BAYESIAN ANALYSIS OF TIME SERIES USING LINDLEY'S APPROXIMATION

by

Karina Perilioglu

Submitted to Graduate School of Natural and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Mathematics

Yeditepe University
2016

BAYESIAN ANALYSIS OF TIME SERIES USING LINDLEY'S APPROXIMATION

APPROVED BY:

Prof. Dr. Alexandros Papadopoulos
(Thesis Supervisor) ....

Prof. Dr. Müjgan Tez ....

Assoc. Prof. Dr. Mustafa Nadar ....

Assist. Prof. Dr. Elif Çiğdem Kaspar ....

Assist. Prof. Dr. Oğul Esen ....

DATE OF APPROVAL: ..../..../2016

ACKNOWLEDGEMENTS

I am sincerely and heartily grateful to my thesis supervisor, Prof. Dr. Alexandros Papadopoulos, for the continuous support of my Ph.D study and related study, for his patience, motivation, and immense knowledge. I am sure it would have not been possible without his help.

Besides my thesis supervisor, I would like to thank the rest of my thesis committee, especially Doc. Dr. Mustafa Nadar and Dr. Assist. Prof. Dr. Oğul Esen for their insightful comments and suggestions for improvements.

I owe a lot to my parents, brothers and parents-in-law, who encouraged, helped me and longed to see this achievement come true.

I am very much indebted to my family, my husband Ahmet and daughter Eira, for their endless love and support, which makes everything more beautiful.

ABSTRACT

BAYESIAN ANALYSIS OF TIME SERIES USING LINDLEY'S APPROXIMATION

The autoregressive model of order p and moving-average model of order q are analyzed when the parameters and the precision of the error term are random variables. In the analysis the squared error (SE) and linear exponential (LINEX) loss functions are utilized. Using four different priors, the Bayes estimators of the parameters are derived. Under independent truncated normal priors for the parameters and gamma or improper priors for the precision of the error term, the Bayes estimators are found not to be in a closed form. Therefore, Lindley's approximation is used to obtain the approximate estimators. A computer simulation study compares the maximum likelihood and the Bayes estimates obtained using Lindley's approximation and Markov chain Monte Carlo (MCMC) techniques, in particular, Gibbs sampler. Under independent Uniform priors for the parameters and gamma or improper priors for the precision of the error term, the posterior distributions of the model parameters and the precision of the error term have a closed form, however using the LINEX loss function, the Bayes estimators of the model parameters are intractable. In this case, the truncated normal approximation is used to derive the approximate Bayes estimators. A computer simulation study is employed to compare the maximum likelihood and the Bayes estimates. Examples are given to illustrate the findings.

ÖZET

LİNDLEY'İN YAKLAŞIMI İLE BAYES ZAMAN SERİSİ ANALİZİ

Model parametrelerinin ve hata terimlerinin rassal değişken olduğu durumlarda, otoregresif modelinin p derecesi ve hareketli ortalama modelinin q derecesi analiz edilmiştir. Analizde karesel hata (SE) ve dogrusal üstel (LINEX) kayıp fonksiyonları kullanılmıştır. Dört degişik önsel kullanarak, parametrelerin Bayes tahmin edicileri türetilmiştir. Bayes tahmin edicilerinin kapalı formda olmadığı, parametreler için bağımsız kesilmiş normal önseller ve hata terimlerinin kesinligi için gama veya belirsiz önseller olması durumlarında bulunmuştur. Bundan dolayı Lindley'in yaklaşımı, yaklaşık tahmin edici elde etmek için kullanılmıştır. Bilgisayar simulasyon çalışmasi ile en çok olabilirlik ve Bayes tahminleri karşılaştırılmaktadır. Lindley'in yaklaşımı ve Markov zinciri Monte Carlo (MCMC) Gibbs örneklemesi kullanılarak Bayes tahminleri elde edilmiştir. Parametreler için bağımsız tekdüze önseller ve hata terimlerinin kesinliği için gama veya belirsiz önseller olması durumlarında, model parameterlerinin sonsal dağılımlarının ve hata terimlerinin kesinliğinin kapalı formu olduğu bulunmaktadır. Fakat LINEX kayıp fonksiyonlarında, model parametrelerinin Bayes tahmin edicileri uygun formda değildir. Bu durumda yaklaşık Bayes tahmin edicilerinin türetilmesi için kesikli normal yaklaşım kullanılmıştır. Bilgisayar simulasyon çalışması en çok olabilirlik ve Bayes tahminlerini karşılaştırmak için kullanılmıştır. Bulguları göstermek için örnekler verilmiştir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

A time series is a sequence of observations taken sequentially in time. An intrinsic feature of a time series is that, typically, adjacent observations are dependent. Examples of time series are daily rainfall, monthly levels of unemployment, weekly share prices. Time series data are used in statistics, econometrics, mathematical finance, medicine, meteorology and many other areas.

The literature about the theoretical and methodological aspects of the time series models is vast and most of it is non-Bayesian. For non-Bayesian, theory and methodology see books by Box and Jenkins [1], Priestley [2], Brokwell and Davis [3], Hamilton [4], Chatfield [5], among others. The subject of Bayesian Time Series was pioneered by Zellner [6] and Broemeling [7]. The books by Pole et al. [8] and Barber et al. [9] are a good introduction to a Bayesian time series analysis. Kitagawa and Gersch [10] analyzed the problems of modeling stationary and non-stationary time series using Bayesian stochastic regression treatment, so called "smoothness prior" approach. The fundamental statistical ideas for this method are the likelihood of the Bayesian model and its use as a measure of the goodness of fit of the model. West and Harrison [11] focused on Dynamic Linear Models (DLM) that are a class of Bayesian Forecasting Models and their uses in forecasting and time series analysis. Autoregressive and moving average (AR, MA) processes, that we focus on in this study, can also be written in a DLM form and analyzed using the aforementioned theory.

The Bayesian analysis of time series consists of determining the posterior distribution of the parameters of the model and the predictive distribution of future observations. The use of Bayes theorem allows us to update the prior information we often possess before seeing the data into a posterior distribution by incorporating the information, called likelihoods, provided by the observed data. The use of prior information is very helpful, especially for the inference for the small-size data. This improves the precision of the estimators of the model parameters predicted values.

The interest in Bayesian methods grew rapidly during the last four decades. This is mostly caused by the increased appreciation of the advantages that Bayesian inference involves (see Steel [12]). Berger [13] provided a concise overview of Bayesian activity up to the year 2000.

Both authors note the enormous impact of the development of the computational facilities and the evolution of Markov Chain Monte Carlo (MCMC) methods on Bayesian statistics. As Berger [13] states, "it would be hard to find an area of human investigation in which there does not exist some level of Bayesian work". Bayesian methods are being used more and more extensively in many applications fields, see [14]. Economics and econometrics ([15], [16]), finance ([17], [18], [19], [20], [21]), engineering ([22], [23]), genetics ([24], [25], [26]), medicine ([27], [28]) and physical sciences ([29], [30]) are only a few application areas, more references and application fields are given in [13]. Below we provide an overview of the main components of Bayesian inference and methods that can be employed in the analysis.

How to choose the prior is an important issue in Bayesian analysis. There are several approaches to this problem, the most popular are the subjective, objective and empirical methods. The empirical Bayes approach assumes that the prior can be estimated from the data itself, this method is often seen as a bridge between the classical frequentists approach and Bayesian inference. In the subjective Bayes approach the prior measures what is known or believed before the experiment takes place, whereas in the objective Bayes approach, the prior should be chosen based on mathematical properties, such as reference priors that in some sense maximize information gain. Berger [31] briefly describes all of these methods and provides good arguments for the use of the objective approach.

Any and all of these approaches can be used, even within the same model. For example, a subjective prior on parameters may be employed where there is a considerable amount of prior knowledge and a reference prior on other parameters that are less important or less understood. The impact of the prior declines as the data sample increases, when there is a lot of data, different priors tend to lead to very similar conclusions.

In the subjective Bayes approach, the prior reflects researcher's believes and the information that is available before the analysis, and depending on this, it can take different functional form. Nevertheless, in the vast of the Bayesian analysis non-informative and conjugate priors are utilized. This choice leads to posterior and predictive distributions that can be derived analytically. Zellner [6] obtained the posterior and predictive distributions for AR(1) and AR(2) processes using Jeffrey's improper prior. Broemeling [7] and Shaarawy and Broemeling [32] discussed Bayesian analysis of invertible and stationary ARMA processes. Using conditional likelihood function and Normal-Gamma and Jeffrey's improper priors for the model parameters they derived the posterior distributions for the model parameters and one-step ahead predictive distributions. Schervish and Tsay [33] gave a fully Bayesian analysis of autoregressive models with or without exogenous variables. In this study, we consider four different sets of priors: Truncated Normal and Uniform priors for AR and MA models parameters and Gamma and Improper priors for the precision parameter.

As noted by Fan and Yao [34], Zeithammer and Lenk [35], among others, it can be shown that the Bayesian analysis method for AR model is equivalent to some Bayesian multivariate regressive analysis. There has been increasing interest in Bayesian methods in application to various multivariate time series models, such as vector autoregressive (VAR), vector autoregressive-moving average (VARMA) and multivariate generalized autoregressive conditional heteroskedasticity (MGARCH) models. Generally, multivariate models require estimation of a large number of parameters, thus over-parameterization is often a problem - sample size is not sufficient to estimate the parameters of the model. Bayesian methods can be employed to reduce the number of parameters and make the use model feasible by utilizing priors that provide a logical and consistent method of imposing parameter restrictions. Sevinc and Ergun [36] studied the VAR model using Bayesian point of view. In the Bayesian VAR analysis of the Turkish unemployment rate and industrial production index, they employed five different priors for the model parameters, namely Minessota, Diffuse, Normal - Wishard, Normal-Diffuse and Extended Natural Conjugate (ENC) priors. Using the first three priors, the posterior distributions have a closed form but when the Normal-Diffuse and ENC priors are employed, the posterior distributions are not in a known form, hence the MCMC methods are utilized.

As it was previously noted, the posterior and predictive distributions are analytically intractable for most of the prior distributions of model parameters. Therefore, an approximation is required. As it was pointed out by Mahmoud [37], the use of numerical computer routines may not converge for a given set of data. Thus, when the posterior is complicated, researchers have usually resorted to the Markov Chain Monte Carlo (MCMC) method and successfully achieve a satisfactory computational answer; see [38] and [39]. A development of Markov chain Monte Carlo methods enabled sampling from a posterior distribution by constructing a Markov chain which converges to the posterior distribution in a given statistical inference problem. There exist different ways to generate the required Markov chain, the most popular being the Gibbs sampler, introduced by Geman and Geman [40] and developed by Gelfand

and Smith [41]; and the Metropolis-Hastings sampler, proposed in Metropolis et al. [42] and generalized by Hastings [43]. Using the MCMC methods, the Bayesian analysis can be applied to a wide variety of models and prior distributions, for the description of the methods and their applications, see Chib and Greenberg [44] and Tsay [45], among others.

As alternative to the MCMC methods, one can use Tierney's and Kadane's approximation (see [46]) or Lindley's method. Lindley [47] developed approximation for the ratio of integrals that can be used to approximate the posterior and predictive distributions and relevant expectations. Singh et al. [48] employed the LINEX loss function and derived the Bayes estimators of the generalized-exponential distribution parameters using Lindley's approximation. They compared the obtained approximate Bayes estimates to the ML estimates using different combinations of prior parameters and found that the approximate Bayes estimators perform better in terms of the loss in most cases. Nadar et al. [49] used Lindley's approximation in the Bayesian estimation of P(Y<X) for Kumaraswamy's distribution. Anderson et al. [50] applied this approximation for the Bayesian estimation in non-homogeneous AR(1) process assuming autoregression of coefficients. In this study, we apply Lindley's approximation for the AR and MA processes with normally distributed errors when the choice of the prior distributions of parameters results in analytically intractable posterior and predictive distributions.

There exist different criteria for estimating unknown distribution parameters. Bayesian estimators are based on minimization of the expected loss function. Although there do not exist some certain rules one could follow to choose the loss function, this choice is fundamental to construct an optimal forecast. Depending on the study goal, different loss functions should be employed. In this study we consider two loss functions: symmetric squared error (SE) and asymmetric linear - exponential (LINEX). The SE loss function is widely used due to its mathematical tractability. However, there are no reasons why the consequences of under-predicting should be the same as the costs from over-predicting. For example, a life insurer prefers not to underestimate mortality, since it would lead to decreased profitability of the product by having to pay out more death benefit claims than it was expected. Granger [51] also questions the use of symmetric loss function and gives some more examples where an asymmetric loss function should be employed. A few examples from his text are the following. The cost of arriving 10 min early in the airport is quite different from arriving 10 min late. The cost of having a computer that is 10 per cent too small for a task is different than being 10 per cent too big.

Satchell et al. [52] applied the LINEX loss function to various of volatility models, including GARCH models, and derived one-step ahead volatility forecasts. Their empirical study on Glaxo Wellcome company return volatility data using GARCH(1, 1) model suggests that the forecasts derived using the LINEX loss function outperform the conventional GARCH estimates. Singh et.al. [53] studied the inverse Gaussian distribution and obtained the Bayes estimators using the SE and general entropy loss functions. They compared the derived parameter estimators to the ML estimators and observed that, generally, the Bayes estimators derived using the asymmetric general entropy loss function results in lower losses than that of the ML estimators.

The present study is organized as follows. Section 2 reviews the study objectives and methodology that will be used. In Section 3 we introduce some concepts and fundamental results that will be used in the study. In Section 4 we consider an ARMA (p, q) process and assume the model which was analyzed by Shaarawy and Broemeling [32]. We summarize their results and then using them find the Bayes estimators of model parameters and one-step ahead forecast under the SE and LINEX loss functions. Sections 5 and 7 give the Bayesian setting for AR(p) and MA(q) models, respectively. Independent truncated normal and independent uniform priors for the AR(p) and MA(q) coefficients and gamma and improper priors for the precision parameter are used. Using conditional likelihood function and Lindley's approximation the approximate Bayesian estimators under the SE and LINEX loss functions are derived. In Sections 6 and 8 computer simulation study compares the maximum likelihood and the Bayesian estimators for AR(p) and MA(q) models, respectively. Real time series examples are given to illustrate the findings. Finally, in Section 9 we overview the findings and future study.

2. STUDY OBJECTIVES AND METHODOLOGY

In this section we overview the present study objectives and methodology that will be used.

2.1. OBJECTIVES

This study has the following major objectives:

(i) Determine the Bayes estimators of AR(p) and MA(q) parameters and one-step ahead prediction of a future observation using the normal-gamma conjugate prior under the LINEX loss function. We expand the results of Shaarawy and Broemeling [32] by using different loss function.

(ii) Determine the Bayes estimators of AR(p), $p = 1, 2,$ model parameters using four different priors: independent uniform priors for AR(p) coefficients and gamma prior for the precision, independent uniform priors for AR(p) coefficients and improper prior for the precision, independent truncated normal priors for AR(p) coefficients and gamma prior for the precision, independent truncated normal priors for AR(p) coefficients and improper prior for the precision.

(iii) Determine the Bayes estimators of one-step ahead prediction of the future observation for AR(p), $p = 1, 2$ model using four different priors: independent uniform priors for AR(p) coefficients and gamma prior for the precision, independent uniform priors for AR(p) coefficients and improper prior for the precision, independent truncated normal priors for AR(p) coefficients and gamma prior for the precision, independent truncated normal priors for AR(p) coefficients and improper prior for the precision.

(iv) Determine the Bayes estimators of MA(q), $q = 1, 2,$ model parameters using four different priors: independent uniform priors for MA(q) coefficients and gamma prior for the precision, independent uniform priors for MA(q) coefficients and improper prior for the precision, independent truncated normal priors for MA(q) coefficients and gamma prior for the precision, independent truncated normal priors for MA(q) coefficients and improper prior for the precision.

(v) Determine the Bayes estimators of one-step ahead prediction of the future observation for MA(q), $q = 1, 2,$ model using four different priors: independent uniform priors for MA(q) coefficients and gamma prior for the precision, independent uniform priors for MA(q) coefficients and improper prior for the precision, independent truncated normal priors for MA(q) coefficients and gamma prior for the precision, independent truncated normal priors for MA(q) coefficients and improper prior for the precision.

(vi) Using a computer simulation study compare the Bayes and maximum likelihood estimates of AR(p) and MA(q) parameters and one-step ahead forecasts.

(vii) Obtain empirical time series that follow AR(p) and MA(q) $p, q = 1, 2,$ models, calculate the Bayes and maximum likelihood estimates of the model parameters and one-step ahead prediction assuming independent truncated normal priors for the coefficients and improper prior for the precision. Then compare the Bayes and maximum likelihood estimates.

The SE and LINEX loss functions will be employed.

2.2. METHODOLOGY

To address the study questions we will analyse the AR(p) and MA(q) models. For the first objective we will mainly use the article of Shaarawy and Broemeling [32] and the methods described there. The Bayes estimator under the LINEX loss function involves the posterior moment generating function. Using the normal-gamma prior, Shaarawy and Broemeling [32] showed that the posterior distributions of the coefficients' parameters and a predictive distribution of a one-step ahead forecast are t-distributions. The moment generating function of the t-distribution is undefined which prohibits the use of a direct formula. Thus, a proper approximation is required. To approximate the t-distribution we will use the normal distribution with proper mean and variance parameters. Since the degrees of freedom of t-distribution is sufficient large, this approximation is valid.

For the uniform prior case for the coefficient parameters, we will show that the posterior distribution of the coefficients' parameters follow a truncated t-distribution. Similarly, to the t-distribution case, the moment generating function does not have a tractable form. Therefore, under the LINEX loss function, the normal approximation will be employed and approximate estimators will be defined

When the resulting posterior or predictive are intractable, we will apply Lindley's approximation and will obtain the approximate estimators.

To perform the numerical study, we will follow the procedures given below.

Given the autoregressive process of order p, AR(p),

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + \epsilon_t, \;\; t = 1, 2, \ldots,$$

where $\phi_1, \ldots, \phi_p$ are real unknown parameters, and $\{\epsilon_t\}$ is a sequence of independent and normally distributed random variables with mean zero and precision τ.

(i) Simulate τ.

(ii) Simulate $\phi_i, i = 1, \ldots, p$.

(iii) Generate the AR(p) series : $y_1, \ldots, y_n$ for some n.

(iv) Calculate the ML estimates for the parameters and find the error.

(v) Calculate the Bayes estimates for the parameters under the SE and LINEX loss functions and find the error.

(vi) Calculate the ML and Bayes estimates of one step ahead forecasts under the SE and LINEX loss functions and find the errors.

(vii) Repeat the above procedures N times and calculate the mean errors under the SE and LINEX loss functions.

These procedures described above are suitable for the AR(p) model. For the MA(q) model the procedures will be analogous.

For the empirical study the required zero-mean stationary series will be obtained using differentiation of the original data and/or similar procedures .

2.3. MATERIALS

To perform numerical analysis a numerical computing and programming language MATLAB and R will be used.

3. BACKGROUND MATERIALS AND PRELIMINARIES

This chapter introduces some concepts and fundamental results that are used in the present study. For the stationarity we follow definitions Brokwell and Davis [3], for the autoregressive and moving average processes - by Box and Jenkins [1].

3.1. STATIONARITY

A stationary time series is defined as follows.

Definition 1. *(Stationarity) The time series $\{Y_t, t \in \mathbb{Z}\}$, with index set $\mathbb{Z}$, is said to be stationary if*

(i) $\mathrm{E}(Y_t)^2 < \infty$ *for all $t \in \mathbb{Z}$,*

(ii) $Y_t = m$, *where m is some constant, for all $t \in \mathbb{Z}$,*

(iii) $\mathrm{Cov}(Y_s, Y_t) = \mathrm{Cov}(Y_{s+\tau}, Y_{t+\tau})$ *for all $s, t, \tau \in \mathbb{Z}$.*

Stationarity as just defined is frequently referred to as second order or weak stationarity. Weak stationarity means that a stochastic process $\{Y_t\}$ has a finite variation, constant first moment and that the second moment $\mathrm{Cov}(Y_s, Y_t)$ only depends on $t - s$ and does not depend on s or t.

3.2. AUTOREGRESSIVE PROCESS

Autoregressive processes are often used in describing situations in which the present value of a time series depends on its preceding values plus a random shock. We define the autoregressive process of order p, AR(p), by

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + \epsilon_t, \quad t = 1, 2, \ldots, \tag{3.1}$$

where $\phi_1, \ldots, \phi_p$ are real unknown parameters, and $\{\epsilon_t\}$ is a sequence of independent and identically distributed random variables with mean zero and precision τ. Namely, an autoregressive model of order p states that Y_t is the linear function of the previous p values of the series plus an error term. Alternatively, AR(p) process can be given by

$$\phi_p(B)Y_t = \epsilon_t, \tag{3.2}$$

where $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$. For the AR(p) process to be stationary, the roots of $\phi_p(B) = 0$ must lie outside of the unit circle.

3.3. MOVING AVERAGE PROCESS

Moving average processes are useful in describing phenomena in which events produce an immediate effect that only lasts for a short periods of time.

We define the moving average process of order q, MA(q), by

$$Y_t = \epsilon_t - \sum_{i=1}^{q} \theta_i \epsilon_{t-i}, \ \ t = 1, 2, \ldots, \tag{3.3}$$

where $\theta_1, \ldots, \theta_q$ are real unknown parameters, and $\{\epsilon_t\}$ is a sequence of independent and identically distributed random variables with mean zero and precision τ. The MA(q) process can also be written in the following equivalent form

$$Y_t = \theta_q(B)\epsilon_t, \tag{3.4}$$

where $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$. Because $1 + \theta_1^2 + \cdots + \theta_q^2 < \infty$, a finite moving average process is always stationary.

3.4. AUTOREGRESSIVE - MOVING AVERAGE PROCESS

The autoregressive moving average process, ARMA(p, q), is defined by

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} - \sum_{i=1}^{q} \theta_i \epsilon_{t-i} + \epsilon_t \quad (3.5)$$

where $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ are real unknown parameters, and $\{\epsilon_t\}$ is a sequence of independent and identically distributed random variables with mean zero and precision τ. The ARMA(p,q) process can also be written in the following equivalent form

$$\phi_p(B) Y_t = \theta_q(B) \epsilon_t, \quad (3.6)$$

where $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$. The autoregressive moving average process is stationary provided that the characteristic equation $\phi_p(B) = 0$ has all its roots lying outside the unit circle.

3.5. LOSS FUNCTIONS AND BAYES ESTIMATORS

The well-known squared-error loss function (SE) is given by

$$l_{SE}(\omega, \delta) = (\omega - \delta)^2, \quad (3.7)$$

where ω is a univariate parameter and δ is its estimate. This loss function is symmetric, thus it has an implicit assumption that the costs from under-predicting and over-predicting are same. It can be easily shown that the value $\delta(\omega)$ that minimizes the posterior expectation of l_{SE} in Equation 3.7, given a sample $S_n = (Y_1, \ldots, Y_n)'$, is the posterior expectation

$$\hat{\delta}_{SE} = \mathrm{E}_{\Omega|S_n}(\delta|S_n) \quad (3.8)$$

One can use the asymmetric linear-exponential (LINEX) loss function which was first proposed by Varian [54] and is given as

$$l_{LINEX}(\omega,\delta) = \mathrm{e}^{\gamma(\delta-\omega)} - \gamma(\delta-\omega) - 1 \tag{3.9}$$

where ω is a univariate parameter and $\gamma \neq 0$. The parameter γ is known and gives the degree of asymmetry. If $\gamma > 0$ and the errors $\delta - \omega$ are positive, the LINEX loss function is almost exponential and for negative errors almost linear, in this situation overestimation is a more important than underestimation. Where $\gamma < 0$, underestimation is more important than overestimation. Let $M_{\Omega|S_n}(t) = \mathrm{E}_{\Omega|S_n}(\mathrm{e}^{t\omega})$ denote the moment generating function of the posterior density function of Ω given S_n. It can be easily verified that the value of $\delta(\omega)$ that minimizes $\mathrm{E}_{\Omega|S_n}(l(\omega,\delta_{LINEX}(\omega))$ Equation 3.9 is

$$\hat{\delta}_{LINEX} = -\frac{1}{\gamma}\ln(M_{\Omega|S_n})(-\gamma) \tag{3.10}$$

provided that $M_{\Omega|S_n}(\cdot)$ exists and is finite.

3.6. LINDLEY'S APPROXIMATION

Lindley [47] developed approximate procedures for the evaluation of the ratio of two integrals which are in the form

$$\frac{\int w(\boldsymbol{\theta})\exp(L(\boldsymbol{\theta}))\mathrm{d}\boldsymbol{\theta}}{\int g(\boldsymbol{\theta})\exp(L(\boldsymbol{\theta}))\mathrm{d}\boldsymbol{\theta}}$$

where $\boldsymbol{\theta} = (\theta_1,\theta_2,\ldots,\theta_n)$, $L(\boldsymbol{\theta})$ is the logarithm of the likelihood function, and $g(\boldsymbol{\theta})$ and $w(\boldsymbol{\theta}) = v(\boldsymbol{\theta})g(\boldsymbol{\theta})$ are arbitrary functions of $\boldsymbol{\theta}$. The posterior expectation of the function $v(\boldsymbol{\theta})$, for given sample $\mathbf{y}$, is

$$\mathrm{E}[v(\boldsymbol{\theta})|\mathbf{y}] = \frac{\int v(\boldsymbol{\theta})\exp(L(\boldsymbol{\theta})+\rho(\boldsymbol{\theta}))\mathrm{d}\boldsymbol{\theta}}{\int \exp(L(\boldsymbol{\theta})+\rho(\boldsymbol{\theta}))\mathrm{d}\boldsymbol{\theta}} \tag{3.11}$$

where $L(\boldsymbol{\theta})+\rho(\boldsymbol{\theta})$ is the logarithm of the posterior distribution of $\boldsymbol{\theta}$ except for the normalizing constant and $\rho(\boldsymbol{\theta})$ is the logarithm of $g(\boldsymbol{\theta})$. Expanding $L(\boldsymbol{\theta}) + \rho(\boldsymbol{\theta})$ in Equation 3.11 into a Taylor series expansion about the ML estimates of $\boldsymbol{\theta}$, Lindley obtained the required expression for $\mathrm{E}[v(\boldsymbol{\theta})|\mathbf{y}]$ and Equation 3.11 asymptotically is estimated by

$$v^* = v + \frac{1}{2}\sum_i \sum_j [v_{ij} + 2v_i\rho_j]\sigma_{ij} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_l L_{ijk}\sigma_{ij}\sigma_{kl}v_l, \tag{3.12}$$

where $i, j, k, l = 1, 2, \ldots, N$ and

$$v = v(\boldsymbol{\theta}),\ v_i = \frac{\partial v}{\partial \theta_i},\ v_{ij} = \frac{\partial^2 v}{\partial \theta_i \partial \theta_j},\ L_{ijk} = \frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_k},\ \rho_j = \frac{\partial \rho}{\partial \theta_j},$$

and σ_{ij} is the (i,j)th element of the inverse matrix $\{-L_{ij}\}$ and all are evaluated at the ML estimates of the parameters.

For AR(p) model $N = p + 1$ and for MA(q) model $N = q + 1$.

3.7. TRUNCATED NORMAL APPROXIMATION

Lemma 1. *t-distribution approaches normal distribution as $\nu \to \infty$.*

This is a well-know fact. Using the limits $\lim_{k\to\infty} \frac{\gamma(k+\frac{1}{2})}{\sqrt{2\pi n}\gamma(k)} = \frac{1}{2\pi}$ and $\lim_{k\to\infty}(1+\frac{x^2}{k})^{-\frac{k+1}{2}} = \mathrm{e}^{-\frac{X^2}{2}}$, it can be shown that $f_\nu(x) \to \phi(x)$, as $\nu \to \infty$ for every value of $x \in (-\infty, \infty)$. Here $f_\nu(x)$ denotes p.d.f. of a standard t-distribution and $\phi(x)$ denotes p.d.f. of a standard normal distribution.

Theorem 3.1. *(Scheffe's Theorem) Suppose that X_n has density function $f_n, n \geq 1$, and X has density function f. Then $f_n(x) \to f(x)$ (for all but a countable number of x) implies that $X_n \sim X$.*

Lemma 2. *Truncated t-distribution approaches truncated normal distribution as $\nu \to \infty$.*

Proof. Let $f_\nu(x; \mu, \sigma, a, b)$ be probability density function of a truncated t-distribution and $\phi(x; \mu, \sigma, a, b)$ be probability density function of a truncated normal distribution, where the truncation interval is (a,b). Since $X \sim Tt_\nu(0,1,a,b)$ implies that $Y = \mu + \sigma X \sim Tt_\nu(\mu,\sigma,a,b)$ and same is valid for a truncated normal distribution, it is sufficient to show that $f_\nu(x;0,1,a,b) \to \phi(x;0,1,a,b), \nu \to \infty$.

As $\nu \to \infty$,

$$\begin{aligned} f_\nu(x;0,1,a,b) &= \frac{1}{T(b|\nu) - T(a|\nu)} f_\nu(x) I(a < x < b), \\ \phi(x;0,1,a,b) &= \frac{1}{\Phi(b) - \Phi(a)} \phi(x) I(a < x < b), \\ \frac{f_\nu(x;0,1,a,b)}{\phi(x;0,1,a,b)} &= \frac{\Phi(b) - \Phi(a)}{T(b|\nu) - T(a|\nu)} \times \frac{f_\nu(x)}{\phi(x)}, \\ \frac{f_\nu(x)}{\phi(x)} &\to 1. \end{aligned}$$

We need to show that $\frac{\Phi(b)-\Phi(a)}{T(b|\nu)-T(a|\nu)} \to 1$, as $\nu \to \infty$. But $f_\nu \to \phi(x)$, as $\nu \to \infty$, by Scheffe's Theorem, implies that a t-distributed random variable converges to a normal distributed random variable in distribution, i.e. a c.d.f. of a t-distribution approaches a c.d.f. of a normal distribution, as $\nu \to \infty$. This fact was also mentioned in [57], page 282.

3.8. GIBBS SAMPLER

The Gibbs sampler is an iterative Monte Carlo method designed to extract marginal distributions from intractable joint distributions. Consider a p-dimensional with probability density function $p(x)$, where $x = (x_1, x_2, \ldots, x_p)$. Suppose the complete conditional distributions, $p(x_1|x_2, x_3, \ldots, x_p)$, $p(x_2|x_1, x_3, \ldots, x_p)$, ..., $p(x_p|x_1, x_2, \ldots, x_{p-1})$ have a much simpler form and are easily sampled. Then the algorithm for obtaining a draw $(x_1, x_2, \ldots, x_p)$ proceeds as follows.

(i) Specify initial value $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_p^{(0)})$.

(ii) Successively generate values $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ in the following way. For each $t = 1, 2, \ldots$, using the value $x^{(t)}$, update the sample to $x^{(t+1)}$:

- draw a new value of $x_1^{(t+1)}$ from $p(x_1|x_2^{(t)}, x_3^{(t)}, \ldots, x_p^{(t)})$;
- continue through new draws of $x_i^{(t+1)}$ from $p(x_i|x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, x_{i+1}^{(t)}, \ldots, x_p^{(t)})$, where $i = 2, 3, \ldots, p-1$;
- complete the re-sampling by drawing $x_p^{(t+1)}$ from $p(x_p|x_1^{(t+1)}, x_2^{(t+1)}, \ldots, x_{p-1}^{(t+1)})$.

After M iterations of the above scheme, the sample $x^{(M)} = (x_1^{(M)}, x_2^{(M)}, \ldots, x_p^{(M)})$ is obtained. Under regularity conditions, for example, see Tierney [55], as $M \to \infty$, the sampled values converge in distribution to the relevant marginal and joint distribution. For M large, so that the desired convergence in distribution has been attained, the $N - M$ values $x^{(M+1)}, x^{(M+2)}, \ldots, x^{(N)}$ are a sample from $p(x)$.

Then the expectation of a function, $g(x)$, of the parameters is estimated via the sample average

$$\hat{\mathrm{E}}(g(x)) = \frac{1}{N-M} \sum_{j=M+1}^{N} g(x^{(j)}).$$

4. BAYESIAN ANALYSIS OF AUTOREGRESSIVE - MOVING AVERAGE PROCESSES

In this section we consider an ARMA (p, q) process, defined by Equation 3.5, which is invertible and stationary. We assume normal - gamma conjugate prior for the parameters. This model was analyzed by Shaarawy and Broemeling [32]. They derived posterior distributions of the model parameters and one-step ahead predictive distribution. Here we summarize their results and then using them find Bayes estimators of model parameters and one-step ahead forecast under the SE and LINEX loss functions.

In the model 3.5 we assume the parameters have conjugate normal - gamma prior. That is,

$$\xi(\phi,\theta,\tau)=\xi_1(\phi,\theta|\tau)\xi_2(\tau),\ \phi\in\mathbb{R}^p,\ \theta\in\mathbb{R}^q,\tau>0,$$

where the marginal prior density of τ is the gamma distribution

$$\xi_2(\tau)\propto\tau^{\alpha-1}\mathrm{e}^{-\tau\beta}$$

and ξ_1 is the normal density $N(\mu,\tau^{-1}Q^{-1})$. So ξ is the normal - gamma density function with parameters μ,α,β, and Q which is positive definite matrix of order $p+q$.

Suppose in the model 3.5 we have n observations $S_n=(Y_1,Y_2,\ldots,Y_n)'$, then the residuals are given by

$$\epsilon_t=Y_t-\sum_{i=1}^{p}\phi_iY_{t-i}+\sum_{j=1}^{q}\theta_j\epsilon_{t-j}, \tag{4.1}$$

where $t=p,p+1,\ldots,n$.

By conditioning on the first p observations and assuming that $\epsilon_1=\cdots=\epsilon_p=\epsilon_0=\cdots=\epsilon_{p-q-1}=0$, where $q>p+1$ (see Tiao and Box [56], p.809; Priestley [2], p.360), we approximate the likelihood function by

$$L(\phi, \theta, \tau | S_n) \propto \tau^{\frac{n-p}{2}} \mathrm{e}^{-\frac{\tau}{2} \sum_{t=p+1}^{n} \epsilon_t^2}, \tag{4.2}$$

where $\phi \in \mathbb{R}^p,\ \theta \in \mathbb{R}^q, \tau > 0$, and $\epsilon_t, t = p+1, \ldots, n$ are given by Equation 4.1. The residuals can be estimated by

$$\hat{\epsilon}_t = Y_t - \sum_{i=1}^{p} \hat{\phi}_i Y_{t-i} + \sum_{j=1}^{q} \hat{\theta}_j \hat{\epsilon}_{t-j},$$

where $t = p+1, \ldots, n, \hat{\epsilon}_{p-1} = \ldots, \hat{\epsilon}_{p-q-1} = 0$, and $\hat{\phi}_j$ and $\hat{\theta}_j$ are the nonlinear least squares estimates of ϕ_i and θ_j and are found by minimizing the conditional sum of squares

$$SS(\phi, \theta) = \sum_{t=p+1}^{n} \epsilon_t^2$$

with respect to ϕ and θ over the region of invertibility and stationarity. The conditional least square estimates are found by a nonlinear regression algorithm explained by Harvey [58]. Thus, the approximate likelihood function has the following form

$$L^*(\phi, \theta, \tau | S_n) \propto \tau^{\frac{n-p}{2}} \mathrm{e}^{-\frac{\tau}{2} \sum_{t=p+1}^{n} \left[Y_t - \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \hat{\epsilon}_{t-j}\right]^2}, \tag{4.3}$$

where $\phi \in \mathbb{R}^p,\ \theta \in \mathbb{R}^q, \tau > 0$, and $\hat{\epsilon}_{p-1} = \ldots, \hat{\epsilon}_{p-q-1} = 0$. Employing Bayes Theorem, the posterior density of ϕ, θ and τ is

$$\begin{aligned} \xi(\phi, \theta, \tau | S_n) &\propto \tau^{\frac{p+q}{2}+\alpha-1} \mathrm{e}^{-\frac{\tau}{2}(2\beta + (\psi-\mu)' Q (\psi-\mu)} \tau^{\frac{n-p}{2}} \mathrm{e}^{-\frac{\tau}{2} \sum_{t=p+1}^{n} \left[Y_t - \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \hat{\epsilon}_{t-j}\right]^2} \\ &\propto \tau^{\frac{n+q+2\alpha}{2}-1} \mathrm{e}^{-\frac{\tau}{2}(2\beta + (\psi-\mu)' Q (\psi-\mu) + \sum_{t=p+1}^{n} \left[Y_t - \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \hat{\epsilon}_{t-j}\right]^2}, \end{aligned} \tag{4.4}$$

where $\psi = (\phi, \theta)'$. Shaarawy and Broemeling [32] derived the marginal posterior distributions for ϕ, θ and τ. Their results are concluded in the following theorems.

Theorem 4.1. *The posterior distribution of ϕ and θ is a (p+q)-dimensional t-distribution* *with* $\nu = n - p + 2\alpha$ *degrees of freedom, location vector*

$$c = (A + Q)^{-1}(B + Q\mu)$$

and scale matrix

$$\Sigma(\psi|S_n) = \frac{(C - (B + Q\mu)'(A + Q)^{-1}(B + Q\mu))(A + Q)^{-1}}{n - p + 2\alpha},$$

where $\psi = (\phi, \theta)'$*, A is symmetric and of order* $p + q$ *and*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Furthermore, A_{11} *is of order p and has* ith *diagonal element* $\sum_{t=p+1}^{n} Y_{t-i}^2$ *and* $ikth$ *off-diagonal element* $\sum_{t=p+1}^{n} Y_{t-i}Y_{t-k}$. A_{22} *is of order q and has* $jkth$ *element* $\sum_{t=p+1}^{n} \hat{\epsilon}_{t-j}\hat{\epsilon}_{t-k}$*.he* $p \times q$ *matrix* A_{12} *has* $ijth$ *element* $\sum_{t=p+1}^{n} Y_{t-i}\hat{\epsilon}_{t-j}$*. The* $p + q$ *column vector B is*

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where B_1 *is of order p and has* ith *element* $\sum_{t=p+1}^{n} Y_{t-i}Y_t$, B_2 *is of order q and has* jth *element* $\sum_{t=p+1}^{n} \hat{\epsilon}_{t-j}Y_t$*. Finally, the scalar C is*

$$C = 2\beta + \mu' Q\mu + \sum_{t=p+1}^{n} Y_t^2.$$

Thus the posterior density of the parameters is given by

$$\psi|S_n \sim t_{p+q}(c, \Sigma(\psi|S_n), \nu).$$

Also, the marginal density of τ is gamma with parameters α_0 and β_0, where

$$\alpha_0 = \frac{n-p+2\alpha}{2}$$

and

$$\beta_0 = \frac{1}{2}\Big(C - (B+Q\mu)'(A+Q)^{-1}(B+Q\mu)\Big).$$

So we have

Corollary 4.1. *The marginal posterior distribution for individual parameter is*

$$\psi_k|S_n \sim t_1(c_k, s_k, \nu),$$

where c_k is the diagonal element of location vector c, s_k is the diagonal element of scale matrix $\Sigma(\psi|S_n)$, $k = 1, \ldots, p+q$ and $\psi_k = \phi_k, k = 1, \ldots, p, \psi_{p+j} = \theta_j, j = 1, \ldots, q$.

Under the SE loss function, the Bayes estimator of a given function is the posterior mean of that function. Under the LINEX loss function, the Bayes estimator of $v = v(\psi, \tau)$ is equal to

$$\hat{v}_{B_{LINEX}} = -\frac{1}{\gamma}\log\Big(\mathrm{E}(\mathrm{e}^{-\gamma v(\psi,\tau)}|S_n)\Big)$$

Theorem 4.2. *Under the SE loss function, the Bayes estimator of ψ_i, $i = 1, \ldots, p+q$ is equal to*

$$\hat{\psi}_{i(B_{SE})} = c_i, \tag{4.5}$$

where c_i is the ith element of location vector.

Under the LINEX loss function, the Bayes estimator involves the moment generating function. For the t-distribution, the moment generating function is undefined. However, since the degrees of freedom is large, the t-distribution can be well approximated by the normal distribution whose moment generating function exists. Using this approximation, we get the following result

Theorem 4.3. *Under the LINEX loss function, the approximate Bayes estimator of* ψ_i*,* $i = 1, \ldots, p$ *is equal to*

$$\hat{\psi}_{i(B_{LINEX})} = c_i - \frac{\gamma s_i}{2}, \tag{4.6}$$

where the parameters c_i, s_i*, are as defined in the Theorems 4.1 and 4.2 .*

For the parameter τ the Bayes estimators are given in the following theorem.

Theorem 4.4. (i) *Under the SE loss function, the Bayes estimator of* τ *is*

$$\hat{\tau}_{B_{SE}} = \frac{\alpha_0}{\beta_0}.$$

(ii) *Under the LINEX loss function, the Bayes estimator of* τ *is equal to*

$$\hat{\tau}_{B_{LINEX}} = -\frac{\alpha_0}{\gamma} \log\Big(1 + \frac{\gamma}{\beta_0}\Big).$$

where α_0, β_0 *are as defined in the Theorem 4.1.*

In the next theorems we give the Bayesian predictive distribution of a future observation $W_1 = Y_{n+1}$ which was derived by Shaarawy and Broemeling [32] and the approximate Bayes estimators of W_1 under the SE and LINEX loss functions.

Theorem 4.5. *The predictive distribution of* W_1 *is a univariate t-distribution with* $\nu = n - p + 2\alpha$ *degrees of freedom, location vector*

$$c_0 = (1 - B_0' A_0^{-1} B_0)^{-1} B_0' A_0^{-1} (B + Q\mu)$$

and scale matrix

$$\Sigma(W_1|S_n) = \frac{(C - (B + Q\mu)' A_0^{-1}(B + Q\mu)}{(1 - B_0' A_0^{-1} B_0)n - p + 2\alpha},$$

where B_0 *is of order* $p + q$*,*

$$B_0 = \begin{pmatrix} B_{01} \\ B_{02} \end{pmatrix},$$

where B_{01} *is of order* p *and has* i*th element* Y_{n+1-i}*,* B_{02} *is of order* q *and has* j*th element* $-\hat{\epsilon}_{n+1-j}$*. The matrix* A_0 *is*

$$A_0 = A + A_1 + Q,$$

where A, Q *are defined in the Theorem 4.1 and* A_1 *is the symmetric matrix*

$$A_1 = \begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix}.$$

Furthermore, A_{11}^* *is of order p and has* ik*th element* $Y_{n+1-i}Y_{n+1-k}$*.* A_{22}^* *is of order q and has* jk*th element* $\hat{\epsilon}_{n+1-j}\hat{\epsilon}_{n+1-k}$*.The* $p \times q$ *matrix* A_{12}^* *has* ij*th element* $-Y_{n+1-i}\hat{\epsilon}_{n+1-j}$*.*

Since the predictive distribution is a univariate t-distribution, in order to find the Bayes estimator of W_1 under the LINEX loss function, normal approximation is used.

Theorem 4.6. (i) *Under the SE loss function, the Bayes estimator of* W_1 *is*

$$\hat{W}_{1_{B_{SE}}} = c_0.$$

(ii) *Under the LINEX loss function, the approximate Bayes estimator of* τ *is equal to*

$$\hat{W}_{1_{B_{LINEX}}} = c_0 - \frac{\gamma s_0}{2}$$

where $s_0 = \Sigma(W_1|S_n)$ *and* $c_0, \Sigma(W_1|S_n)$ *are as defined in the Theorem 4.5.*

5. BAYESIAN ANALYSIS OF AUTOREGRESSIVE PROCESSES

In this section we consider the AR(p) model. We assume four different priors for the parameters.

The AR(p) model is defined by Equation 3.1. We can rewrite it as

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} Y_0 & Y_{-1} & \dots & Y_{1-p} \\ \vdots & \vdots & \dots & \vdots \\ Y_{n-1} & Y_{n-2} & \dots & Y_{n-p} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} := X\phi + \epsilon, \tag{5.1}$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_n)' \sim N(0, \tau^{-1} I)$ $(\tau = \frac{1}{\sigma^2} > 0)$. Y_t is the observation at time t. The model parameters ϕ and the precision τ are considered to be random variables. $Y_0, Y_{-1}, \dots, Y_{1-p}$ are treated as "initial", known constants.

For the latter model we study the following questions.

(i) How to estimate parameters ϕ and τ given a sample $S_n = (Y_1, \dots, Y_n)'$?

(ii) How to forecast a future observation $W_1 = Y_{n+1}$ given a sample $S_n = (Y_1, \dots, Y_n)'$?

5.1. INDEPENDENT TRUNCATED NORMAL - GAMMA PRIOR

In the model 5.1 we assume the parameters $\phi_i, i = 1, \dots, p$, have independent truncated normal priors on intervals $(a_i, b_i), a_i, b_i \in \mathbb{R}$, for all $i = 1, \dots, p$, respectively, with the parameters $\phi_1 \sim TN(\mu_1, \sigma_1^2), \dots, \phi_p \sim TN(\mu_p, \sigma_p^2)$, and the precision has independent gamma prior with the parameters α and β, i.e. $\tau \sim Gamma(\alpha, \beta)$. That is, the joint prior

$$\xi(\phi, \tau) = \xi_1(\phi)\xi_2(\tau),$$

where the marginal prior density of τ is gamma distribution

$$\xi_2(\tau) \propto \tau^{\alpha-1} e^{-\tau\beta}, \tau > 0,$$

and the marginal prior density of ϕ is

$$\xi_1(\phi) \propto \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{p}(\frac{\phi_i-\mu_i}{\sigma_i})^2} = \mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}, \phi \in \mathbb{R}^p,$$

that is, $\xi_1(\phi) \sim N(\mu, Q^{-1})$. So the joint prior density function of parameters

$$\xi(\phi,\tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}.$$

The likelihood function for the model 5.1

$$L(\phi,\tau|S_n) \propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}.$$

Employing Bayes Theorem, the posterior density of ϕ and τ is

$$\begin{aligned}\xi(\phi,\tau|S_n) &= \frac{L(\phi,\tau|S_n)\xi(\phi,\tau)}{\int_\Phi\int_0^\infty L(\phi,\tau|S_n)\xi(\phi,\tau)\mathrm{d}\phi\mathrm{d}\tau} \\ &= \frac{\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}}{\int_\Phi\int_0^\infty \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}. \qquad (5.2)\end{aligned}$$

Under the SE loss function, the Bayes estimator of $v = v(\phi,\tau)$ is the posterior mean of the function and is given by the ratio of two integrals which can be written as

$$\begin{aligned}\hat{v}_{B_{SE}} &= \mathrm{E}(v(\phi,\tau)|S_n) \\ &= \frac{\int_\Phi\int_0^\infty v(\phi,\tau)L(\phi,\tau|S_n)\xi(\phi,\tau)\mathrm{d}\phi\mathrm{d}\tau}{\int_\Phi\int_0^\infty L(\phi,\tau|S_n)\xi(\phi,\tau)\mathrm{d}\phi\mathrm{d}\tau} \\ &= \frac{\int_\Phi\int_0^\infty v(\phi,\tau)\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}{\int_\Phi\int_0^\infty \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}. \qquad (5.3)\end{aligned}$$

Under the LINEX loss function, the Bayes estimator of $v = v(\phi, \tau)$ is equal to

$$\begin{aligned}\hat{v}_{B_{LINEX}} &= -\frac{1}{\gamma}\log\Big(\mathrm{E}(\mathrm{e}^{-\gamma v(\phi,\tau)}|S_n)\Big)\\ &= -\frac{1}{\gamma}\log\Big(\frac{\int_{\Phi}\int_0^{\infty}\mathrm{e}^{-\gamma v(\phi,\tau)}L(\phi,\tau|S_n)\xi(\phi,\tau)\mathrm{d}\phi\mathrm{d}\tau}{\int_{\Phi}\int_0^{\infty}L(\phi,\tau|S_n)\xi(\phi,\tau)\mathrm{d}\phi\mathrm{d}\tau}\Big)\\ &= -\frac{1}{\gamma}\log\Big(\frac{\int_{\Phi}\int_0^{\infty}\mathrm{e}^{-\gamma v(\phi,\tau)}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}{\int_{\Phi}\int_0^{\infty}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}\Big).\end{aligned} \tag{5.4}$$

These ratios of two integrals cannot be solved analytically. Hence, we use Lindley's approximation, for more details see Section 3.6. In the next two sections we apply this model to the AR(1) and AR(2) processes and find the Bayes estimators of their parameters and one-step ahead forecasts.

5.1.1. AR(1) model

In the $p = 1$ case, under the SE loss function, the Bayes estimator of function $v = v(\phi_1, \tau)$ Equation 5.3 can be written as

$$\hat{v}_{B_{SE}} = \frac{\int_{a_1}^{b_1}\int_0^{\infty} v(\phi_1,\tau)L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_0^{\infty} L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}. \tag{5.5}$$

Similarly, under the LINEX loss function, the Bayes estimator of function $v = v(\phi_1, \tau)$ can be written as

$$\hat{v}_{B_{LINEX}} = -\frac{1}{\gamma}\log\Big(\frac{\int_{a_1}^{b_1}\int_0^{\infty}\mathrm{e}^{-\gamma v(\phi_1,\tau)}L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_0^{\infty} L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}\Big). \tag{5.6}$$

By conditioning on the first observation (see [56] and [2]), we may approximate the likelihood function by

$$L(\phi_1, \tau|S_n) \propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2}. \quad (5.7)$$

Then the Bayes estimator of function $v = v(\phi_1, \tau)$, under the SE loss function, is approximately

$$\hat{u}_{B_{SE}} = \frac{\int_{a_1}^{b_1}\int_0^\infty u(\phi_1,\tau)\tau^{\frac{n-1}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\phi_1\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_0^\infty \tau^{\frac{n-1}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\phi_1\mathrm{d}\tau}, \quad (5.8)$$

where $u = v$. Under the LINEX loss function,

$$\hat{u}_{B_{LINEX}} = -\frac{1}{\gamma}\log\Big(\frac{\int_{a_1}^{b_1}\int_0^\infty u(\phi_1,\tau)\tau^{\frac{n-1}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\phi_1\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_0^\infty \tau^{\frac{n-1}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\phi_1\mathrm{d}\tau}\Big), \quad (5.9)$$

where $u = \mathrm{e}^{-\gamma v}$. Still we cannot find analytical expressions for $\hat{u}_{B_{SE}}$ and $\hat{u}_{B_{LINEX}}$. Thus, we apply Lindley's approximation for the ratios of two integrals. For the two parameters case, Lindley's approximation leads to

$$\begin{aligned}\hat{u}_{B_{SE}} &= u + \frac{1}{2}\sum_{i,j=1}^{2}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}(L_{30}[u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}] \\ &+ L_{21}[3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)] \\ &+ L_{12}[u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}] \\ &+ L_{03}[u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2]), \quad (5.10)\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{e}^{-\gamma \hat{u}_{B_{LINEX}}} &= u + \frac{1}{2}\sum_{i,j=1}^{2}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}(L_{30}[u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}] \\
&+ L_{21}[3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)] \\
&+ L_{12}[u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}] \\
&+ L_{03}[u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2]), \qquad (5.11)
\end{aligned}
$$

where $u = u(\phi_1, \tau)$, $u_1 = \frac{\partial u}{\partial \phi_1}$, $u_2 = \frac{\partial u}{\partial \tau}$, $u_{11} = \frac{\partial^2 u}{\partial \phi_1^2}$, $u_{12} = u_{21} = \frac{\partial^2 u}{\partial \phi_1 \partial \tau}$, $u_{22} = \frac{\partial^2 u}{\partial \tau^2}$, $L(\phi_1, \tau)$ is the logarithm of the likelihood function, $\rho(\phi_1, \tau)$ is the logarithm of the joint prior density function, $L_{30} = \frac{\partial^3 L}{\partial \phi_1^3}$, $L_{21} = \frac{\partial^3 L}{\partial \phi_1^2 \partial \tau}$, $L_{12} = \frac{\partial^3 L}{\partial \phi_1 \partial \tau^2}$, $L_{03} = \frac{\partial^3 L}{\partial \tau^3}$, $\rho_1 = \frac{\partial \rho}{\partial \phi_1}$, $\rho_2 = \frac{\partial \rho}{\partial \tau}$ and σ_{ij} is the (i, j)th element of the inverse of the matrix

$$
A = \begin{pmatrix} -\frac{\partial^2 L}{\partial \phi_1^2} & -\frac{\partial^2 L}{\partial \phi_1 \partial \tau} \\ -\frac{\partial^2 L}{\partial \phi_1 \partial \tau} & -\frac{\partial^2 L}{\partial \tau_1^2} \end{pmatrix}
$$

all evaluated at the ML estimates of the parameters. For the prior distribution

$$
\xi(\phi_1, \tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta - \frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}}
$$

we have

$$
\rho = \log(\xi(\phi_1, \tau)) = constant + (\alpha - 1)\log(\tau) - \tau\beta - \frac{(\phi_1 - \mu_1)^2}{2\sigma_1^2},
$$
$$
\hat{\rho_1} = -\frac{\hat{\phi_1} - \mu_1}{\sigma_1^2}, \hat{\rho_2} = \frac{\alpha - 1}{\hat{\tau}} - \beta.
$$

From Equation 5.7,

$$
L_{10} = \frac{\partial L}{\partial \phi_1} = \tau \sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1})Y_{t-1},
$$
$$
L_{01} = \frac{\partial L}{\partial \tau} = \frac{n-1}{2\tau} - \frac{1}{2}\sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1})^2.
$$

Then the ML estimators of ϕ_1 and τ are

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=2}^{n} Y_{t-1}^2}, \quad \hat{\tau} = \frac{n-1}{\sum_{t=2}^{n}(Y_t - \hat{\phi}_1 Y_{t-1})^2}$$

and

$$L_{20} = \frac{\partial^2 L}{\partial \phi_1^2} = -\tau \sum_{t=2}^{n} Y_{t-1}^2,$$
$$L_{11} = \frac{\partial^2 L}{\partial \phi_1 \partial \tau} = \sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1}) Y_{t-1},$$
$$L_{02} = \frac{\partial^2 L}{\partial \tau^2} = -\frac{n-1}{2\tau^2}.$$

Hence,

$$\hat{L}_{30} = \frac{\partial^3 L}{\partial \phi_1^3} = 0, \quad \hat{L}_{21} = \frac{\partial^3 L}{\partial \phi_1^2 \partial \tau} = -\sum_{t=2}^{n} Y_{t-1}^2,$$
$$\hat{L}_{12} = \frac{\partial^3 L}{\partial \phi_1 \partial \tau^2} = 0, \quad \hat{L}_{03} = \frac{\partial^3 L}{\partial \tau^3} = \frac{n-1}{\hat{\tau}^3}.$$

The matrix

$$A = \begin{pmatrix} \hat{\tau} \sum_{t=2}^{n} Y_{t-1}^2 & 0 \\ 0 & \frac{n-1}{2\hat{\tau}^2} \end{pmatrix}. \tag{5.12}$$

Its inverse is

$$A^{-1} = \begin{pmatrix} \frac{1}{\hat{\tau} \sum_{t=2}^{n} Y_{t-1}^2} & 0 \\ 0 & \frac{2\hat{\tau}^2}{n-1} \end{pmatrix}. \tag{5.13}$$

Therefore,

$$\hat{\sigma}_{11} = \frac{1}{\hat{\tau} \sum_{t=2}^{n} Y_{t-1}^2}, \quad \hat{\sigma}_{12} = \hat{\sigma}_{21} = 0, \quad \hat{\sigma}_{22} = \frac{2\hat{\tau}^2}{n-1}.$$

Under the SE loss function, we obtain the following results.

Proposition 5.1. *Under the SE loss function, the approximate Bayes estimator of the parameter ϕ_1 is*

$$\hat{\phi}_{1(B_{SE})} = \hat{\phi}_1 + \hat{\rho}_1\hat{\sigma}_{11}. \tag{5.14}$$

Proof. We use $u(\phi_1, \tau) = \phi_1$. Then

$$u_1 = 1, u_2 = u_{11} = u_{12} = u_{21} = u_{22} = 0$$

and the result follows.

Proposition 5.2. *Under the SE loss function, the approximate Bayes estimator of τ is*

$$\hat{\tau}_{B_{SE}} = \hat{\tau} + \hat{\rho}_2\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{21}\hat{\sigma}_{11}\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{03}\hat{\sigma}_{22}^2. \tag{5.15}$$

Proof. We use $u(\phi_1, \tau) = \tau$. Then

$$u_2 = 1, u_2 = u_{11} = u_{12} = u_{21} = u_{22} = 0$$

and the result follows.

Finally, to get the Bayes estimator of one-step ahead forecast, we use

$$u(\phi_1, \tau) = \mathrm{E}(Y_{n+1}|S_n, \phi_1, \tau) = \phi_1 Y_n. \tag{5.16}$$

Proposition 5.3. *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\hat{W}_{1(B_{SE})} = \hat{\phi}_{1(B_{SE})} Y_n. \tag{5.17}$$

Proof. We substitute Equation 5.16 to Equation 5.5, then

$$\begin{aligned}\hat{W}_{1(B_{SE})} &= \frac{\int_{a_1}^{b_1}\int_0^\infty \phi_1 Y_n L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_0^\infty L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}\\ &= Y_n\frac{\int_{a_1}^{b_1}\int_0^\infty \phi_1 L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_0^\infty L(\phi_1,\tau|S_n)\xi(\phi_1,\tau)\mathrm{d}\phi_1\mathrm{d}\tau}\\ &= \hat{\phi}_{1(B_{SE})}Y_n.\end{aligned}$$

Under the LINEX loss function, we get the following results:

Proposition 5.4. *Under the LINEX loss function, the approximate Bayes estimator of the parameter* ϕ_1 *is*

$$\hat{\phi}_{1(B_{LINEX})} = \hat{\phi}_1 - \frac{1}{\gamma}\log\Big(1+\frac{\gamma^2}{2}\hat{\sigma}_{11} - \gamma\hat{\rho}_1\hat{\sigma}_{11}\Big). \tag{5.18}$$

Proof. We substitute $u(\phi_1,\tau) = \mathrm{e}^{-\gamma\phi_1}$ to Equation 5.11. Then

$$u_1 = -\gamma\mathrm{e}^{-\gamma\phi_1}, u_{11} = \gamma^2\mathrm{e}^{-\gamma\phi_1}, u_2 = u_{12} = u_{21} = u_{22} = 0$$

and the result follows.

Proposition 5.5. *Under the LINEX loss function, the approximate Bayes estimator of the parameter* τ *is*

$$\hat{\tau}_{B_{LINEX}} = \hat{\tau} - \frac{1}{\gamma}\log\Big(1+\frac{\gamma^2}{2}\hat{\sigma}_{22} - \gamma\Big(\hat{\rho}_2\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{21}\hat{\sigma}_{11}\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{03}\hat{\sigma}_{22}^2\Big)\Big). \tag{5.19}$$

Proof. We substitute $u(\phi_1,\tau) = \mathrm{e}^{-\gamma\tau}$ to Equation 5.11. Then

$$u_2 = -\gamma\mathrm{e}^{-\gamma\tau}, u_{22} = \gamma^2\mathrm{e}^{-\gamma\tau}, u_1 = u_{11} = u_{12} = u_{21} = 0$$

and the result follows.

Proposition 5.6. *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample* $S_n = (Y_1, \dots, Y_n)'$ *is*

$$\begin{aligned}\hat{W}_{1(B_{LINEX})} &= \hat{\phi}_1 Y_n - \frac{\gamma}{2\hat{\tau}} - \frac{1}{\gamma}\log\Big(1 + \frac{1}{2}\Big[\gamma^2 Y_n^2 \hat{\sigma}_{11} + \frac{\gamma^2}{\hat{\tau}^3}\Big(1 + \frac{\gamma^2}{4\hat{\tau}}\Big)\hat{\sigma}_{22}\Big] \\ &- \gamma Y_n \hat{\rho}_1 \hat{\sigma}_{11} - \frac{\gamma^2}{2\hat{\tau}^2}\hat{\rho}_2\hat{\sigma}_{22} \\ &- \frac{\gamma^2}{4\hat{\tau}^2}\Big(\hat{L}_{21}\hat{\sigma}_{11}\hat{\sigma}_{22} + \hat{L}_{03}\hat{\sigma}_{22}^2\Big)\Big).\end{aligned} \tag{5.20}$$

Proof. We set $u(\phi_1, \tau) = \mathrm{E}(\mathrm{e}^{-\gamma Y_{n+1}}|S_n, \phi_1, \tau) = \mathrm{e}^{-\gamma\phi_1 Y_n + \frac{\gamma^2}{2\tau}}$ and substitute it to Equation 5.11. Then

$$u_1 = -\gamma Y_n \mathrm{e}^{-\gamma\phi_1 Y_n + \frac{\gamma^2}{2\tau}}, \quad u_2 = -\frac{\gamma^2}{2\tau^2}\mathrm{e}^{-\gamma\phi_1 Y_n + \frac{\gamma^2}{2\tau}},$$

$$u_{11} = \gamma^2 Y_n^2 \mathrm{e}^{-\gamma\phi_1 Y_n + \frac{\gamma^2}{2\tau}}, \quad u_{12} = u_{21} = \frac{\gamma^3 Y_n}{2\tau^2}\mathrm{e}^{-\gamma\phi_1 Y_n + \frac{\gamma^2}{2\tau}}, \quad u_{22} = \frac{\gamma^2}{\tau^3}\Big(1 + \frac{\gamma^2}{4\tau}\Big)\mathrm{e}^{-\gamma\phi_1 Y_n + \frac{\gamma^2}{2\tau}}.$$

and the result follows.

Bayes estimation using the LINEX loss function involves the logarithm of the moment generating function which is approximated using Lindley's method. However, Lindley's approximation is of order n^{-1} and includes only three terms of the Taylor series expansion (see Section 3.6), thus the approximated value of the moment generating function is not guaranteed to be positive. Therefore, for a given LINEX loss function parameter γ, a small proportion of the Bayes estimates under LINEX loss function is expected to be undefined.

