{\pi}\iint_{\mathbb{D}}\frac{d\xi d\eta}{(\overline{\tilde{\zeta}-\zeta})(1-z\bar{\zeta})}$$

$$=\frac{\tilde{\zeta}-z}{1-z\bar{\tilde{\zeta}}}+\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\zeta-z}{1-z\bar{\zeta}}\frac{d\bar{\zeta}}{\overline{\zeta-\tilde{\zeta}}}=\frac{\tilde{\zeta}-z}{1-z\bar{\tilde{\zeta}}},$$

$$\frac{1}{\pi}\iint_{\mathbb{D}}\frac{d\xi d\eta}{(\overline{\tilde{\zeta}-\zeta})(\overline{\zeta-z})}$$

$$=-\frac{1}{\pi}\iint_{\mathbb{D}}\frac{1}{\overline{\tilde{\zeta}-z}}\left(\frac{1}{\overline{\zeta-\tilde{\zeta}}}-\frac{1}{\overline{\zeta-z}}\right)d\xi d\eta=\frac{\tilde{\zeta}-z}{\bar{\tilde{\zeta}}-\bar{z}}.$$

So, we get the solution and solvability conditions. □

3.2.1 Dirichlet Problem for Linear Differential Equations

In this section we will extend the notion of solution of Dirichlet Problem for model equations to a linear differential equation. We will try to find a solution to second order linear differential equation

$$w_{\bar{z}\bar{z}}+q_1(z)w_{z\bar{z}}+q_2(z)\overline{w_{z\bar{z}}}+q_3(z)w_{zz}+q_4(z)\overline{w_{zz}}+r_1(z)w_{\bar{z}}$$

$$+ r_2(z)\overline{w_{\bar{z}}} + r_3(z)w_z + r_4(z)\overline{w_z} + r_5(z)w + r_6(z)\bar{w} = f(z) \text{ in } \mathbb{D} \tag{3.61}$$

satisfying the Dirichlet boundary condition

$$w(z) = \gamma_0(z),\ w_{\bar{z}} = \gamma_1(z) \text{ on } \partial\mathbb{D}, \tag{3.62}$$

where

$$r_1(z), r_2(z), r_3(z), r_4(z), r_5(z), r_6(z), f(z) \in L_p(\mathbb{D}), \tag{3.63}$$

$q_1(z), q_2(z), q_3(z), q_4(z), r_1(z), r_2(z), r_3(z), r_4(z), r_5(z), r_6(z)$ are measurable bounded functions and satisfying

$$\mid q_1(z) \mid + \mid q_2(z) \mid + \mid q_3(z) \mid + \mid q_4(z) \mid \leq q_0 < 1. \tag{3.64}$$

Lemma 3.2.7. *The Dirichlet problem (3.61) and (3.62) is equivalent to the singular integral equation*

$$(I + \widehat{\Pi} + \widehat{K})g = f(z) - q_1\psi_{1z} - q_2(z)\overline{\psi_{1z}} - q_3(z)\psi_{0zz} - q_3(z)\bar{z}\psi_{1zz} - q_4\overline{\psi_{0zz}} - q_4 z\overline{\psi_{1zz}}$$

$$-r_1(z)\psi_1 - r_2(z)\overline{\psi_1} - r_3(z)\psi_{0z} - r_3(z)\bar{z}\psi_{1z} - r_4(z)\overline{\psi_{0z}} - r_4(z)z\overline{\psi_{1z}} - r_5(z)\psi_0 - r_5(z)\bar{z}\psi_1$$

$$- r_6(z)\overline{\psi_0} - r_6(z)z\overline{\psi_1}, \tag{3.65}$$

where $w(z) = \psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}g(z)$,

$$\widehat{\Pi}g = q_1 T_{-1,1}g(z) + q_2\overline{T_{-1,1}g(z)} + q_3 T_{-2,2}g + q_4\overline{T_{-2,2}g(z)}$$

$$= q_1\Pi g(z) + q_2\overline{\Pi g(z)} + q_3\Pi_2 g + q_4\overline{\Pi_2 g} \tag{3.66}$$

and

$$\widehat{K}g = r_1(z)T_{0,1}g(z) + r_2(z)\overline{T_{0,1}g(z)} + r_3(z)T_{-1,2}g(z) + r_4(z)\overline{T_{-1,2}g(z)} + r_5(z)T_{0,2}g(z)$$

$$+ r_6(z)\overline{T_{0,2}g(z)}. \tag{3.67}$$

Proof. We write (3.61) as

$$w_{\bar{z}\bar{z}} = f(z) - q_1(z)w_{z\bar{z}} - q_2(z)\overline{w_{z\bar{z}}} - q_3(z)w_{zz} - q_4(z)\overline{w_{zz}} - r_1(z)w_{\bar{z}}$$

$$- r_2(z)\overline{w_{\bar{z}}} - r_3(z)w_z - r_4(z)\overline{w_z} - r_5(z)w - r_6(z)\bar{w}. \tag{3.68}$$

Then

$$w_{\bar{z}\bar{z}} = \widetilde{f(z)} \quad \text{in } \mathbb{D}, \tag{3.69}$$

$$w(z) = \gamma_0(z),\ w_{\bar{z}} = \gamma_1(z) \text{ on } \partial\mathbb{D}, \tag{3.70}$$

where

$$\widetilde{f(z)} = f(z) - (q_1(z)w_{z\bar{z}} + q_2(z)\overline{w_{z\bar{z}}} + q_3(z)w_{zz} + q_4(z)\overline{w_{zz}}$$

$$+r_1(z)w_{\bar{z}} + r_2(z)\overline{w_{\bar{z}}} + r_3(z)w_z + r_4(z)\overline{w_z} + r_5(z)w + r_6(z)\bar{w}).$$

By Theorem 3.2.4, solution of (3.61) and (3.62) is

$$w(z) = \psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}\widetilde{f(z)}.$$

Let $w_{\bar{z}\bar{z}} = g(z)$. Then

$$g(z) = (\psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}\widetilde{f(z)})_{\bar{z}\bar{z}}, \tag{3.71}$$

$$g(z) + q_1(z)w_{z\bar{z}} + q_2(z)\overline{w_{z\bar{z}}} + q_3(z)w_{zz} + q_4(z)\overline{w_{zz}}$$

$$+r_1(z)w_{\bar{z}} + r_2(z)\overline{w_{\bar{z}}} + r_3(z)w_z + r_4(z)\overline{w_z}$$

$$r_5(z)w + r_6(z)\bar{w} = f(z) \text{ in } \mathbb{D}. \tag{3.72}$$

Since $w_{\bar{z}\bar{z}} = g(z)$ in $\mathbb{D}$,

$$w(z) = \psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}g(z), \bar{w} = \overline{\psi_0(z)} + z\overline{\psi_1(z)} + \overline{T_{0,2}g(z)}, w_z$$

$$= \psi_{0z} + \bar{z}\psi_{1z} + T_{-1,2}g(z), \overline{w_z} = \overline{\psi_{0z}} + z\overline{\psi_{1z}} + \overline{T_{-1,2}g(z)}, w_{\bar{z}}$$

$$= \psi_1 + T_{0,1}g, \overline{w_{\bar{z}}} = \overline{\psi_1} + \overline{T_{0,1}g}, w_{zz} = \psi_{0zz} + \bar{z}\psi_{1zz} + T_{-2,2}g(z), \overline{w_{zz}}$$

$$= \overline{\psi_{0zz}} + z\overline{\psi_{1zz}} + \overline{T_{-2,2}g(z)}, w_{z\bar{z}} = \psi_{1z} + T_{-1,1}g(z), \overline{w_{z\bar{z}}}$$

$$= \overline{\psi_{1z}} + \overline{T_{-1,1}g(z)}.$$

After substituting these derivatives the result follows

$$g(z) + q_1T_{-1,1}g(z) + q_2\overline{T_{-1,1}g(z)} + q_3T_{-2,2}g + q_4\overline{T_{-2,2}g(z)}r_1(z)T_{0,1}g(z)$$

$$+r_2(z)\overline{T_{0,1}g(z)} + r_3(z)T_{-1,2}g(z) + r_4(z)\overline{T_{-1,2}g(z)} + r_5(z)T_{0,2}g(z)$$

$$+r_6(z)\overline{T_{0,2}g(z)} = f - q_1\psi_{1z} - q_2(z)\overline{\psi_{1z}} - q_3(z)\psi_{0zz} - q_3(z)\bar{z}\psi_{1zz}$$

$$-q_4\overline{\psi_{0zz}} - q_4z\overline{\psi_{1zz}} - r_1(z)\psi_1 - r_2(z)\overline{\psi_1} - r_3(z)\psi_{0z} - r_3(z)\bar{z}\psi_{1z}$$

$$- r_4(z)\overline{\psi_{0z}} - r_4(z)z\overline{\psi_{1z}} - r_5(z)\psi_0 - r_5(z)\bar{z}\psi_1 - r_6(z)\overline{\psi_0} - r_6(z)z\overline{\psi_1}. \tag{3.73}$$

Therefore,

$$g = (I + \widehat{\Pi} + \widehat{K})^{-1}(q_1\psi_{1z} - q_2(z)\overline{\psi_{1z}} - q_3(z)\psi_{0zz} - q_3(z)\bar{z}\psi_{1zz}$$

$$-q_4\overline{\psi_{0zz}} - q_4 z\overline{\psi_{1zz}} - r_1(z)\psi_1 - r_2(z)\overline{\psi_1} - r_3(z)\psi_{0z} - r_3(z)\bar{z}\psi_{1z}$$

$$-r_4(z)\overline{\psi_{0z}} - r_4(z)z\overline{\psi_{1z}} - r_5(z)\psi_0 - r_5(z)\bar{z}\psi_1 - r_6(z)\overline{\psi_0} - r_6(z)z\overline{\psi_1})$$

satisfies (3.65) if and only if $w(z) = \psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}g(z)$ satisfies (3.61) with the boundary condition (3.62). □

Solvability of the Singular Integral Equation

Lemma 3.2.8. *If*

$$q_0 \parallel \Pi \parallel_{L_p(\mathbb{D})} < 1 \tag{3.74}$$

is satisfied for $p > 1$, *then the operator* $I + \widehat{\Pi}$ *is invertible.*

Proof. By the properties of norm, we obtain

$$\parallel \widehat{\Pi} g \parallel_{L_p(\mathbb{D})} = \parallel q_1\Pi g + q_2\overline{\Pi g} + q_3\Pi_2 g + q_4\overline{\Pi_2 g} \parallel_{L_p(\mathbb{D})}$$

$$\leq \parallel q_1\Pi g \parallel_{L_p(\mathbb{D})} + \parallel q_2\overline{\Pi g} \parallel_{L_p(\mathbb{D})} + \parallel q_3\Pi_2 g \parallel_{L_p(\mathbb{D})}$$

$$+ \parallel q_4\overline{\Pi_2 g} \parallel_{L_p(\mathbb{D})} = | q_1(z) | \parallel \Pi g \parallel_{L_p(\mathbb{D})} + | q_2(z) | \parallel \overline{\Pi g(z)} \parallel_{L_p(\mathbb{D})}$$

$$+ | q_3(z) | \parallel \Pi_2 g \parallel_{L_p(\mathbb{D})} + | q_4(z) | \parallel \overline{\Pi_2 g} \parallel_{L_p(\mathbb{D})}$$

$$= (| q_1(z) | + | q_2(z) | + | q_3(z) | + | q_4(z) |)$$

$$\times \parallel \Pi g \parallel_{L_p(\mathbb{D})} \leq q_0 \parallel \Pi g \parallel_{L_p(\mathbb{D})} < 1.$$

If condition $q_0 \parallel \Pi \parallel_{L_p(\mathbb{D})} < 1$ holds, then we get $\parallel \widehat{\Pi} \parallel_{L_p(\mathbb{D})} < 1$. By Theorem 2.1.3, the operator $I + \widehat{\Pi}$ is invertible. □

Lemma 3.2.9. *For bounded functions* $r_i(z)$, $i = 1, \ldots, 6$ *and for* $p > 2$, *the operator* $\widehat{K}$ *is a compact operator.*

Proof. Firstly, we will consider the boundedness of $\widehat{K}$.

$$|\widehat{K}g| = |r_1(z)T_{0,1}g(z) + r_2(z)\overline{T_{0,1}g(z)} + r_3(z)T_{-1,2}g(z)$$

$$+r_4(z)\overline{T_{-1,2}g(z)} + r_5(z)T_{0,2}g(z) + r_6(z)\overline{T_{0,2}g(z)}|$$

$$\leq |r_1(z)T_{0,1}g(z)| + |r_2(z)\overline{T_{0,1}g(z)}| + |r_3(z)T_{-1,2}g(z)|$$

$$+|r_4(z)\overline{T_{-1,2}g(z)}| + |r_5(z)T_{0,2}g(z)| + |r_6(z)\overline{T_{0,2}g(z)}|.$$

By Theorem 2.2.11, we get

$$\leq |r_1(z)| M(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})} + |r_2(z)| M(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})}$$

$$+|r_3(z)| M(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})} + |r_4(z)| M(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})}$$

$$+|r_5(z)| M(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})} + |r_6(z)| M(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})}$$

$$= (|r_1(z)| + |r_2(z)| + |r_3(z)| + |r_4(z)| + |r_5(z)| + |r_6(z)|) M(p,\mathbb{D})$$

$$\times \|g\|_{L_p(\mathbb{D})} \leq C(p,\mathbb{D}) \|g\|_{L_p(\mathbb{D})},$$

where $M(p,\mathbb{D}), C(p,\mathbb{D})$ are always nonnegative constants, depending on the quantities in the parentheses.

By Theorem 2.2.12, the operators in $\widehat{K}$ are Hölder continuous, in particular, they are uniformly continuous. So, by Arzela-Ascoli Theorem, the operators in $\widehat{K}g$ are compact operators. □

Now, we can apply the Fredholm alternative.

