2-DIMENSIONAL TOPOLOGICAL QUANTUM FIELD THEORY AND FROBENIUS ALGEBRAS

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ABSTRACT

2-DIMENSIONAL TOPOLOGICAL QUANTUM FIELD THEORY AND FROBENIUS ALGEBRAS

In this project, we define two dimensional topological quantum field theories (TQFTs) and show the relation between Frobenius algebras. More precisely, equivalence between the category $2dTQFT_k$, of symmetric monoidal functors from the category, $2Cob$, of two-dimensional cobordisms to the category, $Vect_k$, of vector spaces over a field and the category of, cFA_k, of commutative Frobenius algebras will be demonstrated.

The work begins with basic properties of category theory. Moreover, a symmetric monoidal categories are explained. In the next three chapters, the category of cobordisms are constructed to define Frobenius algebras clearly. The category of Frobenius algebras are shown by the cobordism notation. At the end of the work, we use Atiyah axioms to explore TQFTs. His comment makes a relation between cobordisms and Frobenius algebras.

ÖZET

2-BOYUTLU TOPOLOJİK KUANTUM ALAN KURAMI VE FROBENIUS CEBİRLERİ

Projenin amacı, iki boyutlu topolojik kuantum alan kuramı ile Frobenius cebirleri arasındaki bağlantıyı açıklamaktır. Daha kesin olarak, iki boyutlu topolojik kuantum alan kuramı kategorisi ile Frobenius cebirlerinin kategorisi arasındaki kategorilerinin denk olduğu gösterilmiştir. Çalışmamıza kategori teorisinin özelliklerinden başlayarak simetrik monodial kategorisi ile devam ediyoruz. Diğer üç bölümde, cobordism kategorisini oluşturarak Frobenius cebirlerini tanımlıyoruz. Frobenis cebir kategorisini cobordism notasyonlarını kullanarak gösteriyoruz. Çalışmamızın sonunda, Atiyah aksiyomlarını kullanarak topolojik kuantum alan kuramını ortaya çıkarıyoruz. Atiyah'ın yorumu cobordism ve Frobenius cebirleri arasındaki ilişkiyi oluşturuyor.

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LIST OF SYMBOLS/ABBREVIATIONS

g	Genus
$H_f(p)$	Hessian matrix of f at point p
k	Index of critical point
m	In-boundaries
n	Out-boundaries
T_pM	Tangent space to M at point p
V^*	Dual vector space
$Vect_k$	k-vector spaces
Z	Functor
α	Natural transformation
β	Pairing
γ	Co-pairing
δ	Co-multiplication
ϵ	Linear form
μ	Multiplication

1. INTRODUCTION

The definition of topological quantum field theories (TQFTs) was explained by Michael Atiyah in 1988. A topological quantum field theory is a symmetric monoidal functor; $Z : nCob \rightarrow Vect_k$. This means that, A TQFT is defined as a functor from the category of cobordisms to the category of vector spaces. This project wants show the relation between TQFTs and commutative Frobenius algebras (cFA_k).

This work uses both algebra and topology. Also category theory, symmetric monoidal category and functors take an important part in this thesis. The category of topological quantum field theory is explained with functor. More precisely, there is an equivalence between categories of two dimensional topological quantum field theory and Frobenius algebra. We will show; $2TQFT_k \simeq cFA_k$ in the thesis.

This thesis consists of four chapters. In the first chapter, we explain general properties of category theory. Moreover, a symmetric monoidal categories have an important role in other chapters. The second chapter explains the properties of cobordism theory. The last two chapters, Frobenius algebras are defined with cobordisms. In addition to this, Atiyah's axioms show their relations clearly. His axioms are the main part in this thesis. Frobenius algebras and cobordisms are used to understand TQFT obviously.

2. MONOIDAL CATEGORIES

In this first chapter we need the categorical concepts and tools for the definition of TQFTs. First of all, we start with some basic concepts of Category Theory, such as categories, functors and natural transformations. Next, we define monoidal categories with some specific properties, that have a very important place in the chapters. We use some structures such as monoidal functors and a symmetric structure to explain it clearly.

2.1. BASIC CONCEPTS FROM CATEGORY THEORY

In this section we can give several properties of the Category Theory.

Definition 2.1.1. A category is the mathematical structure and arrows between them. A category $\mathcal{C}$ consist of

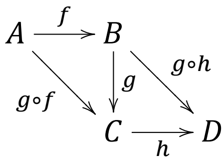
(i) a collection of objects, $Ob(\mathcal{C})$: A, B, C ...

(ii) a collection of morphisms, $Ar(\mathcal{C})$: f, g, h ... (sometimes it is called arrows)
If we work diagrammatically; an arrow $f \in \mathcal{C}(A, B)$ is shown as,
$f : A \to B$ we call A; the domain **(source)** of f and B; the co-domain **(target)** of f.

(iii) Given arrows $f : A \to B$, $g : B \to C$ there exist morphism $g \circ f : A \to C$ called the **"composition"**

(iv) For each object A there is identity morphism; $1_A : A \to A$
These data should satisfy the associative and unit laws which are giving with diagram;

(v) Associative law: $(h \circ g) \circ f = h \circ (g \circ f)$ for every $f : A \to B$, $g : B \to C$ and $h : C \to D$ then we have



(vi) Unit law: $(f \circ 1_A) = f = (1_B \circ f)$ for all $f : A \to B$ then we have

Examples of Categories

(i) **Set**, the category that has sets as objects and set maps as morphisms,

(ii) **Grp**, the category that has groups as objects and group homomorphisms as morphisms,

(iii) **Ab**, the category that has abelian groups as objects and group homomorphisms as morphisms,

(iv) **Rng**, the category that has rings as objects and ring homomorphisms as morphisms,

(v) $\mathbf{Vect_k}$, the category that has k-vector spaces as objects and k-linear maps as morphisms,

(vi) **Top**, the category that has topological spaces as objects and continuous maps as morphisms,

(vii) **Man**, the category that has smooth manifolds as objects and smooth (infinitely differentiable) maps as morphisms,

Remark 2.1.2. *For each pair of categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathcal{C} \times \mathcal{D}$ defined as fallows: its objects are pairs (A, B) where; $A \in Ob(\mathcal{C})$, $B \in Ob(\mathcal{D})$. Also morphism $(A, B) \to (A', B')$ of $\mathcal{C} \times \mathcal{D}$ is pair (f, g) where $f : A \to A'$ morphism of $\mathcal{C}$, $f : B \to B'$ morphism of $\mathcal{D}$.*

Finite Categories

We begin by looking at some examples of simple finite categories such that;

(i) The category $\mathbf{0}$ looks like this: It has no objects and arrows.

(ii) **1** looks like this:

$$*$$

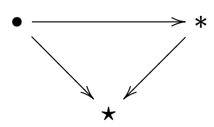
It has one object and its identity arrow, which we do not draw.

(iii) **2** looks like this:

$$* \longrightarrow \star$$

It has two objects,their required identity arrows and exactly one arrow between the objects.

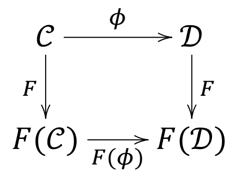
(iv) **3** looks like this: The diagonal arrow is the composite of the vertical and horizontal ones.



Definition 2.1.3. Functor is an assignment between two categories $\mathcal{C}$ and $\mathcal{D}$ such that, a transformation F must take objects and morphisms of $\mathcal{C}$ to objects and morphisms of $\mathcal{D}$. Functors are mappings between categories that preserve the structure. They take a category and embedded it in another category or modelling a category inside an another category. They are shown in the following diagram;

$$F_{ob} : ob(\mathcal{C}) \to ob(\mathcal{D})$$

$$F_{ar} : ar(\mathcal{C}) \to ar(\mathcal{D})$$



Definition 2.1.4. Given two categories $\mathcal{B}$ and $\mathcal{C}$ a natural transformation $\alpha : F \Longrightarrow G$ between functors $F, G : \mathcal{B} \to \mathcal{C}$, consists;

(i) a function α mapping each object $X \in \mathcal{C}$ to a morphism $\alpha_X : F(X) \to G(X)$ in $\mathcal{D}$ such that:

(ii) for any morphism $f : X \to Y$ in $\mathcal{B}$, this diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \alpha_X}\downarrow & & \uparrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

Natural transformations are defined as mapping between functors. They have to preserve structure. Here, two categories $\mathcal{B}$ and $\mathcal{C}$ and we have two functors between these categories that we want to compare. First, take a single object X and one functor let's call it F, maps this object into $F(X)$. Second functor G maps the same object into $G(X)$. If we want to map one functor to another functor, we need to mapping these two $F(X)$ and $G(X)$.

2.2. MONOIDAL CATEGORY

The properties of monoidal categories with functors are studied to classify topological field theories as mentioned in the beginning of the chapter. Some thematic definitions and concepts are given in this section. In a monoidal category, we would like to define things that, what does it mean to multiply two objects. We want to define multiplication of objects. When we are talking about monoids, we talk about operation on elements of set or on morphisms. What is a good name for the product that could be a co-product or bi-functor? The good name is the tensor product. Hence monoidal category has a tensor product. We will see later. First the definition of monoid is given by the following definition.

Definition 2.2.1. A monoid itself can be thought of as a category with one object just M itself. We can represent monoid as a single object category. We also have a morphism. A monoid M is a set equipped with an associative binary operation and a neutral element. More precisely, a monoid is a triple (M, μ, η) where,

(i) M is a set,

(ii) $\mu : M \times M \to M$ is a map (the binary operation)

(iii) $\eta : 1 \to M$ is a map, being 1 a set with one element (whose image by η acts as neutral element),

such that the following diagrams commute:

$$\begin{array}{ccc} M\times M\times M & \xrightarrow{\mu\times id_M} & M\times M \\ {\scriptstyle id_M\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M\times M & \xrightarrow[\mu]{} & M \end{array}$$

$$\begin{array}{ccccc} 1\times M & \xrightarrow{\eta\times id_M} & M\times M & \xleftarrow{id_M\times\eta} & M\times 1 \\ & \searrow & \downarrow{\scriptstyle\mu} & \swarrow & \\ & & M & & \end{array}$$

Definition 2.2.2. Monoidal category is also equivalent to categories with tensors so, a tensor - $\otimes$ - on a category is; a way of combining two objects or arrows to make a new object or arrows of the same category like;

(i) Objects: Given X, Y we can form a new object; $X \otimes Y$.

(ii) Arrows: Given f, g (morphisms) we can form new morphism $f \otimes g$.

Definition 2.2.3. A monoidal category is a sextuple $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ where:

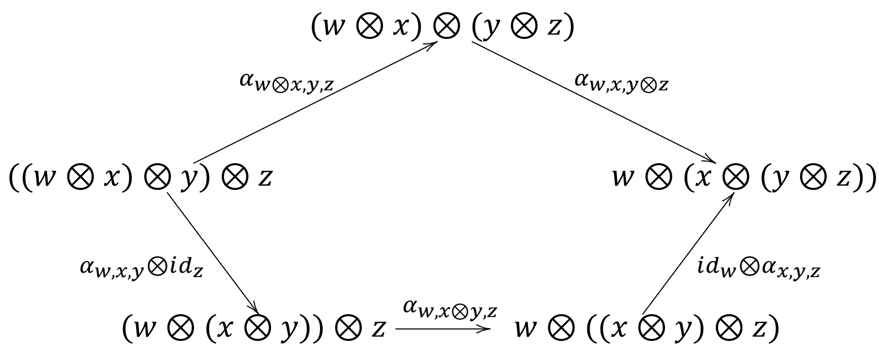
(i) $\mathcal{C}$ is a category

(ii) a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product.

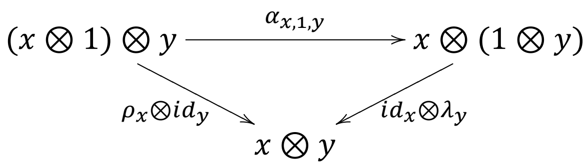
(iii) an object called the **identity object** $1 \in \mathcal{C}$

(iv) a natural isomorphisms called the **associator**:
$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$,
$\lambda(x) : 1 \otimes x \to x$ **left unit**
$\rho(x) : x \otimes 1 \to x$ **right unit**
such that the following diagrams commute for all objects $x, y, z, w \in Ob(\mathcal{C})$:

(v) the **pentagon equation**:



(vi) And natural isomorphisms, ρ and λ, such that the following triangle diagram is commutative,



for all $x, y \in Ob(\mathcal{C})$.

(Recall: $\rho_x : x \otimes I \to x$ and $\lambda_x : I \otimes x \to x$)

Theorem 2.2.4 (Mac Lane's Theorem)**.** *We define*

$$A_1 \otimes A_2 \otimes \dots \otimes A_n = (\dots(A_1 \otimes A_2) \otimes \dots) \otimes A_n \tag{2.1}$$

The coherence theorem Mac Lane says that all diagrams whose morphisms are formed using α, λ, ρ identities, inverses, tensor products and compositions ***commute****.*

Example 2.2.5. *The traditional definition of a monoid is in terms of sets, a monoid is just a set of elements; for example set of numbers so set is a basic example for the monoidal category. It admits a monoidal structure given by the Cartesian product $\times$, the unit object is the singleton set $*$ and the natural isomorphisms are the obvious ones.*

Example 2.2.6. *In monoidal category, we have categorical product and terminal objects. Actually we can say co-product has the same property. It is also monoidal thing; it is associative upto isomorphism. In general it has also unit; the empty set.*

Example 2.2.7. *Given two manifolds Σ and Σ', we can form their disjoint union $\Sigma \sqcup \Sigma'$,* *which is again a manifold. If* Σ *and* Σ' *are oriented, then there is an orientation on* $\Sigma \amalg \Sigma'$ *and the maps should preserve orientation.We can say,* $\amalg$ *is the co-product in the category of (oriented) manifolds. In general, we can have a monoidal category and only requirement for a monoidal category is that, it has a tensor product, it takes one object with another object and produces third object.*

Definition 2.2.8. We say that a monodial category $(\mathcal{C}, \otimes, \alpha, 1, \lambda, \rho)$ is **strict** whenever the natural isomorphisms α, λ, ρ are composed by identities morphisms. In the case, we represent the monoidal category by $(\mathcal{C}, \otimes, 1)$.

In order to simplify our subject we will use strict monoidal categories in the next sections. Hence, whenever we tell the concepts of monoidal category, we refer to strict versions.

Definition 2.2.9. A **strict monoidal functor**

$$F : (\mathcal{B}, \otimes, \alpha, 1, \rho, \lambda) \to (\mathcal{C}, \otimes^{'}, \alpha^{'}, 1^{'}, \rho^{'}, \lambda^{'})$$

between two monoidal categories, is a functor on the underlying categories $F : B \to C$ such that:

(i) $F(x \otimes y) = F(x) \otimes^{'} F(y)$, for any two objects $x, y \in \mathcal{B}$;

(ii) $F(f \otimes g) = F(f) \otimes^{'} F(g)$, for any two morphisms f, g of $\mathcal{B}$;

(iii) $F(1) = 1^{'}$;

(iv) $F(\alpha_{x,y,z}) = \alpha^{'}_{F(x),F(y),F(z)}$, for all $x, y, z \in Ob(\mathcal{B})$;

(v) $F(\rho_x) = \rho^{'}_{F(x)}$, for all $x \in Ob(\mathcal{B})$;

(vi) $F(\lambda_x) = \lambda^{'}_{F(x)}$, for all $x \in Ob(\mathcal{B})$;

2.3. SYMMETRIC MONOIDAL CATEGORIES

Definition 2.3.1. A **braided monoidal category** consists of:

(i) a monoidal category $\mathcal{C}$.