5.1.2. AR(2) model

In the $p = 2$ case, under the SE loss function, the Bayes estimator of function $v = v(\phi_1, \phi_2, \tau)$ can be written as

$$\hat{v}_{B_{SE}} = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} v(\phi_1,\phi_2,\tau)L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}. \tag{5.21}$$

Similarly, under the LINEX loss function, the Bayes estimator of function $v = v(\phi_1, \phi_2, \tau)$ can be written as

$$\begin{aligned} \hat{v}_{B_{LINEX}} &= -\frac{1}{\gamma}\log\Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} \mathrm{e}^{-\gamma v(\phi_1,\phi_2,\tau)}L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau \\ &/ \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau\Big). \end{aligned} \tag{5.22}$$

By conditioning on the first two observations, we may approximate the likelihood function by

$$L(\phi_1,\phi_2,\tau|S_n) \propto \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2}. \tag{5.23}$$

Then the Bayes estimator of function $v = v(\phi_1, \phi_2, \tau)$, under the SE loss function, is approximately

$$\begin{aligned} \hat{u}_{B_{SE}} &= \Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} u(\phi_1,\phi_2,\tau)\tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\sum_{i=1}^{2}\phi_iY_{t-i})^2} \\ &\times \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\sum_{i=1}^{2}\frac{(\phi_i-\mu_i)^2}{2\sigma_i^2}}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau\Big) \\ &/ \Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\sum_{i=1}^{2}\phi_iY_{t-i})^2} \\ &\times \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\sum_{i=1}^{2}\frac{(\phi_i-\mu_i)^2}{2\sigma_i^2}}\mathrm{d}\phi_1\mathrm{d}\phi_1\mathrm{d}\tau\Big) \end{aligned} \tag{5.24}$$

where $u = v$. Under the LINEX loss function,

$$\begin{aligned}
\mathrm{e}^{-\gamma\hat{u}_{B_{LINEX}}} &= (\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} u(\phi_1,\phi_2,\tau)\tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\sum_{i=1}^{2}\phi_i Y_{t-i})^2} \\
&\times \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\sum_{i=1}^{2}\frac{(\phi_i-\mu_i)^2}{2\sigma_i^2}}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau) \\
&/ \ (\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\sum_{i=1}^{2}\phi_i Y_{t-i})^2} \\
&\times \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\sum_{i=1}^{2}\frac{(\phi_i-\mu_i)^2}{2\sigma_i^2}}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau),
\end{aligned} \tag{5.25}$$

where $u = \mathrm{e}^{-\gamma v}$. Still we cannot find analytical expressions for $\hat{u}_{B_{SE}}$ and $\hat{u}_{B_{LINEX}}$. Thus, we apply Lindley's approximation for the ratios of two integrals. For the three parameters case, Lindley's approximation leads to

$$\begin{aligned}
\hat{u}_{B_{SE}} &= u + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) \\
&+ \frac{1}{2}\Big[B_1(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B_2(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\
&+ B_3(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\Big]
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
\mathrm{e}^{-\gamma\hat{u}_{B_{LINEX}}} &= u + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) \\
&+ \frac{1}{2}\Big[B_1(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B_2(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\
&+ B_3(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\Big],
\end{aligned} \tag{5.27}$$

all evaluated at the ML estimates of the parameters, where

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1,2,3,$$
$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}, \quad a_5 = \frac{1}{2}\Big(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}\Big),$$

$$B_1 = \sigma_{11}L_{300} + 2\sigma_{12}L_{210} + 2\sigma_{13}L_{201} + 2\sigma_{23}L_{111} + \sigma_{22}L_{120} + \sigma_{33}L_{102},$$
$$B_2 = \sigma_{11}L_{210} + 2\sigma_{12}L_{120} + 2\sigma_{13}L_{111} + 2\sigma_{23}L_{021} + \sigma_{22}L_{030} + \sigma_{33}L_{012},$$
$$B_3 = \sigma_{11}L_{201} + 2\sigma_{12}L_{111} + 2\sigma_{13}L_{102} + 2\sigma_{23}L_{012} + \sigma_{22}L_{021} + \sigma_{33}L_{003},$$

$u = u(\phi_1, \phi_2, \tau)$, $u_1 = \frac{\partial u}{\partial \phi_1}$, $u_2 = \frac{\partial u}{\partial \phi_2}$, $u_3 = \frac{\partial u}{\partial \tau}$, $u_{11} = \frac{\partial^2 u}{\partial \phi_1^2}$, $u_{22} = \frac{\partial^2 u}{\partial \phi_2^2}$, $u_{33} = \frac{\partial^2 u}{\partial \tau^2}$, $u_{12} = u_{21} = \frac{\partial^2 u}{\partial \phi_1 \partial \phi_2}$, $u_{13} = u_{31} = \frac{\partial^2 u}{\partial \phi_1 \partial \tau}$, $u_{23} = u_{32} = \frac{\partial^2 u}{\partial \phi_2 \partial \tau}$, $L(\phi_1, \phi_2, \tau)$ is the logarithm of the likelihood function, $\rho(\phi_1, \phi_2, \tau)$ is the logarithm of the joint prior density function, $L_{300} = \frac{\partial^3 L}{\partial \phi_1^3}$, $L_{210} = \frac{\partial^3 L}{\partial \phi_1^2 \partial \phi_2}$, $L_{201} = \frac{\partial^3 L}{\partial \phi_1^2 \partial \tau}$, $L_{120} = \frac{\partial^3 L}{\partial \phi_1 \partial \phi_2^2}$, $L_{102} = \frac{\partial^3 L}{\partial \phi_1 \partial \tau^2}$, $L_{021} = \frac{\partial^3 L}{\partial \phi_2^2 \partial \tau}$, $L_{012} = \frac{\partial^3 L}{\partial \phi_2 \partial \tau^2}$, $L_{030} = \frac{\partial^3 L}{\partial \phi_2^3}$, $L_{003} = \frac{\partial^3 L}{\partial \tau^3}$, $\rho_1 = \frac{\partial \rho}{\partial \phi_1}$, $\rho_2 = \frac{\partial \rho}{\partial \phi_2}$, $\rho_3 = \frac{\partial \rho}{\partial \tau}$ and σ_{ij} is the (i, j)th element of the inverse of the matrix

$$A = \begin{pmatrix} -\frac{\partial^2 L}{\partial \phi_1^2} & -\frac{\partial^2 L}{\partial \phi_1 \partial \phi_2} & -\frac{\partial^2 L}{\partial \phi_1 \partial \tau} \\ -\frac{\partial^2 L}{\partial \phi_1 \partial \phi_2} & -\frac{\partial^2 L}{\partial \phi_2^2} & -\frac{\partial^2 L}{\partial \phi_2 \partial \tau} \\ -\frac{\partial^2 L}{\partial \phi_1 \partial \phi_2} & -\frac{\partial^2 L}{\partial \phi_2 \partial \tau} & -\frac{\partial^2 L}{\partial \tau_1^2} \end{pmatrix}.$$

For the prior distribution

$$\xi(\phi_1, \phi_2, \tau) \propto \tau^{\alpha-1} \mathrm{e}^{-\tau\beta - \frac{(\phi_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(\phi_2 - \mu_2)^2}{2\sigma_2^2}}$$

we have

$$\rho = \log(\xi(\phi_1, \phi_2, \tau)) = constant + (\alpha - 1)\log(\tau) - \tau\beta - \frac{(\phi_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(\phi_2 - \mu_2)^2}{2\sigma_2^2},$$
$$\hat{\rho}_1 = -\frac{\hat{\phi}_1 - \mu_1}{\sigma_1^2}, \quad \hat{\rho}_2 = -\frac{\hat{\phi}_2 - \mu_2}{\sigma_2^2}, \quad \hat{\rho}_3 = \frac{\alpha - 1}{\hat{\tau}} - \beta.$$

From Equation 5.23,

$$L_{100} = \frac{\partial L}{\partial \phi_1} = \tau \sum_{t=3}^{n} (Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2}) Y_{t-1},$$
$$L_{010} = \frac{\partial L}{\partial \phi_2} = \tau \sum_{t=3}^{n} (Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2}) Y_{t-2},$$

$$L_{001} = \frac{\partial L}{\partial \tau} = \frac{n-2}{2\tau} - \frac{1}{2}\sum_{t=3}^{n}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2.$$

Then the ML estimators of ϕ_1, ϕ_2 and τ are

$$\hat{\phi}_1 = \frac{(\sum_{t=3}^{n} Y_t Y_{t-1})(\sum_{t=3}^{n} Y_{t-2}^2) - (\sum_{t=3}^{n} Y_t Y_{t-2})(\sum_{t=3}^{n} Y_{t-1} Y_{t-2})}{(\sum_{t=3}^{n} Y_{t-1}^2)(\sum_{t=3}^{n} Y_{t-2}^2) - (\sum_{t=3}^{n} Y_{t-1} Y_{t-2})^2},$$
$$\hat{\phi}_2 = \frac{(\sum_{t=3}^{n} Y_t Y_{t-2})(\sum_{t=3}^{n} Y_{t-1}^2) - (\sum_{t=3}^{n} Y_t Y_{t-1})(\sum_{t=3}^{n} Y_{t-1} Y_{t-2})}{(\sum_{t=3}^{n} Y_{t-1}^2)(\sum_{t=3}^{n} Y_{t-2}^2) - (\sum_{t=3}^{n} Y_{t-1} Y_{t-2})^2},$$
$$\hat{\tau} = \frac{n-2}{\sum_{t=3}^{n}(Y_t - \hat{\phi}_1 Y_{t-1} - \hat{\phi}_2 Y_{t-2})^2}$$

and

$$L_{200} = \frac{\partial^2 L}{\partial \phi_1^2} = -\tau \sum_{t=3}^{n} Y_{t-1}^2, \quad L_{110} = \frac{\partial^2 L}{\partial \phi_1 \partial \phi_2} = -\tau \sum_{t=3}^{n} Y_{t-1} Y_{t-2},$$
$$L_{101} = \frac{\partial^2 L}{\partial \phi_1 \partial \tau} = \sum_{t=3}^{n}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2}) Y_{t-1}, \quad L_{020} = \frac{\partial^2 L}{\partial \phi_2^2} = -\tau \sum_{t=3}^{n} Y_{t-2}^2,$$
$$L_{011} = \frac{\partial^2 L}{\partial \phi_2 \partial \tau} = \sum_{t=3}^{n}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2}) Y_{t-2}, \quad L_{002} = \frac{\partial^2 L}{\partial \tau^2} = -\frac{n-2}{2\tau^2}.$$

Hence,

$$\hat{L}_{300} = \hat{L}_{210} = \hat{L}_{120} = \hat{L}_{102} = \hat{L}_{030} = \hat{L}_{012} = 0,$$
$$\hat{L}_{201} = -\sum_{t=3}^{n} Y_{t-1}^2, \quad \hat{L}_{111} = -\sum_{t=3}^{n} Y_{t-1} Y_{t-2}, \quad \hat{L}_{021} = -\sum_{t=3}^{n} Y_{t-2}^2, \quad \hat{L}_{003} = \frac{n-2}{\hat{\tau}^3}.$$

The matrix

$$A = \begin{pmatrix} -\hat{L}_{200} & -\hat{L}_{110} & 0 \\ -\hat{L}_{110} & -\hat{L}_{020} & 0 \\ 0 & 0 & -\hat{L}_{002} \end{pmatrix}. \tag{5.28}$$

Its inverse is

$$A^{-1} = \frac{1}{D}\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \tag{5.29}$$

where

$$\begin{gathered} D := \det(A) = \hat{L}_{002}(-\hat{L}_{200}\hat{L}_{020} + \hat{L}_{110}^2), \\ A_{11} = \hat{L}_{020}\hat{L}_{002}, \quad A_{12} = A_{21} = -\hat{L}_{002}\hat{L}_{110}, \\ A_{13} = A_{31} = 0, \quad A_{22} = \hat{L}_{002}\hat{L}_{200}, \\ A_{23} = A_{32} = 0, \quad A_{33} = \hat{L}_{200}\hat{L}_{020} - \hat{L}_{110}^2. \end{gathered}$$

Therefore,

$$\begin{gathered} \hat{\sigma}_{11} = \frac{A_{11}}{D}, \quad \hat{\sigma}_{12} = \hat{\sigma}_{21} = \frac{A_{12}}{D}, \quad \hat{\sigma}_{13} = \hat{\sigma}_{31} = \frac{A_{13}}{D} = 0, \\ \hat{\sigma}_{22} = \frac{A_{22}}{D}, \quad \hat{\sigma}_{23} = \hat{\sigma}_{32} = \frac{A_{23}}{D} = 0, \quad \hat{\sigma}_{33} = \frac{A_{33}}{D}. \end{gathered}$$

Under the SE loss function, we obtain the following results.

Proposition 5.7. *Under the SE loss function, the approximate Bayes estimator of the parameter ϕ_1 is*

$$\hat{\phi}_{1(B_{SE})} = \hat{\phi}_1 + \hat{\rho}_1\hat{\sigma}_{11} + \hat{\rho}_2\hat{\sigma}_{12}. \tag{5.30}$$

Proof. We use $u(\phi_1, \phi_2, \tau) = \phi_1$ and substitute it to Equation 5.21. Then

$$u_1 = 1, u_2 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 5.8. *Under the SE loss function, the approximate Bayes estimator of the param-*

eter ϕ_2 is

$$\hat{\phi}_{2(B_{SE})} = \hat{\phi}_2 + \hat{\rho}_1\hat{\sigma}_{21} + \hat{\rho}_2\hat{\sigma}_{22}. \tag{5.31}$$

Proof. We use $u(\phi_1, \phi_2, \tau) = \phi_2$ and substitute it to Equation 5.21. Then

$$u_2 = 1, u_1 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 5.9. *Under the SE loss function, the approximate Bayes estimator of τ is*

$$\hat{\tau}_{B_{SE}} = \hat{\tau} + \hat{\rho}_3\hat{\sigma}_{33} + \frac{1}{2}(\hat{L}_{201}\hat{\sigma}_{11} + 2\hat{L}_{111}\hat{\sigma}_{12} + \hat{L}_{021}\hat{\sigma}_{22} + \hat{L}_{003}\hat{\sigma}_{33})\hat{\sigma}_{33}. \tag{5.32}$$

Proof. We use $u(\phi_1, \phi_2, \tau) = \tau$ and substitute it to Equation 5.21. Then

$$u_3 = 1, u_1 = u_2 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Finally, to derive the Bayes estimator of one-step ahead forecast, we use

$$u(\phi_1, \phi_2, \tau) = \mathrm{E}(Y_{n+1}|S_n, \phi_1, \phi_2, \tau) = \phi_1 Y_n + \phi_2 Y_{n-1}. \tag{5.33}$$

Proposition 5.10. *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\hat{W}_{1(B_{SE})} = \hat{\phi}_{1(B_{SE})} Y_n + \hat{\phi}_{2(B_{SE})} Y_{n-1}. \tag{5.34}$$

Proof. Substitute Equation 5.33 to Equation 5.21, then

$$
\begin{aligned}
\hat{W}_{1(B_{SE})} &= \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty}(\phi_1 Y_n + \phi_2 Y_{n-1})L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau} \\
&= Y_n\Big(\frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty}\phi_1 L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}\Big) \\
&+ Y_{n-1}\Big(\frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty}\phi_2 L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_0^{\infty} L(\phi_1,\phi_2,\tau|S_n)\xi(\phi_1,\phi_2,\tau)\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\tau}\Big) \\
&= \hat{\phi}_{1(B_{SE})}Y_n + \hat{\phi}_{2(B_{SE})}Y_{n-1}.
\end{aligned}
$$

Under the LINEX loss function, we get the following results.

Proposition 5.11. *Under the LINEX loss function, the approximate Bayes estimator of the parameter* ϕ_1 *is*

$$\hat{\phi}_{1(B_{LINEX})} = \hat{\phi}_1 - \frac{1}{\gamma}\log\Big(1 + \frac{\gamma^2}{2}\hat{\sigma}_{11} - \gamma\Big(\hat{\rho}_1\hat{\sigma}_{11} + \hat{\rho}_2\hat{\sigma}_{12}\Big)\Big). \tag{5.35}$$

Proof. We substitute $u(\phi_1,\phi_2,\tau) = \mathrm{e}^{-\gamma\phi_1}$ to Equation 5.27. Then

$$u_1 = -\gamma\mathrm{e}^{-\gamma\phi_1}, u_{11} = \gamma^2\mathrm{e}^{-\gamma\phi_1},$$

$$u_2 = u_3 = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 5.12. *Under the LINEX loss function, the approximate Bayes estimator of the parameter* ϕ_2 *is*

$$\hat{\phi}_{2(B_{LINEX})} = \hat{\phi}_2 - \frac{1}{\gamma}\log\Big(1 + \frac{\gamma^2}{2}\hat{\sigma}_{22} - \gamma\Big(\hat{\rho}_1\hat{\sigma}_{21} + \hat{\rho}_2\hat{\sigma}_{22}\Big)\Big). \tag{5.36}$$

Proof. We substitute $u(\phi_1, \phi_2, \tau) = \mathrm{e}^{-\gamma\phi_2}$ to the Equation 5.27. Then

$$u_2 = -\gamma \mathrm{e}^{-\gamma\phi_2}, u_{22} = \gamma^2 \mathrm{e}^{-\gamma\phi_2},$$
$$u_1 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 5.13. *Under the LINEX loss function, the approximate Bayes estimator of the parameter τ is*

$$\begin{aligned} \hat{\tau}_{B_{LINEX}} &= \hat{\tau} - \frac{1}{\gamma} \log\Big(1 + \frac{\gamma^2}{2}\hat{\sigma}_{33} - \gamma\Big(\hat{\rho}_3\hat{\sigma}_{33} + \frac{1}{2}(\hat{L}_{201}\hat{\sigma}_{11} \\ &+ 2\hat{L}_{111}\hat{\sigma}_{12} + \hat{L}_{021}\hat{\sigma}_{22} + \hat{L}_{003}\hat{\sigma}_{33})\hat{\sigma}_{33}\Big)\Big). \end{aligned} \tag{5.37}$$

Proof. We substitute $u(\phi_1, \phi_2, \tau) = \mathrm{e}^{-\gamma\tau}$ to Equation 5.27. Then

$$u_3 = -\gamma \mathrm{e}^{-\gamma\tau}, u_{33} = \gamma^2 \mathrm{e}^{-\gamma\tau},$$
$$u_2 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = 0$$

and the result follows.

Proposition 5.14. *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\begin{aligned} \hat{W}_{1(B_{LINEX})} &= \hat{\phi}_1 Y_n + \hat{\phi}_2 Y_{n-1} - \frac{\gamma}{2\hat{\tau}} - \frac{1}{\gamma}\log\Big(1 + \Big(-\gamma Y_n(\hat{\rho}_1\hat{\sigma}_{11} + \hat{\rho}_2\hat{\sigma}_{12}) \\ &- \gamma Y_{n-1}(\hat{\rho}_1\hat{\sigma}_{21} + \hat{\rho}_2\hat{\sigma}_{22}) - \frac{\gamma^2}{2\hat{\tau}^2}\hat{\rho}_3\hat{\sigma}_{33} + \gamma^2 Y_n Y_{n-1}\hat{\sigma}_{12} \\ &+ \frac{\gamma^2}{2}\Big(Y_n^2\hat{\sigma}_{11} + Y_{n-1}^2\hat{\sigma}_{22} + \frac{1}{\hat{\tau}^3}\Big(1 + \frac{\gamma^2}{4\hat{\tau}}\Big)\hat{\sigma}_{33}\Big) \\ &- \frac{\gamma^2}{4\hat{\tau}^2}(\hat{L}_{201}\hat{\sigma}_{11} + 2\hat{L}_{111}\hat{\sigma}_{12} + \hat{L}_{021}\hat{\sigma}_{22} + \hat{L}_{003}\hat{\sigma}_{33})\hat{\sigma}_{33}\Big). \end{aligned} \tag{5.38}$$

Proof. We set $u(\phi_1, \phi_2, \tau) = \mathrm{E}(\mathrm{e}^{-\gamma Y_{n+1}} | S_n, \phi_1, \phi_2, \tau) = \mathrm{e}^{-\gamma(\phi_1 Y_n + \phi_2 Y_{n-1}) + \frac{\gamma^2}{2\tau}}$ and substitute it to Equation 5.27. Then

$$u_1 = -\gamma Y_n u, \quad u_2 = -\gamma Y_{n-1} u, \quad u_3 = -\frac{\gamma^2}{2\tau^2} u,$$
$$u_{11} = \gamma^2 Y_n^2 u, \quad u_{12} = u_{21} = \gamma^2 Y_n Y_{n-1} u, \quad u_{13} = u_{31} = \frac{\gamma^3 Y_n}{2\tau^2} u,$$
$$u_{22} = \gamma^2 Y_{n-1}^2 u, \quad u_{23} = u_{32} = \frac{\gamma^3 Y_{n-1}}{2\tau^2} u, \quad u_{33} = \frac{\gamma^2}{\tau^3}\Big(1 + \frac{\gamma^2}{4\tau}\Big)$$

and the result follows.

Bayes estimation using the LINEX loss function involves the logarithm of the moment generating function which is approximated using Lindley's method. However, Lindley's approximation is of order n^{-1} and includes only three terms of the Taylor series expansion (see Section 3.6), thus the approximated value of the moment generating function is not guaranteed to be positive. Therefore, for a given LINEX loss function parameter γ, a small proportion of the Bayes estimates under LINEX loss function is expected to be undefined.

5.2. FULL CONDITIONAL DISTRIBUTIONS OF PARAMETERS UNDER INDEPENDENT TRUNCATED NORMAL - GAMMA PRIOR

5.2.1. AR(1) model

In this section we obtain the full conditional distributions for AR(1) model parameters ϕ_1, τ and the one-step prediction Y_{n+1}. The following propositions provide the required distributions.

Proposition 5.15. *The full conditional distribution of Bayes estimator of ϕ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{\tau \sum_{t=2}^{n} Y_t Y_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{5.39}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{5.40}$$

Proof.

$$\begin{aligned}
\xi(\phi_1|\tau, S_n) &\propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\sum_{t=2}^{n} Y_t^2 - 2\phi_1 \sum_{t=2}^{n} Y_t Y_{t-1} + \phi_1^2 \sum_{t=2}^{n} Y_{t-1}^2) - \frac{\phi_1^2 - 2\phi_1\mu_1 + \mu_1^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{1}{2}((\tau\sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2})\phi_1^2 - 2(\tau\sum_{t=2}^{n} Y_t Y_{t-1} + \frac{\mu_1}{\sigma_1^2})\phi_1)} \\
&\propto \mathrm{e}^{-\frac{\left(\tau\sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}\right)}{2}\left(\phi_1 - \frac{\tau\sum_{t=2}^{n} Y_t Y_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau\sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned} \tag{5.41}$$

Proposition 5.16. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \alpha + \frac{n-1}{2} \tag{5.42}$$

and

$$\tilde{\beta} = \beta + \frac{\sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1})^2}{2}. \tag{5.43}$$

Proof.

$$\begin{aligned}
\xi(\tau|\phi_1, S_n) &\propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta} \\
&= \tau^{\alpha+\frac{n-1}{2}-1} \mathrm{e}^{-\tau(\beta+\frac{1}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2)}.
\end{aligned} \tag{5.44}$$

Proposition 5.17. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}$ and precision τ, where*

$$\tilde{\mu} = \phi_1 Y_n. \tag{5.45}$$

Proof.

$$\begin{aligned} Y_{n+1}|\phi_1, \tau, S_n &= (\phi_1 Y_n + \epsilon_{n+1})|\phi_1, \tau, S_n \\ &= \phi_1 Y_n + (\epsilon_{n+1}|\tau). \end{aligned} \tag{5.46}$$

5.2.2. AR(2) model

Proposition 5.18. *The full conditional distribution of Bayes estimator of ϕ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{\tau(\sum_{t=3}^n Y_t Y_{t-1} - \phi_2 \sum_{t=3}^n Y_{t-1} Y_{t-2}) + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=3}^n Y_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{5.47}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=3}^n Y_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{5.48}$$

Proof.

$$\begin{aligned}
\xi(\phi_1|\phi_2,\tau,S_n) &\propto \tau^{\frac{n-2}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1 Y_{t-1}-\phi_2 Y_{t-2})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(\phi_1^2 Y_{t-1}^2-2\phi_1 Y_t Y_{t-1}+2\phi_1\phi_2 Y_{t-1}Y_{t-2})} \mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\phi_1^2(\sum_{t=3}^{n} Y_{t-1}^2)-2\phi_1(\sum_{t=3}^{n} Y_t Y_{t-1})+2\phi_1\phi_2(\sum_{t=3}^{n} Y_{t-1}Y_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\phi_1^2-2\mu_1\phi_1)}{2\sigma_1^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\phi_1^2(\tau(\sum_{t=3}^{n} Y_{t-1}^2)+\frac{1}{\sigma_1^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\phi_1(\tau[(\sum_{t=3}^{n} Y_t Y_{t-1})-\phi_2(\sum_{t=3}^{n} Y_{t-1}Y_{t-2})]+\frac{\mu_1}{\sigma_1^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=3}^{n} Y_{t-1}^2)+\frac{1}{\sigma_1^2}}{2}\left(\phi_1-\frac{\tau[(\sum_{t=3}^{n} Y_t Y_{t-1})-\phi_2(\sum_{t=3}^{n} Y_{t-1}Y_{t-2})]+\frac{\mu_1}{\sigma_1^2}}{\tau(\sum_{t=3}^{n} Y_{t-1}^2)+\frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned}$$

Proposition 5.19. *The full conditional distribution of Bayes estimator of ϕ_2 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_2$ and variance $\tilde{\sigma}_2^2$, where*

$$\tilde{\mu}_2 = \frac{\tau(\sum_{t=3}^{n} Y_t Y_{t-2} - \phi_1 \sum_{t=3}^{n} Y_{t-1}Y_{t-2}) + \frac{\mu_2}{\sigma_2^2}}{\tau \sum_{t=3}^{n} Y_{t-2}^2 + \frac{1}{\sigma_2^2}} \tag{5.49}$$

and

$$\tilde{\sigma}_2^2 = \frac{1}{\tau \sum_{t=3}^{n} Y_{t-2}^2 + \frac{1}{\sigma_2^2}}. \tag{5.50}$$

Proof.

$$\begin{aligned}
\xi(\phi_2|\phi_1,\tau,S_n) &\propto \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(\phi_2^2Y_{t-2}^2-2\phi_2Y_tY_{t-2}+2\phi_1\phi_2Y_{t-1}Y_{t-2})}\mathrm{e}^{-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\phi_2^2(\sum_{t=3}^{n}Y_{t-2}^2)-2\phi_2(\sum_{t=3}^{n}Y_tY_{t-2})+2\phi_1\phi_2(\sum_{t=3}^{n}Y_{t-1}Y_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\phi_2^2-2\mu_2\phi_2)}{2\sigma_2^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\phi_2^2(\tau(\sum_{t=3}^{n}Y_{t-2}^2)+\frac{1}{\sigma_2^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\phi_2(\tau[(\sum_{t=3}^{n}Y_tY_{t-2})-\phi_1(\sum_{t=3}^{n}Y_{t-1}Y_{t-2})]+\frac{\mu_2}{\sigma_2^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=3}^{n}Y_{t-2}^2)+\frac{1}{\sigma_2^2}}{2}\left(\phi_2-\frac{\tau[(\sum_{t=3}^{n}Y_tY_{t-2})-\phi_1(\sum_{t=3}^{n}Y_{t-1}Y_{t-2})]+\frac{\mu_2}{\sigma_2^2}}{\tau(\sum_{t=3}^{n}Y_{t-2}^2)+\frac{1}{\sigma_2^2}}\right)^2}.
\end{aligned}$$

Proposition 5.20. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \alpha + \frac{n-2}{2} \tag{5.51}$$

and

$$\tilde{\beta} = \beta + \frac{\sum_{t=3}^{n}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2}{2}. \tag{5.52}$$

Proof.

$$\begin{aligned}
\xi(\tau|\phi_1,\phi_2,S_n) &\propto \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta} \\
&= \tau^{\alpha+\frac{n-2}{2}-1}\mathrm{e}^{-\tau(\beta+\frac{1}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2)}.
\end{aligned} \tag{5.53}$$

Proposition 5.21. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample*

$S_n = (Y_1, \ldots, Y_n)'$ *is normal with mean* $\tilde{\mu}$ *and precision* τ*, where*

$$\tilde{\mu} = \phi_1 Y_n + \phi_2 Y_{n-1}. \tag{5.54}$$

Proof.

$$\begin{aligned} Y_{n+1}|\phi_1, \phi_2, \tau, S_n &= (\phi_1 Y_n + \phi_2 Y_{n-1} + \epsilon_{n+1})|\phi_1, \phi_2, \tau, S_n \\ &= \phi_1 Y_n + \phi_2 Y_{n-1} + (\epsilon_{n+1}|\tau). \end{aligned} \tag{5.55}$$

5.3. INDEPENDENT TRUNCATED NORMAL - IMPROPER PRIOR

In the model 5.1 we assume the parameters $\phi_i, i = 1, \ldots, p$, have independent truncated normal priors on intervals $(a_i, b_i), a_i, b_i \in \mathbb{R}$, for all $i = 1, \ldots, p$, respectively, with the parameters $\phi_1 \sim TN(\mu_1, \sigma_1^2), \ldots, \phi_p \sim TN(\mu_p, \sigma_p^2)$, and the precision has independent improper prior. In this model,

$$\xi(\phi, \tau) = \xi_1(\phi)\xi_2(\tau),$$

where the marginal prior density of τ is

$$\xi_2(\tau) \propto \frac{1}{\tau}, \tau > 0,$$

and the marginal prior density of ϕ is

$$\xi_1(\phi) \propto \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{p}(\frac{\phi_i - \mu_i}{\sigma_i})^2} = \mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}, \phi \in \mathbb{R}^p,$$

that is, $\xi_1(\phi) \sim N(\mu, Q^{-1})$. So the joint prior density function of parameters

$$\xi(\phi,\tau) \propto \tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}.$$

The likelihood function for the model 5.1

$$L(\phi,\tau|S_n) \propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\ .$$

Employing Bayes Theorem, the posterior density of ϕ and τ is

$$\begin{aligned}\xi(\phi,\tau|S_n) &= \frac{L(\phi,\tau|S_n)\xi(\phi,\tau)}{\int_\Phi\int_0^\infty L(\phi,\tau|S_n)\xi(\phi,\tau)\mathrm{d}\phi\mathrm{d}\tau}\\ &= \frac{\tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}}{\int_\Phi\int_0^\infty \tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau} \qquad (5.56)\end{aligned}$$

Under the SE loss function, the Bayes estimator of $v = v(\phi,\tau)$ is the posterior mean of the function and is given by the ratio of two integrals which can be written as

$$\hat{v}_{B_{SE}} = \frac{\int_\Phi\int_0^\infty v(\phi,\tau)\tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}{\int_\Phi\int_0^\infty \tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau} \qquad (5.57)$$

Under the LINEX loss function, the approximate Bayes estimator of $v = v(\phi,\tau)$ is equal to

$$\hat{v}_{B_{LINEX}} = -\frac{1}{\gamma}\log\Big(\frac{\int_\Phi\int_0^\infty \mathrm{e}^{-\gamma v(\phi,\tau)}\tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}{\int_\Phi\int_0^\infty \tau^{-1}\mathrm{e}^{-\frac{1}{2}(\phi-\mu)'Q(\phi-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}\mathrm{d}\phi\mathrm{d}\tau}\Big) \qquad (5.58)$$

Similarly to independent truncated normal-gamma prior case, we cannot find analytical expressions of the ratios of these integrals, again we use Lindley's approximation. In the next two sections we apply this model to the AR(1) and AR(2) processes and find the Bayes estimators of their parameters and one-step ahead forecasts.

5.3.1. AR(1) model

In the $p = 1$ case, using the SE loss function, the Bayes estimator of function $\upsilon = \upsilon(\phi_1, \tau)$ is defined by Equation 5.5 and employing the LINEX loss function, is defined by Equation 5.6. If we approximate the likelihood function by the conditional likelihood function defined by Equation 5.7, under the SE loss function, the Bayes estimator of the function $\upsilon = \upsilon(\phi_1, \tau)$ is defined by Equation 5.8 and the Bayes estimator under the LINEX loss function is defined by Equation 5.9. If we apply Lindley's approximation to this model, we get the following result.

Proposition 5.22. *In the AR(1) model, we assume the parameters follow independent truncated normal - improper model. Then*

(i) *Under the SE loss function, the approximate Bayes estimators of ϕ_1, τ are defined by Equations 5.14 and 5.15, respectively.*

(ii) *Under the LINEX loss function, the approximate Bayes estimators of ϕ_1, τ are defined by Equations 5.18 and 5.19, respectively.*

(iii) *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by Equation 5.17.*

(iv) *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by Equation 5.20.*

Where

$$\rho = \log(\xi(\phi_1, \tau)) = constant - \log(\tau) - \frac{(\phi_1 - \mu_1)^2}{2\sigma_1^2},$$
$$\hat{\rho}_1 = -\frac{\hat{\phi}_1 - \mu_1}{\sigma_1^2}, \quad \hat{\rho}_2 = -\frac{1}{\hat{\tau}},$$

and other functions are same as defined in the Section 5.1.1.

5.3.2. AR(2) model

In the $p = 2$ case, under the SE loss function, the Bayes estimator of function $v = v(\phi_1, \phi_2, \tau)$ is defined by Equation 5.21 and under the LINEX loss function, is defined by Equation 5.22. If we approximate the likelihood function by the conditional likelihood function defined by Equation 5.23, under the SE loss function, the Bayes estimator of the function $v = v(\phi_1, \phi_2, \tau)$ is defined by Equation 5.24 and the Bayes estimator under the LINEX loss function is defined by Equation 5.25. If we apply Lindley's approximation to this model, we get the following result.

Proposition 5.23. *In the AR(2) model, we assume the parameters follow independent truncated normal - improper model. Then*

(i) *Under the SE loss function, the approximate Bayes estimators of ϕ_1, ϕ_2 and τ are defined by Equations 5.30, 5.31 and 5.32, respectively.*

(ii) *Under the LINEX loss function, the approximate Bayes estimators of ϕ_1, ϕ_2 and τ are defined by Equations 5.35, 5.36 and 5.37, respectively.*

(iii) *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by Equation 5.34.*

(iv) *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by Equation 5.38.*

Where

$$\rho = \log(\xi(\phi_1, \phi_2, \tau)) = constant - \log(\tau) - \frac{(\phi_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(\phi_2 - \mu_2)^2}{2\sigma_2^2},$$
$$\hat{\rho_1} = -\frac{\hat{\phi}_1 - \mu_1}{\sigma_1^2}, \quad \hat{\rho_2} = -\frac{\hat{\phi}_2 - \mu_2}{\sigma_2^2}, \quad \hat{\rho_3} = -\frac{1}{\hat{\tau}},$$

and other functions are same as defined in the Section 5.1.2.

5.4. FULL CONDITIONAL DISTRIBUTIONS OF PARAMETERS UNDER INDEPENDENT TRUNCATED NORMAL - IMPROPER PRIOR

5.4.1. AR(1) model

In this section we obtain the full conditional distributions for AR(1) model parameters ϕ_1, τ and the one-step prediction Y_{n+1}. The following propositions provide the required distributions.

Proposition 5.24. *The full conditional distribution of Bayes estimator of ϕ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{\tau \sum_{t=2}^{n} Y_t Y_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{5.59}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{5.60}$$

Proof.

$$\begin{aligned}
\xi(\phi_1|\tau, S_n) &\propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2} \tau^{-1} \mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\sum_{t=2}^{n} Y_t^2 - 2\phi_1 \sum_{t=2}^{n} Y_t Y_{t-1} + \phi_1^2 \sum_{t=2}^{n} Y_{t-1}^2) - \frac{\phi_1^2 - 2\phi_1\mu_1 + \mu_1^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{1}{2}((\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2})\phi_1^2 - 2(\tau \sum_{t=2}^{n} Y_t Y_{t-1} + \frac{\mu_1}{\sigma_1^2})\phi_1)} \\
&\propto \mathrm{e}^{-\frac{\left(\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}\right)}{2}\left(\phi_1 - \frac{\tau \sum_{t=2}^{n} Y_t Y_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=2}^{n} Y_{t-1}^2 + \frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned} \tag{5.61}$$

Proposition 5.25. *The full conditional distribution of Bayes estimator of τ given a sample*

$S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where

$$\tilde{\alpha} = \frac{n-1}{2} \tag{5.62}$$

and

$$\tilde{\beta} = \frac{\sum_{t=2}^{n}(Y_t - \phi_1 Y_{t-1})^2}{2}. \tag{5.63}$$

Proof.

$$\begin{aligned} \xi(\tau|\phi_1, S_n) &\propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2} \tau^{-1} \mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}} \\ &\propto \tau^{\frac{n-1}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2} \tau^{-1} \\ &= \tau^{\frac{n-1}{2}-1} \mathrm{e}^{-\tau(\frac{1}{2}\sum_{t=2}^{n}(Y_t-\phi_1 Y_{t-1})^2)}. \end{aligned} \tag{5.64}$$

Proposition 5.26. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}$ and precision τ, where*

$$\tilde{\mu} = \phi_1 Y_n. \tag{5.65}$$

Proof.

$$\begin{aligned} Y_{n+1}|\phi_1, \tau, S_n &= (\phi_1 Y_n + \epsilon_{n+1})|\phi_1, \tau, S_n \\ &= \phi_1 Y_n + (\epsilon_{n+1}|\tau). \end{aligned} \tag{5.66}$$

5.4.2. AR(2) model

Proposition 5.27. *The full conditional distribution of Bayes estimator of ϕ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{\tau(\sum_{t=3}^n Y_t Y_{t-1} - \phi_2 \sum_{t=3}^n Y_{t-1}Y_{t-2}) + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=3}^n Y_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{5.67}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=3}^n Y_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{5.68}$$

Proof.

$$\begin{aligned}
\xi(\phi_1|\phi_2, \tau, S_n) &\propto \tau^{\frac{n-2}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^n (Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2} \tau^{-1} \mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2} - \frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^n (\phi_1^2 Y_{t-1}^2 - 2\phi_1 Y_t Y_{t-1} + 2\phi_1\phi_2 Y_{t-1}Y_{t-2})} \mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\phi_1^2(\sum_{t=3}^n Y_{t-1}^2) - 2\phi_1(\sum_{t=3}^n Y_t Y_{t-1}) + 2\phi_1\phi_2(\sum_{t=3}^n Y_{t-1}Y_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\phi_1^2 - 2\mu_1\phi_1)}{2\sigma_1^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\phi_1^2(\tau(\sum_{t=3}^n Y_{t-1}^2) + \frac{1}{\sigma_1^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\phi_1(\tau[(\sum_{t=3}^n Y_t Y_{t-1}) - \phi_2(\sum_{t=3}^n Y_{t-1}Y_{t-2})] + \frac{\mu_1}{\sigma_1^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=3}^n Y_{t-1}^2) + \frac{1}{\sigma_1^2}}{2}\left(\phi_1 - \frac{\tau[(\sum_{t=3}^n Y_t Y_{t-1}) - \phi_2(\sum_{t=3}^n Y_{t-1}Y_{t-2})] + \frac{\mu_1}{\sigma_1^2}}{\tau(\sum_{t=3}^n Y_{t-1}^2) + \frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned}$$

Proposition 5.28. *The full conditional distribution of Bayes estimator of ϕ_2 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_2$ and variance $\tilde{\sigma}_2^2$, where*

$$\tilde{\mu}_2 = \frac{\tau(\sum_{t=3}^n Y_t Y_{t-2} - \phi_1 \sum_{t=3}^n Y_{t-1}Y_{t-2}) + \frac{\mu_2}{\sigma_2^2}}{\tau \sum_{t=3}^n Y_{t-2}^2 + \frac{1}{\sigma_2^2}} \tag{5.69}$$

and

$$\tilde{\sigma}_2^2 = \frac{1}{\tau \sum_{t=3}^{n} Y_{t-2}^2 + \frac{1}{\sigma_2^2}}. \tag{5.70}$$

Proof.

$$\begin{aligned}
\xi(\phi_2|\phi_1,\tau,S_n) &\propto \tau^{\frac{n-2}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1 Y_{t-1}-\phi_2 Y_{t-2})^2} \tau^{-1} \mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(\phi_2^2 Y_{t-2}^2 - 2\phi_2 Y_t Y_{t-2} + 2\phi_1\phi_2 Y_{t-1} Y_{t-2})} \mathrm{e}^{-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\phi_2^2(\sum_{t=3}^{n} Y_{t-2}^2) - 2\phi_2(\sum_{t=3}^{n} Y_t Y_{t-2}) + 2\phi_1\phi_2(\sum_{t=3}^{n} Y_{t-1} Y_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\phi_2^2-2\mu_2\phi_2)}{2\sigma_2^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\phi_2^2(\tau(\sum_{t=3}^{n} Y_{t-2}^2)+\frac{1}{\sigma_2^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\phi_2(\tau[(\sum_{t=3}^{n} Y_t Y_{t-2}) - \phi_1(\sum_{t=3}^{n} Y_{t-1} Y_{t-2})] + \frac{\mu_2}{\sigma_2^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=3}^{n} Y_{t-2}^2)+\frac{1}{\sigma_2^2}}{2}\left(\phi_2 - \frac{\tau[(\sum_{t=3}^{n} Y_t Y_{t-2}) - \phi_1(\sum_{t=3}^{n} Y_{t-1} Y_{t-2})] + \frac{\mu_2}{\sigma_2^2}}{\tau(\sum_{t=3}^{n} Y_{t-2}^2)+\frac{1}{\sigma_2^2}}\right)^2}.
\end{aligned}$$

Proposition 5.29. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \frac{n-2}{2} \tag{5.71}$$

and

$$\tilde{\beta} = \frac{\sum_{t=3}^{n}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2}{2}. \tag{5.72}$$

Proof.

$$\begin{aligned}\xi(\tau|\phi_1,\phi_2,S_n) &\propto \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2}\tau^{-1}\mathrm{e}^{-\frac{(\phi_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\phi_2-\mu_2)^2}{2\sigma_2^2}}\\ &\propto \tau^{\frac{n-2}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2}\tau^{-1}\\ &= \tau^{\frac{n-2}{2}-1}\mathrm{e}^{-\tau(\frac{1}{2}\sum_{t=3}^{n}(Y_t-\phi_1Y_{t-1}-\phi_2Y_{t-2})^2)}. \end{aligned} \quad (5.73)$$

Proposition 5.30. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}$ and precision τ, where*

$$\tilde{\mu} = \phi_1 Y_n + \phi_2 Y_{n-1}. \quad (5.74)$$

Proof.

$$\begin{aligned} Y_{n+1}|\phi_1,\phi_2,\tau,S_n &= (\phi_1Y_n+\phi_2Y_{n-1}+\epsilon_{n+1})|\phi_1,\phi_2,\tau,S_n\\ &= \phi_1Y_n+\phi_2Y_{n-1}+(\epsilon_{n+1}|\tau). \end{aligned} \quad (5.75)$$

5.5. INDEPENDENT UNIFORM - GAMMA PRIOR

In the model 5.1 we assume the parameters $\phi_i, i = 1, \ldots, p$, have independent uniform priors on intervals $(a_i, b_i), a_i, b_i \in \mathbb{R}$, for all $i = 1, \ldots, p$, respectively, that is, $\phi_1 \sim U(a_1, b_1), \ldots, \phi_p \sim U(a_p, b_p)$, and the precision has independent gamma prior with the parameters α and β, i.e. $\tau \sim Gamma(\alpha, \beta)$. That is,

$$\xi(\phi, \tau) = \xi_1(\phi)\xi_2(\tau),$$

where the marginal prior density of τ is gamma distribution

$$\xi_2(\tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta}, \tau > 0,$$

and the marginal prior density of ϕ is

$$\xi_1(\phi) \propto 1.$$

So the joint prior density function of parameters

$$\xi(\phi,\tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta}.$$

The likelihood function for the model 5.1

$$L(\phi,\tau|S_n) \propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y-X\phi)'(Y-X\phi)}.$$

We approximate the likelihood function by the conditional likelihood function

$$\begin{aligned} L^*(\phi,\tau|S_n) &\propto \tau^{\frac{n-p}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=p+1}^{n}(Y_t-\sum_{i=1}^{p}\phi_i Y_{t-i})^2} \\ &:= \tau^{\frac{n-p}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y^*-X^*\phi)'(Y^*-X^*\phi)}. \end{aligned} \tag{5.76}$$

Employing Bayes Theorem, the posterior density of ϕ and τ is

$$\xi(\phi,\tau|S_n) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta}\tau^{\frac{n-p}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y^*-X^*\phi)'(Y^*-X^*\phi)}. \tag{5.77}$$

Now we can find the posterior distribution for ϕ. We have

$$\xi(\phi,\tau|S_n) \propto \tau^{a-1}\mathrm{e}^{-\frac{\tau}{2}b},$$

where $a := \frac{2\alpha+n-p}{2}$, $b := 2\beta + (Y^* - X^*\phi)'(Y^* - X^*\phi)$. Then the posterior density of ϕ

$$\begin{aligned}\xi(\phi|S_n) &\propto \int_0^\infty \tau^{a-1} \mathrm{e}^{-\frac{\tau}{2}b} \mathrm{d}\tau \\ &= \frac{2}{b}^a \Gamma(a) \\ &\propto b^{-a}.\end{aligned}$$

We rewrite b as follows

$$\begin{aligned} b &= 2\beta + (Y^* - X^*\phi)'(Y^* - X^*\phi) \\ &= 2\beta + (\phi - (X^{*'}X^*)^{-1}(X^{*'}Y^*))'(X^{*'}X^*)(\phi - (X^{*'}X^*)^{-1}(X^{*'}Y^*)) \\ &- (Y^{*'}X^*)(X^{*'}X^*)^{-1}(X^{*'}Y^*) + Y^{*'}Y^* \\ &:= 2\beta + (\phi - c)'(X^{*'}X^*)(\phi - c) - d + Y^{*'}Y^*, \end{aligned}$$

where $c = (X^{*'}X^*)^{-1}(X^{*'}Y^*)$, $\quad d = (Y^{*'}X^*)(X^{*'}X^*)^{-1}(X^{*'}Y^*)$.
Therefore,

$$\begin{aligned} \xi(\phi|S_n) &\propto (2\beta + (\phi - c)'(X^{*'}X^*)(\phi - c) - d + Y^{*'}Y^*)^{-a} \\ &\propto \Big(1 + \frac{(\phi - c)'(X^{*'}X^*)(\phi - c)}{2\beta - d + Y^{*'}Y^*}\Big)^{-a} \\ &= \Big(1 + \frac{(\phi - c)'\Big[\frac{(2\beta - d + Y^{*'}Y^*)(X^{*'}X^*)^{-1}}{\nu}\Big]^{-1}(\phi - c)}{\nu}\Big)^{-\frac{\nu+p}{2}}, \end{aligned}$$

where $\nu = n + 2\alpha - 2p$. We get the following result.

Proposition 5.31. *The posterior distribution of ϕ is the truncated p-dimensional t-distribution with $\nu = n + 2\alpha - 2p$ degrees of freedom, location vector*

$$c = (X^{*'}X^*)^{-1}(X^{*'}Y^*)$$

and scale matrix

$$\Sigma(\phi|S_n) = \frac{(2\beta - d + Y^{*'}Y^*)(X^{*'}X^*)^{-1}}{n + 2\alpha - 2p},$$

where $d = (Y^{*'}X^*)(X^{*'}X^*)^{-1}(X^{*'}Y^*)$*. That is*

$$\phi|S_n \sim Tt_p(c, \Sigma(\phi|S_n), \nu).$$

So we have

Corollary 5.1. *The marginal posterior distribution for individual parameter is*

$$\phi_i|S_n \sim Tt_1(c_i, s_i, \nu),$$

where s_i *is the diagonal element of scale matrix* $\Sigma(\phi|S_n)$*,* $i = 1, \ldots, p$*.*

The posterior density of τ

$$\begin{aligned}
\xi(\tau|S_n) &\propto \int_\Phi \tau^{a-1} \mathrm{e}^{-\frac{\tau}{2}b} \mathrm{d}\phi \\
&= \int_\Phi \tau^{a-1} \mathrm{e}^{-\frac{\tau}{2}((\phi-c)'(X^{*'}X^*)(\phi-c)+2\beta-d+Y'Y)} \mathrm{d}\phi \\
&= \tau^{a-1} \mathrm{e}^{-\frac{\tau}{2}(2\beta-d+Y'Y)} \int_\Phi \mathrm{e}^{-\frac{\tau}{2}(\phi-c)'(X^{*'}X^*)(\phi-c)} \mathrm{d}\phi \\
&\propto \tau^{a-\frac{p}{2}-1} \mathrm{e}^{-\frac{\tau}{2}(2\beta-d+Y'Y)},
\end{aligned}$$

where $c = (X^{*'}X^*)^{-1}(X^{*'}Y), \quad d = (Y'X^*)(X^{*'}X^*)^{-1}(X^{*'}Y)$.

We get the following result

Proposition 5.32. *The posterior distribution of* τ *is the gamma distribution with parameters*

α_0 *and* β_0*, where*

$$\alpha_0 = \frac{n + 2\alpha - 2p}{2} \tag{5.78}$$

and

$$\beta_0 = \frac{1}{2}(2\beta - (Y'X^*)(X^{*'}X^*)^{-1}(X^{*'}Y) + Y'Y). \tag{5.79}$$

In oder to derive the posterior density for one-step ahead prediction, we denote

$$\begin{aligned} Y_f &= \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ Y_1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ Y_n & Y_{n-1} & \dots & Y_{n-p+1} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{n+1} \end{pmatrix} \\ &:= \begin{pmatrix} X^* \\ P \end{pmatrix} \phi + \epsilon := X_f\phi + \epsilon, \end{aligned} \tag{5.80}$$

where

$$P = \begin{pmatrix} Y_n & Y_{n-1} & \dots & Y_{n-p+1} \end{pmatrix}.$$

The likelihood function for the latter model is

$$L^*(\phi, \tau | S_n) \propto \tau^{\frac{n-p+1}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y - X_f\phi)'(Y_f + X_f\phi)}.$$

The joint prior density function of the parameters

$$\xi(\phi, \tau) \propto \tau^{\alpha-1} \mathrm{e}^{-\tau\beta}.$$

We assume that X_f is a known constant and employ Bayes and Fubini theorems, then

$$\begin{aligned}\xi(W_1|S_n) &= \xi(Y_{n+1}|S_n) \propto \xi(Y_{n+1}, S_n) \\ &\propto \int_{\Phi\times(0,\infty)} \tau^{\frac{n-p+1}{2}+\alpha-1}\mathrm{e}^{-\frac{\tau}{2}(2\beta+(Y_f-X_f\phi)'(Y_f-X_f\phi))}\mathrm{d}\phi\mathrm{d}\tau \\ &= \int_0^\infty \tau^{\frac{n-p+1}{2}+\alpha-1}\mathrm{e}^{-\tau\beta}\mathrm{d}\tau \int_\Phi \mathrm{e}^{-\frac{\tau}{2}(Y_f-X_f\phi)'(Y_f-X_f\phi)}\mathrm{d}\phi \qquad (5.81)\end{aligned}$$

Rewrite

$$\begin{aligned}(Y_f - X_f\phi)'(Y_f - X_f\phi) &= (\phi-\phi_0)'(X_f'X_f)(\phi-\phi_0) \qquad (5.82)\\ &- (X_f'Y_f)'(X_f'X_f)^{-1}(X_f'Y_f) + Y_f'Y_f \qquad (5.83)\end{aligned}$$

where $\phi_0 = (X_f'X_f)^{-1}(X_f'Y_f)$.

Hence

$$\begin{aligned}\xi(W_1|S_n) &\propto \int_0^\infty \tau^{\frac{n-p+1}{2}+\alpha-1}\mathrm{e}^{-\tau\beta}\mathrm{e}^{-\frac{\tau}{2}(-(X_f'Y_f)'(X_f'X_f)^{-1}(X_f'Y_f)+Y_f'Y_f)}\mathrm{d}\tau \\ &\times \int_\Phi \mathrm{e}^{-\frac{\tau}{2}((\phi-\phi_0)'(X_f'X_f)(\phi-\phi_0))}\mathrm{d}\phi \\ &\propto \int_0^\infty \tau^{\frac{n-p+2\alpha+1-p}{2}-1}\mathrm{e}^{-\frac{\tau}{2}(2\beta-(X_f'Y_f)'(X_f'X_f)^{-1}(X_f'Y_f)+Y_f'Y_f)}\mathrm{d}\tau \\ &\propto (2\beta-(X_f'Y_f)'(X_f'X_f)^{-1}(X_f'Y_f)+Y_f'Y_f)^{-\frac{n-2p+2\alpha+1}{2}} \qquad (5.84)\end{aligned}$$

We notice that

$$\begin{aligned}(X_f'Y_f)'(X_f'X_f)^{-1}(X_f'Y_f) &= (Y'X^* + Y_{n+1}P)(X_f'X_f)^{-1}(X^{*\prime}Y + Y_{n+1}P') \\ &= (Y'X^*)(X_f'X_f)^{-1}(X^{*\prime}Y) \\ &+ (Y'X^*)(X_f'X_f)^{-1}Y_{n+1}P' \\ &+ Y_{n+1}P(X_f'X_f)^{-1}(X^{*\prime}Y) \\ &+ Y_{n+1}^2 P(X_f'X_f)^{-1}P' \end{aligned} \tag{5.85}$$

and $Y_f'Y_f = Y'Y + Y_{n+1}^2$.

We use the following lemma

Lemma 3. *If random variable Z has a probability density function* $f_Z(z) \propto (az^2 - 2bz + c)^{-d}, a > 0, ac - b^2 \neq 0$*, then* $Z \sim t(\frac{b}{a}, \frac{ac-b^2}{a^2(2d-1)}, 2d-1)$.

Thus,

$$\xi(W_1|S_n) \quad \propto \quad (E + 2DY_{n+1} + CY_{n+1}^2)^{-\frac{n-2p+2\alpha+1}{2}}, \tag{5.86}$$

where

$$E = 2\beta - (Y'X^*)(X_f'X_f)^{-1}(X^{*\prime}Y) + Y'Y, \tag{5.87}$$

$$D = -(Y'A^*)(A_f'A_f)^{-1}P', \tag{5.88}$$

$$C = (1 - P(A_f'A_f)^{-1}P'). \tag{5.89}$$

We get the following result

Proposition 5.33. *The one-step predictive distribution is the t-distribution* $t(\frac{D}{C}, \frac{CE-D^2}{C^2(n+2\alpha-2p)}, n + 2\alpha - 2p)$*, where C, D and E are as defined above.*

Theorem 5.1. *Under the SE loss function, the Bayes estimator of* ϕ_i*,* $i = 1, \ldots, p$ *is equal to*

$$\hat{\phi}_{i(B_{SE})} \quad = \quad c_i + \frac{k\nu\sqrt{s_i}}{\nu - 1}\Big((1 + \frac{(a_i - c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}} - (1 + \frac{(b_i - c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}}\Big), \tag{5.90}$$

where

$$k = \frac{\gamma(\frac{\nu+1}{2})}{k_0\gamma(\frac{\nu}{2})\sqrt{\pi\nu}}, \quad k_0 = F(\frac{(b_i - c_i)}{\sqrt{s_i}}|\nu) - F(\frac{(a_i - c_i)}{\sqrt{s_i}}|\nu).$$

Here $F(\cdot|\nu)$ denotes the c.d.f. of the Student's t-distribution and the parameters ν, c_i, s_i, are as defined in the Proposition 5.31 and Corollary 5.1.

Proof. Under the SE loss function, the Bayes estimator of ϕ_i is the posterior mean. The computation of the moments of the truncated t-distribution can are given in the proof of the Theorem 1in [59].

Theorem 5.2. *Under the SE loss function, the Bayes estimator of τ, is equal to*

$$\hat{\tau}_{(B_{SE})} = \frac{\alpha_0}{\beta_0}, \tag{5.91}$$

where α_0 and β_0 are defined by Equations 5.78 and 5.79, respectively.

Proof. Under the SE loss function, the Bayes estimator of τ is the posterior mean.

Under the LINEX loss function, the Bayes estimator involves the moment generating function. For the truncated t-distribution, the moment generating function is defined but does not have a tractable form. We can approximate it using numeric integration or, since the degrees of freedom is large, the truncated t-distribution can be well approximated by the truncated normal distribution whose moment generating function exists, see Section 3.7 Using this approximation, we get the following result.

Theorem 5.3. *Under the LINEX loss function, the Bayes estimator of ϕ_i, $i = 1, \ldots, p$ is equal to*

$$\hat{\phi}_{i(B_{LINEX})} = c_i - \frac{\gamma s_i}{2} - \frac{1}{\gamma}\log\Big(\frac{\Phi(\frac{b_i - c_i}{\sqrt{s_i}} + s_i\gamma) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}} + s_i\gamma)}{\Phi(\frac{b_i - c_i}{\sqrt{s_i}}) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}})}\Big), \tag{5.92}$$

where the parameters c_i, s_i, are as defined in Proposition 5.31 and Corollary 5.1.

Proof. We substitute the moment generating function of truncated normal distribution into the expression of the Bayes estimator of ϕ_i under the LINEX loss function.

Theorem 5.4. *Under the LINEX loss function, the Bayes estimator of τ, is equal to*

$$\hat{\tau}_{B_{LINEX}} = -\frac{\alpha_0}{\gamma}\log\left(1+\frac{\gamma}{\beta_0}\right) \tag{5.93}$$

where α_0 and β_0 are defined by Equations 5.78 and 5.79, respectively.

Proof. We substitute the moment generating function of the gamma distribution into the expression of the Bayes estimator of τ under the LINEX loss function.

For the t-distribution, the moment generating function is undefined, in order to be able to define the Bayes estimator of the one-step ahead prediction, we approximate the t-distribution by the normal distribution.

Theorem 5.5. *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by*

$$\hat{W}_{1(B_{LINEX})} = \frac{D}{C} - \frac{\gamma(CE - D^2)}{2C^2(n + 2\alpha - 2p)} \tag{5.94}$$

where C, D and E are defined by Equations 5.89, 5.88 and 5.87, respectively.

Proof. We substitute the moment generating function of the normal distribution into the expression of the Bayes estimator of Y_{n+1} under the LINEX loss function.

5.6. INDEPENDENT UNIFORM - IMPROPER PRIOR

In the model 5.1 we assume the parameters $\phi_i, i = 1, \ldots, p$, have independent uniform priors on intervals $(a_i, b_i), a_i, b_i \in \mathbb{R}$, for all $i = 1, \ldots, p$, respectively, that is, $\phi_1 \sim U(a_1, b_1), \ldots, \phi_p \sim U(a_p, b_p)$, and the precision has independent improper prior. In this model,

$$\xi(\phi, \tau) = \xi_1(\phi)\xi_2(\tau),$$

where the marginal prior density of τ is

$$\xi_2(\tau) \propto \frac{1}{\tau}, \tau > 0,$$

and the marginal prior density of ϕ is

$$\xi_1(\phi) \propto 1.$$

So the joint prior density function of parameters

$$\xi(\phi, \tau) \propto \tau^{-1}$$

The likelihood function for the model 5.1

$$L(\phi, \tau | S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y - X\phi)'(Y - X\phi)} .$$

We approximate the likelihood function by the conditional likelihood function

$$\begin{aligned} L^*(\phi, \tau | S_n) &\propto \tau^{\frac{n-p}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=p+1}^{n}(Y_t - \sum_{i=1}^{p}\phi_i Y_{t-i})^2} \\ &:= \tau^{\frac{n-p}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y^* - X^*\phi)'(Y^* - X^*\phi)}. \end{aligned} \tag{5.95}$$

Employing Bayes Theorem, the posterior density of ϕ and τ is

$$\xi(\phi, \tau | S_n) \propto \tau^{-1} \tau^{\frac{n-p}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y^* - X^*\phi)'(Y^* - X^*\phi)}. \tag{5.96}$$

In the same manner as in Section 5.5, we can show that the posterior distribution for $\phi_i,\ i = 1, \ldots, p$, is the truncated t-distribution, the posterior distribution for τ is the gamma distribution and the one-step predictive distribution is the t-distribution. For the independent uniform prior for ϕ and improper prior for τ, we get the following results.

Proposition 5.34. *The posterior distribution of ϕ is the truncated p-dimensional t-distribution with $\nu = n - 2p$ degrees of freedom, location vector*

$$c = (X^{*'}X^{*})^{-1}(X^{*'}Y^{*})$$

and scale matrix

$$\Sigma(\phi|S_n) = \frac{(-d + Y^{*'}Y^{*})(X^{*'}X^{*})^{-1}}{n - 2p},$$

where $d = (Y^{*'}X^{*})(X^{*'}X^{*})^{-1}(X^{*'}Y^{*})$. That is

$$\phi|S_n \sim Tt_p(c, \Sigma(\phi|S_n), \nu).$$

So we have

Corollary 5.2. *The marginal posterior distribution for individual parameter is*

$$\theta_i|S_n \sim Tt_1(c_i, s_i, \nu),$$

where s_i is the diagonal element of scale matrix $\Sigma(\theta|S_n)$, $i = 1, \ldots, q$.

Proposition 5.35. *The posterior distribution of τ is the gamma distribution with parameters α_0 and β_0, where*

$$\alpha_0 = \frac{n - 2p}{2} \tag{5.97}$$

and

$$\beta_0 = \frac{1}{2}(-(Y'X^*)(X^{*'}X^*)^{-1}(X^{*'}Y) + Y'Y). \tag{5.98}$$

Proposition 5.36. *The one-step predictive distribution is the t-distribution* $t(\frac{D}{C}, \frac{CE-D^2}{C^2(n-2p)}, n-2p)$*, where C, D and E are as defined below*

$$E = -(Y'A^*)(A_f'A_f)^{-1}(A^{*'}Y) + Y'Y, \tag{5.99}$$

$$D = -(Y'A^*)(A_f'A_f)^{-1}P', \tag{5.100}$$

$$C = (1 - P(A_f'A_f)^{-1}P'). \tag{5.101}$$

The derived Bayes estimator are given in the theorems below

Theorem 5.6. *Under the SE loss function, the Bayes estimator of* ϕ_i*,* $i = 1, \ldots, p$ *is equal to*

$$\hat{\theta}_{i(B_{SE})} \;=\; c_i + \frac{k\nu\sqrt{s_i}}{\nu - 1}\Big((1 + \frac{(a_i - c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}} - (1 + \frac{(b_i - c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}}\Big), \tag{5.102}$$

where

$$k = \frac{\gamma(\frac{\nu+1}{2})}{k_0\gamma(\frac{\nu}{2})\sqrt{\pi\nu}}, \quad k_0 = F(\frac{(b_i - c_i)}{\sqrt{s_i}}|\nu) - F(\frac{(a_i - c_i)}{\sqrt{s_i}}|\nu).$$

Here $F(\cdot|\nu)$ *denotes the c.d.f. of the Student's t-distribution and the parameters* ν, c_i, s_i*, are as defined in Proposition 5.34 and Corollary 5.2.*

Theorem 5.7. *Under the SE loss function, the Bayes estimator of* τ*, is equal to*

$$\hat{\tau}_{(B_{SE})} \;=\; \frac{\alpha_0}{\beta_0}, \tag{5.103}$$

where α_0 *and* β_0 *are defined by Equations 5.97 and 5.98, respectively.*

Proposition 5.37. *Under the SE loss function, the Bayes estimator of the one-step ahead* *forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\hat{W}_{1(B_{SE})} = \sum_{k=1}^{p} \hat{\phi}_{k(B_{SE})} Y_{n-k+1}. \quad (5.104)$$

By approximating the truncated t-distribution by the truncated normal distribution, we get the following result.

Theorem 5.8. *Under the LINEX loss function, the approximate Bayes estimator of ϕ_i, $i = 1, \ldots, p$ is equal to*

$$\hat{\phi}_{i(B_{LINEX})} = c_i - \frac{\gamma s_i}{2} - \frac{1}{\gamma} \log\Big(\frac{\Phi(\frac{b_i - c_i}{\sqrt{s_i}} + s_i\gamma) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}} + s_i\gamma)}{\Phi(\frac{b_i - c_i}{\sqrt{s_i}}) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}})} \Big), \quad (5.105)$$

where the parameters c_i, s_i are as defined in the Proposition 5.34 and Corollary 5.2.

The Bayes estimator of τ is given below.

Theorem 5.9. *Under the LINEX loss function, the Bayes estimator of τ, is equal to*

$$\hat{\tau}_{B_{LINEX}} = -\frac{\alpha_0}{\gamma} \log\Big(1 + \frac{\gamma}{\beta_0}\Big) \quad (5.106)$$

where α_0 and β_0 are defined by Equations 5.97 and 5.98, respectively.

By approximating the t-distribution by the normal distribution, we get the following result

Theorem 5.10. *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by*

$$\hat{W}_{1(B_{LINEX})} = \frac{D}{C} - \frac{\gamma(CE - D^2)}{2C^2(n + 2\alpha - 2p)} \quad (5.107)$$

where C, D and E are defined by Equations 5.101, 5.100 and 5.99, respectively.

6. NUMERICAL STUDY OF ESTIMATION AND FORECASTING FOR AUTOREGRESSIVE PROCESSES

The following procedures will be used for the statistical calculation for the AR(p) model.

(i) Simulate τ.

(ii) Simulate ϕ.

(iii) Generate the AR(p) series: $y_1, \ldots, y_n$ for some n.

(iv) Calculate the ML estimates for the parameters and compute the error using the SE and LINEX loss functions.

(v) Calculate the Bayes estimates for the parameters using the SE and LINEX loss functions and find the error. For truncated normal prior for ϕ, we calculate approximate Bayes estimates using Lindley's approximation and Gibbs sampling. We run Gibbs sampler for an initial 1,000 iterations that we discard, and then for a further 9,000 iterations of which we store every fifth.

(vi) Calculate the ML and Bayes estimates of one-step ahead forecasts using the SE and LINEX loss functions and find the errors.

(vii) Since the Bayes estimation using Gibbs sampler is computationally expensive, repeat the above procedures 1,000 times for truncated normal and 10,000 for uniform priors for ϕ, and calculate the mean errors under the SE and LINEX loss functions.

The simulation study is undertaken using sample sizes $n = 50, 100, 150, 200$ and LINEX loss function parameters $\gamma = -1.25, -0.75, -0.25, 0.25, 0.75, 1.25$.

In order to compare the ML and Bayes estimators, the average squared errors are used when the Bayes estimates are computed using the SE loss function. The average errors, computed using the LINEX loss function are used when the Bayes estimates are obtained using the LINEX loss function.

6.1. AR(1) MODEL

In this section we study the AR(1) model

$$Y_t = \phi_1 Y_{t-1} + \epsilon_t. \tag{6.1}$$

The AR(1) process is stationary if $-1 < \phi_1 < 1$.

6.1.1. Independent Truncated Normal prior for ϕ_1 and Gamma or Improper priors for τ

We consider a truncated normal prior for ϕ_1 with $\mu_1 = 0.5$, $\sigma_1 = 0.3$, defined over the interval (a_1, b_1), where $a_1 = 0.25$, $b_1 = 0.75$. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$. The obtained AR(1) process is stationary.

Table 6.1 presents the average values of AR(1) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used. Under the SE loss function, the average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overall the Bayes estimates are found to have smaller average estimation errors than the ML estimates, for ϕ_1 the smallest estimation errors are obtained for the Bayes estimates obtained using Lindley's approximation; for τ the Bayes estimation using the Gibbs sampler is found to result in the smallest estimation errors when τ has gamma prior and the Bayes estimation using Lindley's approximation when τ has improper prior; for the one-step prediction all estimates have similar errors, the ones of Bayes estimates being slightly smaller. All estimator performances are reasonably close to each other as the sample size increases.

Under the LINEX loss function, there is a non-zero probability that the Bayes estimates using Lindley's approximation may be undefined (see Section 5.1.1), Table 6.2 shows proportion of undefined Bayes estimates using this approximation. Under our choice of parameters, undefined estimates are obtained only for τ when it has gamma prior and LINEX loss function parameters are $\gamma = -1.25, -0.75$. The proportion of undefined τ estimates decreases significantly as sample size increases. We exclude the simulation where we obtain undefined estimates and calculate average errors where all estimates are defined.

Table 6.3, Table 6.4, Table 6.5, Table 6.6, Table 6.7 and Table 6.8 present the average values of AR(1) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25, -0.75, -0.25, 0.25,$ $0.75, 1.25$, respectively. Under the LINEX loss function, the average estimation errors are also found to decrease with increasing sample size. Again, in general the Bayes estimation has the smallest average errors.

For ϕ_1, the smallest average estimation errors are obtained for the Bayes estimates using Lindley's approximation, the difference between the ML and the Bayes estimates are more noticeable when the LINEX parameter has higher absolute value. For the one-step prediction, the Bayes estimates have significantly smaller average errors than the ML estimates, both Lindley's approximation and Gibbs sampler methods have similar performance.

For τ when the LINEX loss function parameters are positive, the smallest average estimation errors are obtained using the Bayes estimates; where gamma prior is used, the Gibbs sampling method is superior, whereas when improper prior is used, Lindley's approximation performance is better. When $\gamma = -1.25$, if improper prior is used for τ, the ML estimates of τ have smaller average error than the Bayes estimates; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, for sample sizes $n = 50, 100$ the Bayes estimates obtained using Lindley's approximation have the highest average errors, for sample sizes $n = 150, 200$ both Bayes estimates perform better than the ML estimates.

When $\gamma = -0.75$, if improper prior is used for τ, the ML estimates of τ have smaller average error than the Bayes estimates obtained using the Gibbs sampling but higher than the average errors of the Bayes estimates obtained using Lindley's approximation; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, for sample sizes $n = 50, 100$ the Bayes estimates obtained using Lindley's approximation have the highest average errors, for sample size $n = 150, 200$ both Bayes estimates perform better than the ML estimates.

When $\gamma = -0.25$, the Bayes estimates of τ have smaller average errors than the ML estimates. When improper prior is used for τ, the Bayes estimates obtained using Lindley's approximation are found to be superior; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, for sample size $n = 50$ the Bayes estimates obtained using Lindley's approximation has the highest average error, for sample sizes $n = 100, 150, 200$ both Bayes estimates perform better than the ML estimates.