Theorem 3.2.10. *If $q_0 \, \|\Pi\|_{L_p(\mathbb{D})} < 1$ is satisfied, then (3.61) with the boundary condition (3.62) has a solution of the form $w(z) = \psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}g(z)$, where g is a solution of the singular integral equation (3.65) and $\psi_0(z), \psi_1(z)$ are analytic functions in $\mathbb{D}$.*

Proof. If (3.74) is satisfied then by Theorem 2.1.3 $I + \widehat{\Pi}$ is invertible. In Lemma 3.2.9 we have showed that $\widehat{K}$ is compact. By Nikolskii Criterion, the operator $I + \widehat{\Pi} + \widehat{K}$ is Fredholm operator with index zero. Theorem 2.1.5 implies that the singular integral equation (3.65) has the Fredholm alternative, i.e. it has at least a solution. Hence, if g is a solution of (3.65), then $w(z) = \psi_0(z) + \bar{z}\psi_1(z) + T_{0,2}g(z)$ is a solution to (3.61) with the boundary condition (3.62). □

Also, we can find the solution of (3.61) and (3.62) by denoting the solution as $w = w_1 + w_2$ where w_1 is the solution of the problem

$$w_{1\bar{z}\bar{z}} = 0 \quad \text{in } \mathbb{D},\ w_1 = \gamma_0(z),\ w_{1\bar{z}} = \gamma_1(z) \text{ on } \partial\mathbb{D} \tag{3.75}$$

and w_2 is a solution of the problem

$$\begin{gathered} w_{2\bar{z}\bar{z}} + q_1(z)w_{2z\bar{z}} + q_2(z)\overline{w_{2z\bar{z}}} + q_3(z)w_{2zz} + q_4(z)\overline{w_{2zz}} + r_1(z)w_{2\bar{z}} \\ +r_2(z)\overline{w_{2\bar{z}}} + r_3(z)w_{2z} + r_4(z)\overline{w_{2z}} + r_5(z)w_2 + r_6(z)\bar{w_2} = f(z) \\ -q_1(z)w_{1z\bar{z}} - q_2(z)\overline{w_{1z\bar{z}}} - q_3(z)w_{1zz} - q_4(z)\overline{w_{1zz}} - r_1(z)w_{1\bar{z}} \\ - r_2(z)\overline{w_{1\bar{z}}} - r_3(z)w_{1z} - r_4(z)\overline{w_{1z}} - r_5(z)w_1 - r_6(z)\bar{w_1} \quad \text{in } \mathbb{D}, \end{gathered} \tag{3.76}$$

$$w_2 = 0,\ w_{2\bar{z}} = 0 \text{ on } \partial\mathbb{D}. \tag{3.77}$$

By Theorem 3.2.4, the solution of (3.75) is

$$\begin{aligned} w_1(z) &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta \\ &= \psi_0(z) + \bar{z}\psi_1(z). \end{aligned}$$

Then the problem (3.76) and (3.77) becomes

$$\begin{gathered} w_{2\bar{z}\bar{z}} + q_1(z)w_{2z\bar{z}} + q_2(z)\overline{w_{2z\bar{z}}} + q_3(z)w_{2zz} \\ + q_4(z)\overline{w_{2zz}} + r_1(z)w_{2\bar{z}} + r_2(z)\overline{w_{2\bar{z}}} + r_3(z)w_{2z} \qquad (3.78) \\ +r_4(z)\overline{w_{2z}} + r_5(z)w_2 + r_6(z)\bar{w_2} \\ = f(z) - q_1\psi_{1z} - q_2(z)\overline{\psi_{1z}} - q_3(z)\psi_{0zz} - q_3(z)\bar{z}\psi_{1zz} \\ -q_4\overline{\psi_{0zz}} - q_4 z\overline{\psi_{1zz}} - r_1(z)\psi_1 - r_2(z)\overline{\psi_1} \\ -r_3(z)\psi_{0z} - r_3(z)\bar{z}\psi_{1z} - r_4(z)\overline{\psi_{0z}} - r_4(z)z\overline{\psi_{1z}} \\ - r_5(z)\psi_0 - r_5(z)\bar{z}\psi_1 - r_6(z)\overline{\psi_0} - r_6(z)z\overline{\psi_1} \quad \text{in } \mathbb{D}, \end{gathered} \tag{3.79}$$

$$w_2 = 0,\ w_{2\bar{z}} = 0 \text{ on } \partial\mathbb{D}. \tag{3.80}$$

Then

$$w_{2\bar{z}\bar{z}} = \widetilde{f_2} \text{ in } \mathbb{D}, \quad w_2 = 0,\ w_{2\bar{z}} = 0 \text{ on } \partial\mathbb{D}, \tag{3.81}$$

where

$$\widetilde{f_2} = f(z) - q_1\psi_{1z} - q_2(z)\overline{\psi_{1z}} - q_3(z)\psi_{0zz} - q_3(z)\bar{z}\psi_{1zz}$$

$$-q_4\overline{\psi_{0zz}} - q_4 z\overline{\psi_{1zz}} - r_1(z)\psi_1 - r_2(z)\overline{\psi_1} - r_3(z)\psi_{0z}$$

$$-r_3(z)\bar{z}\psi_{1z} - r_4(z)\overline{\psi_{0z}} - r_4(z)z\overline{\psi_{1z}}$$

$$-r_5(z)\psi_0 - r_5(z)\bar{z}\psi_1 - r_6(z)\overline{\psi_0} - r_6(z)z\overline{\psi_1}.$$

By Theorem 3.2.4, we obtain

$$w_2 = T_{0,2}\widetilde{f_2}.$$

By previous method, we conclude that the problem (3.76) and (3.77) is equivalent to the singular integral equation

$$(I + \widehat{\Pi} + \widehat{K})g_2 = f(z) - q_1\psi_{1z} - q_2(z)\overline{\psi_{1z}} - q_3(z)\psi_{0zz}$$

$$-q_3(z)\bar{z}\psi_{1zz} - q_4\overline{\psi_{0zz}} - q_4 z\overline{\psi_{1zz}} - r_1(z)\psi_1 - r_2(z)\overline{\psi_1}$$

$$-r_3(z)\psi_{0z} - r_3(z)\bar{z}\psi_{1z} - r_4(z)\overline{\psi_{0z}} - r_4(z)z\overline{\psi_{1z}} - r_5(z)\psi_0$$

$$- r_5(z)\bar{z}\psi_1 - r_6(z)\overline{\psi_0} - r_6(z)z\overline{\psi_1}, \tag{3.82}$$

where $w_2 = T_{0,2}g_2$.

Example 3.2.11. *Let us consider the problem*

$$w_{\bar{z}\bar{z}} + 2\bar{z}^2 w_{z\bar{z}} + 2z\overline{w_{z\bar{z}}} + \bar{z}w_{zz} + z\overline{w_{zz}} + z^3 w_{\bar{z}} + z^2\overline{w_{\bar{z}}} + 4zw_z + 5\overline{w_z} + 3\bar{z}w + 2z^2\bar{w} = 20\bar{z}^3 + 4z^3$$

$$+24z^2\bar{z}^3 + 24z^2\bar{z}^2 + 12z\bar{z}^3 + 12z^3\bar{z} + 5z^3\bar{z}^4 + 4z^6\bar{z} + 4z^3 + 5z^6 + 4z^3\bar{z}^3 + 4z^2 + 24z^3\bar{z}^2 + 30z^2\bar{z}^2$$

$$+ 3\bar{z}^6 + 6z^3\bar{z}^3 + 12\bar{z}^2 + 2z^7 + 4z^4\bar{z}^3 + 8z^2 \text{ in } \mathbb{D}, \tag{3.83}$$

$$w(z) = \bar{z}^5 + 2z^3\bar{z}^2 + 4\bar{z} \quad w_{\bar{z}} = 5\bar{z}^4 + 4z^3\bar{z} + 4 \quad \text{on } \partial\mathbb{D}. \tag{3.84}$$

Then $w_1 = 4\bar{z}$ *is a solution of the problem*

$$w_{1\bar{z}\bar{z}} = 0 \text{ in } \mathbb{D}, \tag{3.85}$$

$$w_1 = \bar{z}^5 + 2z^3\bar{z}^2 + 4\bar{z}, \quad w_{1\bar{z}} = 5\bar{z}^4 + 4z^3\bar{z} + 4 \text{ on } \partial\mathbb{D} \tag{3.86}$$

and $w_2 = \bar{z}^5 + 2z^3\bar{z}^2$ *is a solution of the problem*

$$w_{2\bar{z}\bar{z}} + 2\bar{z}^2 w_{2z\bar{z}} + 2z\overline{w_{2z\bar{z}}} + \bar{z}w_{2zz} + z\overline{w_{2zz}} + z^3 w_{2\bar{z}} + z^2\overline{w_{2\bar{z}}} + 4zw_{2z} + 5\overline{w_{2z}} + 3\bar{z}w_2 + 2z^2\bar{w}_2$$

$$= 24z^2\bar{z}^3 + 24z^2\bar{z}^2 + 12z\bar{z}^3 + 12z^3\bar{z} + 5z^3\bar{z}^4 + 4z^6\bar{z} + 4z^3 + 5z^6 + 4z^3\bar{z}^3 + 4z^2 + 24z^3\bar{z}^2$$

$$+30z^2\bar{z}^2 + 3\bar{z}^6 + 6z^3\bar{z}^3 + 12\bar{z}^2 + 2z^7 + 4z^4\bar{z}^3 + 4z - 4z^3 - 4z^2 - 12\bar{z}^2 - 8z^3 \quad \textit{in } \mathbb{D}, \quad (3.87)$$

$$w_2 = 0, \quad w_{2\bar{z}} = 0 \textit{ on } \partial\mathbb{D}. \quad (3.88)$$

So, solution of the problem (3.83) - (3.84) is $w(z) = w_1 + w_2 = \bar{z}^5 + 2z^3\bar{z}^2 + 4\bar{z}$.

Now, we consider the linear differential equation

$$w_{zz} + q_1(z)w_{z\bar{z}} + q_2(z)\overline{w_{z\bar{z}}} + q_3(z)w_{\bar{z}\bar{z}}$$

$$+q_4(z)\overline{w_{\bar{z}\bar{z}}} + r_1(z)w_{\bar{z}} + r_2(z)\overline{w_{\bar{z}}} + r_3(z)w_z$$

$$+ r_4(z)\overline{w_z} + r_5(z)w + r_6(z)\bar{w} = f(z) \text{ in } \mathbb{D} \quad (3.89)$$

satisfying the Dirichlet boundary condition

$$w(z) = \gamma_0(z),\ w_z = \gamma_1(z) \text{ on } \partial\mathbb{D}, \quad (3.90)$$

where

$$r_1(z), r_2(z), r_3(z), r_4(z), r_5(z), r_6(z), f(z) \in L_p(\mathbb{D}),$$

and $q_1(z), q_2(z), q_3(z), q_4(z), r_1(z), r_2(z), r_3(z), r_4(z), r_5(z), r_6(z)$ are measurable bounded functions and satisfying

$$|\, q_1(z)\,| + |\, q_2(z)\,| + |\, q_3(z)\,| + |\, q_4(z)\,| \le q_0 < 1. \quad (3.91)$$

Lemma 3.2.12. *The Dirichlet problem (3.89) and (3.90) is equivalent to the singular integral equation*

$$(I + \widehat{\Pi} + \widehat{K})g = f(z) - q_1\overline{\psi_{1z}} - q_2(z)\psi_{1z} - q_3(z)\overline{\psi_{0zz}} - q_3(z)z\overline{\psi_{1zz}} - q_4\psi_{0zz} - q_4\bar{z}\psi_{1zz}$$

$$-r_1(z)\overline{\psi_{0z}} - z\overline{\psi_{1z}} - r_2(z)\psi_{0z} - r_2(z)\bar{z}\psi_{1z} - r_3(z))\overline{\psi_1} - r_4(z)\psi_1 - r_5(z)\overline{\psi_0} - r_5(z)z\overline{\psi_1}$$

$$- r_6(z)\psi_0 - r_6(z)\bar{z}\psi_1, \quad (3.92)$$

where $w(z) = \overline{\psi_0} + z\overline{\psi_1} + T_{2,0}g(z)$,

$$\widehat{\Pi}g = q_1 T_{1,-1}g(z) + q_2\overline{T_{1,-1}g(z)} + q_3 T_{2,-2}g + q_4\overline{T_{2,-2}g(z)}$$

$$= q_1\overline{\Pi}g(z) + q_2\Pi\overline{g(z)} + q_3\overline{\Pi_2}g + q_4\Pi_2\overline{g} \tag{3.93}$$

and

$$\widehat{K}g = r_1(z)T_{2,-1}g(z) + r_2(z)\overline{T_{2,-1}g(z)} + r_3(z)T_{1,0}g(z) + r_4(z)\overline{T_{1,0}g(z)} + r_5(z)T_{2,0}g(z)$$

$$r_6(z)\overline{T_{2,0}g(z)}. \tag{3.94}$$

Proof. We write (3.89) as

$$w_{zz} = f(z) - q_1(z)w_{z\bar{z}} - q_2(z)\overline{w_{z\bar{z}}} - q_3(z)w_{\bar{z}\bar{z}} - q_4(z)\overline{w_{\bar{z}\bar{z}}}$$

$$- r_1(z)w_{\bar{z}} - r_2(z)\overline{w_{\bar{z}}} - r_3(z)w_z - r_4(z)\overline{w_z} - r_5(z)w - r_6(z)\bar{w}. \tag{3.95}$$

Then

$$w_{zz} = \widetilde{f(z)} \quad \text{in } \mathbb{D}, \tag{3.96}$$

$$w(z) = \gamma_0(z),\ w_z = \gamma_1(z) \text{ on } \partial\mathbb{D}, \tag{3.97}$$

where

$$\widetilde{f(z)} = f(z) - q_1(z)w_{z\bar{z}} - q_2(z)\overline{w_{z\bar{z}}} - q_3(z)w_{\bar{z}\bar{z}} - q_4(z)\overline{w_{\bar{z}\bar{z}}}$$

$$-r_1(z)w_{\bar{z}} - r_2(z)\overline{w_{\bar{z}}} - r_3(z)w_z - r_4(z)\overline{w_z} - r_5(z)w - r_6(z)\bar{w}.$$

By Theorem 3.2.6, solution of (3.89) and (3.90) is

$$w(z) = \overline{\psi_0(z)} + z\overline{\psi_1(z)} + T_{2,0}\widetilde{f(z)}.$$

Let $w_{zz} = g(z)$. Then,

$$g(z) = (\overline{\psi_0(z)} + z\overline{\psi_1(z)} + T_{2,0}\widetilde{f(z)})_{zz}, \tag{3.98}$$

$$g(z) + q_1(z)w_{z\bar{z}} + q_2(z)\overline{w_{z\bar{z}}} + q_3(z)w_{\bar{z}\bar{z}} + q_4(z)\overline{w_{\bar{z}\bar{z}}} + r_1(z)w_{\bar{z}}$$

$$+ r_2(z)\overline{w_{\bar{z}}} + r_3(z)w_z + r_4(z)\overline{w_z} + r_5(z)w + r_6(z)\bar{w} = f(z) \text{ in } \mathbb{D}. \tag{3.99}$$

Since $w_{zz} = g(z)$ in $\mathbb{D}$,

$$w(z) = \overline{\psi_0(z)} + z\overline{\psi_1(z)} + T_{2,0}g(z), \bar{w}$$

$$= \psi_0(z) + \bar{z}\psi_1(z) + \overline{T_{2,0}g(z)}, w_z = z\overline{\psi_1(z)} + T_{1,0}g(z),$$

$$\overline{w_z} = \psi_1(z) + \overline{T_{1,0}g(z)}, w_{\bar{z}}$$

$$= \overline{\psi_{0z}} - z\overline{\psi_{1z}} + T_{2,-1}g, \overline{w_{\bar{z}}} = \overline{\psi_{0z}} - z\overline{\psi_{1z}} + T_{2,-1}g, w_{\bar{z}\bar{z}} = \overline{\psi_{0zz}}$$

$$+z\overline{\psi_{1zz}} + T_{2,-2}g(z), \overline{w_{\bar{z}\bar{z}}} = \psi_{0zz} + \bar{z}\psi_{1zz} + \overline{T_{2,-2}g(z)}, w_{z\bar{z}}$$

$$= \overline{\psi_{1z}} + T_{1,-1}g(z), \overline{w_{z\bar{z}}} = \psi_{1z} + \overline{T_{1,-1}g(z)}.$$

