GEOMETRY OF SECOND ORDER DEGENERATE LAGRANGIANS

by

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Submitted to Graduate School of Natural and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Mathematics

Yeditepe University

2017

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Dedicated to my daughter Defne...

ACKNOWLEDGEMENTS

First of all, I would like to thank to my advisor Prof. Dr. Hasan Gümral who accepted me as a PHD student first, for suggestion of a research topic and provide me to find my own path so that have learned a lot from him.

I would be grateful to my current advisor Yrd.Doç.Dr. Oğul Esen for his full guidance and support to complete this thesis work. He spent lots of time reading of all the preliminary versions of thesis. His useful comments and advices made the thesis clear and readable. Without his help, probably, I could not complete this thesis.

I would also thank to the members of my thesis committee Prof. Dr. Muhittin Mungan, Prof. Dr. Ender Abadoğlu and Yrd. Doç. Dr. Serkan Sütlü for their helpful comments.

Thanks to Işık University and the people who have supported me financially as a full time teaching assistantship position during my studies.

Last, but never least, I would like to thank to my family for their love and infinite support, especially to my husband Candaş Uçgun for his unconditional love, understanding and continual support to complete my thesis even when everything seemed hopeless.

Finally, thanks to my beloved daughter, Defne Uçgun, for her smiley face and life energy. Thank you, my pretty girl whom this thesis is dedicated.

ABSTRACT

GEOMETRY OF SECOND ORDER DEGENERATE LAGRANGIANS

The goal of this thesis is to present the Hamiltonian formulations of the dynamical systems generated by the second order Pais-Uhlenbeck, Sarıoğlu-Tekin and Clèment Lagrangians.

Pais-Uhlenbeck Lagrangian is non-degenerate in the sense of Ostrogradsky whereas Sarıoğlu-Tekin and Clèment Lagrangians are degenerate. For the degenerate or/and constraint systems, the Legendre transformation is not possible in a straight forward way. For the degenerate systems, one additionally needs to employ, for example, the Dirac-Bergmann algorithm in order to arrive at the Hamiltonian picture.

We shall follow several alternative methods while arriving at the Hamiltonian representations of Pais-Uhlenbeck, Sarıoğlu-Tekin and Clèment dynamics. At first, we first shall identify the configuration spaces, the tangent and the cotangent bundles. We shall first use Jacobi-Ostragradskii momenta to define the primary sets of constraints. Accordingly, the total Hamiltonian will be written. The Dirac-Bergmann algorithm will be run in order to identify the final constraint submanifold. In each step of the algorithm, we shall revise the total Hamiltonian by adding the secondary constraints. Once the final constraint set is determined, it is immediate to write the Hamilton's equations governing the dynamics. This is the first and most common way. An alternative way arriving at the Hamilton's equations is to construct the Dirac bracket. To do this, we shall first classify the constraints, determining the final constraint submanifold, into two classes, namely the first and the second. Then, using this classification, we shall define the Dirac brackets associated with the physical systems.

There is an alternative way to arrive the Hamilton's equations. In this approach, instead of studying directly with the second order Lagrangians, we shall reduce the second order Pais-Uhlenbeck, Sarıoğlu-Tekin and Clèment Lagrangians to first order Lagrangians by introducing new coordinates and Lagrange multipliers. In this case, the reductions will give degenerate first order Lagrangians even though the second order Lagrangian is non-degenerate. We shall apply the Dirac-Bergmann algorithm for these first order formalisms in order to write the Hamilton's equations.

ÖZET

İKİNCİ DERECE DEJENERE LAGRANGİANLARININ GEOMETRİSİ

Bu tezin amacı ikinci derece Pais-Uhlenbeck, Sarıoğlu-Tekin and Clèment Lagrange fonksiyonları ile üretilen dinamik sistemlerin Hamilton formülasyonlarını elde etmektir.

Pais-Uhlenbeck, Ostrogradsky anlamında yozlaşmamış, fakat Sarıoğlu-Tekin and Clèment yozlaşmış Lagrange fonksiyonlarıdır. Yozlaşmış sistemler için Legendre dönüşümleri direkt olarak Hamilton resmini veremez. Bu tip durumlarda Dirac-Bergmann algoritması uygulanması gerekmektedir.

Pais-Uhlenbeck, Sarıoğlu-Tekin ve Clèment dinamik denklemlerine karşılık gelen Hamilton temsilleri bir kaç alternatif metod izlenerek elde edilcektir. Öncelikle, konfigürasyon uzayları, tanjant ve kotanjant demetleri belirlenecektir. Jacobi-Ostragradskii momentum değişkenleri aracılığıyla öncül kısıt altkatmanı tanımlanacaktır. Toplam Hamilton fonksiyonu yazılacaktır. Dirac-Bergmann algoritması çalıştırılacak ve bu şekilde son kısıt altkatmanı elde edilecektir. Algoritmanın her adımında ikincil kısıtlar eklenerek toplam Hamilton fonksiyonu revize edilecektir. Son kısıt katmanı elde edildiğinde, Hamilton denklemlerine ulaşmak artık kolaydır. Buraya kadar yapılan literatürdeki en temel yaklaşımdır. Hamilton temsile ulaşmak için yapılan alternatif bir yaklaşım ise Dirac çerçevelerini yazmaktır. Son kısıt altkatmanını belirleyen fonksiyonlar ilk ve ikinci sınıf olmak üzere ayrılacak, bu şekilde Dirac çerçevesi tanımlanacaktır.

İkinci derece Lagrange fonksiyonları ile çalışmaktansa, yeni koordinatlar ve Lagrange çarpımları aracılığıyla, ikinci derece Lagrange fonksiyonları birinci derece Lagrnage fonksiyonlarına indiregenecektir. İkinci derece Lagrange fonsiyonu yozlaşmamış olsa bile, indirgenmiş birici derece Lagrange fonksiyonu yozlaşmış olacaktır. Bu durumda kaçınılmaz olarak Dirac-Bergmann algoritması kullanılacak ve Hamilton denklemleri bu şekilde elde edilecektir.

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1. INTRODUCTION

There exist two different but equivalent representations of the Newton's classical dynamics, namely the Lagrangian and the Hamiltonian dynamics. These theories offer two different formulations of the Newton's second law

$$F(q,t) = m\ddot{q} \tag{1.1}$$

governing the motion of a single particle under the conservative force field [1, 2, 3, 4]. The passage between the Lagrangian and the Hamiltonian dynamics is available by means of the Legendre transformations if some non-degeneracy conditions hold. For the degenerate cases, constructing passage is not an easy task [5, 6].

1.1. THE LAGRANGIAN AND THE HAMILTONIAN DYNAMICS

If the configuration space is an n-dimensional manifold Q, then the Lagrangian dynamics is generated by a function(al) L on the tangent bundle TQ which, physically, corresponds the velocity phase space [1, 2, 7]. The dynamics is governed by the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \tag{1.2}$$

where $(q^i, \dot{q}^j)$ is a local chart on TQ those induced from (q^i) on Q. Euler-Lagrange equations are the second order differential equations in n numbers. If, particularly, the Lagrangian function is chosen to be the difference of kinetic and potential energies $L = K - V$ of a single particle then the Euler-Lagrange equations (1.2) equal to the Newton's second law (1.1) assuming that the force $F(q,t)$ is given by minus of the gradient of V.

A Poisson structure on a manifold P is a bilinear skew-symmetric binary operation $\{\bullet, \bullet\}$ on the space $\mathcal{F}(P)$ of smooth functions that satisfies

- Jacobi identity: $\{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_2\}\} = 0,$
- Leibniz identity: $\{F_1 F_2, F_3\} = F_1 \{F_2, F_3\} + \{F_1, F_3\} F_2$

for all F_1, F_2, F_3 in $\mathcal{F}(P)$ [8, 9, 10]. The Hamilton's equations, governed by a Hamiltonian function(al) H, is given by

$$\dot{z} = \{z, H\}\,. \tag{1.3}$$

for a curve $z = z(t)$ in P parameterized by the time variable t.

A manifold M is called symplectic if it is equipped with a non-degenerate closed two-form Ω [1, 11, 7, 12]. Cotangent bundle T^*Q of a manifold Q, which can be assumed to be the momentum phase space of a physical system, carries a canonical symplectic two-form. In the symplectic framework, Hamilton's equations are defined by

$$i_{X_H}\Omega = -dH, \tag{1.4}$$

where i is the interior derivative (contraction), d is the exterior derivative, and X_H is the Hamiltonian vector field associated with the Hamiltonian function H, [13, 14]. The Hamilton's equations (1.4) take the particular form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{\partial H}{\partial q^i} \tag{1.5}$$

on the Darboux' coordinates (q^i, p_j).

A symplectic manifold is necessarily Poisson with the introduction of the non-degenerate Poisson bracket

$$\{H, K\} := \Omega\,(X_H, X_K)\,. \tag{1.6}$$

Using the identification presented in Eq.(1.6), one may show that the Hamilton's equations in (1.3) and (1.4) are coinciding. Note that, inverse of this discussion is not true, that is a Poisson manifold is not necessarily symplectic [10].

Starting with the Euler-Lagrange equations (1.2), to write the Hamilton's equations (1.5), one needs to relate the velocities $(\dot{q}^i)$ with momenta (p_i). This can be achieved by defining the fiber derivative of the Lagrangian function, namely the Legendre transformation,

$$p_i := \frac{\partial L}{\partial \dot{q}^i}\,(q, \dot{q})\,. \tag{1.7}$$

It is evident that, in order to make the transformation (1.7) invertible, one needs to employ a non-degeneracy condition, called the Hessian condition,

$$\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0. \tag{1.8}$$

If a Lagrangian function satisfies the Hessian condition, then it is called non-degenerate (regular). In this case, the velocities $\dot{q}$ can be written as functions of position and momenta (q, p). That is, we have an invertible (one to one) transformation between the tangent and cotangent bundles

$$\mathbb{F}L : TQ \to T^*Q : (q, \dot{q}) \to \left(q, \frac{\partial L}{\partial \dot{q}} \right). \tag{1.9}$$

By defining the canonical Hamiltonian function

$$H(q,p) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}(q,p) - L\left(q, \frac{\partial L}{\partial \dot{q}^i}(q,p) \right),$$

the Hamilton's equations (1.5) become equal with Euler-Lagrange equations (1.2).

1.2. THE LEGENDRE TRANSFORMATION AND DIRAC-BERGMANN CONSTRAINT ALGORITHM

If the Hessian condition (1.8) is not satisfied then the Lagrangian is called degenerate (singular) that is, one cannot solve the velocities in terms of momenta. Instead, one arrives an immersed submanifold C of T^*Q defined by the following constraint functions

$$\Phi_a(q,p) = p_a - \frac{\partial L}{\partial q^a} = 0 \tag{1.10}$$

where a ranging from 1 to the dimension of the kernel of $\mathbb{F}L$. At 1950s, Dirac proposed an algorithm to write the Hamilton's equations under the presence of such constraints [15]. This method nowadays is called as the Dirac-Bergmann theory of constraints. The geometrization of this algorithm was given by Gotay Nester and Hinds [16].

In the Dirac-Bergmann algorithm, the ultimate goal is arrive a final set of constraints, satisfying the consistency conditions, by starting from the primary set constraints (1.10). By this, one obtains a well-defined Poisson submanifold C_f of T^*Q. Using the constraints of the first kind $\{\chi_\alpha, \alpha = 1, ..., r\}$, the Dirac-Poisson bracket is defined by

$$\{F, H\}_{C_f} = \{F, H\} - \{F, \chi_\alpha\}(C^{-1})^{\alpha\beta}\{\chi_\beta, H\} \tag{1.11}$$

on the final constraint manifold in terms of the Poisson bracket on T^*Q. Here,

$$C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}\,, \;\; \alpha, \beta = 1, ..., r$$

is an invertible $r \times r-$matrix [17]. In the main body of the thesis, we shall present the algorithm and the construction of the Dirac bracket more explicitly.