Table 6.1. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using SE loss function

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5045	-	0.5045	-	0.5045	-	0.5045	-
τ	1.6529	-	1.6529	-	1.6529	-	1.6529	-
Y_{n+1}	0.0302	-	-0.0099	-	-0.0124	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4830	0.0157	0.4914	0.0084	0.4946	0.0050	0.4976	0.0038
Lindley's - ϕ_1 with gamma prior for τ	0.4926	0.0111	0.4951	0.0069	0.4966	0.0045	0.4989	0.0035
Lindley's - ϕ_1 with improper prior for τ	0.4926	0.0111	0.4951	0.0069	0.4966	0.0045	0.4989	0.0035
Gibbs - ϕ_1 with gamma prior for τ	0.4907	0.0116	0.4947	0.0070	0.4964	0.0045	0.4988	0.0035
Gibbs - ϕ_1 with improper prior for τ	0.4908	0.0116	0.4946	0.0070	0.4965	0.0045	0.4988	0.0035
MLEs of τ	1.7650	0.1735	1.7063	0.0770	1.6848	0.0462	1.6773	0.0343
Lindley's - τ with gamma prior	1.5723	0.1442	1.6364	0.0553	1.6445	0.0377	1.6486	0.0287
Lindley's - τ with improper prior	1.7290	0.1598	1.6890	0.0738	1.6735	0.0450	1.6689	0.0336
Gibbs - τ with gamma prior	1.6625	0.0914	1.6571	0.0535	1.6530	0.0371	1.6535	0.0285
Gibbs - τ with improper prior	1.7327	0.1609	1.6902	0.0739	1.6737	0.0451	1.6693	0.0338
MLEs of Y_{n+1}	-0.0049	0.6255	-0.0156	0.7089	0.0013	0.6431	-0.0107	0.6866
Lindley's - Y_{n+1} with gamma prior for τ	-0.0048	0.6252	-0.0157	0.7060	0.0014	0.6426	-0.0104	0.6857
Lindley's - Y_{n+1} with improper prior for τ	-0.0048	0.6252	-0.0157	0.7060	0.0014	0.6426	-0.0104	0.6857
Gibbs - Y_{n+1} with gamma prior for τ	-0.0042	0.6275	-0.0167	0.7066	0.0011	0.6434	-0.0103	0.6842
Gibbs - Y_{n+1} with improper prior for τ	-0.0048	0.6260	-0.0153	0.7074	0.0012	0.6423	-0.0104	0.6870

Table 6.2. Proportion of undefined AR(1) model estimates under independent truncated normal prior for ϕ_1

		Gamma prior for τ			**Improper prior for τ**		
LINEX parameter	**Sample size**	ϕ_1	τ	Y_{n+1}	ϕ_1	τ	Y_{n+1}
$\gamma = -1.25$	50	0.0000	0.0790	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0080	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.75$	50	0.0000	0.0320	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.75$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 1.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 6.3. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5046	-	0.5049	-	0.5045	-	0.5045	-
τ	1.5776	-	1.6403	-	1.6506	-	1.6529	-
Y_{n+1}	0.0337	-	-0.0120	-	-0.0134	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4821	0.0128	0.4918	0.0068	0.4945	0.0040	0.4976	0.0030
Lindley's - ϕ_1 with gamma prior for τ	0.5004	0.0086	0.4999	0.0055	0.4996	0.0035	0.5012	0.0027
Lindley's - ϕ_1 with improper prior for τ	0.5004	0.0086	0.4999	0.0055	0.4996	0.0035	0.5012	0.0027
Gibbs - ϕ_1 with gamma prior for τ	0.4980	0.0089	0.4995	0.0055	0.4993	0.0035	0.5010	0.0027
Gibbs - ϕ_1 with improper prior for τ	0.4984	0.0088	0.4993	0.0055	0.4994	0.0035	0.5010	0.0027
MLEs of τ	1.6305	0.0786	1.6889	0.0512	1.6822	0.0339	1.6773	0.0255
Lindley's - τ with gamma prior	1.4927	0.2219	1.6049	0.0687	1.6377	0.0333	1.6465	0.0244
Lindley's - τ with improper prior	1.6046	0.0827	1.6778	0.0529	1.6747	0.0347	1.6717	0.0260
Gibbs - τ with gamma prior	1.6379	0.0640	1.6770	0.0428	1.6735	0.0298	1.6708	0.0229
Gibbs - τ with improper prior	1.6789	0.0882	1.7142	0.0557	1.6979	0.0359	1.6892	0.0269
MLEs of Y_{n+1}	-0.0036	0.7151	-0.0158	0.7620	0.0009	0.6766	-0.0107	0.7314
Lindley's - Y_{n+1} with gamma prior for τ	0.4420	0.5147	0.4111	0.5501	0.4245	0.5167	0.4119	0.5416
Lindley's - Y_{n+1} with improper prior for τ	0.4644	0.5198	0.4196	0.5515	0.4301	0.5182	0.4161	0.5415
Gibbs - Y_{n+1} with gamma prior for τ	0.4390	0.5181	0.4080	0.5523	0.4230	0.5184	0.4115	0.5374
Gibbs - Y_{n+1} with improper prior for τ	0.4675	0.5203	0.4199	0.5540	0.4304	0.5173	0.4157	0.5440

Table 6.4. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5050	-	0.5045	-	0.5045	-	0.5045	-
τ	1.6184	-	1.6529	-	1.6529	-	1.6529	-
Y_{n+1}	0.0300	-	-0.0099	-	-0.0124	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4832	0.0045	0.4914	0.0024	0.4946	0.0014	0.4976	0.0011
Lindley's - ϕ_1 with gamma prior for τ	0.4980	0.0031	0.4977	0.0020	0.4984	0.0013	0.5003	0.0010
Lindley's - ϕ_1 with improper prior for τ	0.4980	0.0031	0.4977	0.0020	0.4984	0.0013	0.5003	0.0010
Gibbs - ϕ_1 with gamma prior for τ	0.4958	0.0032	0.4973	0.0020	0.4982	0.0013	0.5001	0.0010
Gibbs - ϕ_1 with improper prior for τ	0.4961	0.0032	0.4972	0.0020	0.4983	0.0013	0.5001	0.0010
MLEs of τ	1.6995	0.0344	1.7063	0.0200	1.6848	0.0125	1.6773	0.0093
Lindley's - τ with gamma prior	1.4460	0.2183	1.6034	0.0262	1.6315	0.0119	1.6403	0.0085
Lindley's - τ with improper prior	1.6437	0.0340	1.6791	0.0198	1.6667	0.0124	1.6638	0.0093
Gibbs - τ with gamma prior	1.6596	0.0253	1.6763	0.0154	1.6665	0.0106	1.6638	0.0081
Gibbs - τ with improper prior	1.7191	0.0372	1.7152	0.0210	1.6897	0.0129	1.6811	0.0096
MLEs of Y_{n+1}	-0.0026	0.2034	-0.0156	0.2230	0.0013	0.2004	-0.0107	0.2126
Lindley's - Y_{n+1} with gamma prior for τ	0.2590	0.1801	0.2386	0.1981	0.2548	0.1834	0.2425	0.1933
Lindley's - Y_{n+1} with improper prior for τ	0.2697	0.1808	0.2434	0.1983	0.2581	0.1836	0.2451	0.1934
Gibbs - Y_{n+1} with gamma prior for τ	0.2572	0.1809	0.2367	0.1986	0.2539	0.1838	0.2424	0.1925
Gibbs - Y_{n+1} with improper prior for τ	0.2705	0.1811	0.2438	0.1990	0.2582	0.1833	0.2447	0.1939

Table 6.5. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5045	-	0.5045	-	0.5045	-	0.5045	-
τ	1.6529	-	1.6529	-	1.6529	-	1.6529	-
Y_{n+1}	0.0302	-	-0.0099	-	-0.0124	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4830	0.0005	0.4914	0.0003	0.4946	0.0002	0.4976	0.0001
Lindley's - ϕ_1 with gamma prior for τ	0.4943	0.0003	0.4960	0.0002	0.4972	0.0001	0.4993	0.0001
Lindley's - ϕ_1 with improper prior for τ	0.4943	0.0003	0.4960	0.0002	0.4972	0.0001	0.4993	0.0001
Gibbs - ϕ_1 with gamma prior for τ	0.4923	0.0004	0.4956	0.0002	0.4970	0.0001	0.4993	0.0001
Gibbs - ϕ_1 with improper prior for τ	0.4925	0.0004	0.4955	0.0002	0.4971	0.0001	0.4992	0.0001
MLEs of τ	1.7650	0.0051	1.7063	0.0023	1.6848	0.0014	1.6773	0.0011
Lindley's - τ with gamma prior	1.4569	0.0202	1.6039	0.0019	1.6250	0.0012	1.6345	0.0009
Lindley's - τ with improper prior	1.6741	0.0045	1.6626	0.0022	1.6561	0.0014	1.6559	0.0010
Gibbs - τ with gamma prior	1.6734	0.0029	1.6634	0.0017	1.6575	0.0012	1.6569	0.0009
Gibbs - τ with improper prior	1.7509	0.0051	1.6984	0.0023	1.6790	0.0014	1.6732	0.0011
MLEs of Y_{n+1}	-0.0049	0.0200	-0.0156	0.0225	0.0013	0.0203	-0.0107	0.0216
Lindley's - Y_{n+1} with gamma prior for τ	0.0811	0.0196	0.0689	0.0221	0.0858	0.0202	0.0738	0.0214
Lindley's - Y_{n+1} with improper prior for τ	0.0839	0.0196	0.0705	0.0221	0.0869	0.0202	0.0747	0.0214
Gibbs - Y_{n+1} with gamma prior for τ	0.0808	0.0197	0.0677	0.0221	0.0853	0.0202	0.0738	0.0214
Gibbs - Y_{n+1} with improper prior for τ	0.0840	0.0197	0.0709	0.0221	0.0867	0.0202	0.0745	0.0215

Table 6.6. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5045	-	0.5045	-	0.5045	-	0.5045	-
τ	1.6529	-	1.6529	-	1.6529	-	1.6529	-
Y_{n+1}	0.0302	-	-0.0099	-	-0.0124	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4830	0.0005	0.4914	0.0003	0.4946	0.0002	0.4976	0.0001
Lindley's - ϕ_1 with gamma prior for τ	0.4908	0.0003	0.4942	0.0002	0.4960	0.0001	0.4984	0.0001
Lindley's - ϕ_1 with improper prior for τ	0.4908	0.0003	0.4942	0.0002	0.4960	0.0001	0.4984	0.0001
Gibbs - ϕ_1 with gamma prior for τ	0.4890	0.0004	0.4938	0.0002	0.4958	0.0001	0.4984	0.0001
Gibbs - ϕ_1 with improper prior for τ	0.4891	0.0004	0.4937	0.0002	0.4959	0.0001	0.4983	0.0001
MLEs of τ	1.7650	0.0058	1.7063	0.0025	1.6848	0.0015	1.6773	0.0011
Lindley's - τ with gamma prior	1.5157	0.0040	1.5997	0.0018	1.6188	0.0012	1.6290	0.0009
Lindley's - τ with improper prior	1.6410	0.0044	1.6468	0.0022	1.6458	0.0013	1.6482	0.0010
Gibbs - τ with gamma prior	1.6518	0.0028	1.6508	0.0017	1.6486	0.0012	1.6501	0.0009
Gibbs - τ with improper prior	1.7151	0.0050	1.6821	0.0023	1.6684	0.0014	1.6654	0.0011
MLEs of Y_{n+1}	-0.0049	0.0195	-0.0156	0.0224	0.0013	0.0204	-0.0107	0.0219
Lindley's - Y_{n+1} with gamma prior for τ	-0.0906	0.0195	-0.1004	0.0220	-0.0830	0.0200	-0.0947	0.0215
Lindley's - Y_{n+1} with improper prior for τ	-0.0935	0.0195	-0.1020	0.0220	-0.0841	0.0200	-0.0955	0.0215
Gibbs - Y_{n+1} with gamma prior for τ	-0.0891	0.0196	-0.1012	0.0220	-0.0831	0.0201	-0.0945	0.0214
Gibbs - Y_{n+1} with improper prior for τ	-0.0935	0.0195	-0.1015	0.0221	-0.0844	0.0200	-0.0954	0.0215

Table 6.7. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5045	-	0.5045	-	0.5045	-	0.5045	-
τ	1.6529	-	1.6529	-	1.6529	-	1.6529	-
Y_{n+1}	0.0302	-	-0.0099	-	-0.0124	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4830	0.0043	0.4914	0.0023	0.4946	0.0014	0.4976	0.0011
Lindley's - ϕ_1 with gamma prior for τ	0.4873	0.0031	0.4923	0.0019	0.4948	0.0013	0.4975	0.0010
Lindley's - ϕ_1 with improper prior for τ	0.4873	0.0031	0.4923	0.0019	0.4948	0.0013	0.4975	0.0010
Gibbs - ϕ_1 with gamma prior for τ	0.4856	0.0033	0.4921	0.0020	0.4946	0.0013	0.4975	0.0010
Gibbs - ϕ_1 with improper prior for τ	0.4857	0.0033	0.4920	0.0020	0.4947	0.0013	0.4974	0.0010
MLEs of τ	1.7650	0.0607	1.7063	0.0243	1.6848	0.0138	1.6773	0.0101
Lindley's - τ with gamma prior	1.5300	0.0285	1.5951	0.0153	1.6131	0.0105	1.6239	0.0080
Lindley's - τ with improper prior	1.6134	0.0390	1.6323	0.0194	1.6360	0.0120	1.6408	0.0091
Gibbs - τ with gamma prior	1.6309	0.0251	1.6385	0.0147	1.6399	0.0103	1.6434	0.0079
Gibbs - τ with improper prior	1.6814	0.0436	1.6662	0.0206	1.6581	0.0125	1.6577	0.0094
MLEs of Y_{n+1}	-0.0049	0.1868	-0.0156	0.2202	0.0013	0.2036	-0.0107	0.2229
Lindley's - Y_{n+1} with gamma prior for τ	-0.2629	0.1752	-0.2701	0.1972	-0.2520	0.1807	-0.2634	0.1944
Lindley's - Y_{n+1} with improper prior for τ	-0.2717	0.1761	-0.2749	0.1972	-0.2554	0.1806	-0.2659	0.1943
Gibbs - Y_{n+1} with gamma prior for τ	-0.2596	0.1759	-0.2704	0.1972	-0.2517	0.1805	-0.2630	0.1937
Gibbs - Y_{n+1} with improper prior for τ	-0.2721	0.1762	-0.2744	0.1972	-0.2558	0.1804	-0.2654	0.1949

Table 6.8. Average AR(1) model estimates and estimation errors under independent truncated normal prior for ϕ_1 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5045	-	0.5045	-	0.5045	-	0.5045	-
τ	1.6529	-	1.6529	-	1.6529	-	1.6529	-
Y_{n+1}	0.0302	-	-0.0099	-	-0.0124	-	-0.0305	-
ESTIMATES								
MLEs of ϕ_1	0.4830	0.0118	0.4914	0.0063	0.4946	0.0038	0.4976	0.0029
Lindley's - ϕ_1 with gamma prior for τ	0.4837	0.0088	0.4905	0.0054	0.4936	0.0035	0.4966	0.0027
Lindley's - ϕ_1 with improper prior for τ	0.4837	0.0088	0.4905	0.0054	0.4936	0.0035	0.4966	0.0027
Gibbs - ϕ_1 with gamma prior for τ	0.4823	0.0091	0.4903	0.0055	0.4934	0.0035	0.4966	0.0027
Gibbs - ϕ_1 with improper prior for τ	0.4824	0.0091	0.4902	0.0055	0.4935	0.0035	0.4966	0.0027
MLEs of τ	1.7650	0.2061	1.7063	0.0745	1.6848	0.0404	1.6773	0.0293
Lindley's - τ with gamma prior	1.5364	0.0732	1.5905	0.0416	1.6079	0.0288	1.6192	0.0221
Lindley's - τ with improper prior	1.5919	0.1112	1.6193	0.0541	1.6269	0.0332	1.6338	0.0251
Gibbs - τ with gamma prior	1.6107	0.0686	1.6264	0.0403	1.6314	0.0284	1.6367	0.0218
Gibbs - τ with improper prior	1.6496	0.1185	1.6508	0.0570	1.6479	0.0344	1.6501	0.0259
MLEs of Y_{n+1}	-0.0049	0.6083	-0.0156	0.7404	0.0013	0.7074	-0.0107	0.7990
Lindley's - Y_{n+1} with gamma prior for τ	-0.4369	0.4899	-0.4410	0.5420	-0.4219	0.5078	-0.4327	0.5423
Lindley's - Y_{n+1} with improper prior for τ	-0.4521	0.4970	-0.4490	0.5423	-0.4275	0.5072	-0.4370	0.5415
Gibbs - Y_{n+1} with gamma prior for τ	-0.4323	0.4921	-0.4407	0.5416	-0.4208	0.5056	-0.4319	0.5387
Gibbs - Y_{n+1} with improper prior for τ	-0.4548	0.4968	-0.4489	0.5398	-0.4282	0.5060	-0.4358	0.5452

6.1.2. Independent Uniform prior for ϕ_1 and Gamma or Improper priors for τ

We consider a uniform prior for ϕ_1 in the interval (a_1, b_1), where $a_1 = 0.25$, $b_1 = 0.75$. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$. The obtained AR(1) process is stationary.

Table 6.9 presents the average values of AR(1) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used.

Table 6.10, Table 6.11, Table 6.12, Table 6.13, Table 6.14 and Table 6.15 present the average values of AR(1) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25, -0.75, -0.25,$ $0.25, 0.75, 1.25$, respectively.

The average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overall the Bayes estimates are found to have smaller average estimation errors than ML estimates. For ϕ_1 and one-step prediction, the Bayes estimates are found to have smaller average estimation errors under both SE and LINEX loss functions with all parameter γ values. Under SE loss function, for τ the Bayes estimation is found to result in the smallest estimation errors. Under the LINEX loss function, when τ has gamma prior the Bayes estimation has better performance than the ML estimation for all parameter γ values. When when τ has improper prior the ML estimates have smaller average errors when $\gamma = -1.25, -0.75$, when $\gamma = -0.25, 0.25, 0.75, 1.25$ the Bayes estimation is superior. All estimator performances are reasonably close to each other as the sample size increases.

Table 6.9. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using SE loss function

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0157	0.4900	0.0078	0.4934	0.0051	0.4955	0.0037
Bayes - ϕ_1 with gamma prior for τ	0.4995	0.0090	0.4997	0.0055	0.4997	0.0039	0.5002	0.0030
Bayes - ϕ_1 with improper prior for τ	0.4995	0.0090	0.4997	0.0055	0.4997	0.0039	0.5002	0.0030
MLEs of τ	1.7800	0.1702	1.7247	0.0711	1.7072	0.0452	1.6988	0.0332
Bayes - τ with gamma prior	1.6690	0.0865	1.6710	0.0499	1.6717	0.0356	1.6724	0.0278
Bayes - τ with improper prior	1.7437	0.1567	1.7073	0.0681	1.6957	0.0440	1.6903	0.0325
MLEs of Y_{n+1}	-0.0004	0.6689	0.0062	0.6868	-0.0024	0.6686	0.0060	0.6572
Bayes - Y_{n+1} with gamma prior for τ	0.0001	0.6623	0.0060	0.6844	-0.0023	0.6672	0.0059	0.6566
Bayes - Y_{n+1} with improper prior for τ	0.0001	0.6623	0.0060	0.6844	-0.0023	0.6672	0.0059	0.6566

Table 6.10. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0129	0.4900	0.0062	0.4934	0.0040	0.4955	0.0030
Bayes - ϕ_1 with gamma prior for τ	0.4938	0.0109	0.4959	0.0058	0.4971	0.0038	0.4982	0.0028
Bayes - ϕ_1 with improper prior for τ	0.4936	0.0109	0.4958	0.0058	0.4971	0.0038	0.4982	0.0028
MLEs of τ	1.7800	0.1066	1.7247	0.0499	1.7072	0.0330	1.6988	0.0246
Bayes - τ with gamma prior	1.7265	0.0708	1.7039	0.0401	1.6948	0.0283	1.6902	0.0220
Bayes - τ with improper prior	1.8431	0.1315	1.7508	0.0553	1.7235	0.0353	1.7107	0.0259
MLEs of Y_{n+1}	-0.0004	0.7195	0.0062	0.7318	-0.0024	0.7246	0.0060	0.6883
Bayes - Y_{n+1} with gamma prior for τ	0.4124	0.5356	0.4194	0.5322	0.4112	0.5278	0.4197	0.5112
Bayes - Y_{n+1} with improper prior for τ	0.4248	0.5386	0.4264	0.5321	0.4161	0.5280	0.4235	0.5114

Table 6.11. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0045	0.4900	0.0022	0.4934	0.0014	0.4955	0.0011
Bayes - ϕ_1 with gamma prior for τ	0.4899	0.0039	0.4940	0.0021	0.4959	0.0014	0.4973	0.0010
Bayes - ϕ_1 with improper prior for τ	0.4897	0.0039	0.4940	0.0021	0.4959	0.0014	0.4973	0.0010
MLEs of τ	1.7800	0.0411	1.7247	0.0186	1.7072	0.0121	1.6988	0.0090
Bayes - τ with gamma prior	1.7028	0.0250	1.6906	0.0143	1.6855	0.0101	1.6830	0.0079
Bayes - τ with improper prior	1.8012	0.0459	1.7330	0.0196	1.7122	0.0126	1.7025	0.0092
MLEs of Y_{n+1}	-0.0004	0.2104	0.0062	0.2150	-0.0024	0.2113	0.0060	0.2042
Bayes - Y_{n+1} with gamma prior for τ	0.2473	0.1912	0.2541	0.1925	0.2457	0.1891	0.2542	0.1844
Bayes - Y_{n+1} with improper prior for τ	0.2547	0.1918	0.2583	0.1924	0.2487	0.1891	0.2565	0.1844

Table 6.12. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0005	0.4900	0.0002	0.4934	0.0002	0.4955	0.0001
Bayes - ϕ_1 with gamma prior for τ	0.4860	0.0004	0.4921	0.0002	0.4946	0.0002	0.4964	0.0001
Bayes - ϕ_1 with improper prior for τ	0.4859	0.0004	0.4921	0.0002	0.4946	0.0002	0.4964	0.0001
MLEs of τ	1.7800	0.0050	1.7247	0.0022	1.7072	0.0014	1.6988	0.0010
Bayes - τ with gamma prior	1.6801	0.0027	1.6775	0.0016	1.6763	0.0011	1.6759	0.0009
Bayes - τ with improper prior	1.7622	0.0050	1.7158	0.0021	1.7012	0.0014	1.6943	0.0010
MLEs of Y_{n+1}	-0.0004	0.0212	0.0062	0.0217	-0.0024	0.0212	0.0060	0.0207
Bayes - Y_{n+1} with gamma prior for τ	0.0821	0.0210	0.0888	0.0214	0.0803	0.0209	0.0887	0.0205
Bayes - Y_{n+1} with improper prior for τ	0.0846	0.0210	0.0902	0.0214	0.0813	0.0209	0.0895	0.0205

Table 6.13. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0005	0.4900	0.0002	0.4934	0.0002	0.4955	0.0001
Bayes - ϕ_1 with gamma prior for τ	0.4821	0.0004	0.4902	0.0002	0.4934	0.0002	0.4954	0.0001
Bayes - ϕ_1 with improper prior for τ	0.4821	0.0004	0.4902	0.0002	0.4934	0.0002	0.4954	0.0001
MLEs of τ	1.7800	0.0057	1.7247	0.0023	1.7072	0.0014	1.6988	0.0011
Bayes - τ with gamma prior	1.6582	0.0027	1.6646	0.0016	1.6672	0.0011	1.6689	0.0009
Bayes - τ with improper prior	1.7257	0.0048	1.6990	0.0021	1.6903	0.0014	1.6863	0.0010
MLEs of Y_{n+1}	-0.0004	0.0211	0.0062	0.0217	-0.0024	0.0211	0.0060	0.0208
Bayes - Y_{n+1} with gamma prior for τ	-0.0830	0.0208	-0.0764	0.0215	-0.0852	0.0209	-0.0768	0.0205
Bayes - Y_{n+1} with improper prior for τ	-0.0855	0.0208	-0.0778	0.0215	-0.0861	0.0209	-0.0775	0.0205

Table 6.14. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	Average	Average error	Average	Average error	Average	Average error	Average	Average error
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0043	0.4900	0.0022	0.4934	0.0014	0.4955	0.0010
Bayes - ϕ_1 with gamma prior for τ	0.4782	0.0040	0.4883	0.0021	0.4922	0.0014	0.4945	0.0010
Bayes - ϕ_1 with improper prior for τ	0.4782	0.0040	0.4883	0.0021	0.4922	0.0014	0.4945	0.0010
MLEs of τ	1.7800	0.0617	1.7247	0.0223	1.7072	0.0136	1.6988	0.0098
Bayes - τ with gamma prior	1.6371	0.0237	1.6521	0.0138	1.6583	0.0099	1.6620	0.0078
Bayes - τ with improper prior	1.6914	0.0428	1.6828	0.0188	1.6797	0.0122	1.6784	0.0090
MLEs of Y_{n+1}	-0.0004	0.2092	0.0062	0.2162	-0.0024	0.2069	0.0060	0.2064
Bayes - Y_{n+1} with gamma prior for τ	-0.2481	0.1862	-0.2417	0.1933	-0.2506	0.1870	-0.2423	0.1849
Bayes - Y_{n+1} with improper prior for τ	-0.2556	0.1860	-0.2459	0.1933	-0.2535	0.1870	-0.2445	0.1850

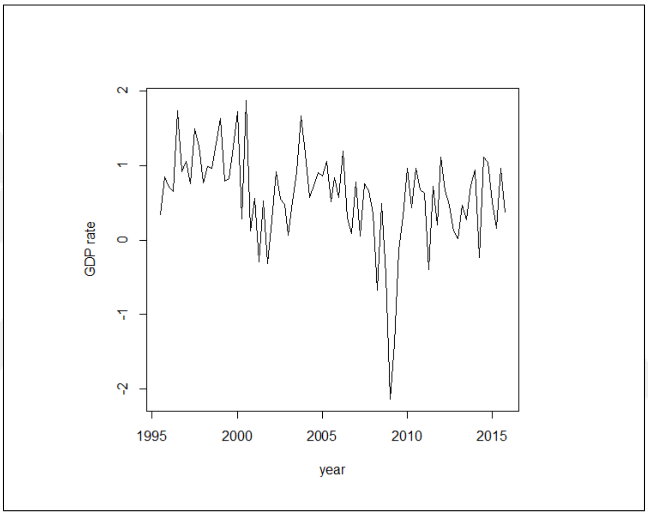
Table 6.15. Average AR(1) model estimates and estimation errors under independent uniform prior for ϕ_1 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.5006	-	0.5006	-	0.5006	-	0.5006	-
τ	1.6751	-	1.6751	-	1.6751	-	1.6751	-
Y_{n+1}	-0.0090	-	0.0021	-	0.0012	-	-0.0019	-
ESTIMATES								
MLEs of ϕ_1	0.4807	0.0118	0.4900	0.0059	0.4934	0.0039	0.4955	0.0029
Bayes - ϕ_1 with gamma prior for τ	0.4743	0.0111	0.4865	0.0058	0.4909	0.0038	0.4936	0.0028
Bayes - ϕ_1 with improper prior for τ	0.4744	0.0111	0.4865	0.0058	0.4909	0.0038	0.4936	0.0028
MLEs of τ	1.7800	0.2314	1.7247	0.0680	1.7072	0.0400	1.6988	0.0284
Bayes - τ with gamma prior	1.6167	0.0646	1.6398	0.0380	1.6495	0.0273	1.6552	0.0214
Bayes - τ with improper prior	1.6591	0.1176	1.6669	0.0515	1.6692	0.0335	1.6706	0.0250
MLEs of Y_{n+1}	-0.0004	0.7176	0.0062	0.7478	-0.0024	0.6936	0.0060	0.7005
Bayes - Y_{n+1} with gamma prior for τ	-0.4133	0.5165	-0.4069	0.5376	-0.4161	0.5162	-0.4078	0.5130
Bayes - Y_{n+1} with improper prior for τ	-0.4256	0.5166	-0.4139	0.5375	-0.4209	0.5168	-0.4115	0.5135

6.1.3. Analysis of Real Gross Domestic Product Data

We consider the growth rates of the U.S. real gross domestic product (GDP) from 1995 first quarter to 2015 second quarter. The series consists of 82 quarterly observations. The original data (series GDPC96) were downloaded from Federal Reserve Bank of St Louis, and the GDP are in millions of 2009 chained dollars. The growth rate is the first differenced series of the log(GDP), expressed in per cents. Figure 6.1 shows the plotted growth rate series.

Figure 6.1. GDP growth rate data

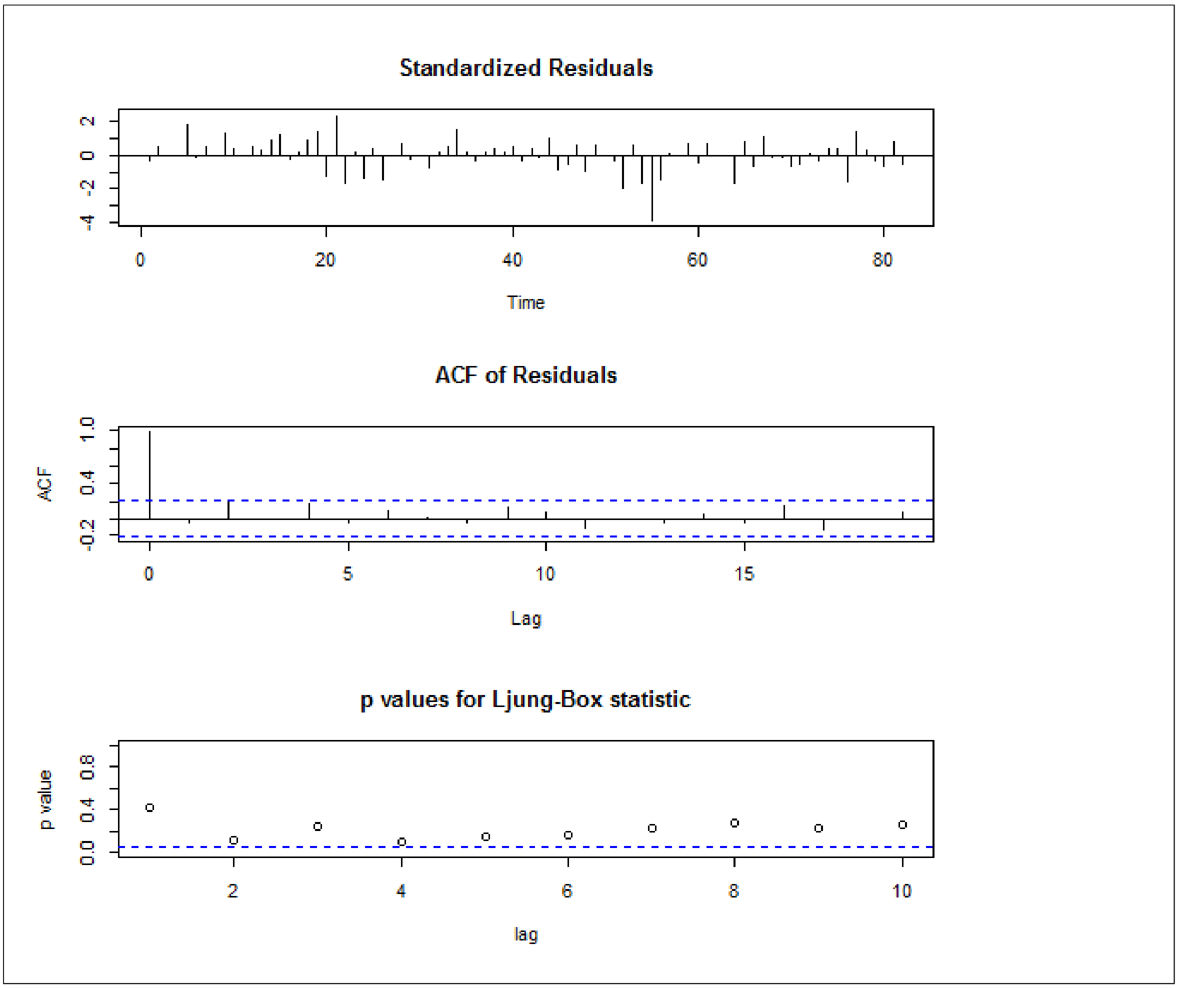


We analyze the data using the independent truncated normal prior for ϕ_1 with mean equal to the sample mean and variance equal to the sample variance and improper prior for τ. The LINEX loss function's parameter are $\gamma = 0.75$ and $\gamma = -0.75$. In order to apply the analysis using the assumed form of the AR(1) model, we need to subtract the series mean from each of the observations to obtain a zero-mean series.

Based on the augmented Dickey-Fuller test, the unit-root hypothesis is rejected. The test statistic is -3.76 with p-value less than 0.01, thus the obtained mean-adjusted GDP rate series is stationary. Model checking shows that the AR(1) model can be fitted to the zero-mean series; see Figure 6.2

We obtain estimates of ϕ_1, τ and one-step predicted value using 72, 73,…, 80, 81 observations. Table 6.16 and Table 6.17 present the estimation and one-step prediction results. It is observed that the prediction errors of the Bayes estimates are smaller than that of the ML es-

Figure 6.2. AR(1) model checking



timates, the Bayes estimates obtained using Lindley's approximation method being superior.

Table 6.16. AR(1) model estimates for empirical data

	SAMPLE SIZE										
ACTUAL	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**AVERAGE**
Y_{n+1}	0.1208	0.3188	0.4072	-0.1009	0.4848	0.4548	0.2226	0.0695	0.4174	0.1609	0.2556
ESTIMATES											
ML											
ϕ_1	0.4547	0.4557	0.4528	0.4541	0.4433	0.4203	0.4241	0.4203	0.4213	0.4140	0.4360
τ	15.3207	15.4938	15.6572	15.8225	15.4397	15.1871	15.3574	15.5127	15.6178	15.6403	15.5049
Y_{n+1}	0.2338	0.1942	0.2847	0.3260	0.0965	0.3526	0.3422	0.2436	0.1777	0.3242	0.2575
SE LOSS FUNCTION											
ϕ_1 Lindley's approx.	0.3932	0.3941	0.3917	0.3928	0.3827	0.3629	0.3669	0.3638	0.3647	0.3583	0.3771
ϕ_1 Gibbs sampling	0.4027	0.4005	0.3972	0.3995	0.3866	0.3685	0.3695	0.3661	0.3716	0.3614	0.3824
τ Lindley's approx.	15.1049	15.2786	15.4427	15.6086	15.2338	14.9873	15.1579	15.3138	15.4201	15.4448	15.2993
τ Gibbs sampling	15.0566	15.2567	15.3471	15.5471	15.2381	15.0179	15.1223	15.2715	15.5228	15.4388	15.2819
Y_{n+1} Lindley's approx.	0.2370	0.2025	0.2809	0.3169	0.1180	0.3395	0.3310	0.2456	0.1883	0.3154	0.2575
Y_{n+1} Gibbs sampling	0.2234	0.1913	0.2800	0.3279	0.1206	0.3325	0.3312	0.2608	0.1965	0.3100	0.2574
LINEX LOSS FUNCTION $\gamma = 0.75$											
ϕ_1 Lindley's approx.	0.3906	0.3916	0.3891	0.3903	0.3801	0.3602	0.3643	0.3612	0.3622	0.3557	0.3745
ϕ_1 Gibbs sampling	0.3989	0.3966	0.3933	0.3959	0.3829	0.3646	0.3660	0.3624	0.3682	0.3579	0.3787
τ Lindley's approx.	13.8464	14.0127	14.1704	14.3298	14.0008	13.7874	13.9504	14.1002	14.2051	14.2362	14.0639
τ Gibbs sampling	13.1169	13.3328	13.3896	13.3974	13.1988	13.1981	13.2946	13.4587	13.5460	13.4783	13.3411
Y_{n+1} Lindley's approx.	0.2107	0.1765	0.2553	0.2915	0.0917	0.3130	0.3049	0.2199	0.1626	0.2899	0.2316
Y_{n+1} Gibbs sampling	0.1971	0.1659	0.2534	0.3026	0.0951	0.3073	0.3049	0.2359	0.1702	0.2848	0.2317
LINEX LOSS FUNCTION $\gamma = -0.75$											
ϕ_1 Lindley's approx.	0.3961	0.3970	0.3945	0.3956	0.3856	0.3659	0.3699	0.3667	0.3676	0.3612	0.3800
ϕ_1 Gibbs sampling	0.4066	0.4043	0.4011	0.4032	0.3902	0.3725	0.3730	0.3697	0.3750	0.3649	0.3861
τ Lindley's approx.	16.6439	16.8250	16.9954	17.1677	16.7308	16.4390	16.6176	16.7797	16.8858	16.9006	16.7986
τ Gibbs sampling	17.9216	18.3392	18.4965	18.5589	18.0012	17.7813	17.7641	17.8860	18.3565	18.1612	18.1266
Y_{n+1} Lindley's approx.	0.2632	0.2284	0.3065	0.3423	0.1442	0.3660	0.3572	0.2714	0.2139	0.3410	0.2834
Y_{n+1} Gibbs sampling	0.2496	0.2166	0.3066	0.3534	0.1463	0.3578	0.3576	0.2858	0.2229	0.3354	0.2832

Table 6.17. AR(1) model errors for empirical data

	SAMPLE SIZE										
ERRORS	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**AVERAGE**
SE LOSS FUNCTION											
Y_{n+1} ML	0.0128	0.0155	0.0150	0.1822	0.1507	0.0105	0.0143	0.0303	0.0575	0.0267	0.0515
Y_{n+1} Lindley's approx.	0.0135	0.0135	0.0159	0.1745	0.1345	0.0133	0.0117	0.0310	0.0525	0.0239	0.0484
Y_{n+1} Gibbs sampling	0.0105	0.0163	0.0162	0.1839	0.1326	0.0149	0.0118	0.0366	0.0488	0.0222	0.0494
LINEX LOSS FUNCTION $\gamma = 0.75$											
Y_{n+1} ML	0.0037	0.0042	0.0041	0.0572	0.0386	0.0029	0.0041	0.0089	0.0152	0.0078	0.0147
Y_{n+1} Lindley's approx.	0.0023	0.0055	0.0062	0.0479	0.0395	0.0055	0.0019	0.0066	0.0172	0.0048	0.0137
Y_{n+1} Gibbs sampling	0.0017	0.0063	0.0064	0.0508	0.0388	0.0059	0.0019	0.0081	0.0162	0.0044	0.0141
LINEX LOSS FUNCTION $\gamma = -0.75$											
Y_{n+1} ML	0.0035	0.0045	0.0043	0.0462	0.0468	0.0030	0.0039	0.0082	0.0172	0.0072	0.0145
Y_{n+1} Lindley's approx.	0.0055	0.0024	0.0029	0.0496	0.0356	0.0023	0.0049	0.0109	0.0123	0.0087	0.0135
Y_{n+1} Gibbs sampling	0.0045	0.0030	0.0029	0.0520	0.0351	0.0027	0.0050	0.0125	0.0112	0.0082	0.0137

6.2. AR(1) MODEL PARAMETER IMPACT ANALYSIS

6.2.1. Independent Truncated Normal prior for ϕ_1 and Gamma or Improper priors for τ

To estimate the impact of parameter α, we use fixed parameters $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\alpha = 10, 20, 30$. Figure 6.3 below compares the average estimation and prediction errors when α varies. We notice that as α increases, the average estimation errors of ϕ_1 slightly increase, the average estimation errors of τ increase more than parameter α and the average prediction errors decrease proportionally to α increase. It suggests that the average estimation errors and α have inverse linear relationship.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $\mu_1 = 0.375$, $\sigma_1 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 6.4 compares the average estimation and prediction errors when β varies. We notice that β and the average estimation errors of ϕ_1 have a weak nonlinear relationship, the average estimation errors of τ and β have nonlinear inverse relationship. As parameter β increases, the average prediction errors increase.

To estimate the impact of parameter μ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\sigma_1 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\mu_1 = 0.125, 0.375, 0.625, 0.875$. Figure 6.5 compares the average estimation and prediction errors when μ_1 varies. We notice that as μ_1 increases, the average estimation errors of ϕ_1 decrease, the changes of average estimation errors of τ and the average prediction errors remain almost unchanged.

To estimate the impact of parameter σ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\sigma_1 = 0.1, 0.2, 0.3$. Figure 6.6 compares the average estimation and prediction errors when σ_1 varies. We notice that as σ_1 increases, the average Bayes estimation errors of ϕ_1 slightly increase, the changes of average ML estimation errors of ϕ_1 remain almost unchanged, the average estimation errors of τ remain unchanged. The average Bayes prediction errors slightly increase when σ_1 increase, whereas the average ML prediction errors do not seem to depend on σ_1.

To estimate the impact of parameter γ, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.3$, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 6.7 compares the average estimation and prediction errors when γ varies. We notice that as γ increases, the average estimation errors of ϕ_1, τ and prediction increase more than the increase in γ.

Figure 6.3. Impact of parameter α on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

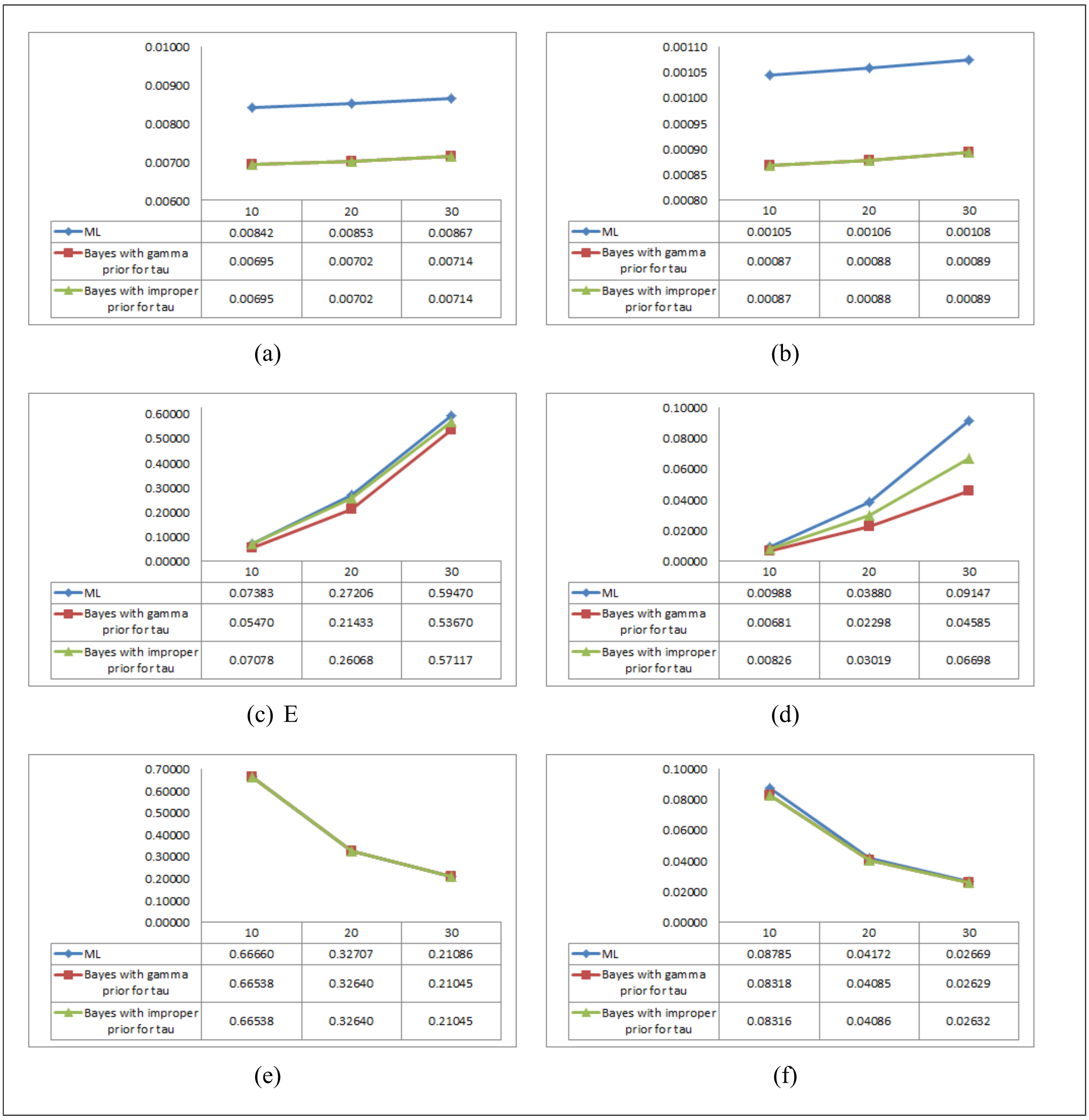


Figure 6.4. Impact of parameter β on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

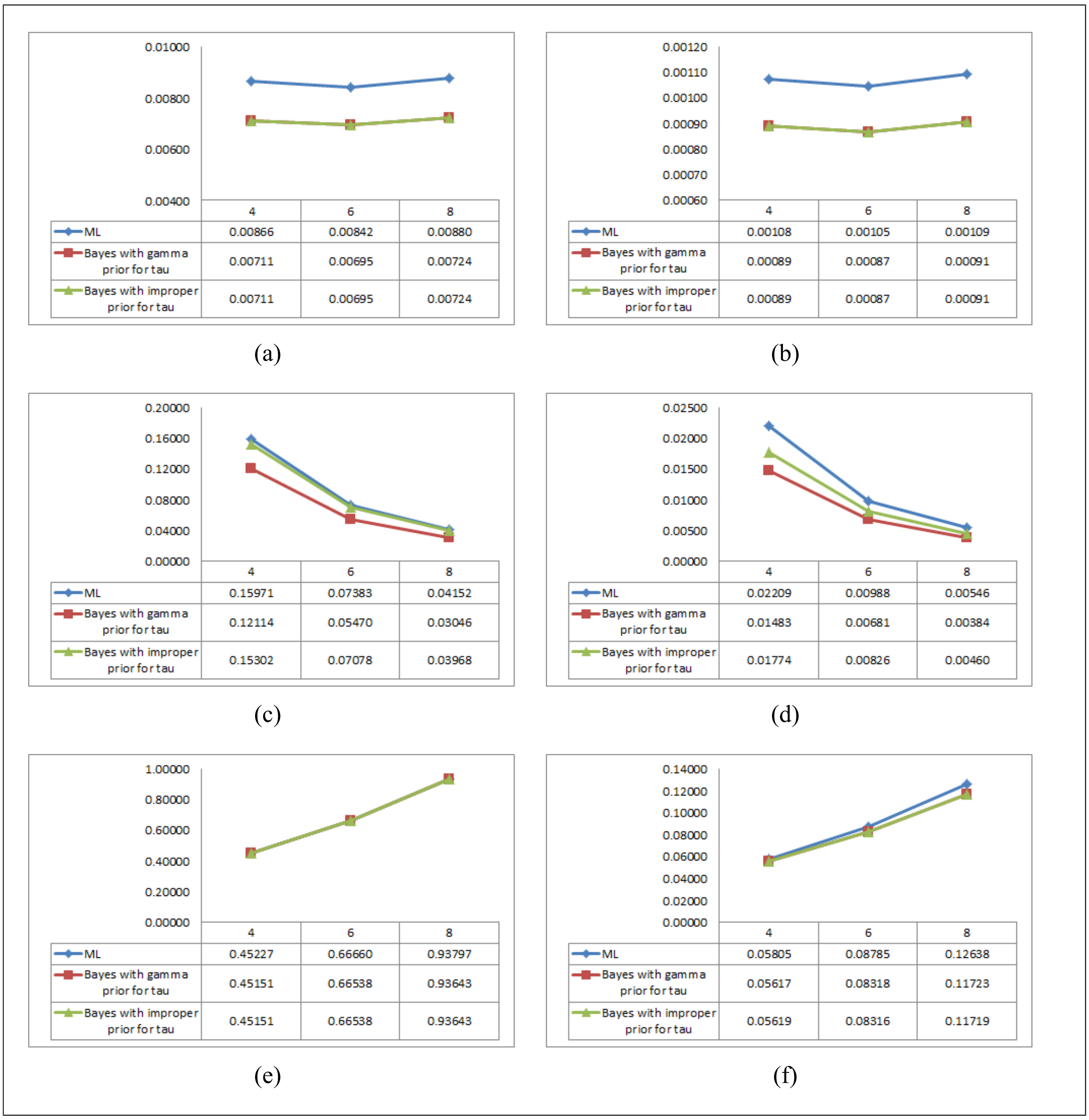


Figure 6.5. Impact of parameter μ_1 on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

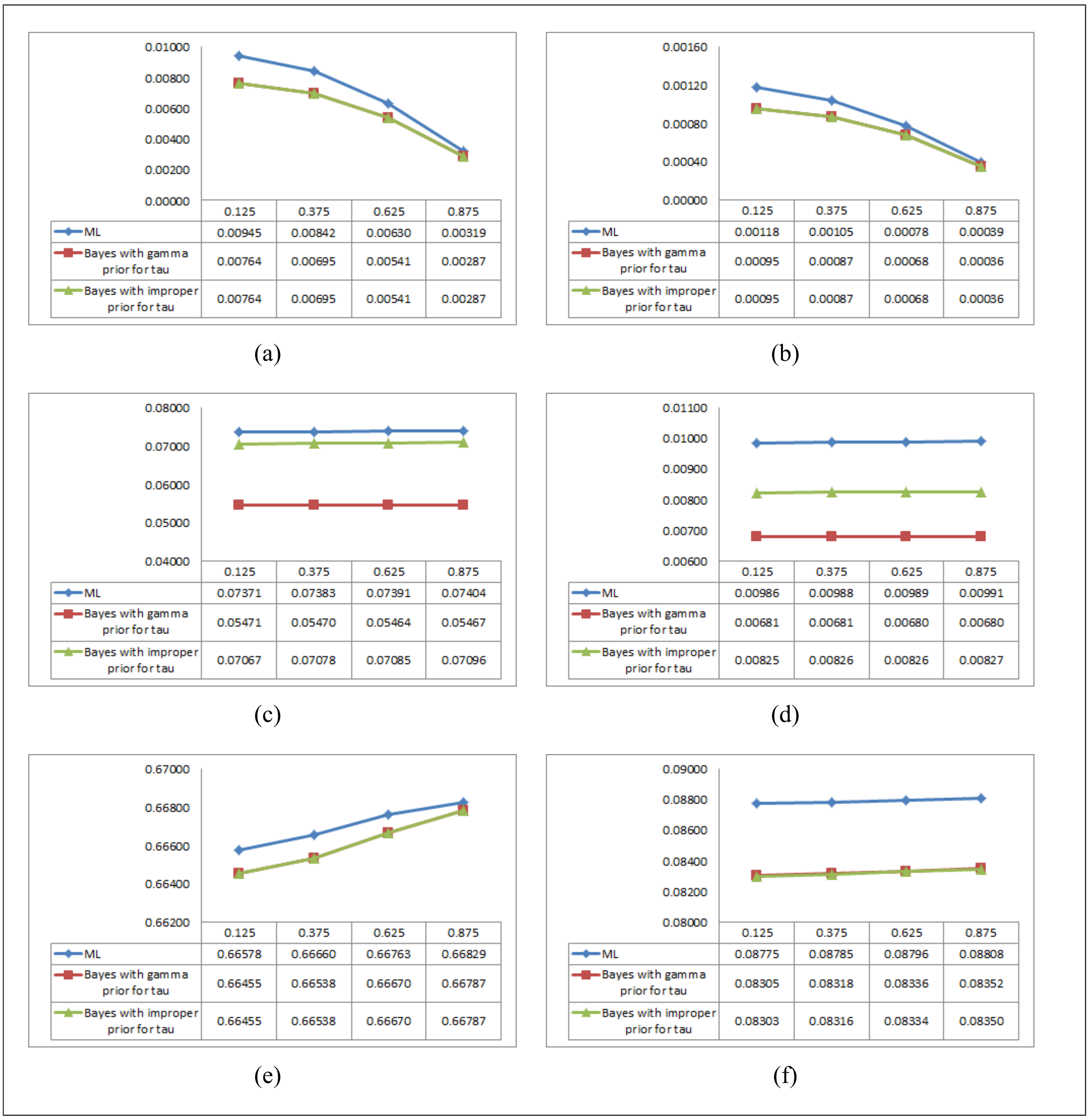
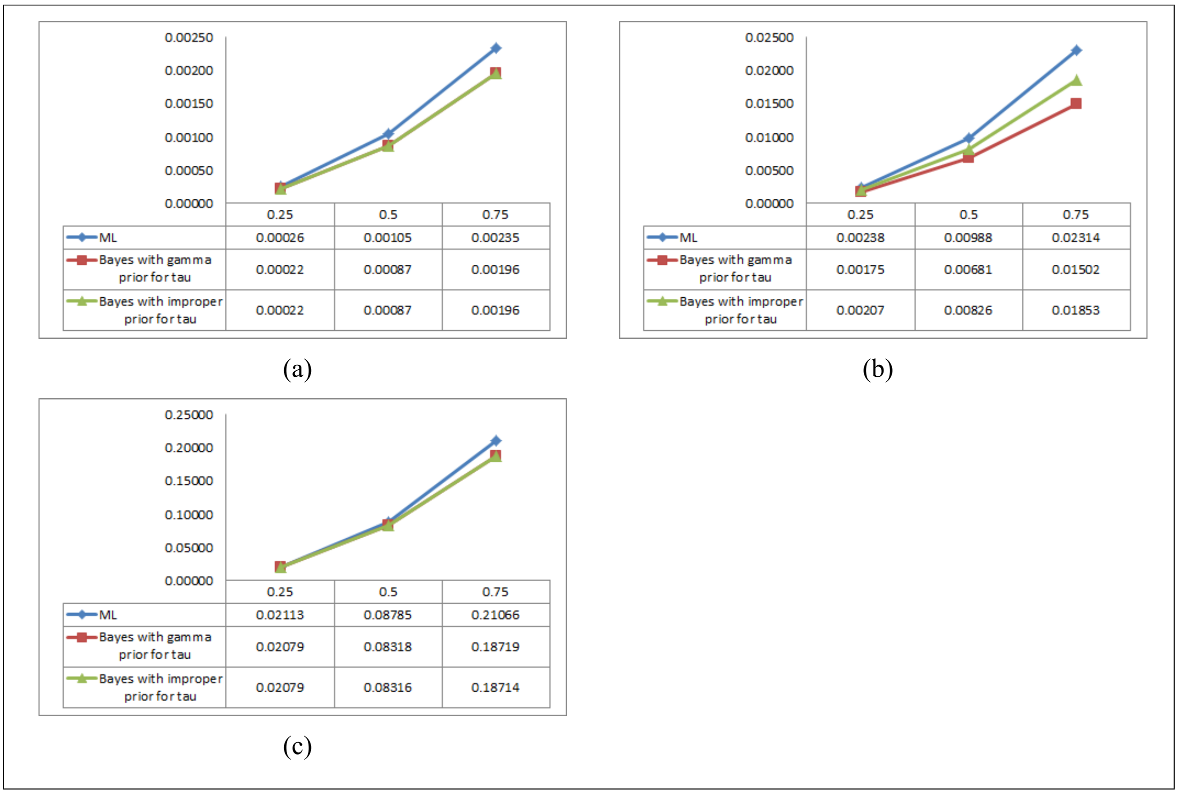


Figure 6.6. Impact of parameter σ_1 on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

Figure 6.7. Impact of parameter γ on average estimation and prediction errors. (a) Estimation of ϕ_1 under LINEX, (b) Estimation of τ under LINEX loss, (c) Estimation of Y_{n+1} under LINEX loss.



6.2.2. Independent Uniform prior for ϕ_1 and Gamma or Improper priors for τ

To estimate the impact of parameter α, we use fixed parameters $\beta = 6$, $(c_1, d_1) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\alpha = 10, 20, 30$. Figure 6.8 below compares the average estimation and prediction errors when α varies. We notice that α has a negligent impact on the average estimation errors of ϕ_1, the average estimation errors of τ increase more than parameter α . The average estimation errors and α have inverse almost linear relationship.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $(c_1, d_1) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 6.9 compares the average estimation and prediction errors when β varies. We notice that as β increases, the average estimation errors of ϕ_1 remain unchanged, the average estimation errors of τ and β have nonlinear inverse relationship. As parameter β increases, the average prediction errors increase.

To estimate the impact of the interval of ϕ_1, we use sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $(c_1, d_1) = (0, 0.25), (0.25, 0.5), (0.5, 0.75), (0.75, 1)$. Figure 6.10 compares the average estimation and prediction errors when the interval of ϕ_1 varies, the interval is represented by its middle point. We notice that as the mean of ϕ_1 increases, the average estimation errors of ϕ_1 decrease, the changes of average estimation errors of τ and the average prediction errors remain almost unchanged.

To estimate the impact of parameter γ, we fix ϕ_1 interval, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 6.11 compares the average estimation and prediction error when γ varies. We notice that as γ increases, the average estimation errors of ϕ_1, τ and prediction increase more than the increase in γ.

Figure 6.12 presents the average estimation errors of ϕ_1 under the LINEX loss function, estimated using the numerical method and the truncated normal approximation. We notice that the Bayes estimation errors for ϕ_1 are significantly smaller when the numerical approach is used (left in the figure); and the difference between the ML and the Bayes estimates becomes more noticeable, the Bayes estimates have the smallest average estimation errors.

Figure 6.8. Impact of parameter α on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

Figure 6.9. Impact of parameter β on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

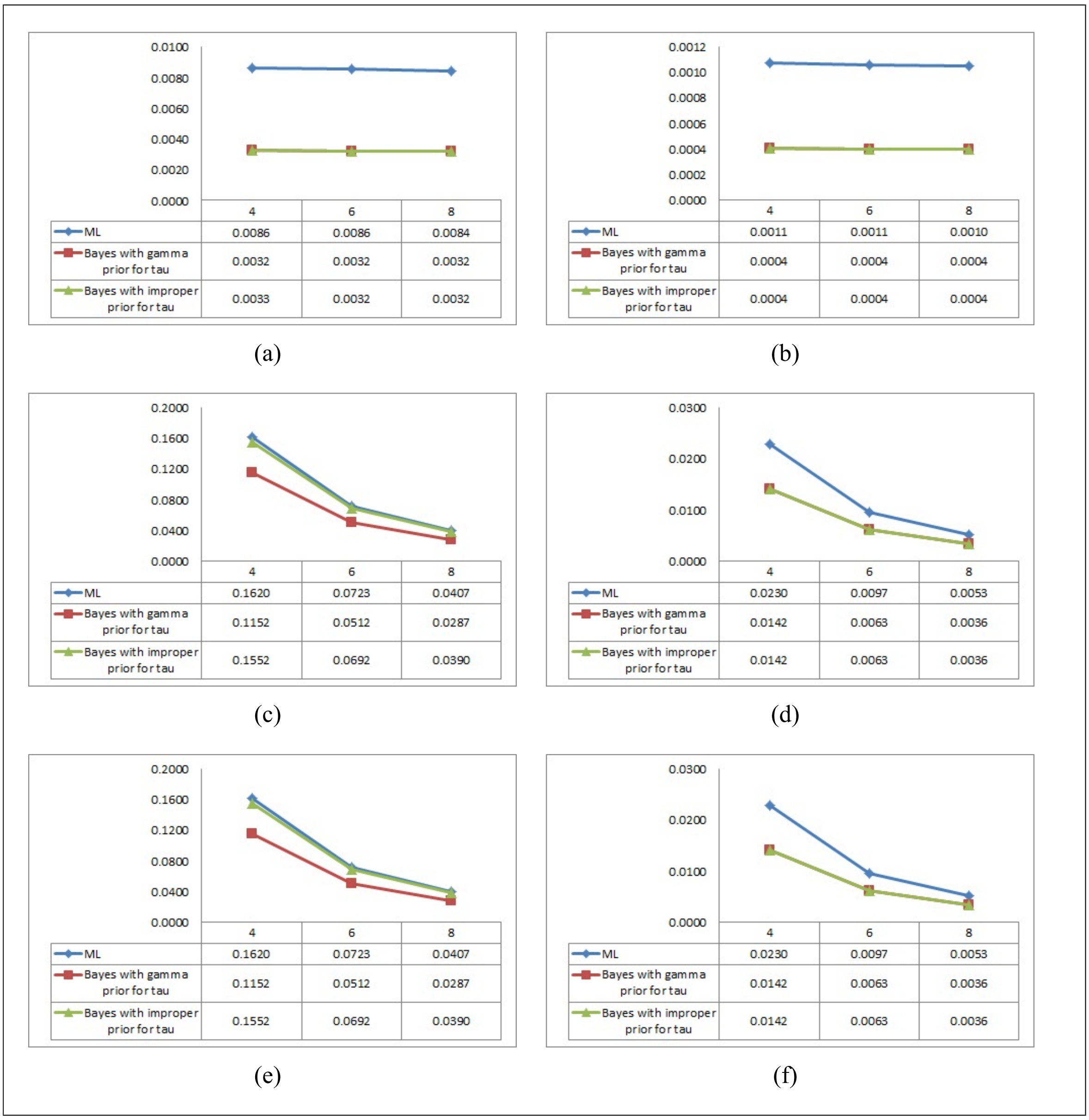


Figure 6.10. Impact of ϕ_1 interval on average estimation and prediction errors. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

Figure 6.11. Impact of parameter γ on average estimation and prediction errors. (a) Estimation of ϕ_1 under LINEX, (b) Estimation of τ under LINEX loss, (c) Estimation of Y_{n+1} under LINEX loss.

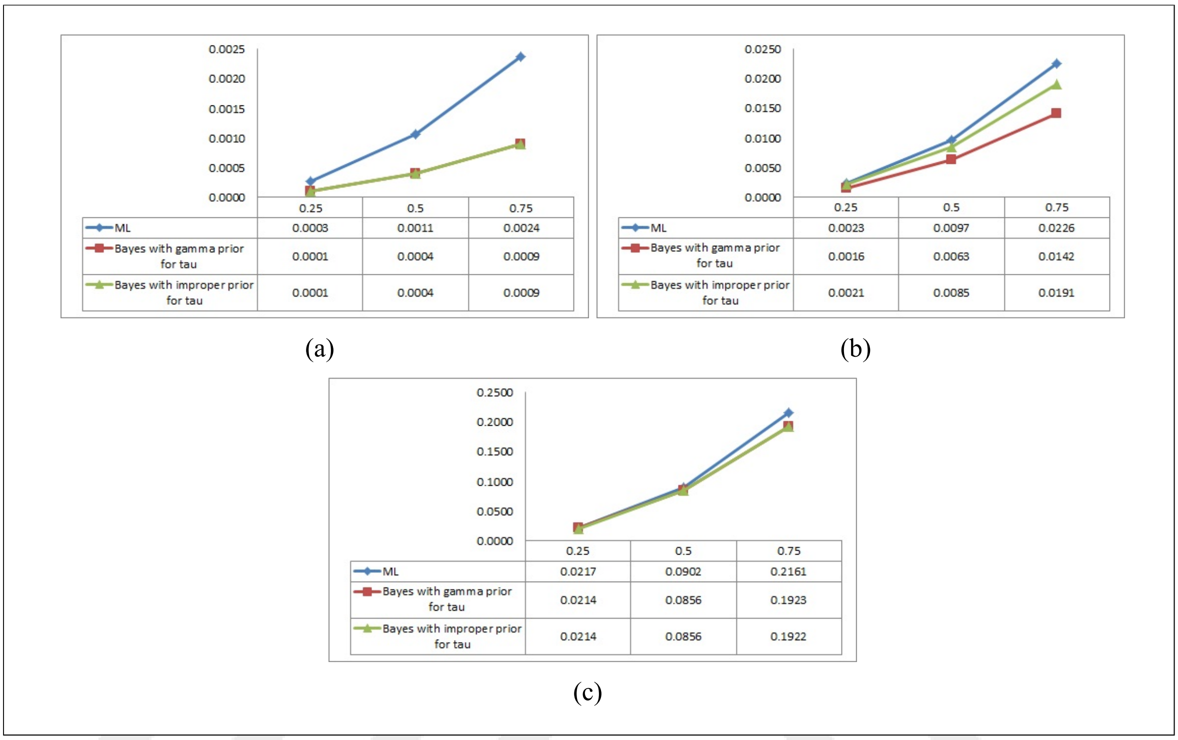


Figure 6.12. Impact of estimation method of ϕ_1 under LINEX loss function on average estimation and prediction errors, (a) Numerical approach, α varies, (b) Approximation, α varies, (c) Numerical approach, β varies, (d) Approximation, β varies, (e) Numerical approach, interval of ϕ_1 varies, (f) Approximation, interval of ϕ_1 varies, (e) Numerical approach, γ varies, (f) Approximation, γ varies.

6.3. AR(2) MODEL

In this section we study the AR(2) model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t. \tag{6.2}$$

The AR(2) process is stationary if $\phi_2 - \phi_1 < 1$, $\phi_2 + \phi_1 < 1$ and $-1 < \phi_2 < 1$.

6.3.1. Independent Truncated Normal prior for ϕ_1, ϕ_2 and Gamma or Improper priors for τ

We consider the independent truncated normal priors for ϕ_1 and ϕ_2 with parameters $\mu_1 = 0.25$, $\sigma_1 = 0.2$, $a_1 = 0$, $b_1 = 0.5$, and $\mu_2 = 0.2$, $\sigma_2 = 0.3$, $a_2 = 0$, $b_2 = 0.4$. The intervals forϕ_1 and ϕ_2 are chosen such that the AR(2) model would be stationary. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$.

Table 6.18 presents the average values of AR(2) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used. Under the SE loss function, the average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overallthe Bayes estimates are found to have smaller average estimation errors than the ML estimates, for ϕ_1 and ϕ_2 the smallest estimation errors are obtained for the Bayes estimates obtained us-ing Lindley's approximation; for τ the Bayes estimation using the Gibbs sampler is found to result in the smallest estimation errors when τ has gamma prior and the Bayes estimation using Lindley's approximation when τ has improper prior; for the one-step prediction all estimates have similar errors, the ones of Bayes estimates being slightly smaller. All estimator performances are reasonably close to each other as the sample size increases.