(ii) a natural isomorphism called the **braiding:**

$$B_{X,Y} : X \otimes Y \to Y \otimes X$$

Definition 2.3.2. A **symmetric monoidal category** is a braided monoidal category $\mathcal{C}$ for which the braiding satisfies $B_{X,Y} = B_{Y,X}^{-1}$ for all objects X and Y.

Definition 2.3.3. A strict monoidal category $(\mathcal{C}, \otimes, 1)$ is said to be symmetric if there exist a natural isomorphism τ, such that, for any pair of objects X, Y, there is a twist map

$$\tau_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

such that the following two conditions are satisfied:

For each pair of arrows $f : X \to X'$ and $g : Y \to Y'$ the following diagram commutes:

(i)

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\tau_{X,Y}} & Y \otimes X \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ X' \otimes Y' & \xrightarrow[\tau_{X',Y'}]{} & Y' \otimes X' \end{array}$$

(ii)

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{\tau_{X,Y} \otimes Z} & Y \otimes Z \otimes X \\ & {\scriptstyle \tau_{X,Y} \otimes\, id_Z} \searrow \quad \nearrow {\scriptstyle id_Y \otimes \tau_{X,Y}} & \\ & Y \otimes X \otimes Z & \end{array}$$

for any objects $X, Y, Z \in Ob(\mathcal{C})$.

Remark 2.3.4. *It can be said that, every monoidal (braided monoidal, symmetric monoidal) category is equivalent to strict one.*

Note. *The symmetric monoidal category in this thesis will be also related with the cobordism category* ***Cob(n)****.*

3. THE CATEGORY OF COBORDISMS

This chapter wants to tell the importance of the symmetric monoidal category of cobordisms. After some basic informations from topology and differential geometry, we define the category of cobordisms and we construct a symmetric monoidal structure on it. At the end of this chapter, a monoidal category is given in terms of generators and relations in 2- dimensional cobordisms. They are used to define functors from such a category to another symmetric monoidal category. Cobordisms play a central role in this thesis. They can be used in the formulation of topological quantum field theory. We are going to see that, cobordisms and Frobenius algebras are closely related in 2 dimensions. We start by describing the notion of a manifold with boundary and its properties of in-boundary and out-boundary in order to give a definition of a cobordism.

3.1. TOPOLOGICAL PROPERTIES ON SMOOTH MANIFOLDS

Let us review some definitions of topological manifolds;

Definition 3.1.1. An n-dimensional *manifold* with boundary is a Hausdorff, second countable topological space equipped with an open covering such that each open set is homeomorphic to $\mathbb{H}^n$.

$$\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$$

(i) The **boundary** of M denoted ∂M and consists of all points in M mapped to the points. $\mathbb{H}_0^n = \{(x^1, \dots, x^{n-1}, 0)\} \in \mathbb{H}^n$ through the coordinate functions.

(ii) A topological space M is called a **d-dimensional topological manifold** if that for every point in the set, there exists an open set $\mathcal{U}$ containing the point p such that there exists a map x from this region open set in the manifold into $x(\mathcal{U}) \subseteq \mathbb{R}^d$. It does not need to be surjective. It goes the image of the domain under the chart x and it is supposed to lie in $\mathbb{R}^d$ such that x is invertible and continuous. Also the inverse of x is continuous.

(iii) For each point $p \in M$,we define the set T_pM is called the **tangent space** to M at the point p. This is simply the connection of all possible tangent vectors to all possible smooth curves through the point.

(iv) A **closed** manifold is a compact manifold with empty boundary. As an example of a manifold with boundary, consider a cylinder; A product between the unit circle denoted S^1 and the compact unit interval $I = [0, 1]$ is a cylinder C in 2 dimensions. $C = S^1 \times I$. The boundary ∂C consists of 2 copies of the unit circle such that; $\partial C_0 = S^1 \times 0$ and $\partial C_1 = S^1 \times 1$. Hence S^1 is an example of a closed manifold.

3.1.1 Orientations on smooth manifolds

Since we start to tell about what is inwards and outwards of the boundary of a manifold. Then the orientation plays an important role on the manifold.

Definition 3.1.2. Let V be a real vector space of dimension n. Some basis are chosen for the vector space V. The basis $\{e_1, ..., e_n\}$ should be ordered. The basis vectors are arranged with an invertible matrix. Then a new basis is constructed for the vector space V. The determinant of the matrix should be non-zero. Its signature determine the orientation. Hence there are two possible orientations. If the sign is plus sign then a basis is positive. Then orientation preserving. If the basis is not positive the orientation reversing. Orientation should be preserved.

The boundary of manifolds can be distinguished the connected components of the boundary as in or out-boundaries. These distinctions can be defined on the following definition.

Definition 3.1.3. Let M be an n-dimensional orientable manifold with boundary ∂M and N be a closed orientable $(n-1)$-dimensional sub-manifold of M. Then we take a tangent space T_xN at $x \in N$. And a positive basis is chosen such that $[v_1, ..., v_{n-1}]$. Also consider a basis $[v_1, ..., v_{n-1}, v_n]$ of T_xM is also positive at the same point $x \in M$. Moreover N is a connected component of the boundary M. It can be thought that V_n points either **inward** or **outward**

with respect to M. If V_n is inward; N is **in-boundary** or if V_n is outward; N is **out-boundary**. Shortly the boundary ∂M of M consists of the union of different in and out- boundaries. ∂M_{in} and ∂M_{out} construct the boundary of M.

Then we can show; in-boundaries can be drawn to the left and out boundaries to the right. For example; a cylinder can be drawn according to this order.

(i) Cylinder has one in-boundary and one out-boundary components it can be drawn as;

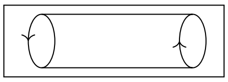


Figure 3.1. The cylinder with boundaries

(ii) When the both boundary components are in-boundaries, then the cylinder can be drawn as;

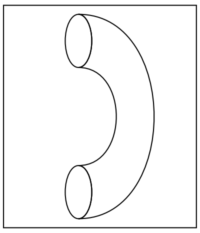


Figure 3.2. The cylinder with in-boundaries

(iii) With two out-boundaries cylinder is drawn as;

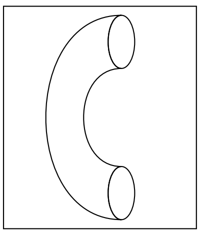


Figure 3.3. The cylinder with out-boundaries

From here on we are ready to give definition of a cobordism.

3.2. THE CATEGORY OF COBORDISMS: $nCob$

The goal of this section is to construct the symmetric monoidal category of n-cobordisms.

3.2.1 Cobordisms

Cobordism theory is related to manifolds that in this category; objects are closed oriented (n-1)-manifolds and morphisms are oriented n-cobordism classes together with in- boundary and out-boundary. In boundary; you are going into cobordism. It can be thought as a domain. For out-boundary we can say it is co-domain. In this section, composition of cobordisms takes an important part in this category which can be defined "gluing". Moreover, the identity morphism of an object O. W is the cobordism class of cylinder $W \times I$.

Definition 3.2.1. A cobordism between closed n-dimensional orientable manifolds M_1 and M_2 is an $(n+1)$-dimensional orientable manifold W with boundary $\partial W = M_1 \amalg \overline{M}_2$ where $\overline{M}_2$ is the n-dimensional manifold M_2 whose orientation is reverse orientation of M_2. So two cobordant manifolds M_1 is cobordant to M_2 when their disjoint union forms the boundary of a manifold boundary W.

One property about cobordism is; oriented cobordism is an equivalence relation on manifolds.

(i) Reflexive: A manifold M is cobordant to itself and a manifold $M \times [0,1]$ can be defined. So a cobordism between M and M for every manifold M.

(ii) Symmetric: A disjoint union of operation is symmetric so if $\partial W = M_1 \amalg M_2$ boundary of a manifold W. We can also say, $\partial W = M_2 \amalg M_1$.

(iii) Transitive: If M_1 is cobordant to M_2 with W_1 and M_2 is cobordant to M_3 with W_2 then W_1 and W_2 can be glued along M_2 to create a cobordism between M_1 and M_3 Transitive condition is a glue for composition of cobordisms.

Note. *One more important thing is; diffeomorphic manifolds are cobordant which represent the same cobordism. The following definition is about this property.*

Definition 3.2.2. Let Σ_1 and Σ_2 be two closed $(n-1)$-dimensional orientable manifolds. An oriented n-cobordism M from Σ_1 to Σ_2 is an oriented n-manifold with boundary M together with two smooth maps $f_1 : \Sigma_1 \rightarrow M$ and $f_2 : \Sigma_2 \rightarrow M$ mapping diffeomorphically Σ_1 onto the in-boundary of M and Σ_2 onto the out-boundary of M and preserving orientations.

$\Sigma_1 \xrightarrow{f_1} M \xleftarrow{f_2} \Sigma_2$ where f_1 is an orientation preserving diffeomorphism of Σ_1 onto $f_1(\Sigma_1) \subset \partial M$ and f_2 is an orientation preserving diffeomorphism of Σ_2 onto $f_2(\Sigma_2) \subset \partial M$ such that $f_1(\Sigma_1)$ and $f_2(\Sigma_2)$ called the in-out boundaries respectively.

Let us look some important examples. A disc D^2 is a two dimensional manifold with boundary. According to a cobordism, the disc D^2 gives us two cases either the boundary component S^1 is an in-boundary or out-boundary.

If S^1 is an in-boundary, the disc becomes a cobordism from S^1 to $(n-1)$-dimensional empty manifold $S^1 \xrightarrow{D^2} \emptyset_{n-1}$.

In this case it is called **right cap** such that,

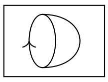


Figure 3.4. The right cap

In other case when S^1 is out-boundary a disc becomes a cobordism from $(n-1)$-dimensional empty manifold to S^1; $\emptyset_{n-1} \xrightarrow{\overline{D}^2} S^1$. It is called **left cap**

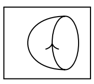


Figure 3.5. The left cap

Moreover, a cobordism M between two circles to one circle; this means that two circles are mapped to the in-boundary and one circle to the out boundary of the cobordism such that $S^1 \sqcup S^1 \xrightarrow{M} S^1$ it is called **left pair of pants**

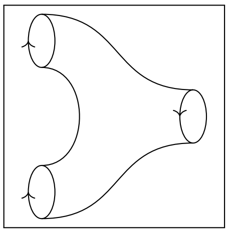


Figure 3.6. The left pair of pants

When two circles are mapped to the out-boundary and one circle to the in-boundary of the cobordism such that $S^1 \xrightarrow{\overline{M}} S^1 \amalg S^1$ it is called *right pair of pants*

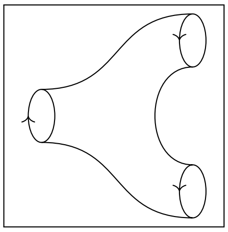
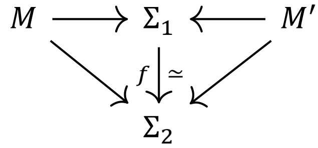


Figure 3.7. The right pair of pants

There are several ways to construct cobordism. The following property shows us that, that two cobordisms are equivalent.

Definition 3.2.3. Two oriented cobordisms M and M' from Σ_1 to Σ_2 are said to be equivalent if there exist an orientation preserving diffeomorphism $\varphi : M \to M'$ such that the following diagram commutes;



A cobordism M of smooth closed $(n-1)$-dimensional manifolds Σ_1 and Σ_2 comes equipped with given diffeomorphisms such that, $f_1 : \Sigma_1 \to \partial M$ and $f_2 : \Sigma_2 \to \partial M$ onto the parts of the boundary. They are pointing into cobordism and pointing out of the cobordism. These notions are determined by the orientations of the boundary components that they are incoming and outgoing parts of the boundary. Hence two cobordisms M and M' represent the same morphism in Cob_n such that $\varphi : M \to M'$.

Example 3.2.4. *Given (orientable) (n-1)-manifolds* Σ_0 *and* Σ_1 *and a diffeomorphism* $\varphi : \Sigma_0 \to \Sigma_1$ *we can construct a cobordism from* Σ_0 *to* Σ_1*. Take the cylinder* $\Sigma_1 \times I$ *with the smooth maps* $f_0 : \Sigma_0 \to \Sigma_1 \times I$ *and* $f_1 : \Sigma_1 \to \Sigma_1 \times I$ *defined by the following compositions:*

$$f_0 : \Sigma_0 \xrightarrow{\varphi} \Sigma_1 \cong \Sigma_1 \times 0 \mapsto \Sigma_1 \times I \tag{3.1}$$

$$f_1 : \Sigma_1 \cong \Sigma_1 \times 1 \mapsto \Sigma_1 \times I$$

3.2.2 Gluing cobordisms

An important property of cobordisms is that, they can be glued together. In order to explain the composition of cobordisms, first we just try to show how to glue simpler things for example topological spaces and topological manifolds. These two things are fundamental properties of gluing cobordisms. We start with gluing of topological spaces.This gluing is explained in the following data step by step.

(i) Let M_0, M_1 and Σ be topological spaces, and let $f_0 : \Sigma \to M_0$, $f_1 : \Sigma \to M_1$ be continuous maps between these topological spaces.

(ii) Consider that M_0, M_1 are disjoint and the maps are injective.

(iii) Now there is an equivalence relation on $M_0 \amalg M_1$ defined as follows: two points are taken such that, $x_0 \in M_0$ and $x_1 \in M_1$ are defined to be equivalent, denoted by $x_0 \sim x_1$ if and only if there exists $x \in \Sigma$ such that $f_0(x) = x_0$ and $f_1(x) = x_1$

(iv) Then, the gluing of M_0 and M_1 by Σ denoted by $M_0 \amalg_\Sigma M_1$ is the quotient set by the equivalence relation $(\sim)$ $M_0 \amalg_\Sigma M_1)/\sim$

(v) Two natural maps are obtained;

$$g_0 : M_0 \to M_0 \amalg_\Sigma M_1$$

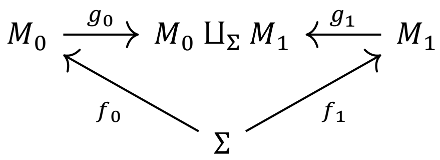
$$g_1 : M_1 \to M_0 \amalg_\Sigma M_1$$

(vi) The topology of $M_0 \amalg_\Sigma M_1$ is defined by saying a subset open if its inverse image in M_0 and M_1 are both open.

A subset $U \subset M_0 \amalg_\Sigma M_1$ is open if and only if $g_0^{-1}(U) \subset M_0$ and $g_1^{-1}(U) \subset M_1$ are both open.

Hence $M_0 \amalg_\Sigma M_1$ is obtained by gluing M_0 and M_1 along Σ.

(vii) We have commutative diagram of continuous maps between topological spaces. It is defined by "Universal Mapping Property" of gluing topological spaces.