1.3. LAGRANGIANS DEPENDING ON ACCELERATIONS

Note that, Euler-Lagrange equations (1.2) are the second order differential equation. The question may arrise that, is there a Lagrangian formalims for higher order differential systems? More concretely, is it possible to write a third (or fourth) order differential equation as an Euler-Lagrange equation?

The answer is positive. The geometrical framework for the third and fourth order systems is the iterated tangent bundle T^2Q of Q, which consists of accelerations addition to the position and velocities, with coordinates $\left(q^i, \dot{q}^j, \ddot{q}^k\right)$. In this case, a Lagrangian function $L = L\left(q, \dot{q}, \ddot{q}\right)$ is defined on T^2Q and generates the second order Euler-Lagrange equations

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \frac{\partial L}{\partial q^i} = 0, \tag{1.12}$$

which is, in general, a set of fourth order differential equations. If L is linear with respect to the acceleration variable $\ddot{q}^i$, then $\partial L/\partial \ddot{q}^i$ is free from $\ddot{q}^i$, hence the second order Euler-Lagrange equations (1.12) give a set of third order differential equations.

1.3.1. Ostrogradsky's Momenta

To write the second order Euler-Lagrange equations (1.12) as in the form of Hamilton's equations (1.5), one proceeds as follows. First, consider the dual bundle T^*TQ with local coordinates $(q^i, \dot{q}^j, p_k, \dot{p}_l)$. On T^*TQ, the canonical Poisson bracket is defined by

$$\{q^i, p_j\} = \{\dot{q}^i, \dot{p}_j\} = \delta^i_j$$

and, all others are zero. A higher dimensional version the fiber derivative (1.7) was introduced by Ostrogradsky around 1850's, and it is given by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}, \ddot{q}) - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}^i}(q, \dot{q}, \ddot{q}), \quad \dot{p}_i = \frac{\partial L}{\partial \ddot{q}^i}(q, \dot{q}, \ddot{q}). \tag{1.13}$$

A second order Lagrangian is called non-degenerate in the sense of Ostrogradsky if one can solve $\ddot{q}$ in terms of $(q, \dot{q}, \dot{p})$ using (1.13). This is possible if the second order Hessian condition is satisfied, namely

$$\det \frac{\partial^2 L}{\partial \ddot{q}^i \ddot{q}^j} \neq 0.$$

In this case, the Hamiltonian formulation follows introduction of the canonical Hamiltonian function

$$H(q, \dot{q}, p, \dot{p}) = p_i \dot{q}^i + \dot{p}_i \ddot{q}^i(q, \dot{q}, \dot{p}) - L(q, \dot{q}, \ddot{q}(q, \dot{q}, \dot{p}))$$

on the iterated cotangent bundle T^*TQ.

When there are degeneracies, the Legendre transformation is not immediate. The direct way to solve this to apply Dirac-Bergmann constraint algorithm to the image space of (1.13). In the literature, there are intensive studies on the Legendre transformation of singular or/and constraint higher order Lagrangian systems, [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. We cite [31, 32, 33] for the case of Ostrogradsky-Legendre transformation on Lie groups. We additionally refer some recent studies on the second order Lagrangians whose dependence on the acceleration are linearly and/or affine [34].

1.3.2. Reduction to a First Order System

It is possible to recast a second order Lagrangian function L as a first order Lagrangian function $\bar{L}$ by calling consecutive time derivatives of initial coordinates as new coordinates. Evidently, this can be done in several different but equivalent ways. One option is to embed the second order tangent bundle T^2Q into the iterated tangent bundle given by

$$T^2Q \simeq \{Z \in TTQ : \tau_{TQ}(Z) = T\tau_Q(Z)\},$$

where τ_{TQ} is the tangent bundle projection $TTQ \to TQ$, and $T\tau_Q$ is the tangent mapping of the projection τ_Q. In this case, a first order Lagrangian function can be written as

$$\bar{L}(q_1, \dot{q}_1, \dot{q}_2, \lambda) = L(q_1, \dot{q}_1, \dot{q}_2) + \lambda_i \left(\dot{q}_1^i - q_2^i\right),$$

where it is assumed that that $q_1 = q$ and $q_2 = \dot{q}$.

Alternatively, by the definitions $q_1 = q$, $q_2 = \dot{q}$ and $q_3 = \ddot{q}$ being made, one may introduce the following first order Lagrangian

$$\bar{L}(q_1, q_2, q_3, \lambda, \beta) = L(q_1, q_2, q_3) + \lambda_i \left(\dot{q}_1^i - q_2^i\right) + \beta_i \left(\dot{q}_2^i - q_3^i\right),$$

where both of λ_i's and β_i's are the Lagrange multiplies.

Note that, absence of the $\dot{\lambda}$ manifests the degeneracy of both of the first order Lagrangians derived in this subsection. Hence, to arrive the Hamiltonian picture, one has to employ Dirac-Bergmann algorithm.

1.4. THE GOAL OF THIS THESIS

Our goal in this thesis is to obtain Hamiltonian formulations of some of the second order Lagrangian formalisms arising in the theory of topological massive gravity, namely Clément, Sarıoğlu-Tekin, and Pais-Uhlenbeck Lagrangians. We record here these Lagrangian densities with some comments on the physical motivations.

The action for topologically massive gravity consists of the action for cosmologocal gravity and the Chern-Simons term. Clément, in his search for particle like solutions for this theory, reduced the action [35, 36, 37] to the second order degenerate Lagrangian density

$$L^C[\mathbf{X}] = -\frac{m}{2}\zeta\dot{\mathbf{X}}^2 - \frac{2m\Lambda}{\zeta} + \frac{\zeta^2}{2\mu m}\mathbf{X}\cdot(\dot{\mathbf{X}}\times\ddot{\mathbf{X}}) \tag{1.14}$$

for three component vector function $\mathbf{X}$ of the independent variable t. The notation $[\mathbf{X}]$ represents three vectors consisting of $\mathbf{X}$, its velocity $\dot{\mathbf{X}}$ and acceleration $\ddot{\mathbf{X}}$, that is $[\mathbf{X}] = (\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}})$. Here, the inner product $\mathbf{X}^2 = T^2 - X^2 - Y^2$ is defined by the Lorentzian metric and the triple product is $\mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) = \epsilon_{ijk} X^i \dot{X}^j \ddot{X}^k$ where ϵ_{ijk} is the completely antisymmetric tensor of rank three. Dot denotes the derivative with respect to the variable t and $\zeta = \zeta(t)$ is a function which allows arbitrary reparametrization of the variable t. Λ and $1/2m$ are cosmological and Einstein gravitational constants, respectively.

In a more recent work [38], Sarıoğlu and Tekin considered an action consisting of Einstein-Hilbert, Chern-Simons and Pauli-Fierz terms and, obtained the reduced Lagrangian density

$$L^{ST}[\mathbf{X}, \mathbf{Y}] = \frac{1}{2} \left[a(\dot{\mathbf{X}}^2 + \dot{\mathbf{Y}}^2) + \frac{2}{\mu} \dot{\mathbf{Y}} \cdot \ddot{\mathbf{X}} - m^2(\mathbf{Y}^2 + \mathbf{X}^2) \right] \tag{1.15}$$

by supressing the spatial part of the theory. Here, a, μ, m are parameters and $\mathbf{X}$, $\mathbf{Y}$ are three-vectors. In the context of higher derivative theories, they also considered Pais-Uhlenbeck oscillator as a nonrelativistic limit. This is described by the nondegenerate Lagrangian density

$$L^{PU}[X] = \frac{1}{2} \left[\ddot{X}^2 - (q^2 + p^2)\dot{X}^2 + p^2 \Omega^2 X^2 \right] \tag{1.16}$$

where X is a real dynamical variable, p and q are positive real parameters [39]. Functionally, Clèment and Sarıoğlu-Tekin Lagrangians are significantly different than the Pais-Uhlenberg Lagrangian since they involve degeneracies.

1.5. CONTENTS

This thesis is organised as follows: In the next chapter, the brief summery of the first order Lagrangian and the Hamiltonian dynamics for non-degenerate and degenerate cases are presented. For the first order degenerate theories, the Dirac-Poisson bracket and the Dirac-Bergmann constraint algorithm are exhibited.

In the third chapter, the second order Lagrangian theory are reviewed. The Jacobi-Ostrogradsky momenta are defined. In progress, the Hamiltonian formalisms of a second order degenerate Lagrangian theory is analysed in two different but equivalent ways. At First, the Dirac-Bergmann constraint algorithm is directly applied to the total Hamiltonian written for the second order Lagrangian. Secondly, the second order Lagrangian is reduced to a first order Lagrangian by defining new coordinates and encoding these new coordinates in to the theory by addition of some Lagrange multipliers. Then, the total Hamiltonian is written for this reduced first order Lagrangian, and accordingly, the Dirac-Bergmann algorithm is applied. As a particular and alternative way, the Legendre transformation is performed to an unconstraint Lagrangian which is obtained by solving the Lagrange multipliers.

In the last chapter, we analyse some concrete second order degenerate Lagrangians, namely Pais-Uhlenbeck [39], Sarıoğlu-Tekin [38] and Clèment [37]. For each of them, the Jacobi-Ostrogradsky momenta are defined and the Dirac-Bergmann constraint algorithm is applied. At the end, the Hamilton's equations are written. Alternatively, the Dirac-Poisson brackets are computed for each of the theories and the Hamilton's equations are rewritten using the Dirac-Poisson bracket in order to make a cross check. The reductions of these second order theories to the proper first order ones are also exhibited. Similar to the second order versions, the Dirac-Bergmann algorithms are applied and the Hamilton's equations are written to these reduced first order systems as well.

2. THE FIRST ORDER THEORY

2.1. LAGRANGIAN DYNAMICS

We start with an $n-$dimensional manifold Q, assumed to be the configuration space of a physical system, and a local coordinate chart

$$q = (q^1, ..., q^n) \in Q.$$

The tangent space to the manifold Q at a point q is denoted by T_qQ. The union of the all tangent spaces constitutes the tangent bundle TQ, which corresponds to the velocity-phase space of the physical system [1, 2, 7]. We equip the tangent bundle TQ with the induced local coordinate system

$$(q, \dot{q}) = (q^1, ..., q^n, \dot{q}^1, ..., \dot{q}^n) \in TQ$$

consisting of the positions and the velocities. The tangent bundle projection $\tau_Q : TQ \to Q$, locally, maps the two-tuple $(q, \dot{q})$ to its first components (q) defining the position.

A first order Lagrangian density $L = L(q, \dot{q})$ is a real valued function on TQ. The corresponding action integral is

$$S_L = \int_{t_1}^{t_2} L(q, \dot{q})dt, \tag{2.1}$$

for two fixed points $q(t_1)$ and $q(t_2)$ in Q [3, 7]. In order to derive the extremum values of the functional S_L, we take variation of the action integral and equate it to the zero, that is

$$\delta S_L = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt = 0. \tag{2.2}$$

Using the commutation of the variation with the time derivative, applying the by parts technique for the second term in the integral (2.2), and employing the boundary conditions $\delta q(t_1) = \delta q(t_2) = 0$, we arrive

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right] \delta q^i dt = 0. \tag{2.3}$$

Note here that, the boundary terms define the Lagrangian one-form

$$\theta_L[q^i] \equiv \frac{\partial L}{\partial \dot{q}^i} dq^i. \tag{2.4}$$

We assume that the variation δq^i is arbitrary, then Eq.(2.3) gives the Euler-Lagrange equations [3]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}, \tag{2.5}$$

which constitute a system of second order differential equations

$$\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} \ddot{q}^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \dot{q}^j. \tag{2.6}$$

For the Lagrangians resulting in the same Euler-Lagrange equations (2.5), Lagrange one-form θ_L in (2.4) is not unique. However, its functional exterior derivative

$$\Omega_L \equiv d\theta_L \tag{2.7}$$

is a well-defined presymplectic two-form on TQ.