Under the LINEX loss function, there is a non-zero probability that the Bayes estimates using Lindley's approximation may be undefined (see Section 5.1.2), Table 6.19 shows proportion of undefined Bayes estimates using this approximation. Under our choice of parameters, undefined estimates are obtained only for τ when it has gamma prior and LINEX loss function parameters are $\gamma = -1.25, -0.75, -0.25$. The proportion of undefined τ estimates decreases significantly as sample size increases. We exclude the simulation where we obtain undefined estimates and calculate average errors where all estimates are defined.

Table 6.20, Table 6.21, Table 6.22, Table 6.23, Table 6.24 and Table 6.25 present the average values of AR(2) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25, -0.75, -0.25, 0.25, 0.75, 1.25$, respectively. Under the LINEX loss function, the average estimation errors are also found to decrease with increasing sample size. Generally, the Bayes estimation has the smallest average errors.

For ϕ_1 and ϕ_2, the smallest average estimation errors are obtained for the Bayes estimates using Lindley's approximation, the difference between the ML and the Bayes estimates are more noticeable when the LINEX parameter has higher absolute value. For the one-step prediction, generally, the Bayes estimates have smaller average errors than the ML estimates, the Gibbs sampler methods have slightly better performance.

For τ, when the LINEX loss function parameters are positive, the smallest average estimation errors are obtained using the Bayes estimates; where gamma prior is used, the Gibbs sampling method is superior, whereas when improper prior is used, Lindley's approximation performance is better. When $\gamma = -1.25, -0.75$, if improper prior is used for τ, the ML estimates of τ have smaller average error than the Bayes estimates; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, when $\gamma = -1.25$ for sample sizes $n = 50, 100, 150$, when $\gamma = -0.75$ for sample sizes $n = 50, 100$ the Bayes estimates obtained using Lindley's approximation have the highest average errors, for other sample size both Bayes estimates perform better than the ML estimates.

When $\gamma = -0.25$, the Bayes estimates of τ have smaller average errors than the ML estimates. When improper prior is used for τ, the Bayes estimates performances are indistinguishable; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, for sample size $n = 50$ the Bayes estimates obtained using Lindley's approximation has the highest average error, for sample sizes $n = 100, 150, 200$ both Bayes estimates perform better than the ML estimates.

Table 6.18. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using SE loss function

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2492	-	0.2492	-	0.2492	-	0.2492	-
ϕ_2	0.2022	-	0.2022	-	0.2022	-	0.2022	-
τ	1.6641	-	1.6641	-	1.6641	-	1.6641	-
Y_{n+1}	-0.0283	-	0.0157	-	0.0380	-	-0.0024	-
SE LOSS FUNCTION								
MLEs of ϕ_1	0.2424	0.0211	0.2486	0.0098	0.2481	0.0060	0.2472	0.0046
Lindley's - ϕ_1 with gamma prior for τ	0.2436	0.0097	0.2483	0.0066	0.2483	0.0046	0.2475	0.0037
Lindley's - ϕ_1 with improper prior for τ	0.2436	0.0097	0.2483	0.0066	0.2483	0.0046	0.2475	0.0037
Gibbs - ϕ_1 with gamma prior for τ	0.2448	0.0111	0.2488	0.0069	0.2484	0.0047	0.2476	0.0038
Gibbs - ϕ_1 with improper prior for τ	0.2447	0.0112	0.2489	0.0069	0.2485	0.0047	0.2477	0.0038
MLEs of ϕ_2	0.1734	0.0196	0.1891	0.0102	0.1964	0.0067	0.1987	0.0049
Lindley's - ϕ_2 with gamma prior for τ	0.2011	0.0119	0.1986	0.0079	0.2016	0.0056	0.2022	0.0043
Lindley's - ϕ_2 with improper prior for τ	0.2011	0.0119	0.1986	0.0079	0.2016	0.0056	0.2022	0.0043
Gibbs - ϕ_2 with gamma prior for τ	0.1913	0.0127	0.1966	0.0081	0.2007	0.0057	0.2017	0.0043
Gibbs - ϕ_2 with improper prior for τ	0.1913	0.0127	0.1964	0.0081	0.2007	0.0057	0.2017	0.0044
MLEs of τ	1.8032	0.1847	1.7388	0.0781	1.7049	0.0450	1.6943	0.0302
Lindley's - τ with gamma prior	1.5523	0.1488	1.6455	0.0577	1.6512	0.0376	1.6563	0.0269
Lindley's - τ with improper prior	1.7281	0.1556	1.7033	0.0712	1.6818	0.0426	1.6772	0.0289
Gibbs - τ with gamma prior	1.6669	0.0910	1.6716	0.0550	1.6623	0.0367	1.6626	0.0265
Gibbs - τ with improper prior	1.7375	0.1575	1.7061	0.0718	1.6832	0.0427	1.6782	0.0290
MLEs of Y_{n+1}	-0.0055	0.6768	0.0091	0.6344	-0.0094	0.7164	-0.0100	0.6850
Lindley's - Y_{n+1} with gamma prior for τ	-0.0082	0.6610	0.0078	0.6349	-0.0097	0.7115	-0.0088	0.6835
Lindley's - Y_{n+1} with improper prior for τ	-0.0082	0.6610	0.0078	0.6349	-0.0097	0.7115	-0.0088	0.6835
Gibbs - Y_{n+1} with gamma prior for τ	-0.0066	0.6611	0.0091	0.6355	-0.0095	0.7133	-0.0092	0.6826
Gibbs - Y_{n+1} with improper prior for τ	-0.0064	0.6647	0.0077	0.6342	-0.0097	0.7121	-0.0083	0.6831

Table 6.19. Proportion of undefined AR(2) model estimates under independent truncated normal prior for ϕ_1 and ϕ_2

		Gamma prior for τ				**Improper prior for τ**			
LINEX parameter	**Sample size**	ϕ_1	ϕ_2	τ	Y_{n+1}	ϕ_1	ϕ_2	τ	Y_{n+1}
$\gamma = -1.25$	50	0.0000	0.0000	0.0620	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0050	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.75$	50	0.0000	0.0000	0.0310	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.25$	50	0.0000	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.75$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 1.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 6.20. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2494	-	0.2491	-	0.2493	-	0.2492	-
ϕ_2	0.2016	-	0.2024	-	0.2023	-	0.2022	-
τ	1.6056	-	1.6562	-	1.6626	-	1.6641	-
Y_{n+1}	-0.0330	-	0.0169	-	0.0385	-	-0.0024	-
LINEX LOSS FUNCTION								
MLEs of ϕ_1	0.2441	0.0167	0.2482	0.0077	0.2481	0.0048	0.2472	0.0036
Lindley's - ϕ_1 with gamma prior for τ	0.2501	0.0078	0.2530	0.0051	0.2519	0.0036	0.2503	0.0029
Lindley's - ϕ_1 with improper prior for τ	0.2501	0.0078	0.2530	0.0051	0.2519	0.0036	0.2503	0.0029
Gibbs - ϕ_1 with gamma prior for τ	0.2541	0.0088	0.2534	0.0054	0.2519	0.0037	0.2503	0.0030
Gibbs - ϕ_1 with improper prior for τ	0.2543	0.0088	0.2536	0.0054	0.2520	0.0037	0.2504	0.0030
MLEs of ϕ_2	0.1716	0.0159	0.1895	0.0083	0.1967	0.0053	0.1987	0.0038
Lindley's - ϕ_2 with gamma prior for τ	0.2091	0.0093	0.2045	0.0062	0.2057	0.0044	0.2051	0.0034
Lindley's - ϕ_2 with improper prior for τ	0.2091	0.0093	0.2045	0.0062	0.2057	0.0044	0.2051	0.0034
Gibbs - ϕ_2 with gamma prior for τ	0.1996	0.0098	0.2024	0.0064	0.2047	0.0045	0.2046	0.0034
Gibbs - ϕ_2 with improper prior for τ	0.2000	0.0098	0.2023	0.0064	0.2047	0.0045	0.2046	0.0034
MLEs of τ	1.6964	0.0864	1.7272	0.0508	1.7021	0.0313	1.6943	0.0222
Lindley's - τ with gamma prior	1.5485	0.2001	1.6595	0.0679	1.6694	0.0317	1.6728	0.0216
Lindley's - τ with improper prior	1.7077	0.0931	1.7340	0.0534	1.7060	0.0324	1.6971	0.0228
Gibbs - τ with gamma prior	1.6630	0.0668	1.6968	0.0427	1.6830	0.0286	1.6802	0.0206
Gibbs - τ with improper prior	1.7197	0.0951	1.7376	0.0540	1.7077	0.0325	1.6984	0.0230
MLEs of Y_{n+1}	-0.0035	0.8519	0.0093	0.6338	-0.0095	0.8480	-0.0100	0.7032
Lindley's - Y_{n+1} with gamma prior for τ	0.6395	0.6027	0.6253	0.5471	0.6108	0.5984	0.6093	0.5897
Lindley's - Y_{n+1} with improper prior for τ	0.6423	0.6105	0.6257	0.5508	0.6116	0.6008	0.6100	0.5913
Gibbs - Y_{n+1} with gamma prior for τ	0.4414	0.5719	0.4341	0.4812	0.4152	0.5681	0.4119	0.5275
Gibbs - Y_{n+1} with improper prior for τ	0.4666	0.5727	0.4432	0.4846	0.4214	0.5662	0.4186	0.5308

Table 6.21. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2486	-	0.2492	-	0.2492	-	0.2492	-
ϕ_2	0.2022	-	0.2022	-	0.2022	-	0.2022	-
τ	1.6348	-	1.6625	-	1.6641	-	1.6641	-
Y_{n+1}	-0.0324	-	0.0162	-	0.0380	-	-0.0024	-
LINEX LOSS FUNCTION								
MLEs of ϕ_1	0.2427	0.0059	0.2484	0.0027	0.2481	0.0017	0.2472	0.0013
Lindley's - ϕ_1 with gamma prior for τ	0.2471	0.0028	0.2511	0.0018	0.2504	0.0013	0.2492	0.0010
Lindley's - ϕ_1 with improper prior for τ	0.2471	0.0028	0.2511	0.0018	0.2504	0.0013	0.2492	0.0010
Gibbs - ϕ_1 with gamma prior for τ	0.2499	0.0031	0.2516	0.0019	0.2505	0.0013	0.2492	0.0011
Gibbs - ϕ_1 with improper prior for τ	0.2499	0.0031	0.2517	0.0019	0.2506	0.0013	0.2493	0.0011
MLEs of ϕ_2	0.1726	0.0057	0.1891	0.0029	0.1964	0.0019	0.1987	0.0014
Lindley's - ϕ_2 with gamma prior for τ	0.2061	0.0034	0.2019	0.0022	0.2039	0.0016	0.2039	0.0012
Lindley's - ϕ_2 with improper prior for τ	0.2061	0.0034	0.2019	0.0022	0.2039	0.0016	0.2039	0.0012
Gibbs - ϕ_2 with gamma prior for τ	0.1965	0.0036	0.1999	0.0023	0.2029	0.0016	0.2034	0.0012
Gibbs - ϕ_2 with improper prior for τ	0.1967	0.0036	0.1997	0.0023	0.2029	0.0016	0.2034	0.0012
MLEs of τ	1.7405	0.0347	1.7362	0.0198	1.7049	0.0119	1.6943	0.0082
Lindley's - τ with gamma prior	1.5245	0.1433	1.6507	0.0427	1.6621	0.0110	1.6659	0.0077
Lindley's - τ with improper prior	1.7205	0.0351	1.7264	0.0199	1.6981	0.0120	1.6891	0.0082
Gibbs - τ with gamma prior	1.6675	0.0247	1.6896	0.0156	1.6759	0.0104	1.6731	0.0074
Gibbs - τ with improper prior	1.7300	0.0356	1.7293	0.0200	1.6995	0.0120	1.6902	0.0082
MLEs of Y_{n+1}	-0.0073	0.2229	0.0091	0.1944	-0.0094	0.2364	-0.0100	0.2123
Lindley's - Y_{n+1} with gamma prior for τ	0.3913	0.2067	0.3974	0.1897	0.3835	0.2103	0.3827	0.2092
Lindley's - Y_{n+1} with improper prior for τ	0.3950	0.2092	0.3991	0.1908	0.3850	0.2110	0.3839	0.2098
Gibbs - Y_{n+1} with gamma prior for τ	0.2546	0.1964	0.2628	0.1753	0.2449	0.2027	0.2432	0.1917
Gibbs - Y_{n+1} with improper prior for τ	0.2674	0.1976	0.2671	0.1755	0.2480	0.2021	0.2473	0.1925

Table 6.22. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2491	-	0.2492	-	0.2492	-	0.2492	-
ϕ_2	0.2021	-	0.2022	-	0.2022	-	0.2022	-
τ	1.6626	-	1.6641	-	1.6641	-	1.6641	-
Y_{n+1}	-0.0274	-	0.0157	-	0.0380	-	-0.0024	-
LINEX LOSS FUNCTION								
MLEs of ϕ_1	0.2420	0.0007	0.2486	0.0003	0.2481	0.0002	0.2472	0.0001
Lindley's - ϕ_1 with gamma prior for τ	0.2446	0.0003	0.2493	0.0002	0.2490	0.0001	0.2481	0.0001
Lindley's - ϕ_1 with improper prior for τ	0.2446	0.0003	0.2493	0.0002	0.2490	0.0001	0.2481	0.0001
Gibbs - ϕ_1 with gamma prior for τ	0.2463	0.0003	0.2498	0.0002	0.2491	0.0001	0.2481	0.0001
Gibbs - ϕ_1 with improper prior for τ	0.2462	0.0003	0.2499	0.0002	0.2492	0.0001	0.2482	0.0001
MLEs of ϕ_2	0.1734	0.0006	0.1891	0.0003	0.1964	0.0002	0.1987	0.0002
Lindley's - ϕ_2 with gamma prior for τ	0.2030	0.0004	0.1997	0.0002	0.2024	0.0002	0.2028	0.0001
Lindley's - ϕ_2 with improper prior for τ	0.2030	0.0004	0.1997	0.0002	0.2024	0.0002	0.2028	0.0001
Gibbs - ϕ_2 with gamma prior for τ	0.1933	0.0004	0.1977	0.0003	0.2014	0.0002	0.2023	0.0001
Gibbs - ϕ_2 with improper prior for τ	0.1934	0.0004	0.1975	0.0003	0.2014	0.0002	0.2023	0.0001
MLEs of τ	1.8000	0.0053	1.7388	0.0024	1.7049	0.0014	1.6943	0.0009
Lindley's - τ with gamma prior	1.5267	0.0092	1.6488	0.0019	1.6548	0.0012	1.6594	0.0008
Lindley's - τ with improper prior	1.7438	0.0049	1.7118	0.0022	1.6872	0.0013	1.6811	0.0009
Gibbs - τ with gamma prior	1.6767	0.0029	1.6781	0.0017	1.6668	0.0011	1.6661	0.0008
Gibbs - τ with improper prior	1.7531	0.0049	1.7146	0.0022	1.6886	0.0013	1.6822	0.0009
MLEs of Y_{n+1}	-0.0052	0.0215	0.0091	0.0200	-0.0094	0.0230	-0.0100	0.0216
Lindley's - Y_{n+1} with gamma prior for τ	0.1300	0.0212	0.1441	0.0200	0.1283	0.0224	0.1284	0.0220
Lindley's - Y_{n+1} with improper prior for τ	0.1320	0.0212	0.1453	0.0201	0.1292	0.0225	0.1291	0.0220
Gibbs - Y_{n+1} with gamma prior for τ	0.0801	0.0209	0.0935	0.0198	0.0752	0.0224	0.0749	0.0214
Gibbs - Y_{n+1} with improper prior for τ	0.0839	0.0210	0.0939	0.0197	0.0761	0.0223	0.0768	0.0214

Table 6.23. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2492	-	0.2492	-	0.2492	-	0.2492	-
ϕ_2	0.2022	-	0.2022	-	0.2022	-	0.2022	-
τ	1.6641	-	1.6641	-	1.6641	-	1.6641	-
Y_{n+1}	-0.0283	-	0.0157	-	0.0380	-	-0.0024	-
LINEX LOSS FUNCTION								
MLEs of ϕ_1	0.2424	0.0007	0.2486	0.0003	0.2481	0.0002	0.2472	0.0001
Lindley's - ϕ_1 with gamma prior for τ	0.2424	0.0003	0.2473	0.0002	0.2476	0.0001	0.2469	0.0001
Lindley's - ϕ_1 with improper prior for τ	0.2424	0.0003	0.2473	0.0002	0.2476	0.0001	0.2469	0.0001
Gibbs - ϕ_1 with gamma prior for τ	0.2431	0.0003	0.2478	0.0002	0.2477	0.0001	0.2470	0.0001
Gibbs - ϕ_1 with improper prior for τ	0.2430	0.0003	0.2479	0.0002	0.2478	0.0001	0.2472	0.0001
MLEs of ϕ_2	0.1734	0.0006	0.1891	0.0003	0.1964	0.0002	0.1987	0.0002
Lindley's - ϕ_2 with gamma prior for τ	0.1991	0.0004	0.1975	0.0002	0.2008	0.0002	0.2016	0.0001
Lindley's - ϕ_2 with improper prior for τ	0.1991	0.0004	0.1975	0.0002	0.2008	0.0002	0.2016	0.0001
Gibbs - ϕ_2 with gamma prior for τ	0.1892	0.0004	0.1955	0.0003	0.1999	0.0002	0.2011	0.0001
Gibbs - ϕ_2 with improper prior for τ	0.1893	0.0004	0.1953	0.0003	0.1999	0.0002	0.2012	0.0001
MLEs of τ	1.8032	0.0062	1.7388	0.0025	1.7049	0.0014	1.6943	0.0010
Lindley's - τ with gamma prior	1.5646	0.0036	1.6421	0.0018	1.6477	0.0012	1.6532	0.0008
Lindley's - τ with improper prior	1.7101	0.0048	1.6950	0.0022	1.6765	0.0013	1.6732	0.0009
Gibbs - τ with gamma prior	1.6559	0.0028	1.6652	0.0017	1.6578	0.0011	1.6591	0.0008
Gibbs - τ with improper prior	1.7193	0.0049	1.6977	0.0022	1.6779	0.0013	1.6743	0.0009
MLEs of Y_{n+1}	-0.0055	0.0214	0.0091	0.0200	-0.0094	0.0223	-0.0100	0.0216
Lindley's - Y_{n+1} with gamma prior for τ	-0.1460	0.0209	-0.1285	0.0203	-0.1476	0.0226	-0.1460	0.0214
Lindley's - Y_{n+1} with improper prior for τ	-0.1479	0.0210	-0.1297	0.0204	-0.1485	0.0227	-0.1467	0.0214
Gibbs - Y_{n+1} with gamma prior for τ	-0.0930	0.0205	-0.0753	0.0199	-0.0942	0.0222	-0.0934	0.0213
Gibbs - Y_{n+1} with improper prior for τ	-0.0965	0.0206	-0.0785	0.0199	-0.0954	0.0222	-0.0933	0.0213

Table 6.24. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2492	-	0.2492	-	0.2492	-	0.2492	-
ϕ_2	0.2022	-	0.2022	-	0.2022	-	0.2022	-
τ	1.6641	-	1.6641	-	1.6641	-	1.6641	-
Y_{n+1}	-0.0283	-	0.0157	-	0.0380	-	-0.0024	-
LINEX LOSS FUNCTION								
MLEs of ϕ_1	0.2424	0.0059	0.2486	0.0028	0.2481	0.0017	0.2472	0.0013
Lindley's - ϕ_1 with gamma prior for τ	0.2401	0.0027	0.2453	0.0018	0.2461	0.0013	0.2458	0.0010
Lindley's - ϕ_1 with improper prior for τ	0.2401	0.0027	0.2453	0.0018	0.2461	0.0013	0.2458	0.0010
Gibbs - ϕ_1 with gamma prior for τ	0.2397	0.0031	0.2458	0.0019	0.2463	0.0013	0.2460	0.0011
Gibbs - ϕ_1 with improper prior for τ	0.2396	0.0031	0.2460	0.0020	0.2465	0.0013	0.2461	0.0011
MLEs of ϕ_2	0.1734	0.0054	0.1891	0.0028	0.1964	0.0019	0.1987	0.0014
Lindley's - ϕ_2 with gamma prior for τ	0.1951	0.0033	0.1952	0.0022	0.1993	0.0016	0.2004	0.0012
Lindley's - ϕ_2 with improper prior for τ	0.1951	0.0033	0.1952	0.0022	0.1993	0.0016	0.2004	0.0012
Gibbs - ϕ_2 with gamma prior for τ	0.1851	0.0036	0.1933	0.0023	0.1984	0.0016	0.2000	0.0012
Gibbs - ϕ_2 with improper prior for τ	0.1852	0.0036	0.1931	0.0023	0.1984	0.0016	0.2000	0.0012
MLEs of τ	1.8032	0.0657	1.7388	0.0249	1.7049	0.0137	1.6943	0.0089
Lindley's - τ with gamma prior	1.5753	0.0268	1.6354	0.0154	1.6410	0.0104	1.6475	0.0075
Lindley's - τ with improper prior	1.6779	0.0428	1.6792	0.0200	1.6662	0.0120	1.6655	0.0081
Gibbs - τ with gamma prior	1.6344	0.0248	1.6526	0.0153	1.6490	0.0103	1.6523	0.0074
Gibbs - τ with improper prior	1.6845	0.0425	1.6814	0.0201	1.6673	0.0120	1.6665	0.0081
MLEs of Y_{n+1}	-0.0055	0.2122	0.0091	0.1956	-0.0094	0.2155	-0.0100	0.2118
Lindley's - Y_{n+1} with gamma prior for τ	-0.4023	0.2006	-0.3817	0.1958	-0.4029	0.2112	-0.4007	0.1993
Lindley's - Y_{n+1} with improper prior for τ	-0.4043	0.2034	-0.3835	0.1972	-0.4044	0.2119	-0.4019	0.1998
Gibbs - Y_{n+1} with gamma prior for τ	-0.2664	0.1848	-0.2446	0.1794	-0.2636	0.1975	-0.2620	0.1886
Gibbs - Y_{n+1} with improper prior for τ	-0.2783	0.1868	-0.2512	0.1794	-0.2670	0.1973	-0.2634	0.1885

Table 6.25. Average AR(2) model estimates and estimation errors under independent truncated normal prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2492	-	0.2492	-	0.2492	-	0.2492	-
ϕ_2	0.2022	-	0.2022	-	0.2022	-	0.2022	-
τ	1.6641	-	1.6641	-	1.6641	-	1.6641	-
Y_{n+1}	-0.0283	-	0.0157	-	0.0380	-	-0.0024	-
LINEX LOSS FUNCTION								
MLEs of ϕ_1	0.2424	0.0164	0.2486	0.0077	0.2481	0.0047	0.2472	0.0036
Lindley's - ϕ_1 with gamma prior for τ	0.2377	0.0075	0.2434	0.0051	0.2447	0.0036	0.2447	0.0029
Lindley's - ϕ_1 with improper prior for τ	0.2377	0.0075	0.2434	0.0051	0.2447	0.0036	0.2447	0.0029
Gibbs - ϕ_1 with gamma prior for τ	0.2363	0.0087	0.2438	0.0054	0.2449	0.0037	0.2449	0.0029
Gibbs - ϕ_1 with improper prior for τ	0.2362	0.0087	0.2440	0.0054	0.2451	0.0037	0.2450	0.0029
MLEs of ϕ_2	0.1734	0.0147	0.1891	0.0078	0.1964	0.0052	0.1987	0.0038
Lindley's - ϕ_2 with gamma prior for τ	0.1911	0.0093	0.1929	0.0061	0.1977	0.0044	0.1993	0.0034
Lindley's - ϕ_2 with improper prior for τ	0.1911	0.0093	0.1929	0.0061	0.1977	0.0044	0.1993	0.0034
Gibbs - ϕ_2 with gamma prior for τ	0.1810	0.0099	0.1911	0.0063	0.1969	0.0044	0.1989	0.0034
Gibbs - ϕ_2 with improper prior for τ	0.1811	0.0100	0.1909	0.0063	0.1969	0.0044	0.1989	0.0034
MLEs of τ	1.8032	0.2252	1.7388	0.0768	1.7049	0.0406	1.6943	0.0257
Lindley's - τ with gamma prior	1.5795	0.0709	1.6291	0.0424	1.6349	0.0287	1.6422	0.0207
Lindley's - τ with improper prior	1.6517	0.1220	1.6648	0.0561	1.6565	0.0334	1.6582	0.0223
Gibbs - τ with gamma prior	1.6138	0.0673	1.6402	0.0424	1.6403	0.0284	1.6455	0.0207
Gibbs - τ with improper prior	1.6517	0.1147	1.6655	0.0558	1.6570	0.0333	1.6587	0.0223
MLEs of Y_{n+1}	-0.0055	0.7301	0.0091	0.6412	-0.0094	0.7189	-0.0100	0.7021
Lindley's - Y_{n+1} with gamma prior for τ	-0.6313	0.5920	-0.6076	0.5666	-0.6298	0.5914	-0.6276	0.5673
Lindley's - Y_{n+1} with improper prior for τ	-0.6293	0.6021	-0.6079	0.5711	-0.6305	0.5936	-0.6283	0.5693
Gibbs - Y_{n+1} with gamma prior for τ	-0.4416	0.5249	-0.4149	0.4966	-0.4336	0.5375	-0.4314	0.5122
Gibbs - Y_{n+1} with improper prior for τ	-0.4648	0.5343	-0.4249	0.4986	-0.4395	0.5365	-0.4342	0.5128

6.3.2. Independent Uniform prior for ϕ_1, ϕ_2 and Gamma or Improper priors for τ

We consider the independent uniform priors for ϕ_1 and ϕ_2 with parameters $a_1 = 0$, $b_1 = 0.5$, and $a_2 = 0, b_2 = 0.4$. The intervals for ϕ_1 and ϕ_2 are chosen such that the AR(2) model would be stationary. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$.

Table 6.26 presents the average values of AR(2) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used.

Table 6.27, Table 6.28, Table 6.29, Table 6.30, Table 6.31 and Table 6.32 present the average values of AR(2) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25$, -0.75, -0.25, $0.25, 0.75, 1.25$, respectively.

The average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overall the Bayes estimates are found to result in smaller average estimation errors than the ML estimates. For ϕ_1, ϕ_2 and one-step prediction, the Bayes estimates are found to have smaller average estimation errors under both SE and LINEX loss functions with all parameter γ values. Under SE loss function, for τ the Bayes estimation is found to result in the smallest estimation errors. Under the LINEX loss function, when τ has gamma prior the Bayes estimation has better performance than the ML estimation for all parameter γ values. When when τ has improper prior the ML estimates have smaller average errors when $\gamma = -1.25, -0.75$, when $\gamma = -0.25, 0.25, 0.75, 1.25$ the Bayes estimation is superior. All estimator performances are reasonably close to each other as the sample size increases.

Table 6.26. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using SE loss function

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0218	0.2466	0.0108	0.2480	0.0071	0.2486	0.0052
Bayes - ϕ_1 with gamma prior for τ	0.2473	0.0100	0.2480	0.0064	0.2488	0.0048	0.2491	0.0038
Bayes - ϕ_1 with improper prior for τ	0.2472	0.0100	0.2480	0.0064	0.2488	0.0048	0.2491	0.0038
MLEs of ϕ_2	0.1688	0.0208	0.1842	0.0103	0.1888	0.0067	0.1916	0.0050
Bayes - ϕ_2 with gamma prior for τ	0.1907	0.0080	0.1933	0.0056	0.1946	0.0042	0.1954	0.0035
Bayes - ϕ_2 with improper prior for τ	0.1907	0.0080	0.1933	0.0056	0.1946	0.0042	0.1954	0.0035
MLEs of τ	1.8169	0.1898	1.7413	0.0757	1.7160	0.0529	1.7036	0.0374
Bayes - τ with gamma prior	1.6656	0.0998	1.6695	0.0541	1.6694	0.0396	1.6691	0.0308
Bayes - τ with improper prior	1.7412	0.1598	1.7058	0.0689	1.6928	0.0496	1.6864	0.0356
MLEs of Y_{n+1}	0.0020	0.6703	-0.0009	0.6834	0.0045	0.6519	0.0015	0.6875
Bayes - Y_{n+1} with gamma prior for τ	0.0020	0.6512	-0.0015	0.6784	0.0038	0.6516	0.0008	0.6854
Bayes - Y_{n+1} with improper prior for τ	0.0020	0.6513	-0.0015	0.6783	0.0038	0.6516	0.0008	0.6854

Table 6.27. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0172	0.2466	0.0084	0.2480	0.0055	0.2486	0.0041
Bayes - ϕ_1 with gamma prior for τ	0.2582	0.0146	0.2529	0.0078	0.2522	0.0053	0.2516	0.0040
Bayes - ϕ_1 with improper prior for τ	0.2580	0.0147	0.2529	0.0078	0.2521	0.0053	0.2516	0.0040
MLEs of ϕ_2	0.1688	0.0170	0.1842	0.0083	0.1888	0.0053	0.1916	0.0040
Bayes - ϕ_2 with gamma prior for τ	0.1852	0.0135	0.1914	0.0074	0.1934	0.0049	0.1950	0.0037
Bayes - ϕ_2 with improper prior for τ	0.1850	0.0136	0.1914	0.0074	0.1934	0.0049	0.1950	0.0037
MLEs of τ	1.8169	0.1227	1.7413	0.0541	1.7160	0.0379	1.7036	0.0271
Bayes - τ with gamma prior	1.7245	0.0860	1.7029	0.0455	1.6926	0.0327	1.6870	0.0248
Bayes - τ with improper prior	1.8451	0.1409	1.7500	0.0577	1.7208	0.0396	1.7069	0.0280
MLEs of Y_{n+1}	0.0020	0.6913	-0.0009	0.8022	0.0045	0.7208	0.0015	0.7580
Bayes - Y_{n+1} with gamma prior for τ	0.4246	0.5286	0.4166	0.5480	0.4212	0.5062	0.4176	0.5377
Bayes - Y_{n+1} with improper prior for τ	0.4380	0.5345	0.4239	0.5444	0.4262	0.5047	0.4215	0.5376

Table 6.28. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0062	0.2466	0.0030	0.2480	0.0020	0.2486	0.0015
Bayes - ϕ_1 with gamma prior for τ	0.2531	0.0053	0.2504	0.0028	0.2505	0.0019	0.2504	0.0014
Bayes - ϕ_1 with improper prior for τ	0.2529	0.0053	0.2504	0.0028	0.2505	0.0019	0.2504	0.0014
MLEs of ϕ_2	0.1688	0.0060	0.1842	0.0029	0.1888	0.0019	0.1916	0.0014
Bayes - ϕ_2 with gamma prior for τ	0.1801	0.0049	0.1889	0.0027	0.1918	0.0018	0.1937	0.0013
Bayes - ϕ_2 with improper prior for τ	0.1800	0.0049	0.1889	0.0027	0.1918	0.0018	0.1937	0.0013
MLEs of τ	1.8169	0.0468	1.7413	0.0200	1.7160	0.0140	1.7036	0.0100
Bayes - τ with gamma prior	1.7003	0.0296	1.6893	0.0158	1.6832	0.0115	1.6798	0.0088
Bayes - τ with improper prior	1.8012	0.0484	1.7319	0.0202	1.7095	0.0141	1.6986	0.0100
MLEs of Y_{n+1}	0.0020	0.2071	-0.0009	0.2248	0.0045	0.2094	0.0015	0.2205
Bayes - Y_{n+1} with gamma prior for τ	0.2556	0.1893	0.2496	0.1959	0.2545	0.1845	0.2512	0.1946
Bayes - Y_{n+1} with improper prior for τ	0.2636	0.1902	0.2540	0.1952	0.2575	0.1840	0.2535	0.1946

Table 6.29. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0007	0.2466	0.0003	0.2480	0.0002	0.2486	0.0002
Bayes - ϕ_1 with gamma prior for τ	0.2480	0.0006	0.2480	0.0003	0.2489	0.0002	0.2492	0.0002
Bayes - ϕ_1 with improper prior for τ	0.2479	0.0006	0.2479	0.0003	0.2489	0.0002	0.2492	0.0002
MLEs of ϕ_2	0.1688	0.0007	0.1842	0.0003	0.1888	0.0002	0.1916	0.0002
Bayes - ϕ_2 with gamma prior for τ	0.1750	0.0005	0.1865	0.0003	0.1902	0.0002	0.1925	0.0001
Bayes - ϕ_2 with improper prior for τ	0.1749	0.0005	0.1865	0.0003	0.1902	0.0002	0.1925	0.0001
MLEs of τ	1.8169	0.0056	1.7413	0.0023	1.7160	0.0016	1.7036	0.0011
Bayes - τ with gamma prior	1.6769	0.0032	1.6761	0.0017	1.6740	0.0012	1.6727	0.0010
Bayes - τ with improper prior	1.7605	0.0051	1.7144	0.0022	1.6983	0.0016	1.6905	0.0011
MLEs of Y_{n+1}	0.0020	0.0211	-0.0009	0.0219	0.0045	0.0208	0.0015	0.0219
Bayes - Y_{n+1} with gamma prior for τ	0.0865	0.0210	0.0826	0.0215	0.0879	0.0205	0.0847	0.0215
Bayes - Y_{n+1} with improper prior for τ	0.0892	0.0210	0.0840	0.0215	0.0889	0.0205	0.0855	0.0215

Table 6.30. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0007	0.2466	0.0003	0.2480	0.0002	0.2486	0.0002
Bayes - ϕ_1 with gamma prior for τ	0.2428	0.0006	0.2455	0.0003	0.2473	0.0002	0.2480	0.0002
Bayes - ϕ_1 with improper prior for τ	0.2428	0.0006	0.2455	0.0003	0.2473	0.0002	0.2480	0.0002
MLEs of ϕ_2	0.1688	0.0006	0.1842	0.0003	0.1888	0.0002	0.1916	0.0002
Bayes - ϕ_2 with gamma prior for τ	0.1698	0.0005	0.1840	0.0003	0.1886	0.0002	0.1913	0.0002
Bayes - ϕ_2 with improper prior for τ	0.1698	0.0005	0.1840	0.0003	0.1886	0.0002	0.1913	0.0002
MLEs of τ	1.8169	0.0063	1.7413	0.0024	1.7160	0.0017	1.7036	0.0012
Bayes - τ with gamma prior	1.6545	0.0031	1.6631	0.0017	1.6648	0.0012	1.6656	0.0010
Bayes - τ with improper prior	1.7225	0.0049	1.6973	0.0021	1.6874	0.0015	1.6824	0.0011
MLEs of Y_{n+1}	0.0020	0.0213	-0.0009	0.0213	0.0045	0.0204	0.0015	0.0215
Bayes - Y_{n+1} with gamma prior for τ	-0.0825	0.0210	-0.0844	0.0212	-0.0788	0.0202	-0.0817	0.0214
Bayes - Y_{n+1} with improper prior for τ	-0.0852	0.0210	-0.0859	0.0212	-0.0798	0.0202	-0.0825	0.0214

Table 6.31. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0062	0.2466	0.0030	0.2480	0.0020	0.2486	0.0015
Bayes - ϕ_1 with gamma prior for τ	0.2377	0.0053	0.2430	0.0028	0.2457	0.0019	0.2468	0.0014
Bayes - ϕ_1 with improper prior for τ	0.2378	0.0053	0.2430	0.0028	0.2457	0.0019	0.2468	0.0014
MLEs of ϕ_2	0.1688	0.0057	0.1842	0.0028	0.1888	0.0019	0.1916	0.0014
Bayes - ϕ_2 with gamma prior for τ	0.1647	0.0050	0.1815	0.0027	0.1870	0.0018	0.1901	0.0014
Bayes - ϕ_2 with improper prior for τ	0.1647	0.0050	0.1815	0.0027	0.1870	0.0018	0.1901	0.0014
MLEs of τ	1.8169	0.0646	1.7413	0.0234	1.7160	0.0162	1.7036	0.0112
Bayes - τ with gamma prior	1.6328	0.0269	1.6503	0.0147	1.6559	0.0109	1.6587	0.0085
Bayes - τ with improper prior	1.6868	0.0417	1.6808	0.0186	1.6767	0.0138	1.6745	0.0100
MLEs of Y_{n+1}	0.0020	0.2153	-0.0009	0.2050	0.0045	0.1946	0.0015	0.2082
Bayes - Y_{n+1} with gamma prior for τ	-0.2515	0.1924	-0.2515	0.1883	-0.2454	0.1783	-0.2482	0.1917
Bayes - Y_{n+1} with improper prior for τ	-0.2596	0.1930	-0.2558	0.1888	-0.2484	0.1787	-0.2505	0.1919

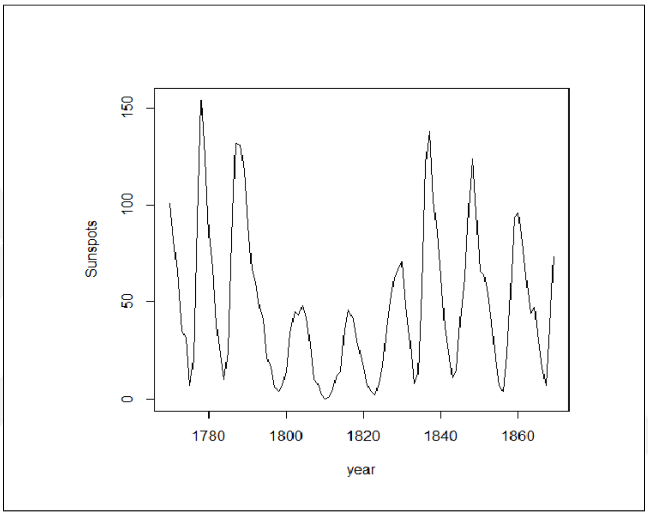
Table 6.32. Average AR(2) model estimates and estimation errors under independent uniform prior for ϕ_1 and ϕ_2 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
ϕ_1	0.2499	-	0.2499	-	0.2499	-	0.2499	-
ϕ_2	0.1989	-	0.1989	-	0.1989	-	0.1989	-
τ	1.6688	-	1.6688	-	1.6688	-	1.6688	-
Y_{n+1}	0.0040	-	0.0079	-	0.0118	-	0.0065	-
ESTIMATES								
MLEs of ϕ_1	0.2451	0.0172	0.2466	0.0085	0.2480	0.0056	0.2486	0.0041
Bayes - ϕ_1 with gamma prior for τ	0.2326	0.0146	0.2406	0.0078	0.2441	0.0053	0.2456	0.0039
Bayes - ϕ_1 with improper prior for τ	0.2327	0.0147	0.2406	0.0078	0.2441	0.0053	0.2456	0.0039
MLEs of ϕ_2	0.1688	0.0157	0.1842	0.0079	0.1888	0.0051	0.1916	0.0039
Bayes - ϕ_2 with gamma prior for τ	0.1596	0.0138	0.1791	0.0074	0.1853	0.0049	0.1889	0.0038
Bayes - ϕ_2 with improper prior for τ	0.1597	0.0139	0.1791	0.0074	0.1853	0.0049	0.1889	0.0038
MLEs of τ	1.8169	0.2112	1.7413	0.0702	1.7160	0.0485	1.7036	0.0330
Bayes - τ with gamma prior	1.6120	0.0728	1.6379	0.0402	1.6470	0.0299	1.6518	0.0235
Bayes - τ with improper prior	1.6533	0.1102	1.6647	0.0503	1.6661	0.0381	1.6666	0.0277
MLEs of Y_{n+1}	0.0020	0.7740	-0.0009	0.6728	0.0045	0.6236	0.0015	0.6833
Bayes - Y_{n+1} with gamma prior for τ	-0.4206	0.5603	-0.4185	0.5164	-0.4121	0.4854	-0.4147	0.5303
Bayes - Y_{n+1} with improper prior for τ	-0.4340	0.5653	-0.4257	0.5191	-0.4171	0.4878	-0.4185	0.5317

6.3.3. Analysis of Wolfer's Sunspot Data

We consider the Wolfer sunspot numbers (that measure the average number of sunspots on sun during a year) for each year during the period 1770-1869. The series consists of 100 yearly observations. The sunspot data were also analyzed by Box and Tiao [1], among the others. Figure 6.13 shows the plotted sunspots series.

Figure 6.13. Sunspots data

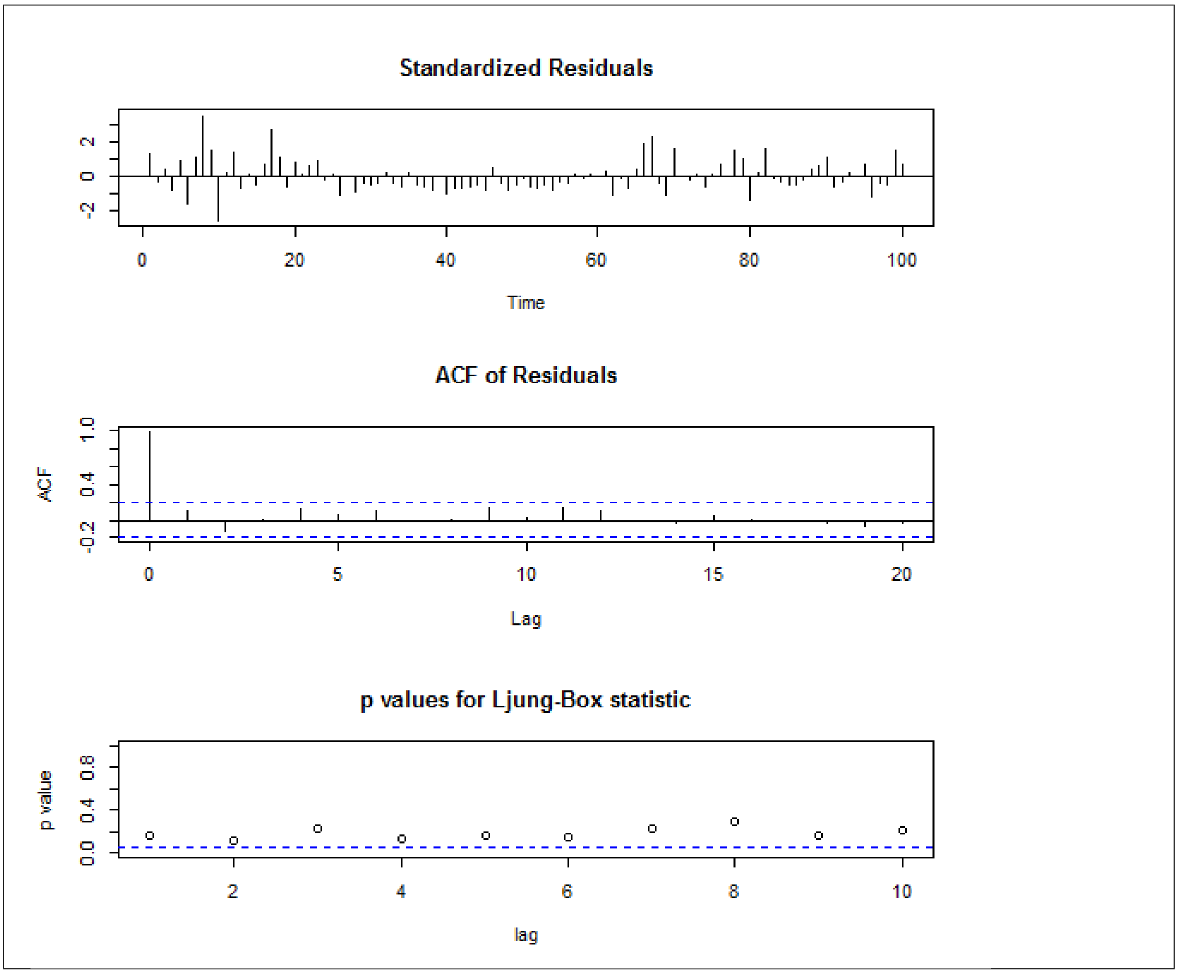


We analyze the data using the independent truncated normal prior for ϕ_1 with mean equal to the sample mean and variance equal to the sample variance, truncated normal prior forϕ_2 and with mean equal to the sample third moment and variance equal to the sample fourth moment, and improper prior for τ. The LINEX loss function's parameter are $\gamma = 0.75$ and $\gamma = -0.75$. In order to apply the analysis using the assumed form of the AR(2) model, we need to subtract the series mean from each of the observations to obtain a zero-mean series.

Based on the augmented Dickey-Fuller test, the unit-root hypothesis is rejected. The test statistic is -4.66 with p-value less than 0.01, thus the obtained mean-adjusted sunpots series is stationary. Model checking shows that the AR(2) model can be fitted to the zero-mean series; see Figure 6.14

We obtain estimates of ϕ_1, ϕ_2, τ and one-step predicted value using 90, 91,…, 98, 99 observations. Table 6.33 and Table 6.34 present the estimation and one-step prediction results. It is observed that under the LINEX loss function the prediction errors of the Bayes estimates

Figure 6.14. AR(2) model checking



are significantly smaller than that of the ML estimates, the Bayes estimates obtained using Gibbs sampling method have the smallest average estimation errors. Under the SE loss function, the ML and Bayes estimates obtained using Lindley's approximation are found to have similar average prediction error, whereas the average prediction error of Bayes estimates obtained using Gibbs sampler is slightly higher.

Table 6.33. AR(2) model estimates for empirical data

	SAMPLE SIZE										
ACTUAL	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**AVERAGE**
Y_{n+1}	96.0000	77.0000	59.0000	44.0000	47.0000	30.0000	16.0000	7.0000	37.0000	74.0000	48.7000
ESTIMATES											
ML											
ϕ_1	1.4193	1.4078	1.4065	1.4055	1.4051	1.4013	1.4002	1.4029	1.4061	1.3978	1.4052
ϕ_2	-0.7158	-0.7072	-0.7080	-0.7056	-0.7051	-0.7009	-0.6995	-0.7018	-0.7033	-0.7020	-0.7049
τ	0.0042	0.0043	0.0043	0.0044	0.0044	0.0044	0.0044	0.0044	0.0045	0.0044	0.0044
Y_{n+1}	107.8991	82.8298	54.6818	42.9141	34.5279	49.3047	23.3434	15.4885	12.4864	60.9982	48.4474
SE LOSS FUNCTION											
ϕ_1 Lindley's approx.	1.4192	1.4078	1.4064	1.4054	1.4051	1.4013	1.4001	1.4029	1.4060	1.3977	1.4052
ϕ_1 Gibbs sampling	1.4192	1.4088	1.4075	1.4116	1.4042	1.4016	1.4004	1.4056	1.4045	1.4014	1.4065
ϕ_2 Lindley's approx.	-0.7158	-0.7071	-0.7080	-0.7055	-0.7050	-0.7009	-0.6995	-0.7017	-0.7032	-0.7020	-0.7049
ϕ_2 Gibbs sampling	-0.7159	-0.7079	-0.7086	-0.7109	-0.7054	-0.7025	-0.7002	-0.7044	-0.7018	-0.7050	-0.7063
τ Lindley's approx.	0.0042	0.0042	0.0042	0.0043	0.0043	0.0043	0.0043	0.0043	0.0044	0.0043	0.0043
τ Gibbs sampling	0.0042	0.0042	0.0042	0.0043	0.0043	0.0043	0.0043	0.0043	0.0044	0.0043	0.0043
Y_{n+1} Lindley's approx.	107.8975	82.8300	54.6829	42.9150	34.5285	49.3045	23.3441	15.4889	12.4865	60.9967	48.4475
Y_{n+1} Gibbs sampling	107.4530	82.9737	54.6166	43.3600	34.5044	50.1869	23.5940	15.4723	12.5148	61.6795	48.6355
LINEX LOSS FUNCTION $\gamma = 0.75$											
ϕ_1 Lindley's approx.	1.4170	1.4057	1.4044	1.4034	1.4031	1.3993	1.3981	1.4009	1.4041	1.3957	1.4032
ϕ_1 Gibbs sampling	1.4169	1.4067	1.4054	1.4094	1.4020	1.3996	1.3982	1.4035	1.4025	1.3993	1.4044
ϕ_2 Lindley's approx.	-0.7179	-0.7092	-0.7100	-0.7076	-0.7070	-0.7029	-0.7014	-0.7037	-0.7052	-0.7039	-0.7069
ϕ_2 Gibbs sampling	-0.7182	-0.7100	-0.7107	-0.7130	-0.7075	-0.7045	-0.7023	-0.7064	-0.7038	-0.7071	-0.7084
τ Lindley's approx.	0.0042	0.0042	0.0042	0.0043	0.0043	0.0043	0.0043	0.0043	0.0044	0.0043	0.0043
τ Gibbs sampling	0.0042	0.0042	0.0042	0.0043	0.0043	0.0043	0.0043	0.0043	0.0044	0.0043	0.0043
Y_{n+1} Lindley's approx.	-0.4941	-25.1322	-52.4236	-63.2774	-70.6862	-55.6035	-82.1625	-89.2746	-91.6187	-44.6038	-57.5277
Y_{n+1} Gibbs sampling	63.8720	29.3755	13.0706	5.4777	-9.1691	6.3575	-37.2485	-30.4573	-34.9658	22.1198	2.8433
LINEX LOSS FUNCTION $\gamma = -0.75$											
ϕ_1 Lindley's approx.	1.4214	1.4099	1.4085	1.4075	1.4071	1.4033	1.4022	1.4049	1.4080	1.3997	1.4072
ϕ_1 Gibbs sampling	1.4215	1.4110	1.4096	1.4138	1.4063	1.4036	1.4026	1.4077	1.4065	1.4035	1.4086
ϕ_2 Lindley's approx.	-0.7137	-0.7050	-0.7059	-0.7035	-0.7031	-0.6989	-0.6975	-0.6998	-0.7013	-0.7000	-0.7029
ϕ_2 Gibbs sampling	-0.7137	-0.7057	-0.7066	-0.7088	-0.7033	-0.7004	-0.6981	-0.7024	-0.6998	-0.7029	-0.7042
τ Lindley's approx.	0.0042	0.0042	0.0042	0.0043	0.0043	0.0043	0.0043	0.0043	0.0044	0.0043	0.0043
τ Gibbs sampling	0.0042	0.0042	0.0042	0.0043	0.0043	0.0043	0.0043	0.0043	0.0044	0.0043	0.0043
Y_{n+1} Lindley's approx.	216.2923	190.7919	161.7873	149.1056	139.7419	154.2129	128.8494	120.2516	116.5915	166.6001	154.4224
Y_{n+1} Gibbs sampling	153.8031	128.0725	100.5126	91.8164	85.9951	89.3893	62.4082	50.8744	53.3288	107.2476	92.3448

Table 6.34. AR(2) model errors for empirical data

	SAMPLE SIZE										
ERRORS	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**AVERAGE**
SE LOSS FUNCTION											
Y_{n+1} ML	141.5882	33.9868	18.6464	1.1792	155.5544	372.6705	53.9258	72.0540	600.9188	169.0481	161.9572
Y_{n+1} Lindley's approx.	141.5494	33.9895	18.6372	1.1773	155.5384	372.6653	53.9354	72.0615	600.9107	169.0849	161.9550
Y_{n+1} Gibbs sampling	131.1716	35.6856	19.2139	0.4096	156.1393	407.5124	57.6692	71.7791	599.5262	151.7947	163.0902
LINEX LOSS FUNCTION $\gamma = 0.75$											
Y_{n+1} ML	7502.4920	73.8584	2.2778	0.2573	8.3542	1940582.3082	240.0419	574.5615	17.3852	8.7514	194901.0288
Y_{n+1} Lindley's approx.	71.3706	75.5992	82.5677	79.4580	87.2647	63.2026	72.6219	71.2060	95.4641	87.9529	78.6708
Y_{n+1} Gibbs sampling	23.0960	34.7184	33.4470	27.8917	41.1268	16.7318	38.9364	27.0930	52.9743	37.9102	33.3926
LINEX LOSS FUNCTION $\gamma = -0.75$											
Y_{n+1} ML	7.9244	3.3850	21.2597	0.4435	11535.8163	13.4785	4.5116	5.3681	9.65×10^7	17167.2778	9.65×10^6
Y_{n+1} Lindley's approx.	89.2192	84.3439	76.0905	77.8292	68.5565	92.1596	83.6370	83.9387	58.6936	68.4501	78.2918
Y_{n+1} Gibbs sampling	42.3523	37.3044	30.1345	34.8623	28.2463	43.5420	33.8061	31.9058	11.2466	23.9357	31.7336

6.4. AR(2) MODEL PARAMETER IMPACT ANALYSIS

6.4.1. Independent truncated normal prior for ϕ_1, ϕ_2 and gamma or improper priors for τ

We have undertaken the parameter impact analysis to define how the average estimation and prediction errors change when model parameters vary. We use fixed parameters $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\alpha_1 = 10, 20, 30$. Figure 6.15 presents the mean errors when the parameter α changes. We notice that as α increases, the average estimation errors of ϕ_1 and ϕ_2 change slightly. The average estimation errors of τ increase more than parameter α and the average prediction errors decrease proportionally to α increase.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 6.16 presents the mean errors when the parameter β changes. We notice that β and the average estimation errors of ϕ_1 and ϕ_2 have a weak nonlinear relationship, the average estimation errors of τ and β have nonlinear inverse relationship. As parameter β increases, the average prediction errors increase.

To estimate the impact of parameter μ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\mu_1 = -0.375, -0.125, 0.125, 0.375$. Figure 6.17 presents the mean errors when the parameter μ_1 changes. We notice that as absolute value of μ_1 increases, the average estimation errors of ϕ_1 and ϕ_2 decrease, the average estimation errors of τ and the average prediction errors remain almost unchanged.

To estimate the impact of parameter μ_2, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$,$\sigma_1 = 0.2$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\mu_2 = -0.375, -0.125, 0.125, 0.375$. Figure 6.18 presents the mean errors when the parameter μ_2 changes. The changes of μ_2 have similar impact to the average estimation and prediction errors as the μ_1 changes.

To estimate the impact of parameter σ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\sigma_1 = 0.1, 0.2, 0.3$. Figure 6.19 presents the mean errors when the parameter σ_1 changes. We notice that as σ_1 increases, the average Bayes estimation errors of ϕ_1 and ϕ_2 increase, the changes of average ML estimation errors of ϕ_1 and ϕ_2 are very small, the average estimation errors of τ remain unchanged. The average Bayes prediction errors slightly increase when σ_1 increase, whereas the average ML prediction errors do not seem to depend on σ_1.

To estimate the impact of parameter σ_2, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\sigma_2 = 0.1, 0.2, 0.3$. Figure 6.20 presents the mean errors when the parameter σ_2 changes. The changes of σ_2 have similar impact to the average estimation and prediction errors as the changes of σ_1.

To estimate the impact of parameter γ, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 6.21 presents the mean errors when the parameter γ changes. We notice that as γ increases, the average estimation errors of ϕ_1, ϕ_2, τ and prediction increase more than the increase in γ.

Figure 6.15. Impact of parameter α on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 6.16. Impact of parameter β on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

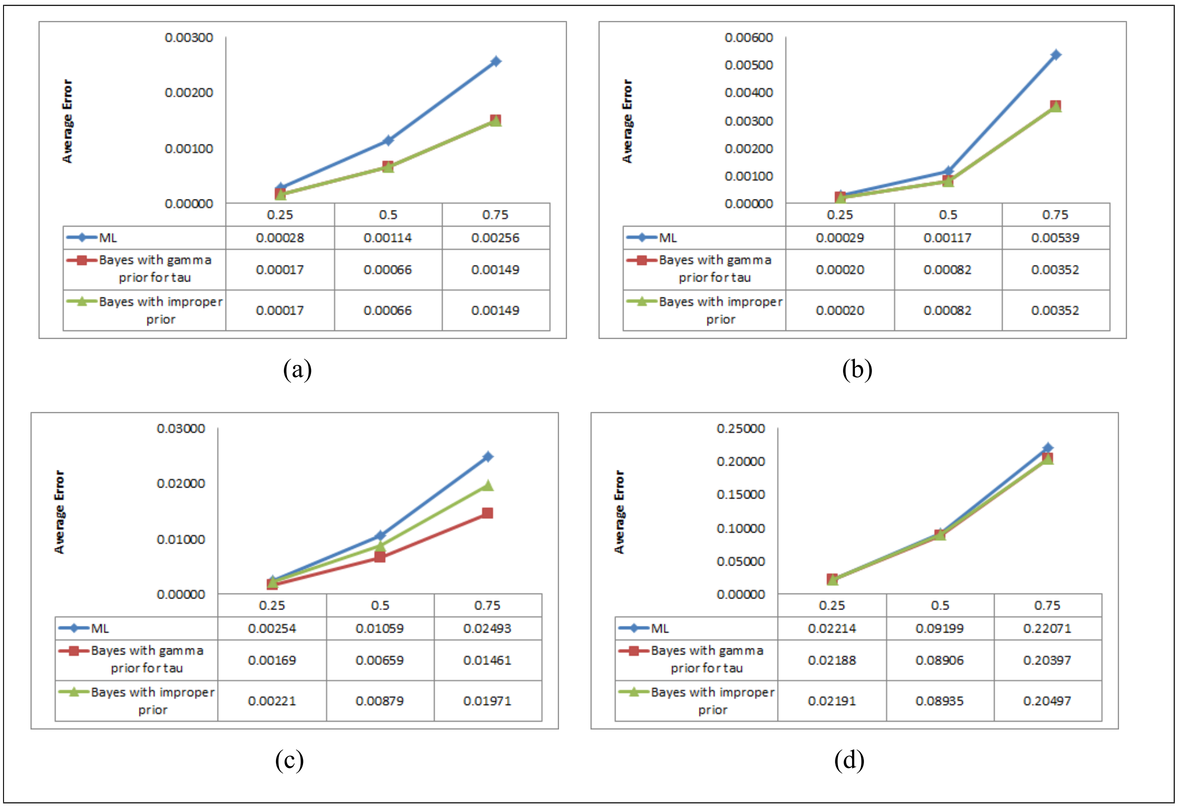
Figure 6.17. Impact of parameter μ_1 on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 6.18. Impact of parameter μ_2 on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 6.19. Impact of parameter σ_1 on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 6.20. Impact of parameter σ_2 on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 6.21. Impact of parameter γ on average estimation and prediction errors for AR(2) independent truncated normal prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under LINEX, (b) Estimation of ϕ_2 under LINEX, (c) Estimation of τ under LINEX loss, (d) Estimation of Y_{n+1} under LINEX loss.



6.4.2. Independent Uniform prior for ϕ_1, ϕ_2 and Gamma or Improper priors for τ

To estimate the impact of parameter α, we use fixed parameters $\beta = 6$, $(c_1, d_1) = (0.25, 0.5)$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\alpha_1 = 10, 20, 30$. Figure 6.22 presents the mean errors when the parameter α changes. We notice that as α increases, the average estimation errors of ϕ_1 and ϕ_2 change only slightly. The average estimation errors of τ increase more than parameter α and the average prediction errors decrease proportionally to α increase.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $(c_1, d_1) = (0.25, 0.5)$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 6.23 presents the mean errors when the parameter β changes. We notice that β does not have any significant impact on the average estimation errors of ϕ_1 and ϕ_2, the average estimation errors of τ decrease as β increases. As parameter β increases, the average prediction errors increase.

To estimate the impact of parameter (c_1, d_1), we use fixed parameters $\alpha = 10$, $\beta = 6$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $(c_1, d_1) = (-0.5, -0.25),\ (-0.25, 0),\ (0, 0.25),\ (0.25, 0.5)$. Figure 6.24 presents the mean errors when (c_1, d_1) changes. The interval is represented by its middle point. We notice that the average estimation errors remain almost unchanged as the interval changes.

To estimate the impact of parameter (c_2, d_2), we use fixed parameters $\alpha = 10$, $\beta = 6$, $(c_1, d_1) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $(c_2, d_2) = (-0.5, -0.25),\ (-0.25, 0),\ (0, 0.25),\ (0.25, 0.5)$. Figure 6.25 presents the mean errors when the parameter (c_2, d_2) changes. The interval is represented by its middle point. We notice that the average estimation errors of ϕ_1 and ϕ_2 increase as the absolute value of interval mean point decreases, the changes of average estimation errors of τ remain almost unchanged. The average prediction errors slightly increase as the mean of ϕ_2 increases.

To estimate the impact of parameter γ, we use fixed parameters $\alpha = 10$, $\beta = 6$,$(c_1, d_1) = (0.25, 0.5)$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 6.26 presents the mean errors when the parameter γ changes. We notice that as γ increases, the average estimation errors of ϕ_1,ϕ_2, τ and prediction increase more than the increase in γ.

Figure 6.27 and Figure 6.28 present the average estimation errors of ϕ_1, ϕ_2 under the LINEX loss function, estimated using the numerical method and the truncated normal approximation. We notice that the Bayes estimation errors for ϕ_1, ϕ_2 are significantly smaller when the numerical approach is used (left in the figure); and the difference between the ML and the Bayes estimates becomes more noticeable, the Bayes estimates have the smallest average estimation errors.

Figure 6.22. Impact of parameter α on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

	10	20	30
ML	0.00911	0.00931	0.00916
Bayes with gamma prior for tau	0.00328	0.00335	0.00328
Bayes with improper prior for tau	0.00328	0.00336	0.00329

(a)

	10	20	30
ML	0.00028	0.00029	0.00029
Bayes with gamma prior for tau	0.00010	0.00010	0.00010
Bayes with improper prior for tau	0.00010	0.00010	0.00010

(b)

	10	20	30
ML	0.00939	0.00937	0.00931
Bayes with gamma prior for tau	0.00334	0.00338	0.00338
Bayes with improper prior for tau	0.00334	0.00339	0.00338

(c)

	10	20	30
ML	0.00029	0.00029	0.00029
Bayes with gamma prior for tau	0.00010	0.00011	0.00011
Bayes with improper prior for tau	0.00010	0.00011	0.00011

(d)

	10	20	30
ML	0.07852	0.29130	0.65825
Bayes with gamma prior for tau	0.05231	0.17103	0.32082
Bayes with improper prior for tau	0.07122	0.26493	0.59539

(e)

	10	20	30
ML	0.00255	0.00973	0.02286
Bayes with gamma prior for tau	0.00163	0.00531	0.00992
Bayes with improper prior for tau	0.00221	0.00817	0.01824

(f)

	10	20	30
ML	0.70001	0.32242	0.20779
Bayes with gamma prior for tau	0.69710	0.32074	0.20669
Bayes with improper prior for tau	0.69726	0.32078	0.20670

(g)

	10	20	30
ML	0.02216	0.01015	0.00649
Bayes with gamma prior for tau	0.02184	0.01009	0.00648
Bayes with improper prior for tau	0.02184	0.01009	0.00648

(h)

Figure 6.23. Impact of parameter β on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 6.24. Impact of (c_1, d_1) on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

(a)

	-0.375	-0.125	0.125	0.375
ML	0.00910	0.00907	0.00908	0.00911
Bayes with gamma prior for tau	0.00329	0.00331	0.00329	0.00328
Bayes with improper prior for tau	0.00329	0.00332	0.00329	0.00328

(b)

	-0.375	-0.125	0.125	0.375
ML	0.00028	0.00028	0.00028	0.00028
Bayes with gamma prior for tau	0.00010	0.00010	0.00010	0.00010
Bayes with improper prior for tau	0.00010	0.00010	0.00010	0.00010

(c)

	-0.375	-0.125	0.125	0.375
ML	0.00914	0.00913	0.00925	0.00939
Bayes with gamma prior for tau	0.00332	0.00331	0.00332	0.00334
Bayes with improper prior for tau	0.00332	0.00331	0.00332	0.00334

(d)

	-0.375	-0.125	0.125	0.375
ML	0.00028	0.00028	0.00029	0.00029
Bayes with gamma prior for tau	0.00010	0.00010	0.00010	0.00010
Bayes with improper prior for tau	0.00010	0.00010	0.00010	0.00010

(e)

	-0.375	-0.125	0.125	0.375
ML	0.07784	0.07791	0.07818	0.07852
Bayes with gamma prior for tau	0.05212	0.05215	0.05223	0.05231
Bayes with improper prior for tau	0.07062	0.07069	0.07093	0.07122

(f)

	-0.375	-0.125	0.125	0.375
ML	0.00252	0.00253	0.00253	0.00255
Bayes with gamma prior for tau	0.00162	0.00162	0.00162	0.00163
Bayes with improper prior for tau	0.00219	0.00219	0.00220	0.00221

(g)

	-0.375	-0.125	0.125	0.375
ML	0.69701	0.69870	0.69991	0.70001
Bayes with gamma prior for tau	0.69077	0.68770	0.69019	0.69710
Bayes with improper prior for tau	0.69080	0.68770	0.69019	0.69726

(h)

	-0.375	-0.125	0.125	0.375
ML	0.02206	0.02212	0.02216	0.02216
Bayes with gamma prior for tau	0.02175	0.02180	0.02184	0.02184
Bayes with improper prior for tau	0.02175	0.02180	0.02184	0.02184

Figure 6.25. Impact of (c_2, d_2) on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

(a)

	-0.375	-0.125	0.125	0.375
ML	0.00870	0.00999	0.01014	0.00911
Bayes with gamma prior for tau	0.00326	0.00343	0.00343	0.00328
Bayes with improper prior	0.00327	0.00343	0.00343	0.00328

(b)

	-0.375	-0.125	0.125	0.375
ML	0.00027	0.00031	0.00032	0.00028
Bayes with gamma prior for tau	0.00010	0.00011	0.00011	0.00010
Bayes with improper prior	0.00010	0.00011	0.00011	0.00010

(c)

	-0.375	-0.125	0.125	0.375
ML	0.00877	0.00994	0.01014	0.00939
Bayes with gamma prior for tau	0.00331	0.00349	0.00347	0.00334
Bayes with improper prior	0.00332	0.00348	0.00348	0.00334

(d)

	-0.375	-0.125	0.125	0.375
ML	0.00027	0.00031	0.00032	0.00029
Bayes with gamma prior for tau	0.00010	0.00011	0.00011	0.00010
Bayes with improper prior	0.00010	0.00011	0.00011	0.00010

(e)

	-0.375	-0.125	0.125	0.375
ML	0.07827	0.07831	0.07840	0.07852
Bayes with gamma prior for tau	0.05225	0.05228	0.05231	0.05231
Bayes with improper prior	0.07098	0.07103	0.07112	0.07122

(f)

	-0.375	-0.125	0.125	0.375
ML	0.00254	0.00254	0.00254	0.00255
Bayes with gamma prior for tau	0.00162	0.00163	0.00163	0.00163
Bayes with improper prior	0.00220	0.00220	0.00221	0.00221

(g)

	-0.375	-0.125	0.125	0.375
ML	0.69570	0.69696	0.69860	0.70001
Bayes with gamma prior for tau	0.68686	0.68724	0.68851	0.69710
Bayes with improper prior	0.68686	0.68722	0.68854	0.69726

(h)

	-0.375	-0.125	0.125	0.375
ML	0.02203	0.02207	0.02212	0.02216
Bayes with gamma prior for tau	0.02170	0.02174	0.02179	0.02184
Bayes with improper prior	0.02170	0.02174	0.02179	0.02184

Figure 6.26. Impact of parameter γ on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Estimation of ϕ_1 under LINEX, (b) Estimation of ϕ_2 under LINEX, (c) Estimation of τ under LINEX loss, (d) Estimation of Y_{n+1} under LINEX loss.