Gluing cobordisms: Every oriented cobordism has a source and target. And we can compose cobordisms by gluing. First we take two cobordisms such that,

$$\Sigma_1 \to M \to \Sigma_2 \text{ and } \Sigma_2 \to M' \to \Sigma_3$$

$$\Sigma_1 \xrightarrow{f_1} M \xleftarrow{f_2} \Sigma_2$$

$$\Sigma_2 \xrightarrow{f_2'} M' \xleftarrow{f_3} \Sigma_3$$

where Σ_2 is non-empty and composite cobordism is formed by gluing M and M' along Σ_2 and then

$$\Sigma_1 \xrightarrow{f_1} M \circ M' \xleftarrow{f_3} \Sigma_3$$

by identifying their common boundary components using;

$$f_2 \circ f_2^{-1} = \partial M_{out} \to \partial M_{in'} \tag{3.2}$$

And the gluing can be pictured as;

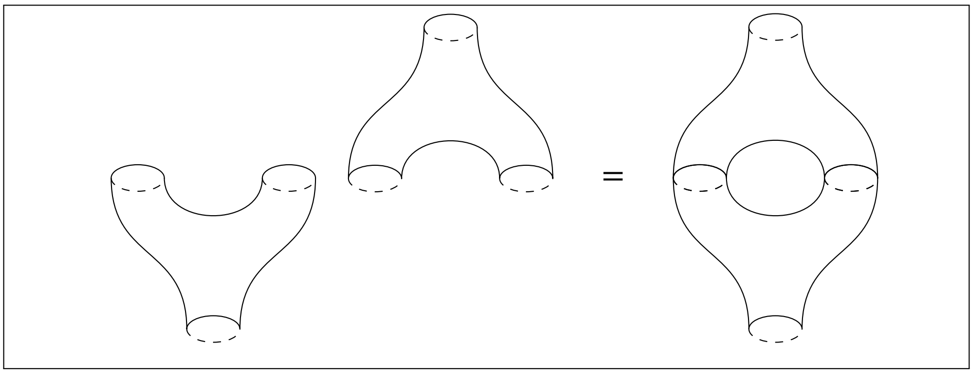


Figure 3.8. The gluing of cobordism

3.2.3 Morse Theory

Let M^n be a smooth manifold, $f : M \to \mathbb{R}$ be a smooth function, $[a, b]$ be an interval such that $f^{-1}[a, b]$ is compact. Morse theory tells us exactly, how to cut-up the manifold into pieces with a single Morse critical point. We know what the manifold looks like locally around that critical point. You can attach a k-handle around this critical point. Then some basic results are obtained from Morse theory;

(i) The Regular Interval Theorem

(ii) The Morse Lemma and Existence of Morse functions

(iii) The fundamental theorem about one-handle attached

Definition 3.2.5. Morse theory shows that locally around only critical point x and f maybe written as $f(X_1, \dots, X_{m+1}) = -X_1^2 - \cdots - X_k^2 + X_{k+1}^2 + \cdots X_{m+1}^2$ according to appropriate local coordinates for some integer k called **Morse index** of x.

Definition 3.2.6. Let $f : M \to \mathbb{R}$ be a real-valued function on a manifold M. The notion of critical point; that is point $p \in M$ where $\frac{\partial f}{\partial X_1}(p) = 0, \frac{\partial f}{\partial X_2}(p) = 0, \cdots, \frac{\partial f}{\partial X_m}(p) = 0$ with respect to local coordinates $(X_1, \dots, X_m)$ about p. Let $H_f(p)$ be a Hessian matrix of f at p with respect to local coordinates $(X_1, \dots, X_m)$ the critical point $p \in M$ is called degenerate. If $\det H_f(p) = 0$ otherwise, that is $H_f(p)$ is invertible, $p \in M$ is called **non-degenerate**.

There some examples about functions whether if it is degenerate or not;

Example 3.2.7. $f(x) = x^2$ $f'(x) = 2x$ $f'(0) = 0$ *(critical point)* $f''(x) = 2$ $f''(0) = 2$ *(not degenerate)*

Example 3.2.8. $f(x) = x^3$ $f'(x) = 3x^2$ $f'(0) = 0$ $f''(x) = 6x$ $f''(0) = 0$ *(degenerate)*

Definition 3.2.9. We say that X is a gradient-like vector field for a Morse function $f : M \to \mathbb{R}$ if the following two conditions hold:

(i) $X.f > 0$ away from the critical points of f

(ii) If p_0 is critical point of f with index k, then p_0 has a neighbourhood V with a suitable coordinate system $(X_1, \ldots, X_m)$ such that f has the standard form

$$f = -X_1^2 - \cdots - X_k^2 + X_{k+1}^2 + \cdots X_m^2 + f(p_0) \tag{3.3}$$

and X can be written as its gradient vector field:

$$X = -2X_1 \frac{\partial}{\partial X_1} - \cdots - 2X_k \frac{\partial}{\partial X_k} + 2X_{k+1} \frac{\partial}{\partial X_{k+1}} + \cdots + 2X_m \frac{\partial}{\partial X_m}$$

Remark 3.2.10. *Almost all smooth functions on M are Morse functions. (All smooth functions on M can not be Morse functions because of the measure. There is a measure on smooth functions of* M*. If the measure is not zero, the function is Morse function. Otherwise it is not.) More precisely, taking* M *to be embedded in a Euclidean space* V *it has two conditions such that, almost all height functions on* M *are Morse and for almost all* $q \in V$ *distance from* q *is a Morse function on* M*.*

Example 3.2.11. *A cylinder can be given an example of degenerate critical points on its side and f be the height function. Then we can say;* $f^{-1}(a)$ *is the bottom line and* $f^{-1}(b)$ *is the top line of the cylinder. All points are degenerate critical points in* $f^{-1}(a)$ *and* $f^{-1}(b)$*.*

Theorem 3.2.12 (Regular Interval Theorem)**.** *Let* $W : M \to N$ *be a cobordism, and* $f : W \to$

$[0,1]$ *be a smooth map without critical points, such that* $M = f^{-1}(0)$ *and* $N = f^{-1}(1)$. *Then there exists a diffeomorphism between* $M \times [0,1]$ *and* W *such that the following diagram commutes:*

$$\begin{array}{ccc} M \times [0,1] & \xrightarrow{\sim} & W \\ & {\scriptstyle p}\searrow & \downarrow \\ & & [0,1] \end{array}$$

where $p : M \times [0,1] \xrightarrow{p} [0,1]$ *is the natural projection, onto the second factor. This proof is also explained in [3].*

Proof. This theorem says that $f^{-1}[a,b]$ has no critical points, the level sets $f^{-1}(c)$ for $c \in [a,b]$ are all diffeomorphic. To show this, we give an importance on the flow of a gradient like vector field for f. The key step is construction a vector field on W. We have a real number p flowing along vector field v for time t we use Riemannian metric.

The gradient flow equation: Let M be a manifold, g be a Riemannian metric on M and $f : M \to \mathbb{R}$ be a Morse function. A gradient flow line is a curve $\eta : (a,b) \to M$ that satisfies the differential equation.

Since f has no critical points and the vector field is defined, $X(x) = \frac{gradf(x)}{|gradf(x)|^2}$ and let $\eta_x(t)$ be a solution curve. It starts at point p of $f^{-1}(a)$. The derivative of map

$$\frac{d}{dt}\eta_x(t) = X(\eta_x(t)) \text{ and } f(\eta_x(t)) = t \tag{3.4}$$

Let I be a maximal interval on which η_x is defined. Assume that $I = [a,b]$. First we should need to show M is compact and $f(\eta_x(I)) = I$ is bounded. Let $d = sup(I)$ by the compactness property of M and there is a point $x \in M$ that is a limit point of $\eta_x\left(\frac{d-1}{n}\right)$. Since from the equation $\eta_x'(t) = X(\eta_x(t))$ is bounded. Limit point should be unique so $lim_{t \to d^-}\eta_x(t) = x$ η_x can be extended to d by making $\eta_x(d) = x$. Now $lim_{t \to d}\eta_x'(t) = lim_{t \to d}X(\eta_x(t)) \to X(\eta_x(d))$. And let v be this limit so $\eta_x'(d) = v$. To show this for every $\epsilon > 0$ there exist a $\delta > 0$ so that for all h with $0 < h < \delta$. Then we can say,

$$\left| \frac{\eta_x(d) - \eta_x(d-h)}{h} - v \right| < \epsilon$$

So let $\epsilon > 0$ be given. From the definition of v there exist a δ_1 so that for all w with

$0 < h < \delta_1$. Then,

$$\left|\eta_x'(d-h) - v\right| < \epsilon \tag{3.5}$$

$$\eta_x(d-h) - \eta_x(d) = \int_{d-h}^{d} \eta_x'(t)dt \tag{3.6}$$

$$\eta_x(d-h) - \eta_x(d) + vh = \int_{d-h}^{d} (\eta_x'(t) - v)dt \tag{3.7}$$

$$\left|\eta_x(d-h) - \eta_x(d) + vh\right| \leq \int_{d-h}^{d} \left|\eta_x'(t) - v\right| dt \tag{3.8}$$

$$\leq \int_{d-h}^{d} tdt \tag{3.9}$$

$$\leq \epsilon h \tag{3.10}$$

$$\left|\frac{\eta_x(d-h) - \eta_x(d)}{h} + v\right| \leq \epsilon \tag{3.11}$$

$$\left|\frac{\eta_x(d-h) - \eta_x(d)}{-h} - v\right| \leq \epsilon \tag{3.12}$$

Hence $\eta_x'(d) = v$ and since $v = X(\eta_x(d))$ then the flow equation is satisfied by η_x at d. By maximality of I, $d \in I$. Similarly with $c = Inf(I)$ we see that $c \in I$ so I is closed. If $\eta_x(s) \notin \partial M$, then by the existence of solution of ordinary differential equation there is an interval $(s-\epsilon, s+\epsilon)$ around s on which η_x satisfies the differential equation $\eta_x'(t) = X(\eta_x(t))$. Therefore, $\eta_x(c)$ and $\eta_x(d)$ are in ∂M. Thus, $c = f(\eta_x(c))$ and $d = f(\eta_x(d))$ maybe either a and b. Since, $\eta_x : I \to M$ and $f : M \to [a,b]$ the derivation of $f \circ \eta_x$ is 1. $c = a$ and $d = b$ so $I = [a,b]$. Since $x \in M$ is arbitrary and $a \leq f(x) \leq b$ we can say that $f(M) = [a,b]$ because there exists $x_1, x_2 \in M$ such that $f(x_1) = a, f(x_2) = b$. Moreover $x \notin \partial M$, η_x is defined in a small neighbourhood of $t = f(x)$ so $a < f(x) < b$. Hence, $f^{-1}(a)$ and $f^{-1}(b)$ are unions of the boundary components. Here $F : f^{-1}(a) \times [a,b] \to M$. From the formula $F(x,t) = \eta_x$. Now take $G : M \to f^{-1}(a) \times [a,b], G(x) = (\eta_x(a), f(x))$. Then $f((\eta_x(t)) = t$. We have $F(G(x)) = x$ and $G(F(x,t)) = (x,t)$ proves that F is a diffeomorphism. □

Remark 3.2.13. *We define a Morse function on cobordism M with a smooth f which has only non-degenerate critical points. If M is a cylinder then f must have at least one points. First we need to remember index to find its critical points.*

We can compute the "index" k in another way. There is a local coordinate system $(y^1, \ldots, y^n)$

in a neighbourhood U of p with $y^i(p) = 0$ for all $i \in \{1, \dots, n\}$ and $q \in U$. Then,

$$f(q) = f(p) - (y^1(q))^2 - \cdots - (y^k(q))^2 + (y^{k+1}(q))^2 + \cdots + (y^n(q))^2. \qquad (3.13)$$

We can arrange index in local coordinates to understand the following theorem clearly. For example, the index of torus can be find of each critical points a, b, c. Again torus is considered as a height function such that; $f : \mathbb{T} \to \mathbb{R}$ by $f((x, y, z)) = z$ and the in critical point c, the function of f in local coordinate is $f = c + x^2 - y^2$ and the index is 1.

Definition 3.2.14. Given $f : M \to \mathbb{R}$ be a smooth function and p be a non-degenerate critical point with index k. If $f(p) = c$ assume that $f^{-1}([c - \epsilon, c + \epsilon])$ is compact and does not contain a critical point of f other than p for some $\epsilon > 0$. Then for all small ϵ the set $f^{-1}([-\infty, c + \epsilon])$ has the homotopy type of $f^{-1}([-\infty, c - \epsilon])$ with a k-cell attached.

Because the index of c is one, then the homotopy type of $f^{-1}([-\infty, f(c) + \epsilon])$ is a disk with a one-cell attached. And $f^{-1}([-\infty, f(c) + \epsilon])$ is homeomorphic to a cylinder.
If we show this using the cobordism theory first we should know what "ascending cobordism" is?

Definition 3.2.15. Let f be Morse function on (W, M, M') with $f^{-1}(a) = M$,$f^{-1}(b) = M'$ then we define the ascending cobordism to be $W_c = f^{-1}[a, c]$ for $c \in [a, b]$. W_c is a cobordism between $f^{-1}(a) = M$,$f^{-1}(c) = M'$

There is a connection between Morse theory and k-handle decomposition. Morse function on your cobordism, you can construct an ascending cobordism. They can form open dense subset of set of all smooth functions on cobordism. We can introduce an ascending cobordism so we can take some point c in a co-domain $[0, 1]$. Then we can look the pre-image of the closed interval $W_c = f^{-1}[0, c]$. This gives us an ascending cobordism. There is no critical values of f in some closed interval $[c, c']$. We can say W_c and $W_{c'}$ are diffeomorphic to each other. The critical point is index k and you can attach k-handle. Assume, $[0, 1]$ is regular values for f. Otherwise, it has no critical points on boundary W. We can find collar neighbourhood inside W. It is diffeomorphic to $M \times [0, \epsilon]$. Morse function can be used to attach a k-handle.

If we return to our example again;torus is a height function and some topological properties can be given for an ascending cobordism;

(i) If there are no critical points in $f^{-1}([c,c'])$ then W_c is diffeomorphic to $W_{c'}$

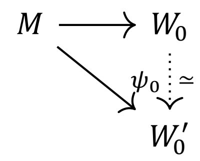
(ii) If there is a single critical point of index k in $f^{-1}([c,c'])$ then $W_{c'}$ is diffeomorphic to W_c with k-cell $D^k \times D^{(m+1)-k}$ attached

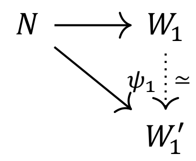
We call this k-handle attached. $W_{c'}$ is obtained from W_c by attaching a k-handle. In torus, again when we attached 0-cell $D^0 \times D^2$ and we obtain a disc. The ascending cobordism has a first saddle point x_1 where a 1-handle $D^1 \times D^1$ is attached. Also second saddle point x_2 is found. This is another 1-handle attached.

Proposition 3.2.16. *Given a cobordism* $W_0 : M \to N$ *and* $W_1 : N \to P$ *and Morse functions are* $f_0 : W_0 \to [0,1]$ *and* $f_1 : W_1 \to [1,2]$ *and we have a topological manifold* $W_0 \sqcup_N W_1$ *with continuous map* $W_0 \sqcup_N W_1 \to [0,2]$. *By regular interval theorem, choose* $\epsilon > 0$. *Take two intervals* $[1-\epsilon,1]$ *and* $[1,1+\epsilon]$ *are regular for* f_0 *and* f_1. *Then* $f_0^{-1}([1-\epsilon,1])$ *and* $f_1^{-1}([1,1+\epsilon])$ *are diffeomorphic to cylinder* C_1 *and* C_2. *From gluing of cylinder we can say that* W_0W_1 *of two cobordism* $W_0 : M \to N$ *and* $W_1 : N \to P$. *Then we can define homeomorphism* $\phi : f_0^{-1}([1-\epsilon,1]) \sqcup f_1^{-1}([1,1+\epsilon]) \to N \times [0,2]$

3.2.4 Construction of n-Cobordism

Since we now know how to glue cobordisms. In this part, we can compose equivalence cobordism classes. Two cobordisms are in the same equivalence class if they are homeomorphic to each other. We only need to check that, this gluing extends to the diffeomorphism classes of cobordisms. More precisely, the composition of two cobordisms does not only depend on the cobordism which is chosen, but it is also depend on their equivalence classes. We want to show that, the cobordism obtained by gluing W_0 and W_1 by the following diagram. It is diffeomorphic to W_0' and W_1' which is also one obtained by gluing are in the same cobordism class.