After substituting these derivatives the result follows

$$g(z) + q_1 T_{1,-1}g(z) + q_2\overline{T_{1,-1}g(z)} + q_3 T_{2,-2}g$$

$$+q_4\overline{T_{2,-2}g(z)} + r_1(z)T_{2,-1}g(z) + r_2(z)\overline{T_{2,-1}g(z)}$$

$$+r_3(z)T_{1,0}g(z) + r_4(z)\overline{T_{1,0}g(z)} + r_5(z)T_{2,0}g(z) + r_6(z)\overline{T_{2,0}g(z)}$$

$$= f(z) - q_1\overline{\psi_{1z}} - q_2(z)\psi_{1z} - q_3(z)\overline{\psi_{0zz}} - q_3(z)z\overline{\psi_{1zz}} - q_4\psi_{0zz}$$

$$-q_4\bar{z}\psi_{1zz} - r_1(z)\overline{\psi_{0z}} - z\overline{\psi_{1z}} - r_2(z)\psi_{0z} - r_2(z)\bar{z}\psi_{1z} - r_3(z)\overline{\psi_1}$$

$$- r_4(z)\psi_1 - r_5(z)\overline{\psi_0} - r_5(z)z\overline{\psi_1} - r_6(z)\psi_0 - r_6(z)\bar{z}\psi_1. \tag{3.100}$$

Therefore,

$$g = (I + \hat{\Pi} + \hat{K})^{-1}(f - q_1\overline{\psi_{1z}} - q_2(z)\psi_{1z}$$

$$-q_3(z)\overline{\psi_{0zz}} - q_3(z)z\overline{\psi_{1zz}} - q_4\psi_{0zz} - q_4\bar{z}\psi_{1zz} - r_1(z)\overline{\psi_{0z}}$$

$$-z\overline{\psi_{1z}} - r_2(z)\psi_{0z} - r_2(z)\bar{z}\psi_{1z} - r_3(z))\overline{\psi_1} - r_4(z)\psi_1$$

$$-r_5(z)\overline{\psi_0} - r_5(z)z\overline{\psi_1} - r_6(z)\psi_0 - r_6(z)\bar{z}\psi_1)$$

satisfies (3.92) if and only if $w(z) = \overline{\psi_0(z)} + z\overline{\psi_1(z)} + T_{2,0}g(z)$ satisfies (3.89) with the boundary condition (3.90). □

Solvability of the Singular Integral Equation

Lemma 3.2.13. *If*

$$q_0 \mid\mid \Pi \mid\mid_{L_p(\mathbb{D})} < 1 \tag{3.101}$$

for $p > 1$, then the operator $I + \hat{\Pi}$ is invertible.

Proof. By the properties of norm, we obtain

$$\| \widehat{\Pi} g \|_{L_p(\mathbb{D})} = \| q_1 \overline{\Pi} g + q_2 \Pi \overline{g} + q_3 \overline{\Pi}_2 g + q_4 \Pi_2 \overline{g} \|_{L_p(\mathbb{D})}$$

$$\leq \| q_1 \overline{\Pi} g \|_{L_p(\mathbb{D})} + \| q_2 \Pi \overline{g} \|_{L_p(\mathbb{D})} + \| q_3 \overline{\Pi}_2 g \|_{L_p(\mathbb{D})} + \| q_4 \Pi_2 \overline{g} \|_{L_p(\mathbb{D})}$$

$$= | q_1(z) | \| \overline{\Pi} g \|_{L_p(\mathbb{D})} + | q_2(z) | \| \Pi \overline{g} \|_{L_p(\mathbb{D})}$$

$$+ | q_3(z) | \| \overline{\Pi}_2 g \|_{L_p(\mathbb{D})} + | q_4(z) | \| \overline{\Pi_2 g} \|_{L_p(\mathbb{D})}$$

$$= (| q_1(z) | + | q_2(z) | + | q_3(z) | + | q_4(z) |) \| \Pi_2 \overline{g} \|_{L_p(\mathbb{D})}$$

$$\leq q_0 \| \Pi g \|_{L_p(\mathbb{D})} < 1.$$

If condition $q_0 \ \| \ \Pi \ \|_{L_p(\mathbb{D})} < 1$ holds, then we get $\| \ \widehat{\Pi} \ \|_{L_p(\mathbb{D})} < 1$. By Theorem 2.1.3, the operator $I + \widehat{\Pi}$ is invertible. □

Lemma 3.2.14. *For bounded functions* $r_i(z), \quad i = 1, \ldots, 6$ *and for* $p > 2$*, the operator* $\widehat{K}$ *is a compact operator.*

Proof. Firstly, we will consider the boundedness of $\widehat{K}$.

$$| \widehat{K} g | = | r_1(z) T_{2,-1} g(z) + r_2(z) \overline{T_{2,-1} g(z)} + r_3(z) T_{1,0} g(z)$$

$$+ r_4(z) \overline{T_{1,0} g(z)} + r_5(z) T_{2,0} g(z) + r_6(z) \overline{T_{2,0} g(z)} | \leq | r_1(z) T_{2,-1} g(z) |$$

$$+ | r_2(z) \overline{T_{2,-1} g(z)} | + | r_3(z) T_{1,0} g(z) | + | r_4(z) \overline{T_{1,0} g(z)} |$$

$$+ | r_4(z) \overline{T_{1,0} g(z)} | + | r_5(z) T_{2,0} g(z) | + | r_6(z) \overline{T_{2,0} g(z)} |$$

By Theorem 2.2.11, we get

$$| \widehat{K} g | \leq | r_1(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})} + | r_2(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})}$$

$$+ | r_3(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})} + | r_4(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})}$$

$$+ | r_5(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})} + | r_6(z) | M(p, \mathbb{D}) \| g \|_{L_p(\mathbb{D})}$$

$$= (| r_1(z) | + | r_2(z) | + | r_3(z) | + | r_4(z) | + | r_5(z) |$$

$$+ \mid r_6(z) \mid) M(p,\mathbb{D}) \parallel g\|_{L_p(\mathbb{D})} \leq C(p,\mathbb{D}) \parallel g\|_{L_p(\mathbb{D})},$$

where $M(p,\mathbb{D}), C(p,\mathbb{D})$ are always nonnegative constants, depending on the quantities in the parentheses.

By Theorem 2.2.12, the operators in $\widehat{K}$ are Hölder continuous, in particular, they are uniformly continuous. So, by Arzela-Ascoli Theorem, the operators in $\widehat{K}g$ are compact operators. □

Now, we can apply the Fredholm alternative.

Theorem 3.2.15. *If* $q_0 \parallel \Pi \parallel_{L_p(\mathbb{D})} < 1$ *is satisfied, then (3.89) with the boundary condition (3.90) has a solution of the form* $w(z) = \overline{\psi_0(z)} + z\overline{\psi_1(z)} + T_{2,0}g(z)$*, where* g *is a solution of the singular integral equation (3.92) and* $\psi_0(z), \psi_1(z)$ *are analytic functions in* $\mathbb{D}$.

Proof. If (3.101) is satisfied then by Lemma 2.1.3, $I + \widehat{\Pi}$ is invertible. In Lemma 3.2.14, we have showed that $\widehat{K}$ is compact. By Nikolskii Criterion, the operator $I + \widehat{\Pi} + \widehat{K}$ is Fredholm operator with index zero. Theorem 2.1.5 implies that the singular integral equation (3.92) has the Fredholm alternative, i.e. it has at least a solution. Hence, if g is a solution of (3.92) , then $w(z) = \overline{\psi_0(z)} + z\overline{\psi_1(z)} + T_{2,0}g(z)$ is a solution to the (3.89) with the boundary condition (3.90) . □

Also we can investigate the boundary value problem by separating into the boundary value problems

$$w_{1zz} = 0 \quad \text{in } \mathbb{D},$$

$$w_1 = \gamma_0(z), \quad w_{1z} = \gamma_1(z) \quad \text{on} \quad \partial\mathbb{D}$$

and

$$w_{2zz} + q_1(z)w_{2z\bar{z}} + q_2(z)\overline{w_{2z\bar{z}}} + q_3(z)w_{2\bar{z}\bar{z}}$$

$$+q_4(z)\overline{w_{2\bar{z}\bar{z}}} + r_1(z)w_{2\bar{z}} + r_2(z)\overline{w_{2\bar{z}}} + r_3(z)w_{2z}$$

$$+r_4(z)\overline{w_{2z}} + r_5(z)w_2 + r_6(z)\bar{w}_2$$

$$= f - q_1(z)w_{1z\bar{z}} - q_2(z)\overline{w_{1z\bar{z}}} - q_3(z)w_{1\bar{z}\bar{z}} - q_4(z)\overline{w_{1\bar{z}\bar{z}}} - r_1(z)w_{1\bar{z}}$$

$$-r_2(z)\overline{w_{1\bar{z}}} - r_3(z)w_{1z} - r_4(z)\overline{w_{1z}} - r_5(z)w_1 - r_6(z)\bar{w}_1 \quad \text{in } \mathbb{D},$$

$$w_1 = 0, \quad w_{1z} = 0 \quad \text{on} \quad \partial\mathbb{D}.$$

3.3. DIRICHLET PROBLEM FOR HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section, we will investigate the Dirichlet problem for higher order complex partial differential equations.

Lemma 3.3.1. *The Dirichlet problem*

$$\partial_{\bar{z}}^k w = 0 \ in \ \mathbb{D}, \ w = 0 \ on \ \partial\mathbb{D}, \ k \in \mathbb{Z}^+, \ k > 2$$

has infinitely many linearly independent solutions over $\mathbb{C}$.

Proof. From the differential equation $w_{\bar{z}^k} = 0$ it follows that $w_{\bar{z}^{k-1}}$ is the analytic function. By integration we have that

$$w(z) = \sum_{n=1}^{k} \bar{z}^{n-1}\varphi_n(z), \text{ where } \varphi_i, \ i = 1, \dots, k \text{ are analytic in } \mathbb{D}.$$

Then

$$\sum_{n=1}^{k} z^{k-n}\varphi_n(z) = 0 \text{ on } \partial\mathbb{D}.$$

Since $\varphi_k(z)$ is analytic in $\mathbb{D}$, we have that

$$\varphi_k(z) = -\sum_{n=1}^{k-1} z^{k-n}\varphi_n(z) \text{ on } \partial\mathbb{D}.$$

Applying Theorem 3.1.3, we get $\varphi_k(z) = -\sum_{n=1}^{k-1} z^{k-n}\varphi_n(z)$ in $\mathbb{D}$. Hence, we have the following solution.

$$w(z) = \sum_{n=1}^{k-1} \bar{z}^{n-1}\varphi_n(z)(1 - |z|^{2(k-n)}).$$

Then the expected result is obtained. □

This means that the Dirichlet problem for the higher order complex partial differential equations is ill-posed. We can make it well-posed problem, taking additional conditions $w_{\bar{z}^m},\ m = 1, \dots, k-1$ on $\partial\mathbb{D}$. We have the following theorem.

Theorem 3.3.2. *[2] The Dirichlet problem*

$$\partial_{\bar{z}}^k w = f(z) \ in\ \mathbb{D},\ \partial_{\bar{z}}^\nu w = \gamma_\nu, on\ \partial\mathbb{D}, 0 \le \nu \le k-1$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C}),\ 0 \le \nu \le k-1$, *if and only if for* $0 \le \nu \le k-1$,

$$\sum_{\lambda=\nu}^{k-1} \frac{\bar{z}}{2\pi i} \int_{\partial\mathbb{D}} (-1)^{\lambda-\nu} \frac{\gamma_{\lambda(\zeta)}}{1-\bar{z}\zeta} \frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!} d\zeta + \frac{(-1)^{k-\nu}\bar{z}}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{1-\bar{z}\zeta} \frac{(\overline{\zeta - z})^{k-1-\nu}}{(k-1-\nu)!} d\xi d\eta = 0.$$

The solution then is uniquely given by the formula

$$w(z) = \sum_{\nu=0}^{k-1} \frac{(-1)^\nu}{2\pi i} \int_{\partial\mathbb{D}} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{(\overline{\zeta - z})^\nu}{\zeta - z} d\zeta + \frac{(-1)^k}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{(k-1)!} \frac{(\overline{\zeta - z})^{k-1}}{\zeta - z} d\xi d\eta.$$

Moreover, we have the following result.

Lemma 3.3.3. *The Dirichlet problem*

$$\partial_{\bar{z}}^k w = 0 \ in\ \mathbb{D},\ w = 0 \ on\ \partial\mathbb{D},\ k \in \mathbb{Z}^+,\ k > 2$$

has infinitely many linearly independent solutions over $\mathbb{C}$.

Proof. Integrating $w(z)$ with respect to z we obtain

$$w(z) = \sum_{m=1}^{k} z^{m-1} \overline{\varphi_m(z)}, \tag{3.102}$$

where $\varphi_i(z),\ i = 1, \dots, k$ are analytic functions in $\mathbb{D}$. Since

$$w \equiv 0 \text{ on } \partial\mathbb{D},$$

$$\sum_{m=1}^{k} \bar{z}^{k-m}\overline{\varphi_m(z)} = 0 \text{ on } \partial\mathbb{D},$$

we have that

$$\varphi_k(z) = -\sum_{m=1}^{k-1} \bar{z}^{k-m}\overline{\varphi_m(z)} \text{ on } \partial\mathbb{D}$$

and $\varphi_k(z)$ is analytic function in $\mathbb{D}$, Theorem 3.1.4 can be applied. Then the expected result is obtained. In a similar way, we can make a unique solution in $\mathbb{D}$ problem for this equation, taking additional conditions $w_{z^m},\ m = 1, \dots, k-1$ on $\partial\mathbb{D}$. □

We have the following theorem.