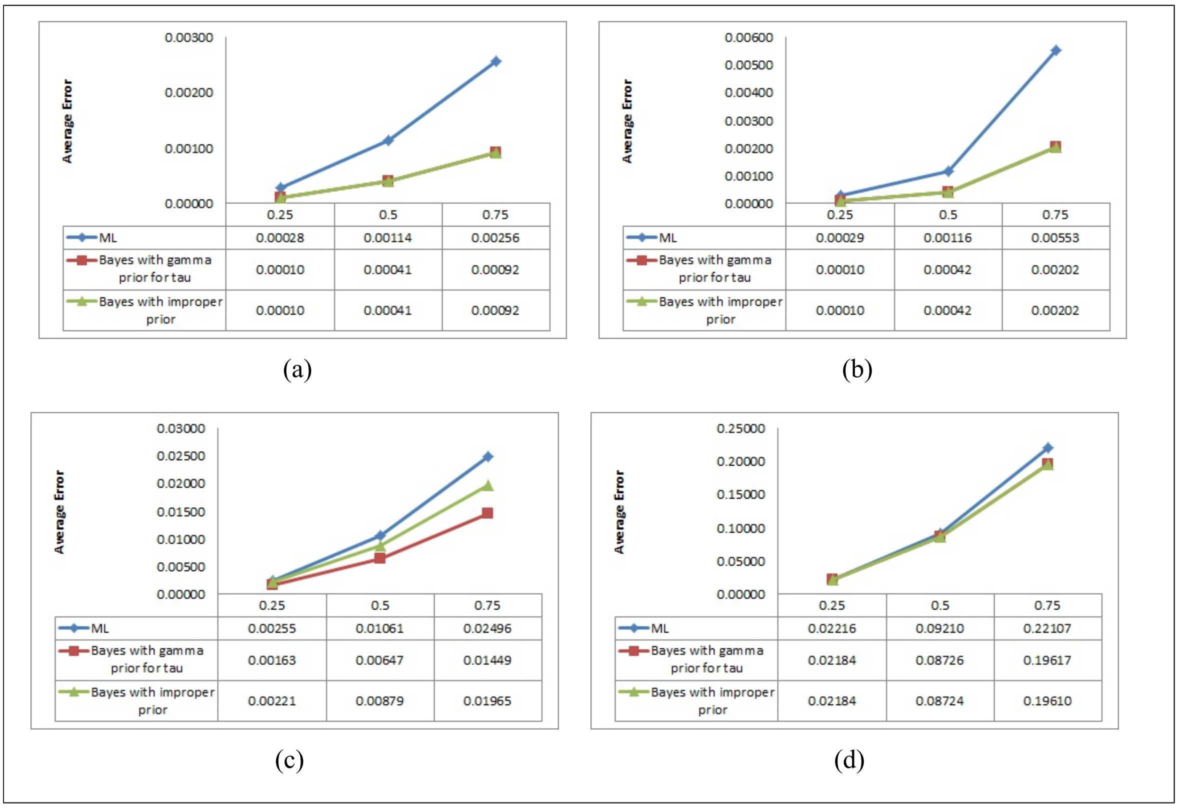


Figure 6.27. Impact of estimation method of ϕ_1 under LINEX loss function on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Numerical approach, α varies, (b) Approximation, α varies, (c) Numerical approach, β varies, (d) Approximation, β varies, (e) Numerical approach, interval of ϕ_1 varies, (f) Approximation, interval of ϕ_1 varies, (g) Numerical approach, γ varies, (h) Approximation, γ varies.

Figure 6.28. Impact of estimation method of ϕ_2 under LINEX loss function on average estimation and prediction errors for AR(2) independent uniform prior for ϕ_1 and ϕ_2. (a) Numerical approach, α varies, (b) Approximation, α varies, (c) Numerical approach, β varies, (d) Approximation, β varies, (e) Numerical approach, interval of ϕ_1 varies, (f) Approximation, interval of ϕ_1 varies, (g) Numerical approach, γ varies, (h) Approximation, γ varies.

7. BAYESIAN INFERENCES AND FORECASTING OF MOVING AVERAGE PROCESSES

In this section we consider the MA(q) model.

We can write MA(q) model as

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} - \begin{pmatrix} \epsilon_0 & \epsilon_{-1} & \dots & \epsilon_{1-q} \\ \vdots & \vdots & \dots & \vdots \\ \epsilon_{n-1} & \epsilon_{n-2} & \dots & \epsilon_{n-q} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_q \end{pmatrix} := -A\theta + \epsilon, \tag{7.1}$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_n)' \sim N(0, \tau^{-1}I)$ $(\tau = \frac{1}{\sigma^2} > 0)$. Y_t is the observation at time t. The parameters $\theta \in \mathbb{R}^q$ in the model and precision τ are considered to be random variables.

For the latter model we study the following questions.

(i) How to estimate parameters θ and τ given a sample $S_n = (Y_1, \dots, Y_n)'$?

(ii) How to forecast a future observation $W_1 = Y_{n+1}$ given a sample $S_n = (Y_1, \dots, Y_n)'$?

7.1. INDEPENDENT TRUNCATED NORMAL - GAMMA PRIOR

In the model 7.1 we assume that the parameters $\theta_i, i = 1, \dots, q$, have independent truncated normal priors on intervals $(c_i, d_i), c_i, d_i \in \mathbb{R}$, for all $i = 1, \dots, q$, respectively, with the parameters $\theta_1 \sim TN(\mu_1, \sigma_1^2), \dots, \theta_q \sim TN(\mu_q, \sigma_q^2)$, and the precision has independent gamma prior with the parameters α and β, i.e. $\tau \sim Gamma(\alpha, \beta)$. That is,

$$\xi(\theta, \tau) = \xi_1(\theta)\xi_2(\tau),$$

where the marginal prior density of τ is gamma distribution

$$\xi_2(\tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta}, \tau > 0,$$

and the marginal prior density of θ is

$$\xi_1(\theta) \propto \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{q}(\frac{\theta_i-\mu_i}{\sigma_i})^2} = \mathrm{e}^{-\frac{1}{2}(\theta-\mu)'Q(\theta-\mu)}, \theta \in \mathbb{R}^q,$$

that is, $\xi_1(\theta) \sim N(\mu, Q^{-1})$. So the joint prior density function of parameters

$$\xi(\theta,\tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{1}{2}(\theta-\mu)'Q(\theta-\mu)}.$$

The likelihood function for this model

$$L(\theta,\tau|S_n) \propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y+A\theta)'(Y+A\theta)},$$

where

$$A = \begin{pmatrix} \epsilon_0 & \epsilon_{-1} & \dots & \epsilon_{1-q} \\ \vdots & \vdots & \dots & \vdots \\ \epsilon_{n-1} & \epsilon_{n-2} & \dots & \epsilon_{n-q} \end{pmatrix}.$$

We approximate the likelihood function by letting $\epsilon_i = 0$, for all $i \leq 0$. This approximation was used by Box and Jenkins [1], Hillmer and Tiao [60] and many other authors. In order to use the similar procedures used for the AR(p) model to find the Bayes estimators of θ and τ and W_1, we need to approximate the residuals ϵ_t. The residuals can be estimated by

$$\epsilon_t^* = Y_t - \sum_{j=1}^{q}\theta_j^*\epsilon_{t-j}^*,$$

where $t = 1,\dots,n, \epsilon_{-1}^* = \dots, \epsilon_{-q-1}^* = 0$, and θ_j^* are the nonlinear least squares estimates of θ_j obtained by minimizing the conditional sum of squares

$$SS(\theta) = \sum_{t=1}^{n}(\epsilon_t^*)^2$$

with respect to θ over the region of invertibility. Thus,

$$A^* = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \epsilon_1^* & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \epsilon_{n-1}^* & \epsilon_{n-2}^* & \dots & \epsilon_{n-q}^* \end{pmatrix}.$$

and hence

$$L^*(\theta, \tau | S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y + A^*\theta)'(Y + A^*\theta)}. \tag{7.2}$$

Employing Bayes Theorem, the approximate posterior density of θ and τ is

$$\begin{aligned} \xi(\theta, \tau | S_n) &= \frac{L^*(\theta, \tau | S_n)\xi(\theta, \tau)}{\int_\Theta \int_0^\infty L^*(\theta, \tau | S_n)\xi(\theta, \tau)\mathrm{d}\theta\mathrm{d}\tau} \\ &= \frac{\tau^{\alpha-1}\mathrm{e}^{-\tau\beta - \frac{1}{2}(\theta-\mu)'Q(\theta-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)}}{\int_\Theta \int_0^\infty \tau^{\alpha-1}\mathrm{e}^{-\tau\beta - \frac{1}{2}(\theta-\mu)'Q(\theta-\mu)}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)}\mathrm{d}\theta\mathrm{d}\tau}. \end{aligned} \tag{7.3}$$

Using SE and LINEX loss functions, the Bayes estimators of $v = v(\theta, \tau)$ cannot be found analytically. Hence, we use Lindley's approximation and Gibbs sampling. In the next two sections we apply this model to the MA(1) and MA(2) processes and find the approximate Bayes estimators of their parameters and one-step ahead forecasts.

7.1.1. MA(1) model

In the $q = 1$ case, under the SE loss function, the Bayes estimator of function $v = v(\theta_1, \tau)$ can be written as

$$\hat{v}_{B_{SE}} = \frac{\int_{c_1}^{d_1} \int_0^\infty v(\theta_1, \tau)L^*(\theta_1, \tau | S_n)\xi(\theta_1, \tau)\mathrm{d}\theta_1\mathrm{d}\tau}{\int_{c_1}^{d_1} \int_0^\infty L^*(\theta_1, \tau | S_n)\xi(\theta_1, \tau)\mathrm{d}\theta_1\mathrm{d}\tau}, \tag{7.4}$$

where

$$L^*(\theta,\tau|S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon^*_{t-1})^2}. \tag{7.5}$$

Similarly, under the LINEX loss function, the Bayes estimator of function $v = v(\theta_1, \tau)$ can be written as

$$\hat{v}_{B_{LINEX}} = -\frac{1}{\gamma}\log\Big(\frac{\int_{c_1}^{d_1}\int_0^\infty \mathrm{e}^{-\gamma v(\theta_1,\tau)} L^*(\theta_1,\tau|S_n)\xi(\theta_1,\tau)\mathrm{d}\theta_1\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_0^\infty L^*(\theta_1,\tau|S_n)\xi(\theta_1,\tau)\mathrm{d}\theta_1\mathrm{d}\tau}\Big). \tag{7.6}$$

Then the Bayes estimator of function $v = v(\theta_1, \tau)$, under the SE loss function, is approximately

$$\hat{u}_{B_{SE}} = \frac{\int_{c_1}^{d_1}\int_0^\infty u(\theta_1,\tau)\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon^*_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\theta_1\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_0^\infty \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon^*_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\theta_1\mathrm{d}\tau}, \tag{7.7}$$

where $u = v$. Under the LINEX loss function,

$$\hat{u}_{B_{LINEX}} = -\frac{1}{\gamma}\log\Big(\frac{\int_{c_1}^{d_1}\int_0^\infty u(\theta_1,\tau)\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon^*_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\theta_1\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_0^\infty \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon^*_{t-1})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}}\mathrm{d}\theta_1\mathrm{d}\tau}\Big), \tag{7.8}$$

where $u = \mathrm{e}^{-\gamma v}$. Since we cannot find analytical expressions for $\hat{u}_{B_{SE}}$ and $\hat{u}_{B_{LINEX}}$, we apply Lindley's approximation for the ratios of two integrals. For the two parameters case, Lindley's approximation leads to

$$\begin{aligned}\hat{u}_{B_{SE}} &= u + \frac{1}{2}\sum_{i,j=1}^{2}(u_{ij}+2u_i\rho_j)\sigma_{ij} + \frac{1}{2}(L_{30}[u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}] \\ &+ L_{21}[3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22}+2\sigma_{12}^2)] \\ &+ L_{12}[u_1(\sigma_{11}\sigma_{22}+2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}] \\ &+ L_{03}[u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2]),\end{aligned} \tag{7.9}$$

and

$$\begin{aligned}
e^{-\gamma \hat{u}_{B_{LINEX}}} &= u + \frac{1}{2}\sum_{i,j=1}^{2}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}(L_{30}[u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}] \\
&+ L_{21}[3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)] \\
&+ L_{12}[u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}] \\
&+ L_{03}[u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2]), \qquad (7.10)
\end{aligned}$$

where $u = u(\theta_1, \tau)$, $u_1 = \frac{\partial u}{\partial \theta_1}$, $u_2 = \frac{\partial u}{\partial \tau}$, $u_{11} = \frac{\partial^2 u}{\partial \theta_1^2}$, $u_{12} = u_{21} = \frac{\partial^2 u}{\partial \theta_1 \partial \tau}$, $u_{22} = \frac{\partial^2 u}{\partial \tau^2}$, $L(\theta_1, \tau)$ is the logarithm of the approximate likelihood function, $\rho(\phi_1, \tau)$ is the logarithm of the joint prior density function, $L_{30} = \frac{\partial^3 L}{\partial \theta_1^3}$, $L_{21} = \frac{\partial^3 L}{\partial \theta_1^2 \partial \tau}$, $L_{12} = \frac{\partial^3 L}{\partial \theta_1 \partial \tau^2}$, $L_{03} = \frac{\partial^3 L}{\partial \tau^3}$, $\rho_1 = \frac{\partial \rho}{\partial \theta_1}$, $\rho_2 = \frac{\partial \rho}{\partial \tau}$ and σ_{ij} is the (i, j)th element of the inverse of the matrix

$$E = \begin{pmatrix} -\frac{\partial^2 L}{\partial \theta_1^2} & -\frac{\partial^2 L}{\partial \theta_1 \partial \tau} \\ -\frac{\partial^2 L}{\partial \theta_1 \partial \tau} & -\frac{\partial^2 L}{\partial \tau_1^2} \end{pmatrix}$$

all evaluated at the ML estimates of the parameters. For the prior distribution

$$\xi(\theta_1, \tau) \propto \tau^{\alpha-1} e^{-\tau\beta - \frac{(\theta_1 - \mu_1)^2}{2\sigma_1^2}}$$

we have

$$\rho = \log(\xi(\theta_1, \tau)) = constant + (\alpha - 1)\log(\tau) - \tau\beta - \frac{(\theta_1 - \mu_1)^2}{2\sigma_1^2},$$
$$\hat{\rho_1} = -\frac{\hat{\theta_1} - \mu_1}{\sigma_1^2}, \hat{\rho_2} = \frac{\alpha - 1}{\hat{\tau}} - \beta.$$

From Equation 7.5,

$$L_{10} = \frac{\partial L}{\partial \theta_1} = -\tau \sum_{t=1}^{n}(Y_t + \theta_1 \epsilon_{t-1}^*)\epsilon_{t-1}^*,$$
$$L_{01} = \frac{\partial L}{\partial \tau} = \frac{n}{2\tau} - \frac{1}{2}\sum_{t=1}^{n}(Y_t + \theta_1 \epsilon_{t-1}^*)^2.$$

Then the ML estimators of θ_1 and τ are

$$\hat{\theta}_1 = -\frac{\sum_{t=1}^{n} Y_t \epsilon_{t-1}^*}{\sum_{t=1}^{n} (\epsilon_{t-1}^*)^2}, \quad \hat{\tau} = \frac{n}{\sum_{t=1}^{n} (Y_t + \hat{\theta}_1 \epsilon_{t-1}^*)^2}$$

and

$$L_{20} = \frac{\partial^2 L}{\partial \theta_1^2} = -\tau \sum_{t=1}^{n} (\epsilon_{t-1}^*)^2,$$
$$L_{11} = \frac{\partial^2 L}{\partial \theta_1 \partial \tau} = -\sum_{t=1}^{n} (Y_t + \theta_1 \epsilon_{t-1}^*) \epsilon_{t-1}^*,$$
$$L_{02} = \frac{\partial^2 L}{\partial \tau^2} = -\frac{n}{2\tau^2}.$$

Hence,

$$\hat{L}_{30} = \frac{\partial^3 L}{\partial \theta_1^3} = 0, \quad \hat{L}_{21} = \frac{\partial^3 L}{\partial \theta_1^2 \partial \tau} = -\sum_{t=1}^{n} (\epsilon_{t-1}^*)^2,$$
$$\hat{L}_{12} = \frac{\partial^3 L}{\partial \theta_1 \partial \tau^2} = 0, \quad \hat{L}_{03} = \frac{\partial^3 L}{\partial \tau^3} = \frac{n}{\hat{\tau}^3}.$$

The matrix

$$E = \begin{pmatrix} \hat{\tau} \sum_{t=1}^{n} (\epsilon_{t-1}^*)^2 & \sum_{t=1}^{n} (Y_t + \hat{\theta}_1 \epsilon_{t-1}^*) \epsilon_{t-1}^* \\ \sum_{t=1}^{n} (Y_t + \hat{\theta}_1 \epsilon_{t-1}^*) \epsilon_{t-1}^* & \frac{n}{2\hat{\tau}^2} \end{pmatrix} = \begin{pmatrix} \hat{\tau} \sum_{t=1}^{n} (\epsilon_{t-1}^*)^2 & 0 \\ 0 & \frac{n}{2\hat{\tau}^2} \end{pmatrix}. \quad (7.11)$$

Its inverse is

$$E^{-1} = \begin{pmatrix} \frac{1}{\hat{\tau} \sum_{t=1}^{n} (\epsilon_{t-1}^*)^2} & 0 \\ 0 & \frac{2\hat{\tau}^2}{n} \end{pmatrix}. \quad (7.12)$$

Hence,

$$\hat{\sigma}_{11} = \frac{1}{\hat{\tau} \sum_{t=1}^{n} (\epsilon_{t-1}^*)^2}, \quad \hat{\sigma}_{12} = \hat{\sigma}_{21} = 0, \quad \hat{\sigma}_{22} = \frac{2\hat{\tau}^2}{n}.$$

Therefore, under the SE loss function, we obtain the following results.

Proposition 7.1. *Under the SE loss function, the approximate Bayes estimator of the parameter θ_1 is*

$$\hat{\theta}_{1(B_{SE})} = \hat{\theta}_1 + \hat{\rho}_1 \hat{\sigma}_{11}. \tag{7.13}$$

Proof. We use $u(\theta_1, \tau) = \theta_1$. Then

$$u_1 = 1, u_2 = u_{11} = u_{12} = u_{21} = u_{22} = 0$$

and the result follows.

Proposition 7.2. *Under the SE loss function, the approximate Bayes estimator of τ is*

$$\hat{\tau}_{B_{SE}} = \hat{\tau} + \hat{\rho}_2 \hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{21}\hat{\sigma}_{11}\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{03}\hat{\sigma}_{22}^2. \tag{7.14}$$

Proof. We use $u(\theta_1, \tau) = \tau$. Then

$$u_2 = 1, u_2 = u_{11} = u_{12} = u_{21} = u_{22} = 0$$

and the result follows.

Finally, to get the Bayes estimator of one-step ahead forecast, we use

$$u(\theta_1, \tau) = \mathrm{E}(Y_{n+1} | S_n, \theta_1, \tau) = -\theta_1 \epsilon_n^*. \tag{7.15}$$

Proposition 7.3. *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\hat{W}_{1(B_{SE})} = -\hat{\theta}_{1(B_{SE})} \epsilon_n^*. \tag{7.16}$$

Proof. Substitute Equation 7.15 to Equation 7.4, then

$$\begin{aligned}\hat{W}_{1(B_{SE})} &= \frac{\int_{c_1}^{d_1}\int_0^\infty -\theta_1\epsilon_n^* L^*(\theta_1,\tau|S_n)\xi(\theta_1,\tau)\mathrm{d}\theta_1\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_0^\infty L^*(\theta_1,\tau|S_n)\xi(\theta_1,\tau)\mathrm{d}\theta_1\mathrm{d}\tau}\\ &= -\epsilon_n^*\frac{\int_{c_1}^{d_1}\int_0^\infty \theta_1 L^*(\theta_1,\tau|S_n)\xi(\theta_1,\tau)\mathrm{d}\theta_1\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_0^\infty L^*(\theta_1,\tau|S_n)\xi(\theta_1,\tau)\mathrm{d}\theta_1\mathrm{d}\tau}\\ &= -\hat{\theta}_{1(B_{SE})}\epsilon_n^*.\end{aligned}$$

Under the LINEX loss function, we get the following results

Proposition 7.4. *Under the LINEX loss function, the approximate Bayes estimator of the parameter θ_1 is*

$$\hat{\theta}_{1(B_{LINEX})} = \hat{\theta}_1 - \frac{1}{\gamma}\log\Big(1+\frac{\gamma^2}{2}\hat{\sigma}_{11} - \gamma\hat{\rho}_1\hat{\sigma}_{11}\Big). \tag{7.17}$$

Proof. We substitute $u(\theta_1,\tau) = \mathrm{e}^{-\gamma\theta_1}$ to Equation 7.8. Then

$$u_1 = -\gamma\mathrm{e}^{-\gamma\theta_1}, u_{11} = \gamma^2\mathrm{e}^{-\gamma\theta_1}, u_2 = u_{12} = u_{21} = u_{22} = 0$$

and the result follows.

Proposition 7.5. *Under the LINEX loss function, the approximate Bayes estimator of the parameter τ is*

$$\hat{\tau}_{B_{LINEX}} = \hat{\tau} - \frac{1}{\gamma}\log\Big(1+\frac{\gamma^2}{2}\hat{\sigma}_{22} - \gamma\Big(\hat{\rho}_2\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{21}\hat{\sigma}_{11}\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{03}\hat{\sigma}_{22}^2\Big)\Big). \tag{7.18}$$

Proof. We substitute $u(\theta_1,\tau) = \mathrm{e}^{-\gamma\tau}$ to Equation 7.8. Then

$$u_2 = -\gamma\mathrm{e}^{-\gamma\tau}, u_{22} = \gamma^2\mathrm{e}^{-\gamma\tau}, u_1 = u_{11} = u_{12} = u_{21} = 0$$

and the result follows.

Proposition 7.6. *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\begin{aligned}\hat{W}_{1(B_{LINEX})} &= -\hat{\theta}_1\epsilon_n^* - \frac{\gamma}{2\hat{\tau}} - \frac{1}{\gamma}\log\Big(1 + \frac{1}{2}\Big[\gamma^2(\epsilon_n^*)^2\hat{\sigma}_{11} + \frac{\gamma^2}{\hat{\tau}^3}\Big(1 + \frac{\gamma^2}{4\hat{\tau}}\Big)\hat{\sigma}_{22}\Big] \\ &+ \gamma\epsilon_n^*\hat{\rho}_1\hat{\sigma}_{11} - \frac{\gamma^2}{2\hat{\tau}^2}\hat{\rho}_2\hat{\sigma}_{22} - \frac{\gamma^2}{4\hat{\tau}^2}\Big(\hat{L}_{21}\hat{\sigma}_{11}\hat{\sigma}_{22} + \hat{L}_{03}\hat{\sigma}_{22}^2\Big)\Big).\end{aligned} \tag{7.19}$$

Proof. We set $u(\theta_1, \tau) = \mathrm{E}(\mathrm{e}^{-\gamma Y_{n+1}}|S_n, \theta_1, \tau) = \mathrm{e}^{\gamma\theta_1\epsilon_n^* + \frac{\gamma^2}{2\tau}}$ and substitute it to Equation 7.8. Then

$$u_1 = \gamma\epsilon_n^*\mathrm{e}^{\gamma\theta_1\epsilon_n^* + \frac{\gamma^2}{2\tau}}, \quad u_2 = -\frac{\gamma^2}{2\tau^2}\mathrm{e}^{\gamma\theta_1\epsilon_n^* + \frac{\gamma^2}{2\tau}},$$
$$u_{11} = \gamma^2(\epsilon_n^*)^2\mathrm{e}^{\gamma\theta_1\epsilon_n^* + \frac{\gamma^2}{2\tau}}, \quad u_{12} = u_{21} = -\frac{\gamma^3\epsilon_n^*}{2\tau^2}\mathrm{e}^{\gamma\theta_1\epsilon_n^* + \frac{\gamma^2}{2\tau}}, \quad u_{22} = \frac{\gamma^2}{\tau^3}\Big(1 + \frac{\gamma^2}{4\tau}\Big)\mathrm{e}^{\gamma\theta_1\epsilon_n^* + \frac{\gamma^2}{2\tau}}$$

and the result follows.

Bayes estimation using the LINEX loss function involves the logarithm of the moment generating function which is approximated using Lindley's method. However, Lindley's approximation is of order n^{-1} and includes only three terms of the Taylor series expansion (see Section 3.6), thus the approximated value of the moment generating function is not guaranteed to be positive. Therefore, for a given LINEX loss function parameter γ, a small proportion of the Bayes estimates under LINEX loss function is expected to be undefined.

7.1.2. MA(2) model

In the $q = 2$ case, under the SE loss function, the Bayes estimator of function $v = v(\theta_1, \theta_2, \tau)$ can be written as

$$\hat{v}_{B_{SE}} = \frac{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^\infty v(\theta_1, \theta_2, \tau)L^*(\theta_1, \theta_2, \tau|S_n)\xi(\theta_1, \theta_2, \tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^\infty L^*(\theta_1, \theta_2, \tau|S_n)\xi(\theta_1, \theta_2, \tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau} \tag{7.20}$$

where

$$L^*(\theta,\tau|S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon^*_{t-1}+\theta_2\epsilon^*_{t-2})^2} \tag{7.21}$$

Similarly, under the LINEX loss function, the Bayes estimator of function $v = v(\theta_1, \theta_2, \tau)$ can be written as

$$\begin{aligned} \hat{v}_{B_{LINEX}} &= -\frac{1}{\gamma}\log\Big(\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^\infty \mathrm{e}^{-\gamma v(\theta_1,\theta_2,\tau)} L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau \\ &/ \int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^\infty L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau\Big). \end{aligned} \tag{7.22}$$

Then the Bayes estimator of function $v = v(\theta_1, \theta_2, \tau)$, under the SE loss function, is approximately

$$\begin{aligned} \hat{u}_{B_{SE}} &= \int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^\infty u(\theta_1,\theta_2,\tau)\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\sum_{i=1}^{2}\theta_i\epsilon^*_{t-i})^2} \\ &\times \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\sum_{i=1}^{2}\frac{(\theta_i-\mu_i)^2}{2\sigma_i^2}}\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau \\ &/ \int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^\infty \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\sum_{i=1}^{2}\theta_i\epsilon^*_{t-i})^2} \\ &\times \tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\sum_{i=1}^{2}\frac{(\theta_i-\mu_i)^2}{2\sigma_i^2}}\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau, \end{aligned} \tag{7.23}$$

where $u = v$. Under the LINEX loss function,

$$\begin{aligned} \mathrm{e}^{-\gamma \hat{u}_{B_{LINEX}}} &= \int_{c_1}^{d_1} \int_{c_2}^{d_2} \int_0^{\infty} u(\theta_1, \theta_2, \tau) \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2} \sum_{t=1}^{n} (Y_t + \sum_{i=1}^{2} \theta_i \epsilon_{t-i}^*)^2} \\ &\times \tau^{\alpha-1} \mathrm{e}^{-\tau\beta - \sum_{i=1}^{2} \frac{(\theta_i - \mu_i)^2}{2\sigma_i^2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\tau \\ &/ \int_{c_1}^{d_1} \int_{c_2}^{d_2} \int_0^{\infty} \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2} \sum_{t=1}^{n} (Y_t + \sum_{i=1}^{2} \theta_i \epsilon_{t-i}^*)^2} \\ &\times \tau^{\alpha-1} \mathrm{e}^{-\tau\beta - \sum_{i=1}^{2} \frac{(\theta_i - \mu_i)^2}{2\sigma_i^2}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\tau, \end{aligned} \quad (7.24)$$

where $u = \mathrm{e}^{-\gamma v}$. We cannot find analytical expressions for $\hat{u}_{B_{SE}}$ and $\hat{u}_{B_{LINEX}}$. Thus, we apply Lindley's approximation for the ratios of two integrals. For the three parameters case, Lindley's approximation leads to

$$\begin{aligned} \hat{u}_{B_{SE}} &= u + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) \\ &+ \frac{1}{2}\Big[B_1(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B_2(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\ &+ B_3(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\Big] \end{aligned} \quad (7.25)$$

and

$$\begin{aligned} \mathrm{e}^{-\gamma \hat{u}_{B_{LINEX}}} &= u + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) \\ &+ \frac{1}{2}\Big[B_1(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B_2(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \\ &+ B_3(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\Big] \end{aligned} \quad (7.26)$$

all evaluated at the ML estimates of the parameters, where

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \quad i = 1, 2, 3,$$
$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}, \quad a_5 = \frac{1}{2}\Big(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}\Big),$$

$$B_1 = \sigma_{11}L_{300} + 2\sigma_{12}L_{210} + 2\sigma_{13}L_{201} + 2\sigma_{23}L_{111} + \sigma_{22}L_{120} + \sigma_{33}L_{102},$$
$$B_2 = \sigma_{11}L_{210} + 2\sigma_{12}L_{120} + 2\sigma_{13}L_{111} + 2\sigma_{23}L_{021} + \sigma_{22}L_{030} + \sigma_{33}L_{012},$$
$$B_3 = \sigma_{11}L_{201} + 2\sigma_{12}L_{111} + 2\sigma_{13}L_{102} + 2\sigma_{23}L_{012} + \sigma_{22}L_{021} + \sigma_{33}L_{003},$$

$u = u(\theta_1, \theta_2, \tau)$, $u_1 = \frac{\partial u}{\partial \theta_1}$, $u_2 = \frac{\partial u}{\partial \theta_2}$, $u_3 = \frac{\partial u}{\partial \tau}$, $u_{11} = \frac{\partial^2 u}{\partial \theta_1^2}$, $u_{22} = \frac{\partial^2 u}{\partial \theta_2^2}$, $u_{33} = \frac{\partial^2 u}{\partial \tau^2}$, $u_{12} = u_{21} = \frac{\partial^2 u}{\partial \theta_1 \partial \theta_2}$, $u_{13} = u_{31} = \frac{\partial^2 u}{\partial \theta_1 \partial \tau}$, $u_{23} = u_{32} = \frac{\partial^2 u}{\partial \theta_2 \partial \tau}$, $L(\theta_1, \theta_2, \tau)$ is the logarithm of the approximate likelihood function, $\rho(\theta_1, \theta_2, \tau)$ is the logarithm of the joint prior density function, $L_{300} = \frac{\partial^3 L}{\partial \theta_1^3}$, $L_{210} = \frac{\partial^3 L}{\partial \theta_1^2 \partial \theta_2}$, $L_{201} = \frac{\partial^3 L}{\partial \theta_1^2 \partial \tau}$, $L_{120} = \frac{\partial^3 L}{\partial \theta_1 \partial \theta_2^2}$, $L_{102} = \frac{\partial^3 L}{\partial \theta_1 \partial \tau^2}$, $L_{021} = \frac{\partial^3 L}{\partial \theta_2^2 \partial \tau}$, $L_{012} = \frac{\partial^3 L}{\partial \theta_2 \partial \tau^2}$, $L_{030} = \frac{\partial^3 L}{\partial \theta_2^3}$, $L_{003} = \frac{\partial^3 L}{\partial \tau^3}$, $\rho_1 = \frac{\partial \rho}{\partial \theta_1}$, $\rho_2 = \frac{\partial \rho}{\partial \theta_2}$, $\rho_3 = \frac{\partial \rho}{\partial \tau}$ and σ_{ij} is the (i, j)th element of the inverse of the matrix

$$E = \begin{pmatrix} -\frac{\partial^2 L}{\partial \theta_1^2} & -\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} & -\frac{\partial^2 L}{\partial \theta_1 \partial \tau} \\ -\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} & -\frac{\partial^2 L}{\partial \theta_2^2} & -\frac{\partial^2 L}{\partial \theta_2 \partial \tau} \\ -\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} & -\frac{\partial^2 L}{\partial \theta_2 \partial \tau} & -\frac{\partial^2 L}{\partial \tau_1^2} \end{pmatrix}.$$

For the prior distribution

$$\xi(\theta_1, \theta_2, \tau) \propto \tau^{\alpha-1} \mathrm{e}^{-\tau\beta - \frac{(\theta_1-\mu_1)^2}{2\sigma_1^2} - \frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}}$$

we have

$$\rho = \log(\xi(\theta_1, \theta_2, \tau)) = constant + (\alpha - 1)\log(\tau) - \tau\beta - \frac{(\theta_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(\theta_2 - \mu_2)^2}{2\sigma_2^2},$$
$$\hat{\rho}_1 = -\frac{\hat{\theta}_1 - \mu_1}{\sigma_1^2}, \quad \hat{\rho}_2 = -\frac{\hat{\theta}_2 - \mu_2}{\sigma_2^2}, \quad \hat{\rho}_3 = \frac{\alpha - 1}{\hat{\tau}} - \beta.$$

From Equation 7.21,

$$L_{100} = \frac{\partial L}{\partial \theta_1} = -\tau \sum_{t=1}^{n} (Y_t + \theta_1 \epsilon_{t-1}^* + \theta_2 \epsilon_{t-2}^*) \epsilon_{t-1}^*,$$
$$L_{010} = \frac{\partial L}{\partial \theta_2} = -\tau \sum_{t=1}^{n} (Y_t + \theta_1 \epsilon_{t-1}^* + \theta_2 \epsilon_{t-2}^*) \epsilon_{t-2}^*,$$

$$L_{001} = \frac{\partial L}{\partial \tau} = \frac{n}{2\tau} - \frac{1}{2}\sum_{t=1}^{n}(Y_t + \theta_1 \epsilon_{t-1}^* + \theta_2 \epsilon_{t-2}^*)^2.$$

Then the ML estimators of θ_1, θ_2 and τ are

$$\hat{\theta}_1 = -\frac{(\sum_{t=1}^n Y_t \epsilon_{t-1}^*)(\sum_{t=1}^n (\epsilon_{t-2}^*)^2) - (\sum_{t=1}^n Y_t \epsilon_{t-2}^*)(\sum_{t=1}^n \epsilon_{t-1}^* \epsilon_{t-2}^*)}{(\sum_{t=1}^n (\epsilon_{t-1}^*)^2)(\sum_{t=1}^n (\epsilon_{t-2}^*)^2) - (\sum_{t=1}^n \epsilon_{t-1}^* \epsilon_{t-2}^*)^2},$$
$$\hat{\theta}_2 = -\frac{(\sum_{t=1}^n Y_t \epsilon_{t-2}^*)(\sum_{t=1}^n (\epsilon_{t-1}^*)^2) - (\sum_{t=1}^n Y_t \epsilon_{t-1}^*)(\sum_{t=1}^n \epsilon_{t-1}^* \epsilon_{t-2}^*)}{(\sum_{t=1}^n (\epsilon_{t-1}^*)^2)(\sum_{t=1}^n (\epsilon_{t-2}^*)^2) - (\sum_{t=1}^n \epsilon_{t-1}^* \epsilon_{t-2}^*)^2},$$
$$\hat{\tau} = \frac{n}{\sum_{t=1}^n (Y_t + \hat{\theta}_1 \epsilon_{t-1}^* + \hat{\theta}_2 \epsilon_{t-2}^*)^2}$$

and

$$L_{200} = \frac{\partial^2 L}{\partial \theta_1^2} = -\tau \sum_{t=1}^{n} (\epsilon_{t-1}^*)^2, \quad L_{110} = \frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} = -\tau \sum_{t=1}^{n} \epsilon_{t-1}^* \epsilon_{t-2}^*,$$
$$L_{101} = \frac{\partial^2 L}{\partial \theta_1 \partial \tau} = -\sum_{t=1}^{n} (Y_t + \theta_1 \epsilon_{t-1}^* + \theta_2 \epsilon_{t-2}^*)\epsilon_{t-1}^*, \quad L_{020} = \frac{\partial^2 L}{\partial \theta_2^2} = -\tau \sum_{t=1}^{n} (\epsilon_{t-2}^*)^2,$$
$$L_{011} = \frac{\partial^2 L}{\partial \theta_2 \partial \tau} = -\sum_{t=1}^{n} (Y_t + \theta_1 \epsilon_{t-1}^* + \theta_2 \epsilon_{t-2}^*)\epsilon_{t-2}^*, \quad L_{002} = \frac{\partial^2 L}{\partial \tau^2} = -\frac{n}{2\tau^2}.$$

Hence,

$$\hat{L}_{300} = \hat{L}_{210} = \hat{L}_{120} = \hat{L}_{102} = \hat{L}_{030} = \hat{L}_{012} = 0,$$
$$\hat{L}_{201} = -\sum_{t=1}^{n} (\epsilon_{t-1}^*)^2, \quad \hat{L}_{111} = -\sum_{t=1}^{n} \epsilon_{t-1}^* \epsilon_{t-2}^*, \quad \hat{L}_{021} = -\sum_{t=1}^{n} (\epsilon_{t-2}^*)^2, \quad \hat{L}_{003} = \frac{n}{\hat{\tau}^3}.$$

The matrix

$$E = \begin{pmatrix} -\hat{L}_{200} & -\hat{L}_{110} & -\hat{L}_{101} \\ -\hat{L}_{110} & -\hat{L}_{020} & -\hat{L}_{011} \\ -\hat{L}_{101} & -\hat{L}_{011} & -\hat{L}_{002} \end{pmatrix} = \begin{pmatrix} -\hat{L}_{200} & -\hat{L}_{110} & 0 \\ -\hat{L}_{110} & -\hat{L}_{020} & 0 \\ 0 & 0 & -\hat{L}_{002} \end{pmatrix}. \tag{7.27}$$

Its inverse is

$$E^{-1} = \frac{1}{D} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}, \tag{7.28}$$

where

$$D := \det(E) = -\hat{L}_{200}\hat{L}_{020}\hat{L}_{002} + \hat{L}_{002}\hat{L}_{110}^2,$$
$$E_{11} = \hat{L}_{020}\hat{L}_{002}, \quad E_{12} = E_{21} = -\hat{L}_{002}\hat{L}_{110},$$
$$E_{13} = E_{31} = 0, \quad E_{22} = \hat{L}_{200}\hat{L}_{002},$$
$$E_{23} = E_{32} = 0, \quad E_{33} = \hat{L}_{200}\hat{L}_{020} - \hat{L}_{110}^2.$$

Therefore,

$$\hat{\sigma}_{11} = \frac{E_{11}}{D}, \quad \hat{\sigma}_{12} = \hat{\sigma}_{21} = \frac{E_{12}}{D}, \quad \hat{\sigma}_{13} = \hat{\sigma}_{31} = \frac{E_{13}}{D} = 0,$$
$$\hat{\sigma}_{22} = \frac{E_{12}}{D}, \quad \hat{\sigma}_{23} = \hat{\sigma}_{32} = \frac{E_{23}}{D} = 0, \quad \hat{\sigma}_{33} = \frac{E_{33}}{D}.$$

Therefore, under the SE loss function, we obtain the following results

Proposition 7.7. *Under the SE loss function, the approximate Bayes estimator of the parameter θ_1 is*

$$\hat{\theta}_{1(B_{SE})} = \hat{\theta}_1 + \hat{\rho}_1\hat{\sigma}_{11} + \hat{\rho}_2\hat{\sigma}_{12}. \tag{7.29}$$

Proof. We use $u(\theta_1, \theta_2, \tau) = \theta_1$. Then

$$u_1 = 1, u_2 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 7.8. *Under the SE loss function, the approximate Bayes estimator of the param-*

eter θ_2 is

$$\hat{\theta}_{2(B_{SE})} = \hat{\theta}_2 + \hat{\rho}_1\hat{\sigma}_{21} + \hat{\rho}_2\hat{\sigma}_{22}. \tag{7.30}$$

Proof. We use $u(\theta_1, \theta_2, \tau) = \theta_2$. Then

$$u_2 = 1, u_1 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 7.9. *Under the SE loss function, the approximate Bayes estimator of τ is*

$$\hat{\tau}_{B_{SE}} = \hat{\tau} + \hat{\rho}_3\hat{\sigma}_{33} + \frac{1}{2}B_3\hat{\sigma}_{33}, \tag{7.31}$$

where

$$B_3 = \hat{L}_{201}\hat{\sigma}_{11} + 2\hat{L}_{111}\hat{\sigma}_{12} + \hat{L}_{021}\hat{\sigma}_{22} + \hat{L}_{003}\hat{\sigma}_{33}. \tag{7.32}$$

Proof. We use $u(\theta_1, \theta_2, \tau) = \tau$. Then

$$u_3 = 1, u_1 = u_2 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Finally, to get the Bayes estimator of one-step ahead forecast, we use

$$u(\theta_1, \theta_2, \tau) = \mathrm{E}(Y_{n+1}|S_n, \theta_1, \theta_2, \tau) = -\theta_1\epsilon_n^* - \theta_2\epsilon_{n-1}^*. \tag{7.33}$$

Proposition 7.10. *Under the SE loss function, the Bayes estimator of the one-step ahead* *forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\hat{W}_{1(B_{SE})} = -\hat{\theta}_{1(B_{SE})}\epsilon_n^* - \hat{\theta}_{2(B_{SE})}\epsilon_{n-1}^*. \tag{7.34}$$

Proof. Substitute Equation 7.33 to Equation 7.20, then

$$\begin{aligned}
\hat{W}_{1(B_{SE})} &= \frac{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^{\infty}(-\theta_1\epsilon_n^* - \theta_2\epsilon_{n-1}^*)L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^{\infty}L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau} \\
&= -\epsilon_n^*\Big(\frac{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^{\infty}\theta_1 L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^{\infty}L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau}\Big) \\
&- \epsilon_{n-1}^*\Big(\frac{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^{\infty}\theta_2 L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau}{\int_{c_1}^{d_1}\int_{c_2}^{d_2}\int_0^{\infty}L^*(\theta_1,\theta_2,\tau|S_n)\xi(\theta_1,\theta_2,\tau)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\tau}\Big) \\
&= -\hat{\theta}_{1(B_{SE})}\epsilon_n^* - \hat{\theta}_{2(B_{SE})}\epsilon_{n-1}^*.
\end{aligned}$$

Under the LINEX loss function, we get the following results

Proposition 7.11. *Under the LINEX loss function, the approximate Bayes estimator of the parameter θ_1 is*

$$\hat{\theta}_{1(B_{LINEX})} = \hat{\theta}_1 - \frac{1}{\gamma}\log\Big(1 + \frac{\gamma^2}{2}\hat{\sigma}_{11} - \gamma\big(\hat{\rho}_1\hat{\sigma}_{11} + \hat{\rho}_2\hat{\sigma}_{12}\big)\Big). \tag{7.35}$$

Proof. We substitute $u(\theta_1, \theta_2, \tau) = \mathrm{e}^{-\gamma\theta_1}$ to Equation 7.26. Then

$$u_1 = -\gamma\mathrm{e}^{-\gamma\theta_1}, u_{11} = \gamma^2\mathrm{e}^{-\gamma\theta_1},$$
$$u_2 = u_3 = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 7.12. *Under the LINEX loss function, the approximate Bayes estimator of the parameter θ_2 is*

$$\hat{\theta}_{2(B_{LINEX})} = \hat{\theta}_2 - \frac{1}{\gamma}\log\Big(1 + \frac{\gamma^2}{2}\hat{\sigma}_{22} - \gamma\Big(\hat{\rho}_1\hat{\sigma}_{21} + \hat{\rho}_2\hat{\sigma}_{22}\Big)\Big). \quad (7.36)$$

Proof. We substitute $u(\theta_1, \theta_2, \tau) = \mathrm{e}^{-\gamma\theta_2}$ to Equation 7.26. Then

$$u_2 = -\gamma \mathrm{e}^{-\gamma\theta_2}, u_{22} = \gamma^2 \mathrm{e}^{-\gamma\theta_2},$$

$$u_1 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{23} = u_{31} = u_{32} = u_{33} = 0$$

and the result follows.

Proposition 7.13. *Under the LINEX loss function, the approximate Bayes estimator of the parameter τ is*

$$\hat{\tau}_{B_{LINEX}} = \hat{\tau} - \frac{1}{\gamma}\log\Big(1 + \frac{\gamma^2}{2}\hat{\sigma}_{33} - \gamma\Big(\hat{\rho}_3\hat{\sigma}_{33} + \frac{1}{2}B_3\hat{\sigma}_{33}\Big)\Big), \quad (7.37)$$

where B_3 is defined by Equation 7.32.

Proof. We substitute $u(\theta_1, \theta_2, \tau) = \mathrm{e}^{-\gamma\tau}$ to Equation 7.26. Then

$$u_3 = -\gamma \mathrm{e}^{-\gamma\tau}, u_{33} = \gamma^2 \mathrm{e}^{-\gamma\tau},$$

$$u_2 = u_3 = u_{11} = u_{12} = u_{13} = u_{21} = u_{22} = u_{23} = u_{31} = u_{32} = 0$$

and the result follows.

Proposition 7.14. *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is*

$$\begin{aligned}\hat{W}_{1(B_{LINEX})} &= -\hat{\theta}_1\epsilon_n^* - \hat{\theta}_2\epsilon_{n-1}^* - \frac{\gamma}{2\hat{\tau}} - \frac{1}{\gamma}\log\Big(1 + \Big(\gamma\epsilon_n^* a_1 + \gamma\epsilon_{n-1}^* a_2 - \frac{\gamma^2}{2\hat{\tau}^2}a_3 + a_4 \\ &+ a_5\Big) - B_3\frac{\gamma^2}{4\hat{\tau}^2}\hat{\sigma}_{33}\Big) \quad (7.38)\end{aligned}$$

where B_3 is defined by Equation 7.32.

$$a_i = \hat{\rho}_1\hat{\sigma}_{i1} + \hat{\rho}_2\hat{\sigma}_{i2} + \hat{\rho}_3\hat{\sigma}_{i3}, \quad i = 1,2,3,$$
$$a_4 = \gamma^2\epsilon_n^*\epsilon_{n-1}^*\hat{\sigma}_{12} - \frac{\gamma^3\epsilon_n^*}{2\hat{\tau}^2}\hat{\sigma}_{13} - \frac{\gamma^3\epsilon_{n-1}^*}{2\hat{\tau}^2}\hat{\sigma}_{23},$$
$$a_5 = \frac{1}{2}\Big(\gamma^2(\epsilon_n^*)^2\hat{\sigma}_{11} + \gamma^2(\epsilon_{n-1}^*)^2\hat{\sigma}_{22} + \frac{\gamma^2}{\hat{\tau}^3}\Big(1 + \frac{\gamma^2}{4\hat{\tau}}\Big)\hat{\sigma}_{33}\Big)\Big).$$

Proof. We set $u(\theta_1,\theta_2,\tau) = \mathrm{E}(\mathrm{e}^{-\gamma Y_{n+1}}|S_n,\theta_1,\theta_2,\tau) = \mathrm{e}^{\gamma(\theta_1\epsilon_n^*+\theta_2\epsilon_{n-1}^*)+\frac{\gamma^2}{2\tau}}$ and substitute it to Equation 7.26. Then

$$u_1 = \gamma\epsilon_n^* u, \quad u_2 = \gamma\epsilon_{n-1}^* u, \quad u_3 = -\frac{\gamma^2}{2\tau^2}u,$$
$$u_{11} = \gamma^2(\epsilon_n^*)^2 u, \quad u_{12} = u_{21} = \gamma^2\epsilon_n^*\epsilon_{n-1}^* u, \quad u_{13} = u_{31} = -\frac{\gamma^3\epsilon_n^*}{2\tau^2}u,$$
$$u_{22} = \gamma^2(\epsilon_{n-1}^*)^2 u, \quad u_{23} = u_{32} = -\frac{\gamma^3\epsilon_{n-1}^*}{2\tau^2}u, \quad u_{33} = \frac{\gamma^2}{\tau^3}\Big(1 + \frac{\gamma^2}{4\tau}\Big)u$$

and the result follows.

Bayes estimation using the LINEX loss function involves the logarithm of the moment generating function which is approximated using Lindley's method. However, Lindley's approximation is of order n^{-1} and includes only three terms of the Taylor series expansion (see Section 3.6), thus the approximated value of the moment generating function is not guaranteed to be positive. Therefore, for a given LINEX loss function parameter γ, a small proportion of the Bayes estimates under LINEX loss function is expected to be undefined.

7.2. FULL CONDITIONAL DISTRIBUTIONS OF PARAMETERS UNDER INDEPENDENT TRUNCATED NORMAL - GAMMA PRIOR

7.2.1. MA(1) model

In this section we obtain the full conditional distributions for MA(1) model parameters θ_1, τ and the one-step prediction Y_{n+1}. The following propositions provide the required distributions.

Proposition 7.15. *The full conditional distribution of Bayes estimator of θ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{-\tau \sum_{t=1}^{n} Y_t \epsilon_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=1}^{n} \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{7.39}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=1}^{n} \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{7.40}$$

Proof.

$$\begin{aligned}
\xi(\theta_1|\tau, S_n) &\propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta - \frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\sum_{t=1}^{n} Y_t^2 + 2\theta_1 \sum_{t=1}^{n} Y_t\epsilon_{t-1} + \theta_1^2 \sum_{t=1}^{n}\epsilon_{t-1}^2) - \frac{\theta_1^2 - 2\theta_1\mu_1 + \mu_1^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{1}{2}((\tau\sum_{t=1}^{n}\epsilon_{t-1}^2 + \frac{1}{\sigma_1^2})\theta_1^2 - 2(-\tau\sum_{t=1}^{n} Y_t\epsilon_{t-1} + \frac{\mu_1}{\sigma_1^2})\theta_1)} \\
&\propto \mathrm{e}^{-\frac{\left(\tau\sum_{t=1}^{n}\epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}\right)}{2}\left(\theta_1 - \frac{-\tau\sum_{t=1}^{n} Y_t\epsilon_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau\sum_{t=1}^{n}\epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned} \tag{7.41}$$

Proposition 7.16. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \alpha + \frac{n}{2} \tag{7.42}$$

and

$$\tilde{\beta} = \beta + \frac{\sum_{t=1}^{n}(Y_t + \theta_1\epsilon_{t-1})^2}{2}. \tag{7.43}$$

Proof.

$$\begin{aligned}
\xi(\tau|\theta_1, S_n) &\propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1})^2} \tau^{\alpha-1} \mathrm{e}^{-\tau\beta} \\
&= \tau^{\alpha+\frac{n}{2}-1} \mathrm{e}^{-\tau(\beta+\frac{1}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1})^2)}. \qquad (7.44)
\end{aligned}$$

Proposition 7.17. *The full conditional distribution of Bayes estimator of / Y_{n+1} given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}$ and precision τ, where*

$$\tilde{\mu} = -\theta_1\epsilon_n. \qquad (7.45)$$

Proof.

$$\begin{aligned}
Y_{n+1}|\theta_1, \tau, S_n &= (-\theta_1\epsilon_n + \epsilon_{n+1})|\theta_1, \tau, S_n \\
&= -\theta_1\epsilon_n + (\epsilon_{n+1}|\tau). \qquad (7.46)
\end{aligned}$$

7.2.2. MA(2) model

Proposition 7.18. *The full conditional distribution of Bayes estimator of θ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{-\tau(\sum_{t=1}^{n} Y_t\epsilon_{t-1} + \theta_2\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2}) + \frac{\mu_1}{\sigma_1^2}}{\tau\sum_{t=1}^{n}\epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}} \qquad (7.47)$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau\sum_{t=1}^{n}\epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}}. \qquad (7.48)$$

Proof.

$$
\begin{aligned}
\xi(\theta_1|\theta_2,\tau,S_n) &\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(\theta_1^2\epsilon_{t-1}^2+2\theta_1Y_t\epsilon_{t-1}+2\theta_1\theta_2\epsilon_{t-1}\epsilon_{t-2})}\mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\theta_1^2(\sum_{t=1}^{n}\epsilon_{t-1}^2)+2\theta_1(\sum_{t=1}^{n}Y_t\epsilon_{t-1})+2\theta_1\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\theta_1^2-2\mu_1\theta_1)}{2\sigma_1^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\theta_1^2(\tau(\sum_{t=1}^{n}\epsilon_{t-1}^2)+\frac{1}{\sigma_1^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\theta_1(-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-1})+\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_1}{\sigma_1^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=1}^{n}\epsilon_{t-1}^2)+\frac{1}{\sigma_1^2}}{2}\left(\theta_1-\frac{-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-1})+\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_1}{\sigma_1^2}}{\tau(\sum_{t=1}^{n}\epsilon_{t-1}^2)+\frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned}
$$

.

Proposition 7.19. *The full conditional distribution of Bayes estimator of θ_2 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_2$ and variance $\tilde{\sigma}_2^2$, where*

$$\tilde{\mu}_2 = \frac{-\tau(\sum_{t=1}^{n} Y_t\epsilon_{t-2} + \theta_1 \sum_{t=1}^{n} \epsilon_{t-1}\epsilon_{t-2}) + \frac{\mu_2}{\sigma_2^2}}{\tau \sum_{t=1}^{n} \epsilon_{t-2}^2 + \frac{1}{\sigma_2^2}} \tag{7.49}$$

and

$$\tilde{\sigma}_2^2 = \frac{1}{\tau \sum_{t=1}^{n} \epsilon_{t-2}^2 + \frac{1}{\sigma_2^2}}. \tag{7.50}$$

Proof.

$$
\begin{aligned}
\xi(\theta_2|\theta_1,\tau,S_n) &\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(\theta_2^2\epsilon_{t-2}^2+2\theta_2Y_t\epsilon_{t-2}+2\theta_1\theta_2\epsilon_{t-1}\epsilon_{t-2})}\mathrm{e}^{-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\theta_2^2(\sum_{t=1}^{n}\epsilon_{t-2}^2)+2\theta_2(\sum_{t=1}^{n}Y_t\epsilon_{t-2})+2\theta_1\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\theta_2^2-2\mu_2\theta_2)}{2\sigma_2^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\theta_2^2(\tau(\sum_{t=1}^{n}\epsilon_{t-2}^2)+\frac{1}{\sigma_2^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\theta_2(-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-2})+\theta_1(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_2}{\sigma_2^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=1}^{n}\epsilon_{t-2}^2)+\frac{1}{\sigma_2^2}}{2}\left(\theta_2-\frac{-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-2})+\theta_1(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_2}{\sigma_2^2}}{\tau(\sum_{t=1}^{n}\epsilon_{t-2}^2)+\frac{1}{\sigma_2^2}}\right)^2}.
\end{aligned}
$$

Proposition 7.20. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \alpha + \frac{n}{2} \tag{7.51}$$

and

$$\tilde{\beta} = \beta + \frac{\sum_{t=1}^{n}(Y_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2})^2}{2}. \tag{7.52}$$

Proof.

$$
\begin{aligned}
\xi(\tau|\theta_1,\theta_2,S_n) &\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{\alpha-1}\mathrm{e}^{-\tau\beta} \\
&= \tau^{\alpha+\frac{n}{2}-1}\mathrm{e}^{-\tau(\beta+\frac{1}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2)}.
\end{aligned} \tag{7.53}
$$

Proposition 7.21. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample*

$S_n = (Y_1, \ldots, Y_n)'$ *is normal with mean* $\tilde{\mu}$ *and precision* τ*, where*

$$\tilde{\mu} = -\theta_1\epsilon_n - \theta_2\epsilon_{n-1}. \tag{7.54}$$

Proof.

$$\begin{aligned} Y_{n+1}|\theta_1, \theta_2, \tau, S_n &= (-\theta_1\epsilon_n - \theta_2\epsilon_{n-1} + \epsilon_{n+1})|\theta_1, \theta_2, \tau, S_n \\ &= -\theta_1\epsilon_n - \theta_2\epsilon_{n-1} + (\epsilon_{n+1}|\tau). \end{aligned} \tag{7.55}$$

7.3. INDEPENDENT TRUNCATED NORMAL - IMPROPER PRIOR

In the model 7.1 we assume the parameters $\theta_i, i = 1, \ldots, q$, have independent truncated normal priors on intervals $(c_i, d_i), c_i, d_i \in \mathbb{R}$, for all $i = 1, \ldots, q$, respectively, with the parameters $\theta_1 \sim TN(\mu_1, \sigma_1^2), \ldots, \theta_q \sim TN(\mu_q, \sigma_q^2)$, and the precision has independent improper prior. In this model,

$$\xi(\theta, \tau) = \xi_1(\theta)\xi_2(\tau),$$

where the marginal prior density of τ is

$$\xi_2(\tau) \propto \frac{1}{\tau}, \tau > 0,$$

and the marginal prior density of θ is

$$\xi_1(\theta) \propto \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{q}(\frac{\theta_i - \mu_i}{\sigma_i})^2} = \mathrm{e}^{-\frac{1}{2}(\theta-\mu)'Q(\theta-\mu)}, \theta \in \mathbb{R}^q,$$

that is, $\xi_1(\theta) \sim N(\mu, Q^{-1})$. So the joint prior density function of parameters

$$\xi(\phi, \tau) \propto \tau^{-1} \mathrm{e}^{-\frac{1}{2}(\theta-\mu)' Q(\theta-\mu)}.$$

The approximate likelihood function for the model

$$L^*(\phi, \tau | S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*X\theta)},$$

where A^* is as defined in the Section 7.1. Employing Bayes Theorem, the approximate posterior density of θ and τ is

$$\begin{aligned} \xi(\theta, \tau | S_n) &= \frac{L^*(\theta, \tau | S_n)\xi(\theta, \tau)}{\int_\Theta \int_0^\infty L^*(\theta, \tau | S_n)\xi(\theta, \tau) \mathrm{d}\theta \mathrm{d}\tau} \\ &= \frac{\tau^{-1} \mathrm{e}^{-\frac{1}{2}(\theta-\mu)' Q(\theta-\mu)} \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A\theta)}}{\int_\Theta \int_0^\infty \tau^{-1} \mathrm{e}^{-\frac{1}{2}(\theta-\mu)' Q(\theta-\mu)} \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)} \mathrm{d}\theta \mathrm{d}\tau}. \end{aligned} \tag{7.56}$$

Similarly to independent Truncated Normal-Gamma prior case, we cannot find analytical expressions of the approximate Bayes estimators under both the SE and LINEX loss functions, again we use Lindley's approximation. In the next two sections we apply this model to the MA(1) and MA(2) processes and find the approximate Bayes estimators of their parameters and one-step ahead forecasts.

7.3.1. MA(1) model

In the $q = 1$ case, under the SE loss function, the Bayes estimator of function $v = v(\theta_1, \tau)$ is defined by Equation 7.4, under the LINEX loss function, it is defined by Equation 7.6. If we approximate the likelihood function by the approximate likelihood function defined by Equation 7.5, under the SE loss function, the Bayes estimator of the function $v = v(\theta_1, \tau)$ is defined by Equation 7.7 and the Bayes estimator under the LINEX loss function is defined by Equation 7.8. If we apply Lindley's approximation to this model, we get the following result

Proposition 7.22. *In the MA(1) model, we assume the parameters follow Independent TN - improper model. Then*

(i) *Under the SE loss function, the approximate Bayes estimators of θ_1, τ are defined by Equations 7.13 and 7.14, respectively.*

(ii) *Under the LINEX loss function, the approximate Bayes estimators of θ_1, τ are defined by Equations 7.17 and 7.18, respectively.*

(iii) *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by Equation 7.16.*

(iv) *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by Equation 7.19.*

Where

$$\rho = \log(\xi(\theta_1, \tau)) = constant - \log(\tau) - \frac{(\theta_1 - \mu_1)^2}{2\sigma_1^2},$$
$$\hat{\rho_1} = -\frac{\hat{\theta}_1 - \mu_1}{\sigma_1^2}, \quad \hat{\rho_2} = -\frac{1}{\hat{\tau}},$$

and other functions are same as defined in the Section 7.1.1.

7.3.2. MA(2) model

In the $q = 2$ case, under the SE loss function, the Bayes estimator of function $v = v(\theta_1, \theta_2, \tau)$ is defined by Equation 7.20, under the LINEX loss function, it is defined by Equation 7.22. If we approximate the likelihood function by the approximate likelihood function defined by Equation 7.21, under the SE loss function, the Bayes estimator of the function $v = v(\theta_1, \theta_2, \tau)$ is defined by Equation 7.23 and the Bayes estimator under the LINEX loss function is defined by Equation 7.24. If we apply Lindley's approximation to this model, we get the following result.

Proposition 7.23. *In the MA(2) model, we assume the parameters follow Independent TN - Improper model. Then*

(i) *Under the SE loss function, the approximate Bayes estimators of* θ_1, θ_2 *and* τ *are defined by Equations 7.29, 7.30 and 7.31, respectively.*

(ii) *Under the LINEX loss function, the approximate Bayes estimators of* θ_1, θ_2 *and* τ *are defined by Equations 7.35, 7.36 and 7.37, respectively.*

(iii) *Under the SE loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample* $S_n = (Y_1, \ldots, Y_n)'$ *is given by Equation 7.34.*

(iv) *Under the LINEX loss function, the approximate Bayes estimator of the one-step ahead forecast given a sample* $S_n = (Y_1, \ldots, Y_n)'$ *is given by Equation 7.38.*

Where

$$\rho = \log(\xi(\theta_1, \theta_2, \tau)) = constant - \log(\tau) - \frac{(\theta_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(\theta_2 - \mu_2)^2}{2\sigma_2^2},$$

$$\hat{\rho}_1 = -\frac{\hat{\theta}_1 - \mu_1}{\sigma_1^2}, \quad \hat{\rho}_2 = -\frac{\hat{\theta}_2 - \mu_2}{\sigma_2^2}, \quad \hat{\rho}_3 = -\frac{1}{\hat{\tau}},$$

and other functions are same as defined in the Section 7.1.2.

7.4. FULL CONDITIONAL DISTRIBUTIONS OF PARAMETERS UNDER INDEPENDENT TRUNCATED NORMAL - IMPROPER PRIOR

7.4.1. MA(1) model

In this section we obtain the full conditional distributions for MA(1) model parameters θ_1, τ and the one-step prediction Y_{n+1}. The following propositions provide the required distributions.