If the determinant of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \tag{2.8}$$

is not zero or, equivalently, if the rank r of the Hessian matrix (2.8) is equal to the dimension n of Q, then the Lagrangian L is called non-degenerate [40, 41, 42]. In this case, the accelerations $\ddot{q}$'s are uniquely determined by the positions q and the velocities $\dot{q}$. If the rank of the Hessian matrix (2.8) is smaller then n, the Lagrangian is called degenerate. In this case, it is not possible to determine $\ddot{q}$'s uniquely in terms of q and $\dot{q}$. So that, a solution of the Euler-Lagrange Eq.(2.5) may contain arbitrary functions.

2.2. SYMPLECTIC AND POISSON MANIFOLDS, HAMILTONIAN DYNAMICS

A manifold M equipped with a closed and non-degenerate 2-form ω_M is called a symplectic manifold. A symplectic manifold is denoted by (M, ω_M). The non-degeneracy of the symplectic two-form ω_M enables us to define a 1-1 correspondence between the gradients of the functions and the vector fields on M. This isomorphism leads us to write the Hamilton's equation of motion in a coordinate invariant form. Let H be a the Hamiltonian function on M, then the Hamilton's equations are

$$i_{X_H}\Omega = -dH \tag{2.9}$$

where i is the interior derivative (contraction), d is the exterior derivative, and X_H is the Hamiltonian vector field associated with H, [13, 14]. The triplet (M, ω_M, X_H) is called a Hamiltonian system.

To see the Hamilton's equations (2.9) in coordinates, we first introduce the cotangent bundle T^*Q which is generic for all symplectic manifolds. Physically, one may regard T^*Q as the momentum-phase space of the system. On T^*Q, there is a distinguished set of local coordinates

$$(q, p) = (q^1, ..., q^n, p_1, ..., p_n)$$

called the Darboux' coordinates which enables us to write the symplectic structure in the form of

$$\omega_{T^*Q} = dp_i \wedge dq^i.$$

In this local picture, for a Hamiltonian function H on T^*Q, the Hamiltonian vector field is computed to be

$$X_H(q, p) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^j} \right), \tag{2.10}$$

so that the Hamilton's equations (2.9) turn out to be

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.11}$$

Alternatively, one may represent the Hamilton's equations (2.11) in terms of a Poisson bracket. To arrive this, start with a function $F = F(q, p)$, and take the derivative of F with respect to time. This gives

$$\frac{dF}{dt} = \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i \tag{2.12}$$

$$= \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} =: \{F, H\}\,. \tag{2.13}$$

Here, $\{F, H\}$ is the canonical Poisson bracket of two functions. In this notation, the Hamilton's equations (2.11) can be written as

$$\dot{z} = \{z, H\}.$$

More general, a Poisson structure on a manifold P is a skew-symmetric bilinear map which takes two differentiable functions H and F to a new one

$$\{.,.\} : \mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P}), \tag{2.14}$$

satisfying both of the Jacobi identity

$$\{F, \{H, G\}\} + \{H, \{G, F\}\} + \{G, \{F, H\}\} \tag{2.15}$$

and the Leibnitz identity

$$\{F_1F_2, G\} = \{F_1, G\}\,F_2 + \{F_2, G\}\,F_1. \tag{2.16}$$

A manifold equipped with a Poisson bracket is called a Poisson manifold.

A symplectic manifold is necessarily a Poisson manifold with the introduction of the non-degenerate Poisson bracket

$$\{H, K\} := \Omega\,(X_H, X_K)\,. \tag{2.17}$$

A Poisson manifold is not a symplectic manifold necessarily due to the non-degeneracy condition on the two-form. Actually, a local picture of a Poisson manifold foliates into a product space whose leafs are symplectic [10].

2.3. THE LEGENDRE TRANSFORMATION

At least in a theoretical level, one expects that the Euler-Lagrange equations (2.5) and the Hamilton's equations (2.11) be in relation. Unfortunately, in practice, constructing the passage between the Lagrangian and Hamiltonian formulations is not so straight forward. This may be achieved by the Legendre transformation defined in terms of the fiber derivative

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2.18}$$

Geometrically, the fibre derivative maps the tangent bundle TQ into the cotangent bundle T^*Q, the phase space of Hamiltonian mechanics

$$\mathbb{F}L : TQ \rightarrow T^*Q : \left(q^i, \dot{q}^j\right) \rightarrow \left(q^i, \frac{\partial L}{\partial \dot{q}^j}\right). \tag{2.19}$$

In equation (2.18), all velocities $\dot{q}^i$ can be expressed uniquely in terms of momenta if the non-degeneracy condition holds that is if the rank of the Hessian matrix (2.8) is full. In this case, we arrive a Hamiltonian function

$$H(q^i, p_i) = p_j \dot{q}^j(q^i, p_i) - L(q^i, \dot{q}^i(q^j, p_j)). \tag{2.20}$$

depending on (q^i, p_i). The Euler-Lagrange equations (2.5) and the The Hamilton's equations (2.11) for the Hamiltonian function H presented in Eq.(2.20) coincide. To see this, take exterior derivative of the right hand side of equation (2.20). That is

$$d(p_i \dot{q}^i - L) = \dot{q}^i dp_i + p_i d\dot{q}^i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \tag{2.21}$$

using $p_i = \frac{\partial L}{\partial \dot{q}^i}$, the second and fourth term cancel so we have

$$d(p_i \dot{q}^i - L) = \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i. \tag{2.22}$$

On the other hand exterior derivative of left hand side of the equation (2.20)

$$dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i \tag{2.23}$$

Equivalence of differentials terms dp_i and dq^i in equations (2.22) and (2.23) give the Hamilton equations (2.11).

Obviously, for degenerate cases, all these calculations can not be performed. In the following section, we study Dirac-Bergmann constraint algorithm to arrive a Hamiltonian formulation for degenerate Lagrangian systems.

2.4. DIRAC-BERGMANN ALGORITHM

A (generalized) Hamiltonian formulation of degenerate first order Lagrangian systems was developed by Dirac [15], nowadays the method he proposed is called the Dirac-Bergmann constraint algorithm [40, 42, 43, 44].

For a given Lagrangian L, if the rank r of the hessian matrix in Eq.(2.8) is less then dimension n of Q, then all of the velocities cannot be solved in terms of the momentum variables. So that, the momentum variables are not all independent. Instead, there exists some relations

$$\Phi_m(q,p) = 0, \quad m = 1, ..., n-r, \tag{2.24}$$

called as primary constraints, [43]. They define a $n+r$ dimensional submanifold C, called the primary constraint submanifold, of $2n$ dimensional phase space T^*Q.

Hamiltonian $H(q^i, p_i)$ for the degenerate cases is far from being unique. One may add arbitrary linear combinations of the primary constraints to the Hamiltonian function. This leads to the total Hamiltonian

$$H_T(q,p) = H(q,p) + u^m(q,p)\Phi_m(q,p) \tag{2.25}$$

for arbitrary functions u^m called the Lagrange multipliers [15, 40, 41]. Note that, on the constraint submanifold C, the canonical Hamiltonian H and the total Hamiltonian H_T are coinciding.

Thus, in dealing with dynamics of total Hamiltonian H_T we need to evaluate quantities at $\Phi_m(q,p) = 0$. Note that, as it is stated in [40], we don't use constraints before working out Poisson brackets since primary constraints may have non-zero Poisson brackets with some canonical variables. To remind this, it is customary to use the weak equality symbol $\approx$. Accordingly, the Hamilton's equations of motion for H_T become

$$\dot{q}^i = \frac{\partial H_T}{\partial p_i} \approx \frac{\partial H}{\partial p_i} + u^m \frac{\partial \Phi_m}{\partial p_i} \tag{2.26}$$

$$\dot{p}_i = \frac{\partial H_T}{\partial q^i} \approx -\frac{\partial H}{\partial q^i} - u^m \frac{\partial \Phi_m}{\partial q^i} \tag{2.27}$$

where we set $\phi_m \approx 0$ on right hand sides after taking derivatives. Using the canonical Poisson bracket, we can rewrite the Hamilton equations (2.26) and (2.27) as

$$\dot{q}^i = \{q^i, H_T\} \approx \{q^i, H\} + u^m \{q^i, \Phi_m\} \tag{2.28}$$

$$\dot{p}_i = \{p^i, H_T\} \approx \{p_i, H\} + u^m \{p_i, \Phi_m\}. \tag{2.29}$$

More generally, the evolution of an arbitrary function $F(q^i, p_i)$ on the phase space is

$$\begin{aligned} \dot{F} &\approx \frac{\partial F}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} \\ &\approx \frac{\partial F}{\partial q^i}\left(\frac{\partial H}{\partial p_i} + u^m \frac{\partial \Phi_m}{\partial p_i}\right) + \frac{\partial F}{\partial p_i}\left(-\frac{\partial H}{\partial q^i} - u^m \frac{\partial \Phi_m}{\partial q^i}\right) \\ &\approx \{F, H\} + u^m \{F, \Phi_m\} \end{aligned} \tag{2.30}$$

using (2.26) and (2.27).

2.4.1. Consistency Conditions

In particular, taking F in (2.30) to be a primary constraint Φ_m, we obtain a consistency condition

$$\dot{\Phi}_m \approx \{\Phi_m, H\} + u^j \{\Phi_m, \Phi_j\} \approx 0, \tag{2.31}$$

see, for example, [15, 40, 41, 42]. By repeating this for $m = 1, ..., n - r$, we arrive a set of nonhomogeneous linear equations in the unknown u^j's. The following cases may arise from these equations:

(i) They may be inconsistent $1 \approx 0$ and do not possess any solution for u^j.

(ii) They may be a set of equations $0 \approx 0$.

(iii) They may be a set of equations independent of u^j's and Φ_m's. In this case, we have that $\det(\{\Phi_m, \Phi_j\}) \approx 0$ and $\{\Phi_m, H\} \neq 0$. Thus, Eq.(2.31) define new constraints

$$\phi_r(q^i, p_i) \approx 0 \tag{2.32}$$

called the secondary constraints. Note that, by definition, the secondary constraints are independent from the primary ones. Add these secondary constraints into the total Hamiltonian H_T in (2.25), and define

$$H_T^1 = H + u^m \Phi_m + v^r \phi_r \tag{2.33}$$

with Lagrange multipliers v^r. The set of consistency conditions must be extended to include secondary constraints. Consistency of secondary constraints leads to

$$\{\phi_r, H_T^1\} = \{\phi_r, H\} + u^m \{\phi_r, \Phi_m\}^+ v^s \{\phi_r, \phi_s\} \approx 0 \tag{2.34}$$

which may either imply new (tertiary) constraint or may restrict the multipliers u^m's or v^r's. Repeating this process, one enlarges the primary constraint set with the new (secondary, tertiary, ...) constraints, redefines H_T by introducing new Lagrange multipliers for new constraints and by repeating the consistency computations. This process ends when no more new constraint arises.

(iv) They may be a set of equations for the unknown multipliers u^j's.

(i) $\det(\{\Phi_m, \Phi_j\}) \neq 0$ and $\{\Phi_m, H\} \neq 0$.

In this case u^j's are uniquely fixed to be

$$u^j \approx -\left(M^{-1}\right)^{jm} \{\phi_m, H\} \tag{2.35}$$

where $M = \{\Phi_m, \Phi_j\}$. The equations of motion become

$$\dot{F} \approx \{F, H\} - \{F, \Phi_m\} \left(M^{-1}\right)^{mj} \{\Phi_j, H\} \tag{2.36}$$

for any function F.

(ii) det $(\{\Phi_m, \Phi_j\}) \approx 0$ and $\{\Phi_m, H\} \approx 0$.

In this case a homogeneous system of equations for u^js are obtained and a non-trivial solution exists.