(i) W_0 and W_0' are in the same cobordism class.

(ii) There exist a diffeomorphisms between cobordisms such that

$$\psi_0 : W_0 \xrightarrow{\cong} W_0'$$

$$\psi_1 : W_1 \xrightarrow{\cong} W_1'$$

(iii) W_0W_1 and $W_0'W_1'$ can be glued as smooth manifold.

(iv) ψ_0 and ψ_1 can be glued as continuous maps.

(v) ψ is an homeomorphism between the gluing cobordisms

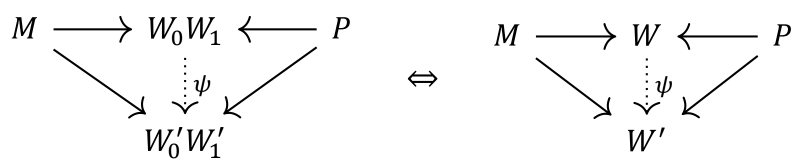
$$\psi : W_0W_1 \xrightarrow{\cong} W_0'W_1'$$

(vi) Now define a new smooth structure on $W_0'W_1'$ with ψ of the smooth structure of W_0W_1 such that,

$$W = W_0W_1, W' = W_0'W_1' \text{ and } \psi = \psi_0 \amalg \psi_1 : W \xrightarrow{\cong} W'$$

$$\psi : W_0W_1 \xrightarrow{\cong} W_0'W_1'$$

We can draw the following diagram.



(vii) If these two smooth structures on $W_0'W_1'$ are not same, they are at least diffeomorphic. So W_0W_1 and $W_0'W_1'$ are equivalent.

(viii) Two cobordism classes which are composed can be defined like this,
$W_0 : M \to N$ and $W_1 : N \to P$ glue this cobordism class $W_0 W_1 : M \to P$
$W_0 \circ W_1 : M \to P$ where $W_0 \amalg_N W_1 : M \to P$

(ix) This composition is associative, then take 3 cobordism classes,

$$M \xrightarrow{W_0} N, N \xrightarrow{W_1} P, P \xrightarrow{W_2} R$$

(x) According to the associative rule,

$$W_2 \circ (W_1 \circ W_0) = (W_2 \circ W_1) \circ W_0 \tag{3.14}$$

(xi) Universal property is used to have a canonical homeomorphism

$$(W_0 \amalg_N W_1) \amalg_P W_2 \xrightarrow{\cong} W_0 \amalg_N (W_1 \amalg_P W_2)$$

(xii) The smooth structure $W_0 \amalg_N W_1 \amalg_P W_2$ is obtained by replacing the charts on the neighbourhood of N and P by the ones "cylinder construction"

(xiii) Hence $W_0 \amalg_N W_1 \amalg_P W_2$ are diffeomorphic.

(xiv) Construction of the category of $nCob$, the objects are closed oriented $(n-1)$-dimensional manifolds, arrows cobordism classes between them in this category.

(xv) Composition in this category consists of gluing identity arrows are "cylinders". It is constructed as the product of $W \times [0, 1]$.

(xvi) The cylinder is indeed the identity, choose a cobordism $M \xrightarrow{W_0} N$ the cylinder C over M.

(xvii) Decompose W_0 in two parts; $W_0 = W_{0[\epsilon,1]} \circ W_{0[0.\epsilon]}$
$W_0 \circ C = [W_{0[\epsilon,1]} \circ W_{0[0.\epsilon]}] \circ C = [W_{0[\epsilon,1]} \circ (W_{0[0.\epsilon]} \circ C) = W_{0[\epsilon,1]} \circ W_{0[0.\epsilon]} = W_0$.
Hence $W_0 C = W_0$

(xviii) $M \xrightarrow{W_0} N$ is equivalent to $N \xrightarrow{W_1} P$ with C over M.

$$W_0 = W_1 \circ C$$

$$W_0 = W_1 \circ (M \times I)$$

$$W_0 \circ (M \times I) = (W_1 \circ (M \times I)) = W_1 \circ ((M \times I) \circ (M \times I))$$

$$W_1 \circ (M \times I)$$

Remark 3.2.17. *What are the isomorphisms in $nCob$? This question can be answered with* ***invertible morphism****. Because in cobordism category, every morphism is not invertible. For example, the reverse pair of pants is not invertible. Because, if it is invertible, their composition should be equivalent to a cylinder. When any cobordism is glued with left pair of pants their composition is not a cylinder. Any cobordism which is glued with it, a hole can be appeared. They can not compose to each other.*

Definition 3.2.18. Taken two differentiable manifolds X,Y, we say they are homotopy equivalent if there exist smooth maps $f : X \to Y, g : Y \to X$, such that $f \circ g \simeq id_y$ and $g \circ f \simeq id_x$. Then we say that f and g are homotopy equivalences.

Definition 3.2.19. Let $W : M_0 \to M_1$ be a cobordism between closed n dimensional manifolds. We say that W is an h cobordism if the inclusion maps $M_0 \hookrightarrow W \hookleftarrow M_1$ are homotopy equivalences.

Cobordism category is also related with a monoidal structure. If Σ and Σ' are two (n-1)-dimensional manifolds then the disjoint union is again $(n-1)$ manifold. Given two cobordism $M : \Sigma_1 \to \Sigma_2$ and $M' : \Sigma_1' \to \Sigma_2'$ their disjoint union $M \amalg M'$ is a cobordism from $\Sigma_1 \amalg \Sigma_1' \to \Sigma_2 \amalg \Sigma_2'$. Also the empty manifold is an $(n-1)$-dimensional manifold, $\emptyset_n$ is a cobordism $\emptyset_{n-1} \to \emptyset_{n-1}$ act as unit. Therefore, these conditions satisfy the axioms of monoidal category.

3.2.5 The Twist Cobordism

The symmetry of the disjoint union is defined by twist cobordism. We can define the twist diffeomorphism ; $\tau : \Sigma \amalg \Sigma' \to \Sigma' \amalg \Sigma$. The canonical identification between $\Sigma \amalg \Sigma'$ and $\Sigma' \amalg \Sigma$ and it maps $x \in \Sigma \subset \Sigma \amalg \Sigma'$ to $x \in \Sigma \subset \Sigma' \amalg \Sigma$. Then the diffeomorphism τ defines a cobordism such that,

$$T_{\Sigma,\Sigma'} : \Sigma \amalg \Sigma' \to \Sigma' \amalg \Sigma.$$

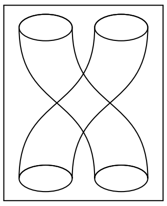


Figure 3.9. The twist cobordism

We can take elements m, m' and $m \in \Sigma$, $m' \in \Sigma'$. Then m_1 and m'_2 are found in $\Sigma \amalg \Sigma'$. Similarly, m_2 and m'_1 are found in $\Sigma' \amalg \Sigma$. Therefore, m_1 is mapped to m_2 and m'_2 is mapped to m'_1 by the twist diffeomorphism $\tau_{x,y}$. When this diffeomorphism is related with cobordism, $T_{\Sigma,\Sigma'}$ is called a twist cobordism. We can obtain a cylinder with the composition of $T_{\Sigma,\Sigma'}$ and $T_{\Sigma',\Sigma}$. This new twist can be compared with the cylinder, first we should know that, $\Sigma \amalg \Sigma'$ and $\Sigma' \amalg \Sigma$ are not same. Then the twist satisfies this condition such that, $T_{\Sigma',\Sigma} \circ T_{\Sigma',\Sigma} = (\Sigma \amalg \Sigma') \times [0, 1]$. Then we obtain a manifold with their composition. Therefore, in the symmetric monoidal category twist cobordism acts as a twist map. We can also say for twist cobordism, two in-boundary components and two out-boundary components can be permuted. The important thing is, when two twists are composed you can obtain an identity cobordism by the following picture.

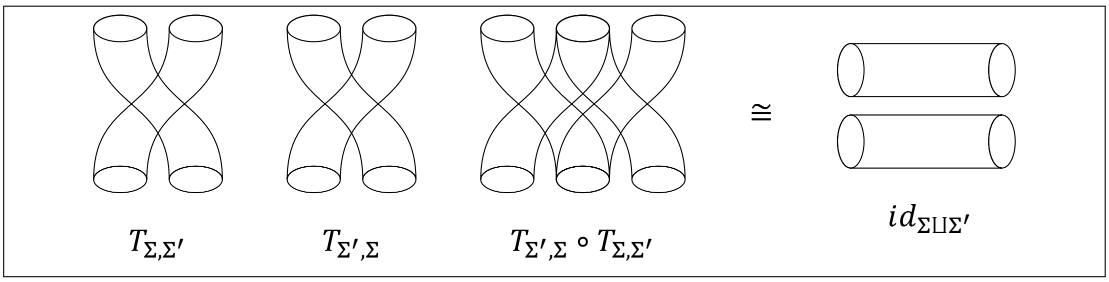


Figure 3.10. Composition of two twists

3.2.6 The Geometry of Surfaces

We will in this section focus ourselves to two dimensions and we only study with **2Cob** which is the main theme of this text. The surfaces are compact and oriented. Classification theorem says that, surfaces can be appeared in many different forms. According to this theorem, every compact surface is equivalent to one representative surface. This surface is also called **normal form**. Every surface can be transformed into a normal form. The objects in 2Cob are closed oriented 1-manifolds. Every closed, oriented 1-manifold is diffeomorphic to a finite disjoint union of circles. We should study the surface because cobordism is determined by its genus and number of in-boundaries and out-boundaries. We can use four cobordisms such that, left-right cap and left-right pair of pants to construct cobordism M with genus, in-boundaries and out-boundaries.

Classification theorem says that every two connected closed oriented surface diffeomorphic if and only if they have the same genus. In 2Cob the surfaces will have oriented boundaries which are diffeomorphic to a finite disjoint union of circles. First, we should distinguish in and out- boundaries. Then we can say the following theorem.

Theorem 3.2.20. *Two connected, compact oriented surfaces with oriented boundary are diffeomorphic if and only if they have the same genus and same number of in-boundaries and the number of out-boundaries.*

Normal form of a connected surface: Normal form of connected surface can be constructed with **m** in-boundaries and **n** out-boundaries with genus **g**. The surface can be decomposed into a number of basic cobordisms. The normal form has three parts;
First part is called the **in-part** which consists of a cobordism from m circles to 1 circle ($m \rightarrow 1$)
Middle part is called the **topological part** which consists of a cobordism from one circle to one circle ($1 \rightarrow 1$)
Third part is called the **out-part** which consists of a cobordism from one circle to n circle

$(1 \to n)$

Let us describe the in-part;

Suppose $m > 0$ take $m-1$ copies of left pair of pants and glue them together with the number of cylinders. Then, the output of one left pair of pants connects to the lower input hole of the following left pair of pants in a same way. The involved cylinders always come on top of the pair of pants in the disjoint union. Therefore, each out-boundary of pair of pants is glued to the lower in-boundary of the next pair of pants. In the case $m = 0$, the in-part only consists of a left-cap instead of any pair of pants.

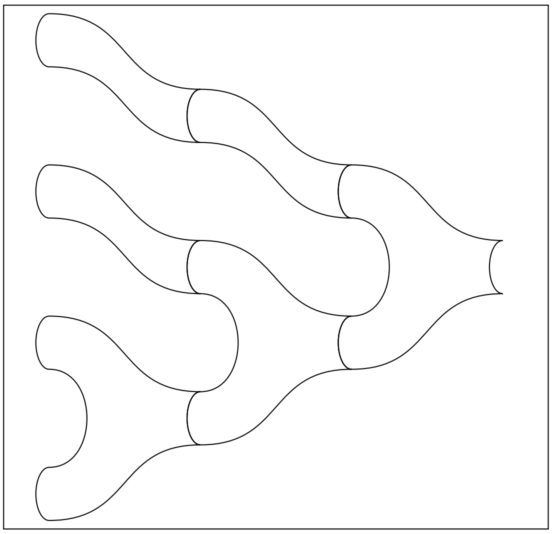


Figure 3.11. The in-part for the case $m = 4$

The topological part of the normal form consists of all the holes; the topological part can be constructed from g left-pair of pants and g right-pair of pants so that a hole is created with connection between in-boundaries of the left pair of pants and out-boundaries of the right pair of pants.

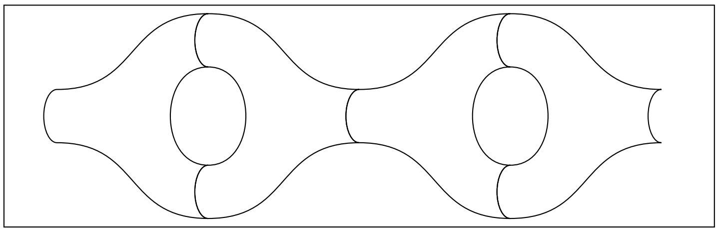


Figure 3.12. The middle part for the case $g = 2$

For the out-part suppose that $n > 0$; the composition of $(n-1)$ copies of right pair of pants composed in a similar way; the lower output hole of each piece is connected to the input hole of its sequel. The involved cylinders always come on the top of the right pair of pants. When $n = 0$, the out-part consists of a single copy of right-cap.

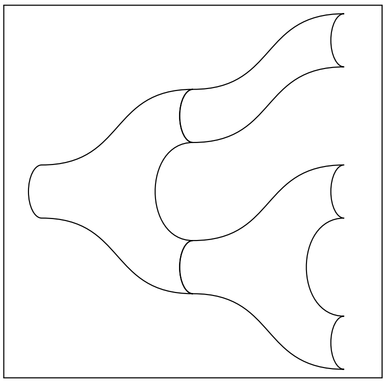


Figure 3.13. The out-part for the case $n = 3$

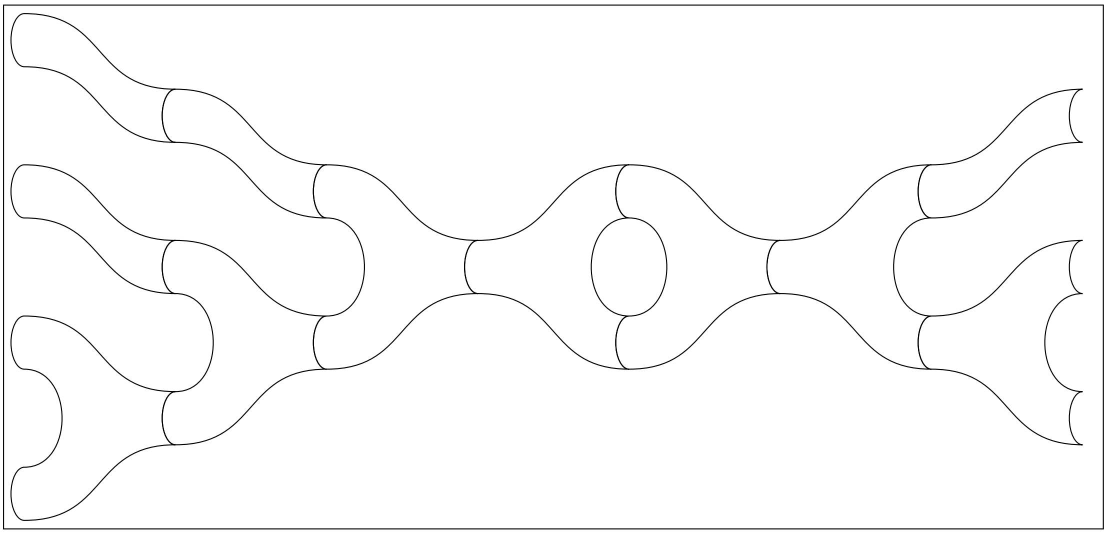


Figure 3.14. The case with in-boundaries, genus and out-boundaries

3.2.7 Generators and Relations of 2Cob

Generators and relations are known from the group theory. If we make a short summary; we can say that groups can be described with using generators and relations. Every group

G is isomorphic to a quotient of a free group. A set of generators and relations are used to define G. The free group F represents generators and $G \cong F/H$ represents relations. It is the normal subgroup of H of F. There is a group homomorphism such that, $f : G \to G'$, images are generators and satisfy relations. If generators and relations are explained on the category theory, we define a functor $F : A \to B$; generating morphisms are in A and relation is preserved. But we interested in monoidal functors so we define 2Cob in terms of generators and relations for a monoidal category. This section is explained in [1].