Proposition 7.24. *The conditional distribution of Bayes estimator of* θ_1 *given a sample* $S_n = (Y_1, \ldots, Y_n)'$ *is normal with mean* $\tilde{\mu}_1$ *and variance* $\tilde{\sigma}_1^2$*, where*

$$\tilde{\mu}_1 = \frac{-\tau \sum_{t=1}^{n} Y_t \epsilon_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=1}^{n} \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{7.57}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=1}^n \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{7.58}$$

Proof.

$$\begin{aligned}
\xi(\theta_1|\tau, S_n) &\propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^n (Y_t+\theta_1\epsilon_{t-1})^2} \tau^{-1} \mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\sum_{t=1}^n Y_t^2 + 2\theta_1 \sum_{t=1}^n Y_t\epsilon_{t-1} + \theta_1^2 \sum_{t=1}^n \epsilon_{t-1}^2) - \frac{\theta_1^2 - 2\theta_1\mu_1 + \mu_1^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{1}{2}((\tau \sum_{t=1}^n \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2})\theta_1^2 - 2(-\tau \sum_{t=1}^n Y_t\epsilon_{t-1} + \frac{\mu_1}{\sigma_1^2})\theta_1)} \\
&\propto \mathrm{e}^{-\frac{\left(\tau \sum_{t=1}^n \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}\right)}{2}\left(\theta_1 - \frac{-\tau \sum_{t=1}^n Y_t\epsilon_{t-1} + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=1}^n \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned} \tag{7.59}$$

Proposition 7.25. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \frac{n}{2} \tag{7.60}$$

and

$$\tilde{\beta} = \frac{\sum_{t=1}^n (Y_t + \theta_1\epsilon_{t-1})^2}{2}. \tag{7.61}$$

Proof.

$$\begin{aligned}
\xi(\tau|\theta_1, S_n) &\propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^n (Y_t+\theta_1\epsilon_{t-1})^2} \tau^{-1} \mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^n (Y_t+\theta_1\epsilon_{t-1})^2} \tau^{-1} \\
&= \tau^{\frac{n}{2}-1} \mathrm{e}^{-\tau(\frac{1}{2}\sum_{t=1}^n (Y_t+\theta_1\epsilon_{t-1})^2)}.
\end{aligned} \tag{7.62}$$

Proposition 7.26. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}$ and precision τ, where*

$$\tilde{\mu} = -\theta_1 \epsilon_n. \tag{7.63}$$

Proof.

$$\begin{aligned} Y_{n+1}|\theta_1, \tau, S_n &= (-\theta_1\epsilon_n + \epsilon_{n+1})|\theta_1, \tau, S_n \\ &= -\theta_1\epsilon_n + (\epsilon_{n+1}|\tau). \end{aligned} \tag{7.64}$$

7.4.2. MA(2) model

Proposition 7.27. *The full conditional distribution of Bayes estimator of θ_1 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_1$ and variance $\tilde{\sigma}_1^2$, where*

$$\tilde{\mu}_1 = \frac{-\tau(\sum_{t=1}^n Y_t\epsilon_{t-1} + \theta_2 \sum_{t=1}^n \epsilon_{t-1}\epsilon_{t-2}) + \frac{\mu_1}{\sigma_1^2}}{\tau \sum_{t=1}^n \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}} \tag{7.65}$$

and

$$\tilde{\sigma}_1^2 = \frac{1}{\tau \sum_{t=1}^n \epsilon_{t-1}^2 + \frac{1}{\sigma_1^2}}. \tag{7.66}$$

Proof.

$$\begin{aligned}
\xi(\theta_1|\theta_2,\tau,S_n) &\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{-1}\mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(\theta_1^2\epsilon_{t-1}^2+2\theta_1Y_t\epsilon_{t-1}+2\theta_1\theta_2\epsilon_{t-1}\epsilon_{t-2})}\mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\theta_1^2(\sum_{t=1}^{n}\epsilon_{t-1}^2)+2\theta_1(\sum_{t=1}^{n}Y_t\epsilon_{t-1})+2\theta_1\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\theta_1^2-2\mu_1\theta_1)}{2\sigma_1^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\theta_1^2(\tau(\sum_{t=1}^{n}\epsilon_{t-1}^2)+\frac{1}{\sigma_1^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\theta_1(-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-1})+\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_1}{\sigma_1^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=1}^{n}\epsilon_{t-1}^2)+\frac{1}{\sigma_1^2}}{2}\left(\theta_1-\frac{-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-1})+\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_1}{\sigma_1^2}}{\tau(\sum_{t=1}^{n}\epsilon_{t-1}^2)+\frac{1}{\sigma_1^2}}\right)^2}.
\end{aligned}$$

Proposition 7.28. *The full conditional distribution of Bayes estimator of θ_2 given a sample $S_n = (Y_1, \ldots, Y_n)'$ is normal with mean $\tilde{\mu}_2$ and variance $\tilde{\sigma}_2^2$, where*

$$\tilde{\mu}_2 = \frac{-\tau\big(\sum_{t=1}^{n}Y_t\epsilon_{t-2}+\theta_1\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2}\big)+\frac{\mu_2}{\sigma_2^2}}{\tau\sum_{t=1}^{n}\epsilon_{t-2}^2+\frac{1}{\sigma_2^2}} \tag{7.67}$$

and

$$\tilde{\sigma}_2^2 = \frac{1}{\tau\sum_{t=1}^{n}\epsilon_{t-2}^2+\frac{1}{\sigma_2^2}}. \tag{7.68}$$

Proof.

$$\begin{aligned}
\xi(\theta_2|\theta_1,\tau,S_n) &\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{-1}\mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(\theta_2^2\epsilon_{t-2}^2+2\theta_2Y_t\epsilon_{t-2}+2\theta_1\theta_2\epsilon_{t-1}\epsilon_{t-2})}\mathrm{e}^{-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \mathrm{e}^{-\frac{\tau}{2}(\theta_2^2(\sum_{t=1}^{n}\epsilon_{t-2}^2)+2\theta_2(\sum_{t=1}^{n}Y_t\epsilon_{t-2})+2\theta_1\theta_2(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2}))} \\
&\times \mathrm{e}^{-\frac{(\theta_2^2-2\mu_2\theta_2)}{2\sigma_2^2}} \\
&= \mathrm{e}^{-\frac{1}{2}(\theta_2^2(\tau(\sum_{t=1}^{n}\epsilon_{t-2}^2)+\frac{1}{\sigma_2^2}))} \\
&\times \mathrm{e}^{-\frac{1}{2}(-2\theta_2(-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-2})+\theta_1(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_2}{\sigma_2^2}))} \\
&\propto \mathrm{e}^{-\frac{\tau(\sum_{t=1}^{n}\epsilon_{t-2}^2)+\frac{1}{\sigma_2^2}}{2}\left(\theta_2-\frac{-\tau[(\sum_{t=1}^{n}Y_t\epsilon_{t-2})+\theta_1(\sum_{t=1}^{n}\epsilon_{t-1}\epsilon_{t-2})]+\frac{\mu_2}{\sigma_2^2}}{\tau(\sum_{t=1}^{n}\epsilon_{t-2}^2)+\frac{1}{\sigma_2^2}}\right)^2}.
\end{aligned}$$

Proposition 7.29. *The full conditional distribution of Bayes estimator of τ given a sample $S_n = (Y_1, \ldots, Y_n)'$ is gamma with parameters $\tilde{\alpha}$ and variance $\tilde{\beta}$, where*

$$\tilde{\alpha} = \frac{n}{2} \tag{7.69}$$

and

$$\tilde{\beta} = \frac{\sum_{t=1}^{n}(Y_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2})^2}{2}. \tag{7.70}$$

Proof.

$$\begin{aligned}
\xi(\tau|\theta_1,\theta_2,S_n) &\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{-1}\mathrm{e}^{-\frac{(\theta_1-\mu_1)^2}{2\sigma_1^2}-\frac{(\theta_2-\mu_2)^2}{2\sigma_2^2}} \\
&\propto \tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2}\tau^{-1} \\
&= \tau^{\frac{n}{2}-1}\mathrm{e}^{-\tau(\frac{1}{2}\sum_{t=1}^{n}(Y_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2})^2)} \qquad (7.71)
\end{aligned}$$

.

Proposition 7.30. *The full conditional distribution of Bayes estimator of Y_{n+1} given a sample*

$S_n = (Y_1, \ldots, Y_n)'$ *is normal with mean* $\tilde{\mu}$ *and precision* τ*, where*

$$\tilde{\mu} = -\theta_1\epsilon_n - \theta_2\epsilon_{n-1}. \tag{7.72}$$

Proof.

$$\begin{aligned} Y_{n+1}|\theta_1, \theta_2, \tau, S_n &= (-\theta_1\epsilon_n - \theta_2\epsilon_{n-1} + \epsilon_{n+1})|\theta_1, \theta_2, \tau, S_n \\ &= -\theta_1\epsilon_n - \theta_2\epsilon_{n-1} + (\epsilon_{n+1}|\tau). \end{aligned} \tag{7.73}$$

7.5. INDEPENDENT UNIFORM - GAMMA PRIOR

In the model 7.1 we assume the parameters $\theta_i, i = 1, \ldots, q$, have independent uniform priors on intervals $(c_i, d_i), c_i, d_i \in \mathbb{R}$, for all $i = 1, \ldots, q$, respectively, that is, $\theta_1 \sim U(c_1, d_1), \ldots, \theta_q \sim U(c_q, d_q)$, and the precision has independent gamma prior with the parameters $\tau \sim Gamma(\alpha, \beta)$. That is,

$$\xi(\theta, \tau) = \xi_1(\theta)\xi_2(\tau),$$

where the marginal prior density of τ is gamma distribution

$$\xi_2(\tau) \propto \tau^{\alpha-1}\mathrm{e}^{-\tau\beta}, \tau > 0,$$

and the marginal prior density of θ is

$$\xi_1(\theta) \propto 1.$$

So the joint prior density function of parameters

$$\xi(\theta, \tau) \propto \tau^{\alpha-1} \mathrm{e}^{-\tau\beta}.$$

The approximate likelihood function for this model

$$L^*(\theta, \tau | S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)}, \tag{7.74}$$

where A^* is as defined in the Section 7.1

Employing Bayes Theorem, the posterior density of θ and τ is

$$\xi(\theta, \tau | S_n) \propto \tau^{\alpha-1} \mathrm{e}^{-\tau\beta} \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)}. \tag{7.75}$$

Now we can find the posterior distribution for θ. We have

$$\xi(\theta, \tau | S_n) \propto \tau^{a-1} \mathrm{e}^{-\frac{\tau}{2} b},$$

where $a := \frac{n+2\alpha}{2}$, $b := 2\beta + (Y + A^*\theta)'(Y + A^*\theta)$.

Then the posterior density of θ

$$\begin{aligned}
\xi(\theta | S_n) &\propto \int_0^\infty \tau^{a-1} \mathrm{e}^{-\frac{\tau}{2} b} \mathrm{d}\tau \\
&= \frac{2}{b}^a \Gamma(a) \\
&\propto b^{-a}.
\end{aligned}$$

We rewrite b as follows

$$\begin{aligned}
b &= 2\beta + (Y + A^*\theta)'(Y + A^*\theta) \\
&= 2\beta + Y'Y + Y'A^*\theta + \theta' A^{*'} Y + \theta' A^{*'} A^* \theta \\
&:= 2\beta + (\theta - c)'(A^{*'} A^*)(\theta - c) - d + Y'Y,
\end{aligned}$$

where $c = -(A^{*'}A^*)^{-1}(A^{*'}Y), \quad d = (Y'A^*)(A^{*'}A^*)^{-1}(A^{*'}Y)$.

Therefore,

$$\begin{aligned}\xi(\theta|S_n) &\propto (2\beta + (\theta - c)'(A^{*'}A^*)(\theta - c) - d + Y'Y)^{-a} \\ &\propto \Big(1 + \frac{(\theta - c)'(A^{*'}A^*)(\theta - c)}{2\beta - d + Y'Y}\Big)^{-a} \\ &= \Big(1 + \frac{(\theta - c)'\Big[\frac{(2\beta - d + Y'Y)(A^{*'}A^*)^{-1}}{\nu}\Big]^{-1}(\theta - c)}{\nu}\Big)^{-\frac{\nu+q}{2}},\end{aligned}$$

where $\nu = n + 2\alpha - q$. We get the following result.

Proposition 7.31. *The posterior distribution of θ is the truncated q-dimensional t-distribution with $\nu = n + 2\alpha - q$ degrees of freedom, location vector*

$$c = -(A^{*'}A^*)^{-1}(A^{*'}Y)$$

and scale matrix

$$\Sigma(\theta|S_n) = \frac{(2\beta - d + Y'Y)(A^{*'}A^*)^{-1}}{n + 2\alpha - q},$$

where $d = (Y'A^*)(A^{*'}A^*)^{-1}(A^{*'}Y)$. That is

$$\theta|S_n \sim Tt_q(c, \Sigma(\theta|S_n), \nu).$$

So we have

Corollary 7.1. *The marginal posterior distribution for individual parameter is*

$$\theta_i|S_n \sim Tt_1(c_i, s_i, \nu),$$

where s_i is the diagonal element of scale matrix $\Sigma(\theta|S_n)$, $i = 1, \ldots, q$.

The posterior density of τ

$$\begin{aligned}\xi(\tau|S_n) &\propto \int_\Theta \tau^{a-1}\mathrm{e}^{-\frac{\tau}{2}b}\mathrm{d}\theta \\ &= \int_\Theta \tau^{a-1}\mathrm{e}^{-\frac{\tau}{2}((\theta-c)'(A^{*'}A^*)(\theta-c)+2\beta-d+Y'Y)}\mathrm{d}\theta \\ &= \tau^{a-1}\mathrm{e}^{-\frac{\tau}{2}(2\beta-d+Y'Y)}\int_\Theta \mathrm{e}^{-\frac{\tau}{2}(\theta-c)'(A^{*'}A^*)(\theta-c)}\mathrm{d}\theta \\ &\propto \tau^{a-\frac{q}{2}-1}\mathrm{e}^{-\frac{\tau}{2}(2\beta-d+Y'Y)},\end{aligned}$$

where $c = -(A^{*'}A^*)^{-1}(A^{*'}Y), \quad d = (Y'A^*)(A^{*'}A^*)^{-1}(A^{*'}Y)$.

We get the following result.

Proposition 7.32. *The posterior distribution of τ is the gamma distribution with parameters α_0 and β_0, where*

$$\alpha_0 = \frac{n+2\alpha-q}{2} \tag{7.76}$$

and

$$\beta_0 = \frac{1}{2}(2\beta - (Y'A^*)(A^{*'}A^*)^{-1}(A^{*'}Y) + Y'Y). \tag{7.77}$$

In oder to derive the posterior density for one-step ahead prediction, we denote

$$\begin{aligned}Y_f &= \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n+1}\end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{n+1}\end{pmatrix} - \begin{pmatrix} 0 & 0 & \dots & 0 \\ \epsilon_1^* & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \epsilon_n^* & \epsilon_{n-1}^* & \dots & \epsilon_{n-q+1}^*\end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_q\end{pmatrix} \\ &:= -\begin{pmatrix} A^* \\ P\end{pmatrix}\theta + \epsilon := -A_f\theta + \epsilon,\end{aligned} \tag{7.78}$$

where

$$P = \begin{pmatrix} \epsilon_n^* & \epsilon_{n-1}^* & \dots & \epsilon_{n-q+1}^* \end{pmatrix}.$$

The likelihood function for the latter model is

$$L^*(\theta, \tau | S_n) \propto \tau^{\frac{n+1}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A_f\theta)'(Y_f+A_f\theta)}.$$

The joint prior density function of parameters

$$\xi(\theta, \tau) \propto \tau^{\alpha-1} \mathrm{e}^{-\tau\beta}.$$

We assume that A_f is a known constant and employ Bayes and Fubini theorems, then

$$\begin{aligned} \xi(W_1|S_n) &= \xi(Y_{n+1}|S_n) \propto \xi(Y_{n+1}, S_n) \\ &\propto \int_{\Theta\times(0,\infty)} \tau^{\frac{n+1}{2}+\alpha-1} \mathrm{e}^{-\frac{\tau}{2}(2\beta+(Y_f+A_f\theta)'(Y_f+A_f\theta))} \mathrm{d}\theta \mathrm{d}\tau \\ &= \int_0^\infty \tau^{\frac{n+1}{2}+\alpha-1} \mathrm{e}^{-\tau\beta} \mathrm{d}\tau \int_\Theta \mathrm{e}^{-\frac{\tau}{2}(Y_f+A_f\theta)'(Y_f+A_f\theta)} \mathrm{d}\theta. \end{aligned} \tag{7.79}$$

Rewrite

$$(Y_f + A_f\theta)'(Y_f + A_f\theta) = (\theta - \theta_0)'(A_f'A_f)(\theta - \theta_0) \tag{7.80}$$

$$- (A_f'Y_f)'(A_f'A_f)^{-1}(A_f'Y_f) + Y_f'Y_f, \tag{7.81}$$

where $\theta_0 = -(A_f'A_f)^{-1}(A_f'Y_f)$.

Hence

$$\begin{aligned}
\xi(W_1|S_n) &\propto \int_0^\infty \tau^{\frac{n+1}{2}+\alpha-1}\mathrm{e}^{-\tau\beta}\mathrm{e}^{-\frac{\tau}{2}(-(A_f'Y_f)'(A_f'A_f)^{-1}(A_f'Y_f)+Y_f'Y_f)}\mathrm{d}\tau \\
&\times \int_\Theta \mathrm{e}^{-\frac{\tau}{2}((\theta-\theta_0)'(A_f'A_f)(\theta-\theta_0))}\mathrm{d}\theta \\
&\propto \int_0^\infty \tau^{\frac{n+2\alpha+1-q}{2}-1}\mathrm{e}^{-\frac{\tau}{2}(2\beta-(A_f'Y_f)'(A_f'A_f)^{-1}(A_f'Y_f)+Y_f'Y_f)}\mathrm{d}\tau \\
&\propto (2\beta-(A_f'Y_f)'(A_f'A_f)^{-1}(A_f'Y_f)+Y_f'Y_f)^{-\frac{n+2\alpha+1-q}{2}}. \qquad (7.82)
\end{aligned}$$

We notice that

$$\begin{aligned}
(A_f'Y_f)'(A_f'A_f)^{-1}(A_f'Y_f) &= (Y'A^*+Y_{n+1}P)(A_f'A_f)^{-1}(A^{*\prime}Y+Y_{n+1}P') \\
&= (Y'A^*)(A_f'A_f)^{-1}(A^{*\prime}Y) \\
&+ (Y'A^*)(A_f'A_f)^{-1}Y_{n+1}P' \\
&+ Y_{n+1}P(A_f'A_f)^{-1}(A^{*\prime}Y) \\
&+ Y_{n+1}^2P(A_f'A_f)^{-1}P' \qquad (7.83)
\end{aligned}$$

and $Y_f'Y_f = Y'Y + Y_{n+1}^2$.

Thus,

$$\xi(W_1|S_n) \propto (E+2DY_{n+1}+CY_{n+1}^2)^{-\frac{n+2\alpha+1-q}{2}}, \qquad (7.84)$$

where

$$E = 2\beta-(Y'A^*)(A_f'A_f)^{-1}(A^{*\prime}Y)+Y'Y, \qquad (7.85)$$

$$D = -(Y'A^*)(A_f'A_f)^{-1}P', \qquad (7.86)$$

$$C = (1-P(A_f'A_f)^{-1}P'). \qquad (7.87)$$

We get the following result.

Proposition 7.33. *The one-step predictive distribution is t-distribution* $t(\frac{D}{C}, \frac{CE-D^2}{C^2(n+2\alpha-q)}, n+2\alpha-q)$*, where C, D and E are as defined above.*

Theorem 7.1. *Under the SE loss function, the Bayes estimator of* θ_i*,* $i = 1, \ldots, q$ *is equal to*

$$\hat{\theta}_{i(B_{SE})} = c_i + \frac{k\nu\sqrt{s_i}}{\nu - 1}\Big((1 + \frac{(a_i - c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}} - (1 + \frac{(b_i - c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}}\Big), \quad (7.88)$$

where

$$k = \frac{\gamma(\frac{\nu+1}{2})}{k_0\gamma(\frac{\nu}{2})\sqrt{\pi\nu}}, \quad k_0 = F(\frac{(b_i - c_i)}{\sqrt{s_i}}|\nu) - F(\frac{(a_i - c_i)}{\sqrt{s_i}}|\nu).$$

Here $F(\cdot|\nu)$ *denotes the c.d.f. of the Student's t-distribution and the parameters* ν, c_i, s_i*, are as defined in the Proposition 7.31 and Corollary 7.1.*

Proof. Under the SE loss function, the Bayes estimator of θ_i is the posterior mean. See [59], the proof of the Theorem 1, for the computation of the moments of the truncated t-distribution.

Theorem 7.2. *Under the SE loss function, the Bayes estimator of* τ*, is equal to*

$$\hat{\tau}_{(B_{SE})} = \frac{\alpha_0}{\beta_0}, \quad (7.89)$$

where α_0 *and* β_0 *are defined by Equations 7.76 and 7.77, respectively.*

Proof. Under the SE loss function, the Bayes estimator of τ is the posterior mean.

Proposition 7.34. *Under the SE loss function, the Bayes estimator of the one-step ahead forecast given a sample* $S_n = (Y_1, \ldots, Y_n)'$ *is*

$$\hat{W}_{1(B_{SE})} = -\sum_{k=1}^{q} \hat{\theta}_{k(B_{SE})}\epsilon^*_{n-k+1}. \quad (7.90)$$

Proof. Similar to the proof of Proposition 7.10.

Under the LINEX loss function, the Bayes estimator involves the moment generating function. For the truncated t-distribution, moment generating function is defined but does not have a tractable form. We can either use numerical integration methods or since the degrees of freedom is large, truncated t-distribution can be well approximated by the truncated normal distribution whose moment generating function exists. Using this approximation, we get the following result

Theorem 7.3. *Under the LINEX loss function, the Bayes approximate estimator of θ_i, $i = 1, \ldots, q$ is equal to*

$$\hat{\theta}_{i(B_{LINEX})} = c_i - \frac{\gamma s_i}{2} - \frac{1}{\gamma} \log \Big(\frac{\Phi(\frac{b_i - c_i}{\sqrt{s_i}} + s_i \gamma) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}} + s_i \gamma)}{\Phi(\frac{b_i - c_i}{\sqrt{s_i}}) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}})} \Big), \tag{7.91}$$

where the parameters ν, c_i, s_i, are as defined in the Proposition 7.31 and Corollary 7.1.

Proof. We substitute the moment generating function of truncated normal distribution into the expression of the Bayes estimator of θ_i under the LINEX loss function.

Theorem 7.4. *Under the LINEX loss function, the Bayes estimator of τ is equal to*

$$\hat{\tau}_{(B_{LINEX})} = -\frac{\alpha_0}{\gamma} \log \Big(1 + \frac{\gamma}{\beta_0} \Big) \tag{7.92}$$

where α_0 and β_0 are defined by Equations 7.76 and 7.77, respectively.

Proof. We substitute the moment generating function of the gamma distribution into the expression of the Bayes estimator of τ under the LINEX loss function.

Theorem 7.5. *Under the LINEX loss function, the Bayes estimator of the one-step ahead forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by*

$$\hat{W}_{1(B_{LINEX})} = \frac{D}{C} - \frac{\gamma(CE - D^2)}{2C^2(n + 2\alpha - q)} \tag{7.93}$$

where C, D and E are defined by Equations 7.87, 7.86 and 7.85, respectively.

Proof. We substitute the moment generating function of the gamma distribution into the expression of the Bayes estimator of τ under the LINEX loss function.

7.6. INDEPENDENT UNIFORM - IMPROPER PRIOR

In the model 7.1 we assume the parameters $\theta_i, i = 1, \dots, q$, have independent uniform priors on intervals $(c_i, d_i), c_i, d_i \in \mathbb{R}$, for all $i = 1, \dots, q$, respectively, that is, $\theta_1 \sim U(c_1, d_1), \dots, \theta_q \sim U(c_q, d_q)$, and the precision has independent improper prior. In this model,

$$\xi(\theta, \tau) = \xi_1(\theta)\xi_2(\tau),$$

where the marginal prior density of τ is

$$\xi_2(\tau) \propto \frac{1}{\tau}, \tau > 0,$$

and the marginal prior density of θ is

$$\xi_1(\theta) \propto 1.$$

So the joint prior density function of parameters

$$\xi(\theta, \tau) \propto \tau^{-1}$$

The likelihood function for this model

$$L^*(\theta, \tau|S_n) \propto \tau^{\frac{n}{2}} \mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)}, \tag{7.94}$$

where A^* is as defined in the Section 7.1.

Employing Bayes Theorem, the approximate posterior density of θ and τ is

$$\xi(\theta, \tau|S_n) \propto \tau^{-1}\tau^{\frac{n}{2}}\mathrm{e}^{-\frac{\tau}{2}(Y+A^*\theta)'(Y+A^*\theta)}. \tag{7.95}$$

In the same manner as in Section 7.5, we can show that the posterior distribution for $\theta_i,\ i = 1, \ldots, q$, is the truncated t-distribution, the posterior distribution for τ is the gamma distribution and the one-step predictive distribution is the t-distribution. For the independent uniform prior for θ and improper prior for τ, we get the following results.

Proposition 7.35. *The posterior distribution of θ is the truncated q-dimensional t-distribution with $\nu = n - q$ degrees of freedom, location vector*

$$c = -(A^{*'}A^*)^{-1}(A^{*'}Y)$$

and scale matrix

$$\Sigma(\theta|S_n) = \frac{(-d + Y'Y)(A^*A^{*'})^{-1}}{n-q},$$

where $d = (Y'A^*)(A^{*'}A^*)^{-1}(A^{*'}Y)$. That is

$$\theta|S_n \sim Tt_q(c, \Sigma(\theta|S_n), \nu).$$

So we have

Corollary 7.2. *The marginal posterior distribution for individual parameter is*

$$\theta_i|S_n \sim Tt_1(c_i, s_i, \nu),$$

where s_i is the diagonal element of scale matrix $\Sigma(\theta|S_n)$, $i = 1, \ldots, q$.

We get the following result

Proposition 7.36. *The posterior distribution of τ is the gamma distribution with parameters α_0 and β_0, where*

$$\alpha_0 = \frac{n-q}{2} \tag{7.96}$$

and

$$\beta_0 = \frac{1}{2}(-(Y'A^*)(A^{*'}A^*)^{-1}(A^{*'}Y) + Y'Y). \tag{7.97}$$

We get the following result

Proposition 7.37. *The one-step predictive distribution is the t-distribution $t(\frac{D}{C}, \frac{CE-D^2}{C^2(n-q)}, n-q)$, where C, D and E are as defined below*

$$E = -(Y'A^*)(A_f'A_f)^{-1}(A^{*'}Y) + Y'Y, \tag{7.98}$$

$$D = -(Y'A^*)(A_f'A_f)^{-1}P', \tag{7.99}$$

$$C = (1 - P(A_f'A_f)^{-1}P'). \tag{7.100}$$

Theorem 7.6. *Under the SE loss function, the Bayes estimator of θ_i, $i = 1, \ldots, q$ is equal to*

$$\hat{\theta}_{i(B_{SE})} = c_i + \frac{k\nu\sqrt{s_i}}{\nu-1}\Big((1+\frac{(a_i-c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}} - (1+\frac{(b_i-c_i)^2}{s_i\nu})^{-\frac{\nu-1}{2}}\Big), \tag{7.101}$$

where

$$k = \frac{\gamma(\frac{\nu+1}{2})}{k_0\gamma(\frac{\nu}{2})\sqrt{\pi\nu}}, \quad k_0 = F(\frac{(b_i-c_i)}{\sqrt{s_i}}|\nu) - F(\frac{(a_i-c_i)}{\sqrt{s_i}}|\nu).$$

Here $F(\cdot|\nu)$ denotes the c.d.f. of the Student's t-distribution and the parameters ν, c_i, s_i, are as defined in the Proposition 7.35 and Corollary 7.2.

Theorem 7.7. *Under the SE loss function, the Bayes estimator of* τ*, is equal to*

$$\hat{\tau}_{(B_{SE})} \quad = \quad \frac{\alpha_0}{\beta_0}, \tag{7.102}$$

where α_0 *and* β_0 *are defined by Equations 7.96 and 7.97, respectively.*

Proposition 7.38. *Under the SE loss function, the Bayes estimator of the one-step ahead forecast given a sample* $S_n = (Y_1, \ldots, Y_n)'$ *is*

$$\hat{W}_{1(B_{SE})} = -\sum_{k=1}^{q} \hat{\theta}_{k(B_{SE})} \epsilon^*_{n-k+1}. \tag{7.103}$$

Under the LINEX loss function, the Bayes estimator involves the moment generating function. For the truncated t-distribution, the moment generating function is defined but is intractable. One can estimate this moment generating function either using numeric integration or by employing some approximation. Since the degrees of freedom is large, the truncated t-distribution can be well approximated by the truncated normal distribution whose moment generating function has a closed form. Using this approximation, we get the following result.

Theorem 7.8. *Under the LINEX loss function, the Bayes estimator of* θ_i*,* $i = 1, \ldots, q$ *is equal to*

$$\hat{\theta}_{i(B_{LINEX})} \quad = \quad c_i - \frac{\gamma s_i}{2} - \frac{1}{\gamma} \log \Big(\frac{\Phi(\frac{b_i - c_i}{\sqrt{s_i}} + s_i\gamma) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}} + s_i\gamma)}{\Phi(\frac{b_i - c_i}{\sqrt{s_i}}) - \Phi(\frac{a_i - c_i}{\sqrt{s_i}})} \Big), \tag{7.104}$$

where and the parameters $\nu, c_i, s_i,$ *are as defined in the Proposition 7.35 and Corollary 7.2.*

Theorem 7.9. *Under the LINEX loss function, the Bayes estimator of* τ *is equal to*

$$\hat{\tau}_{i(B_{LINEX})} \quad = \quad -\frac{\alpha_0}{\gamma} \log \Big(1 + \frac{\gamma}{\beta_0} \Big) \tag{7.105}$$

where α_0 *and* β_0 *are defined by Equations 7.96 and 7.97, respectively.*

Theorem 7.10. *Under the LINEX loss function, the Bayes estimator of the one-step ahead* *forecast given a sample $S_n = (Y_1, \ldots, Y_n)'$ is given by*

$$\hat{W}_{1(B_{LINEX})} = \frac{D}{C} - \frac{\gamma(CE - D^2)}{2C^2(n-q)} \tag{7.106}$$

where C, D and E are defined by Equations 7.100, 7.99 and 7.98, respectively.

8. NUMERICAL STUDY OF ESTIMATION AND FORECASTING FOR MOVING-AVERAGE PROCESSES

We study the MA(q) model. The following procedures for statistical calculation are used.

(i) Simulate τ.

(ii) Simulate θ.

(iii) Generate the MA(q) series: $y_1, \ldots, y_n$ for some n.

(iv) Calculate the ML estimates for the parameters and find the error under the SE and LINEX loss functions.

(v) Calculate the Bayes estimates for the parameters under the SE and LINEX loss functions and find the error. For truncated normal prior for θ, we calculate approximate Bayes estimates using Lindley's approximation and Gibbs sampling. We run the Gibbs sampler for an initial 1,000 iterations that we discard, and then for a further 9,000 iterations of which we store every fifth.

(vi) Calculate the ML and Bayes estimates of one step ahead forecasts under the SE and LINEX loss functions and find the errors.

(vii) Since the estimation using Gibbs sampler is computationally expensive, repeat the above procedures 1,000 times for truncated normal and 10,000 times for uniform priors for θ respectively, and calculate the mean errors under the SE and LINEX loss functions.

The simulation study is undertaken using sample sizes $n = 50, 100, 150, 200$ and LINEX loss function parameters $\gamma = -1.25, -0.75, -0.25, 0.25, 0.75, 1.25$.

In order to compare the ML and Bayes estimators, the average squared errors are used when the Bayes estimates are computed using the SE loss function. The average errors, computed using the LINEX loss function are used when the Bayes estimates are obtained using the LINEX loss function.

8.1. MA(1) model

In this section we study the MA(1) model

$$Y_t = -\theta_1 \epsilon_{t-1} + \epsilon_t. \tag{8.1}$$

8.1.1. Independent Truncated Normal prior for θ_1 and Gamma or Improper priors for τ

We consider a truncated normal prior for θ_1 with $\mu_1 = 0.5$, $\sigma_1 = 0.3$, defined over the interval (c_1, d_1), where $c_1 = 0.25$, $d_1 = 0.75$. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$.

Table 8.1 presents the average values of MA(1) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used. Under the SE loss function, the average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overall the Bayes estimates are found to have smaller average estimation errors than the ML estimates, for θ_1 the smallest estimation errors are obtained for the Bayes estimates obtained using Lindley's approximation; for τ the Bayes estimation using the Gibbs sampler is found to result in the smallest estimation errors; for the one-step prediction all estimates have similar errors, the ones of Bayes estimates being slightly smaller. All estimator performances are reasonably close to each other as the sample size increases.

Under the LINEX loss function, there is a non-zero probability that the Bayes estimates using Lindley's approximation may be undefined (see Section 7.1.1), Table 8.2 shows proportion of undefined Bayes estimates using this approximation. Under our choice of parameters, undefined estimates are obtained only for τ when it has gamma prior and LINEX loss function parameters are $\gamma = -1.25, -0.75, -0.25$. The proportion of undefined τ estimates decreases as sample size increases and when parameter γ becomes less negative. We exclude the simulation where we obtain undefined estimates and calculate average errors where all estimates are defined.

Table 8.3, Table 8.4, Table 8.5, Table 8.6, Table 8.7 and Table 8.8 present the average values of MA(1) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25, -0.75, -0.25,$
$0.25, 0.75, 1.25$, respectively. Under the LINEX loss function, the average estimation errors are also found to decrease with increasing sample size. Generally, the Bayes estimates have smaller average errors than the ML estimates.

For θ_1, the smallest average estimation errors are obtained for the Bayes estimates using Lindley's approximation, the difference between the ML and the Bayes estimates are more noticeable when the LINEX parameter has higher absolute value. For the one-step prediction, the Bayes estimates have significantly smaller average errors than the ML estimates, both Lindley's approximation and Gibbs sampler methods have similar performance.

For τ when the LINEX loss function parameters are positive, the smallest average estimation errors are obtained using the Bayes estimates; where gamma prior is used, the Gibbs sampling method is superior, whereas when improper prior is used, Lindley's approximation performance is better in most cases.

When $\gamma = -1.25, -0.75$, if improper prior is used for τ, the ML estimates of τ have smaller average error than the Bayes estimates; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance but the Bayes estimates obtained using Lindley's approximation perform worse than the ML estimates.

When $\gamma = -0.25$, if improper prior is used for τ, the average Bayes τ estimates errors are smaller than the ML estimates errors, the Bayes estimates obtained using Lindley's approximation have the best performance; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler are superior, for sample sizes $n = 50, 100$ the Bayes estimates obtained using Lindley's approximation have the highest average errors, for sample size $n = 150, 200$ both Bayes estimates perform better than the ML estimates.

Table 8.1. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using SE loss function

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4964	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6734	-	1.6734	-	1.6734	-	1.6734	-
Y_{n+1}	0.0361	-	0.0188	-	0.0108	-	-0.0368	-
SE LOSS FUNCTION								
MLEs of θ_1	0.4730	0.0232	0.4782	0.0117	0.4803	0.0081	0.4810	0.0060
Lindley's - θ_1 with gamma prior for τ	0.4795	0.0162	0.4808	0.0100	0.4819	0.0073	0.4821	0.0056
Lindley's - θ_1 with improper prior for τ	0.4795	0.0162	0.4808	0.0100	0.4819	0.0073	0.4821	0.0056
Gibbs - θ_1 with gamma prior for τ	0.4784	0.0170	0.4805	0.0101	0.4818	0.0074	0.4821	0.0056
Gibbs - θ_1 with improper prior for τ	0.4784	0.0170	0.4806	0.0101	0.4817	0.0074	0.4821	0.0056
MLEs of τ	1.7353	0.1419	1.6931	0.0584	1.6866	0.0395	1.6801	0.0301
Lindley's - τ with gamma prior	1.5677	0.1378	1.6310	0.0522	1.6475	0.0349	1.6524	0.0278
Lindley's - τ with improper prior	1.7006	0.1329	1.6762	0.0568	1.6754	0.0388	1.6717	0.0297
Gibbs - τ with gamma prior	1.6472	0.0797	1.6492	0.0474	1.6559	0.0336	1.6571	0.0273
Gibbs - τ with improper prior	1.7040	0.1318	1.6777	0.0566	1.6758	0.0388	1.6720	0.0296
MLEs of Y_{n+1}	0.0123	0.6555	-0.0011	0.6504	-0.0068	0.5975	0.0056	0.6859
Lindley's - Y_{n+1} with gamma prior for τ	0.0117	0.6469	-0.0020	0.6494	-0.0069	0.5966	0.0053	0.6855
Lindley's - Y_{n+1} with improper prior for τ	0.0117	0.6469	-0.0020	0.6494	-0.0069	0.5966	0.0053	0.6855
Gibbs - Y_{n+1} with gamma prior for τ	0.0110	0.6469	-0.0026	0.6494	-0.0069	0.5955	0.0059	0.6850
Gibbs - Y_{n+1} with improper prior for τ	0.0114	0.6486	-0.0016	0.6485	-0.0076	0.5952	0.0052	0.6868

Table 8.2. Proportion of undefined MA(1) model estimates under independent truncated normal prior for θ_1

		Gamma prior for τ			**Improper prior for τ**		
LINEX parameter	**Sample size**	θ_1	τ	Y_{n+1}	θ_1	τ	Y_{n+1}
$\gamma = -1.25$	50	0.0000	0.0580	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0030	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.75$	50	0.0000	0.0240	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.25$	50	0.0000	0.0020	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.75$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 1.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 8.3. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4971	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6126	-	1.6690	-	1.6734	-	1.6734	-
Y_{n+1}	0.0442	-	0.0198	-	0.0108	-	-0.0368	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.4744	0.0187	0.4781	0.0095	0.4803	0.0065	0.4810	0.0048
Lindley's - θ_1 with gamma prior for τ	0.4919	0.0127	0.4867	0.0079	0.4859	0.0058	0.4852	0.0044
Lindley's - θ_1 with improper prior for τ	0.4919	0.0127	0.4867	0.0079	0.4859	0.0058	0.4852	0.0044
Gibbs - θ_1 with gamma prior for τ	0.4899	0.0133	0.4861	0.0080	0.4857	0.0058	0.4851	0.0044
Gibbs - θ_1 with improper prior for τ	0.4901	0.0133	0.4863	0.0079	0.4857	0.0058	0.4851	0.0044
MLEs of τ	1.6340	0.0710	1.6867	0.0421	1.6866	0.0300	1.6801	0.0230
Lindley's - τ with gamma prior	1.4857	0.8350	1.6050	0.1352	1.6418	0.0355	1.6510	0.0237
Lindley's - τ with improper prior	1.6088	0.0760	1.6755	0.0439	1.6791	0.0309	1.6743	0.0235
Gibbs - τ with gamma prior	1.6408	0.0574	1.6761	0.0370	1.6783	0.0266	1.6744	0.0214
Gibbs - τ with improper prior	1.6821	0.0780	1.7119	0.0450	1.7025	0.0315	1.6917	0.0237
MLEs of Y_{n+1}	0.0146	0.9180	-0.0008	0.7210	-0.0068	0.6438	0.0056	0.6583
Lindley's - Y_{n+1} with gamma prior for τ	0.4573	0.6006	0.4246	0.5167	0.4138	0.4776	0.4237	0.5154
Lindley's - Y_{n+1} with improper prior for τ	0.4784	0.5930	0.4327	0.5165	0.4190	0.4785	0.4275	0.5160
Gibbs - Y_{n+1} with gamma prior for τ	0.4543	0.6063	0.4223	0.5150	0.4124	0.4774	0.4229	0.5145
Gibbs - Y_{n+1} with improper prior for τ	0.4772	0.5915	0.4323	0.5171	0.4188	0.4781	0.4281	0.5178

Table 8.4. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4967	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6448	-	1.6734	-	1.6734	-	1.6734	-
Y_{n+1}	0.0405	-	0.0188	-	0.0108	-	-0.0368	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.4729	0.0066	0.4782	0.0034	0.4803	0.0023	0.4810	0.0017
Lindley's - θ_1 with gamma prior for τ	0.4862	0.0046	0.4844	0.0028	0.4843	0.0021	0.4839	0.0016
Lindley's - θ_1 with improper prior for τ	0.4862	0.0046	0.4844	0.0028	0.4843	0.0021	0.4839	0.0016
Gibbs - θ_1 with gamma prior for τ	0.4846	0.0048	0.4839	0.0029	0.4841	0.0021	0.4839	0.0016
Gibbs - θ_1 with improper prior for τ	0.4847	0.0048	0.4841	0.0029	0.4841	0.0021	0.4839	0.0016
MLEs of τ	1.6830	0.0286	1.6931	0.0158	1.6866	0.0109	1.6801	0.0083
Lindley's - τ with gamma prior	1.4687	0.2817	1.6045	0.0230	1.6350	0.0111	1.6446	0.0084
Lindley's - τ with improper prior	1.6283	0.0292	1.6660	0.0160	1.6685	0.0110	1.6665	0.0084
Gibbs - τ with gamma prior	1.6507	0.0223	1.6679	0.0134	1.6692	0.0095	1.6674	0.0077
Gibbs - τ with improper prior	1.7014	0.0304	1.7020	0.0164	1.6916	0.0112	1.6837	0.0084
MLEs of Y_{n+1}	0.0133	0.2297	-0.0011	0.2104	-0.0068	0.1895	0.0056	0.2029
Lindley's - Y_{n+1} with gamma prior for τ	0.2739	0.1958	0.2525	0.1854	0.2450	0.1694	0.2559	0.1885
Lindley's - Y_{n+1} with improper prior for τ	0.2847	0.1944	0.2573	0.1852	0.2480	0.1695	0.2582	0.1886
Gibbs - Y_{n+1} with gamma prior for τ	0.2713	0.1959	0.2509	0.1851	0.2444	0.1692	0.2559	0.1883
Gibbs - Y_{n+1} with improper prior for τ	0.2835	0.1947	0.2568	0.1851	0.2477	0.1691	0.2584	0.1890

Table 8.5. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4965	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6714	-	1.6734	-	1.6734	-	1.6734	-
Y_{n+1}	0.0362	-	0.0188	-	0.0108	-	-0.0368	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.4732	0.0007	0.4782	0.0004	0.4803	0.0003	0.4810	0.0002
Lindley's - θ_1 with gamma prior for τ	0.4819	0.0005	0.4820	0.0003	0.4827	0.0002	0.4827	0.0002
Lindley's - θ_1 with improper prior for τ	0.4819	0.0005	0.4820	0.0003	0.4827	0.0002	0.4827	0.0002
Gibbs - θ_1 with gamma prior for τ	0.4806	0.0005	0.4816	0.0003	0.4826	0.0002	0.4827	0.0002
Gibbs - θ_1 with improper prior for τ	0.4807	0.0005	0.4818	0.0003	0.4825	0.0002	0.4827	0.0002
MLEs of τ	1.7275	0.0037	1.6931	0.0018	1.6866	0.0012	1.6801	0.0009
Lindley's - τ with gamma prior	1.4828	0.0082	1.6001	0.0019	1.6282	0.0012	1.6385	0.0009
Lindley's - τ with improper prior	1.6398	0.0035	1.6501	0.0017	1.6580	0.0012	1.6587	0.0009
Gibbs - τ with gamma prior	1.6544	0.0025	1.6553	0.0015	1.6603	0.0011	1.6605	0.0009
Gibbs - τ with improper prior	1.7133	0.0037	1.6857	0.0018	1.6810	0.0012	1.6759	0.0009
MLEs of Y_{n+1}	0.0125	0.0213	-0.0011	0.0208	-0.0068	0.0190	0.0056	0.0213
Lindley's - Y_{n+1} with gamma prior for τ	0.0981	0.0205	0.0827	0.0204	0.0769	0.0187	0.0887	0.0213
Lindley's - Y_{n+1} with improper prior for τ	0.1011	0.0205	0.0843	0.0204	0.0780	0.0187	0.0895	0.0213
Gibbs - Y_{n+1} with gamma prior for τ	0.0965	0.0205	0.0818	0.0204	0.0768	0.0186	0.0892	0.0212
Gibbs - Y_{n+1} with improper prior for τ	0.1005	0.0205	0.0843	0.0204	0.0774	0.0186	0.0895	0.0213

Table 8.6. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4964	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6734	-	1.6734	-	1.6734	-	1.6734	-
Y_{n+1}	0.0361	-	0.0188	-	0.0108	-	-0.0368	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.4730	0.0007	0.4782	0.0004	0.4803	0.0003	0.4810	0.0002
Lindley's - θ_1 with gamma prior for τ	0.4772	0.0005	0.4796	0.0003	0.4811	0.0002	0.4815	0.0002
Lindley's - θ_1 with improper prior for τ	0.4772	0.0005	0.4796	0.0003	0.4811	0.0002	0.4815	0.0002
Gibbs - θ_1 with gamma prior for τ	0.4762	0.0005	0.4793	0.0003	0.4810	0.0002	0.4815	0.0002
Gibbs - θ_1 with improper prior for τ	0.4763	0.0005	0.4795	0.0003	0.4810	0.0002	0.4815	0.0002
MLEs of τ	1.7353	0.0048	1.6931	0.0019	1.6866	0.0012	1.6801	0.0009
Lindley's - τ with gamma prior	1.5108	0.0039	1.5941	0.0017	1.6219	0.0011	1.6329	0.0009
Lindley's - τ with improper prior	1.6161	0.0039	1.6348	0.0017	1.6478	0.0012	1.6511	0.0009
Gibbs - τ with gamma prior	1.6368	0.0025	1.6430	0.0015	1.6515	0.0010	1.6537	0.0009
Gibbs - τ with improper prior	1.6874	0.0041	1.6699	0.0018	1.6706	0.0012	1.6681	0.0009
MLEs of Y_{n+1}	0.0123	0.0203	-0.0011	0.0202	-0.0068	0.0188	0.0056	0.0221
Lindley's - Y_{n+1} with gamma prior for τ	-0.0743	0.0200	-0.0867	0.0201	-0.0908	0.0187	-0.0782	0.0216
Lindley's - Y_{n+1} with improper prior for τ	-0.0773	0.0201	-0.0883	0.0202	-0.0918	0.0187	-0.0790	0.0216
Gibbs - Y_{n+1} with gamma prior for τ	-0.0743	0.0200	-0.0870	0.0201	-0.0907	0.0186	-0.0774	0.0216
Gibbs - Y_{n+1} with improper prior for τ	-0.0773	0.0201	-0.0875	0.0201	-0.0927	0.0186	-0.0792	0.0216

Table 8.7. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4964	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6734	-	1.6734	-	1.6734	-	1.6734	-
Y_{n+1}	0.0361	-	0.0188	-	0.0108	-	-0.0368	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.4730	0.0064	0.4782	0.0033	0.4803	0.0022	0.4810	0.0017
Lindley's - θ_1 with gamma prior for τ	0.4726	0.0046	0.4772	0.0028	0.4794	0.0021	0.4803	0.0016
Lindley's - θ_1 with improper prior for τ	0.4726	0.0046	0.4772	0.0028	0.4794	0.0021	0.4803	0.0016
Gibbs - θ_1 with gamma prior for τ	0.4720	0.0048	0.4770	0.0028	0.4794	0.0021	0.4803	0.0016
Gibbs - θ_1 with improper prior for τ	0.4721	0.0048	0.4772	0.0028	0.4794	0.0021	0.4803	0.0016
MLEs of τ	1.7353	0.0544	1.6931	0.0174	1.6866	0.0115	1.6801	0.0087
Lindley's - τ with gamma prior	1.5207	0.0281	1.5883	0.0147	1.6160	0.0100	1.6276	0.0080
Lindley's - τ with improper prior	1.5899	0.0367	1.6207	0.0153	1.6381	0.0105	1.6438	0.0082
Gibbs - τ with gamma prior	1.6167	0.0219	1.6310	0.0133	1.6429	0.0094	1.6470	0.0076
Gibbs - τ with improper prior	1.6556	0.0382	1.6545	0.0155	1.6603	0.0107	1.6605	0.0082
MLEs of Y_{n+1}	0.0123	0.1917	-0.0011	0.1921	-0.0068	0.1818	0.0056	0.2283
Lindley's - Y_{n+1} with gamma prior for τ	-0.2469	0.1775	-0.2565	0.1780	-0.2588	0.1689	-0.2454	0.1970
Lindley's - Y_{n+1} with improper prior for τ	-0.2562	0.1794	-0.2613	0.1785	-0.2619	0.1692	-0.2476	0.1968
Gibbs - Y_{n+1} with gamma prior for τ	-0.2458	0.1776	-0.2562	0.1782	-0.2585	0.1687	-0.2441	0.1970
Gibbs - Y_{n+1} with improper prior for τ	-0.2559	0.1796	-0.2596	0.1783	-0.2630	0.1689	-0.2481	0.1975

Table 8.8. Average MA(1) model estimates and estimation errors under independent truncated normal prior for θ_1 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4964	-	0.4964	-	0.4964	-	0.4964	-
τ	1.6734	-	1.6734	-	1.6734	-	1.6734	-
Y_{n+1}	0.0361	-	0.0188	-	0.0108	-	-0.0368	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.4730	0.0177	0.4782	0.0090	0.4803	0.0062	0.4810	0.0046
Lindley's - θ_1 with gamma prior for τ	0.4680	0.0127	0.4748	0.0078	0.4778	0.0057	0.4791	0.0044
Lindley's - θ_1 with improper prior for τ	0.4680	0.0127	0.4748	0.0078	0.4778	0.0057	0.4791	0.0044
Gibbs - θ_1 with gamma prior for τ	0.4677	0.0132	0.4747	0.0079	0.4779	0.0057	0.4791	0.0043
Gibbs - θ_1 with improper prior for τ	0.4678	0.0132	0.4749	0.0078	0.4778	0.0057	0.4791	0.0044
MLEs of τ	1.7353	0.2325	1.6931	0.0511	1.6866	0.0331	1.6801	0.0248
Lindley's - τ with gamma prior	1.5243	0.0750	1.5829	0.0391	1.6106	0.0270	1.6228	0.0218
Lindley's - τ with improper prior	1.5691	0.1196	1.6081	0.0422	1.6291	0.0289	1.6368	0.0227
Gibbs - τ with gamma prior	1.5972	0.0601	1.6192	0.0366	1.6344	0.0259	1.6404	0.0212
Gibbs - τ with improper prior	1.6255	0.1127	1.6395	0.0424	1.6503	0.0292	1.6529	0.0227
MLEs of Y_{n+1}	0.0123	0.6150	-0.0011	0.6133	-0.0068	0.5972	0.0056	0.8221
Lindley's - Y_{n+1} with gamma prior for τ	-0.4214	0.4893	-0.4275	0.4851	-0.4277	0.4733	-0.4131	0.5556
Lindley's - Y_{n+1} with improper prior for τ	-0.4372	0.4986	-0.4355	0.4883	-0.4329	0.4749	-0.4170	0.5541
Gibbs - Y_{n+1} with gamma prior for τ	-0.4197	0.4893	-0.4267	0.4857	-0.4268	0.4738	-0.4111	0.5575
Gibbs - Y_{n+1} with improper prior for τ	-0.4377	0.4991	-0.4333	0.4883	-0.4343	0.4746	-0.4178	0.5587

8.1.2. Independent Uniform prior for θ_1 and Gamma or Improper priors for τ

We consider a uniform prior for θ_1 in the interval (c_1, d_1), where $c_1 = 0.25$, $d_1 = 0.75$. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$.

Table 8.9 presents the average values of MA(1) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used.

Table 8.10, Table 8.11, Table 8.12, Table 8.13, Table 8.14 and Table 8.15 present the average values of MA(1) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25, -0.75, -0.25,$ $0.25, 0.75, 1.25$, respectively.

The average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overall the Bayes estimates are found to be perform better than the ML estimates. For θ_1 and one-step prediction, the Bayes estimates are found to have smaller average estimation errors under both SE and LINEX loss functions with all parameter γ values. Under SE loss function, for τ the Bayes estimation is found to result in the smallest estimation errors. Under the LINEX loss function, when τ has gamma prior the Bayes estimation has better performance than the ML estimation for all parameter γ values. When when τ has improper prior the ML estimates have smaller average errors when $\gamma = -1.25, -0.75$, when $\gamma = -0.25, 0.25, 0.75, 1.25$ the Bayes estimation is superior. All estimator performances are reasonably close to each other as the sample size increases.

Table 8.9. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using SE loss function

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0225	0.4793	0.0114	0.4812	0.0079	0.4824	0.0061
Bayes - θ_1 with gamma prior for τ	0.4891	0.0121	0.4865	0.0082	0.4866	0.0063	0.4869	0.0051
Bayes - θ_1 with improper prior for τ	0.4891	0.0120	0.4864	0.0082	0.4866	0.0063	0.4869	0.0051
MLEs of τ	1.7505	0.1552	1.7014	0.0710	1.6897	0.0458	1.6826	0.0332
Bayes - τ with gamma prior	1.6522	0.0905	1.6530	0.0548	1.6570	0.0387	1.6580	0.0296
Bayes - τ with improper prior	1.7155	0.1450	1.6844	0.0690	1.6785	0.0450	1.6742	0.0328
MLEs of Y_{n+1}	0.0045	0.6919	-0.0005	0.6794	-0.0008	0.6761	0.0068	0.6766
Bayes - Y_{n+1} with gamma prior for τ	0.0044	0.6838	-0.0011	0.6762	-0.0011	0.6746	0.0063	0.6764
Bayes - Y_{n+1} with improper prior for τ	0.0044	0.6837	-0.0011	0.6762	-0.0011	0.6746	0.0063	0.6764

Table 8.10. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0182	0.4793	0.0092	0.4812	0.0063	0.4824	0.0049
Bayes - θ_1 with gamma prior for τ	0.4927	0.0155	0.4863	0.0085	0.4858	0.0060	0.4858	0.0047
Bayes - θ_1 with improper prior for τ	0.4926	0.0155	0.4863	0.0085	0.4858	0.0060	0.4858	0.0047
MLEs of τ	1.7505	0.1031	1.7014	0.0519	1.6897	0.0344	1.6826	0.0253
Bayes - τ with gamma prior	1.7076	0.0755	1.6850	0.0449	1.6796	0.0314	1.6754	0.0237
Bayes - τ with improper prior	1.8090	0.1226	1.7262	0.0557	1.7055	0.0360	1.6941	0.0261
MLEs of Y_{n+1}	0.0045	0.7415	-0.0005	0.7259	-0.0008	0.7532	0.0068	0.7464
Bayes - Y_{n+1} with gamma prior for τ	0.4213	0.5455	0.4170	0.5374	0.4166	0.5359	0.4245	0.5363
Bayes - Y_{n+1} with improper prior for τ	0.4351	0.5476	0.4249	0.5386	0.4220	0.5355	0.4286	0.5360

Table 8.11. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0064	0.4793	0.0033	0.4812	0.0023	0.4824	0.0017
Bayes - θ_1 with gamma prior for τ	0.4876	0.0056	0.4838	0.0030	0.4841	0.0022	0.4846	0.0017
Bayes - θ_1 with improper prior for τ	0.4875	0.0056	0.4838	0.0030	0.4841	0.0022	0.4846	0.0017
MLEs of τ	1.7505	0.0389	1.7014	0.0190	1.6897	0.0125	1.6826	0.0092
Bayes - τ with gamma prior	1.6849	0.0264	1.6720	0.0158	1.6704	0.0111	1.6684	0.0085
Bayes - τ with improper prior	1.7698	0.0427	1.7091	0.0198	1.6945	0.0128	1.6860	0.0093
MLEs of Y_{n+1}	0.0045	0.2157	-0.0005	0.2128	-0.0008	0.2162	0.0068	0.2133
Bayes - Y_{n+1} with gamma prior for τ	0.2546	0.1946	0.2500	0.1922	0.2496	0.1921	0.2574	0.1915
Bayes - Y_{n+1} with improper prior for τ	0.2629	0.1950	0.2547	0.1924	0.2529	0.1920	0.2599	0.1915

Table 8.12. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0007	0.4793	0.0004	0.4812	0.0002	0.4824	0.0002
Bayes - θ_1 with gamma prior for τ	0.4825	0.0006	0.4813	0.0003	0.4825	0.0002	0.4834	0.0002
Bayes - θ_1 with improper prior for τ	0.4825	0.0006	0.4813	0.0003	0.4825	0.0002	0.4834	0.0002
MLEs of τ	1.7505	0.0046	1.7014	0.0022	1.6897	0.0014	1.6826	0.0010
Bayes - τ with gamma prior	1.6629	0.0029	1.6593	0.0017	1.6615	0.0012	1.6615	0.0009
Bayes - τ with improper prior	1.7330	0.0046	1.6925	0.0022	1.6838	0.0014	1.6781	0.0010
MLEs of Y_{n+1}	0.0045	0.0218	-0.0005	0.0215	-0.0008	0.0215	0.0068	0.0214
Bayes - Y_{n+1} with gamma prior for τ	0.0879	0.0216	0.0830	0.0213	0.0827	0.0212	0.0903	0.0212
Bayes - Y_{n+1} with improper prior for τ	0.0907	0.0216	0.0846	0.0213	0.0838	0.0212	0.0911	0.0212

Table 8.13. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0007	0.4793	0.0004	0.4812	0.0002	0.4824	0.0002
Bayes - θ_1 with gamma prior for τ	0.4774	0.0006	0.4788	0.0003	0.4808	0.0002	0.4821	0.0002
Bayes - θ_1 with improper prior for τ	0.4774	0.0006	0.4788	0.0003	0.4808	0.0002	0.4821	0.0002
MLEs of τ	1.7505	0.0051	1.7014	0.0023	1.6897	0.0015	1.6826	0.0010
Bayes - τ with gamma prior	1.6418	0.0028	1.6468	0.0017	1.6526	0.0012	1.6546	0.0009
Bayes - τ with improper prior	1.6986	0.0045	1.6764	0.0021	1.6732	0.0014	1.6703	0.0010
MLEs of Y_{n+1}	0.0045	0.0220	-0.0005	0.0215	-0.0008	0.0213	0.0068	0.0214
Bayes - Y_{n+1} with gamma prior for τ	-0.0788	0.0217	-0.0840	0.0212	-0.0843	0.0211	-0.0768	0.0211
Bayes - Y_{n+1} with improper prior for τ	-0.0816	0.0217	-0.0856	0.0212	-0.0853	0.0211	-0.0776	0.0211

Table 8.14. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0063	0.4793	0.0032	0.4812	0.0022	0.4824	0.0017
Bayes - θ_1 with gamma prior for τ	0.4723	0.0056	0.4763	0.0030	0.4792	0.0021	0.4809	0.0017
Bayes - θ_1 with improper prior for τ	0.4724	0.0056	0.4763	0.0030	0.4792	0.0021	0.4809	0.0017
MLEs of τ	1.7505	0.0525	1.7014	0.0216	1.6897	0.0136	1.6826	0.0097
Bayes - τ with gamma prior	1.6213	0.0246	1.6347	0.0151	1.6439	0.0107	1.6478	0.0082
Bayes - τ with improper prior	1.6661	0.0388	1.6607	0.0190	1.6628	0.0125	1.6625	0.0091
MLEs of Y_{n+1}	0.0045	0.2224	-0.0005	0.2153	-0.0008	0.2087	0.0068	0.2139
Bayes - Y_{n+1} with gamma prior for τ	-0.2455	0.1980	-0.2511	0.1920	-0.2512	0.1885	-0.2439	0.1911
Bayes - Y_{n+1} with improper prior for τ	-0.2538	0.1980	-0.2558	0.1920	-0.2544	0.1885	-0.2463	0.1911

Table 8.15. Average MA(1) model estimates and estimation errors under independent uniform prior for θ_1 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.4994	-	0.4994	-	0.4994	-	0.4994	-
τ	1.6747	-	1.6747	-	1.6747	-	1.6747	-
Y_{n+1}	-0.0018	-	-0.0054	-	0.0086	-	0.0019	-
ESTIMATES								
MLEs of θ_1	0.4783	0.0173	0.4793	0.0087	0.4812	0.0060	0.4824	0.0047
Bayes - θ_1 with gamma prior for τ	0.4672	0.0156	0.4738	0.0084	0.4775	0.0060	0.4796	0.0046
Bayes - θ_1 with improper prior for τ	0.4673	0.0156	0.4738	0.0084	0.4775	0.0060	0.4796	0.0046
MLEs of τ	1.7505	0.1741	1.7014	0.0647	1.6897	0.0394	1.6826	0.0276
Bayes - τ with gamma prior	1.6016	0.0671	1.6227	0.0412	1.6353	0.0294	1.6411	0.0226
Bayes - τ with improper prior	1.6354	0.1042	1.6455	0.0521	1.6527	0.0344	1.6549	0.0252
MLEs of Y_{n+1}	0.0045	0.7913	-0.0005	0.7644	-0.0008	0.7102	0.0068	0.7425
Bayes - Y_{n+1} with gamma prior for τ	-0.4122	0.5634	-0.4181	0.5398	-0.4182	0.5210	-0.4110	0.5359
Bayes - Y_{n+1} with improper prior for τ	-0.4261	0.5630	-0.4259	0.5391	-0.4235	0.5214	-0.4151	0.5360

8.1.3. Analysis of Overshorts Data

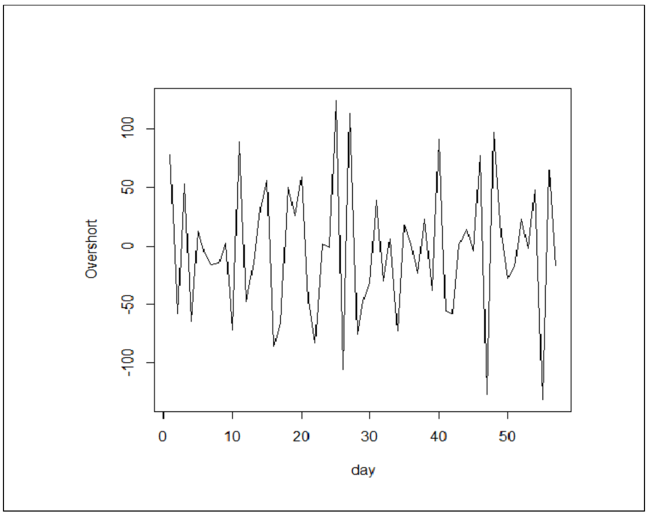
We consider the overshorts data from an underground gasoline tank in Colorado. The series consists of 57 daily observations. The same data were analyzed by Brockwell and Davis, see Example 3.2.8, [3]. Overshort for day t is

$$\begin{aligned} Z_t &= (amount\ of\ fuel\ at\ end\ of\ day\ t) \\ &- (amount\ of\ fuel\ at\ end\ of\ day\ t-1) \\ &- (amount\ of\ fuel\ delivered\ during\ day\ t) \\ &+ (amount\ of\ fuel\ sold\ during\ day\ t) \end{aligned}$$

With no measurement error and no tank leaks, $Z_t = 0$.

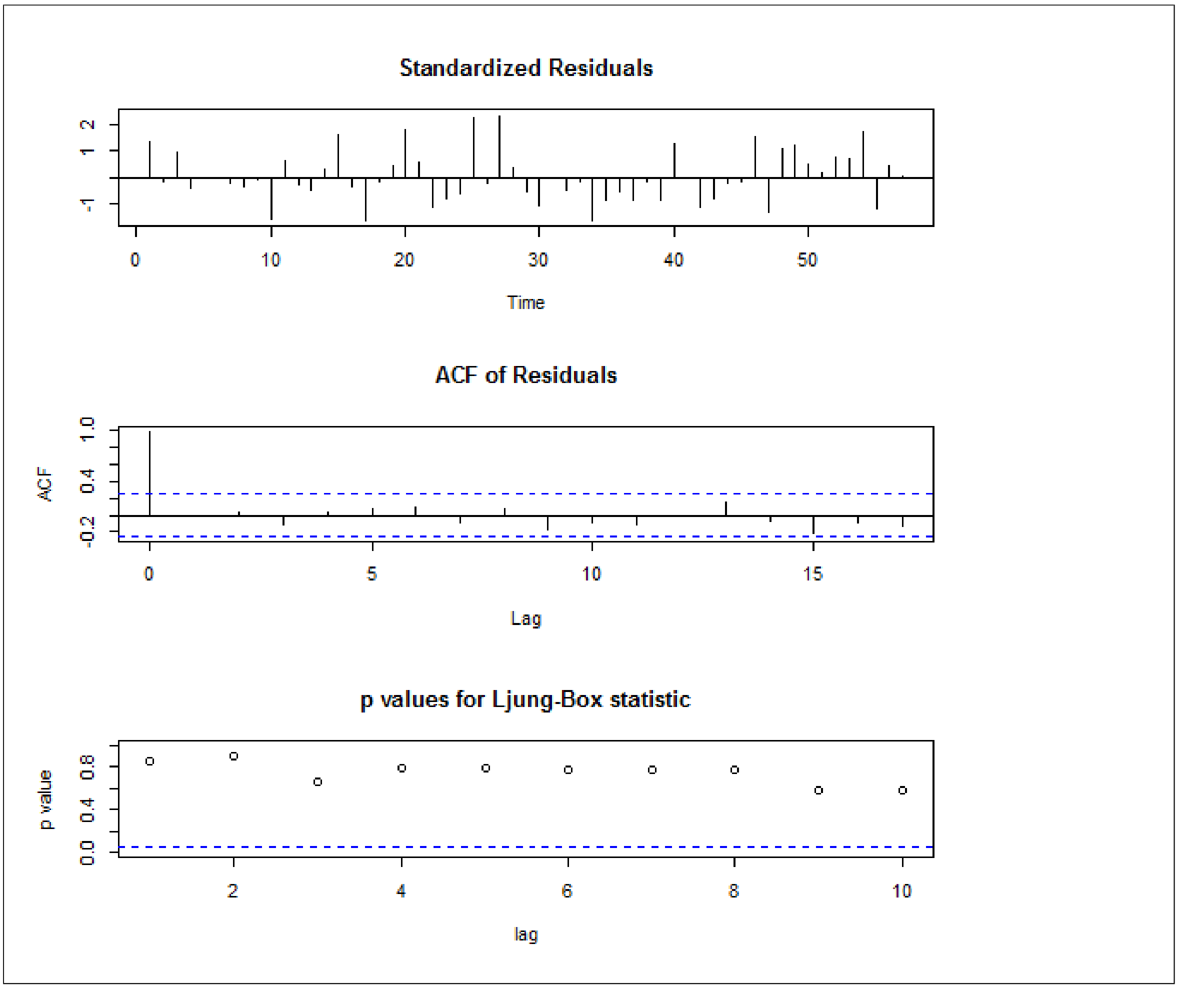
Figure 8.1 shows the plotted overshorts series.

Figure 8.1. Overshorts data



We analyze the data using the independent truncated normal prior for θ_1 with mean equal to the sample mean and variance equal to the sample variance and improper prior for τ. The LINEX loss function's parameter are $\gamma = 0.25$ and $\gamma = -0.25$. In order to apply the analysis using the assumed form of the MA(1) model, we need to subtract the series mean from each of the observations to obtain a zero-mean series. Model checking shows that the MA(1) model can be fitted to the zero-mean series; see Figure 8.2

Figure 8.2. MA(1) model checking.



We obtain estimates of θ_1, τ and one-step predicted value using 47, 48,…, 55, 56 observations. Table 8.16 and Table 8.17 present the estimation and one-step prediction results. It is observed that under the LINEX loss function the prediction errors of the Bayes estimates are significantly smaller than that of the ML estimates, the Bayes estimates obtained using Gibbs sampling method have the smallest average estimation errors. Under the SE loss function, the ML and Bayes estimates obtained using Lindley's approximation are found to have similar average prediction error, whereas the average prediction error of Bayes estimates obtained using Gibbs sampler is slightly higher.