At the end, we are faced with the whole set of constraints (primary, secondary etc...)

$$\Psi_j = \Phi_m \cup \phi_l \approx 0 \quad j = 1, ..., n - r + k = J \tag{2.37}$$

and total Hamiltonian with determined and undetermined Lagrange multipliers. Using this final total Hamiltonian we can find equation of motion using (2.28) and (2.29). One has to be careful about not to use constraints before evaluating the equations of motion, that means, first, we have to evaluate equations of motion, then we can use the constraints.

2.4.2. Dirac Bracket

We may classify the set of all constraints (2.37) into two classes by evaluating the Poisson bracket of the constraints. If the Poisson brackets of $F(q^i, p_i)$ with all $\{\Psi_j\}$ vanish modulo the constraint

$$\{F, \Psi_j\} \approx 0, \quad j = 1, ..., J \tag{2.38}$$

then $F(q^i, p_i)$ is a called first class constraint. Otherwise, it is called a second class constraint, [15, 40]. Surely, this classification is possible only after all constraints have been found.

- *Second Class Constraints*

  Assume that, there is no first class constraints, that is consider the case where all constraints are the second class. Let us denote them by (Ψ_s). The Poisson bracket of these constraints will form a nonsingular matrix $M = [M_{st}]$ with

  $$M_{st} = \{\Psi_s, \Psi_t\}.$$

  Accordingly, the Dirac bracket is defined by

  $$\{F, G\}_{DB} = \{F, G\} - \{F, \Psi_s\} M^{st} \{\Psi_t, G\} \tag{2.39}$$

where M^{st} is components of the inverse matrix of $[M_{st}]$. In this formulation, one can use the constraints before evaluating the bracket. This means that, the weak equality becomes a strong equality, since $\{\Psi_j, F\}_{DB} = 0$. With the help of Dirac bracket (2.39) the equations of motion can be written as

$$\dot{z} \approx \{z, H\}_{DB} \tag{2.40}$$

- *First Class Constraints*

  Contrary, assume that, there is no second class constraint. In this case, we can divide the whole first class constraints as primary first class Ψ_m and the others Ψ_a. As it is stated in [40], the first class constraints do not change the state. They just lead to arbitrary functions in the general solution of the equations of motion. These unwanted degrees of freedom can be eliminated by using the Dirac bracket and by redefining total Hamiltonian H_E.

- *First and Second Class Constraints*

  Assume that the rank R of the Poisson bracket matrix of all constraints Ψ_j (primary and secondary) in (2.37)

$$\{\Psi_i, \Psi_j\} \approx N_{mn} \neq 0 \tag{2.41}$$

is less than J. That is, we have R number of second class and $J - R$ number of first class constraints. Due to Dirac [15, 40], try to make a transformations of second class

$$\Psi_s^* \approx \nu_s^{s'} \Psi_{s'}, \quad s = 1, ..., R \tag{2.42}$$

so as to bring as many second class constraints as possible into the first class. Let us call second class constraints which cannot bring into first class as Ψ_α and their Poisson bracket with each other leads to non singular matrix $N_{\alpha\beta} \approx \{\Psi_\alpha, \Psi_\beta\}$, thus we get Euler-Lagrange equations of motion using inverse of this matrix and Dirac bracket as it is done when all constraints are second class case.

3. THE SECOND ORDER THEORY

3.1. THE SECOND ORDER LAGRANGIAN DYNAMICS

Let Q be an n-dimensional differentiable manifold with coordinates (q^i). The 2nd order tangent bundle

$$T^2Q = \cup T_q^2 Q$$

is $3n$-dimensional manifold with induced coordinates consisting of positions, velocities and accelerations. The induced local chart looks like

$$(q, \dot{q}, \ddot{q},) = (q^1, ..., q^n, \dot{q}^1, ..., \dot{q}^n, \ddot{q}^1, ..., \ddot{q}^n) \in T^2Q.$$

There are projections given by

$$ {}^2_0\tau_Q \quad : \quad T^2Q \to Q : (q, \dot{q}, \ddot{q}) \to (q) \tag{3.1}$$

$$ {}^2_1\tau_Q \quad : \quad T^2Q \to TQ : (q, \dot{q}, \ddot{q}) \to (q, \dot{q})\,. \tag{3.2}$$

Note that T^2Q can be embedded into the iterated tangent bundle $T(TQ)$ with coordinates $(q^i, V^i, \dot{q}^i, \dot{V}^i)$ through the identification $V^i = \dot{q}^i$.

A second order Lagrangian density $L[q] = L(q, \dot{q}, \ddot{q})$ is a function on the second order tangent bundle T^2Q. The functional differential of $L[q]$ with respect to q is

$$\begin{aligned} d(L[q]dt) &= \left(\frac{\partial L}{\partial q^i} q^i + \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i + \frac{\partial L}{\partial \ddot{q}^i}\ddot{q}^i\right) \\ &= \varepsilon_{q^i}(L[q])dq^i + \frac{d}{dt}\theta_L[q] \end{aligned} \tag{3.3}$$

where the first term gives Euler-Lagrange equations

$$\varepsilon_{q^i}(L[q]dt) \equiv \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^i} = 0 \tag{3.4}$$

and the boundary term is the total derivative of the functional one-form

$$\theta_L[q^i] \equiv \frac{\partial L}{\partial \ddot{q}^i} d\dot{q}^i + (\frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}^i}) q^i. \tag{3.5}$$

Similar to the first order theory, for the Lagrangians resulting in the Euler-Lagrange equations (3.4), θ_L is not unique. However, its functional exterior derivative

$$\Omega_L[q] \equiv d\theta_L[q] \tag{3.6}$$

is a well defined two-form on T^2M. A second order Lagrangian L is called to be degenerate if the extended or generalized Hessian matrix

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \ddot{q}^i \partial \ddot{q}^j} \tag{3.7}$$

is a singular matrix, with rank $r < n$ otherwise it is non-degenerate.

3.2. JACOBI-OSTROGRADSKY MOMENTA

On the dual picture, the momentum phase space T^*TQ is a canonical symplectic manifold with coordinates $(q^i, \dot{q}^i, p_i^0, p_i^1)$. Hence it is endowed with the canonical Poisson bracket which results in the fundamental Poisson bracket relations

$$\{q^i, p_j^0\} = \{\dot{q}^i, p_j^1\} = \delta_j^i \tag{3.8}$$

and, all the others are zero.

The form of the Lagrangian one-form θ_L in (3.5) already suggests that we can introduce the momenta for a second order Lagrangian as

$$\begin{aligned} p_i^0[q] &= \frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}^i} \\ &= \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial^2 L}{\partial \ddot{q}^i \partial q^j}\dot{q}^j - \frac{\partial^2 L}{\partial \ddot{q}^i \partial \dot{q}^j}\ddot{q}^j - \frac{\partial^2 L}{\partial \ddot{q}^i \partial \ddot{q}^j}\dddot{q}^j \end{aligned} \tag{3.9}$$

$$p_i^1[q] = \frac{\partial L}{\partial \ddot{q}^i}, \tag{3.10}$$

which are called the Jacobi-Ostrogradsky momenta [45]. Conjugated respectively to

q^i and $\dot{q}^i$. Lagrangian one-form

$$\theta_L[q] \equiv p_j^0[q]dq^j + p_j^1[q]d\dot{q}^j \tag{3.11}$$

is the pull back of the canonical (Liouville) one-form θ_{T^*TQ} by the Legendre map,

$$\mathbb{F}L : T^3Q \longrightarrow T^*TQ \tag{3.12}$$

where T^3Q carries the local coordinates $(q^i, \dot{q}^i, \ddot{q}^i, \dddot{q}^i)$.

If the extended Hessian matrix (3.7) is non-singular, then we can express $\ddot{q}^i$ and $\dddot{q}^i$ as functions of the Ostrogradsky momenta given by

$$\ddot{q}^i = \ddot{q}^i(q^i, \dot{q}^i, p_i^1) \tag{3.13}$$

$$\dddot{q}^i = \dddot{q}^i(q^i, \dot{q}^i, p_i^0, p_i^1) \tag{3.14}$$

The canonical Hamiltonian H for a second order non-degenerate Lagrangian is given by

$$H \equiv p_i^0\dot{q}^i + p_i^1\ddot{q}^i - L(q^i, \dot{q}^i, \ddot{q}^i). \tag{3.15}$$

whereas the Hamilton's equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i^0}, \quad \ddot{q}^i = \frac{\partial H}{\partial p_i^1} \tag{3.16}$$

$$\dot{p}_i^0 = -\frac{\partial H}{\partial q^i}, \quad \dot{p}_i^1 = -\frac{\partial H}{\partial \dot{q}^i}. \tag{3.17}$$

are equivalent to fourth order Lagrange equations of motion (3.4) [45]

3.2.1. Second Order Degenerate Lagrangians

Assume that rank of the extended Hessian matrix W_{ij} is $r < n$, this means it is possible to solve only r number of $\ddot{q}^i$'s, in other words, $n - r$ number of the momenta are dependent according to the set of the primary constraints

$$\Phi_\alpha(q^i, \dot{q}^i, p^1) \approx 0, \quad \alpha = 1, ..., n-r \tag{3.18}$$

follow from (3.10). It is also possible to arrive some constraints $\eta(q^i, \dot{q}^i, p_i^0) \approx 0$ from equation (3.9) if they are independent of $\ddot{q}^i$. But all such constraints can also be derived as secondary constraints [62]. We remark that, if p^0 depends additionally on $\ddot{q}$, then $\ddot{q}$ should be solved as a function of $(q, \dot{q}, p^0)$.

Using primary constraints, total Hamiltonian is

$$H_T = H + u^\alpha \Phi_\alpha \tag{3.19}$$

where u^α's are the Lagrange multipliers, and H is the canonical Hamiltonian in (3.15). Procedure after this point is the same with first order singular theory, c.f. the section 2.4. Check the consistency of each primary constraint Φ_α to get secondary constraint or determine Lagrange multipliers. Once the Dirac-Bergmann algorithm is ended after the substitution of the determined Lagrange multipliers u^α into the total Hamiltonian (3.19), equations of motion are written as

$$\dot{q}^i \approx \{q^i, H_T\}, \quad \ddot{q}^i \approx \{\dot{q}^i, H_T\} \tag{3.20}$$

$$\dot{p}_i^0 \approx \{p_i^0, H_T\}, \quad \dot{p}_i^1 \approx \{p_i^1, H_T\} \tag{3.21}$$

on the final constraint submanifold. In this higher order case, it is also possible to define Dirac Poisson bracket as well after the complete set of constraints is determined c.f. section 2.4.2.