Proposition 3.2.21. *The category 2-Cob is generated under composition and disjoint union by the following morphisms;*

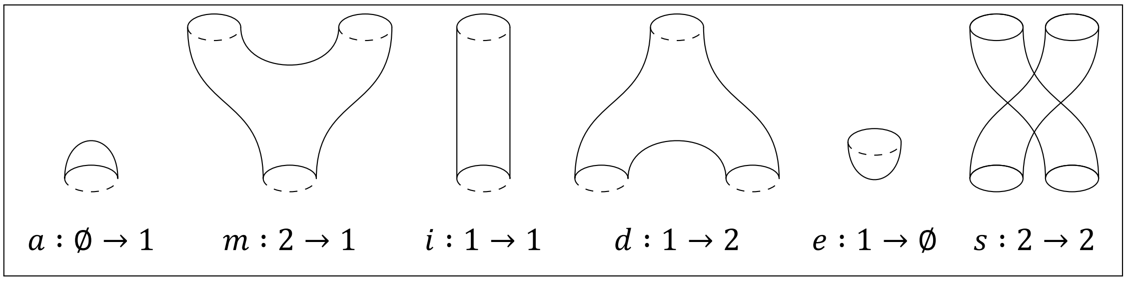


Figure 3.15. Generating morphisms with cobordisms

This theorem wants to prove that, every cobordism is diffeomorphic to the composition and disjoint union of this generators with morphisms. We should define a normal form for connected cobordisms with using number of in-boundaries, out-boundaries and genus.Our aim is to find a connection between connected cobordism and normal form because, they are diffeomorphic to each other. Moreover, every cobordism can be defined in terms of the disjoint union of connected components. According to the theorem of normal form, connected cobordisms can be explained with using number of in-boundaries and out-boundaries components. Then we can give a new definition about connected cobordism.

Definition 3.2.22. Two connected two cobordisms are diffeomorphic if and only if they have the same number of in-boundaries, out-boundaries and genus. We define normal form with using m,n and g and it is explained in three parts. From the above pictures that is given explains normal form.

The first part can be defined as; $(\coprod_{n-2} i \coprod_m) \circ (\coprod_{n-3} i \coprod_m) \circ \ldots \circ (i \coprod m) \circ m : n \to 1$. This constructs n in-boundaries.

The middle part we define as; $(m \circ d) \circ (m \circ d) \circ \ldots \circ (m \circ d)$ where we have g copies of $(m \circ d)$ This gives us g holes.

The third part we define as; $d \circ (i \coprod d) \circ (i \coprod i \coprod d) \circ \ldots (\coprod_{m-2} i \coprod_d) : 1 \to m$. This constructs m out-boundaries.

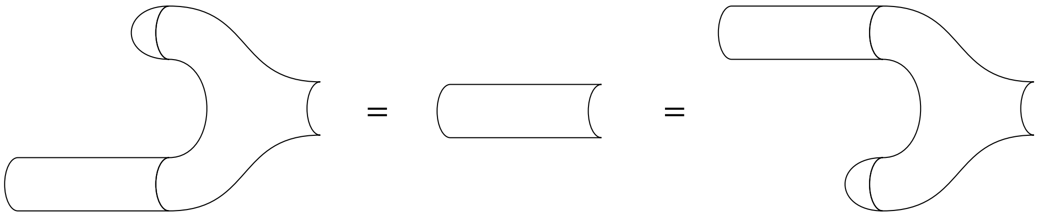
Therefore, connected cobordism can be constructed with the composition of three parts with m, n and g. Generating morphisms i,m,d are used to explain a diffeomorphism class of a connected two cobordisms. Cobordisms are the disjoint union of connected cobordisms. Hence, diffeomorphism class does not change according to the boundary components. Next proposition shows that, M becomes a disjoint union by composing it with the permutation cobordisms.

Proposition 3.2.23. *Every connected two cobordisms can be constructed from gluing and the disjoint union of right, left cap with left and right pair of pants together with cylinder.*

Proof. Given cobordism $M : \Sigma \to \Sigma'$ with the same number of m, n and g. It can be shown that $\Sigma : \Sigma_1 \cup \cdots \cup \Sigma_m$ and $\Sigma' : \Sigma_1' \cup \cdots \cup \Sigma_n'$. M_1 and M_2 are two connected components of M. Let $\sigma_1 \subseteq \Sigma$ be the collection of in-boundaries of M_1. Also $\sigma_2 \subseteq \Sigma$ be the collection of in-boundaries of M_2. Similarly let $\sigma_1' \subseteq \Sigma'$be the collection of out-boundaries for M_1. Also $\sigma_2' \subseteq \Sigma'$ be the collection of out-boundaries for M_2. The circles are permuted until the cobordism is factorized into $M = T \circ M' \circ S$. Here, M' consists disjoint union of connected components M_1' and M_2'. They are in-boundaries of M'. Out-boundary components and in-boundary components of Σ' are permutation of the in and out boundary components of Σ. T and S are called permutation cobordisms which are gluing disjoint union of twist cobordism and cylinders to the in-boundary and out-boundary of M'. Hence every two cobordisms can be constructed by a permutation cobordism. □

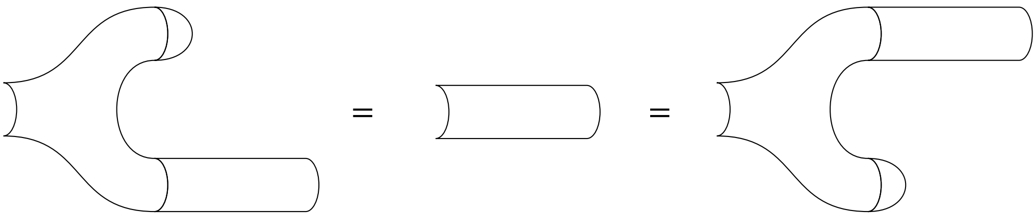
Lemma 3.2.24. *Some relations are also used to define a monoidal functors on $2Cob$. Then a generating morphisms are used with cobordism notations to show the relations between Frobenius algebras.*

Proof. Unit relations with generating morphisms can be drawn. Generating morphisms represent each caps and pair of pants in the following order.



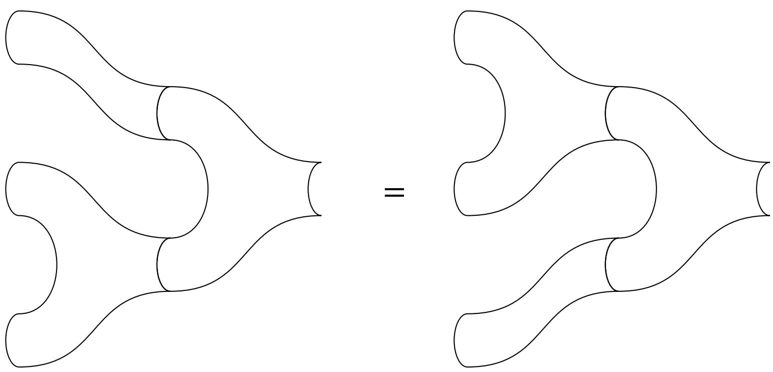
$$m \circ (i \amalg a) = i = m \circ (a \amalg i)$$

Co-unit relations with generating morphisms are also drawn like unit condition.



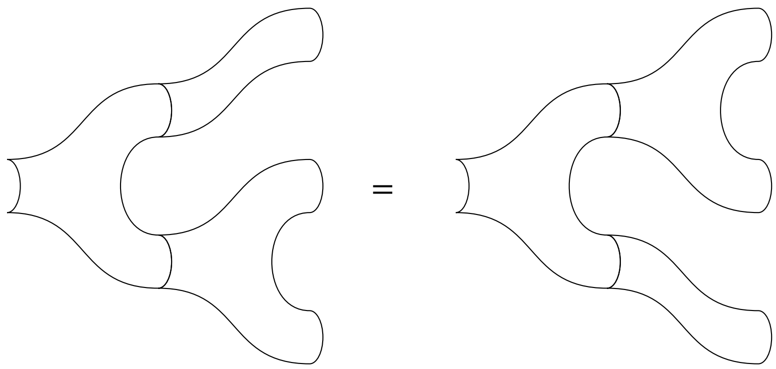
$$(i \amalg e) \circ d = i = (e \amalg i) \circ d$$

Associativity relation with generating morphisms



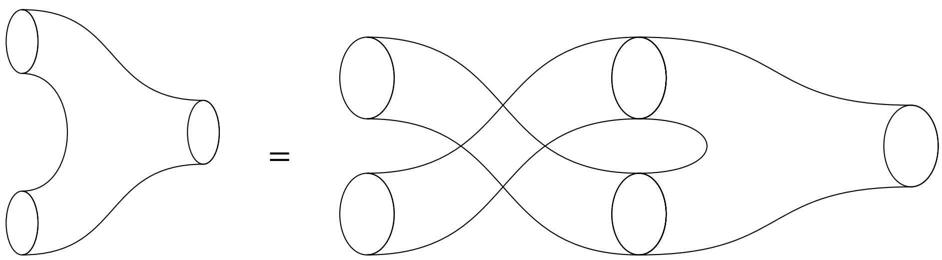
$$m \circ (i \amalg m) = m \circ (m \amalg i)$$

Co-associativity relation is similar with associativity.



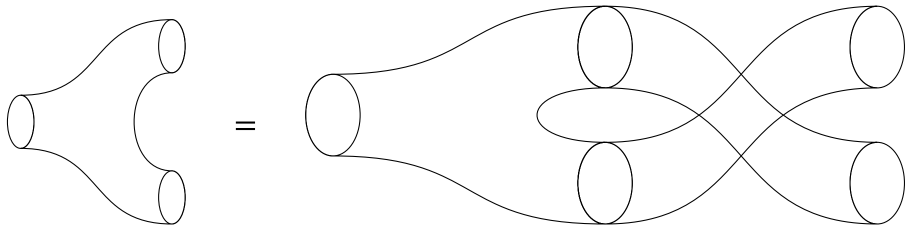
$$(d \amalg i) \circ d = (i \amalg d) \circ d$$

Commutativity relation also can be drawn with generating morphisms.



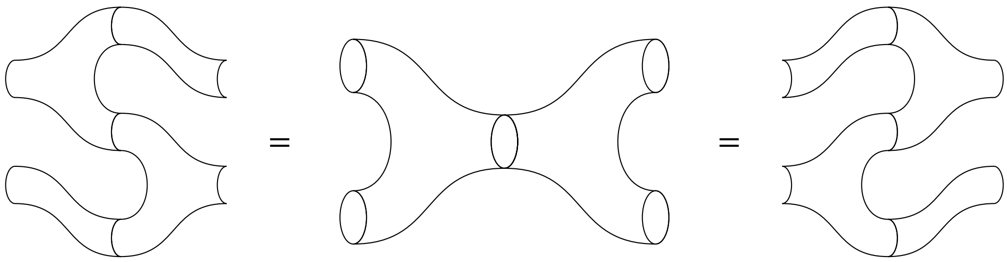
$$m = (m \circ s)$$

Co-commutativity relation is used like commutativity.



$$d = (s \circ d)$$

Then we get Frobenius condition with these generating morphisms.



$$(i \sqcup m) \circ (d \sqcup i) = (d \sqcup m) = (m \sqcup i) \circ (i \sqcup d)$$

□

4. FROBENIUS ALGEBRAS

This chapter interests on the study of commutative Frobenius algebras to understand their relation with topological quantum field theory in last chapter. Before giving the relation between Frobenius algebras and TQFT first we start with some basic definitions on vector spaces and then some important definitions of Frobenius algebras with examples. Graphical notation for linear maps leads to understand relation between TQFTs and Frobenius algebras easily. Moreover, we define a symmetric monoidal structure on Frobenius algebras. More details about this chapter can be found in [1] and [6].

4.1. BASIC DEFINITIONS

Some basic algebraic informations on vector spaces and linear maps are given.

Definition 4.1.1. The category of vector spaces over a field k, called $Vect_k$ has vector spaces over k as objects and k-linear maps between them as morphisms.

Definition 4.1.2. A pairing between two vector spaces V and W, is a linear map $\beta : V \otimes W \to k$. The elements can be written $v \otimes w \mapsto \langle v, w \rangle$. Co-pairing is a linear map $\gamma : k \to V \otimes W$.

Definition 4.1.3. Let A be a k-vector space.

(i) A linear form in A is a linear map $A \to k$

(ii) A pairing in A is a linear map $A \otimes A \to k$, and a co-pairing is a linear map $k \to A \otimes A$.

(iii) A pairing $\beta : A \otimes A \to k$ is called **right** non-degenerate if there exists a right co-pairing $\gamma_r : k \to A \otimes A$ such that the composition

$$A \cong k \otimes A \xrightarrow{\gamma_r \otimes id_A} A \otimes A \otimes A \xrightarrow{id_A \otimes \beta} A \otimes k \cong A \tag{4.1}$$

is the identity on A.

(iv) A pairing $\beta : A \otimes A \to k$ is called **left** non-degenerate if there exists a left co-pairing $\gamma_l : k \to A \otimes A$ such that the composition

$$A \cong A \otimes k \xrightarrow{id_A \otimes \gamma_l} A \otimes A \otimes A \xrightarrow{\beta \otimes id_A} k \otimes A \cong A \tag{4.2}$$

is the identity on A.

(v) A pairing is called non-degenerate if it is both right and left non-degenerate.

If we summarize the definitions briefly;

A linear map $\beta : V \otimes W \to k$ is said to be non degenerate pairing with respect to V if $(\beta \otimes id_V) \circ (id_V \otimes \gamma) = id_V$ and if there exists a linear map $\gamma : k \to W \otimes V$, called copairing, such that:

$$V \xrightarrow{\sim} V \otimes k \xrightarrow{id_V \otimes \gamma} V \otimes (W \otimes V) \xrightarrow{\sim} (V \otimes W) \otimes V \xrightarrow{\beta \otimes id_V} k \otimes V \xrightarrow{\sim} V$$

We can also define non degenerate pairing with respect to W such that:

$$W \xrightarrow{\gamma \otimes id_W} W \otimes V \otimes W \xrightarrow{id_W \otimes \beta} W$$

is the identity on W.