Table 8.16. MA(1) model estimates for empirical data

	SAMPLE SIZE										
ACTUAL	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**AVERAGE**
Y_{n+1}	97.0000	10.0000	-28.0000	-17.0000	23.0000	-2.0000	48.0000	-131.0000	65.0000	-17.0000	4.8000
ESTIMATES											
ML											
θ_1	0.6593	0.7022	0.6580	0.6587	0.6574	0.6590	0.6557	0.6472	0.7012	0.7041	0.6703
τ	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
Y_{n+1}	46.0685	-45.0271	-31.3555	-1.9591	5.0355	-16.5333	-10.8727	-38.3675	61.7445	-17.0909	-4.8358
SE LOSS FUNCTION											
θ_1 Lindley's approx.	0.6593	0.7022	0.6580	0.6587	0.6574	0.6590	0.6557	0.6472	0.7012	0.7041	0.6703
θ_1 Gibbs sampling	0.6594	0.7022	0.6594	0.6557	0.6557	0.6616	0.6640	0.6471	0.7031	0.7012	0.6709
τ Lindley's approx.	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
τ Gibbs sampling	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
Y_{n+1} Lindley's approx.	46.0681	-45.0269	-31.3553	-1.9591	5.0355	-16.5332	-10.8727	-38.3673	61.7441	-17.0908	-4.8358
Y_{n+1} Gibbs sampling	47.4545	-46.9673	-32.1848	-1.8223	5.8638	-17.2539	-11.3590	-36.4190	63.3628	-17.9196	-4.7245
LINEX LOSS FUNCTION $\gamma = 0.25$											
θ_1 Lindley's approx.	0.6565	0.6996	0.6555	0.6563	0.6550	0.6566	0.6534	0.6449	0.6988	0.7020	0.6679
θ_1 Gibbs sampling	0.6564	0.6992	0.6567	0.6533	0.6533	0.6591	0.6616	0.6446	0.7007	0.6990	0.6684
τ Lindley's approx.	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
τ Gibbs sampling	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
Y_{n+1} Lindley's approx.	-270.4496	-359.2885	-350.7894	-314.5892	-302.3340	-319.0745	-308.5450	-339.6435	-249.4175	-322.3521	-313.6483
Y_{n+1} Gibbs sampling	-90.2968	-195.5979	-169.5654	-140.8540	-123.9859	-174.5222	-156.9113	-169.3913	-69.3697	-161.6458	-145.2140
LINEX LOSS FUNCTION $\gamma = -0.25$											
θ_1 Lindley's approx.	0.6620	0.7048	0.6605	0.6611	0.6598	0.6613	0.6580	0.6495	0.7035	0.7063	0.6727
θ_1 Gibbs sampling	0.6624	0.7051	0.6620	0.6582	0.6582	0.6641	0.6664	0.6495	0.7055	0.7035	0.6735
τ Lindley's approx.	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
τ Gibbs sampling	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
Y_{n+1} Lindley's approx.	362.5865	269.2343	288.0784	310.6710	312.4051	286.0079	286.7997	262.9086	372.9064	288.1703	303.9768
Y_{n+1} Gibbs sampling	180.8496	103.5195	124.6484	132.6926	151.9238	117.5090	119.1664	119.6699	239.5728	131.6559	142.1208

Table 8.17. MA(1) model errors for empirical data

	SAMPLE SIZE										
ERRORS	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**AVERAGE**
SE LOSS FUNCTION											
Y_{n+1} ML	2594.0227	3027.9815	11.2592	226.2293	322.7219	211.2174	3465.9946	8580.7837	10.5984	0.0083	1845.0817
Y_{n+1} Lindley's approx.	2594.0563	3027.9553	11.2582	226.2290	322.7238	211.2153	3465.9899	8580.8209	10.6007	0.0082	1845.0857
Y_{n+1} Gibbs sampling	2454.7559	3245.2716	17.5122	230.3622	293.6498	232.6815	3523.4945	8945.5678	2.6803	0.8457	1894.6822
LINEX LOSS FUNCTION $\gamma = 0.25$											
Y_{n+1} ML	11.7329	12.7568	0.2711	38.1981	3.5023	2.6598	13.7182	1.14×10^{11}	0.2570	0.0003	1.14×10^{10}
Y_{n+1} Lindley's approx.	90.8624	91.3221	79.6973	73.3973	80.3335	78.2686	88.1363	51.1609	77.6044	75.3380	78.6121
Y_{n+1} Gibbs sampling	45.8242	50.3995	34.3914	29.9635	35.7465	42.1305	50.2278	8.5979	32.5924	35.1614	36.5035
LINEX LOSS FUNCTION $\gamma = -0.25$											
Y_{n+1} ML	338692.1297	942940.7496	0.4749	2.7835	83.7298	33.2053	2466148.3919	22.1581	0.4428	0.0003	374792.4066
Y_{n+1} Lindley's approx.	65.3966	63.8086	78.0196	80.9178	71.3513	71.0020	58.6999	97.4771	75.9766	75.2926	73.7942
Y_{n+1} Gibbs sampling	19.9624	22.3799	37.1621	36.4231	31.2309	28.8772	16.7916	61.6675	42.6432	36.1640	33.3302

8.2. MA(1) MODEL PARAMETER IMPACT ANALYSIS

8.2.1. Independent Truncated Normal prior for θ_1 and Gamma or Improper priors for τ

To estimate the impact of parameter α, we use fixed parameters $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\alpha = 10, 20, 30$. Figure 8.3 below compares the average estimation and prediction errors when α varies. We notice that as α increases, the average estimation errors of θ_1 slightly increase, the average estimation errors of τ increase more than parameter α increase and the average prediction errors decrease proportionally to α increase. It suggests that the average estimation errors and α have inverse linear relationship.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $\mu_1 = 0.375$, $\sigma_1 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 8.4 compares the average estimation and prediction errors when β varies. We notice that β and the average estimation errors of θ_1 have a weak nonlinear relationship, the average estimation errors of τ and β have nonlinear inverse relationship. As parameter β increases, the average prediction errors increase.

To estimate the impact of parameter μ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\sigma_1 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\mu_1 = 0.125, 0.375, 0.625, 0.875$. Figure 8.5 compares the average estimation and prediction errors when μ_1 varies. We notice that as μ_1 increases, the average estimation errors of θ_1 decrease. The changes of average estimation errors of τ remain almost unchanged. The average prediction errors slightly increase as μ_1 increases.

To estimate the impact of parameter σ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\sigma_1 = 0.1, 0.2, 0.3$. Figure 8.6 compares the average estimation and prediction errors when σ_1 varies. We notice that as σ_1 increases, the average estimation errors of θ_1 increase, the average estimation errors of τ remain unchanged. Under the SE loss function, the average Bayes prediction errors and σ_1 have a nonlinear relationship, whereas the average ML prediction errors increase slightly as σ_1 increases. Under the LINEX loss function, the average prediction errors change insignificantly as σ_1 varies.

To estimate the impact of parameter γ, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.3$, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 8.7 compares the average estimation and prediction errors when γ varies. We notice that as γ increases, the average estimation errors of θ_1, τ and prediction increase more than the increase in γ.

Figure 8.3. Impact of parameter α on average estimation and prediction errors for MA(1) independent truncated normal prior for θ_1. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

Figure 8.4. Impact of parameter β on average estimation and prediction errors for MA(1) independent truncated normal prior for θ_1, (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

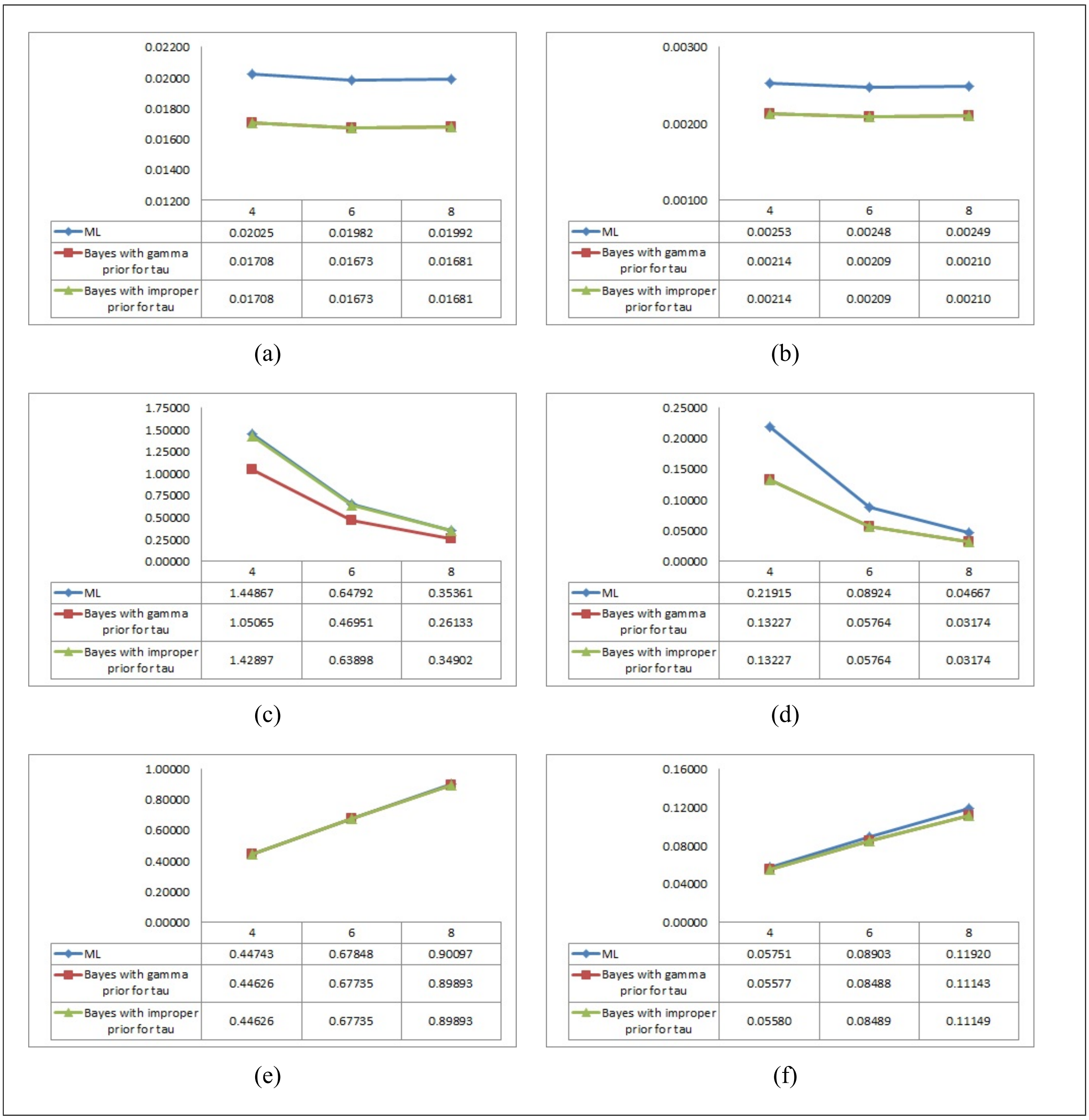


Figure 8.5. Impact of parameter μ_1 on average estimation and prediction errors for MA(1) independent truncated normal prior for θ_1. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

(a)

	0.125	0.375	0.625	0.875
ML	0.02003	0.01982	0.01763	0.01674
Bayes with gamma prior for tau	0.01690	0.01673	0.01504	0.01432
Bayes with improper prior for tau	0.01690	0.01673	0.01504	0.01432

(b)

	0.125	0.375	0.625	0.875
ML	0.00251	0.00248	0.00219	0.00205
Bayes with gamma prior for tau	0.00211	0.00209	0.00188	0.00178
Bayes with improper prior for tau	0.00211	0.00209	0.00188	0.00178

(c)

	0.125	0.375	0.625	0.875
ML	0.65031	0.64792	0.64193	0.62861
Bayes with gamma prior for tau	0.46972	0.46951	0.46953	0.47085
Bayes with improper prior for tau	0.64117	0.63898	0.63362	0.62190

(d)

	0.125	0.375	0.625	0.875
ML	0.08982	0.08924	0.08771	0.08403
Bayes with gamma prior for tau	0.05775	0.05764	0.05737	0.05684
Bayes with improper prior for tau	0.08099	0.08057	0.07949	0.07694

(e)

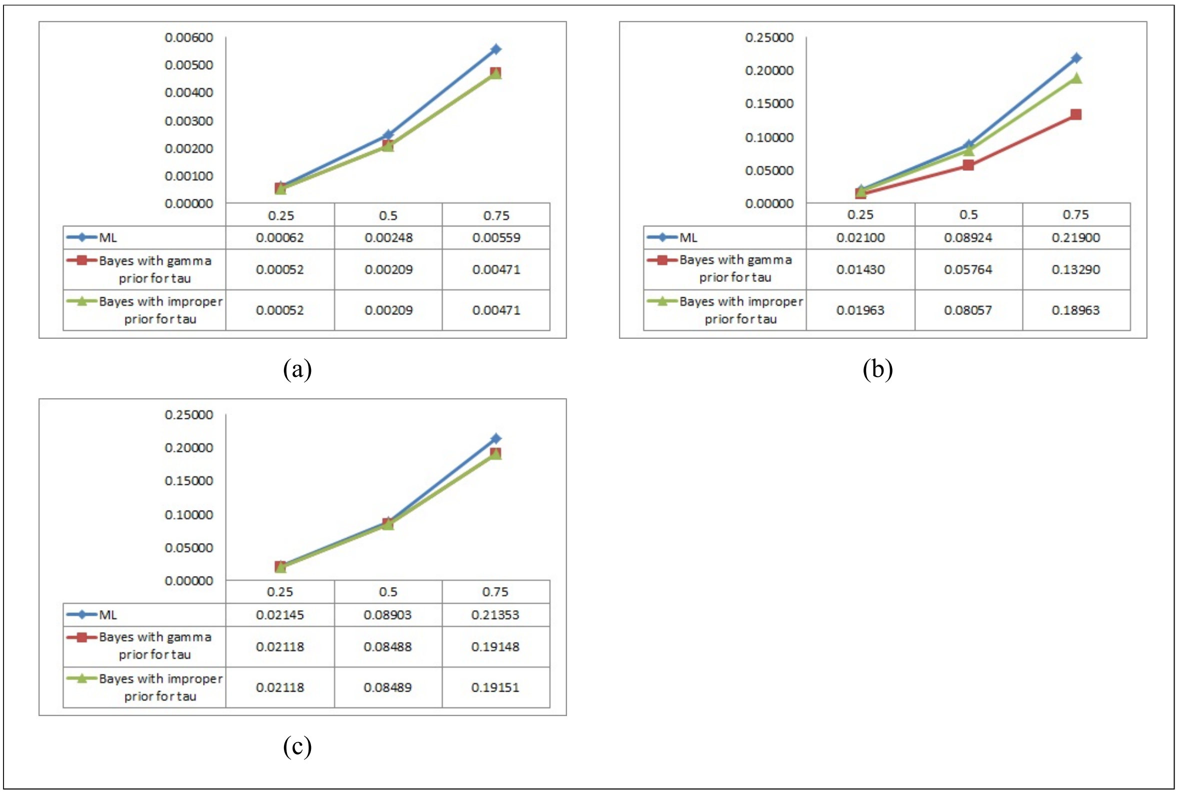
	0.125	0.375	0.625	0.875
ML	0.67735	0.67848	0.68525	0.70447
Bayes with gamma prior for tau	0.67628	0.67735	0.68425	0.70337
Bayes with improper prior for tau	0.67628	0.67735	0.68425	0.70337

(f)

	0.125	0.375	0.625	0.875
ML	0.08885	0.08903	0.09013	0.09280
Bayes with gamma prior for tau	0.08476	0.08488	0.08582	0.08830
Bayes with improper prior for tau	0.08477	0.08489	0.08581	0.08830

Figure 8.6. Impact of parameter σ_1 on average estimation and prediction errors for MA(1) independent truncated normal prior for θ_1. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

Figure 8.7. Impact of parameter γ on average estimation and prediction errors for MA(1) independent truncated normal prior for θ_1. (a) Estimation of θ_1 under LINEX, (b) Estimation of τ under LINEX loss, (c) Estimation of Y_{n+1} under LINEX loss.



8.2.2. Independent Uniform prior for θ_1 and Gamma or Improper priors for τ

To estimate the impact of parameter α, we use fixed parameters $\beta = 6$, $(c_1, d_1) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\alpha = 10, 20, 30$. Figure 8.8 below compares the average estimation and prediction errors when α varies. We notice that as α increases, the average estimation errors of θ_1 increase less that the original increase in α , the average estimation errors of τ increase more than parameter α . The average estimation errors and α have inverse almost linear relationship.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $(c_1, d_1) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 8.9 compares the average estimation and prediction errors when β varies. We notice that as β increases, the average estimation errors of θ_1 remain almost unchanged, the average estimation errors of τ and β have nonlinear inverse relationship. As parameter β increases, the average prediction errors increase.

To estimate the impact of the interval of θ_1, we use sample size of 100 and LINEX loss function parameter $\gamma = 0.5$ and obtain the average estimation errors when $(c_1, d_1) = (0, 0.25), (0.25, 0.5), (0.5, 0.75), (0.75, 1)$. Figure 8.10 compares the average estimation and prediction errors when the interval of θ_1 varies, the interval is represented by its middle point. We notice that as the mean of θ_1 increases, the average estimation errors of θ_1 increase, the average estimation errors of τ and the average prediction errors also increase when the mean of θ_1 increases.

To estimate the impact of parameter γ, we fix θ_1 interval, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 8.11 compares the average estimation and prediction error when γ varies. We notice that as γ increases, the average estimation errors of θ_1, τ and prediction increase more than the increase in γ.

Figure 8.12 below presents the average estimation errors of θ_1 under the LINEX loss function, estimated using the numerical method and the truncated normal approximation. We notice that the Bayes estimation errors for θ_1 are significantly smaller when the numerical approach is used (left in the figure); and the difference between the ML and the Bayes estimates becomes more noticeable, the Bayes estimates have the smallest average estimation errors.

Figure 8.8. Impact of parameter α on average estimation and prediction errors for MA(1) independent uniform prior for θ_1. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

Figure 8.9. Impact of parameter β on average estimation and prediction errors for MA(1) independent uniform prior for θ_1. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.

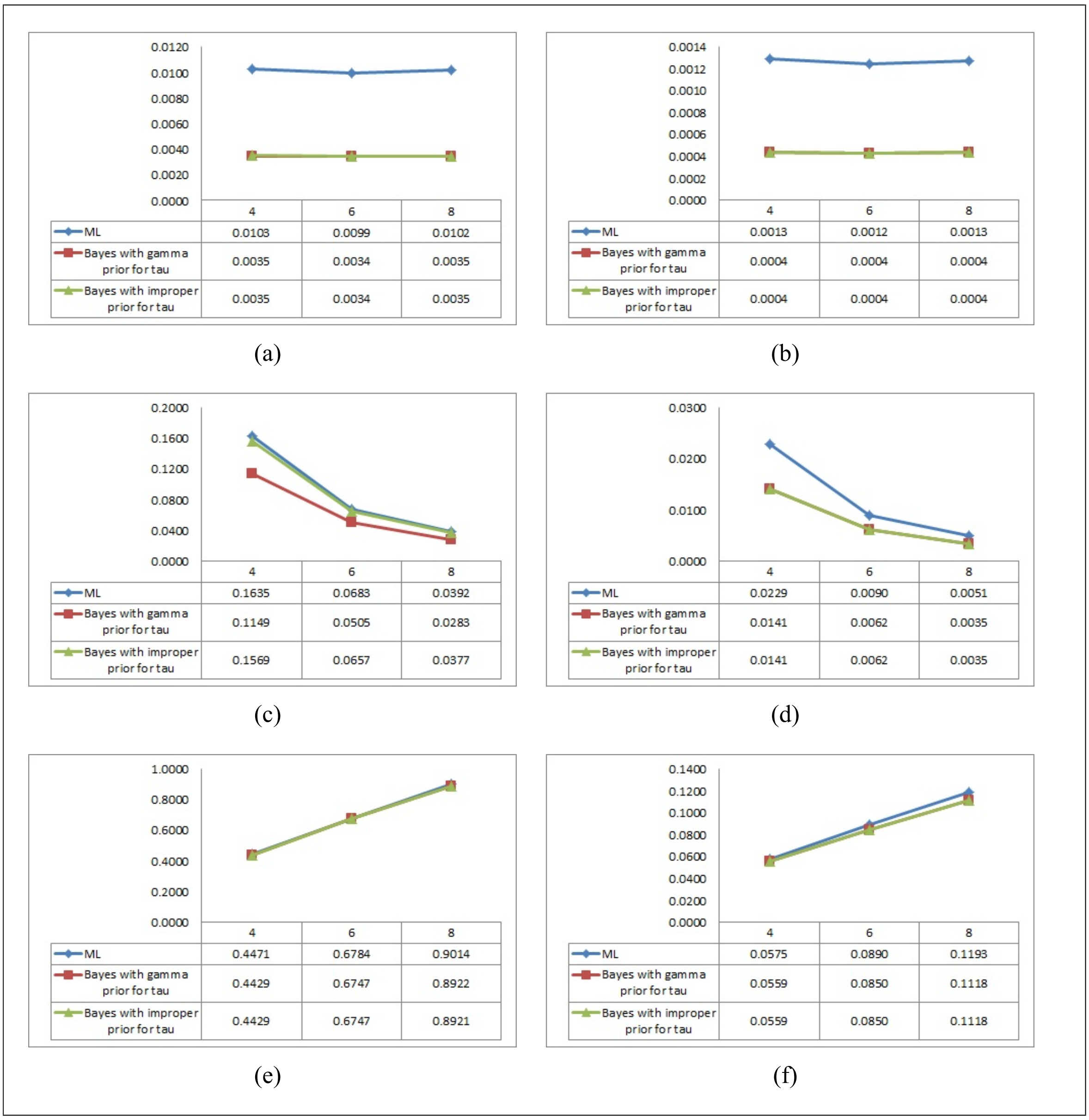
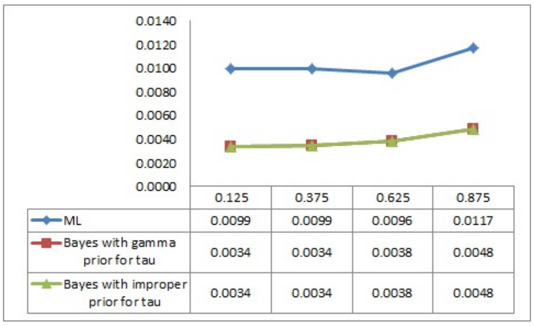
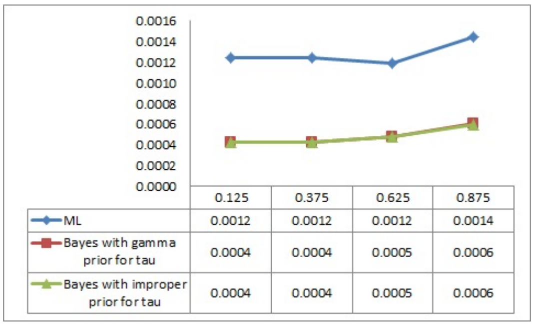


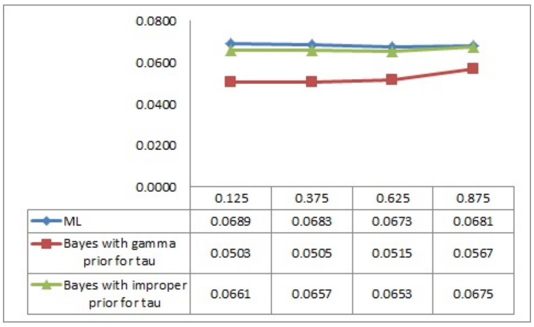
Figure 8.10. Impact of θ_1 interval on average estimation and prediction errors for MA(1) independent uniform prior for θ_1. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of τ under SE loss, (d) Estimation of τ under LINEX loss, (e) Estimation of Y_{n+1} under SE loss, (f) Estimation of Y_{n+1} under LINEX loss.



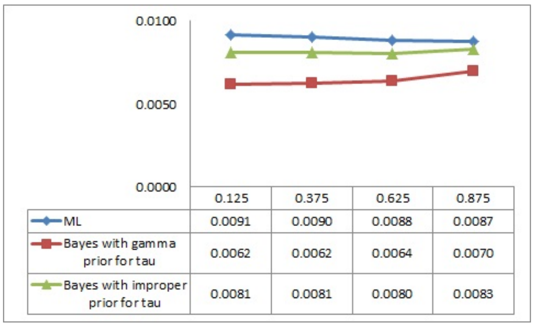
(a)



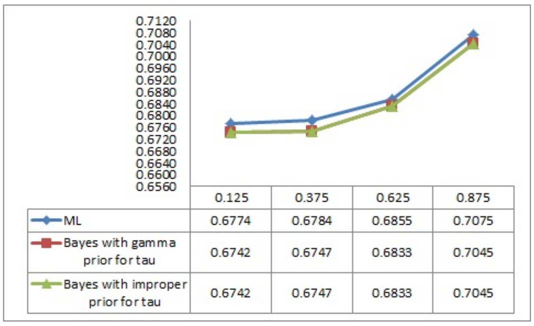
(b)



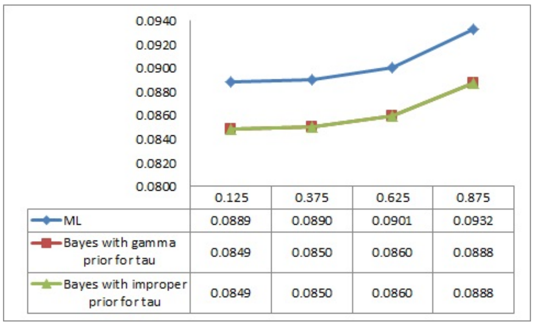
(c)



(d)

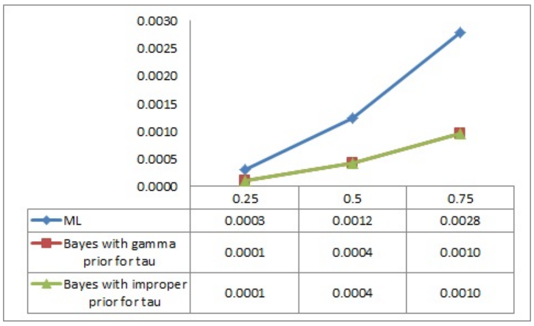


(e)



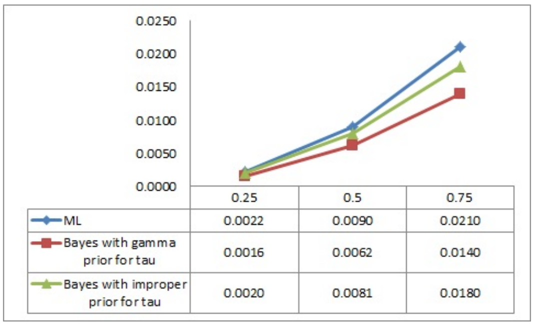
(f)

Figure 8.11. Impact of parameter γ on average estimation and prediction errors for MA(1) independent uniform prior for θ_1. (a) Estimation of θ_1 under LINEX, (b) Estimation of τ under LINEX loss, (c) Estimation of Y_{n+1} under LINEX loss.



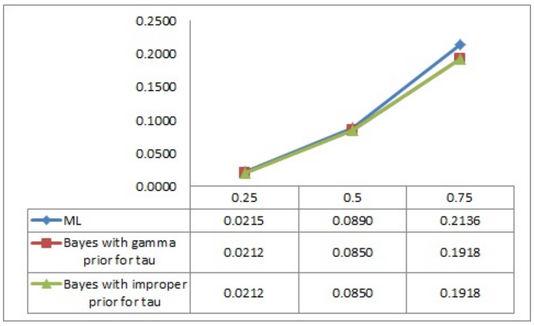
	0.25	0.5	0.75
ML	0.0003	0.0012	0.0028
Bayes with gamma prior for tau	0.0001	0.0004	0.0010
Bayes with improper prior for tau	0.0001	0.0004	0.0010

(a)



	0.25	0.5	0.75
ML	0.0022	0.0090	0.0210
Bayes with gamma prior for tau	0.0016	0.0062	0.0140
Bayes with improper prior for tau	0.0020	0.0081	0.0180

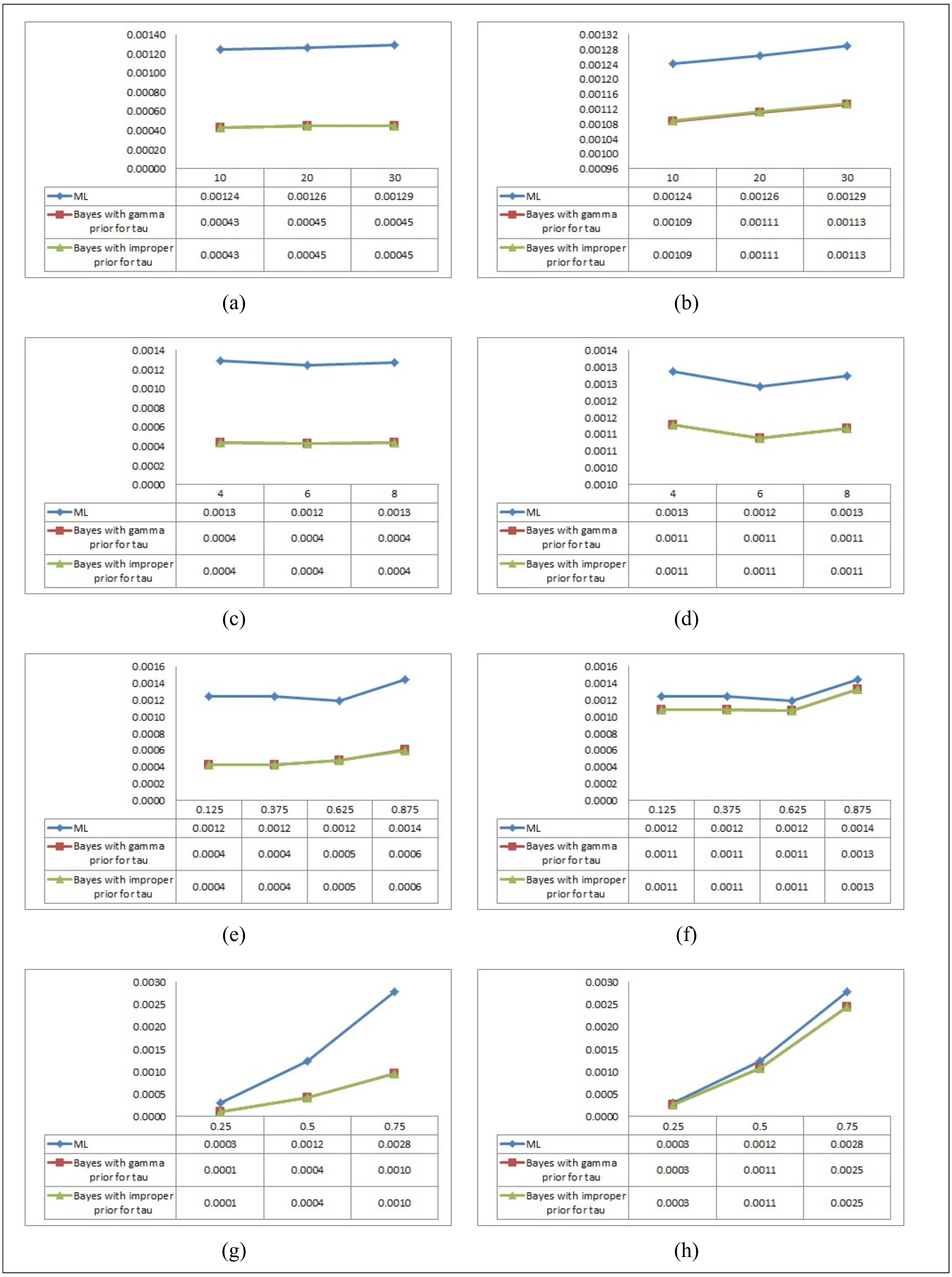
(b)



	0.25	0.5	0.75
ML	0.0215	0.0890	0.2136
Bayes with gamma prior for tau	0.0212	0.0850	0.1918
Bayes with improper prior for tau	0.0212	0.0850	0.1918

(c)

Figure 8.12. Impact of estimation method of θ_1 under LINEX loss function on average estimation and prediction errors for MA(1) independent uniform prior for θ_1. (a) Numerical approach, α varies, (b) Approximation, α varies, (c) Numerical approach, β varies, (d) Approximation, β varies, (e) Numerical approach, interval of θ_1 varies, (f) Approximation, interval of θ_1 varies, (e) Numerical approach, γ varies, (f) Approximation, γ varies.



8.3. MA(2) MODEL

In this section we study the MA(2) model

$$Y_t = -\theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2} + \epsilon_t. \tag{8.2}$$

8.3.1. Independent Truncated Normal prior for θ_1, θ_2 and Gamma or Improper priors for τ

We consider the independent truncated normal priors for θ_1 and θ_2 with parameters $\mu_1 = 0.25$, $\sigma_1 = 0.2$, $c_1 = 0$, $d_1 = 0.5$, and $\mu_2 = 0.2$, $\sigma_2 = 0.3$, $c_2 = 0$, $d_2 = 0.4$. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$.

Table 8.18 presents the average values of MA(2) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used. Under the SE loss function, the average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Generally, the Bayes estimates are found to have smaller average estimation errors than the ML estimates, for θ_1 and θ_2 the smallest estimation errors are obtained for the Bayes estimates obtained using Lindley's approximation; for τ the Bayes estimation using the Gibbs sampler is found to result in the smallest estimation errors when τ has gamma prior and the Bayes estimation using Lindley's approximation when τ has improper prior; for the one-step prediction all estimates have similar errors, the ones of Bayes estimates being slightly smaller. All estimator performances are reasonably close to each other as the sample size increases.

Under the LINEX loss function, there is a non-zero probability that the Bayes estimates using Lindley's approximation may be undefined (see Section 7.1.2), Table 8.19 shows proportion of undefined Bayes estimates using this approximation. Under our choice of parameters, undefined estimates are obtained only for τ when it has gamma prior and LINEX loss function parameters are $\gamma = -1.25, -0.75, -0.25$. The proportion of undefined τ estimates decreases significantly as sample size increases. We exclude the simulation where we obtain undefined estimates and calculate average errors where all estimates are defined.

Table 8.20, Table 8.21, Table 8.22, Table 8.23, Table 8.24 and Table 8.25 present the average values of MA(2) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters$\gamma = -1.25, -0.75, -0.25,$ $0.25, 0.75, 1.25$, respectively. Under the LINEX loss function, the average estimation errors are also found to decrease with increasing sample size. enerally, the Bayes estimation has the smallest average errors.

For θ_1 and θ_2, the smallest average estimation errors are obtained for the Bayes estimates using Lindley's approximation, the difference between the ML and the Bayes estimates are more noticeable when the LINEX parameter has higher absolute value. For the one-step prediction, generally, the Bayes estimates have smaller average errors than the ML estimates, the Gibbs sampler methods have slightly better performance.

For τ when the LINEX loss function parameters are positive, the smallest average estimation errors are obtained using the Bayes estimates; where gamma prior is used, the Gibbs sampling method is superior, whereas when improper prior is used, Lindley's approximation performance is better. When $\gamma = -1.25, -0.75$, if improper prior is used for τ, the ML estimates of τ have smaller average error than the Bayes estimates; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, for sample sizes $n = 50, 100$ the Bayes estimates obtained using Lindley's approximation have the highest average errors, for sample size$n = 150, 200$ both Bayes estimates perform better than the ML estimates.

When $\gamma = -0.25$, the Bayes estimates of τ have smaller average errors than the ML estimates. When improper prior is used for τ, the Bayes estimates performances are indistinguishable except for $n = 50$, where Lindley's approximation method results in smaller average error; if gamma prior is used for τ, the Bayes estimates obtained using Gibbs sampler have the best performance, for sample size $n = 50$ the Bayes estimates obtained using Lindley's approximation has the highest average error, for sample sizes $n = 100, 150, 200$ both Bayes estimates perform better than the ML estimates.

Table 8.18. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using SE loss function

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2453	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1976	-	0.1976	-	0.1976	-	0.1976	-
τ	1.6746	-	1.6746	-	1.6746	-	1.6746	-
Y_{n+1}	-0.0315	-	-0.0370	-	-0.0340	-	-0.0458	-
SE LOSS FUNCTION								
MLEs of θ_1	0.2256	0.0216	0.2351	0.0107	0.2385	0.0068	0.2385	0.0051
Lindley's - θ_1 with gamma prior for τ	0.2372	0.0104	0.2386	0.0073	0.2402	0.0053	0.2397	0.0042
Lindley's - θ_1 with improper prior for τ	0.2372	0.0104	0.2386	0.0073	0.2402	0.0053	0.2397	0.0042
Gibbs - θ_1 with gamma prior for τ	0.2335	0.0120	0.2379	0.0076	0.2400	0.0054	0.2396	0.0042
Gibbs - θ_1 with improper prior for τ	0.2338	0.0119	0.2380	0.0076	0.2400	0.0054	0.2397	0.0042
MLEs of θ_2	0.2128	0.0247	0.2105	0.0124	0.2103	0.0077	0.2091	0.0058
Lindley's - θ_2 with gamma prior for τ	0.2055	0.0147	0.2080	0.0097	0.2089	0.0066	0.2081	0.0051
Lindley's - θ_2 with improper prior for τ	0.2055	0.0147	0.2080	0.0097	0.2089	0.0066	0.2081	0.0051
Gibbs - θ_2 with gamma prior for τ	0.2081	0.0160	0.2084	0.0099	0.2091	0.0066	0.2081	0.0051
Gibbs - θ_2 with improper prior for τ	0.2078	0.0159	0.2084	0.0099	0.2091	0.0066	0.2082	0.0051
MLEs of τ	1.7944	0.1540	1.7350	0.0725	1.7116	0.0432	1.7035	0.0316
Lindley's - τ with gamma prior	1.5654	0.1419	1.6438	0.0530	1.6577	0.0353	1.6651	0.0277
Lindley's - τ with improper prior	1.7227	0.1305	1.7003	0.0664	1.6888	0.0409	1.6865	0.0303
Gibbs - τ with gamma prior	1.7093	0.0856	1.6956	0.0499	1.6877	0.0347	1.6857	0.0273
Gibbs - τ with improper prior	1.7327	0.1328	1.7032	0.0667	1.6904	0.0411	1.6875	0.0304
MLEs of Y_{n+1}	0.0076	0.7024	-0.0103	0.6476	-0.0040	0.6650	0.0032	0.6605
Lindley's - Y_{n+1} with gamma prior for τ	0.0050	0.6896	-0.0102	0.6409	-0.0046	0.6636	0.0031	0.6602
Lindley's - Y_{n+1} with improper prior for τ	0.0050	0.6896	-0.0102	0.6409	-0.0046	0.6636	0.0031	0.6602
Gibbs - Y_{n+1} with gamma prior for τ	0.0058	0.6922	-0.0106	0.6438	-0.0039	0.6634	0.0030	0.6604
Gibbs - Y_{n+1} with improper prior for τ	0.0052	0.6922	-0.0096	0.6429	-0.0054	0.6641	0.0045	0.6589

Table 8.19. Proportion of undefined MA(2) model estimates under independent truncated normal prior for θ_1 and θ_2

		Gamma prior for τ				**Improper prior for τ**			
LINEX parameter	**Sample size**	θ_1	θ_2	τ	Y_{n+1}	θ_1	θ_2	τ	Y_{n+1}
$\gamma = -1.25$	50	0.0000	0.0000	0.0800	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0090	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.75$	50	0.0000	0.0000	0.0450	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = -0.25$	50	0.0000	0.0000	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 0.75$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma = 1.25$	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	200	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 8.20. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2432	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1993	-	0.1978	-	0.1978	-	0.1976	-
τ	1.6138	-	1.6675	-	1.6726	-	1.6746	-
Y_{n+1}	-0.0293	-	-0.0352	-	-0.0333	-	-0.0458	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.2243	0.0169	0.2353	0.0084	0.2385	0.0053	0.2385	0.0040
Lindley's - θ_1 with gamma prior for τ	0.2435	0.0080	0.2439	0.0057	0.2439	0.0042	0.2427	0.0033
Lindley's - θ_1 with improper prior for τ	0.2435	0.0080	0.2439	0.0057	0.2439	0.0042	0.2427	0.0033
Gibbs - θ_1 with gamma prior for τ	0.2412	0.0092	0.2431	0.0059	0.2436	0.0043	0.2424	0.0033
Gibbs - θ_1 with improper prior for τ	0.2418	0.0091	0.2433	0.0059	0.2436	0.0043	0.2425	0.0033
MLEs of θ_2	0.1993	0.0189	0.1978	0.0096	0.1978	0.0060	0.1976	0.0045
Lindley's - θ_2 with gamma prior for τ	0.2115	0.0115	0.2108	0.0076	0.2104	0.0052	0.2091	0.0040
Lindley's - θ_2 with improper prior for τ	0.2162	0.0115	0.2144	0.0076	0.2131	0.0052	0.2112	0.0040
Gibbs - θ_2 with gamma prior for τ	0.2162	0.0125	0.2144	0.0077	0.2131	0.0052	0.2112	0.0040
Gibbs - θ_2 with improper prior for τ	0.2179	0.0123	0.2144	0.0077	0.2132	0.0052	0.2111	0.0040
MLEs of τ	1.6138	0.0795	1.6675	0.0477	1.6726	0.0310	1.6746	0.0233
Lindley's - τ with gamma prior	1.6934	0.2447	1.7247	0.0652	1.7089	0.0293	1.7035	0.0221
Lindley's - τ with improper prior	1.5554	0.0858	1.6570	0.0505	1.6754	0.0322	1.6816	0.0240
Gibbs - τ with gamma prior	1.7039	0.0621	1.7313	0.0391	1.7129	0.0275	1.7064	0.0215
Gibbs - τ with improper prior	1.7053	0.0880	1.7216	0.0509	1.7086	0.0324	1.7035	0.0240
MLEs of Y_{n+1}	-0.0293	0.6747	-0.0352	0.6805	-0.0333	0.6580	-0.0458	0.6357
Lindley's - Y_{n+1} with gamma prior for τ	0.0064	0.5734	-0.0100	0.5631	-0.0037	0.5724	0.0032	0.5893
Lindley's - Y_{n+1} with improper prior for τ	0.6378	0.5793	0.5995	0.5659	0.6010	0.5745	0.6047	0.5915
Gibbs - Y_{n+1} with gamma prior for τ	0.6409	0.5135	0.6003	0.5109	0.6018	0.5139	0.6053	0.5218
Gibbs - Y_{n+1} with improper prior for τ	0.4354	0.5145	0.4061	0.5154	0.4103	0.5162	0.4151	0.5211

Table 8.21. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2441	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1978	-	0.1978	-	0.1976	-	0.1976	-
τ	1.6533	-	1.6726	-	1.6746	-	1.6746	-
Y_{n+1}	-0.0310	-	-0.0368	-	-0.0340	-	-0.0458	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.2243	0.0061	0.2351	0.0030	0.2385	0.0019	0.2385	0.0014
Lindley's - θ_1 with gamma prior for τ	0.2408	0.0029	0.2416	0.0020	0.2425	0.0015	0.2415	0.0012
Lindley's - θ_1 with improper prior for τ	0.2408	0.0029	0.2416	0.0020	0.2425	0.0015	0.2415	0.0012
Gibbs - θ_1 with gamma prior for τ	0.2379	0.0033	0.2409	0.0021	0.2422	0.0015	0.2413	0.0012
Gibbs - θ_1 with improper prior for τ	0.2382	0.0033	0.2410	0.0021	0.2422	0.0015	0.2413	0.0012
MLEs of θ_2	0.1978	0.0068	0.1978	0.0035	0.1976	0.0022	0.1976	0.0016
Lindley's - θ_2 with gamma prior for τ	0.2118	0.0041	0.2107	0.0027	0.2103	0.0019	0.2091	0.0014
Lindley's - θ_2 with improper prior for τ	0.2118	0.0041	0.2119	0.0027	0.2113	0.0019	0.2099	0.0014
Gibbs - θ_2 with gamma prior for τ	0.2118	0.0045	0.2119	0.0028	0.2113	0.0019	0.2099	0.0014
Gibbs - θ_2 with improper prior for τ	0.2139	0.0044	0.2121	0.0028	0.2115	0.0019	0.2099	0.0014
MLEs of τ	1.6533	0.0342	1.6726	0.0181	1.6746	0.0116	1.6746	0.0086
Lindley's - τ with gamma prior	1.7565	0.3112	1.7313	0.0291	1.7116	0.0105	1.7035	0.0079
Lindley's - τ with improper prior	1.5166	0.0348	1.6502	0.0183	1.6684	0.0117	1.6747	0.0086
Gibbs - τ with gamma prior	1.7377	0.0239	1.7217	0.0141	1.7050	0.0098	1.6984	0.0077
Gibbs - τ with improper prior	1.7222	0.0354	1.7130	0.0184	1.7014	0.0117	1.6963	0.0086
MLEs of Y_{n+1}	-0.0310	0.2077	-0.0368	0.2019	-0.0340	0.2016	-0.0458	0.1983
Lindley's - Y_{n+1} with gamma prior for τ	0.0075	0.2006	-0.0100	0.1946	-0.0040	0.2001	0.0032	0.2054
Lindley's - Y_{n+1} with improper prior for τ	0.3936	0.2022	0.3728	0.1954	0.3756	0.2007	0.3808	0.2060
Gibbs - Y_{n+1} with gamma prior for τ	0.3965	0.1886	0.3746	0.1834	0.3770	0.1861	0.3819	0.1886
Gibbs - Y_{n+1} with improper prior for τ	0.2581	0.1883	0.2385	0.1838	0.2439	0.1865	0.2500	0.1881

Table 8.22. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2453	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1978	-	0.1976	-	0.1976	-	0.1976	-
τ	1.6726	-	1.6746	-	1.6746	-	1.6746	-
Y_{n+1}	-0.0309	-	-0.0370	-	-0.0340	-	-0.0458	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.2255	0.0007	0.2351	0.0003	0.2385	0.0002	0.2385	0.0002
Lindley's - θ_1 with gamma prior for τ	0.2386	0.0003	0.2396	0.0002	0.2410	0.0002	0.2403	0.0001
Lindley's - θ_1 with improper prior for τ	0.2386	0.0003	0.2396	0.0002	0.2410	0.0002	0.2403	0.0001
Gibbs - θ_1 with gamma prior for τ	0.2352	0.0004	0.2390	0.0002	0.2407	0.0002	0.2402	0.0001
Gibbs - θ_1 with improper prior for τ	0.2355	0.0004	0.2390	0.0002	0.2408	0.0002	0.2402	0.0001
MLEs of θ_2	0.1978	0.0008	0.1976	0.0004	0.1976	0.0002	0.1976	0.0002
Lindley's - θ_2 with gamma prior for τ	0.2132	0.0005	0.2105	0.0003	0.2103	0.0002	0.2091	0.0002
Lindley's - θ_2 with improper prior for τ	0.2081	0.0005	0.2092	0.0003	0.2097	0.0002	0.2087	0.0002
Gibbs - θ_2 with gamma prior for τ	0.2081	0.0005	0.2092	0.0003	0.2097	0.0002	0.2087	0.0002
Gibbs - θ_2 with improper prior for τ	0.2105	0.0005	0.2096	0.0003	0.2099	0.0002	0.2087	0.0002
MLEs of τ	1.6726	0.0045	1.6746	0.0022	1.6746	0.0013	1.6746	0.0010
Lindley's - τ with gamma prior	1.7913	0.0137	1.7350	0.0018	1.7116	0.0011	1.7035	0.0009
Lindley's - τ with improper prior	1.5454	0.0041	1.6460	0.0021	1.6612	0.0013	1.6682	0.0010
Gibbs - τ with gamma prior	1.7373	0.0027	1.7086	0.0016	1.6942	0.0011	1.6904	0.0009
Gibbs - τ with improper prior	1.7190	0.0042	1.7021	0.0021	1.6922	0.0013	1.6892	0.0010
MLEs of Y_{n+1}	-0.0309	0.0217	-0.0370	0.0205	-0.0340	0.0208	-0.0458	0.0207
Lindley's - Y_{n+1} with gamma prior for τ	0.0078	0.0215	-0.0103	0.0203	-0.0040	0.0211	0.0032	0.0213
Lindley's - Y_{n+1} with improper prior for τ	0.1383	0.0215	0.1224	0.0204	0.1271	0.0212	0.1337	0.0214
Gibbs - Y_{n+1} with gamma prior for τ	0.1400	0.0213	0.1236	0.0202	0.1279	0.0207	0.1343	0.0208
Gibbs - Y_{n+1} with improper prior for τ	0.0892	0.0213	0.0722	0.0202	0.0786	0.0207	0.0853	0.0208

Table 8.23. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2453	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1976	-	0.1976	-	0.1976	-	0.1976	-
τ	1.6746	-	1.6746	-	1.6746	-	1.6746	-
Y_{n+1}	-0.0315	-	-0.0370	-	-0.0340	-	-0.0458	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.2256	0.0007	0.2351	0.0003	0.2385	0.0002	0.2385	0.0002
Lindley's - θ_1 with gamma prior for τ	0.2358	0.0003	0.2375	0.0002	0.2395	0.0002	0.2392	0.0001
Lindley's - θ_1 with improper prior for τ	0.2358	0.0003	0.2375	0.0002	0.2395	0.0002	0.2392	0.0001
Gibbs - θ_1 with gamma prior for τ	0.2318	0.0004	0.2369	0.0002	0.2393	0.0002	0.2390	0.0001
Gibbs - θ_1 with improper prior for τ	0.2321	0.0004	0.2370	0.0002	0.2393	0.0002	0.2391	0.0001
MLEs of θ_2	0.1976	0.0008	0.1976	0.0004	0.1976	0.0002	0.1976	0.0002
Lindley's - θ_2 with gamma prior for τ	0.2128	0.0005	0.2105	0.0003	0.2103	0.0002	0.2091	0.0002
Lindley's - θ_2 with improper prior for τ	0.2032	0.0005	0.2068	0.0003	0.2080	0.0002	0.2075	0.0002
Gibbs - θ_2 with gamma prior for τ	0.2032	0.0005	0.2068	0.0003	0.2080	0.0002	0.2075	0.0002
Gibbs - θ_2 with improper prior for τ	0.2059	0.0005	0.2072	0.0003	0.2083	0.0002	0.2075	0.0002
MLEs of τ	1.6746	0.0051	1.6746	0.0023	1.6746	0.0014	1.6746	0.0010
Lindley's - τ with gamma prior	1.7944	0.0035	1.7350	0.0016	1.7116	0.0011	1.7035	0.0009
Lindley's - τ with improper prior	1.5740	0.0040	1.6408	0.0021	1.6543	0.0013	1.6621	0.0009
Gibbs - τ with gamma prior	1.7056	0.0027	1.6922	0.0016	1.6835	0.0011	1.6825	0.0009
Gibbs - τ with improper prior	1.6983	0.0041	1.6892	0.0021	1.6832	0.0013	1.6822	0.0009
MLEs of Y_{n+1}	-0.0315	0.0227	-0.0370	0.0205	-0.0340	0.0212	-0.0458	0.0210
Lindley's - Y_{n+1} with gamma prior for τ	0.0076	0.0221	-0.0103	0.0203	-0.0040	0.0208	0.0032	0.0203
Lindley's - Y_{n+1} with improper prior for τ	-0.1279	0.0222	-0.1428	0.0203	-0.1363	0.0209	-0.1274	0.0203
Gibbs - Y_{n+1} with gamma prior for τ	-0.1295	0.0220	-0.1439	0.0200	-0.1371	0.0208	-0.1280	0.0204
Gibbs - Y_{n+1} with improper prior for τ	-0.0775	0.0220	-0.0934	0.0200	-0.0864	0.0208	-0.0793	0.0204

Table 8.24. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2453	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1976	-	0.1976	-	0.1976	-	0.1976	-
τ	1.6746	-	1.6746	-	1.6746	-	1.6746	-
Y_{n+1}	-0.0315	-	-0.0370	-	-0.0340	-	-0.0458	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.2256	0.0060	0.2351	0.0030	0.2385	0.0019	0.2385	0.0014
Lindley's - θ_1 with gamma prior for τ	0.2330	0.0029	0.2354	0.0021	0.2379	0.0015	0.2380	0.0012
Lindley's - θ_1 with improper prior for τ	0.2330	0.0029	0.2354	0.0021	0.2379	0.0015	0.2380	0.0012
Gibbs - θ_1 with gamma prior for τ	0.2284	0.0034	0.2349	0.0022	0.2378	0.0015	0.2379	0.0012
Gibbs - θ_1 with improper prior for τ	0.2286	0.0034	0.2349	0.0022	0.2378	0.0015	0.2380	0.0012
MLEs of θ_2	0.1976	0.0071	0.1976	0.0035	0.1976	0.0022	0.1976	0.0016
Lindley's - θ_2 with gamma prior for τ	0.2128	0.0042	0.2105	0.0027	0.2103	0.0018	0.2091	0.0014
Lindley's - θ_2 with improper prior for τ	0.1987	0.0042	0.2043	0.0027	0.2064	0.0018	0.2062	0.0014
Gibbs - θ_2 with gamma prior for τ	0.1987	0.0045	0.2043	0.0028	0.2064	0.0019	0.2062	0.0014
Gibbs - θ_2 with improper prior for τ	0.2016	0.0045	0.2049	0.0028	0.2067	0.0019	0.2063	0.0014
MLEs of τ	1.6746	0.0515	1.6746	0.0230	1.6746	0.0130	1.6746	0.0093
Lindley's - τ with gamma prior	1.7944	0.0256	1.7350	0.0140	1.7116	0.0097	1.7035	0.0077
Lindley's - τ with improper prior	1.5810	0.0347	1.6346	0.0185	1.6478	0.0113	1.6565	0.0085
Gibbs - τ with gamma prior	1.6750	0.0237	1.6767	0.0139	1.6733	0.0097	1.6749	0.0077
Gibbs - τ with improper prior	1.6769	0.0349	1.6766	0.0185	1.6742	0.0114	1.6753	0.0085
MLEs of Y_{n+1}	-0.0315	0.2362	-0.0370	0.2009	-0.0340	0.2134	-0.0458	0.2065
Lindley's - Y_{n+1} with gamma prior for τ	0.0076	0.2135	-0.0103	0.1917	-0.0040	0.1967	0.0032	0.1837
Lindley's - Y_{n+1} with improper prior for τ	-0.3783	0.2164	-0.3925	0.1930	-0.3847	0.1974	-0.3745	0.1841
Gibbs - Y_{n+1} with gamma prior for τ	-0.3802	0.2046	-0.3942	0.1788	-0.3861	0.1881	-0.3756	0.1782
Gibbs - Y_{n+1} with improper prior for τ	-0.2449	0.2058	-0.2593	0.1790	-0.2516	0.1882	-0.2443	0.1782

Table 8.25. Average MA(2) model estimates and estimation errors under independent truncated normal prior for θ_1 and θ_2 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2453	-	0.2453	-	0.2453	-	0.2453	-
θ_2	0.1976	-	0.1976	-	0.1976	-	0.1976	-
τ	1.6746	-	1.6746	-	1.6746	-	1.6746	-
Y_{n+1}	-0.0315	-	-0.0370	-	-0.0340	-	-0.0458	-
LINEX LOSS FUNCTION								
MLEs of θ_1	0.2256	0.0167	0.2351	0.0083	0.2385	0.0052	0.2385	0.0039
Lindley's - θ_1 with gamma prior for τ	0.2302	0.0082	0.2333	0.0057	0.2364	0.0042	0.2368	0.0033
Lindley's - θ_1 with improper prior for τ	0.2302	0.0082	0.2333	0.0057	0.2364	0.0042	0.2368	0.0033
Gibbs - θ_1 with gamma prior for τ	0.2249	0.0094	0.2329	0.0060	0.2364	0.0042	0.2368	0.0033
Gibbs - θ_1 with improper prior for τ	0.2251	0.0093	0.2328	0.0060	0.2364	0.0042	0.2368	0.0033
MLEs of θ_2	0.1976	0.0199	0.1976	0.0099	0.1976	0.0061	0.1976	0.0046
Lindley's - θ_2 with gamma prior for τ	0.2128	0.0117	0.2105	0.0076	0.2103	0.0051	0.2091	0.0040
Lindley's - θ_2 with improper prior for τ	0.1942	0.0117	0.2019	0.0076	0.2048	0.0051	0.2050	0.0040
Gibbs - θ_2 with gamma prior for τ	0.1942	0.0125	0.2019	0.0077	0.2048	0.0052	0.2050	0.0040
Gibbs - θ_2 with improper prior for τ	0.1972	0.0125	0.2026	0.0077	0.2052	0.0052	0.2051	0.0040
MLEs of τ	1.6746	0.1657	1.6746	0.0706	1.6746	0.0380	1.6746	0.0270
Lindley's - τ with gamma prior	1.7944	0.0655	1.7350	0.0385	1.7116	0.0267	1.7035	0.0213
Lindley's - τ with improper prior	1.5831	0.0955	1.6287	0.0515	1.6418	0.0313	1.6512	0.0235
Gibbs - τ with gamma prior	1.6497	0.0649	1.6626	0.0384	1.6636	0.0267	1.6675	0.0213
Gibbs - τ with improper prior	1.6563	0.0927	1.6642	0.0507	1.6655	0.0312	1.6685	0.0234
MLEs of Y_{n+1}	-0.0315	0.8543	-0.0370	0.6653	-0.0340	0.7488	-0.0458	0.6792
Lindley's - Y_{n+1} with gamma prior for τ	0.0076	0.6262	-0.0103	0.5510	-0.0040	0.5696	0.0032	0.5143
Lindley's - Y_{n+1} with improper prior for τ	-0.6051	0.6378	-0.6178	0.5555	-0.6093	0.5719	-0.5984	0.5157
Gibbs - Y_{n+1} with gamma prior for τ	-0.6037	0.5914	-0.6183	0.4934	-0.6101	0.5328	-0.5990	0.4766
Gibbs - Y_{n+1} with improper prior for τ	-0.4146	0.5980	-0.4261	0.4960	-0.4176	0.5319	-0.4098	0.4774

8.3.2. Independent Uniform prior for θ_1, θ_2 and Gamma or Improper priors for τ

We consider the independent uniform priors for θ_1 and θ_2 with parameters $c_1 = 0$, $d_1 = 0.5$, and $c_2 = 0$, $d_2 = 0.4$. The prior for τ is either improper or gamma prior with parameters $\alpha = 10$, $\beta = 6$.

Table 8.26 presents the average values of MA(2) parameters, their estimates, predicted values, estimation and prediction errors when the SE loss function is used.

Table 8.27, Table 8.28, Table 8.29, Table 8.30, Table 8.31 and Table 8.32 present the average values of MA(2) parameters, their estimates, predicted values, estimation and prediction errors when the LINEX loss function is used with parameters $\gamma = -1.25, -0.75, -0.25,$ $0.25, 0.75, 1.25$, respectively.

The average estimation errors of both ML and Bayes estimates decrease, as the sample size increases. This verifies the consistency property of these estimators. Overall the Bayes estimates are found to result in smaller average estimation errors than the ML estimates. For θ_1, θ_2 and one-step prediction, the Bayes estimates are found to have smaller average estimation errors under both SE and LINEX loss functions for all parameter γ values. Under SE loss function, for τ the Bayes estimation is found to result in the smallest estimation errors. Under the LINEX loss function, when τ has gamma prior the Bayes estimation has better performance than the ML estimation for all parameter γ values. When when τ has improper prior the ML estimates have smaller average errors when $\gamma = -1.25, -0.75$, when $\gamma = -0.25, 0.25, 0.75, 1.25$ the Bayes estimation is superior. All estimator performances are reasonably close to each other as the sample size increases.

Table 8.26. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using SE loss function

	SAMPLE SIZE							
	50		**100**		**150**		**200**	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0217	0.2395	0.0106	0.2413	0.0069	0.2418	0.0052
Bayes - θ_1 with gamma prior for τ	0.2428	0.0109	0.2435	0.0071	0.2439	0.0052	0.2440	0.0041
Bayes - θ_1 with improper prior for τ	0.2428	0.0109	0.2435	0.0071	0.2439	0.0052	0.2440	0.0041
MLEs of θ_2	0.2074	0.0244	0.2066	0.0121	0.2072	0.0079	0.2079	0.0060
Bayes - θ_2 with gamma prior for τ	0.2022	0.0086	0.2034	0.0061	0.2041	0.0046	0.2052	0.0037
Bayes - θ_2 with improper prior for τ	0.2022	0.0087	0.2033	0.0061	0.2041	0.0046	0.2052	0.0037
MLEs of τ	1.7867	0.1849	1.7178	0.0753	1.6964	0.0465	1.6881	0.0339
Bayes - τ with gamma prior	1.6507	0.0885	1.6519	0.0507	1.6531	0.0361	1.6556	0.0279
Bayes - τ with improper prior	1.7152	0.1578	1.6834	0.0695	1.6737	0.0442	1.6712	0.0325
MLEs of Y_{n+1}	-0.0004	0.7086	0.0011	0.6935	0.0054	0.6746	0.0018	0.6831
Bayes - Y_{n+1} with gamma prior for τ	0.0005	0.6921	0.0004	0.6874	0.0051	0.6738	0.0018	0.6805
Bayes - Y_{n+1} with improper prior for τ	0.0006	0.6921	0.0004	0.6874	0.0051	0.6738	0.0018	0.6805

Table 8.27. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using LINEX loss function with $\gamma = -1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0174	0.2395	0.0084	0.2413	0.0055	0.2418	0.0041
Bayes - θ_1 with gamma prior for τ	0.2487	0.0147	0.2463	0.0078	0.2457	0.0052	0.2451	0.0039
Bayes - θ_1 with improper prior for τ	0.2486	0.0147	0.2463	0.0078	0.2457	0.0052	0.2451	0.0039
MLEs of θ_2	0.2074	0.0189	0.2066	0.0094	0.2072	0.0061	0.2079	0.0047
Bayes - θ_2 with gamma prior for τ	0.2200	0.0158	0.2127	0.0086	0.2112	0.0058	0.2109	0.0045
Bayes - θ_2 with improper prior for τ	0.2199	0.0158	0.2127	0.0086	0.2112	0.0058	0.2109	0.0045
MLEs of τ	1.7867	0.1133	1.7178	0.0521	1.6964	0.0336	1.6881	0.0250
Bayes - τ with gamma prior	1.7069	0.0731	1.6841	0.0409	1.6757	0.0287	1.6730	0.0222
Bayes - τ with improper prior	1.8112	0.1310	1.7256	0.0558	1.7008	0.0352	1.6911	0.0259
MLEs of Y_{n+1}	-0.0004	0.7539	0.0011	0.7518	0.0054	0.7160	0.0018	0.7493
Bayes - Y_{n+1} with gamma prior for τ	0.4259	0.5451	0.4235	0.5400	0.4261	0.5307	0.4215	0.5397
Bayes - Y_{n+1} with improper prior for τ	0.4403	0.5482	0.4315	0.5399	0.4316	0.5314	0.4256	0.5397

Table 8.28. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using LINEX loss function with $\gamma = -0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0062	0.2395	0.0030	0.2413	0.0020	0.2418	0.0015
Bayes - θ_1 with gamma prior for τ	0.2435	0.0053	0.2437	0.0028	0.2440	0.0019	0.2439	0.0014
Bayes - θ_1 with improper prior for τ	0.2434	0.0053	0.2437	0.0028	0.2440	0.0019	0.2439	0.0014
MLEs of θ_2	0.2074	0.0068	0.2066	0.0034	0.2072	0.0022	0.2079	0.0017
Bayes - θ_2 with gamma prior for τ	0.2146	0.0057	0.2102	0.0031	0.2095	0.0021	0.2096	0.0016
Bayes - θ_2 with improper prior for τ	0.2146	0.0057	0.2102	0.0031	0.2095	0.0021	0.2096	0.0016
MLEs of τ	1.7867	0.0440	1.7178	0.0195	1.6964	0.0124	1.6881	0.0092
Bayes - τ with gamma prior	1.6838	0.0257	1.6710	0.0145	1.6665	0.0103	1.6660	0.0079
Bayes - τ with improper prior	1.7708	0.0459	1.7084	0.0199	1.6898	0.0126	1.6831	0.0092
MLEs of Y_{n+1}	-0.0004	0.2219	0.0011	0.2190	0.0054	0.2098	0.0018	0.2162
Bayes - Y_{n+1} with gamma prior for τ	0.2554	0.1976	0.2545	0.1951	0.2578	0.1900	0.2536	0.1932
Bayes - Y_{n+1} with improper prior for τ	0.2640	0.1979	0.2593	0.1950	0.2611	0.1901	0.2561	0.1932

Table 8.29. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using LINEX loss function with $\gamma = -0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0007	0.2395	0.0003	0.2413	0.0002	0.2418	0.0002
Bayes - θ_1 with gamma prior for τ	0.2382	0.0006	0.2412	0.0003	0.2423	0.0002	0.2426	0.0002
Bayes - θ_1 with improper prior for τ	0.2382	0.0006	0.2412	0.0003	0.2423	0.0002	0.2426	0.0002
MLEs of θ_2	0.2074	0.0008	0.2066	0.0004	0.2072	0.0002	0.2079	0.0002
Bayes - θ_2 with gamma prior for τ	0.2093	0.0006	0.2076	0.0003	0.2078	0.0002	0.2083	0.0002
Bayes - θ_2 with improper prior for τ	0.2093	0.0006	0.2076	0.0003	0.2078	0.0002	0.2083	0.0002
MLEs of τ	1.7867	0.0054	1.7178	0.0023	1.6964	0.0014	1.6881	0.0010
Bayes - τ with gamma prior	1.6615	0.0028	1.6582	0.0016	1.6575	0.0011	1.6590	0.0009
Bayes - τ with improper prior	1.7331	0.0050	1.6916	0.0022	1.6790	0.0014	1.6751	0.0010
MLEs of Y_{n+1}	-0.0004	0.0224	0.0011	0.0220	0.0054	0.0213	0.0018	0.0217
Bayes - Y_{n+1} with gamma prior for τ	0.0849	0.0221	0.0856	0.0217	0.0895	0.0211	0.0857	0.0214
Bayes - Y_{n+1} with improper prior for τ	0.0878	0.0221	0.0872	0.0217	0.0906	0.0211	0.0866	0.0214

Table 8.30. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using LINEX loss function with $\gamma = 0.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0007	0.2395	0.0003	0.2413	0.0002	0.2418	0.0002
Bayes - θ_1 with gamma prior for τ	0.2330	0.0006	0.2386	0.0003	0.2407	0.0002	0.2413	0.0002
Bayes - θ_1 with improper prior for τ	0.2330	0.0006	0.2386	0.0003	0.2407	0.0002	0.2413	0.0002
MLEs of θ_2	0.2074	0.0008	0.2066	0.0004	0.2072	0.0002	0.2079	0.0002
Bayes - θ_2 with gamma prior for τ	0.2039	0.0006	0.2050	0.0003	0.2061	0.0002	0.2071	0.0002
Bayes - θ_2 with improper prior for τ	0.2040	0.0006	0.2050	0.0003	0.2061	0.0002	0.2071	0.0002
MLEs of τ	1.7867	0.0062	1.7178	0.0024	1.6964	0.0015	1.6881	0.0011
Bayes - τ with gamma prior	1.6401	0.0027	1.6457	0.0016	1.6487	0.0011	1.6522	0.0009
Bayes - τ with improper prior	1.6978	0.0049	1.6754	0.0022	1.6685	0.0014	1.6673	0.0010
MLEs of Y_{n+1}	-0.0004	0.0224	0.0011	0.0219	0.0054	0.0214	0.0018	0.0216
Bayes - Y_{n+1} with gamma prior for τ	-0.0856	0.0222	-0.0834	0.0216	-0.0788	0.0211	-0.0821	0.0213
Bayes - Y_{n+1} with improper prior for τ	-0.0885	0.0222	-0.0850	0.0217	-0.0799	0.0211	-0.0830	0.0213

Table 8.31. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using LINEX loss function with $\gamma = 0.75$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0061	0.2395	0.0030	0.2413	0.0019	0.2418	0.0014
Bayes - θ_1 with gamma prior for τ	0.2277	0.0053	0.2361	0.0028	0.2390	0.0019	0.2401	0.0014
Bayes - θ_1 with improper prior for τ	0.2278	0.0053	0.2361	0.0028	0.2390	0.0019	0.2401	0.0014
MLEs of θ_2	0.2074	0.0069	0.2066	0.0034	0.2072	0.0022	0.2079	0.0017
Bayes - θ_2 with gamma prior for τ	0.1986	0.0057	0.2024	0.0031	0.2044	0.0021	0.2058	0.0016
Bayes - θ_2 with improper prior for τ	0.1987	0.0057	0.2024	0.0031	0.2044	0.0021	0.2058	0.0016
MLEs of τ	1.7867	0.0683	1.7178	0.0239	1.6964	0.0141	1.6881	0.0100
Bayes - τ with gamma prior	1.6194	0.0241	1.6335	0.0140	1.6400	0.0101	1.6454	0.0078
Bayes - τ with improper prior	1.6646	0.0432	1.6595	0.0193	1.6581	0.0123	1.6596	0.0091
MLEs of Y_{n+1}	-0.0004	0.2241	0.0011	0.2159	0.0054	0.2134	0.0018	0.2127
Bayes - Y_{n+1} with gamma prior for τ	-0.2561	0.2014	-0.2524	0.1941	-0.2470	0.1900	-0.2500	0.1917
Bayes - Y_{n+1} with improper prior for τ	-0.2648	0.2014	-0.2572	0.1943	-0.2503	0.1900	-0.2525	0.1917

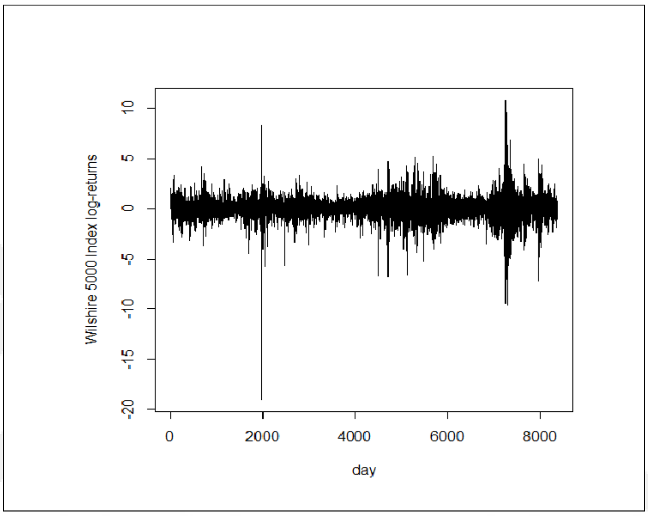
Table 8.32. Average MA(2) model estimates and estimation errors under independent uniform prior for θ_1 and θ_2 using LINEX loss function with $\gamma = 1.25$

	SAMPLE SIZE							
	50		100		150		200	
ACTUAL	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**	**Average**	**Average error**
θ_1	0.2477	-	0.2477	-	0.2477	-	0.2477	-
θ_2	0.2001	-	0.2001	-	0.2001	-	0.2001	-
τ	1.6628	-	1.6628	-	1.6628	-	1.6628	-
Y_{n+1}	0.0109	-	0.0023	-	-0.0014	-	0.0074	-
ESTIMATES								
MLEs of θ_1	0.2343	0.0168	0.2395	0.0082	0.2413	0.0054	0.2418	0.0040
Bayes - θ_1 with gamma prior for τ	0.2225	0.0147	0.2335	0.0077	0.2373	0.0052	0.2388	0.0039
Bayes - θ_1 with improper prior for τ	0.2226	0.0148	0.2335	0.0077	0.2373	0.0052	0.2388	0.0039
MLEs of θ_2	0.2074	0.0195	0.2066	0.0096	0.2072	0.0063	0.2079	0.0048
Bayes - θ_2 with gamma prior for τ	0.1932	0.0159	0.1998	0.0086	0.2027	0.0058	0.2045	0.0045
Bayes - θ_2 with improper prior for τ	0.1934	0.0159	0.1999	0.0086	0.2027	0.0058	0.2045	0.0045
MLEs of τ	1.7867	0.2545	1.7178	0.0739	1.6964	0.0415	1.6881	0.0291
Bayes - τ with gamma prior	1.5995	0.0657	1.6215	0.0384	1.6314	0.0277	1.6387	0.0215
Bayes - τ with improper prior	1.6333	0.1181	1.6442	0.0530	1.6479	0.0339	1.6520	0.0250
MLEs of Y_{n+1}	-0.0004	0.7772	0.0011	0.7310	0.0054	0.7323	0.0018	0.7227
Bayes - Y_{n+1} with gamma prior for τ	-0.4267	0.5634	-0.4213	0.5366	-0.4153	0.5271	-0.4179	0.5326
Bayes - Y_{n+1} with improper prior for τ	-0.4411	0.5636	-0.4294	0.5379	-0.4208	0.5271	-0.4220	0.5328

8.3.3. Analysis of Wilshire 5000 Index Data

We consider Wilshire 5000 index log-return data. The series consists of 8385 daily observations. The data can be downloaded from Federal Reserve Bank of St Louis (FRED). Figure 8.13 shows the plotted Wilshire 5000 index log-return series expressed in per cents.