3.2.2. Reduction to First Order Formalism

A second order Lagrangian on T^2Q may be treated as a constraint first order Lagrangian on the iterated tangent bundle TTQ equipped with a local frame

$$q^i \equiv Q_0^i, \quad \dot{q}^i \equiv \dot{Q}_0^i \equiv Q_1^i, \ (Q_0^i, Q_1^i) \in TQ \equiv M. \tag{3.22}$$

Tangent and cotangent bundles are $TM = TTQ$ and $T^*M = T^*TQ$ with local coordinates

$$\begin{aligned}(Q_0^i, Q_1^i, \dot{Q}_0^i, \dot{Q}_1^i) &= (q^i, \dot{q}^i, \dot{q}^i, \ddot{q}^i) \in TM \\ (Q_0^i, Q_1^i, P_i^0, P_i^1) &= (Q, P) \in T^*M.\end{aligned}$$

The Lagrangian L on T^2Q will become one of the first order degenerate Lagrangians

$$L_c^0(Q_0^i, Q_1^i, \dot{Q}_0^i, \dot{Q}_1^i, \lambda) \equiv L^0(Q_0^i, \dot{Q}_0^i, \dot{Q}_1^i) + \lambda_i^0(\dot{Q}_0^i - Q_1^i) \tag{3.23}$$

$$L_c^1(Q_0^i, Q_1^i, \dot{Q}_0^i, \dot{Q}_1^i, \lambda) \equiv L^1(Q_0^i, Q_1^i, \dot{Q}_1^i) + \lambda_i^1(\dot{Q}_0^i - Q_1^i) \tag{3.24}$$

on TM with the same constraint $Q_1^i - \dot{Q}_0^i = 0$ and with different Lagrange multipliers λ^0 or λ^1 depending on the variables $\dot{Q}_0$ or Q_1 adapted for the second order Lagrangian L. [21, 23, 26, 30]. Variations of both of the Lagrangian densities L_c^0 and L_c^1 with respect to Q_0^i give Euler-Lagrange equations (3.4)

$$\frac{\delta L^0}{\delta Q_0} = \frac{\partial L^0}{\partial Q_0^i} - \frac{d}{dt}\frac{\partial L^0}{\partial \dot{Q}_0^i} - \frac{d\lambda_i^0}{dt} = 0 \tag{3.25}$$

$$\frac{\delta L^1}{\delta Q_1} = \frac{\partial L^1}{\partial Q_0^i} - \frac{d\lambda_i^1}{dt} = 0 \tag{3.26}$$

in the variable $q^i = Q_0^i$, upon the use of constraint $Q_1^i = \dot{Q}_0^i$ and Lagrange multipliers λ_i^0, λ_i^1

$$\lambda_i^0 = -\frac{d}{dt}\frac{\partial L^0}{\partial \dot{Q}_1^i}, \quad \lambda_i^1 = \frac{\partial L^1}{\partial Q_1^i} - \frac{d}{dt}\frac{\partial L^1}{\partial \dot{Q}_1^i}. \tag{3.27}$$

obtained from variation with respect to Q_1^i. Note that the equations (3.25) and (3.26) give Euler-Lagrange equations of motion (3.4) but the definitions of the Lagrange multiplier λ_i^0 and λ_i^1 are changing. Finally variations of L_c^0 and L_c^1 with respect to λ_i^0 and λ_i^1 give constraint $\dot{Q}_0^i - Q_1^i$. Both L_c^0 and L_c^1 are degenerate since derivative of λ is not included. So for the Hamiltonian formalism, we have to apply Dirac analysis.

3.2.3. Hamiltonian Formalism for Reduced First Order Lagrangians

Hamiltonian Formalisms for L_c^1: Canonical momenta for L_c^1 are defined as

$$\Pi_i^0 = \frac{\partial L_c^1}{\partial \dot{Q}_0^i} = \lambda_i^1 \tag{3.28}$$

$$\Pi_i^1 = \frac{\partial L_c^1}{\partial \dot{Q}_1^i} = \frac{\partial L^1}{\partial \dot{Q}_1^i} \tag{3.29}$$

$$\Pi_\lambda^i = \frac{\partial L_c^1}{\partial \dot{\lambda}_i^1} = 0. \tag{3.30}$$

These are identical to Jacobi-Ostrogradsky momenta (3.9) and (3.10) using definition of λ_i^1 in (3.27) and $q^i = Q_0^i, \dot{Q}_0^i = Q_1^i$. The Lagrangian phase space is transformed to Hamiltonian phase space $T^*(TQ \times \mathbb{R}^n)$ with canonical coordinates $(Q_0^i, Q_1^i, \lambda_i^1, \Pi_i^0, \Pi_i^1, \Pi_\lambda^i)$ and satisfy the canonical Poisson bracket relations

$$\{Q_\alpha^i, \Pi_j^\beta\} = \delta_\alpha^\beta \delta_j^i, \quad \{\lambda_i^1, \Pi_\lambda^j\} = \delta_i^j. \tag{3.31}$$

$T^*(TQ \times \mathbb{R}^n)$ is canonical symplectic with the symplectic two form

$$\begin{aligned} \Omega_{T^*(TQ\times\mathbb{R}^n)} &= dQ_0^j \wedge d\Pi_j^0 + dQ_1^j \wedge d\Pi_j^1 + d\lambda_j^1 \wedge d\Pi_\lambda^j \\ &= d\theta_{T^*(TQ\times\mathbb{R}^n)} \end{aligned} \tag{3.32}$$

where $\theta_{T^*(Q\times\mathbb{R}^n)} = \Pi_j^0 dQ_0^j + \Pi_j^1 dQ_1^j + \Pi_\lambda^j d\lambda_j^1$ is the canonical one form.

Definition of momenta leads to primary constraints

$$\Phi_i^0 = \Pi_i^0 - \lambda_i^1 = 0 \tag{3.33}$$

$$\Phi_i^1(Q_0^i, Q_1^i, \Pi_i^1) = 0 \tag{3.34}$$

$$\Phi_\lambda^i = \Pi_\lambda^i = 0 \tag{3.35}$$

since neither of momenta is invertible as a function of canonical coordinates and momenta. The canonical Hamiltonian function for L_c^1 is

$$\begin{aligned} H &= \Pi_i^0 \dot{Q}_0^i + \Pi_i^1 \dot{Q}_1^i + \Pi_\lambda^i \dot{\lambda}_i^1 - L_c^1 \\ &= \Pi_i^1 \dot{Q}_1^i + \Pi_i^0 Q_1^i - L^1 \end{aligned} \tag{3.36}$$

using primary constraints. This canonical Hamiltonian function is also independent from $\dot{Q}_1^i$(For the proof see [62]). Then the total Hamiltonian is

$$H_T = H + u_0^i \phi_i^0 + u_1^i \phi_i^1 + u_i^\lambda \phi_\lambda^i \tag{3.37}$$

where u_0^i, u_1^i and u_i^λ are Lagrange multipliers. After this point, there is no differ much from first order singular theory. Apply Dirac procedure: first check the consistency condition for each primary constraint ϕ_i^0, ϕ_i^1 and ϕ_λ^i to find new constraints or to determine Lagrange multipliers u_i as it is explained in section 2.4 .

As an another interpretation, stated in the ref [30], consistency condition of ϕ_i^0 implies $u_0^i = 0$. Constraints $\phi_\lambda^i = \Pi_\lambda^i$ only effects the equation of motion for λ_i^1, that means we don't need to add this constraint to canonical Hamiltonian function H. Hence we are faced with the Hamiltonian

$$H_T = H + u_1^i \phi_i^1. \tag{3.38}$$

This H_T is completely equivalent to the Hamiltonian (3.19). So consistency of primary constraint will give same secondary constraint.

Hamiltonian Formalisms for L_c^0: For the alternative first order Hamiltonian formalism, Canonical momenta for L_c^0 are

$$\pi_i^0 = \frac{\partial L_c^0}{\partial \dot{Q}_0^i} = \frac{\partial L^0}{\partial \dot{Q}_0^i} + \lambda_i^0 \tag{3.39}$$

$$\pi_i^1 = \frac{\partial L_c^0}{\partial \dot{Q}_1^i} = \frac{\partial L^0}{\partial \dot{Q}_1^i} \tag{3.40}$$

$$\pi_\lambda^i = \frac{\partial L_c^0}{\partial \dot{\lambda}_i^0} = 0. \tag{3.41}$$

When we compare momenta Π and π, it is easy to see $\Pi_i^1 = \pi_i^1, \Pi_\lambda^i = \pi_\lambda^i, \Pi_i^0 = \pi_i^0$

using definitions of λ_i^0, λ_i^1 and constraint $\dot{Q}_0^i = Q_1^i$.

The canonical Hamiltonian for L_c^0

$$H = \pi_i^0 \dot{Q}_0^i + \pi_i^1 \dot{Q}_1^i + \pi_\lambda^i \dot{\lambda}_i^0 - L_c^0 \tag{3.42}$$

is equivalent to the Hamiltonian (3.36) using $\dot{Q}_0^i = Q_1^i$.

After this point, we have to discuss whether there exit any primary constraint or not from the definitions of momenta. It is exact to get constraint π_λ, since L_c^0 is not a function of $\dot{\lambda}^0$ it is not possible to solve $\dot{\lambda}^0$. But on the other hand, if it is possible to solve the velocities $\dot{Q}_0^i$ and $\dot{Q}_1^i$ from π^0 and π_i^1 then the total Hamiltonian is

$$H_T = H + u_i^\lambda \phi_\lambda^i. \tag{3.43}$$

Otherwise, if the momenta $\pi_i^0, \pi_i^1, \pi_\lambda^i$ can not be solved for velocities $\dot{Q}_0^i, \dot{Q}_1^i$ and $\dot{\lambda}_i^1$ there exist primary constraints

$$\phi_i^0 = \pi_i^0 - \frac{\partial L^0}{\partial \dot{Q}_0^i} - \lambda_i^0 = 0 \tag{3.44}$$

$$\phi_i^1(Q_0^i, Q_1^i, \lambda_i^0, \pi_i^0, \pi_i^1) = 0 \tag{3.45}$$

$$\phi_\lambda^i = \pi_\lambda^i = 0. \tag{3.46}$$

Then total Hamiltonian is

$$H_T = H + u_0^i \phi_i^0 + u_1^i \phi_i^1 + u_i^\lambda \phi_\lambda^i. \tag{3.47}$$

Using the total Hamiltonian (3.43) or (3.47) and the primary constraints we have to apply Dirac-Bergmann algorithm as it is given in the section 2.4.

3.2.4. Unconstraint Variational Formalism

It is also possible to express first order Lagrangians L_c^0 and L_c^1 in $(3.23)-(3.24)$ in λ free form on TTQ

$$L_U^0 \equiv L^0(Q_0, \dot{Q}_0, \dot{Q}_1) - (\frac{d}{dt}\frac{\partial L^0}{\partial \dot{Q}_1^i})(\dot{Q}_0^i - Q_1^i) \tag{3.48}$$

$$L_U^1 \equiv L^1(Q_0, Q_1, \dot{Q}_1) + (\frac{\partial L^1}{\partial Q_1^i} - \frac{d}{dt}\frac{\partial L^1}{\partial \dot{Q}_1^i})(\dot{Q}_0^i - Q_1^i) \tag{3.49}$$

substituting λ^0 and λ^1 in (3.27) into L_c^0 and L_c^1 in $(3.23)-(3.24)$. Note that, the constraint $\dot{Q}_0^i - Q_1^i = 0$ must not be used in L_U^0 or L_U^1, if it is done, unconstraint Lagrangian does not give consistent Euler-Lagrange equations. If second order Lagrangian L is nondegenerate, unconstraint L_U^0 and L_U^1 contains $\ddot{Q}_1$, thus reduction to first order does not mean anything. On the other hand if L is degenerate Lagrangian, L_U^0 and L_U^1 will be of the first order.

Thus, the second order degenerate Lagrangian L with the third order Euler-Lagrange equations (3.4) for q^i is reduced to a first order Lagrangian L_U^0 and L_U^1 with two second order Euler-Lagrange equations obtained from variational derivative of L_U^0 and L_U^1 for $Q_0^i = q^i$ and $Q_1^i = \dot{Q}_0^i$. Note that variational derivative with respect to Q_1^i is satisfied identically and variational derivative with respect to Q_0^i gives the Euler-Lagrange equations (3.4) for $Q_0^i = q^i$.