We use these maps in terms of cobordism notation. It is showed in the following picture.

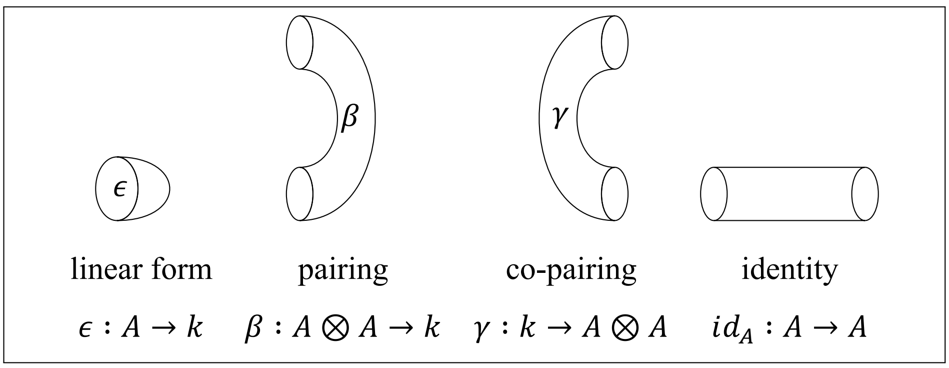


Figure 4.1. Linear maps with cobordism notation

In the picture, the circles represent domain and co-domain in the right and left side. Each circle represents of A, and a column of n circles represents the n^{th} tensor product of A. We showed that composition of two maps such that $g \circ f$ represents two cobordisms. Some circles for the domain of g and the co-domain of f are used to glue cobordisms. We can draw a map for example; $(\epsilon \otimes id_A) \circ \gamma$ represented by the following cobordism notation:

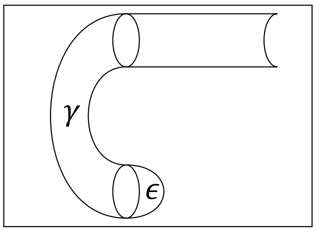


Figure 4.2. Composition of co-pairing,linear form and identity

Definition 4.1.4 (Associative pairing)**.** A pairing, $\beta : M \otimes N \to k$ is called associative if $\langle xa|y\rangle = \langle x|ay\rangle$, for all $x \in M, a \in A, y \in N$. all k algebras.

Lemma 4.1.5. *If β is a non-degenerate pairing, then left and right co-pairings are equal. Also in this case, β admits a* ***unique*** *associated co-pairing.*

Proof. It is known that,

$$A \cong A \otimes k \xrightarrow{id_A \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes id_A} k \otimes A \cong A \tag{4.3}$$

$$A \cong k \otimes A \xrightarrow{\gamma \otimes id_A} A \otimes A \otimes A \xrightarrow{id_A \otimes \beta} A \otimes k \cong A \tag{4.4}$$

$$\begin{array}{ccc} k & \xrightarrow{\gamma_r} & A \otimes A \\ & \searrow^{\gamma_r \otimes \gamma_l} & \downarrow{\scriptstyle id_A \otimes_A \otimes \gamma_l} \\ & & A \otimes A \otimes A \otimes A \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\gamma_l} & A \otimes A \\ & \searrow^{\gamma_r \otimes \gamma_l} & \downarrow{\scriptstyle \gamma_r \otimes\, id_A \otimes_A} \\ & & A \otimes A \otimes A \otimes A \end{array}$$

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\gamma_r \otimes id_A \otimes_A} & A \otimes A \otimes A \otimes A \\ & {\scriptstyle id_A \otimes_A} \searrow \quad \swarrow {\scriptstyle id_A \otimes \beta \otimes id_A} & \\ & A \otimes A & \end{array}$$

$$\begin{array}{ccc} A \otimes A & \xrightarrow{id_A \otimes_A \otimes \gamma_l} & A \otimes A \otimes A \otimes A \\ & {\scriptstyle id_A \otimes_A} \searrow \quad \swarrow {\scriptstyle id_A \otimes \beta \otimes id_A} & \\ & A \otimes A & \end{array}$$

First and second diagram commute. Right and left non-degeneracy conditions satisfy the commutativity. Then third and forth commute. Hence the diagrams are commutative so; $\gamma_l = \gamma_r$. □

Definition 4.1.6. A k-algebra is a k-vector space A equipped with a linear maps $\mu : A \otimes A \to A, (a,b) \mapsto a \cdot b$ called multiplication, and a linear map $\eta : k \to A$, called the unit, such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

$$\begin{array}{ccc} k \otimes A & \xrightarrow{\eta \otimes \mathrm{id}_A} & A \otimes A \\ & \searrow & \downarrow {\scriptstyle \mu} \\ & & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \eta} & A \otimes k \\ {\scriptstyle \mu} \downarrow & \swarrow & \\ A & & \end{array}$$

These diagrams tell us that the multiplication is associative and $\eta(1)$ acts as a unit for the multiplication. μ sends $a \otimes b \to ab$ and unit η sends $1_k \to 1_A$. These conditions satisfy as follows:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \ \text{ for all } \ a, b, c \in A$$

$$1_A \cdot a = a = a \cdot 1_A$$

The morphism of the k-algebra unit η and multiplication μ are represented with cobordism notation.

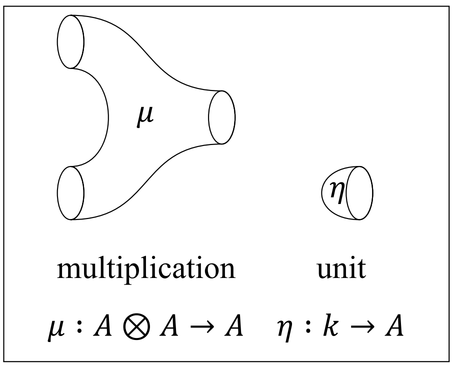


Figure 4.3. Multiplication and unit

Associative and unit condition can be drawn in terms of cobordism notation such that,

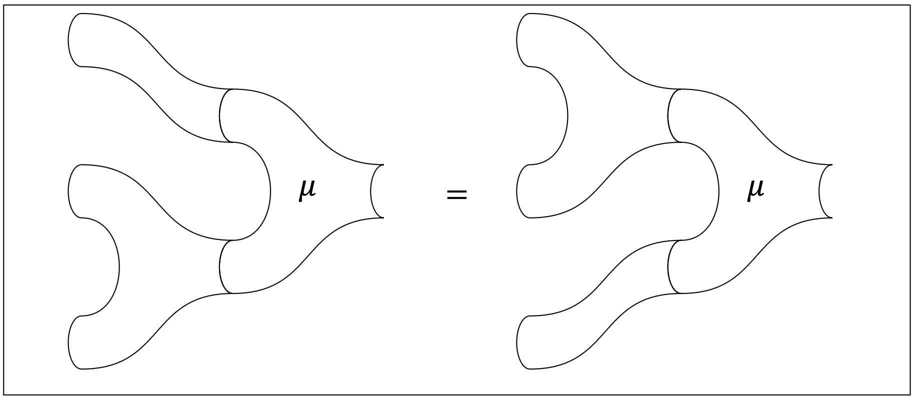


Figure 4.4. Associativity axiom

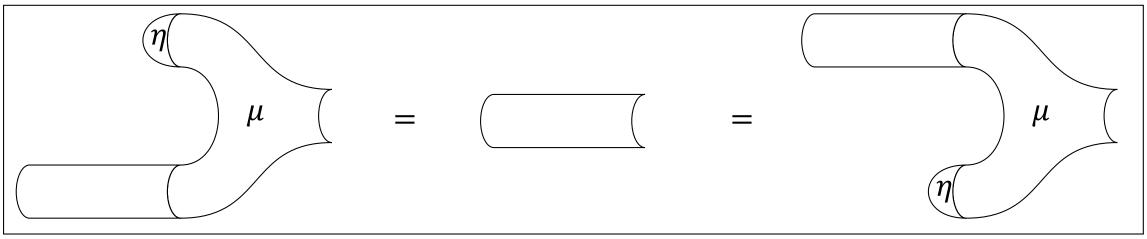


Figure 4.5. Unit axiom

Definition 4.1.7. Let A be a k-algebra with multiplication μ. A pairing $\beta : A \otimes A \to k$ is associative if the following diagram commutes:

$$\begin{array}{ccccc} A \otimes A & \xleftarrow{id_A \otimes \mu} & A \otimes A \otimes A & \xrightarrow{\mu \otimes id_A} & A \otimes A \\ & \searrow^{\beta} & & \swarrow^{\beta} & \\ & & k & & \end{array}$$

Then, we can write in cobordism notation.

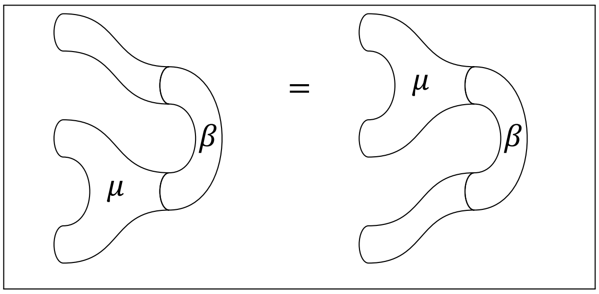


Figure 4.6. Pairing with multiplication

A linear form $\epsilon : A \to k$ determines a pairing β in the following order;

$$\beta : A \otimes A \xrightarrow{\mu} A \xrightarrow{\epsilon} k.$$

These can be drawn in cobordism notation such that;

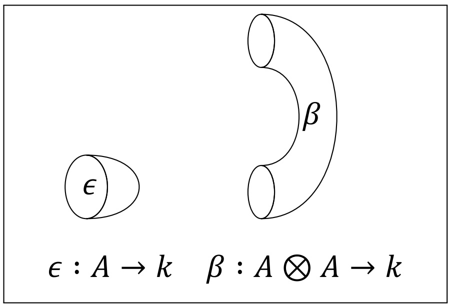


Figure 4.7. Linear form and pairing

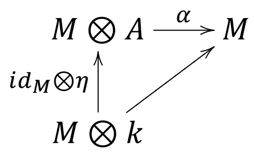
Definition 4.1.8. If V is a vector space over a field K, then the dual space denoted V^* is the set of k-linear maps $V^* \to K$

Definition 4.1.9. A **right A-module** is a vector space M over k with a map $\alpha : M \otimes A \to M$ such that α represents multiplication in A. We can write an action $x \otimes a \mapsto x \cdot a$. Then these properties make the following diagram commute:

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{\mathrm{id}_M \otimes \mu} & M \otimes A \\ {\scriptstyle \alpha \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \alpha} \\ A \otimes A & \xrightarrow[\alpha]{} & A \end{array}$$

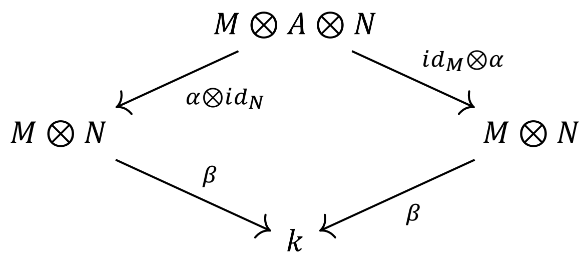
Definition 4.1.10. A **left A- module** is defined similarly with a map $\alpha : A \otimes M \to M$ that commutes with the unit and the multiplication maps. It can be written $a \otimes x \mapsto a \cdot x$ satisfied axioms similar those of right A-modules,

$a \cdot (b \cdot x) = (ab) \cdot x$ and $1 \cdot x = x$. It can be also said that A is both left and right A-module by replacing M.



Definition 4.1.11 (Associative Non-degenerate Pairing)**.** Let A be a k-algebra, M be a right A-module and N be a left A-module. An associative pairing $\beta : M \otimes N \to k$ is such that the following diagram commutes:

That is, the pairing is associative whenever $x \otimes y \mapsto \langle xa|y \rangle = \langle x|ay \rangle$, for all $x \in M, a \in A, y \in N$.



4.2. FROBENIUS ALGEBRAS

Definition 4.2.1. A Frobenius algebra A is a k-algebra with a linear functional $\epsilon : A \to k$, such that the associative pairing, $\beta : A \otimes A \to k$, is non degenerate. The map ϵ is called a Frobenius form or a trace form (A, ϵ) such that $\epsilon(ab) = 0$ for all $a \in A$ implies $b = 0$

Definition 4.2.2. The category of commutative Frobenius algebras is denoted by, cFA_k, has objects; commutative Frobenius algebras over a field k and morphisms; Frobenius algebra homomorphisms between them.

Examples of Frobenius Algebras

Example 4.2.3. *The field $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$, by considering the Frobenius form*

$$\mathbb{C} \xrightarrow{\epsilon} \mathbb{R}$$

$$\epsilon(a + ib) = \frac{19a - b}{5} \tag{4.5}$$

Here, the field of complex numbers together with unit map; $\eta : \mathbb{R} \rightarrow \mathbb{C}$ is a Frobenius form and it is defined by with linear functional like ϵ above. The multiplication $\mu : \mathbb{C} \otimes \mathbb{C}$ is finite dimensional algebra. To prove this Frobenius form; first a canonical basis is taken for $\mathbb{C}$ like $e_1 = 1$ and $e_2 = i$. These basis are determined for $Hom(\mathbb{C}, \mathbb{R})$ there are two conditions; $\eta(e_1) = e_1(\eta) = id_R$ and $\eta(e_2) = e_2(\eta) = 0$. For example; $\epsilon(2 + 3i) = 7$ and $\epsilon(1 - i) = 4$. Let $\lambda(2 + 3i) + \mu(1 - i) = a + ib$. Then, solve this equations to find μ and λ in terms of a and b. We can obtain;$\lambda = \frac{a+b}{5}$ and $\mu = \frac{3a-b}{5}$. We know that, ϵ is linear. Therefore we can say; $\epsilon(a + ib) = \epsilon\left(\frac{a+b}{5}(2 + 3i) + \frac{3a-2b}{5}(1 - i)\right)$. Then, $\epsilon(a + ib) = \left(\frac{a+b}{5}\right)7 + \left(\frac{3a-2b}{5}\right)4$. Hence, we get $\epsilon(a + ib) = \frac{19a-b}{5}$. Then we get the Frobenius form.