Figure 8.13. Wilshire 5000 index log-returns data



We analyze the data using the independent truncated normal prior for θ_1 with mean equal to the sample mean and variance equal to the sample variance, truncated normal prior forθ_2 and with mean equal to the sample third moment and variance equal to the sample fourth moment, and improper prior for τ. The LINEX loss function's parameter are $\gamma = 0.25$ and $\gamma = -0.25$. In order to apply the analysis using the assumed form of the MA(2) model, we need to subtract the series mean from each of the observations to obtain a zero-mean series. Model checking shows that the MA(2) model can be fitted to the zero-mean series; see Figure 8.14.

We obtain estimates of θ_1, θ_2, τ and one-step predicted value using 8375, 8377,..., 8383, 8384 observations. Table 8.33 and Table 8.34 present the estimation and one-step prediction results. It is observed that the prediction errors of the Bayes estimates are slightly smaller than that of the ML estimates, the Bayes estimates obtained using Gibbs sampling method have the smallest average estimation errors. This is consistent with the fact that the Bayes and ML estimators performances are similar when sample size is large.

Figure 8.14. MA(2) model checking

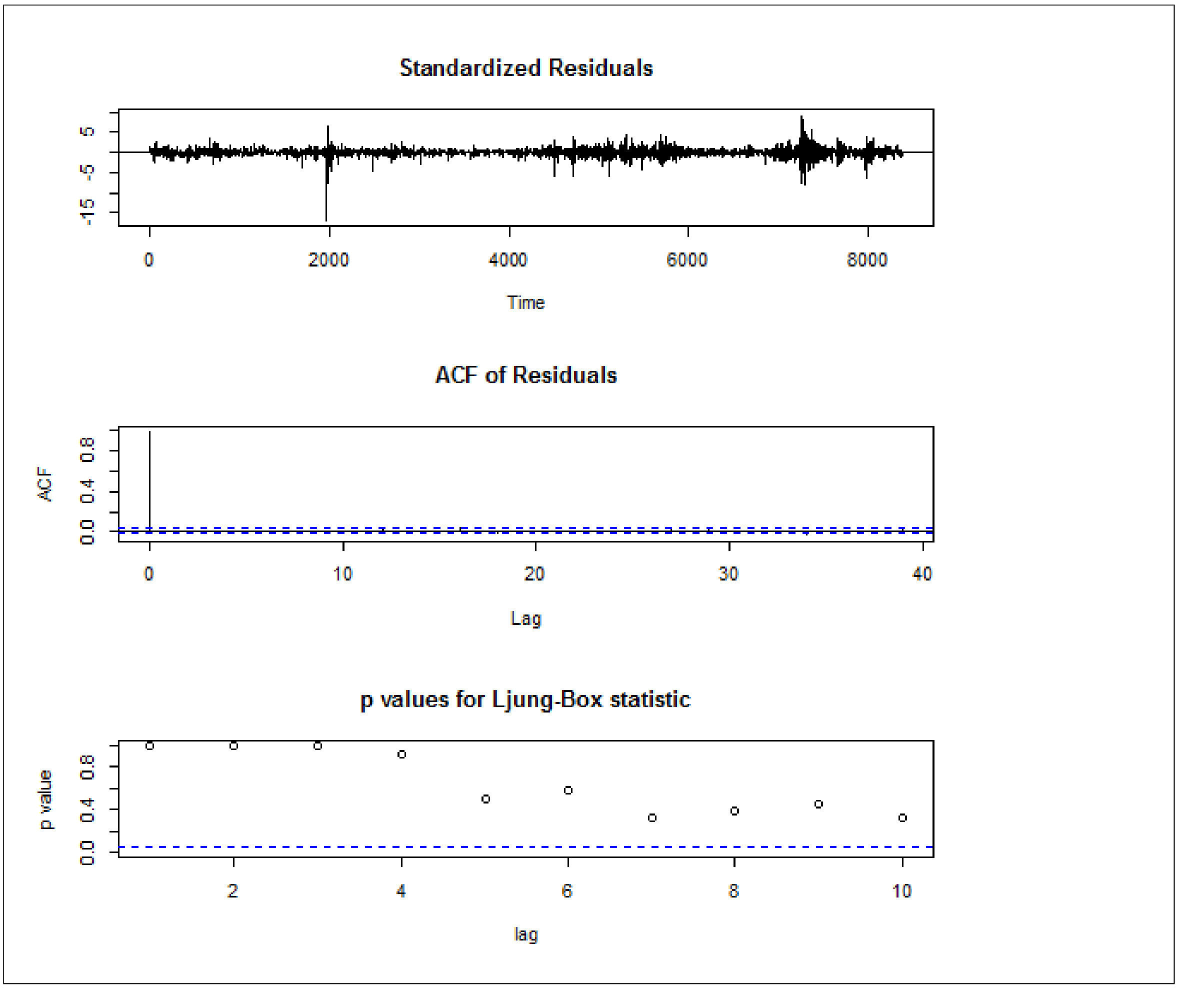


Table 8.33. MA(2) model estimates for empirical data

	SAMPLE SIZE										
ACTUAL	**8375**	**8376**	**8377**	**8378**	**8379**	**8380**	**8381**	**8382**	**8383**	**8384**	**AVERAGE**
Y_{n+1}	65.8100	65.2700	65.6900	65.4900	65.9800	65.9700	66.2300	65.8800	66.1100	65.3700	65.7800
ESTIMATES											
ML											
θ_1	-1.0387	-1.0387	-1.0388	-1.0389	-1.0390	-1.0390	-1.0390	-1.0389	-1.0389	-1.0389	-1.0389
θ_2	-0.9614	-0.9613	-0.9612	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611
τ	5.1901	5.1892	5.1879	5.1871	5.1868	5.1859	5.1858	5.1855	5.1850	5.1847	5.1868
Y_{n+1}	65.3400	65.8286	65.2558	65.6883	65.5065	65.9717	66.0027	66.2086	65.8986	66.0827	65.7784
SE LOSS FUNCTION											
θ_1 Lindley's approx.	-1.0387	-1.0387	-1.0388	-1.0389	-1.0390	-1.0390	-1.0390	-1.0389	-1.0389	-1.0389	-1.0389
θ_1 Gibbs sampling	-1.0386	-1.0386	-1.0393	-1.0394	-1.0390	-1.0395	-1.0391	-1.0388	-1.0397	-1.0393	-1.0391
θ_2 Lindley's approx.	-0.9614	-0.9613	-0.9612	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611
θ_2 Gibbs sampling	-0.9614	-0.9614	-0.9607	-0.9607	-0.9610	-0.9605	-0.9610	-0.9612	-0.9603	-0.9607	-0.9609
τ Lindley's approx.	5.1888	5.1879	5.1867	5.1858	5.1856	5.1847	5.1846	5.1843	5.1838	5.1835	5.1856
τ Gibbs sampling	5.1900	5.1882	5.1880	5.1925	5.1872	5.1840	5.1822	5.1859	5.1852	5.1863	5.1870
Y_{n+1} Lindley's approx.	65.3400	65.8286	65.2558	65.6883	65.5065	65.9717	66.0027	66.2086	65.8986	66.0827	65.7784
Y_{n+1} Gibbs sampling	65.3391	65.8324	65.2624	65.6939	65.4943	65.9800	66.0060	66.2070	65.9054	66.0602	65.7781
LINEX LOSS FUNCTION $\gamma = 0.25$											
θ_1 Lindley's approx.	-1.0387	-1.0387	-1.0388	-1.0389	-1.0390	-1.0390	-1.0390	-1.0389	-1.0389	-1.0389	-1.0389
θ_1 Gibbs sampling	-1.0386	-1.0386	-1.0393	-1.0394	-1.0390	-1.0396	-1.0391	-1.0388	-1.0397	-1.0393	-1.0391
θ_2 Lindley's approx.	-0.9614	-0.9613	-0.9612	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611
θ_2 Gibbs sampling	-0.9614	-0.9614	-0.9607	-0.9607	-0.9610	-0.9605	-0.9610	-0.9612	-0.9603	-0.9607	-0.9609
τ Lindley's approx.	5.1880	5.1871	5.1859	5.1850	5.1848	5.1839	5.1838	5.1835	5.1830	5.1827	5.1848
τ Gibbs sampling	5.1892	5.1874	5.1872	5.1917	5.1864	5.1832	5.1814	5.1851	5.1844	5.1855	5.1862
Y_{n+1} Lindley's approx.	65.3150	65.8036	65.2308	65.6632	65.4815	65.9467	65.9777	66.1836	65.8736	66.0577	65.7533
Y_{n+1} Gibbs sampling	65.3150	65.8071	65.2383	65.6689	65.4694	65.9557	65.9801	66.1828	65.8822	66.0365	65.7536
LINEX LOSS FUNCTION $\gamma = -0.25$											
θ_1 Lindley's approx.	-1.0386	-1.0387	-1.0388	-1.0389	-1.0390	-1.0390	-1.0390	-1.0389	-1.0389	-1.0389	-1.0389
θ_1 Gibbs sampling	-1.0386	-1.0386	-1.0393	-1.0393	-1.0390	-1.0395	-1.0391	-1.0388	-1.0397	-1.0393	-1.0391
θ_2 Lindley's approx.	-0.9614	-0.9613	-0.9612	-0.9611	-0.9610	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611	-0.9611
θ_2 Gibbs sampling	-0.9614	-0.9614	-0.9607	-0.9607	-0.9610	-0.9605	-0.9610	-0.9612	-0.9603	-0.9607	-0.9609
τ Lindley's approx.	5.1896	5.1888	5.1875	5.1866	5.1864	5.1855	5.1854	5.1851	5.1846	5.1843	5.1864
τ Gibbs sampling	5.1908	5.1890	5.1888	5.1933	5.1879	5.1849	5.1831	5.1867	5.1860	5.1871	5.1878
Y_{n+1} Lindley's approx.	65.3650	65.8536	65.2808	65.7133	65.5316	65.9967	66.0278	66.2336	65.9236	66.1078	65.8034
Y_{n+1} Gibbs sampling	65.3634	65.8578	65.2867	65.7189	65.5192	66.0043	66.0318	66.2312	65.9286	66.0839	65.8026

Table 8.34. MA(2) model errors for empirical data

	SAMPLE SIZE										
ERRORS	**8375**	**8376**	**8377**	**8378**	**8379**	**8380**	**8381**	**8382**	**8383**	**8384**	**AVERAGE**
SE LOSS FUNCTION											
Y_{n+1} ML	0.2209	0.3120	0.1885	0.0393	0.2242	0.0000	0.0517	0.1080	0.0447	0.5080	0.1697
Y_{n+1} Lindley's approx.	0.2209	0.3120	0.1885	0.0393	0.2242	0.0000	0.0517	0.1080	0.0447	0.5080	0.1697
Y_{n+1} Gibbs sampling	0.2218	0.3163	0.1829	0.0416	0.2359	0.0001	0.0502	0.1069	0.0418	0.4763	0.1674
LINEX LOSS FUNCTION $\gamma = 0.25$											
Y_{n+1} ML	0.0066	0.0102	0.0057	0.0012	0.0067	0.0000	0.0016	0.0035	0.0014	0.0169	0.0054
Y_{n+1} Lindley's approx.	0.0074	0.0093	0.0063	0.0010	0.0075	0.0000	0.0019	0.0030	0.0017	0.0157	0.0054
Y_{n+1} Gibbs sampling	0.0073	0.0094	0.0061	0.0010	0.0078	0.0000	0.0019	0.0029	0.0016	0.0147	0.0053
LINEX LOSS FUNCTION $\gamma = -0.25$											
Y_{n+1} ML	0.0072	0.0093	0.0061	0.0012	0.0073	0.0000	0.0016	0.0033	0.0014	0.0150	0.0052
Y_{n+1} Lindley's approx.	0.0064	0.0101	0.0054	0.0015	0.0065	0.0000	0.0013	0.0038	0.0011	0.0160	0.0052
Y_{n+1} Gibbs sampling	0.0065	0.0103	0.0053	0.0016	0.0069	0.0000	0.0012	0.0037	0.0010	0.0150	0.0052

8.4. MA(2) MODEL PARAMETER IMPACT ANALYSIS

8.4.1. Independent Truncated Normal prior for θ_1, θ_2 and Gamma or Improper priors for τ

We have undertaken the parameter impact analysis to define how the average estimation and prediction errors change when model parameters vary. We use fixed parameters $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\alpha_1 = 10, 20, 30$. Figure 8.15 presents the mean errors when the parameter α changes. We notice that as α increases, the average estimation errors of θ_1 and θ_2 increase slightly. The average estimation errors of τ increase more than parameter α and the average prediction errors decrease proportionally to α increase.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 8.16 presents the mean errors when the parameter β changes. We notice that β and the average estimation errors of θ_1 and θ_2 have a weak nonlinear relationship, the average estimation errors of τ and β have nonlinear inverse relationship. As parameter β increases, the average prediction errors increase.

To estimate the impact of parameter μ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\mu_1 = -0.375, -0.125, 0.125, 0.375$. Figure 8.17 presents the mean errors when the parameter μ_1 changes. We notice that as absolute mean value of μ_1 increases, the average estimation errors of θ_1 and θ_2 and the average prediction errors increase, the changes of average estimation errors of τ remain almost unchanged.

To estimate the impact of parameter μ_2, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$,$\sigma_1 = 0.2$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\mu_2 = -0.375, -0.125, 0.125, 0.375$. Figure 8.18 presents the mean errors when the parameter μ_2 changes. We notice that as mean value of μ_2 increases, the average estimation errors of θ_1 and θ_2 and the average prediction errors increase, the average τ estimation errors increase with increased absolute mean value of μ_2.

To estimate the impact of parameter σ_1, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\sigma_1 = 0.1, 0.2, 0.3$. Figure 8.19 presents the mean errors when the parameter σ_1 changes. We notice that as σ_1 increases, the average Bayes estimation errors of θ_1 and θ_2 increase, the changes of average ML estimation errors of θ_1 and θ_2 have nonlinear relationship with σ_1. The average estimation errors of τ remain almost unchanged. The average prediction errors have a very low dependency on σ_1.

To estimate the impact of parameter σ_2, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\sigma_2 = 0.1, 0.2, 0.3$. Figure 8.20 presents the mean errors when the parameter σ_2 changes. The changes of σ_2 have similar impact to the average estimation and prediction errors as the changes of σ_1.

To estimate the impact of parameter γ, we use fixed parameters $\alpha = 10$, $\beta = 6$, $\mu_1 = 0.375$, $\sigma_1 = 0.2$, $\mu_2 = 0.375$, $\sigma_2 = 0.3$, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 8.21 presents the mean errors when the parameter γ changes. We notice that as γ increases, the average estimation errors of θ_1, θ_2,τ and prediction increase more than the increase in γ.

Figure 8.15. Impact of parameter α on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

	10	20	30
ML	0.01154	0.01156	0.01160
Bayes with gamma prior for tau	0.00707	0.00709	0.00714
Bayes with improper prior for tau	0.00707	0.00709	0.00714

(a)

	10	20	30
ML	0.00036	0.00036	0.00036
Bayes with gamma prior for tau	0.00022	0.00022	0.00022
Bayes with improper prior for tau	0.00022	0.00022	0.00022

(b)

	10	20	30
ML	0.01324	0.01300	0.01302
Bayes with gamma prior for tau	0.01048	0.01028	0.01031
Bayes with improper prior for tau	0.01048	0.01028	0.01031

(c)

	10	20	30
ML	0.00042	0.00041	0.00041
Bayes with gamma prior for tau	0.00033	0.00032	0.00032
Bayes with improper prior for tau	0.00033	0.00032	0.00032

(d)

	10	20	30
ML	0.07247	0.27308	0.61331
Bayes with gamma prior for tau	0.05799	0.23030	0.56441
Bayes with improper prior for tau	0.06790	0.25700	0.57365

(e)

	10	20	30
ML	0.00233	0.00898	0.02081
Bayes with gamma prior for tau	0.00175	0.00626	0.01249
Bayes with improper prior for tau	0.00211	0.00793	0.01762

(f)

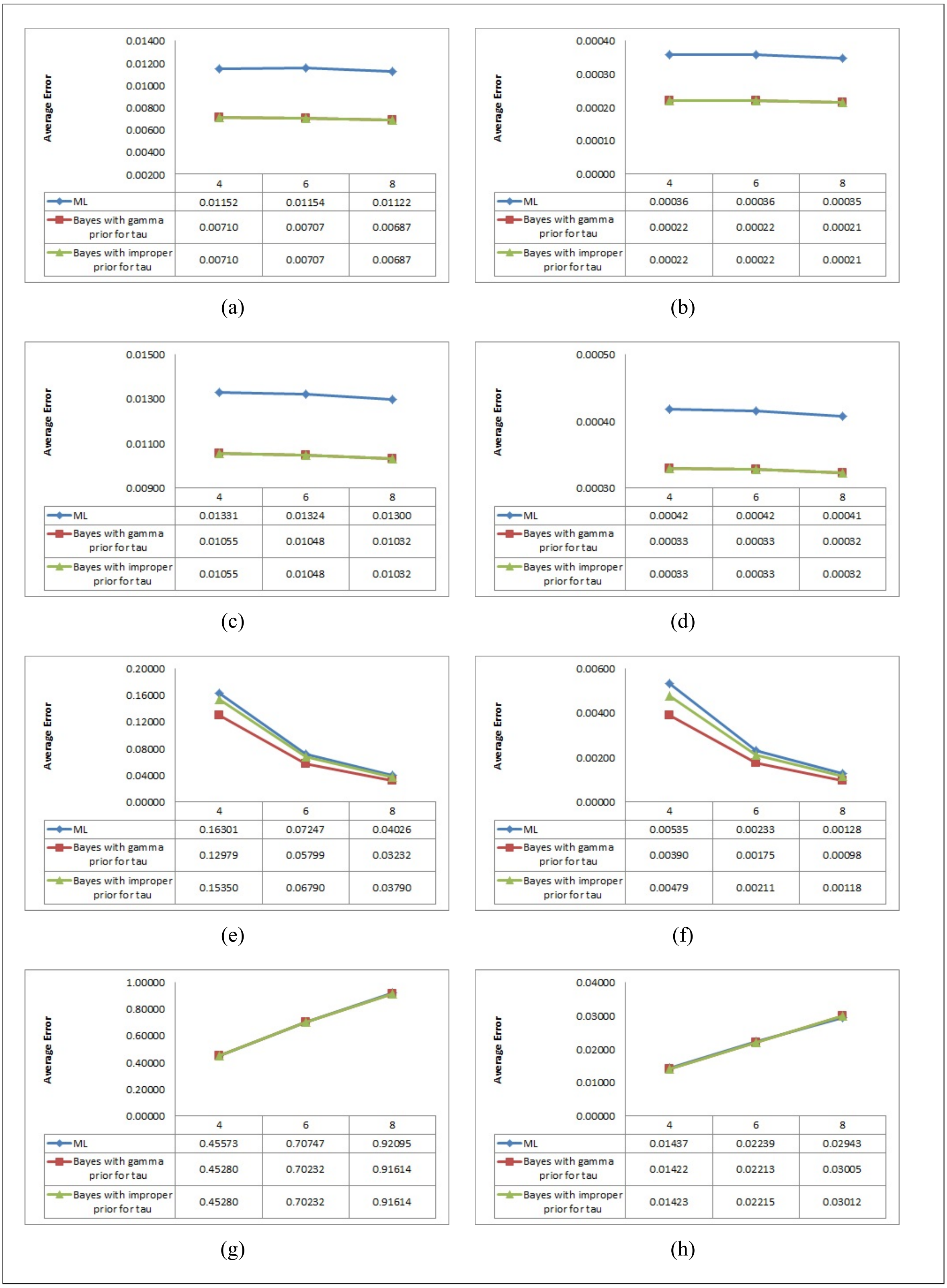
	10	20	30
ML	0.70747	0.32754	0.21125
Bayes with gamma prior for tau	0.70232	0.32541	0.20987
Bayes with improper prior for tau	0.70232	0.32541	0.20987

(g)

	10	20	30
ML	0.02239	0.01032	0.00661
Bayes with gamma prior for tau	0.02213	0.01018	0.00655
Bayes with improper prior for tau	0.02215	0.01018	0.00655

(h)

Figure 8.16. Impact of parameter β on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.



(a)

	4	6	8
ML	0.01152	0.01154	0.01122
Bayes with gamma prior for tau	0.00710	0.00707	0.00687
Bayes with improper prior for tau	0.00710	0.00707	0.00687

(b)

	4	6	8
ML	0.00036	0.00036	0.00035
Bayes with gamma prior for tau	0.00022	0.00022	0.00021
Bayes with improper prior for tau	0.00022	0.00022	0.00021

(c)

	4	6	8
ML	0.01331	0.01324	0.01300
Bayes with gamma prior for tau	0.01055	0.01048	0.01032
Bayes with improper prior for tau	0.01055	0.01048	0.01032

(d)

	4	6	8
ML	0.00042	0.00042	0.00041
Bayes with gamma prior for tau	0.00033	0.00033	0.00032
Bayes with improper prior for tau	0.00033	0.00033	0.00032

(e)

	4	6	8
ML	0.16301	0.07247	0.04026
Bayes with gamma prior for tau	0.12979	0.05799	0.03232
Bayes with improper prior for tau	0.15350	0.06790	0.03790

(f)

	4	6	8
ML	0.00535	0.00233	0.00128
Bayes with gamma prior for tau	0.00390	0.00175	0.00098
Bayes with improper prior for tau	0.00479	0.00211	0.00118

(g)

	4	6	8
ML	0.45573	0.70747	0.92095
Bayes with gamma prior for tau	0.45280	0.70232	0.91614
Bayes with improper prior for tau	0.45280	0.70232	0.91614

(h)

	4	6	8
ML	0.01437	0.02239	0.02943
Bayes with gamma prior for tau	0.01422	0.02213	0.03005
Bayes with improper prior for tau	0.01423	0.02215	0.03012

Figure 8.17. Impact of parameter μ_1 on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX los, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 8.18. Impact of parameter μ_2 on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 8.19. Impact of parameter σ_1 on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

(a)

	0.1	0.2	0.3
ML	0.01149	0.01154	0.01153
Bayes with gamma prior for tau	0.00426	0.00707	0.00927
Bayes with improper prior for tau	0.00426	0.00707	0.00927

(b)

	0.1	0.2	0.3
ML	0.00036	0.00036	0.00036
Bayes with gamma prior for tau	0.00013	0.00022	0.00029
Bayes with improper prior for tau	0.00013	0.00022	0.00029

(c)

	0.1	0.2	0.3
ML	0.01326	0.01324	0.01318
Bayes with gamma prior for tau	0.01040	0.01048	0.01049
Bayes with improper prior for tau	0.01040	0.01048	0.01049

(d)

	0.1	0.2	0.3
ML	0.00042	0.00042	0.00041
Bayes with gamma prior for tau	0.00033	0.00033	0.00033
Bayes with improper prior for tau	0.00033	0.00033	0.00033

(e)

	0.1	0.2	0.3
ML	0.07236	0.07247	0.07235
Bayes with gamma prior for tau	0.05793	0.05799	0.05802
Bayes with improper prior for tau	0.06779	0.06790	0.06780

(f)

	0.1	0.2	0.3
ML	0.00232	0.00233	0.00232
Bayes with gamma prior for tau	0.00174	0.00175	0.00175
Bayes with improper prior for tau	0.00210	0.00211	0.00210

(g)

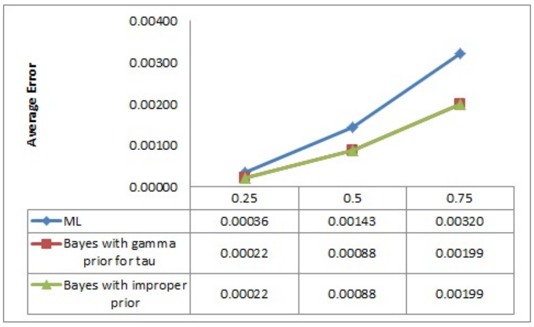
	0.1	0.2	0.3
ML	0.70624	0.70747	0.70687
Bayes with gamma prior for tau	0.69937	0.70232	0.70355
Bayes with improper prior for tau	0.69937	0.70232	0.70355

(h)

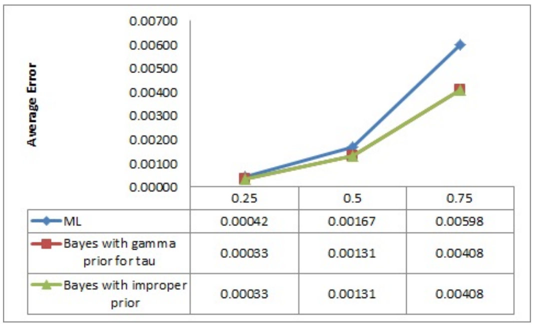
	0.1	0.2	0.3
ML	0.02235	0.02239	0.02238
Bayes with gamma prior for tau	0.02204	0.02213	0.02215
Bayes with improper prior for tau	0.02206	0.02215	0.02218

Figure 8.20. Impact of parameter σ_2 on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under SE loss, (b) Estimation of ϕ_1 under LINEX, (c) Estimation of ϕ_2 under SE loss, (d) Estimation of ϕ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

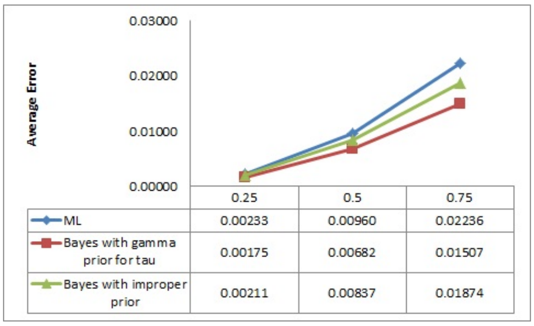
Figure 8.21. Impact of parameter γ on average estimation and prediction errors for MA(2) independent truncated normal prior for θ_1 and θ_2. (a) Estimation of ϕ_1 under LINEX, (b) Estimation of ϕ_2 under LINEX, (c) Estimation of τ under LINEX loss, (d) [Estimation of Y_{n+1} under LINEX loss.



(a)

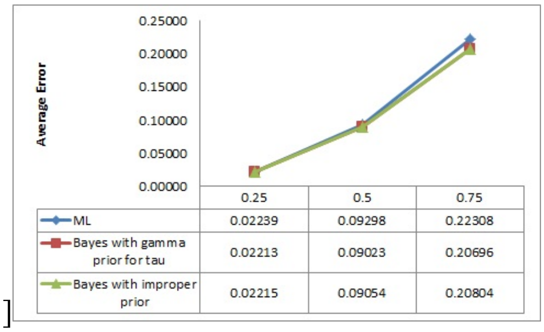


(b)



(c)

]



8.4.2. Independent Uniform prior for θ_1, θ_2 and Gamma or Improper priors for τ

To estimate the impact of parameter α, we use fixed parameters $\beta = 6$, $(c_1, d_1) = (0.25, 0.5)$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\alpha_1 = 10, 20, 30$. Figure 8.22 presents the mean errors when the parameter α changes. We notice that as α increases, the average estimation errors of θ_1 and θ_2 change only slightly. The average estimation errors of τ increase more than parameter α and the average prediction errors decrease proportionally to α increase.

To estimate the impact of parameter β, we use fixed parameters $\alpha = 10$, $(c_1, d_1) = (0.25, 0.5)$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $\beta = 10, 20, 30$. Figure 8.23 presents the mean errors when the parameter β changes. We notice that β does not have a significant impact on the average estimation errors of θ_1 and θ_2, the average estimation errors of τ decrease as β increases. As parameter β increases, the average prediction errors increase.

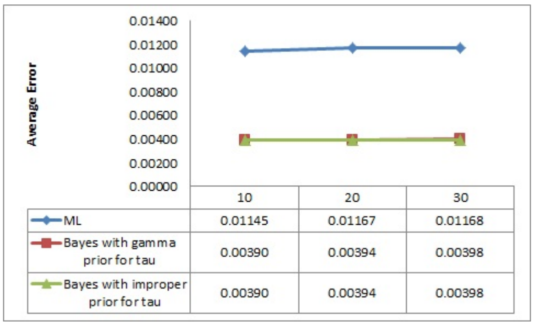
To estimate the impact of parameter (c_1, d_1), we use fixed parameters $\alpha = 10$, $\beta = 6$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $(c_1, d_1) = (-0.5, -0.25)$, $(-0.25, 0)$, $(0, 0.25)$, $(0.25, 0.5)$. Figure 8.24 presents the mean errors when (c_1, d_1) changes. The interval is represented by its middle point. We notice that the average estimation errors of θ_1 and θ_2 decrese as the absolute value of the interval middle point decreases, the similar pattern can be noticed for the average prediction errors. The interval changes have a very limited impact on the average estimation errors of τ.

To estimate the impact of parameter (c_2, d_2), we use fixed parameters $\alpha = 10$, $\beta = 6$, $(c_1, d_1) = (0.25, 0.5)$, sample size of 100 and LINEX loss function parameter $\gamma = 0.25$ and obtain the average estimation errors when $(c_2, d_2) = (-0.5, -0.25)$, $(-0.25, 0)$, $(0, 0.25)$, $(0.25, 0.5)$. Figure 8.25 presents the mean errors when the parameter (c_2, d_2) changes. The interval is represented by its middle point. We notice that the average estimation errors of θ_1 and θ_2 increase as the absolute value of interval mean point decreases, the changes of average estimation errors of τ remain almost unchanged. The average prediction errors slightly increase as the mean of θ_2 increases.

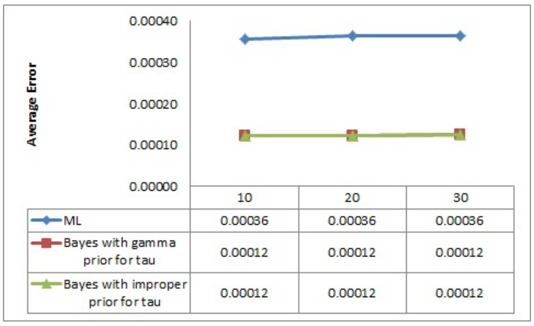
To estimate the impact of parameter γ, we use fixed parameters $\alpha = 10$, $\beta = 6$,$(c_1, d_1) = (0.25, 0.5)$, $(c_2, d_2) = (0.25, 0.5)$, sample size of 100 and obtain the average estimation errors when $\gamma = 0.25, 0.5, 0.75$. Figure 8.26 presents the mean errors when the parameter γ changes. We notice that as γ increases, the average estimation errors of θ_1,θ_2, τ and prediction increase more than the increase in γ.

Figure 8.27 and Figure 8.28 present the average estimation errors of θ_1, θ_2 under the LINEX loss function, estimated using the numerical method and the truncated normal approximation. We notice that the Bayes estimation errors for θ_1, θ_2 are significantly smaller when the numerical approach is used (left in the figure); and the difference between the ML and the Bayes estimates becomes more noticeable, the Bayes estimates have the smallest average estimation errors.

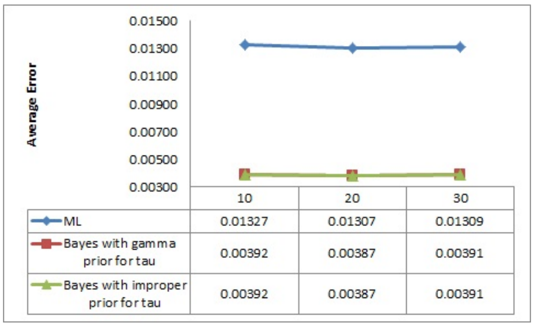
Figure 8.22. Impact of parameter α on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of θ_2 under SE loss, (d) Estimation of θ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.



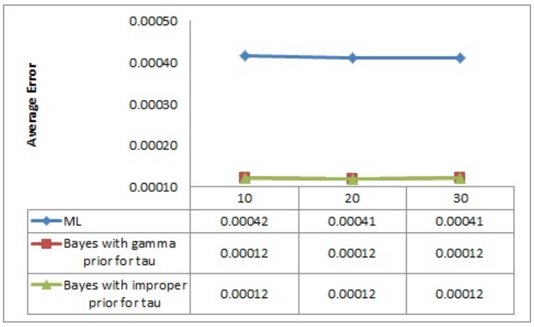
(a)



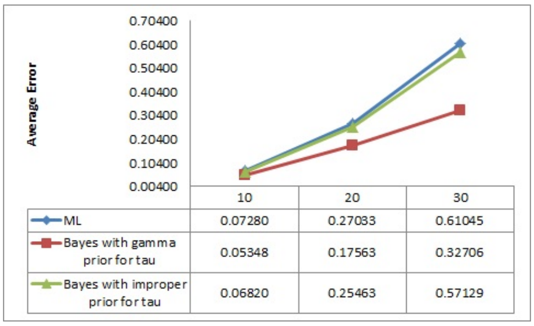
(b)



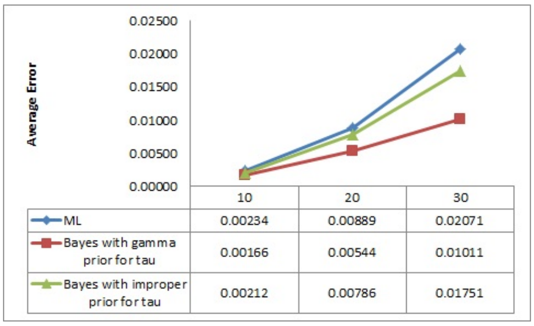
(c)



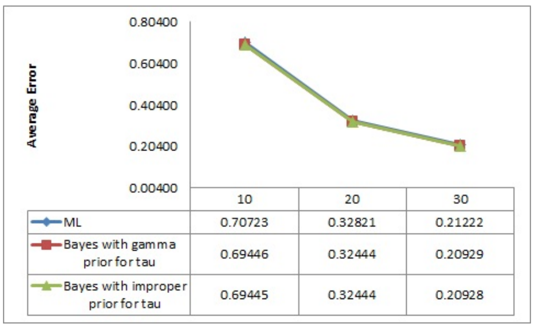
(d)



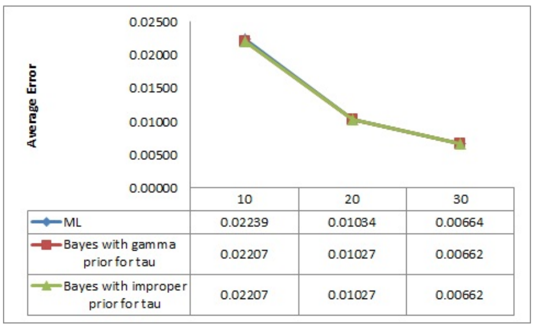
(e)



(f)



(g)



(h)

Figure 8.23. Impact of parameter β on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of θ_2 under SE loss, (d) Estimation of θ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 8.24. Impact of (c_1, d_1) on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of θ_2 under SE loss, (d) Estimation of θ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

Figure 8.25. Impact of (c_2, d_2) on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Estimation of θ_1 under SE loss, (b) Estimation of θ_1 under LINEX, (c) Estimation of θ_2 under SE loss, (d) Estimation of θ_2 under LINEX, (e) Estimation of τ under SE loss, (f) Estimation of τ under LINEX loss, (g) Estimation of Y_{n+1} under SE loss, (h) Estimation of Y_{n+1} under LINEX loss.

(a)

	-0.375	-0.125	0.125	0.375
ML	0.01023	0.01000	0.00990	0.01145
Bayes with gamma prior for tau	0.00346	0.00344	0.00345	0.00390
Bayes with improper prior	0.00346	0.00344	0.00345	0.00390

(b)

	-0.375	-0.125	0.125	0.375
ML	0.00032	0.00031	0.00031	0.00036
Bayes with gamma prior for tau	0.00011	0.00011	0.00011	0.00012
Bayes with improper prior	0.00011	0.00011	0.00011	0.00012

(c)

	-0.375	-0.125	0.125	0.375
ML	0.01357	0.01247	0.01316	0.01327
Bayes with gamma prior for tau	0.00395	0.00360	0.00359	0.00392
Bayes with improper prior	0.00394	0.00360	0.00359	0.00392

(d)

	-0.375	-0.125	0.125	0.375
ML	0.00043	0.00039	0.00041	0.00042
Bayes with gamma prior for tau	0.00012	0.00011	0.00011	0.00012
Bayes with improper prior	0.00012	0.00011	0.00011	0.00012

(e)

	-0.375	-0.125	0.125	0.375
ML	0.07467	0.07533	0.07458	0.07280
Bayes with gamma prior for tau	0.05200	0.05142	0.05159	0.05348
Bayes with improper prior	0.06857	0.06873	0.06836	0.06820

(f)

	-0.375	-0.125	0.125	0.375
ML	0.00241	0.00244	0.00241	0.00234
Bayes with gamma prior for tau	0.00162	0.00160	0.00160	0.00166
Bayes with improper prior	0.00213	0.00213	0.00212	0.00212

(g)

	-0.375	-0.125	0.125	0.375
ML	0.70239	0.70064	0.70163	0.70723
Bayes with gamma prior for tau	0.69103	0.68834	0.68944	0.69446
Bayes with improper prior	0.69103	0.68835	0.68942	0.69445

(h)

	-0.375	-0.125	0.125	0.375
ML	0.02224	0.02219	0.02222	0.02239
Bayes with gamma prior for tau	0.02191	0.02186	0.02189	0.02207
Bayes with improper prior	0.02191	0.02186	0.02189	0.02207

Figure 8.26. Impact of parameter γ on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Estimation of θ_1 under LINEX, (b) Estimation of θ_2 under LINEX, (c) Estimation of τ under LINEX loss, (d) Estimation of Y_{n+1} under LINEX loss.

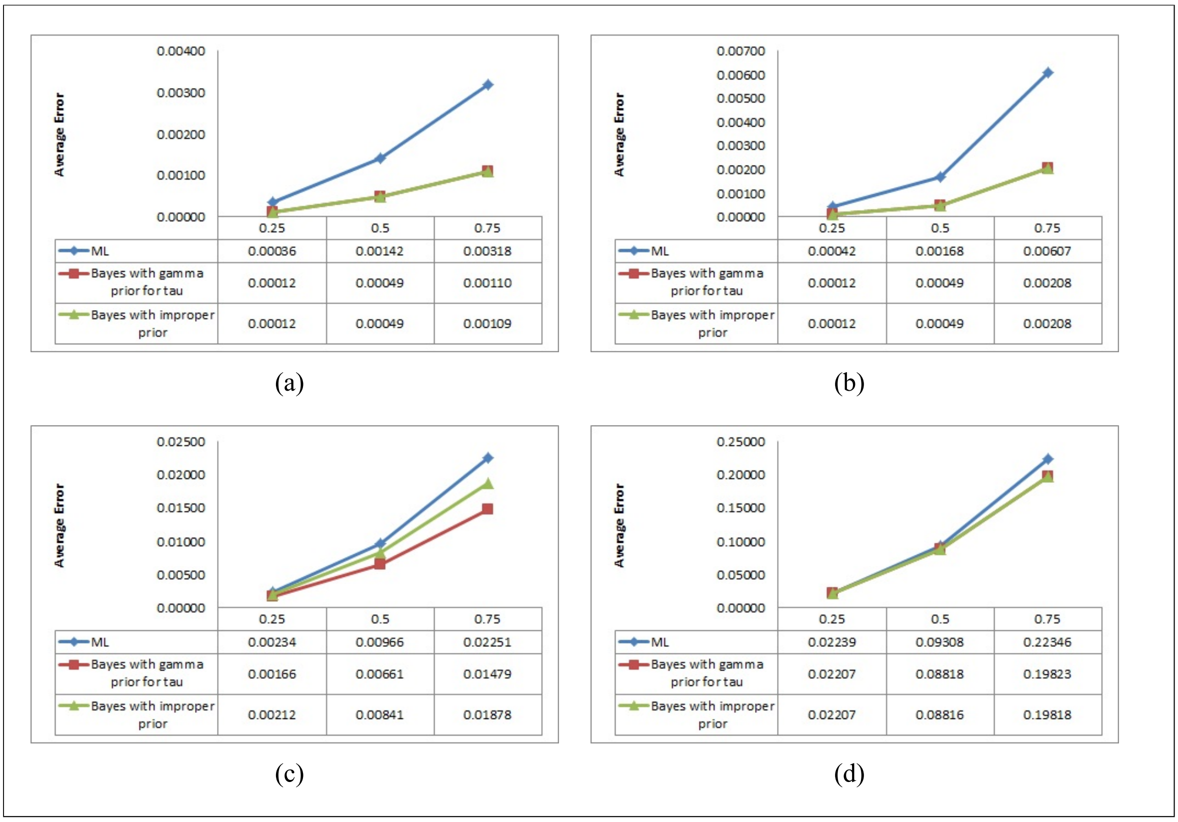
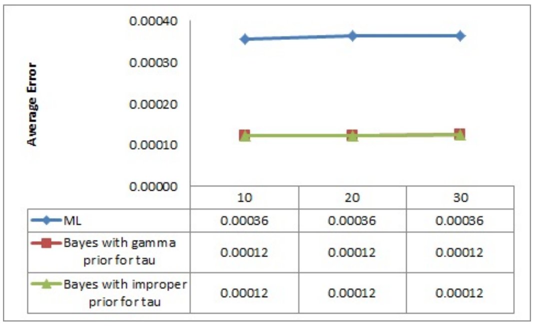
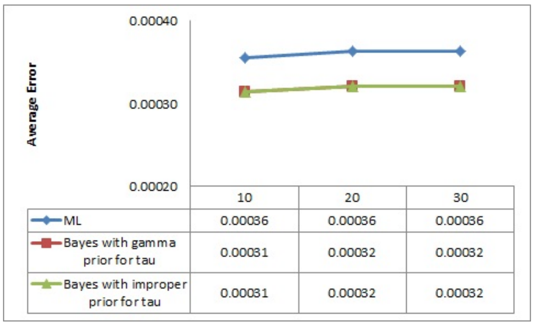


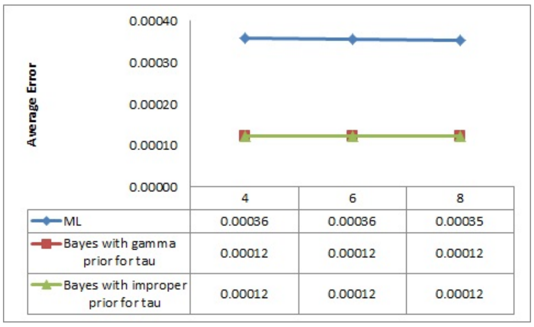
Figure 8.27. Impact of estimation method of θ_1 under LINEX loss function on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Numerical approach, α varies, (b) Approximation, α varies, (c) Numerical approach, β varies, (d) Approximation, β varies, (e) Numerical approach, interval of θ_1 varies, (f) Approximation, interval of θ_1 varies, (g) Numerical approach, γ varies, (h) Approximation, γ varies.



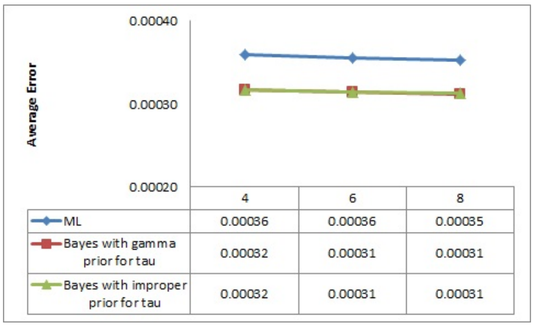
(a)



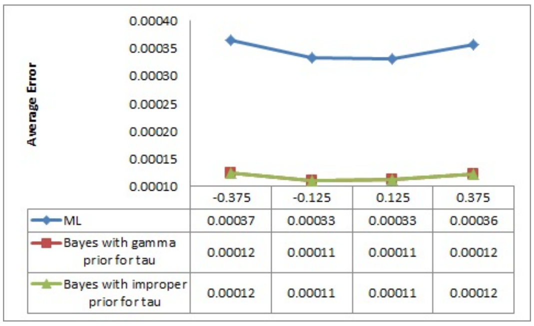
(b)



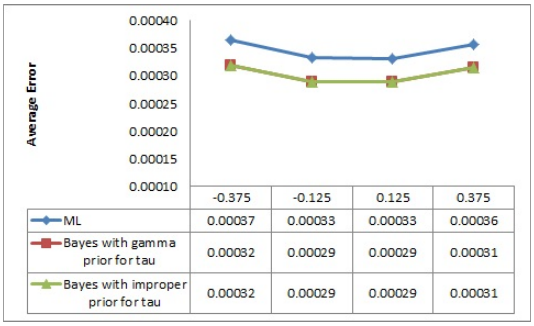
(c)



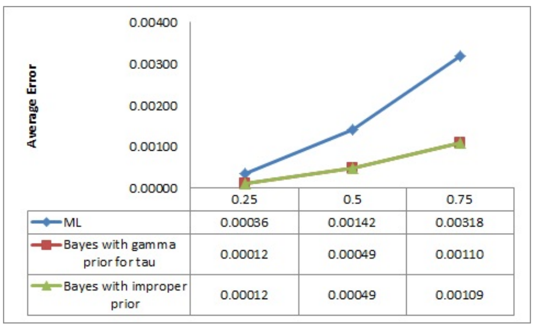
(d)



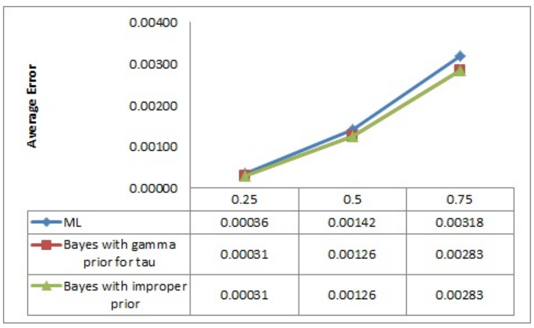
(e)



(f)



(g)



(h)

Figure 8.28. Impact of estimation method of θ_2 under LINEX loss function on average estimation and prediction errors for MA(2) independent uniform prior for θ_1 and θ_2. (a) Numerical approach, α varies, (b) Approximation, α varies, (c) Numerical approach, β varies, (d) Approximation, β varies, (e) Numerical approach, interval of θ_1 varies, (f) Approximation, interval of θ_1 varies, (g) Numerical approach, γ varies, (h) Approximation, γ varies.

(a)

	10	20	30
ML	0.00042	0.00041	0.00041
Bayes with gamma prior for tau	0.00012	0.00012	0.00012
Bayes with improper prior for tau	0.00012	0.00012	0.00012

(b)

	10	20	30
ML	0.00042	0.00041	0.00041
Bayes with gamma prior for tau	0.00036	0.00036	0.00036
Bayes with improper prior for tau	0.00036	0.00036	0.00036

(c)

	4	6	8
ML	0.00042	0.00042	0.00041
Bayes with gamma prior for tau	0.00012	0.00012	0.00012
Bayes with improper prior for tau	0.00012	0.00012	0.00012

(d)

	4	6	8
ML	0.00042	0.00042	0.00041
Bayes with gamma prior for tau	0.00036	0.00036	0.00036
Bayes with improper prior for tau	0.00036	0.00036	0.00036

(e)

	-0.375	-0.125	0.125	0.375
ML	0.00041	0.00034	0.00034	0.00042
Bayes with gamma prior for tau	0.00012	0.00011	0.00011	0.00012
Bayes with improper prior for tau	0.00012	0.00011	0.00011	0.00012

(f)

	-0.375	-0.125	0.125	0.375
ML	0.00041	0.00034	0.00034	0.00042
Bayes with gamma prior for tau	0.00036	0.00029	0.00029	0.00036
Bayes with improper prior for tau	0.00036	0.00029	0.00029	0.00036

(g)

	0.25	0.5	0.75
ML	0.00042	0.00168	0.00607
Bayes with gamma prior for tau	0.00012	0.00049	0.00208
Bayes with improper prior	0.00012	0.00049	0.00208

(h)

	0.25	0.5	0.75
ML	0.00042	0.00168	0.00607
Bayes with gamma prior for tau	0.00036	0.00145	0.00552
Bayes with improper prior	0.00036	0.00145	0.00552

9. CONCLUSIONS

In this study, we presented Bayesian inferences for the autoregressive model of order p and moving average model of order q under two loss functions: squared error and linear exponential. Using independent Truncated Normal - Gamma, Truncated Normal - Improper, Uniform - Gamma and Uniform-Improper priors, the Bayes estimators of the parameters are derived. Under independent truncated normal priors for the parameters, the Bayes estimators are found not to be in a closed form and hence Lindley's approximation is used to obtain the approximate estimators. We undertook a computer simulation study to compare the maximum likelihood and the Bayes estimates obtained using Lindley's approximation and Markov chain Monte Carlo techniques, in particular, Gibbs sampler. As expected, the Bayes estimates are found to have lower estimation errors than the ML estimates. Our simulation study indicates that the Bayes estimators obtained using Lindley's approximation have very similar performances (in terms of average estimation error) to the Bayes estimators obtained using the Gibbs sampler. Moreover, the computational time needed to obtain the Bayes estimates using Lindley's approximation is significantly lower (average time to obtain one set of estimates for the autoregressive and moving average models of order one is 0.0000025 seconds compared to 3.9 seconds using the Gibbs sampler, and 0.0000032 seconds compared to 5.1 seconds for the autoregressive and moving average models of order two). Thus, in order to derive approximate Bayes estimates, Lindley's approximation is preferred over the Gibbs sampler, similar results were found by Kızılaslan and Nadar [61].

The following ideas will be developed in the future research:

- To generalize the Bayesian analysis for the AR and MA models by applying it to the ARMA model.
- To complete the Bayesian analysis for the AR and MA models using other loss functions.
- To apply Lindley's approximation using different priors that lead to posteriors that are not in a closed form.
- To utilize Lindley's approximation for different time series models. In this study we considered univariate AR and MA models, this analysis can be expanded for multi-

variate AR and MA models. However, in practice many time series are nonlinear. To model nonlinear behaviour in time series, the existence of different states of the world or regimes and different dynamics in different regimes can be assumed, i.e. threshold AR, self-exciting threshold AR and smooth transition AR, bilinear models. The present study can be broaden by applying Lindley's approximation to nonlinear models.

- To apply different approximation when the posterior distribution is intractable. As it was pointed out in our present analysis, Lindley's approximation involves only three terms and is of order n^{-1}. Under LINEX loss function this lead to a small proportion of Bayes estimates that are undefined. We recommend expanding Lindley's approximation to consider more terms or to employ some different approximation, i.e. approximation proposed by Tierney and Kadane [46].

REFERENCES

1. G. E. P. Box and G. M. Jenkins. *Time Series Analysis: Forecasting and Control*, Holden-Day, San Francisco, 1970.

2. M. B. Priestley. *Spectral Analysis and Time Series*, Academic Press, New York, 1981.

3. P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods*, Springer-Verlag, New York, 1991.

4. J. D. Hamilton. *Time Series Analysis*, Princeton University Press, Princeton, New Jersey, 1994.

5. C. Chatfield. *The Analysis of Time Series: An Introduction*, Chapman and Hall/CRC Press, Boca Raton, Florida, 2003.

6. A. Zellner. *An Introduction to Bayesian Inference in Econometrics*, Wiley, New York, 1971.

7. L. D. Broemeling. *The Bayesian Methods of Linear Models*, Marcel Dekker, New York, 1985.

8. A. Pole, M. West and J. Harrison. *Applied Bayesian Forecasting and Time Series Analysis*, Chapman and Hall/CRC Press, Boca Raton, Florida, 1994.

9. D. Barber, A. T. Cemgil and S. Chiappa. *Bayesian Time Series Models*, Cambridge University Press, New York, 2011.

10. G. Kitagawa and W. Gersch. *Smoothness Priors Analysis of Time Series* (Lecture Notes in Statistics, Number 116), Springer-Verlag, New York, 1996.

11. M. West and P. J. Harrison. *Bayesian Forecasting and Dynamic Models*, Springer-Verlag, New York, 1997.

12. M. F. J. Steel. Bayesian Time Series Analysis. In: S. N. Durlauf, L. E. Blume, editors. *The New Palgrave Dictionary of Economics* Palgrave Macmillan, 2008.

13. J. O. Berger. Bayesian Analysis: A Look at Today and Thoughts of Tomorrow. *Journal of the American Statistical Association*, 95: 1269-1276, 2000.

14. D. J. Poirier. The Growth of Bayesian Methods in Statistics and Economics Since 1970, *Bayesian Analysis*, 1: 969-980, 2006.

15. G. Baio. *Bayesian Methods in Health Economics*, Chapman and Hall/CRC Press, Boca Raton, Florida, 2012.

16. X. Du, L. Y. Cindy and D. J. Hayes. Speculation and Volatility Spillover in the Crude Oil and Agricultural Commodity Markets: A Bayesian Analysis. *Energy Economics*, 33: 497-503, 2011.

17. O. Aguilar and M.West. Bayesian Dynamic Factor Models and Portfolio Allocation, *Journal of Business and Economic Statistics*, 18:338-357, 2000.

18. K. P. Baks, A. Metrick and J. Wachter. Should Investors Avoid All Actively Managed Mutual Funds? A Study in Bayesian Performance Evaluation, *The Journal of Finance*, 56: 45-85, 2001.

19. A. Brav. Inference in Long-Horizon Event Studies: A Bayesian Approach with Application to Initial Public Offerings, *The Journal of Finance*, 55: 1979-2016, 2000.

20. U. Herold and R. Maurer. Bayesian Asset Allocation and U.S. Domestic Bias, *Financial Analyst Journal*, 59: 54-64, 2003.

21. N. G. Polson and B. V. Tew. Bayesian Portfolio Selection: An Empirical Analysis of the Standard and Poor's 500 Index 1970-1996, *Journal of Business and Economic Statistics*, 18: 164-173, 2000.

22. K.V. Yuen. *Bayesian Methods for Structural Dynamics and Civil Engineering*, John Wiley and Sons Asia, Singapore, 2010.

23. M. Wang and T. Takada. A Bayesian Framework for Prediction of Seismic Ground Motion. *Bulletin of the Seismological Society of America*, 99: 2348-2364, 2009.

24. M. Stephens and D. J. Balding. Bayesian Statistical Methods for Genetic Association Studies. *Nature Reviews Genetics*, 10: 681-690, 2009.

25. J. Corander, P. Waldmann and M. J. Sillanpää. Bayesian Analysis of Genetic Differentiation Between Populations. *Genetics*, 163: 367-74, 2003.

26. S.Ogino, R. B. Wilson, B. Gold, P. Hawley and W. W. Grody. Bayesian Analysis for Cystic Fibrosis Risks in Prenatal and Carrier Screening. *Genetics in Medicine*, 6: 439–449, 2004.

27. L. C. Gurrin, J. J. Kurinczuk and P. R. Burton. Bayesian Statistics in Medical Research: An Intuitive Alternative to Conventional Data Analysis. *Journal of Evaluation in Clinical Practice*, 6: 193-204, 2000.

28. E. Tzala and N. Best. Bayesian Latent Variable Modelling of Multivariate Spatio-Temporal Variation in Cancer Mortality. *Statistical Methods in Medical Research*, 17: 97-118, 2008.

29. P. Gregory. *Bayesian Logical Data Analysis for the Physical Sciences*, Cambridge University Press New York, New York, 2005.

30. Y. Yu, V. L. Kashyap, D. A. van Dyk and A.Young. A Bayesian Analysis of the Correlations Among Sunspot Cycles. *Solar Physics*, 281: 847-862, 2012.

31. J. Berger. The Case for Objective Bayesian Analysis. *Bayesian Analysis*, 1: 385-402, 2006.

32. S. Shaarawy and L. Broemeling. Inference and Prediction with ARMA Processes. *Communications in Statistics - Theorem and Methods*, 14: 1231-1238, 1985.

33. M. J. Schervish and R. S. Tsay. Bayesian Modeling and Forecasting in Large Scale Time Series. In: J. C. Spall, editor. *Bayesian Analysis of Time Series and Dynamic Models*, Marcel Dekker, New York, 1988.

34. C. Fan and S. Yao. Bayesian Approach for ARMA Process and Its Application. *International Business Research*, 1: 49-55, 2008.

35. R. Zeithammer and P. Lenk. Bayesian Analysis of Multivariate Normal Models when Dimensions are Absent. *Quantitative Marketing and Economics*, 4: 241-265, 2006.

36. V. Sevinç and G. Ergün. Usage of Different Prior Distributions in Bayesian Vector Autoregressive Models. *Hacettepe Journal of Mathematics and Statistics*, 38: 85 -93, 2009.

37. M. Mahmoud. Bayesian Estimation of the 3-parameter Inverse Gaussian Distribution. *Trabajos de Estadistica*, 6: 45-62, 1991.

38. C. Robert and G. Casella. *Monte Carlo Statistical Methods*. Springer-Verlag, New York, 1999.

39. J. Li, C. M. Zhang, E. V. Nordheim and C. E. Lehner. On the Multivariate Predictive Distribution of Multi-Dimensional Effective Dose: A Bayesian Approach.*Journal of Statistical Computation and Simulation*, 78: 429-442, 2008.

40. S. Geman and D. Geman. (1984). Stochastic Relaxation, Gibbs Distributions and the Bayesian Restoration of Images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12: 609-628, 1984.

41. A. E. Gelfand and A. F. M. Smith. Sampling-Based Approaches to Calculating Marginal Densities. *Journal of the American Statistical Association*, 85: 398-409, 1990.

42. N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H.Teller and E. Teller. Equations of State Calculations by Fast Computing Machines. *Journal of Chemical Physics*, 21: 1087-1092, 1953.

43. W. K. Hastings. Monte Carlo Sampling Methods using Markov Chains and their Applications, *Biometrika*, 57: 97-109, 1970.

44. S. Chib and E. Greenberg. Understanding the Metropolis-Hastings Algorithm. *The American Statistician*, 49: 327-335, 1995.

45. R.S. Tsay. *Analysis of Financial Time Series*, Wiley, New York, 2010.

46. L. Tierney and J. B. Kadane. Accurate Approximations for Posterior Moments and Marginal Densities. *Journal of the American Statistical Association*, 81: 82-86, 1986.

47. D.V. Lindley. Approximate Bayesian Methods. *Trabajos de Estadistica*, 21: 223-237, 1980.

48. R. Singh, S.K Singh, U. Singh and G.P. Singh. Bayes Estimator of Generalized-Exponential Parameters under LINEX Loss Function using Lindley's Approximation. *Data Science Journal*, 7: 65-75, 2008.

49. M. Nadar, A. Papadopoulos and F. Kizilaslan. Classical and Bayesian Estimation of $P(Y < X)$ for Kumaraswamy's Distribution. *Journal of Statistical Computation and Simulation*, 84: 150-1529, 2012.

50. T. W. Anderson, N. D.Singpurwalla and R. Soyer. *Bayesian Analyses of Nonhomogeneous Autoregressive Processes*, Department of Statistics, Stanford University, 1986.

51. C. W. J. Granger. Outline of Forecast Theory using Generalized Cost Functions. *Spanish Economic Review*, 1: 161-173, 1999.

52. S. Satchell, J. Knight and S. Hwang. Forecasting Volatility using LINEX Loss Functions, Working Papers 99-20, Warwick Business School, Finance Group, 1999.

53. P. K. Singh , S. K. Singh and U. Singh. Bayes Estimator of Inverse Gaussian Parameters Under General Entropy Loss Function using Lindley's Approximation. *Communications in Statistics - Simulation and Computation*, 37: 1750-1762, 2008.

54. H. R. Varian. A Bayesian Approach to Real Estate Assesment. In: S. N. Fienberg, A. Zellner, editors, *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*.North-Holland, Amsterdam, 1975.

55. L. Tierney. Markov Chains for Exploring Posterior Distributions. *The Annals of Statistics*, 22: 1701-1728, 1994.

56. G. C. Tiao and G. E. P. Box. Modeling Multiple Time Series with Applications. *Journal of the American Statistical Association*, 76: 802-816, 1981.

57. A. DasGupta. *Probability for Statistics and Machine Learning: Fundamentals and Advanced Topics*, Springer, New York, 2011.

58. A.C. Harvey. *The Econometric Analysis of Time Series*, Wiley, New York, 1981.

59. H. J. Ho, T. I. Lin, H. Y. Chen and W. L. Wang. Some Results on the Truncated Multivariate t-distribution. *Journal of Statistical Planning and Inference*, 142: 25-40, 2012.

60. S. C. Hillmer and G. C. Tiao. Likelihood Function of Stationary Multiple Autoregressive Moving Average Models. *Journal of the American Statistical Association*, 74: 652-660, 1979.

61. F. Kızılaslan and M. Nadar. Estimation and Prediction of the Kumaraswamy Distribution Based on Record Values and Inter-record Times. *Journal of Statistical Computation and Simulation*, 0:1-23, 2015.