We can apply Hamiltonian theory for first order unconstraint Lagrangians L_U^0 and L_U^1 as it is given in Section 2.3 and 2.4. Canonical momenta for L_U^0 and L_U^1 are defined as

$$p_i^0 = \frac{\partial L_U^0}{\partial \dot{Q}_0^i}, \quad p_i^1 = \frac{\partial L_U^0}{\partial \dot{Q}_1^i} \tag{3.50}$$

$$s_i^0 = \frac{\partial L_U^1}{\partial \dot{Q}_0^i}, \quad s_i^1 = \frac{\partial L_U^1}{\partial \dot{Q}_1^i}. \tag{3.51}$$

On the Hamiltonian phase space, canonical coordinates $(Q_0^i, Q_1^i, p_i^0, p_i^1)$ and $(Q_0^i, Q_1^i, s_i^0, s_i^1)$ satisfy canonical Poisson bracket relations

$$\{Q_0^i, p_j^0\} = \{Q_1^i, p_j^1\} = \{Q_0^i, s_j^0\} = \{Q_1^i, s_j^1\} = \delta_j^i. \tag{3.52}$$

The canonical Hamiltonian function for L_U^0 and L_U^1 can be written as

$$H = p_i^0 \dot{Q}_0^i + p_i^1 \dot{Q}_1^i - L_U^0, \quad H = s_i^0 \dot{Q}_0^i + s_i^1 \dot{Q}_1^i - L_U^1. \tag{3.53}$$

After this point, we have discuss if it is possible to solve velocities from definition of momenta p^i and s^i. Using these velocities we can write the Hamiltonian function and then the Hamilton equations of motion. Otherwise there exit primary constraints, we have to apply Dirac-Bergmann constraint algorithm to find the Hamilton equations of motion.

4. APPLICATIONS: THE TOPOLOGICAL MASSIVE GRAVITY

As an application of second order nondegenerate and degenerate theory, we will first study Pais-Uhlenbeck oscillator whose dynamics is described by the nondegenerate Lagrangians. We will then analyse degenerate Sarıoğlu-Tekin and Clement Lagrangians

4.1. PAIS-UHLENBECK OSCILLATOR

4.1.1. General Setting

To construct the geometric framework for the Pais-Uhlenbeck Oscillator (PUO) [39] , we start with a one-dimensional manifold M and the introduction of the following local coordinates

$$
\begin{aligned}
X &\in M, \\
(X, \dot{X}) &\in TM \\
(X, \dot{X}, \ddot{X}) &\in T^2M \\
(X, \dot{X}, \ddot{X}, \dddot{X}) &\in T^3M \\
(X, \dot{X}, P^0, P^1) &\in T^*TM.
\end{aligned}
$$

The dynamics of classical PUO can be obtained from the second order Lagrangian,

$$L^{PU} = \frac{1}{2}\Big[\ddot{X}^2 - (\Omega^2 + \omega^2)\dot{X}^2 + \omega^2\Omega^2 X^2\Big] \tag{4.1}$$

where ω and Ω are positive real parameters. This is a second order non-degenerate Lagrangian since the rank of $\frac{\partial^2 L^{PU}}{\partial \ddot{X}^2}$ is 1.

The Euler-Lagrange equations of motion (3.4)

$$\ddddot{X} + (\Omega^2 + \omega^2)\ddot{X} + w^2\Omega^2 X = 0 \tag{4.2}$$

are obtained by varying L^{PU} with respect to X.

4.1.2. Jacobi-Ostrogradsky Method

Proposition 4.1.1. *For the second order Lagrangian* (4.1)*, Jacobi-Ostrogradsky momenta* (3.9) *and* (3.10) *become*

$$P^0 = \frac{\partial L^{PU}}{\partial \dot{X}} - \frac{d}{dt}\left(\frac{\partial L^{PU}}{\partial \ddot{X}}\right) = -(\Omega^2 + \omega^2)\dot{X} - \dddot{X} \tag{4.3}$$

$$P^1 = \frac{\partial L^{PU}}{\partial \ddot{X}} = \ddot{X}. \tag{4.4}$$

Note that, the Legendre map is invertible in order to express fiber coordinates $(\ddot{X}, \dddot{X})$ in terms of Jacobi-Ostrogradsky momenta from the equations (4.4) and (4.3).

The Lagrangian one-form (3.11) turns out to be

$$\theta_L[X] \equiv -\left((\Omega^2 + w^2)\dot{X} + \dddot{X}\right) dX + \ddot{X} d\dot{X} \tag{4.5}$$

and the exterior derivative of θ_L is

$$\Omega_L = -(\Omega^2 + w^2) d\dot{X} \wedge dX + d\dddot{X} \wedge dX + d\ddot{X} \wedge d\dot{X}. \tag{4.6}$$

Proposition 4.1.2. *The canonical Hamiltonian function for (PUO) on* T^*TM *is*

$$H^{PU} = P^0 \dot{X} + \frac{1}{2}(P^1)^2 + \frac{1}{2}(\Omega^2 + \omega^2)\dot{X}^2 - \frac{1}{2}\omega^2\Omega^2 X^2. \tag{4.7}$$

The canonical Hamilton's equations are

$$\dot{P}^0 = \omega^2 \Omega^2 X \tag{4.8}$$

$$\dot{P}^1 = -P^0 - (\Omega^2 + \omega^2)\dot{X}. \tag{4.9}$$

Proof. Let us recall the Hamiltonian function defined in (3.15) and calculate

$$\begin{aligned} H^{PU} &= P^0 \dot{X} + P^1 \ddot{X} - L^{PU} \\ &= P^0 \dot{X} + P^1 \ddot{X} - \frac{1}{2}\ddot{X}^2 + \frac{1}{2}(\Omega^2 + \omega^2)\dot{X}^2 - \frac{1}{2}\omega^2\Omega^2 X^2. \end{aligned}$$

By substituting $X^{(2)} = P^1$, we have

$$H^{PU} = P^0\dot{X} + \frac{1}{2}(P^1)^2 + \frac{1}{2}(\Omega^2+\omega^2)\dot{X}^2 - \frac{1}{2}\omega^2\Omega^2X^2.$$

The Hamilton equations of motion are

$$\dot{X} = \{X, H^{PU}\} = \dot{X} \tag{4.10}$$

$$\ddot{X} = \{\dot{X}, H^{PU}\} = P^1 \tag{4.11}$$

$$\dot{P}^0 = \{P^0, H^{PU}\} = \omega^2\Omega^2X \tag{4.12}$$

$$\dot{P}^1 = \{P^1, H^{PU}\} = -P^0 - (\Omega^2+\omega^2)\dot{X} \tag{4.13}$$

From these equations first, second and last one are satisfied identically but equation (4.12) gives Euler-Lagrange equation (4.14)

$$\begin{aligned}\dot{P}^0 &= \omega^2\Omega^2X \\ -(\Omega^2+\omega^2)\ddot{X} - \ddddot{X} &= \omega^2\Omega^2X \end{aligned} \tag{4.14}$$

using the definition of P^0. □

4.1.3. The First Order Formalisms (Constraint Canonical Formalism)

It is possible to reduce the second-order non-degenerate Lagrangian L^{PU} in into two first-order degenerate Lagrangian functions (c.f. (3.23) and (3.24)) as follows

$$L_0^{PU} = \frac{1}{2}\Big[\dot{Q}_1^2 - (\Omega^2+\omega^2)\dot{Q}_0^2 + \omega^2\Omega^2Q_0^2\Big] + \lambda^0(\dot{Q}_0 - Q_1) \tag{4.15}$$

$$L_1^{PU} = \frac{1}{2}\Big[\dot{Q}_1^2 - (\Omega^2+\omega^2)Q_1^2 + \omega^2\Omega^2Q_0^2\Big] + \lambda^1(\dot{Q}_0 - Q_1) \tag{4.16}$$

after the introduction of the coordinates $X = Q_0, \dot{X} = \dot{Q}_0 = Q_1$ with the Lagrange multipliers λ^0 and λ^1. Variations of L_0^{PU} and L_1^{PU} with respect to Q_0

$$\frac{\delta L_0^{PU}}{\delta Q_0} = \frac{\partial L_0^{PU}}{\partial Q_0} - \frac{d}{dt}\frac{\partial L_0^{PU}}{\partial \dot{Q}_0} = \omega^2\Omega^2 Q_0 + (\omega^2+\Omega^2)\ddot{Q}_0 - \dot{\lambda}^0 \tag{4.17}$$

$$\frac{\delta L_1^{PU}}{\delta Q_0} = \frac{\partial L_1^{PU}}{\partial Q_0} - \frac{d}{dt}\frac{\partial L_1^{PU}}{\partial \dot{Q}_0} = \omega^2\Omega^2 Q_0 - \dot{\lambda}^1 \tag{4.18}$$

result with the equations of motion (4.14) using

$$\lambda^0 = -\ddot{Q}_1 \tag{4.19}$$

$$\lambda^1 = -(\omega^2+\Omega^2)Q_1 - \ddot{Q}_1 \tag{4.20}$$

determined from the variations of L_0^{PU} and L_1^{PU} with respect to Q_1.

Hamiltonian Formalism for L_0^{PU}: On the Hamiltonian phase space canonical coordinates $(Q_0, Q_1, \lambda^0, \pi^0, \pi^1, \pi_\lambda)$ satisfy the canonical Poisson bracket relations

$$\{Q_i, \pi^j\} = \delta_i^j, \quad \{\lambda^0, \pi_\lambda\} = 1. \tag{4.21}$$

For the first order Lagrangian L_0^{PU} in Eq.(4.15), the conjugate momenta to the coordinates (Q_0, Q_1, λ^0) are defined by

$$\pi^0 = \frac{\partial L_0^{PU}}{\partial \dot{Q}_0} = \lambda^0 - (\Omega^2+\omega^2)\dot{Q}_0 \tag{4.22}$$

$$\pi^1 = \frac{\partial L_0^{PU}}{\partial \dot{Q}_1} = \dot{Q}_1 \tag{4.23}$$

$$\pi_\lambda = \frac{\partial L_0^{PU}}{\partial \dot{\lambda}^0} = 0. \tag{4.24}$$

Proposition 4.1.3. *The total Hamiltonian function for the first order Lagrangian L_0^{PU} in Eq.(4.15) is given by*

$$\begin{aligned} H_{T^0}^{PU} &= \frac{1}{2}(\pi^1)^2 - \frac{(\pi^0-\lambda^0)^2}{2(\Omega^2+\omega^2)} - \frac{\omega^2\Omega^2}{2}(Q_0)^2 + \lambda^0 Q_1 \\ &+ \pi_\lambda\big[\Omega^2\omega^2 Q_0 - \pi^1(\Omega^2+\omega^2)\big] + \big[\frac{\lambda^0-\pi^0}{(\Omega^2+\omega^2)} - Q_1\big]^2. \end{aligned} \tag{4.25}$$

Proof. From the conjugate momenta π^0 and π^1 it possible to solve the velocities $\dot{Q}_0$ and $\dot{Q}_1$

$$\dot{Q}_0 = \frac{\lambda^0 - \pi^0}{\Omega^2 + \omega^2}, \quad \dot{Q}_1 = \pi^1 \tag{4.26}$$

and the remaining momenta leads to a primary constraint $\Phi_\lambda = \pi_\lambda$. The canonical Hamiltonian function (3.15) for L_0^{PU} is

$$\begin{aligned} H_0^{PU} &= \pi^0\dot{Q}_0 + \pi^1\dot{Q}_1 + \pi_\lambda\dot{\lambda}^0 - L_0^{PU} \\ &= \pi^0\dot{Q}_0 + \pi^1\dot{Q}_1 + \pi_\lambda\dot{\lambda}^0 - \frac{1}{2}\Big[\dot{Q}_1^2 - (\Omega^2+\omega^2)\dot{Q}_0^2 + \omega^2\Omega^2 Q_0^2\Big] - \lambda^0(\dot{Q}_0 - Q_1) \end{aligned}$$

substituting $\dot{Q}_0$ and $\dot{Q}_1$, the Hamiltonian function equals to

$$H_0^{PU} = \frac{1}{2}(\pi^1)^2 - \frac{1}{2}\frac{(\pi^0-\lambda^0)^2}{\Omega^2+\omega^2} - \frac{\omega^2\Omega^2}{2}(Q_0)^2 + \lambda^0 Q_1. \tag{4.27}$$