Theorem 3.3.4. *The Dirichlet problem*

$$\partial_z^k w = f(z) \textit{ in } \mathbb{D},\ \partial_z^\nu w = \gamma_\nu, \textit{on } \partial\mathbb{D}, \quad 0 \le \nu \le k-1$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C}),\ 0 \le \nu \le k-1$, *if and only if for* $0 \le \nu \le k-1$,

$$-\sum_{\lambda=\nu}^{k-1} \frac{z}{2\pi i} \int_{\partial\mathbb{D}} (-1)^{\lambda-\nu} \frac{\gamma_{\lambda(\zeta)}}{1-z\bar{\zeta}} \frac{(\zeta-z)^{\lambda-\nu}}{(\lambda-\nu)!} d\bar{\zeta} + \frac{(-1)^{k-\nu} z}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{1-z\bar{\zeta}} \frac{(\zeta-z)^{k-1-\nu}}{(k-1-\nu)!} d\xi d\eta = 0. \tag{3.103}$$

The solution then is uniquely given by the formula

$$w(z) = \sum_{\nu=0}^{k-1} \frac{(-1)^{\nu+1}}{2\pi i} \int_{\partial\mathbb{D}} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{(\zeta-z)^\nu}{\overline{\zeta-z}} d\bar{\zeta} + \frac{(-1)^k}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{(k-1)!} \frac{(\zeta-z)^{k-1}}{\overline{\zeta-z}} d\xi d\eta. \tag{3.104}$$

Proof. For $k = 1$, the condition (3.103) is just the condition (3.11) in Theorem 3.1.4, and the solution (3.104) is just the solution (3.12) in Theorem 3.1.4. Assuming Theorem 3.3.4 holds for $k-1$, the problem is equivalent to the system

$$\partial_z^{k-1} w = \omega \text{ in } \mathbb{D},\ \partial_z^\nu w = \gamma_\nu, \text{ on } \partial\mathbb{D}\ 0 \le \nu \le k-2,$$

$$\partial_z \omega = f \text{ in } \mathbb{D},\ \partial_z \omega = \gamma_{k-1}, \text{ on } \partial\mathbb{D},$$

with the solvability conditions

$$\sum_{\lambda=\nu}^{k-1} \frac{-z}{2\pi i} \int_{\partial\mathbb{D}} (-1)^{\lambda-\nu} \frac{\gamma_{\lambda(\zeta)}}{1-z\bar{\bar{\zeta}}} \frac{(\zeta-z)^{\lambda-\nu}}{(\lambda-\nu)!} d\bar{\zeta}$$

$$+\frac{(-1)^{k-\nu}z}{\pi} \iint_{\mathbb{D}} \frac{\omega(\zeta)}{1-z\bar{\bar{\zeta}}} \frac{(\zeta-z)^{k-1-\nu}}{(k-1-\nu)!} d\xi d\eta = 0, \quad 0 \le \nu \le k-2, \tag{3.105}$$

$$\frac{-z}{2\pi i} \int_{\partial\mathbb{D}} \gamma_{k-1}(\zeta) \frac{d\bar{\zeta}}{1-z\bar{\bar{\zeta}}} - \frac{-z}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{1-z\bar{\bar{\zeta}}} d\xi d\eta = 0. \tag{3.106}$$

The solutions then are uniquely given by the formulas

$$w(z) = \sum_{\nu=0}^{k-1} \frac{(-1)^{\nu+1}}{2\pi i} \int_{\partial\mathbb{D}} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{(\zeta-z)^\nu}{\overline{\zeta-z}} d\bar{\zeta}$$

$$+\frac{(-1)^k}{\pi} \iint_{\mathbb{D}} \frac{\omega(\zeta)}{(k-1)!} \frac{(\zeta-z)^{k-1}}{\overline{\overline{\zeta-z}}} d\xi d\eta, 0 \le \nu \le k-2, \tag{3.107}$$

$$\omega(z) = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_{k-1}(\zeta) \frac{d\bar{\zeta}}{\overline{\overline{\zeta-z}}} - \frac{1}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{\overline{\overline{\zeta-z}}} d\xi d\eta. \tag{3.108}$$

Inserting ω into (3.105) and (3.108) gives (3.103) with (3.104) on the basis of

$$\frac{1}{\pi} \iint_{\mathbb{D}} \frac{\omega(\zeta)}{1-z\bar{\bar{\zeta}}} \frac{(\zeta-z)^{k-2-\nu}}{(k-2-\nu)!} d\xi d\eta$$

$$= -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_{k-1}(\tilde{\zeta}) \frac{(\overline{\tilde{\zeta}-z})^{k-1-\nu}}{(k-1-\nu)!\,(1-z\bar{\bar{\tilde{\zeta}}})} d\bar{\tilde{\zeta}}$$

$$-\frac{1}{\pi} \iint_{\mathbb{D}} f(\bar{\bar{\tilde{\zeta}}}) \frac{(\overline{\tilde{\zeta}-z})^{k-1-\nu}}{(k-1-\nu)!\,(1-z\bar{\bar{\tilde{\zeta}}})} d\tilde{\xi} d\tilde{\eta}$$

for $0 \le \nu \le k-2$, and

$$\frac{1}{\pi} \iint_{\mathbb{D}} \omega(\zeta) \frac{(\zeta-z)^{k-2}}{(k-2)!\,(\overline{\overline{\zeta-z}})} d\xi d\eta$$

$$= -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_{k-1}(\tilde{\zeta}) \frac{(\tilde{\zeta}-z)^{k-1}}{(k-1)!\,(\overline{\overline{\tilde{\zeta}-z}})} d\bar{\bar{\tilde{\zeta}}}$$

$$-\frac{1}{\pi} \iint_{\mathbb{D}} f(\bar{\bar{\tilde{\zeta}}}) \frac{(\tilde{\zeta}-z)^{k-1}}{(k-1)!\,(\overline{\overline{\tilde{\zeta}-z}})} d\tilde{\xi} d\tilde{\eta}.$$

□

Lemma 3.3.5. *The Dirichlet problem*

$$\partial_z^m \partial_{\bar{z}}^n w = 0 \quad in\ \mathbb{D},\ w = 0\ on\ \partial\mathbb{D},\ m, n \in \mathbb{Z}^+,\ (m, n) \neq (1, 1)$$

has infinitely many linearly independent solutions.

The proof of Lemma 3.3.5 is based on the following 4 lemmas.

Lemma 3.3.6. *Assume that $f(z)$ is analytic function in $\mathbb{D} = \{z : |z| < 1\}$. Then the following formula holds*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \bar{\zeta}^k f(\zeta)\frac{d\zeta}{\zeta - z} = \frac{f(z)}{z^k} - \sum_{m=1}^{k} \frac{1}{z^{k-m+1}}\frac{f^{(m-1)}(0)}{(m-1)!},\ k \geq 0. \tag{3.109}$$

Proof. We denote that

$$I_k = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \bar{\zeta}^k f(\zeta)\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^k(\zeta - z)}d\zeta.$$

Since

$$\frac{1}{\zeta(\zeta - z)} = \left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right)\frac{1}{z},$$

we have that

$$\frac{1}{\zeta^k(\zeta - z)} = \left(\frac{1}{\zeta^{k-1}(\zeta - z)} - \frac{1}{\zeta^k}\right)\frac{1}{z}.$$

Therefore,

$$\begin{aligned} I_k &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\left(\frac{1}{\zeta^{k-1}(\zeta - z)} - \frac{1}{\zeta^k}\right)\frac{1}{z}d\zeta \\ &= \frac{1}{z}I_{k-1} - \frac{1}{z}\frac{f^{(k-1)}(0)}{(k-1)!} \text{ for any } k \geq 1. \end{aligned}$$

It is first order difference equation. Solving it we get

$$I_k = \frac{1}{z^k}I_0 - \sum_{m=1}^{k} \frac{1}{z^{k-m+1}}\frac{f^{(m-1)}(0)}{(m-1)!}, \quad k \geq 0.$$

It is easy to see that

$$I_0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z}d\zeta = f(z).$$

Hence,

$$I_k = \frac{1}{z^k} f(z) - \sum_{m=1}^{k} \frac{1}{z^{k-m+1}} \frac{f^{(m-1)}(0)}{(m-1)!}, \quad k \geq 0.$$

□

Lemma 3.3.7. *Assume that* $f(z)$ *is analytic function in* $\mathbb{D} = \{z : |z| < 1\}$. *Then the following formula holds*

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^k \overline{f(\zeta)} \frac{d\zeta}{\zeta - z} = \sum_{m=0}^{k} \frac{z^{k-m} \overline{f^{(m)}(0)}}{m!}, \ k \geq 0, \tag{3.110}$$

Proof. We have that

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \zeta^k \overline{f(\zeta)} \frac{d\zeta}{\zeta - z} = -\overline{\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^k(\bar{\zeta} - \bar{z})} d\bar{\zeta}}\cdot$$

We denote that

$$J_k = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^k(\bar{\zeta} - \bar{z})} d\bar{\zeta}\cdot$$

Since

$$\frac{1}{\bar{\zeta}(\bar{\zeta} - \bar{z})} = \left(\frac{1}{\bar{\zeta} - \bar{z}} - \frac{1}{\bar{\zeta}}\right) \frac{1}{\bar{z}}$$

we have that

$$\frac{1}{\zeta^k(\bar{\zeta} - \bar{z})} = \left(\frac{1}{\zeta^{k+1}(\bar{\zeta} - \bar{z})} - \frac{1}{\zeta^{k+1}\bar{\zeta}}\right) \frac{1}{\bar{z}}\cdot$$

Then

$$\frac{1}{\zeta^{k+1}(\bar{\zeta} - \bar{z})} = \bar{z} \frac{1}{\zeta^k(\bar{\zeta} - \bar{z})} + \frac{1}{\zeta^k}\cdot$$

Therefore,

$$\begin{aligned} J_k &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^k(\bar{\zeta} - \bar{z})} d\bar{\zeta} \\ &= \bar{z} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^{k-1}(\bar{\zeta} - \bar{z})} d\bar{\zeta} + \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^{k-1}} d\bar{\zeta} \\ &= \bar{z} J_{k-1} + \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^k} \frac{d\bar{\zeta}}{\bar{\zeta}} \\ &= \bar{z} J_{k-1} - \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta \\ &= \bar{z} J_{k-1} - \frac{f^{(k)}(0)}{k!} \text{ for } k \geq 1. \end{aligned}$$

In the same manner in Lemma 3.3.6, we get

$$J_k = \bar{z}^k J_0 - \sum_{m=1}^{k} \frac{\bar{z}^{k-m} f^{(m)}(0)}{m!}.$$

It is easy to see that

$$J_0 = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\bar{\zeta} - \bar{z}} d\bar{\zeta}$$

$$= -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{(\bar{\zeta} - \bar{z})\zeta^2} d\zeta = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{(1 - \bar{z}\zeta)\zeta} d\zeta.$$

Since,

$$\frac{1}{(1 - \bar{z}\zeta)\zeta} = \frac{\bar{z}}{1 - \bar{z}\zeta} + \frac{1}{\zeta}$$

it follows that

$$J_0 = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} f(\zeta) \left(\frac{\bar{z}}{1 - \bar{z}\zeta} + \frac{1}{\zeta} \right) d\zeta = -f(0).$$

Consequently,

$$J_k = -\bar{z}^k f(0) - \sum_{m=1}^{k} \frac{\bar{z}^{k-m} f^{(m)}(0)}{m!} = -\sum_{m=0}^{k} \frac{\bar{z}^{k-m} f^{(m)}(0)}{m!},\ k \geq 0.$$

□

Lemma 3.3.8. *Assume that $f(z)$ is analytic function in $\mathbb{D} = \{z : |z| < 1\}$. Then the following formula holds*

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \bar{\zeta}^k f(\zeta) \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta = \sum_{m=1}^{k} \bar{z}^{k-m+1} \frac{f^{(m-1)}(0)}{(m-1)!},\ k \geq 0. \tag{3.111}$$

Proof. Let

$$P_k = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \bar{\zeta}^k f(\zeta) \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta = \frac{\bar{z}}{2\pi i} \int_{\partial\mathbb{D}} f(\zeta) \frac{d\zeta}{\zeta^k (1 - \bar{z}\zeta)}.$$

On the basis of

$$\frac{1}{\zeta^k (1 - \bar{z}\zeta)} = \frac{\bar{z}}{\zeta^{k-1}(1 - \bar{z}\zeta)} + \frac{1}{\zeta^k}$$

it follows that

$$P_k = \frac{\bar{z}}{2\pi i} \int_{\partial\mathbb{D}} f(\zeta) \left(\frac{\bar{z}}{\zeta^{k-1}(1 - \bar{z}\zeta)} + \frac{1}{\zeta^k} \right) d\zeta$$

$$= \bar{z} P_{k-1} + \bar{z} \frac{f^{(k-1)}(0)}{(k-1)!}$$

and

$$P_k = \bar{z}^k P_0 + \sum_{m=1}^{k} \bar{z}^{k-m+1} \frac{f^{(m-1)}(0)}{(m-1)!}.$$

Here

$$P_0 = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{1 - \bar{z}\zeta} d\zeta = 0.$$

□

Lemma 3.3.9. *Assume that* $f(z)$ *is analytic function in* $\mathbb{D} = \{z : |z| < 1\}$. *Then the following formula holds*

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \zeta^k \overline{f(\zeta)} \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta = \frac{1}{\bar{z}^k} (\overline{-f(z)} + \overline{f(0)}) - \sum_{m=1}^{k} \frac{1}{\bar{z}^{k-m}} \frac{\overline{f^{(m)}(0)}}{m!}, \ k \geq 0. \quad (3.112)$$

Proof. We denote that

$$T_k = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \bar{\zeta}^k f(\zeta) \frac{z}{1 - z\bar{\zeta}} d\bar{\zeta} = \frac{z}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{d\bar{\zeta}}{\zeta^k (1 - z\bar{\zeta})}.$$

Since,

$$\frac{1}{\bar{\zeta}(1 - z\bar{\zeta})} = \frac{z}{1 - z\bar{\zeta}} + \frac{1}{\bar{\zeta}}$$

it follows that

$$\frac{1}{\zeta^k (1 - z\bar{\zeta})} = \frac{z}{\zeta^{k+1} (1 - z\bar{\zeta})} + \frac{1}{\zeta^k}$$

Therefore,

$$\begin{aligned} T_k &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{d\bar{\zeta}}{\zeta^{k-1}(1 - z\bar{\zeta})} - \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{d\bar{\zeta}}{\zeta^{k-1}} \\ &= \frac{1}{z} T_{k-1} + \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{d\zeta}{\zeta^{k+1}} \\ &= \frac{1}{z} T_{k-1} + \frac{f^{(k)}(0)}{k!} \end{aligned}$$

and

$$T_k = \frac{1}{z^k} T_0 + \sum_{m=1}^{k} \frac{1}{z^{k-m}} \frac{f^{(m)}(0)}{m!}$$

It can be computed

$$T_0 = \frac{z}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \frac{d\bar{\zeta}}{1 - z\bar{\zeta}}$$

$$= \frac{-1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta} \right) = -f(z) + f(0)$$

Hence,

$$T_k = \frac{1}{z^k} (-f(z) + f(0)) + \sum_{m=1}^{k} \frac{1}{z^{k-m}} \frac{f^{(m)}(0)}{m!}.$$

□

3.3.1 Dirichlet Problem for Linear Differential Equations

In this section we will extend the notion of solution of Dirichlet Problem for model equations to a linear differential equation. We will try to find a solution to higher order linear differential equation

$$\begin{aligned} &\partial_{\bar{z}}^{k} w + \sum_{j=1}^{k} q_{1j}(z) \partial_{\bar{z}}^{k-j} \partial_{z}^{j} w + \sum_{j=1}^{k} q_{2j}(z) \partial_{\bar{z}}^{j} \partial_{z}^{k-j} \bar{w} \\ &+ \sum_{l=0}^{k-1} \sum_{m=0}^{l} \left[a_{ml}(z) \partial_{\bar{z}}^{l-m} \partial_{z}^{m} w + b_{ml}(z) \partial_{\bar{z}}^{m} \partial_{z}^{l-m} \bar{w} \right] \\ &= f(z) \quad \text{in} \quad \mathbb{D}, \end{aligned} \tag{3.113}$$

with the boundary conditions

$$\partial_{\bar{z}}^{l} w = \gamma_l(z), \quad 0 \le l \le k-1, \tag{3.114}$$

where

$$a_{ml}, b_{ml}, f \in L_p(\mathbb{D}) \tag{3.115}$$

and $q_{1j}(z)$, $q_{2j}(z)$, $j = 1, \dots, k$ are measurable bounded functions satisfying

$$\sum_{j=1}^{k} \left(|\, q_{1j}(z) \,| + |\, q_{2j}(z) \,| \right) \le q_0 < 1. \tag{3.116}$$

Lemma 3.3.10. *The Dirichlet problem (3.113), (3.114) is equivalent to the singular integral equation*

$$\left(I + \widehat{\Pi} + \widehat{K} \right) = f - \Theta(z), \tag{3.117}$$

where $w=\sum_{m=0}^{k-1}\bar{z}^m\psi_m+T_{0,k}g(z)$, $\psi_i, i=0,\dots,k-1$ *are analytic functions,*

$$\widehat{\Pi}g=\sum_{j=1}^{k}q_{1j}(z)\Pi_j g+\sum_{j=1}^{k}q_{2j}(z)\overline{\Pi_j g}, \tag{3.118}$$

$$\widehat{K}g=\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)T_{-m,k-l+m}g(z)+b_{ml}(z)\overline{T_{-m,k-l+m}g(z)}\right] \tag{3.119}$$

and

$$\Theta(z)=\sum_{j=1}^{k}q_{1j}(z)\partial_{\bar{z}}^{k-j}\partial_z^j\sum_{m=0}^{k-1}\bar{z}^m\psi_m+\sum_{j=1}^{k}q_{2j}(z)\partial_{\bar{z}}^j\partial_z^{k-j}\overline{\sum_{m=0}^{k-1}\bar{z}^m\psi_m}$$
$$+\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\partial_{\bar{z}}^{l-m}\partial_z^m\sum_{m=0}^{k-1}\bar{z}^m\psi_m+b_{ml}(z)\partial_{\bar{z}}^m\partial_z^{l-m}\overline{\sum_{m=0}^{k-1}\bar{z}^m\psi_m}\right]. \tag{3.120}$$

Proof. From Theorem 3.3.2, solution of the problem

$$\partial_{\bar{z}}^k w=g \quad in \quad \mathbb{D},$$

$$\partial_{\bar{z}}^l=\gamma_l(z), \quad 0\le l\le k-1$$

is

$$w=\sum_{m=0}^{k-1}\bar{z}^m\psi_m+T_{0,k}g(z).$$

If we differentiate w using the differentiability properties of the Pompeiu operator, and after substituting these derivatives into (3.113), we obtain the singular integral (3.117). Therefore, g satisfies (3.117) if and only if $w=\sum_{m=0}^{k-1}\bar{z}^m\psi_m+T_{0,k}g(z)$ satisfies the problem (3.113) - (3.114). □

So, we should investigate the solvability of the equation (3.117).