Example 4.2.4. *The ring $Mat_n(k)$ of $n \times n$ matrices over a field k, satisfies a Frobenius algebra structure. Trace can be written, $\epsilon : X = (X_{ij}) \longrightarrow tr(X) = \sum_{n}^{i=1} X_{ii}$. Moreover the pairing $\beta : Mat_n(k) \otimes Mat_n(k) \longrightarrow k$. Then we put it into Frobenius form and get; $\beta(x \otimes y) = tr(x \cdot y)$. It is a non-degenerate bilinear pairing which is a well known fact. Therefore, it satisfies Frobenius algebra axioms.*

Example 4.2.5. *Let $G = g_0, g_1, \dots, g_n$ be a finite group with identity element g_0 and $k[G]$ be the vector space over k with basis elements, $g_0, \dots, g_n$ of G. Then we can choose γ which is one of the elements of these basis. Then, $g_0, \dots, g_n \longrightarrow \gamma g_0, \dots, \gamma g_n$ by the k-linear transformation. Hence, we can write matrix of this transformation with respect to basis $g_0, \dots g_n : M_\gamma$. Then, $k[G] \longrightarrow k$ is defined by $\gamma \longrightarrow tr(M_\gamma)$. Therefore, $\epsilon(\gamma \cdot \gamma') = tr(M_{\gamma\gamma'})$ In addition to this, we can choose any $\beta \in k[G]$. Then, $\beta = c_0 g_0 + \dots + c_n g_n \quad c_i \in k$. Similarly by the k-linear transformation we can say, $g_0, \dots, g_n \longrightarrow \beta g_0, \dots, \beta g_n : M_\beta$. Then $k[G] \longrightarrow k$ is defined by $\beta \rightarrow tr(M_\beta)$. Therefore, $\epsilon(\beta \cdot \beta') = tr(M_{\beta\beta'})$. Hence these conditions satisfy Frobenius axioms.*

Definition 4.2.6. A Frobenius algebra (A, ϵ) with the non degenerate pairing β and also co-pairing γ are defined by commutativity property such that the following both diagrams are called **snake relation**.If the snake relation is showed by cobordism notation;

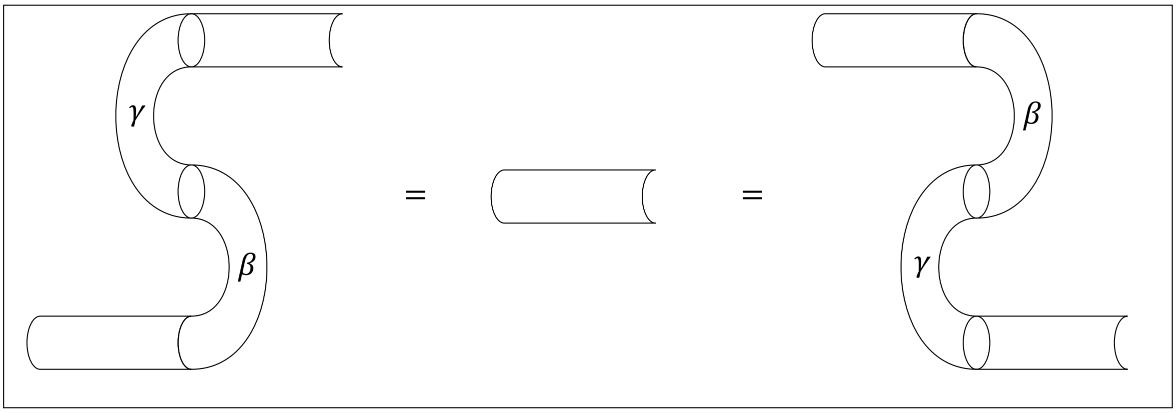


Figure 4.8. The snake relation

Another definition of non-degenerate is; $\beta : A \otimes A \to k$ if there exists $\gamma : k \to A \otimes A$ such that;

$$\begin{array}{ccc} A & \xrightarrow{id_A \otimes \gamma} & A \otimes A \otimes A \\ & \searrow^{id_A} & \downarrow{\scriptstyle \beta \otimes id_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\gamma \otimes _idA} & A \otimes A \otimes A \\ & \searrow^{id_A} & \downarrow{\scriptstyle id_A \otimes \beta} \\ & & A \end{array}$$

Definition 4.2.7. The three point functions φ is defined by $\varphi = \beta \circ (\mu \otimes id_A)$

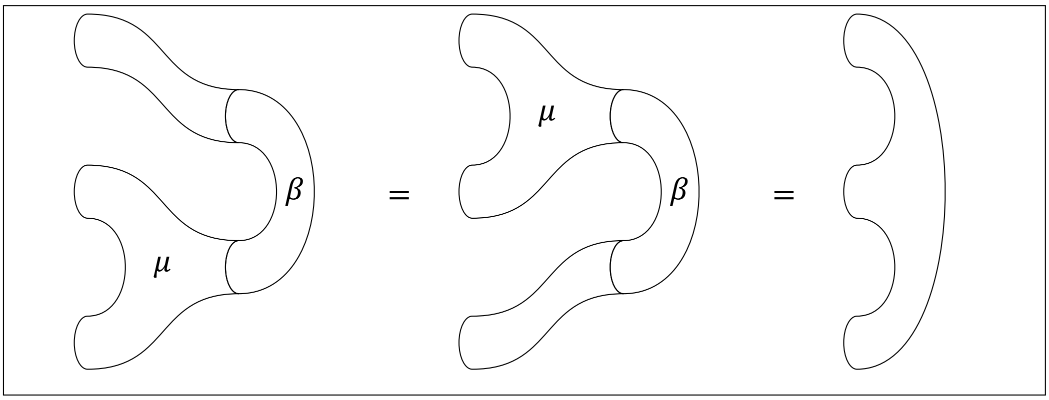


Figure 4.9. The three point functions

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes id_A} & A \otimes A \\ & \searrow^{\varphi} & \downarrow \beta \\ & & k \end{array}$$

Definition 4.2.8. A vector space has multiplication and co-multiplication property with units and it satisfies the multiplication and co-multiplication commutes called Frobenius relation.

4.2.1 Coalgebra structure

Graphical representation can be constructed with these algebraic definitions to see the structures of the Frobenius algebras. It should be known that it has co-algebra structure. Co-unit is formed with Frobenius algebra. The definition of **co-algebra** is given in the following definition.

Definition 4.2.9. A co-algebra over a field k is a k-vector space A with a linear map $\delta : A \to A \otimes A$ called co-multiplication and a linear map $\epsilon : A \to k$ called co-unit such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \delta} & A \otimes A \\ \delta \otimes \mathrm{id}_A \uparrow & & \uparrow \delta \\ A \otimes A & \xleftarrow[\delta]{} & A \end{array}$$

$$\begin{array}{ccc} k \otimes A & \xleftarrow{\varepsilon \otimes \mathrm{id}_A} & A \otimes A \\ & \nwarrow & \uparrow \delta \\ & & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \varepsilon} & A \otimes k \\ \delta \uparrow & \nearrow & \\ A & & \end{array}$$

We will show co-multiplication and co-unit in terms of cobordism notation as follows:

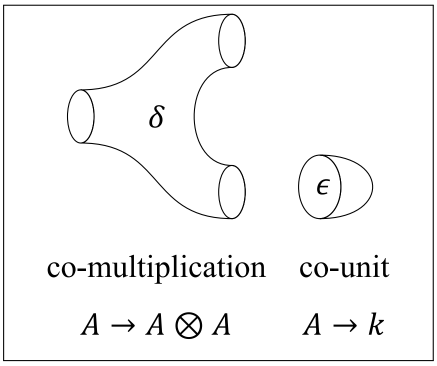


Figure 4.10. Co-multiplication and co-unit

Co-associativity and co-unit axioms are drawn in terms of cobordism such that,

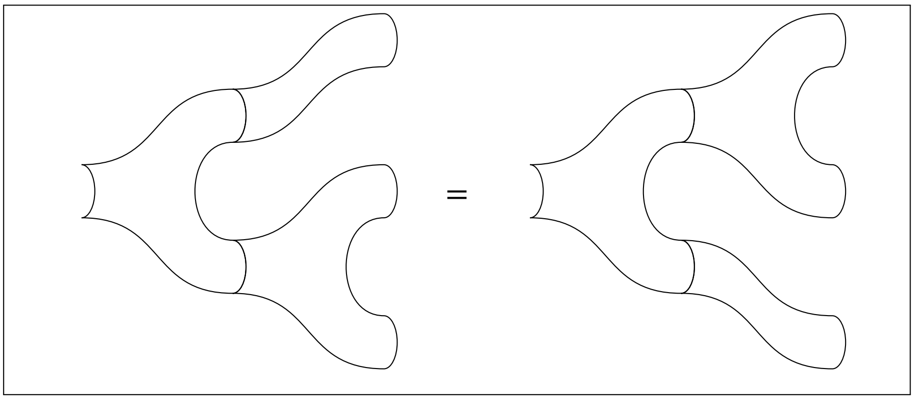


Figure 4.11. Co-associativity axiom

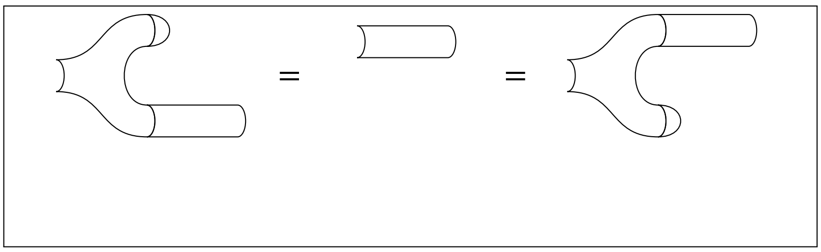


Figure 4.12. Co-unit axiom

Definition 4.2.10. A Frobenius algebra A is a finite dimensional vector space together with linear maps such that $\mu : A \otimes A \to A$, $\delta : A \to A \otimes A$, $\eta : k \to A$ and $\epsilon : A \to k$

(i) $\mu(\eta \otimes id_A) = id_A = \mu(id_A \otimes \eta)$ (Unit)

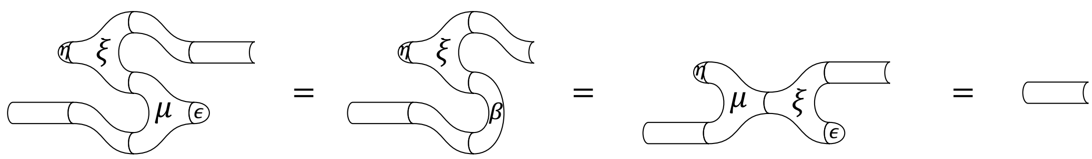
(ii) $(\epsilon \otimes id_A)(\delta) = id_A = (id_A \otimes \epsilon)(\delta)$ (Co-unit)

(iii) $(id_A \otimes \mu)(\delta \otimes id_A) = (\mu \otimes id_A)(id_A \otimes \delta)$ (Frobenius Condition)

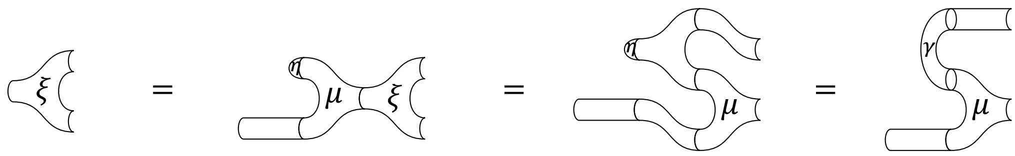
The map left pair of pants is a map $\mu : A \otimes A \to A$. In other words, it is represented as a pair of pants. If A tensored with itself n times say A^n. It can be showed that $A^m \to A^n$ for $n, m \geq 0$. A linear map $\phi : A^m \to A^n$ can be showed by a cobordism with **m** (in-boundaries) and **n** (out-boundaries). The composition of two maps refer to gluing of cobordisms.

Proposition 4.2.11. *Given a Frobenius algebra A, there exists a unique co-associative co-multiplication, δ, whose co-unit if ϵ, satisfying the Frobenius condition.*

Proof. We know that δ is defined as the composition of co-pairing and multiplication. It should be co-associative. Moreover, co-multiplication should satisfy co-associativity. Co-multiplication can be defined in terms of co-pairing and multiplication. From the definition of ϵ is co-unit for δ Frobenius form and pairing are used with the definition of co-multiplication and unit condition. From this definition, we only need to show that co-multiplication δ is unique. Hence we can take another co-associative co-multiplication ξ which has co-unit such that ϵ and also satisfies Frobenius condition. The first step is ξ's Frobenius relation is composed with unit and co-unit. Then we obtain the following picture.



The second equality shows the Frobenius relation. Unit and co-unit relation are shown in the last equality. Then co-pairing γ is unique. Due to the snake relation, we obtain $\xi \circ \eta = \gamma$. ξ can be written in the following picture to get the last equality.

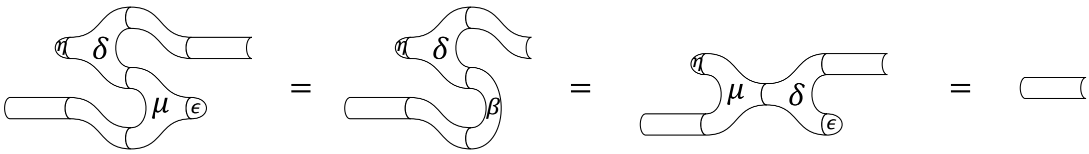


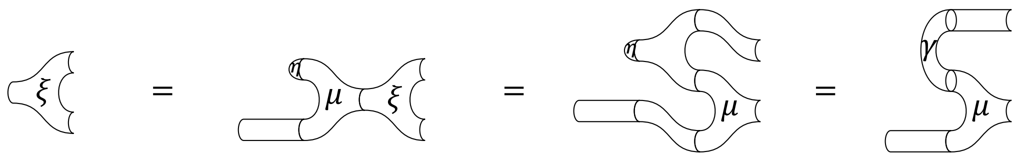
It is clear that ξ is true for the definition of co-multiplication since it can be said that $\xi = \delta$.

□

Proposition 4.2.12. *Given a Frobenius algebra A, there exist a unique co-associative co-multiplication δ, whose co-unit if ϵ satisfying the Frobenius condition.*

Proof.





□

Proposition 4.2.13. *Let A be a vector space with a multiplication $\mu : A \otimes A \to A$, a unit map $\eta : k \to A$, a co-multiplication $\delta : k \to A \otimes A$ and a counit $\epsilon : A \to k$ such that the Frobenius condition holds. Then:*

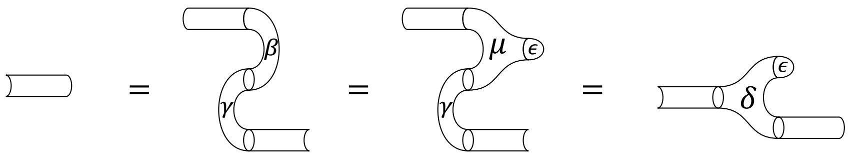
(i) A is a finite dimensional vector space:

(ii) The multiplication μ is associative, i.e., A is a finite dimensional k-algebra;

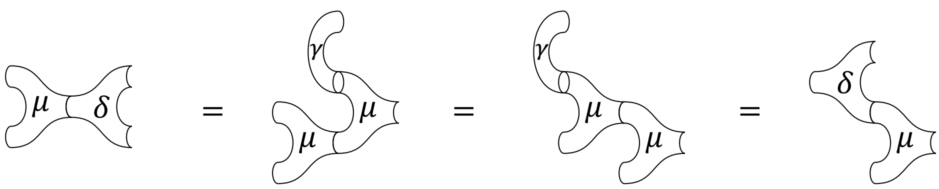
(iii) The co-multiplication is co-associative;

(iv) The co-unit ϵ is a Frobenius form, which means that A is a Frobenius algebra.

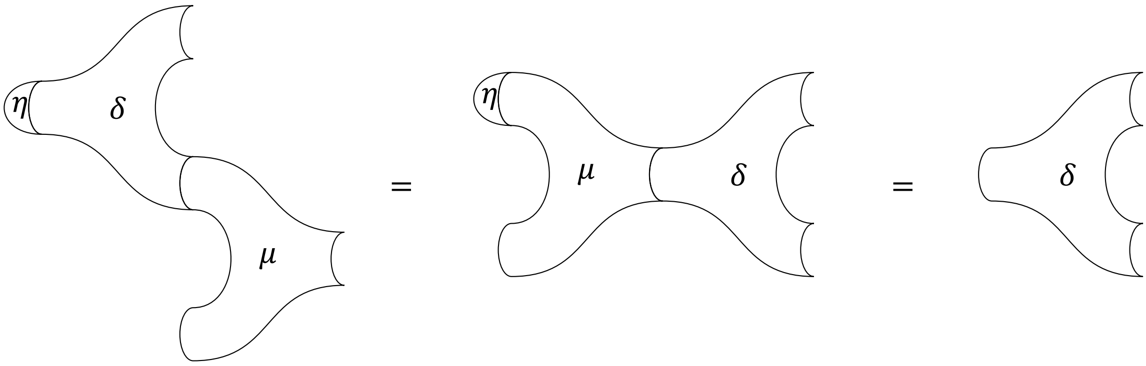
Proof. Assume that (i) holds. The unit condition is satisfied. Co-unit condition is proved with using non-degeneracy condition.