Define the total Hamiltonian

$$H_{T^0}^{PU} = H_0^{PU} + u^\lambda\Phi_\lambda \tag{4.28}$$

by adding primary constraint Φ_λ with undetermined multiplier u^λ. The conservation

$$\begin{aligned} \dot{\Phi}_\lambda = \{\Phi_\lambda, H_{T^0}^{PU}\} &\approx \{\Phi_\lambda, H_0^{PU}\} + u^\lambda\{\Phi_\lambda, \Phi_\lambda\} \\ &\approx \frac{\lambda^0 - \pi^0}{(\Omega^2+\omega^2)} - Q_1 \end{aligned} \tag{4.29}$$

of the primary constraint Φ_λ leads to a secondary constraint

$$\Phi = \frac{\lambda^0 - \pi^0}{(\Omega^2+\omega^2)} - Q_1 \tag{4.30}$$

using canonical Poisson bracket relations (4.21) and $\{\Phi_\lambda, \Phi_\lambda\} = 0$. By adding the secondary constraint Φ with a Lagrange multiplier u, we revise the total Hamiltonian as

$$H_{T^0}^{PU} = H_0^{PU} + u^\lambda\Phi_\lambda + u\Phi. \tag{4.31}$$

The conservation

$$\begin{aligned} \dot{\Phi} = \{\Phi, H_{T^0}^{PU}\} &\approx \{\Phi, H_0^{PU}\} + u^\lambda\{\Phi, \Phi_\lambda\} + u\{\Phi, \Phi\} \\ &\approx -\frac{\Omega^2\omega^2}{\Omega^2+\omega^2}Q_0 - \pi^1 + \frac{u^\lambda}{\Omega^2+\omega^2} \end{aligned} \tag{4.32}$$

of the secondary constraint makes it possible to determine u^λ, and the conservation

$$\begin{aligned}\dot{\Phi}_\lambda = \{\Phi_\lambda, H^{PU}_{T^0}\} &\approx \{\Phi_\lambda, H^{PU}_0\} + u^\lambda\{\Phi_\lambda, \Phi_\lambda\} + u\{\Phi_\lambda, \Phi\} \\ &\approx \Phi - u. \end{aligned} \tag{4.33}$$

of Φ detemines u. The proof will be ended by the substitutions of the determined Lagrange multipliers u and u^λ into the Hamiltonian function (4.31). □

Proposition 4.1.4. *The Hamilton's equations generated by the total Hamiltonian* (4.25) *are*

$$\dot{Q}_0 \approx -\frac{\lambda^0 - \pi^0}{\Omega^2 + \omega^2}, \quad \dot{Q}_1 \approx \pi^1, \quad \dot{\lambda}^0 \approx \Omega^2\omega^2 Q_0 + (\Omega^2 + \omega^2)\pi^1 \tag{4.34}$$

$$\dot{\pi}^0 \approx \Omega^2\omega^2 Q_0, \quad \dot{\pi}^1 \approx -\lambda^0, \quad \dot{\pi}^\lambda \approx 0. \tag{4.35}$$

Proof. We compute

$$\dot{Q}_0 = \{Q_0, H^{PU}_{T^0}\} = -\frac{\lambda^0 - \pi^0}{\Omega^2 + \omega^2} - \Phi \approx -\frac{\lambda^0 - \pi^0}{\Omega^2 + \omega^2} \tag{4.36}$$

$$\dot{Q}_1 = \{Q_1, H^{PU}_{T^0}\} = \pi^1 \tag{4.37}$$

$$\dot{\lambda}^0 = \{\lambda^0, H^{PU}_{T^0}\} = \Omega^2\omega^2 Q_0 - (\Omega^2 + \omega^2)\pi^1 \tag{4.38}$$

$$\dot{\pi}^0 = \{\pi^0, H^{PU}_{T^0}\} = \Omega^2\omega^2 Q_0 - \Omega^2\omega^2 Q_0\pi_\lambda \approx \Omega^2\omega^2 Q_0 \tag{4.39}$$

$$\dot{\pi}^1 = \{\pi^1, H^{PU}_{T^0}\} = -\lambda^0 + \Phi \approx -\lambda^0 \tag{4.40}$$

$$\dot{\pi}^\lambda = \{\pi^\lambda, H^{PU}_{T^0}\} = 0. \tag{4.41}$$

From these equations $\dot{\lambda}^0$ and $\dot{\pi}^0$ give the Euler- Lagrange equations of motion whereas all the others are satisfied identically. □

Dirac Bracket Formalism for L^{PU}_0: All the constraints

$$\Phi_\lambda = \pi^\lambda, \quad \Phi = \frac{\lambda^0 - \pi^0}{(\Omega^2 + \omega^2)} - Q_1 \tag{4.42}$$

are second class since their poisson bracket $\{\Phi_\lambda, \Phi\} = -\frac{1}{\Omega^2+\omega^2}$ is nonzero.

Proposition 4.1.5. *Dirac brackets of coordinates are*

$$\begin{aligned} \{Q_0, \lambda^0\}_{DB} &= 1 && (4.43)\\ \{Q_0, \pi^0\}_{DB} &= 1 && (4.44)\\ \{Q_1, \pi^1\}_{DB} &= 1 && (4.45)\\ \{\lambda^0, \pi^1\}_{DB} &= \Omega^2 + \omega^2 && (4.46) \end{aligned}$$

and all the others zero.

Proof. By recalling the general form of the Dirac bracket presented in (2.39), we compute

$$\{F, G\}_{DB} = \{F, G\} - (\Omega^2 + \omega^2)\,[\{F, \Phi_\lambda\}\{\Phi, G\} - \{F, \Phi\}\{\Phi_\lambda, G\}] \tag{4.47}$$

after the substitution of the inverse of

$$M = \begin{bmatrix} \{\Phi_\lambda, \Phi_\lambda\} & \{\Phi_\lambda, \Phi\} \\ \{\Phi, \Phi_\lambda\} & \{\Phi, \Phi\} \end{bmatrix} = \frac{1}{(\Omega^2 + \omega^2)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

To find the Dirac brackets of the coordinates, we evaluate the Poisson brackets of the coordinates

$$\begin{aligned} \{Q_0, \Phi_\lambda\} &= \{Q_1, \Phi_\lambda\} = \{Q_1, \Phi\} = \{\lambda^0, \Phi\} = 0 \\ \{\pi^0, \Phi_\lambda\} &= \{\pi^0, \Phi\} = \{\pi^1, \Phi_\lambda\} = \{\pi_\lambda, \Phi_\lambda\} = 0 \\ \{Q_0, \Phi\} &= \{\pi_\lambda, \Phi\} = \frac{-1}{\Omega^2 + \omega^2} \\ \{\lambda^0, \Phi_\lambda\} &= \{\pi^1, \Phi\} = 1. \end{aligned}$$

Using these relations and the Dirac bracket (4.47), we evaluate, for instance, the Dirac bracket of Q_0 and Q_0

$$\begin{aligned} \{Q_0, Q_0\}_{DB} &= \{Q_0, Q_0\} - (\Omega^2 + \omega^2)\,[\{Q_0, \Phi_\lambda\}\{\Phi, Q_0\} - \{Q_0, \Phi\}\{\Phi_\lambda, Q_0\}] \\ &= 0 \end{aligned}$$

where we employ $\{Q_0, Q_0\} = \{Q_0, \Phi_\lambda\} = 0$. The Dirac bracket of Q_0 and Q_1 is

$$\begin{aligned}\{Q_0, Q_1\}_{DB} &= \{Q_0, Q_1\} - (\Omega^2 + \omega^2)\left[\{Q_0, \Phi_\lambda\}\{\Phi, Q_1\} - \{Q_0, \Phi\}\{\Phi_\lambda, Q_1\}\right] \\ &= 0\end{aligned}$$

since $\{Q_0, Q_1\} = \{Q_0, \Phi_\lambda\} = \{Q_1, \Phi_\lambda\} = 0$. The Dirac bracket of Q_0 and λ^0 is

$$\begin{aligned}\{Q_0, \lambda^0\}_{DB} &= \{Q_0, \lambda^0\} - (\Omega^2 + \omega^2)\left[\{Q_0, \Phi_\lambda\}\{\Phi, \lambda^0\} - \{Q_0, \Phi\}\{\Phi_\lambda, \lambda^0\}\right] \\ &= 1\end{aligned}$$

since $\{Q_0, \lambda^0\} = \{Q_0, \Phi_\lambda\} = 0$ and $\{Q_0, \Phi\} = \frac{-1}{\Omega^2+\omega^2}$, $\{\Phi_\lambda, \lambda^0\} = 1$. In a similar fashion, one may continue to determine the Dirac brackets of the coordinates. □

Proposition 4.1.6. *By employing the Dirac bracket presented in the proposition (4.1.5), the Hamilton's equations of motion for the canonical Hamiltonian function* (4.27) *are*

$$\dot{Q}_0 = Q_1, \quad \dot{Q}_1 = \pi^1, \quad \dot{\lambda}^0 = (\Omega^2 + \omega^2)\pi^1 + \Omega^2\omega^2 Q_0 \tag{4.48}$$

$$\dot{\pi}^0 = \Omega^2\omega^2 Q_0, \quad \dot{\pi}^1 = -\pi^0 - (\Omega^2 + \omega^2)Q_1, \quad \dot{\pi}_\lambda = 0. \tag{4.49}$$

Proof. A direct calculation results with

$$\begin{aligned}\dot{Q}_0 = \{Q_0, H_0^{PU}\}_{DB} &= \left(\frac{\pi^0 - \lambda^0}{\Omega^2 + \omega^2} + Q_1\right)\{Q_0, \lambda^0\}_{DB} - \frac{\pi^0 - \lambda^0}{\Omega^2 + \omega^2}\{Q_0, \pi^0\}_{DB} \\ &= \left(\frac{\pi^0 - \lambda^0}{\Omega^2 + \omega^2} + Q_1\right) - \frac{\pi^0 - \lambda^0}{\Omega^2 + \omega^2} \\ &= Q_1\end{aligned}$$

since Dirac bracket of Q_0 only with λ^0 and π^0 is nonzero. Similarly, the equation of motion for other coordinates can be computed as

$$
\begin{aligned}
\dot{Q}_1 &= \{Q_1, H_0^{PU}\}_{DB} = \pi^1\{Q_1,\pi^1\}_{DB} = \pi^1 \\
\dot{\lambda}^0 &= \{\lambda^0, H_0^{PU}\}_{DB} = \pi^1\{\lambda^0,\pi^1\}_{DB} - \Omega^2\omega^2 Q_0\{\lambda^0, Q_0\}_{DB} \\
&= (\Omega^2+\omega^2)\pi^1 + \Omega^2\omega^2 Q_0 \\
\dot{\pi}^0 &= \{\pi^0, H_0^{PU}\} = -\Omega^2\omega^2 Q_0\{\pi^0, Q_0\}_{DB} = \Omega^2\omega^2 Q_0 \\
\dot{\pi}^1 &= \{\pi^1, H_0^{PU}\}_{DB} = \left(\frac{-1}{\Omega^2+\omega^2} + Q_1\right)\{\pi^1,\lambda^0\}_{DB} + \lambda^0\{\pi^1, Q_1\}_{DB} \\
&= -\pi^0 - (\Omega^2+\omega^2)Q_1 = -\lambda^0.
\end{aligned}
$$

□

Note that the equations of motion presented in the proposition (4.1.6) are the same with $(4.10)-(4.13)$ after the substitutions $X = Q_0, \dot{X} = Q_1$.