Solvability of the Singular Integral Equation

Lemma 3.3.11. *If*

$$q_0\|\Pi_j\|_{L_p(\mathbb{D}}<1 \tag{3.121}$$

for $p>1$, *then the operator* $I+\widehat{\Pi}$ *is invertible.*

Proof. By the properties of norm, we get

$$\|\widehat{\Pi}g\|_{L_p(\mathbb{D}} = \sum_{j=1}^{k} \|q_{1j}(z)\Pi_j g + q_{2j}(z)\overline{\Pi_j g}\|_{L_p(\mathbb{D})}$$

$$\leq \sum_{j=1}^{k} \left(q_{1j}(z) + q_{2j}(z)\right) \|\Pi_j g\|_{L_p(\mathbb{D})}$$

$$\leq q_0 \|\Pi_j g\|_{L_p(\mathbb{D})} < 1.$$

If (3.121) holds, then we get $\|\widehat{\Pi}\|_{L_p(\mathbb{D})} < 1$. By Theorem 2.1.3, the operator $I + \widehat{\Pi}$ is invertible.

□

Lemma 3.3.12. *$\widehat{K}$ is a compact operator.*

Proof. By Theorem 2.2.11, operators in $\widehat{K}$ are bounded, and also by Theorem 2.2.12, they are Hölder continuous, in particular they are uniformly continuous. So, by Arzela-Ascoli Theorem they are compact. □

In that case, it is easy to apply Fredholm alternative.

Theorem 3.3.13. *If condition $q_0\|\Pi_j\|_{L_p(\mathbb{D}} < 1$ is satisfied, then the problem (3.113), (3.114) has a solution of the form $w = \sum_{m=0}^{k-1} \bar{z}^m \psi_m + T_{0,k}g(z)$, where g is a solution of (3.117).*

Proof. If condition $q_0\|\Pi_j\|_{L_p(\mathbb{D}} < 1$ holds then by Lemma 3.3.11 $I + \widehat{\Pi}$ is invertible. In previous lemma we have proved that $\widehat{K}$ is compact. From the statement of Nikolskii Criterion, the operator $I + \widehat{\Pi} + \widehat{K}$ is Fredholm operator with index zero. Theorem 2.1.5 implies that the singular integral equation 3.117 has the Fredholm alternative, that means it has at least a solution. Therefore, if g is a solution of (3.117), then $w = \sum_{m=0}^{k-1} \bar{z}^m \psi_m + T_{0,k}g(z)$ is a solution to the problem (3.113), (3.114). □

Alternatively, we can find the solution of the problem (3.113), (3.114) by denoting the solution as $w=w_1+w_2$ where w_1 is the solution of the problem

$$\partial_{\bar{z}}^k w_1 = 0 \quad \text{in } \mathbb{D}, \tag{3.122}$$

$$\partial_{\bar{z}}^l w_1 = \gamma_l(z), \quad 0 \le l \le k-1. \tag{3.123}$$

and w_2 is a solution of the problem

$$\partial_{\bar{z}}^k w_2 + \sum_{j=1}^{k} q_{1j}(z)\partial_{\bar{z}}^{k-j}\partial_z^j w_2 + \sum_{j=1}^{k} q_{2j}(z)\partial_{\bar{z}}^j\partial_z^{k-j}\bar{w}_2$$

$$+\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\partial_{\bar{z}}^{l-m}\partial_z^m w_2 + b_{ml}(z)\partial_{\bar{z}}^m\partial_z^{l-m}\bar{w}_2\right]$$

$$= f(z) - \sum_{j=1}^{k} q_{1j}(z)\partial_{\bar{z}}^{k-j}\partial_z^j w_1 - \sum_{j=1}^{k} q_{2j}(z)\partial_{\bar{z}}^j\partial_z^{k-j}\bar{w}_1$$

$$-\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\partial_{\bar{z}}^{l-m}\partial_z^m w_1 + b_{ml}(z)\partial_{\bar{z}}^m\partial_z^{l-m}\bar{w}_1\right] \quad \text{in} \quad \mathbb{D}, \tag{3.124}$$

$$\partial_{\bar{z}}^l w_2 = 0, \quad 0 \le l \le k-1. \tag{3.125}$$

The other Dirichlet problem with linear differential equation is

$$\partial_z^k w + \sum_{j=1}^{k} q_{1j}(z)\partial_z^{k-j}\partial_{\bar{z}}^j w + \sum_{j=1}^{k} q_{2j}(z)\partial_z^j\partial_{\bar{z}}^{k-j}\bar{w}$$

$$+\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\partial_z^{l-m}\partial_{\bar{z}}^m w + b_{ml}(z)\partial_z^m\partial_{\bar{z}}^{l-m}\bar{w}\right] = f(z) \quad \text{in} \quad \mathbb{D}, \tag{3.126}$$

with the boundary conditions

$$\partial_z^l w = \gamma_l(z), \quad 0 \le l \le k-1. \tag{3.127}$$

Solution of (3.126), (3.127) can be attained by the one for the problem (3.113), (3.114) through complex conjugation and using the Theorem 3.3.4.

4. DIRICHLET PROBLEM FOR TWO DIMENSIONAL PARTIAL DIFFERENTIAL EQUATION WITH COMPLEX VARIABLES

In Chapter 4, we carry ideas in Chapter 3 to bidisc $\mathbb{D}^2 := \mathbb{D}_1 \times \mathbb{D}_2 = \{z = (z_1, z_2) : |z_k| < 1, k = 1, 2\}$. Firstly, we derive the solution of the Dirichlet problem of second order model partial differential equations by using the main results in [2]. Then, we extend the boundary value problem to a general linear differential equation. Under suitable solvability conditions, it is seen that the boundary value problem has a unique solution. To reach that conclusion, the problem is reduced into a singular integral equation. After that, applying the Fredholm Theory, we can study the solvability of the singular integral equation.

4.1. DIRICHLET PROBLEM FOR EQUATION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section we study the solution of the Dirichlet problem for the model equation in $\mathbb{D}^2 := \{z = (z_1, z_2) : |z_1| < 1, \quad |z_2| < 1\} = \mathbb{D}_1 \times \mathbb{D}_2$. We have the following main theorem.

Theorem 4.1.1. *The Dirichlet problem defined as*

$$w_{\bar{z}_1\bar{z}_2} = f(z_1, z_2) \quad in \quad \mathbb{D}_1 \times \mathbb{D}_2, \quad (z_1, z_2) \in \mathbb{D}_1 \times \mathbb{D}_2, \tag{4.1}$$

$$w(z_1, z_2) = \gamma_0 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad (z_1, z_2) \in \partial\mathbb{D}_1 \times \mathbb{D}_2, \tag{4.2}$$

$$w_{\bar{z}_1} = \gamma_1 \quad on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \quad (z_1, z_2) \in \partial\mathbb{D}_2 \times \mathbb{D}_1 \tag{4.3}$$

for $\omega, f \in L_1(\mathbb{D}_1 \times \mathbb{D}_2; \mathbb{C})$, $\gamma_0 \in C(\partial\mathbb{D}_1 \times \mathbb{D}_2; \mathbb{C})$ *and* $\gamma_1 \in C(\partial\mathbb{D}_2 \times \mathbb{D}_1; \mathbb{C})$, $| z_1 |< 1$, $| z_2 |< 1$ *is solvable if and only if*

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}_2} \gamma_1(z_1, \zeta_2) \frac{\bar{z}_2 d\zeta_2}{1 - \bar{z}_2\zeta_2} - \frac{1}{\pi} \iint_{\mathbb{D}_2} f(z_1, \zeta_2) \frac{\bar{z}_2 d\xi_2 d\eta_2}{1 - \bar{z}_2\zeta_2} = 0 \tag{4.4}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2-z_2}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}f(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}d\xi_1 d\eta_1=0. \tag{4.5}$$

The solution is uniquely given as

$$w(z_1,z_2)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1-z_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2-z_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}f(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}. \tag{4.6}$$

Proof. We decompose the Dirichlet problem (4.1) - (4.3) into the system of Dirichlet problems of first order:

$$w_{\bar{z}_1}=\omega \quad \text{in} \quad \mathbb{D}_1, \quad w=\gamma_0 \quad \text{on} \quad \partial\mathbb{D}_1\times\mathbb{D}_2, \tag{4.7}$$

$$\omega_{\bar{z}_2}=f \quad \text{in} \quad \mathbb{D}_2, \quad \omega=\gamma_1 \quad \text{on} \quad \partial\mathbb{D}_2\times\mathbb{D}_1, \quad \text{i.e.} \quad w_{\bar{z}_1}=\gamma_1 \quad \text{on} \quad \partial\mathbb{D}_2\times\mathbb{D}_1. \tag{4.8}$$

By Theorem 3.1.3, the Dirichlet problem (4.7) for $w_{\bar{z}_1}\in L_1(\mathbb{D}_1;\mathbb{C})$, $\gamma_0\in C(\partial\mathbb{D}_1\times\mathbb{D}_2;\mathbb{C})$, $|z_1|<1$ is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1}-\frac{1}{\pi}\iint_{\mathbb{D}_1}\omega(\zeta_1,z_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}d\xi_1 d\eta_1=0, \tag{4.9}$$

the solution then is uniquely given by

$$w(z_1,z_2)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1-z_1}-\frac{1}{\pi}\iint_{\mathbb{D}_1}\omega(\zeta_1,z_2)\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\cdot \tag{4.10}$$

Moreover, the Dirichlet problem (4.8) for $f\in L_1(\mathbb{D}_2;\mathbb{C})$, $\gamma_1\in C(\partial\mathbb{D}_2\times\mathbb{D}_1;\mathbb{C})$, $|z_2|<1$ is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_1(z_1,\zeta_2)\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2}-\frac{1}{\pi}\iint_{\mathbb{D}_2}f(z_1,\zeta_2)\frac{\bar{z_2}}{1-\bar{z_2}\zeta_2}d\xi_2 d\eta_2=0, \tag{4.11}$$

the solution then is uniquely given by

$$\omega(z_1,z_2)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_1(z_1,\zeta_2)\frac{d\zeta_2}{\zeta_2-z_2}-\frac{1}{\pi}\iint_{\mathbb{D}_2}f(z_1,\zeta_2)\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}\cdot \tag{4.12}$$

We insert the equation (4.12) into equations (4.9) and (4.10). For this we will obtain $\omega(\zeta_1, z_2)$. By changing the variables, we get

$$\omega(\zeta_1, z_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_2} \gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2} - \frac{1}{\pi}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}.$$

Then it follows that

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} \gamma_0(\zeta_1, z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1}$$

$$-\frac{1}{\pi}\iint_{\mathbb{D}_1}\left\{\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2} - \frac{1}{\pi}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\right\}\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}d\xi_1 d\eta_1 = 0$$

and

$$w(z_1, z_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1, z_2)\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\iint_{\mathbb{D}_1}\left\{\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\right.$$

$$\left. -\frac{1}{\pi}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\right\}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}. \tag{4.13}$$

So, we obtain (4.5) and (4.6), respectively. □

Note. *We can decompose the Dirichlet problem (4.1)-(4.3) into the system of Dirichlet problems of first order as :*

$$w_{\bar{z_2}} = \omega \quad \text{in} \quad \mathbb{D}_2, \quad w = \gamma_0 \quad \text{on} \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \tag{4.14}$$

$$\omega_{\bar{z_1}} = f \quad \text{in} \quad \mathbb{D}_1, \quad \omega = \gamma_1 \quad \text{on} \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad \text{i.e.} \quad w_{\bar{z_2}} = \gamma_1 \quad \text{on} \quad \partial\mathbb{D}_1 \times \mathbb{D}_2. \tag{4.15}$$

Let us consider the Dirichlet problem defined as

$$w_{\bar{z_1}\bar{z_2}} = 2\bar{z_1} \quad \text{in } \mathbb{D}_1 \times \mathbb{D}_2,$$

$$w(z_1, z_2) = \bar{z_1}^2\bar{z_2} \quad \text{on } \partial\mathbb{D}_1 \times \mathbb{D}_2,$$

$$w_{\bar{z_1}} = 2\bar{z_1}\bar{z_2} \quad \text{on } \partial\mathbb{D}_2 \times \mathbb{D}_1.$$

Since,

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_1}\bar{\zeta_1}^2\bar{z_2}\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint\limits_{\mathbb{D}_1}\int\limits_{\partial\mathbb{D}_2}2\bar{\zeta_1}\bar{\zeta_2}\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2-z_2}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^2}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}2\bar{\zeta_1}\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}d\xi_1 d\eta_1=0,$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_2}2\bar{z_1}\bar{\zeta_2}\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2}-\frac{1}{\pi}\iint\limits_{\mathbb{D}_2}2\bar{z_1}\frac{\bar{z_2}d\xi_2 d\eta_2}{1-\bar{z_2}\zeta_2}=0.$$

Solvability conditions are satisfied. So, we can compute the solution of the Dirichlet problem by the following formula

$$w(z_1,z_2)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_1}\bar{\zeta_1}^2\bar{z_2}\frac{d\zeta_1}{\zeta_1-z_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint\limits_{\mathbb{D}_1}\int\limits_{\partial\mathbb{D}_2}2\bar{\zeta_1}\bar{\zeta_2}\frac{d\zeta_2}{\zeta_2-z_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}$$

$$+\frac{1}{\pi^2}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}2\bar{\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}=\bar{z_1}^2\bar{z_2}.$$

4.2. DIRICHLET PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS

In this section we will extend the notion of solution of (4.1)-(4.3) to a linear differential equation. We employ the Laurent Schwarz notations.