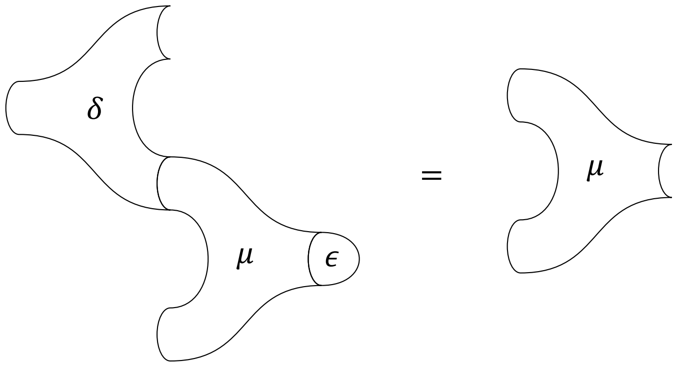
The left hand side of ϵ is co-unit is constructed with using non-degeneracy condition of left hand side. In the second equality, we use $\beta(a \otimes \beta) = \epsilon(ab)$. From the co-multiplication in the third equality, we get last picture. We use co-multiplication for the first equality and use associativity for the second equality to prove Frobenius condition.



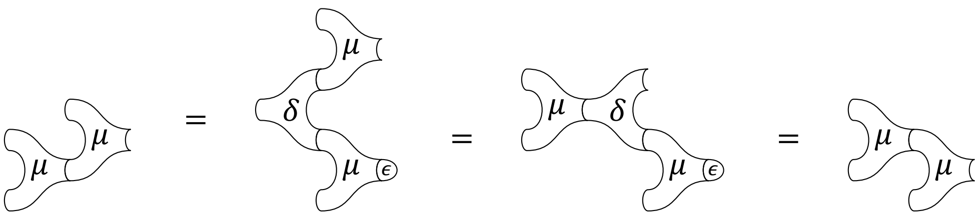
Therefore Frobenius relation of left hand side is proven, for the right hand side co-multiplication is used. Assume (ii) holds. Frobenius condition is used to show associativity of multiplication with caps together. In the picture last equality, we use unit-condition.



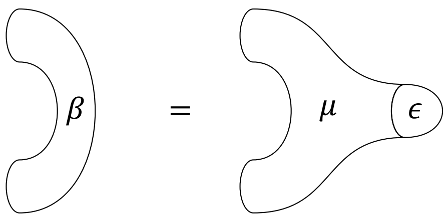
Then we obtain



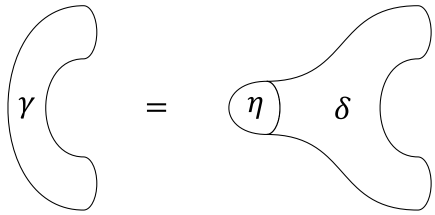
We have



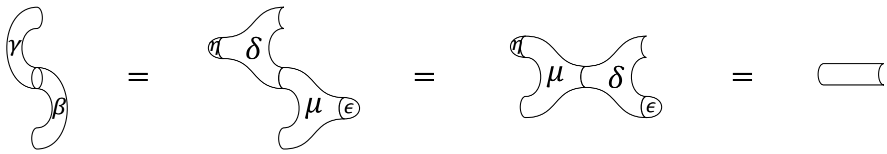
Define pairing



There exist a co-pairing to show non-degeneracy of snake relation. Co-pairing is defined as;



Frobenius condition is used to solve the problem. In the picture Frobenius condition is satisfied by co-multiplication in the second equality. When we look at last equality unit condition and multiplication is satisfied to obtain the right hand side of non-degeneracy condition.



□

More details about proof can be found in [6].

5. TOPOLOGICAL QUANTUM FIELD THEORIES

In this final chapter, we start with giving a general definition of some properties of topological quantum field theories. We want to prove that there is an equivalence of categories between the category $2TQFT_k$ and the category cFA_k. Finally, we show some examples to support our result of this thesis. Moreover, some important conclusion remarks are given to understand the summary of thesis. More details can be found in [5] and [1].

Definition 5.0.1. A Topological Quantum Field Theory (TQFT), of dimension n over a field k, is a symmetric monoidal functor

$$F : nCob \longrightarrow Vect_k.$$

5.1. SOME PROPERTIES OF TQFT (ATIYAH'S AXIOMS)

- A compact $(n-1)$-oriented manifold Σ with a complex vector space $Z(\Sigma)$ for every $(n-1)$ manifold Σ.
- A compact n- oriented manifold M with boundary ∂M, signed a vector $Z(M) \in Z(\partial M)$

(i) Z is functorial with respect to orientation preserving diffeomorphisms of Σ and M.

(ii) Z is involutory such that $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with orientation and $Z(\Sigma)^*$ is the dual vector space of $Z(\Sigma)$

(iii) Z is multiplicative; $Z(\Sigma_1 \amalg \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

(iv) $Z(\emptyset_{n-1}) = \mathbb{C}$

(v) $(\Sigma \times I) = id_{Z(\Sigma)}$ such that I is an interval.

Proof. (i) An orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ induces as an isomorphism $Z(f) : Z(\Sigma) \to Z(\Sigma')$ and it can be say $Z(gf) = Z(g) \cdot Z(f)$ for $g : \Sigma \to \Sigma''$ and if f extends to an orientation preserving diffeomorphism $M \to M'$ with $\partial M = \Sigma$ and $\partial M' = \Sigma'$ then $Z(f)$ takes the element $Z(M)$ to $Z(M')$

(ii) Suppose $\wedge$ is a field, $Z(\Sigma)$ and $Z(\Sigma)^*$ are dual vector spaces. For example, $\mathbb{C}$ or $\mathbb{R}$ is field. Relation between $Z(\Sigma)$ and $Z(\Sigma^*)$ is like a integer homology and cohomology relation.

(iii) We gave a Atiyah' s definition of a topological quantum field theory in the beginning of this chapter. Z is a tensor functor between the category of Cob(d) and the category of complex vector spaces. Tensor functor preserves tensor product so the third axiom says that, the tensor product of the right hand side is the usual tensor product of vector spaces. The tensor product of the left hand side is given by disjoint union of manifolds. Therefore, more concerently **tensor functor** means that if we take disjoint union in of two manifolds Σ_1 and Σ_2 then applies Z result to be suppose tensor product of Z of Σ_1 and Z of Σ_2.

(iv) Not only both tensor product is preserved but also zero unit is preserved. The functor is from the empty manifold is supposed to be identify with the unit to the tensor product of complex vector spaces which is just the complex numbers.

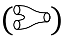
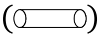
(v) $Z(\Sigma \times I) \in End(Z(\Sigma))$ is an invariant σ and more generally it acts as the identity on the subspace of $Z(\Sigma)$ spanned by all elements $Z(M)$ with $\partial M = \Sigma$ if we replace $Z(\Sigma)$ by its image under the invariant σ. It is easy to see that the axioms are still satisfied.

□

Atiyah's definition explains that, cobordism is related with TQFT axioms;

Definition 5.1.1. The map Z assigns an element $Z(M) \in Z(\Sigma_2)^* \otimes \Sigma_1 = Hom(Z(\Sigma_1, Z(\Sigma_2))$. A functor Z can be obtained from this way. Therefore, $Z : nCob \to Vect_k$ and all axioms satisfy that Z is a symmetric monoidal functor. Conversely, from the symmetric monoidal functor $Z : nCob \to Vect_k$ we use cobordisms to get Atiyah's comment of TQFT. Let $M : \Sigma_1 \to \Sigma_2$ cobordisms $M : \phi \to \Sigma_1^* \amalg \Sigma_2$ and hence obtaining a map $Z : \mathbb{C} \to Z(\Sigma_1 \amalg \Sigma_2)$ which is the same as the giving a vector $Z(1) \in Z(\Sigma_1 \amalg \Sigma_2)$

Question: What is the d-dimensional topological quantum field theory?

Definition 5.1.2. First of all, we have a tensor functor which we can evaluate on the objects on $Cob(d)$. If $d = 2$; so the objects of manifolds on dimension one and there is one manifold namely the circle. If Z is a two-dimensional TQFT then we can evaluate Z of circle and get some vector space which are call A; $Z(S^1) = A$. Here, A is a complex vector space and we can show what Z does on all objects because, every closed one manifold is just the disjoint union of finitely many circles. Also, Z is a tensor functor to take a disjoint unions of tensor products. For example; if we evaluate Z on three circles, then we just get tensor product of three caps of A. It is called the "pair of pants" Z () $= A \otimes A \to A$. It is a cobordism from two circle to one circle. Hence, this gives us a linear map which is called "multiplication". It is denoted as μ. We should know, μ is associative and commutative because there is a diffeomorphism on pair of pants which swaps in-coming circles.
In addition to this, there is also unit with respect to this multiplication. A disc can be drawn by the following way; Z (): This disc is a cobordism from the empty set to one circle, that supposed to give me a linear map from Z of the empty set which we said over there was the complex numbers into Z of circle which is A;$Z(\emptyset) \simeq \mathbb{C} \to A$. Moreover, such a linear map we can distinct of as given elements of A Let take an element which are called 1 and inside of A; $1 \in A$. We can say that, 1 is unit with respect to this multiplication.
Also, we can read the disc in the other way. We can take of the disc as a cobordism from one circle to the empty set; Z () .Z of the disc is going to be a linear map from A into the complex numbers. A linear map can be written $A \xrightarrow{tr} \mathbb{C}$. It is called a trace. What is the trace pairing?
A trace pairing by linear linear form on A which is given by taking two elements of A , first multiplying them and taking a trace. This linear map can be written; $A \otimes A \xrightarrow{\mu} A \xrightarrow{tr} \mathbb{C}$. If we evaluate on this cobordism two circles to the empty set given by a cylinder; Z (). Hence, we have a trace pairing which is not generate pairing. Then A is finite dimensional.

- What happens when you apply TQFT to manifolds?

We have a disjoint union when you apply TQFT and you get a tensor product. We can take $(n-1)$-dimensional manifold to the $(n-1)$-dimensional manifold. For example, take a cobordism from the empty $(n-1)$-dimensional manifold to empty $(n-1)$-dimensional manifold like; $M : \emptyset_{n-1} \rightarrow \emptyset_{n-1}$ when we apply TQFT to this, we map this manifold to category of vector spaces.

$$Z(M) : Z(\emptyset_{n-1}) \rightarrow Z(\emptyset_{n-1})$$

Then, $Z(M) : \mathbb{F} \rightarrow \mathbb{F}$. This map is taken from field to the field. It is scalar. Therefore $Z(M)$ is a topological constant.

Theorem 5.1.3. *There is a natural equivalence of categories* $2TQFT_k \simeq cFA_k$.

Proof. First we will show that;

$$Cat_{\otimes}^{sym}(2Cob_{Sk}, Vect_k) \simeq cFA_k \tag{5.1}$$

and also

$$Cat_{\otimes}^{sym}(2Cob, Vect_k) \simeq Cat_{\otimes}^{sym}(2Cob_{Sk}, Vect_k). \tag{5.2}$$

If we have a three monoidal categories and if $2TQFT_k \simeq 2Cob_{Sk}$ and $2Cob_{Sk} \simeq 2Cob$ and we can say $2TQFT_k \simeq 2Cob$ Then we can prove

$$2TQFT_k = Cat_{\otimes}^{sym}(2Cob, Vect_k) \simeq cFA_k. \tag{5.3}$$

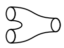
A skeleton of $2Cob$ is obtained as follows; In general, we denote **n**; the disjoint union of n circles, we denote**1**; the circle and **0**;the empty one manifold as an object of $2Cob$. There is a symmetric monoidal natural transformation such as α between two 2-dimensional topological quantum field theories. This transformation is determined by its components $\{\alpha_1, \dots \alpha_n\}_{n\geq 0}$. Then, α_1 can determine these components. A and B are strict monoidal functors and a monoidal natural transformation says, u is morphism between two strict monoidal functors that, $u_{A\otimes B} = u_A \otimes' u_B$. This means that, $\alpha_{n+m} = \alpha_{n\sqcup m} = \alpha_n \otimes \alpha_m$. Hence, the component α_n is the n^{th} tensor product of α_1. Now we will find a functor $F : 2TQFT_k \rightarrow cFA_k$. which is a category equivalence and (also $Cat_{\otimes}^{sym}(2Cob_{Sk}, Vect_k) \rightarrow cFA_k$). Then A be a 2-dimensional TQFT and we can define;

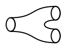
$F(A) = (A(\mathbf{1}), A($  $), A($  $), A($  $), A($  $)).$ (5.4)

First, take a symmetric monoidal natural transformation between two 2-dimensional TQFT as,$\alpha : A \to B$ with $\alpha_n : A(n) \to (n)$. Then, we can say

$$F(\alpha) = \alpha_1 : A(1) \to B(1)$$

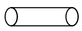
$F(\alpha)$ is a Frobenius algebra morphism so we will show that $F(A)$ is a commutative Frobenius algebra. Then we can say, the composition of natural transformations is composition of morphisms. A is a monoidal functor that respects the relations of $2Cob$. The unit, co-unit,commutativity and co-commutativity satisfy that, $F(A)$ is a Frobenius algebra. Also the component α_n is just the n^{th} tensor product of α_1. Hence α represents the commutativity of algebra or co-algebra morphisms. Then α_1 is a Frobenius algebra morphism. The functor F is an equivalence of categories, we prove this with its inverse functor. $G : cFA_k \to 2TQFT_k$. Let $(A, \mu, \delta, \eta, \epsilon)$ be a commutative Frobenius algebra and $G(A, \mu, \delta, \eta, \epsilon)$ be the symmetric monoidal functor shown as follows:
$G(A, \mu, \delta, \eta, \epsilon) : 2Cob \to Vect_k$. We know that $2Cob$ has generators and carries Frobenius structures so generators can be shown in terms of Frobenius algebras. Then we can show;

 $\mapsto \mu$

 $\mapsto \delta$

 $\mapsto \eta$

 $\mapsto \epsilon$

which also maps $\mathbf{1} \mapsto A, \mathbf{n} \mapsto A^{\otimes n}, \mathbf{0} \mapsto k,$  $\mapsto id_A,$  $\mapsto \tau_{A,A}$. Now there is a Frobenius algebra morphism between two commutative Frobenius algebras A and B; $\phi : A \to B$ and $G(\phi)$ is the monoidal natural transformation. It has components;$G(\phi)_n = \phi^{\otimes n} = \phi \otimes \cdots (n) \cdots \otimes \phi$ then, $G(\phi)_1 = \phi$ and $G(\phi)_0 = id_k$. Now A is a functor $A : 2Cob \to Vect_k$ and we use the generators of $2Cob$:  ,  ,  ,  and  We can say that TQFTs are symmetric monoidal functors.  is the linear twist map between the vector spaces. The component α_n is the n^{th} tensor product of α_1. □

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