Let $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_i\in\mathbb{N}$, $i=1,\dots,4$ and $\partial_\alpha := \frac{\partial^{|\alpha|}}{\partial_{z_1}^{\alpha_1}\partial_{\bar{z_1}}^{\alpha_2}\partial_{z_2}^{\alpha_3}\partial_{\bar{z_2}}^{\alpha_4}}$. We will try to find a solution $w(z_1,z_2)\in W^{p,2}(\mathbb{D}_1\times\mathbb{D}_2)$ to the Dirichlet problem

$$\partial_{\bar{z_1}}\partial_{\bar{z_2}}w(z_1,z_2)+\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}}q_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$+\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}}r_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}+\sum_{|\alpha|\le 1}s_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$+\sum_{|\alpha|\le 1}t_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}=f(z_1,z_2)\quad\text{in}\quad\mathbb{D}_1\times\mathbb{D}_2 \tag{4.16}$$

satisfying the nonhomogeneous boundary conditions

$$w(z_1,z_2)=\gamma_0(z_1,z_2)\quad\text{on}\quad\partial\mathbb{D}_1\times\mathbb{D}_2, \tag{4.17}$$

$$w_{\bar{z}_1} = \gamma_1(z_1, z_2) \quad \text{on} \quad \partial\mathbb{D}_{\mathbb{2}} \times \mathbb{D}_{\mathbb{1}}, \tag{4.18}$$

where

$$s_\alpha, t_\alpha, f \in L_p(\mathbb{D}_{\mathbb{1}} \times \mathbb{D}_{\mathbb{2}})$$

and $q_\alpha, r_\alpha, s_\alpha, t_\alpha$ are measurable bounded functions satisfying the condition

$$+ \sum_{\substack{|\alpha|=2, \alpha_1+\alpha_2<2, \\ \alpha_3+\alpha_4<2, (\alpha_1,\alpha_3)\neq(1,1)}} \mid q_\alpha(z_1, z_2) \mid + \mid r_\alpha(z_1, z_2) \mid$$

$$+ \mid s_{(0,1,0,0)} \mid + \mid s_{(0,0,0,1)} \mid + \mid t_{(0,1,0,0)} \mid + \mid t_{(0,0,0,1)} \mid \leq q_0 < 1. \tag{4.19}$$

Note. *In the linear equation (4.16), we restrict the values of* $\alpha_i, \quad i = 1, \dots, 4$ *i.e. the derivatives of* $w(z_1, z_2)$. *Otherwise, we obtain the unbounded operators, and it is a deep investigation of integro differential equations.*

Lemma 4.2.1. *The Dirichlet problem (4.16)-(4.18) is equivalent to the singular integral equation*

$$(I + \widehat{\Pi} + \widehat{K})g(z_1, z_2) = f(z_1, z_2) - \Theta(z_1, z_2), \tag{4.20}$$

where

$$w(z_1, z_2) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}_{\mathbb{1}}} \gamma_0(\zeta_1, z_2) \frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi} \frac{1}{2\pi i} \iint_{\mathbb{D}_{\mathbb{1}}} \int_{\partial\mathbb{D}_{\mathbb{2}}} \gamma_1(\zeta_1, \zeta_2) \frac{d\zeta_2}{\zeta_2 - z_2} \frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$

$$+ \frac{1}{\pi^2} \iint_{\mathbb{D}_{\mathbb{1}}} \iint_{\mathbb{D}_{\mathbb{2}}} f(\zeta_1, \zeta_2) \frac{d\xi_1 d\eta_1}{\zeta_1 - z_1} \frac{d\xi_2 d\eta_2}{\zeta_2 - z_2} = \psi_{\partial\mathbb{D}_{\mathbb{1}}, \partial\mathbb{D}_{\mathbb{2}}}(\gamma_0, \gamma_1) + T_{\mathbb{D}_{\mathbb{1}};0,1} T_{\mathbb{D}_{\mathbb{2}};0,1} g(z_1, z_2),$$

$$\widehat{\Pi} g = \sum_{\substack{|\alpha|=2, \alpha_1+\alpha_1<2, \\ \alpha_3+\alpha_4<2, (\alpha_1,\alpha_3)\neq(1,1)}} q_\alpha(z_1, z_2) T_{\mathbb{D}_{\mathbb{1}};-\alpha_2, 1-\alpha_1} T_{\mathbb{D}_{\mathbb{2}};-\alpha_4, 1-\alpha_3} g$$

$$+ \sum_{\substack{|\alpha|=2, \alpha_1+\alpha_1<2, \\ \alpha_3+\alpha_4<2, (\alpha_1,\alpha_3)\neq(1,1)}} r_\alpha(z_1, z_2) \overline{T_{\mathbb{D}_{\mathbb{1}};-\alpha_2, 1-\alpha_1} T_{\mathbb{D}_{\mathbb{2}};-\alpha_4, 1-\alpha_3} g} + s_{(0,1,0,0)} T_{\mathbb{D}_{\mathbb{1}};-1,1} T_{\mathbb{D}_{\mathbb{2}};0,1} g$$

$$+ s_{(0,0,0,1)}(z_1, z_2) T_{\mathbb{D}_{\mathbb{1}};0,1} T_{\mathbb{D}_{\mathbb{2}};-1,1} g + t_{(0,1,0,0)} \overline{T_{\mathbb{D}_{\mathbb{1}};-1,1} T_{\mathbb{D}_{\mathbb{2}};0,1} g} + t_{(0,0,0,1)}(z_1, z_2) \overline{T_{\mathbb{D}_{\mathbb{1}};0,1} T_{\mathbb{D}_{\mathbb{2}};-1,1} g},$$

$$\widehat{K} g = s_{(1,0,0,0)} T_{\mathbb{D}_{\mathbb{2}};0,1} g + s_{(0,0,1,0)} T_{\mathbb{D}_{\mathbb{1}};0,1} g + s_{(0,0,0,0)} T_{\mathbb{D}_{\mathbb{1}};0,1} T_{\mathbb{D}_{\mathbb{2}};0,1} g + t_{(1,0,0,0)} \overline{T_{\mathbb{D}_{\mathbb{2}};0,1} g}$$

$$+ t_{(0,0,1,0)} \overline{T_{\mathbb{D}_{\mathbb{1}};0,1} g} + t_{(0,0,0,0)} \overline{T_{\mathbb{D}_{\mathbb{1}};0,1} T_{\mathbb{D}_{\mathbb{2}};0,1} g},$$

$$\Theta(z_1, z_2) = \partial_{\bar{z}_1} \partial_{\bar{z}_2} (\psi_{\partial\mathbb{D}_{\mathbb{1}}, \partial\mathbb{D}_{\mathbb{2}}}(\gamma_0, \gamma_1)) + \sum_{\substack{|\alpha|=2, \alpha_1+\alpha_1<2, \\ \alpha_3+\alpha_4<2, (\alpha_1,\alpha_3)\neq(1,1)}} q_\alpha(z_1, z_2) \partial_\alpha (\psi_{\partial\mathbb{D}_{\mathbb{1}}, \partial\mathbb{D}_{\mathbb{2}}}(\gamma_0, \gamma_1))$$

$$+\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_1<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1))}+\sum_{|\alpha|\leq 1} s_\alpha(z_1,z_2)\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1))$$

$$+\sum_{|\alpha|\leq 1} t_\alpha(z_1,z_2)\overline{\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1))}.$$

Proof. We write the linear differential equation (4.16) as

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}w(z_1,z_2)=f(z_1,z_2)-\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} q_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$-\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}-\sum_{|\alpha|\leq 1} s_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$-\sum_{|\alpha|\leq 1} t_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)} \quad \text{in} \quad \mathbb{D}_1\times\mathbb{D}_2.$$

Then the Dirichlet problem (4.16)-(4.18) turns into

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}w(z_1,z_2)=\tilde{f} \quad \text{in} \quad \mathbb{D}_1\times\mathbb{D}_2, \tag{4.21}$$

$$w(z_1,z_2)=\gamma_0(z_1,z_2) \quad \text{on} \quad \partial\mathbb{D}_1\times\mathbb{D}_2, \tag{4.22}$$

$$w_{\bar{z}_1}=\gamma_1(z_1,z_2) \quad \text{on} \quad \partial\mathbb{D}_2\times\mathbb{D}_1, \tag{4.23}$$

where

$$\tilde{f}=f(z_1,z_2)-\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} q_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$-\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}$$

$$-\sum_{|\alpha|\leq 1} s_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)-\sum_{|\alpha|\leq 1} t_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)} \quad \text{in} \quad \mathbb{D}_1\times\mathbb{D}_2.$$

This type solution of problem (4.21)-(4.23) follows from Theorem 4.1.1, so

$$w(z_1,z_2)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1-z_1}$$

$$-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2-z_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}g(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2} \tag{4.24}$$

or shortly

$$w(z_1,z_2) = \psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g(z_1,z_2). \tag{4.25}$$

Let $\partial_{\bar{z_1}}\partial_{\bar{z_2}}w(z_1,z_2) = g(z_1,z_2)$. Then

$$g = \partial_{\bar{z_1}}\partial_{\bar{z_2}}(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g(z_1,z_2)).$$

It follows that,

$$g + \sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} q_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2) + \sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}$$
$$+ \sum_{|\alpha|\leq 1} s_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2) + \sum_{|\alpha|\leq 1} t_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)} = f(z_1,z_2) \quad \text{in} \quad \mathbb{D}_1\times\mathbb{D}_2, \tag{4.26}$$

Here, the derivatives of $\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1)$ with respect to $\bar{z_1}$ is equal to zero, because $\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1)$ is an analytic function with respect to $\bar{z_1}$. If we integrate

$$\partial_{\bar{z_1}}\partial_{\bar{z_2}}w(z_1,z_2) = g(z_1,z_2),$$

with respect to $\bar{z_1}$, $\bar{z_2}$ respectively, we get

$$w(z_1,z_2) = \psi_{0,\mathbb{D}_1} + T_{\mathbb{D}_1;0,1}(\psi_{1,\mathbb{D}_2}) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g(z_1,z_2),$$

where $\psi_{0,\mathbb{D}_1}$, $\psi_{1,\mathbb{D}_2}$ are analytic functions with respect to $\bar{z_1}$, $\bar{z_2}$,respectively. Then

$$\partial_\alpha w = \partial_\alpha(\psi_{0,\mathbb{D}_1} + T_{\mathbb{D}_1;0,1}(\psi_{1,\mathbb{D}_2}) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g(z_1,z_2)). \tag{4.27}$$

Substituting the derivative (4.27) into (4.26), and arranging the operators, we obtain the singular integral equation (4.20).

Therefore, $g(z_1,z_2)$ is the solution of the singular integral equation (4.20) if and only if $w(z_1,z_2) = \psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g(z_1,z_2)$ is the solution of the Dirichlet problem (4.16)-(4.18). □

Solvability of the Singular Integral Equation

Lemma 4.2.2. *If the condition (4.19) is satisfied, then* $I + \widehat{\Pi}$ *is invertible.*

Proof.

$$\|\widehat{\Pi}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} \le \| \sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} q_\alpha(z_1,z_2)T_{\mathbb{D}_1;-\alpha_2,1-\alpha_1}T_{\mathbb{D}_2;-\alpha_4,1-\alpha_3}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+\| \sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} r_\alpha(z_1,z_2)\overline{T_{\mathbb{D}_1;-\alpha_2,1-\alpha_1}T_{\mathbb{D}_2;-\alpha_4,1-\alpha_3}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+\|s_{(0,1,0,0)}T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;0,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + \|s_{(0,0,0,1)}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+\|t_{(0,1,0,0)}\overline{T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;0,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + \|t_{(0,0,0,1)}\overline{T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$\le \sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} | q_\alpha(z_1,z_2) | \|T_{\mathbb{D}_1;-\alpha_2,1-\alpha_1}T_{\mathbb{D}_2;-\alpha_4,1-\alpha_3}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ \sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} | r_\alpha(z_1,z_2) | \|\overline{T_{\mathbb{D}_1;-\alpha_2,1-\alpha_1}T_{\mathbb{D}_2;-\alpha_4,1-\alpha_3}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | s_{(0,1,0,0)} | \|T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;0,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | s_{(0,0,0,1)} | \|T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | t_{(0,1,0,0)} | \|\overline{T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;0,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | t_{(0,0,0,1)} | \|\overline{T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$= | q_{(1,0,0,1)} | \|T_{\mathbb{D}_2;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | q_{(0,1,1,0)} | \|T_{\mathbb{D}_1;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | q_{(0,1,0,1)} | \|T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | r_{(1,0,0,1)} | \|\overline{T_{\mathbb{D}_2;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | r_{(0,1,1,0)} | \|\overline{T_{\mathbb{D}_1;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | r_{(0,1,0,1)} | \|\overline{T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | s_{(0,1,0,0)} | \|T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;0,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | s_{(0,0,0,1)}(z_1,z_2) | \|T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | t_{(0,1,0,0)} | \|\overline{T_{\mathbb{D}_1;-1,1}T_{\mathbb{D}_2;0,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | t_{(0,0,0,1)}(z_1,z_2) | \|\overline{T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}.$$

Using Theorem 2.2.13, we get

$$\|\widehat{\Pi}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} \le | q_{(1,0,0,1)} | M(p)\|g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | q_{(0,1,1,0)} | M(p)\|g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | q_{(0,1,0,1)} | M(p)\|T_{\mathbb{D}_1;-1,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | r_{(1,0,0,1)} | M(p)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | r_{(0,1,1,0)} | M(p)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | r_{(0,1,0,1} | M(p)\|\overline{T_{\mathbb{D}_1;-1,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | s_{(0,1,0,0)} | M(p)\|T_{\mathbb{D}_2;0,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | s_{(0,0,0,1)} | M(p)\|T_{\mathbb{D}_1;0,1}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ | t_{(0,1,0,0)} | M(p)\|\overline{T_{\mathbb{D}_2;0,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | s_{(0,0,0,1)} | M(p)\|\overline{T_{\mathbb{D}_1;0,1}g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)},$$

where M is a constant depending on the quantities in the parenthesis.

Moreover, by Theorem 2.2.13 and Theorem 2.2.11 , we have that

$$\|\widehat{\Pi}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} \le | q_{(1,0,0,1)} | M(p)\|g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + | q_{(0,1,1,0)} | M(p)\|g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ \mid q_{(0,1,0,1)} \mid M(p)M'(p)\|g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + \mid r_{(1,0,0,1)} \mid M(p)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ \mid r_{(0,1,1,0)} \mid M(p)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + \mid r_{(0,1,0,1} \mid M(p)M'(p)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)}$$

$$+ \mid s_{(0,1,0,0)} \mid M(p)M'(p,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + \mid s_{(0,0,0,1)} \mid M(p)M'(p,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)}$$

$$+ \mid t_{(0,1,0,0)} \mid M(p)M'(p,\mathbb{D}_2)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} + \mid s_{(0,0,0,1)} \mid M(p)M'(p,\mathbb{D}_2)\|\overline{g}\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)},$$

where M' is a constant depending on the quantities in the parenthesis.

$$\|\widehat{\Pi}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} \leq C(p,\mathbb{D}_1,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)}(\mid q_{(1,0,0,1)} \mid + \mid q_{(0,1,1,0)} \mid + \mid q_{(0,1,0,1)} \mid$$

$$+ \mid r_{(1,0,0,1)} \mid + \mid r_{(0,1,1,0)} \mid + \mid r_{(0,1,0,1} \mid + \mid s_{(0,1,0,0)} \mid + \mid s_{(0,0,0,1)} \mid + \mid t_{(0,1,0,0)} \mid + \mid s_{(0,0,0,1)} \mid)$$

$$= C(p,\mathbb{D}_1,\mathbb{D}_2)\left(\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} \mid q_\alpha(z_1,z_2) \mid + \mid r_\alpha(z_1,z_2) \mid + \mid s_{(0,1,0,0)} \mid + \mid s_{(0,0,0,1)} \mid\right.$$

$$\left. + \mid t_{(0,1,0,0)} \mid + \mid t_{(0,0,0,1)} \mid\right) \|g\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)}.$$

From equation (4.19) it follows

$$\|\widehat{\Pi}g\|_{L_p(\mathbb{D}_1\times\mathbb{D}_2)} \leq C(p,\mathbb{D}_1,\mathbb{D}_2)q_0\|g\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)} < 1.$$

By Theorem 2.1.3, $I + \widehat{\Pi}$ is invertible. □

Lemma 4.2.3. *For measurable bounded functions $s_\alpha, t_\alpha, f \in L_p(\mathbb{D}_1 \times \mathbb{D}_2)$ and for $p > 2$ the operator $\widehat{K}$ is a compact operator.*

Proof. We start with the boundedness of $\widehat{K}$.

$$\mid \widehat{K}g \mid = \mid s_{(1,0,0,0)}T_{\mathbb{D}_2;0,1}g + s_{(0,0,1,0)}T_{\mathbb{D}_1;0,1}g + s_{(0,0,0,0)}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g + t_{(1,0,0,0)}\overline{T_{\mathbb{D}_2;0,1}g}$$

$$+t_{(0,0,1,0)}\overline{T_{\mathbb{D}_1;0,1}g} + t_{(0,0,0,0)}\overline{T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g} \mid .$$

By the triangle inequality it follows that,

$$\mid \widehat{K}g \mid \leq \mid s_{(1,0,0,0)}T_{\mathbb{D}_2;0,1}g \mid + \mid s_{(0,0,1,0)}T_{\mathbb{D}_1;0,1}g \mid + \mid s_{(0,0,0,0)}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g \mid$$

$$+ \mid t_{(1,0,0,0)}\overline{T_{\mathbb{D}_2;0,1}g} \mid + \mid t_{(0,0,1,0)}\overline{T_{\mathbb{D}_1;0,1}g} \mid + \mid t_{(0,0,0,0)}\overline{T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g} \mid$$

$$= \mid s_{(1,0,0,0)} \mid\mid T_{\mathbb{D}_2;0,1}g \mid + \mid s_{(0,0,1,0)} \mid\mid T_{\mathbb{D}_1;0,1}g \mid + \mid s_{(0,0,0,0)} \mid\mid T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g \mid$$

$$+ \mid t_{(1,0,0,0)} \mid\mid \overline{T_{\mathbb{D}_2;0,1}g} \mid + \mid t_{(0,0,1,0)} \mid\mid \overline{T_{\mathbb{D}_1;0,1}g} \mid + \mid t_{(0,0,0,0)} \mid\mid \overline{T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}g} \mid .$$

By Theorem 2.2.11, we can write

$$\mid \widehat{K}g \mid < \mid s_{(1,0,0,0)} \mid S(p,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_2)} + \mid s_{(0,0,1,0)} \mid S(p,\mathbb{D}_1)\|g\|_{L_p(\mathbb{D}_1)}$$

$$+ \mid s_{(0,0,0,0)} \mid S(p,\mathbb{D}_1)\|T_{\mathbb{D}_2;0,1}g\|_{L_p(\mathbb{D}_1)} + \mid t_{(1,0,0,0)} \mid S(p,\mathbb{D}_2)\|\overline{g}\|_{L_p(\mathbb{D}_2)}$$

$$+ \mid t_{(0,0,1,0)} \mid S(p,\mathbb{D}_1)\|\overline{g}\|_{L_p(\mathbb{D}_1)} + \mid t_{(0,0,0,0)} \mid S(p,\mathbb{D}_1)\|\overline{T_{\mathbb{D}_2;0,1}g}\|_{L_p(\mathbb{D}_1)},$$

where S is a constant depending on the quantities in the parenthesis.

$$\mid \widehat{K}g \mid < \mid s_{(1,0,0,0)} \mid S(p,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_2)} + \mid s_{(0,0,1,0)} \mid S(p,\mathbb{D}_1)\|g\|_{L_p(\mathbb{D}_1)}$$

$$+ \mid s_{(0,0,0,0)} \mid S(p,\mathbb{D}_1)S'(p,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)} + \mid t_{(1,0,0,0)} \mid S(p,\mathbb{D}_2)\|\overline{g}\|_{L_p(\mathbb{D}_2)}$$

$$+ \mid t_{(0,0,1,0)} \mid S(p,\mathbb{D}_1)\|\overline{g}\|_{L_p(\mathbb{D}_1)} + \mid t_{(0,0,0,0)} \mid S(p,\mathbb{D}_1)S'(p,\mathbb{D}_2)\|\overline{g}\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)}$$

$$\leq C(p,\mathbb{D}_1,\mathbb{D}_2)\|g\|_{L_p(\mathbb{D}_2\times\mathbb{D}_2)},$$

where S', C are constants depending on the quantities in the parenthesis. There are one-dimensional and two-dimensional operators in $\widehat{K}$. In the previous chapter, we have proved that one-dimensional operators are Hölder continuous, in particular, they are uniformly continuous.

We will show that two-dimensional operators in $\widehat{K}$ are Hölder continuous. By Theorem 2.2.12 it follows that

$$\mid T_{\mathbb{D}_i;0,1}T_{\mathbb{D}_j;0,1}g(z_1,z_2) - T_{\mathbb{D}_i;0,1}T_{\mathbb{D}_j;0,1}g(z_1^*,z_2^*) \mid$$

$$\leq M(p,\mathbb{D}_i)\|T_{\mathbb{D}_j;0,1}g\|_{L_p(\mathbb{D}_i)} \mid (z_1,z_2) - (z_1^*,z_2^*) \mid^{\alpha},$$

where $0 < \alpha < 1$, $p > 2$, $(z_1,z_2),(z_1^*,z_2^*) \in \mathbb{D}_1 \times \mathbb{D}_2$ for $i \neq j$, $i,j = 1,2$. Applying Theorem 2.2.12 again, we obtain,

$$\mid T_{\mathbb{D}_i;0,1}T_{\mathbb{D}_j;0,1}g(z_1,z_2) - T_{\mathbb{D}_i;0,1}T_{\mathbb{D}_j;0,1}g(z_1^*,z_2^*) \mid$$

$$\leq M(p,\mathbb{D}_i)C(p,\mathbb{D}_j)\|g\|_{L_p(\mathbb{D}_i\times\mathbb{D}_j)} \mid (z_1,z_2) - (z_1^*,z_2^*) \mid^{\alpha} .$$

Each term in $\widehat{K}$ are uniformly continuous and bounded. By Arzela-Ascoli Theorem, the operators in $\widehat{K}g$ are compact operators. □

Now, we can apply the Fredholm alternative.

Theorem 4.2.4. *If the condition (4.19) is satisfied, then the Dirichlet problem (4.16)-(4.17) -(4.18) has a solution of the form* $w(z_1, z_2) = \psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0, \gamma_1) + T_{\mathbb{D}_1;0,1} T_{\mathbb{D}_2;0,1} g(z_1, z_2)$*, where* $g(z_1, z_2)$ *is a solution of the singular integral equation (4.20).*

Proof. If the condition (4.19) is satisfied then by Lemma 4.2.2, $I + \widehat{\Pi}$ is invertible and $\widehat{K}$ is compact. From Nikolskii Criterion it follows that $I + \widehat{\Pi} + \widehat{K}$ is Fredholm operator with index zero. Theorem 2.1.5 implies that the singular integral equation (4.20) has the Fredholm alternative. Fredholm alternative states that the equation has at least a solution. Therefore, if $g(z_1, z_2)$ is a solution of the singular integral equation (4.20), then $w(z_1, z_2) = \psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0, \gamma_1) + T_{\mathbb{D}_1;0,1} T_{\mathbb{D}_2;0,1} g(z_1, z_2)$ is a solution to the Dirichlet problem (4.16)-(4.18). □

4.3. DIRICHLET PROBLEM FOR EQUATION OF FOURTH ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section, we construct the solution and solvability conditions of the Dirichlet problem for the fourth order differential equation. Applying the main Theorem 4.1.1 in previous section, we obtain the following fundamental theorem.

Theorem 4.3.1. *The Dirichlet problem defined as*

$$w_{\bar{z}_1{}^2\bar{z}_2{}^2} = f(z_1, z_2) \in \mathbb{D}_1 \times \mathbb{D}_2, \quad i.e. \quad (z_1, z_2) \in \mathbb{D}_1 \times \mathbb{D}_2, \tag{4.28}$$

$$w(z_1, z_2) = \gamma_0 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad i.e. \quad (z_1, z_2) \in \partial\mathbb{D}_1 \times \mathbb{D}_2, \tag{4.29}$$

$$w_{\bar{z}_1} = \gamma_1 \quad on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \quad i.e. \quad (z_1, z_2) \in \partial\mathbb{D}_2 \times \mathbb{D}_1, \tag{4.30}$$

$$w_{\bar{z}_1\bar{z}_2} = \gamma_2 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad i.e. \quad (z_1, z_2) \in \partial\mathbb{D}_1 \times \mathbb{D}_2, \tag{4.31}$$

$$w_{\bar{z}_1{}^2\bar{z}_2} = \gamma_3 \quad on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \quad i.e. \quad (z_1, z_2) \in \partial\mathbb{D}_2 \times \mathbb{D}_1 \tag{4.32}$$

for $w_{\bar{z}_1}, w_{\bar{z}_1\bar{z}_2}, w_{\bar{z}_1{}^2\bar{z}_2}, f \in L_1(\mathbb{D}_1 \times \mathbb{D}_2; \mathbb{C})$*,* $\gamma_0, \gamma_2 \in C(\partial\mathbb{D}_1 \times \mathbb{D}_2; \mathbb{C})$ *and* $\gamma_1, \gamma_3 \in C(\partial\mathbb{D}_2 \times$

$\mathbb{D}_1;\mathbb{C})$, $|z_1|<1, |z_2|<1$ *is solvable if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_2(\zeta_1,z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_3(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2-z_2}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}f(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}d\xi_1 d\eta_1=0. \tag{4.33}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_3(z_1,\zeta_2)\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2}-\frac{1}{\pi}\iint_{\mathbb{D}_2}f(z_1,\zeta_2)\frac{\bar{z_2}d\xi_2 d\eta_2}{1-\bar{z_2}\zeta_2}=0, \tag{4.34}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2-z_2}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_1}\gamma_2(\tilde{\zeta_1},\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\tilde{\zeta_1}}{\tilde{\zeta_1}-\zeta_1}d\xi_1 d\eta_1\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}$$

$$-\frac{1}{\pi^3}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_2}\gamma_3(\tilde{\zeta_1},\tilde{\zeta_2})\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\tilde{\zeta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^4}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\iint_{\mathbb{D}_2}f(\tilde{\zeta_1},\tilde{\zeta_2})\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\tilde{\xi_2}d\tilde{\eta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}d\xi_1 d\eta_1=0 \tag{4.35}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_1(z_1,\zeta_2)\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2}-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_1}\gamma_2(\zeta_1,\zeta_2)\frac{\bar{z_2}}{1-\bar{z_2}\zeta_2}\frac{d\zeta_1}{\zeta_1-z_1}d\xi_2 d\eta_2$$

$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_2}\gamma_3(\zeta_1,\tilde{\zeta_2})\frac{\bar{z_2}}{1-\bar{z_2}\zeta_2}\frac{d\tilde{\zeta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}d\xi_2 d\eta_2$$

$$-\frac{1}{\pi^3}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\iint_{\mathbb{D}_2}f(\zeta_1,\tilde{\zeta_2})\frac{\bar{z_2}}{1-\bar{z_2}\zeta_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\tilde{\xi_2}d\tilde{\eta_2}}{\tilde{\zeta_2}-\zeta_2}d\xi_2 d\eta_2=0. \tag{4.36}$$

Then the solution is uniquely given as

$$w(z_1,z_2)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1-z_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2-z_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}$$

$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_1}\gamma_2(\tilde{\zeta_1},\zeta_2)\frac{d\tilde{\zeta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}$$

$$-\frac{1}{\pi^3}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_2}\gamma_3(\tilde{\zeta_1},\tilde{\zeta_2})\frac{d\tilde{\zeta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}$$

$$+\frac{1}{\pi^4}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\iint_{\mathbb{D}_2}f(\tilde{\zeta_1},\tilde{\zeta_2})\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\tilde{\xi_2}d\tilde{\eta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}\cdot \tag{4.37}$$

Proof. We construct a system of Dirichlet problems of second order:

$$w_{\bar{z_1}\bar{z_2}} = \omega \quad \text{in } \mathbb{D}_1 \times \mathbb{D}_2, \tag{4.38}$$

$$w = \gamma_0 \quad \text{on} \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad w_{\bar{z_1}} = \gamma_1 \quad \text{on} \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \tag{4.39}$$

$$\omega_{\bar{z_1}\bar{z_2}} = f \quad \text{in } \mathbb{D}_1 \times \mathbb{D}_2, \tag{4.40}$$

$$\omega = \gamma_2 \quad \text{on} \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad \omega_{\bar{z_1}} = \gamma_3 \quad \text{on} \quad \partial\mathbb{D}_2 \times \mathbb{D}_1. \tag{4.41}$$

By Theorem 4.1.1, the Dirichlet problem (4.38)-(4.39) is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2} \gamma_1(z_1,\zeta_2)\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2} - \frac{1}{\pi}\iint_{\mathbb{D}_2} \omega(z_1,\zeta_2)\frac{\bar{z_2}d\xi_2 d\eta_2}{1-\bar{z_2}\zeta_2} = 0 \tag{4.42}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} \gamma_0(\zeta_1,z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} \gamma_1(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1$$
$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} \omega(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1 = 0. \tag{4.43}$$

The solution then is uniquely given as

$$w(z_1,z_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} \gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} \gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$
$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} \omega(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2} \tag{4.44}$$

and the Dirichlet problem (4.40)-(4.41) is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2} \gamma_3(z_1,\zeta_2)\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2} - \frac{1}{\pi}\iint_{\mathbb{D}_2} f(z_1,\zeta_2)\frac{\bar{z_2}d\xi_2 d\eta_2}{1-\bar{z_2}\zeta_2} = 0 \tag{4.45}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} \gamma_2(\zeta_1,z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} \gamma_3(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1$$
$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1 = 0. \tag{4.46}$$

The solution then is uniquely given as

$$\omega(z_1,z_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} \gamma_2(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} \gamma_3(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}. \tag{4.47}$$

We insert the equation (4.47) into equations (4.42)-(4.44). For this, we will construct $\omega(z_1,\zeta_2)$. Changing of the variables, we get

$$\omega(z_1,\zeta_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_2(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_3(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}. \tag{4.48}$$

Then, it follows

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{\bar{z}_1 d\zeta_1}{1-\bar{z}_1\zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{d\zeta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1 +$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\left\{\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_2(\tilde{\zeta}_1,\tilde{\zeta}_2)\frac{d\tilde{\zeta}_1}{\tilde{\zeta}_1 - \zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_3(\tilde{\zeta}_1,\tilde{\zeta}_2)\frac{d\tilde{\zeta}_2}{\tilde{\zeta}_2 - \zeta_2}\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1 - \zeta_1}\right.$$

$$\left.+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} f(\tilde{\zeta}_1,\tilde{\zeta}_2)\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1 - \zeta_1}\frac{d\tilde{\xi}_2 d\tilde{\eta}_2}{\tilde{\zeta}_2 - \zeta_2}\right\}\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1 = 0$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}\gamma_1(z_1,\zeta_2)\frac{\bar{z}_2 d\zeta_2}{1-\bar{z}_2\zeta_2} - \frac{1}{\pi}\iint_{\mathbb{D}_2}\left\{\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_2(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1 - z_1}\right.$$

$$\left.-\frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_3(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1} + \frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} f(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\right\}$$

$$\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}d\xi_2 d\eta_2 = 0 \tag{4.49}$$

and

$$w(z_1,z_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\left\{\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}\gamma_2(\tilde{\zeta}_1,\tilde{\zeta}_2)\frac{d\tilde{\zeta}_1}{\tilde{\zeta}_1 - \zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2}\gamma_3(\tilde{\zeta}_1,\tilde{\zeta}_2)\frac{d\tilde{\zeta}_2}{\tilde{\zeta}_2 - \zeta_2}\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1 - \zeta_1}\right.$$

$$\left.+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} f(\tilde{\zeta}_1,\tilde{\zeta}_2)\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1 - \zeta_1}\frac{d\tilde{\xi}_2 d\tilde{\eta}_2}{\tilde{\zeta}_2 - \zeta_2}\right\}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}. \tag{4.50}$$

□

Note. *Alternatively, we may decompose the Dirichlet problem defined in Theorem 4.3.1 into the system of Dirichlet problems of second order as:*

$$w_{\bar{z_1}\bar{z_1}} = \omega \quad in \quad \mathbb{D}_1 \times \mathbb{D}_2, \quad w = \gamma_0 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \quad w_{\bar{z_1}} = \gamma_1 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \tag{4.51}$$

$$\omega_{\bar{z_1}\bar{z_1}} = f \quad in \quad \mathbb{D}_1 \times \mathbb{D}_2, \quad w = \gamma_0 \quad on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \quad \omega_{\bar{z_1}} = \gamma_1 \quad on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1. \tag{4.52}$$

Example 4.3.2. *Let us consider the Dirichlet problem defined as*

$$\begin{gathered} w_{\bar{z_1}^2\bar{z_2}^2} = 36\bar{z_1}\bar{z_2} \quad in \quad \mathbb{D}_1 \times \mathbb{D}_2, \\ w(z_1, z_2) = \bar{z_1}^3\bar{z_2}^3 + 2z_2 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \\ w_{\bar{z_1}} = 3\bar{z_1}^2\bar{z_2}^3 on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \\ w_{\bar{z_1}\bar{z_2}} = 9\bar{z_1}^2\bar{z_2}^2 \quad on \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \\ w_{\bar{z_1}^2\bar{z_2}} = 18\bar{z_1}\bar{z_2}^2 \quad on \quad \partial\mathbb{D}_2 \times \mathbb{D}_1. \end{gathered} \tag{4.53}$$

Since,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} 9\bar{\zeta_1}^2\bar{z_2}^2\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} 18\bar{\zeta_1}\bar{\zeta_2}^2\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^2}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2} 36\bar{\zeta_1}\bar{\zeta_2}\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1 = 0, \tag{4.54}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2} 18\bar{z_1}\bar{\zeta_2}^2\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2} - \frac{1}{\pi}\iint_{\mathbb{D}_2} 36\bar{z_1}\bar{\zeta_2}\frac{\bar{z_2}d\xi_2 d\eta_2}{1-\bar{z_2}\zeta_2} = 0, \tag{4.55}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}(\bar{\zeta_1}^3\bar{z_2}^3 + 2z_2)\frac{\bar{z_1}d\zeta_1}{1-\bar{z_1}\zeta_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} 3\bar{\zeta_1}^2\bar{\zeta_2}^3\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\zeta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_1} 9\bar{\zeta_1}^2\bar{\zeta_2}^2\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\tilde{\zeta_1}}{\tilde{\zeta_1}-\zeta_1}d\xi_1 d\eta_1\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}$$

$$-\frac{1}{\pi^3}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_2} 18\bar{\zeta_1}\bar{\zeta_2}^2\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\tilde{\zeta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}d\xi_1 d\eta_1$$

$$+\frac{1}{\pi^4}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\iint_{\mathbb{D}_2} 36\bar{\zeta_1}\bar{\zeta_2}\frac{\bar{z_1}}{1-\bar{z_1}\zeta_1}\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\tilde{\xi_2}d\tilde{\eta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}d\xi_1 d\eta_1 = 0 \tag{4.56}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2} 3\bar{z_1}^2\bar{\zeta_2}^3\frac{\bar{z_2}d\zeta_2}{1-\bar{z_2}\zeta_2} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_1} 9\bar{\zeta_1}^2\bar{\zeta_2}^2\frac{\bar{z_2}}{1-\bar{z_2}\zeta_2}\frac{d\zeta_1}{\zeta_1 - z_1}d\xi_2 d\eta_2$$

$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}\int\limits_{\partial\mathbb{D}_2} 18\bar{\zeta}_1\bar{\zeta}_2^{\,2}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{d\tilde{\zeta}_2}{\tilde{\zeta}_2-\zeta_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}d\xi_2 d\eta_2$$

$$-\frac{1}{\pi^3}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}\iint\limits_{\mathbb{D}_2} 36\bar{\zeta}_1\bar{\zeta}_2\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\tilde{\xi}_2 d\tilde{\eta}_2}{\tilde{\zeta}_2-\zeta_2}d\xi_2 d\eta_2 = 0. \quad (4.57)$$

Since the solvability conditions are satisfied, we can compute the $w(z_1,z_2)$ *by the following formula*

$$w(z_1,z_2)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_1}(\bar{\zeta}_1^{\,3}\bar{z}_2^{\,3}+2z_2)\frac{d\zeta_1}{\zeta_1-z_1}-\frac{1}{\pi}\frac{1}{2\pi i}\iint\limits_{\mathbb{D}_1}\int\limits_{\partial\mathbb{D}_2}3\bar{\zeta}_1^{\,2}\bar{\zeta}_2^{\,3}\frac{d\zeta_2}{\zeta_2-z_2}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}$$

$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}\int\limits_{\partial\mathbb{D}_1}9\bar{\zeta}_1^{\,2}\bar{\zeta}_2^{\,2}\frac{d\zeta_1}{\zeta_1-z_1}\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1-\zeta_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}$$

$$-\frac{1}{\pi^3}\frac{1}{2\pi i}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}\int\limits_{\partial\mathbb{D}_2}18\bar{\zeta}_1\bar{\zeta}_2^{\,2}\frac{d\tilde{\zeta}_2}{\tilde{\zeta}_2-\zeta_2}\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}$$

$$+\frac{1}{\pi^4}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_1}\iint\limits_{\mathbb{D}_2}\iint\limits_{\mathbb{D}_2}36\bar{\zeta}_1\bar{\zeta}_2\frac{d\tilde{\xi}_1 d\tilde{\eta}_1}{\tilde{\zeta}_1-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\tilde{\xi}_2 d\tilde{\eta}_2}{\tilde{\zeta}_2-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}=\bar{z}_1^{\,3}\bar{z}_2^{\,3}+2z_2.$$

4.4. DIRICHLET PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS

In this section we will extend the notion of solution of the model equation in Theorem 4.3.1 to a linear differential equation. We will try to find a solution $w(z_1,z_2)\in W^{p,2}(\mathbb{D}_1\times\mathbb{D}_2)$ to the Dirichlet problem

$$\partial_{\bar{z}_1}^2\partial_{\bar{z}_2}^2 w(z_1,z_2)+\sum_{\substack{|\alpha|=4,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_1,\alpha_3)\neq(2,2)}} q_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$+\sum_{\substack{|\alpha|=4,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_1,\alpha_3)\neq(2,2)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}$$

$$+\sum_{\substack{|\alpha|\le 3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3}} s_\alpha(z_1,z_2)\partial_\alpha w(z_1,z_2)$$

$$+\sum_{\substack{|\alpha|\le 3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3}} t_\alpha(z_1,z_2)\overline{\partial_\alpha w(z_1,z_2)}=f(z_1,z_2)\quad \text{in}\quad \mathbb{D}_1\times\mathbb{D}_2 \quad (4.58)$$

satisfying the nonhomogeneous boundary conditions

$$w(z_1,z_2)=\gamma_0\quad\text{on}\quad\partial\mathbb{D}_1\times\mathbb{D}_2,\quad (z_1,z_2)\in\partial\mathbb{D}_1\times\mathbb{D}_2, \quad (4.59)$$

$$w_{\bar{z_1}} = \gamma_1 \quad \text{on} \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \qquad (z_1, z_2) \in \partial\mathbb{D}_2 \times \mathbb{D}_1, \tag{4.60}$$

$$w_{\bar{z_1}\bar{z_2}} = \gamma_2 \quad \text{on} \quad \partial\mathbb{D}_1 \times \mathbb{D}_2, \qquad (z_1, z_2) \in \partial\mathbb{D}_1 \times \mathbb{D}_2, \tag{4.61}$$

$$w_{\bar{z_1}^2\bar{z_2}} = \gamma_3 \quad \text{on} \quad \partial\mathbb{D}_2 \times \mathbb{D}_1, \quad (z_1, z_2) \in \partial\mathbb{D}_2 \times \mathbb{D}_1, \tag{4.62}$$

where

$$s_\alpha, t_\alpha, f \in L_p(\mathbb{D}_1 \times \mathbb{D}_2)$$

and $q_\alpha, r_\alpha, s_\alpha, t_\alpha$ are measurable bounded functions satisfying the condition

$$\sum_{\substack{|\alpha|=2,\alpha_1+\alpha_2<2,\\ \alpha_3+\alpha_4<2,(\alpha_1,\alpha_3)\neq(1,1)}} |\, q_\alpha(z_1,z_2) \,| + |\, r_\alpha(z_1,z_2) \,| + \sum_{\substack{|\alpha|\leq 3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_2,\alpha_4)\neq(0,0)}} |\, s_\alpha(z_1,z_2) \,| + |\, t_\alpha(z_1,z_2) \,|$$
$$\leq q_0 < 1. \tag{4.63}$$

Note. *In the linear differential equation, we restrict the values of α_i, $\quad i = 1, \ldots, 4$ or the derivatives of $w(z_1, z_2)$. Otherwise, we obtain the unbounded operators, and it is deep investigation of integro differential equations.*

Lemma 4.4.1. *The Dirichlet problem is equivalent to the singular integral equation*

$$(I + \widehat{\Pi} + \widehat{K})g(z_1, z_2) = f(z_1, z_2) - \Theta(z_1, z_2), \tag{4.64}$$

where

$$w(z_1,z_2) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_1} \gamma_0(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\int_{\partial\mathbb{D}_2} \gamma_1(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2 - z_2}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$
$$+\frac{1}{\pi^2}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_1} \gamma_2(\tilde{\zeta_1},\zeta_2)\frac{d\tilde{\zeta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}$$
$$-\frac{1}{\pi^3}\frac{1}{2\pi i}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\int_{\partial\mathbb{D}_2} \gamma_3(\tilde{\zeta_1},\tilde{\zeta_2})\frac{d\tilde{\zeta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}$$
$$+\frac{1}{\pi^4}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_1}\iint_{\mathbb{D}_2}\iint_{\mathbb{D}_2} f(\tilde{\zeta_1},\tilde{\zeta_2})\frac{d\tilde{\xi_1}d\tilde{\eta_1}}{\tilde{\zeta_1}-\zeta_1}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\frac{d\tilde{\xi_2}d\tilde{\eta_2}}{\tilde{\zeta_2}-\zeta_2}\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}, \tag{4.65}$$

or shortly,

$$w(z_1,z_2) = \psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g, \tag{4.66}$$

$$\widehat{\Pi}g = \sum_{\substack{|\alpha|=4,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_1,\alpha_3)\neq(2,2)}} q_\alpha(z_1,z_2)\partial_\alpha(T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g)$$

$$+ \sum_{\substack{|\alpha|=4,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_1,\alpha_3)\neq(2,2)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha(T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g)}$$

$$+ \sum_{\substack{|\alpha|\leq3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_2,\alpha_4)\neq(0,0)}} s_\alpha(z_1,z_2)\partial_\alpha(T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g)$$

$$+ \sum_{\substack{|\alpha|\leq3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_2,\alpha_4)\neq(0,0)}} t_\alpha(z_1,z_2)\overline{\partial_\alpha(T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g)},$$

$$\widehat{K}g = \sum_{\substack{|\alpha|\leq3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_2,\alpha_4)=(0,0)}} s_\alpha(z_1,z_2)\partial_\alpha(T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g)$$

$$+ \sum_{\substack{|\alpha|\leq3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_2,\alpha_4)=(0,0)}} t_\alpha(z_1,z_2)\overline{\partial_\alpha(T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g)},$$

$$\Theta(z_1,z_2) = \partial_{\bar{z}_1}\partial_{\bar{z}_2}(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$$

$$+ \sum_{\substack{|\alpha|=4,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_1,\alpha_3)\neq(2,2)}} q_\alpha(z_1,z_2)\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$$

$$+ \sum_{\substack{|\alpha|=4,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3,(\alpha_1,\alpha_3)\neq(2,2)}} r_\alpha(z_1,z_2)\overline{\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3)}$$

$$+ \sum_{\substack{|\alpha|\leq3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3}} s_\alpha(z_1,z_2)\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$$

$$+ \sum_{\substack{|\alpha|\leq3,\alpha_1+\alpha_2<3,\\ \alpha_3+\alpha_4<3}} t_\alpha(z_1,z_2)\overline{\partial_\alpha(\psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3)}$$

In a similar manner in previous section, we obtain the following result.

Theorem 4.4.2. *If the condition (4.63) is satisfied, then the Dirichlet problem (4.58)-(4.62) has a solution of the form*

$$w(z_1,z_2) = \psi_{\partial\mathbb{D}_1,\partial\mathbb{D}_2}(\gamma_0,\gamma_1,\gamma_2,\gamma_3) + T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_1;0,1}T_{\mathbb{D}_2;0,1}T_{\mathbb{D}_2;0,1}g, \tag{4.67}$$

where g is a solution of the singular integral equation (4.64).

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