Hamiltonian formalism for L_1^{PU}: Similar to L_0^{PU}, it is also possible to derive Hamilton equations of motion for L_1^{PU}. The canonical coordinates $(Q_0, Q_1, \lambda^1, \Pi^0, \Pi^1, \Pi_\lambda)$ on the Hamiltonian phase space satisfy the canonical Poisson bracket relations

$$\{Q_i, \Pi^j\} = \delta_i^j, \quad \{\lambda^1, \Pi_\lambda\} = 1. \tag{4.50}$$

For the first order Lagrangian L_1^{PU} in equation (4.16) , the conjugate momenta to the coordinates Q_0, Q_1 and λ^1 are defined by

$$\Pi^0 = \frac{\partial L_1^{PU}}{\partial \dot{Q}_0} = \lambda^1 \tag{4.51}$$

$$\Pi^1 = \frac{\partial L_1^{PU}}{\partial \dot{Q}_1} = \dot{Q}_1 \tag{4.52}$$

$$\Pi_\lambda = \frac{\partial L_1^{PU}}{\partial \dot{\lambda}} = 0. \tag{4.53}$$

Proposition 4.1.7. *Total Hamiltonian function for the first order Lagrangian in equation* (4.16) *is defined by*

$$H_{T^1}^{PU} = \frac{1}{2}\Big[(\Pi^1)^2 + 2\Pi^0 Q_1 + (\Omega^2+\omega^2){Q_1}^2 - \omega^2\Omega^2 Q_0^2\Big] + \omega^2\Omega^2 Q_0 \Pi^\lambda.$$

Proof. Only Π^1 can be solved for velocity $\dot{Q}_1$ from the conjugate momenta, the other momenta lead to the primary constraints

$$\phi^0 = \Pi^0 - \lambda^1, \quad \phi_\lambda = \Pi_\lambda. \tag{4.54}$$

The canonical Hamiltonian function (3.15) for L_1^{PU} is

$$\begin{aligned} H_1^{PU} &= \Pi_\lambda \dot{\lambda}^1 + \Pi^1 \dot{Q}_1 + \Pi^0 \dot{Q}_0 - L_1^{ST} && (4.55)\\ &= \Pi_\lambda \dot{\lambda}^1 + \Pi^1 \dot{Q}_1 + \Pi^0 \dot{Q}_0 - \frac{1}{2}\Big[\dot{Q}_1^2 - (\Omega^2+\omega^2)Q_1^2 + \omega^2\Omega^2 Q_0^2\Big] \\ &- \lambda^1(\dot{Q}_0 - Q_1) && (4.56) \end{aligned}$$

replacing $\dot{Q}_1$ and $\pi^0 = \lambda^1 = \Pi_\lambda = 0$, the canonical Hamiltonian equals to

$$H_1^{PU} = \frac{1}{2}\Big[(\Pi^1)^2 + 2\Pi^0 Q_1 + (\Omega^2+\omega^2){Q_1}^2 - \omega^2\Omega^2 Q_0^2\Big]. \tag{4.57}$$

the total Hamiltonian is defined as

$$H_{T^1}^{PU} = H_1^{PU} + v_0\phi^0 + v^\lambda\phi_\lambda \tag{4.58}$$

by adding the primary constraints with undetermined multipliers v_0 and v^λ. The requirement that primary constraints are preserved in time lead to

$$\begin{aligned} \dot{\phi}^0 = \{\phi^0, H_{T^1}^{PU}\} &\approx \{\phi^0, H_1^{PU}\} + v_0\{\phi^0,\phi^0\} + v^\lambda\{\phi^0,\phi_\lambda\} \\ &\approx \omega^2\Omega^2 Q_0 - v^\lambda && (4.59)\\ \dot{\phi}^\lambda = \{\phi^\lambda, H_{T^1}^{PU}\} &\approx \{\phi^\lambda, H_1^{PU}\} + v_0\{\phi^\lambda,\phi^0\} + v^\lambda\{\phi^\lambda,\phi_\lambda\} \\ &\approx v_0 && (4.60) \end{aligned}$$

for consistency, thus the multiplier v_0 and v^λ become determined. Substituting these determined Lagrange multipliers into (4.58) the proof will be completed. □

Proposition 4.1.8. *Hamilton equations generated by the total Hamiltonian $H_{T^1}^{PU}$ given in the proposition* 4.1.7 *are*

$$\dot{Q}_0 = Q_1, \quad \dot{Q}_1 = \Pi^1, \quad \dot{\lambda}^1 = \omega^2\Omega^2 Q_0, \quad \dot{\Pi}_\lambda = \{\Pi_\lambda, H_T\} = 0 \tag{4.61}$$

$$\dot{\Pi}^0 \approx -\omega^2\Omega^2 Q_0, \quad \dot{\Pi}^1 = -\Pi^0 - (\Omega^2+\omega^2)Q_1. \tag{4.62}$$

Proof. Hamilton equations for coordinates are

$$\dot{Q}_0 = \{Q_0, H_T\} = Q_1 \tag{4.63}$$

$$\dot{Q}_1 = \{Q_1, H_T\} = \Pi^1 \tag{4.64}$$

$$\dot{\lambda}^1 = \{\lambda^1, H_T\} = \omega^2\Omega^2 Q_0 \tag{4.65}$$

$$\dot{\Pi}^0 = \{\Pi^0, H_T\} = -\omega^2\Omega^2(\Pi_\lambda - Q_0) \approx -\omega^2\Omega^2 Q_0 \tag{4.66}$$

$$\dot{\Pi}^1 = \{\Pi^1, H_T\} = -\Pi^0 - (\Omega^2 + \omega^2)Q_1 \tag{4.67}$$

$$\dot{\Pi}_\lambda = \{\Pi_\lambda, H_T\} = 0. \tag{4.68}$$

From these equations (4.66) and (4.67) give the equations of motion, all the others satisfied identically. □

Dirac Bracket Formalism for L_1^{PU}: All constraints are second class

$$\phi^0 = \Pi^0 - \lambda^1, \quad \phi_\lambda = \Pi_\lambda \tag{4.69}$$

since the Poisson bracket $\{\phi^0, \phi_\lambda\} = -1$ is nonzero.

Proposition 4.1.9. *Nonzero Dirac brackets of the coordinates and the momenta are*

$$\{Q_0, \lambda^1\}_{DB} = 1 \tag{4.70}$$

$$\{Q_0, \Pi^0\}_{DB} = 1 \tag{4.71}$$

$$\{Q_1, \Pi^1\}_{DB} = 1. \tag{4.72}$$

Proof. To prove these, we use the Dirac bracket (2.39)

$$\{F, G\}_{DB} = \{F, G\} - \{F, \phi^0\}\{\phi_\lambda, G\} + \{F, \phi_\lambda\}\{\phi^0, G\} \tag{4.73}$$

substuting inverse of

$$M = \begin{bmatrix} \{\phi^0, \phi^0\} & \{\phi^0, \phi_\lambda\} \\ \{\phi_\lambda, \phi^0\} & \{\phi_\lambda, \phi_\lambda\} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

To get Dirac brackets of the coordinates, we evaluate the Poisson brackets of coordinates with the constraints

$$\{Q_0, \phi^0\} = \{\lambda^1, \phi_\lambda\} = \{\pi_\lambda, \phi^0\} = 1$$

and all others are zero. Using these relations and the Dirac bracket (4.73) we can write Dirac bracket of coordinates starting with Q_0

$$\{Q_0, Q_0\}_{DB} = \{Q_0, Q_0\} - \{Q_0, \phi^0\}\{\phi_\lambda, Q_0\} + \{Q_0, \phi_\lambda\}\{\phi^0, Q_0\} = 0$$

since $\{Q_0, Q_0\} = \{Q_0, \phi_\lambda\} = 0$. Dirac bracket of Q_0 with λ^1 is

$$\{Q_0, \lambda^1\}_{DB} \quad = \quad \{Q_0, \lambda^1\} - \{Q_0, \phi^0\}\{\phi_\lambda, \lambda^1\} - \{Q_0, \phi_\lambda\}\{\phi^0, \lambda^1\} = 1$$

since $\{Q_0, \lambda^1\} = \{Q_0, \phi_\lambda\} = 0$ and $\{Q_0, \phi^0\} = \{\lambda^1, \phi_\lambda\} = 1$. Similarly we can find the other Dirac brackets of coordinates with each other. □

Proposition 4.1.10. *Using the Dirac brackets of the coordinates in the proposition* $(4.1.9)$*, Hamilton's equations generated by the Hamiltonian function* (4.57) *are*

$$\dot{Q}_0 \quad = \quad Q_1, \quad \dot{Q}_1 = \Pi^1, \quad \dot{\lambda}^1 = \Omega^2\omega^2 Q_0 \tag{4.74}$$

$$\dot{\Pi}^0 \quad = \quad \Omega^2\omega^2 Q_0, \quad \dot{\Pi}^1 = -\Pi^0 - (\Omega^2 + \omega^2)Q_1, \quad \dot{\Pi}_\lambda = 0 \tag{4.75}$$

Proof. Let us derive these equations. Using Dirac brackets of the coordinates and the Hamiltonian (4.57), equations of motion for Q_0 is

$$\begin{aligned} \dot{Q}_0 = \{Q_0, H_1^{PU}\}_{DB} &= Q_1\{Q_0, \Pi^0\}_{DB} \\ &= Q_1 \end{aligned}$$

since the Dirac bracket of Q_0 only with λ^1 and Π^0 is nonzero. Similarly, equation of motion for other coordinates are

$$\begin{aligned}\dot{Q}_1 &= \{Q_1, H_1^{PU}\}_{DB} = \Pi^1\{Q_1, \Pi^1\}_{DB} = \Pi^1 \\ \dot{\lambda}^1 &= \{\lambda^1, H_1^{PU}\}_{DB} = -\Omega^2\omega^2 Q_0\{\lambda^1, Q_0\}_{DB} = \Omega^2\omega^2 Q_0 \\ \dot{\Pi}^0 &= \{\Pi^0, H_1^{PU}\} = -\Omega^2\omega^2 Q_0\{\Pi^0, Q_0\}_{DB} = \Omega^2\omega^2 Q_0 \\ \dot{\Pi}^1 &= \{\Pi^1, H_1^{PU}\}_{DB} = \big(\Pi^0 + (\Omega^2+\omega^2)Q_1\big)\,\{\Pi^1, Q_1\}_{DB} \\ &= -\Pi^0 - (\Omega^2+\omega^2)Q_1 \\ \dot{\Pi}_\lambda &= \{\Pi_\lambda, H_1^{PU}\}_{DB} = 0.\end{aligned}$$

□

4.1.4. Unconstraint Variational Formalism

Unconstraint Lagrangians for L_0^{PU} and L_1^{PU} are obtained by replacing determined λ^0 and λ^1 from (4.19) and (4.20)

$$L_{U0}^{PU} = \frac{1}{2}\Big[\dot{Q}_1^2 - (\Omega^2+\omega^2)\dot{Q}_0^2 + \omega^2\Omega^2 Q_0^2\Big] - \ddot{Q}_1(\dot{Q}_0 - Q_1) \tag{4.76}$$

$$L_{U1}^{PU} = \frac{1}{2}\Big[\dot{Q}_1^2 - (\Omega^2+\omega^2)Q_1^2 + \omega^2\Omega^2 Q_0^2\Big] - \big((\omega^2+\Omega^2)Q_1 + \ddot{Q}_1\big)(\dot{Q}_0 - Q_1) \tag{4.77}$$

these unconstraint Lagrangians are second order. It is not possible to express first order L^{PU} in a unconstraint first order form. Since L^{PU} is nondegenerate.

4.2. SARIOĞLU-TEKİN LAGRANGIAN

4.2.1. General Setting

Consider the local coordinates for the six dimensional manifold N

$$
\begin{aligned}
(X^i, Y^i) &\in N, \\
(X^i, Y^i, \dot{X}^i, \dot{Y}^i) &\in TN \\
(X^i, Y^i, \dot{X}^i, \dot{Y}^i, \ddot{X}^i, \ddot{Y}^i) &\in T^2N \\
(X^i, Y^i, \dot{X}^i, \dot{Y}^i, R_i^0, S_i^0, R_i^1, S_i^1) &\in T^*TN,
\end{aligned}
$$

where i runs from 1 to 3. Consider also Sarıoğlu-Tekin Lagrangian on T^2N given by

$$
L^{ST}[X^i, Y^i] = \frac{\delta_{ij}}{2}\left[a(\dot{X}^i\dot{X}^j + \dot{Y}^i\dot{Y}^j) + \frac{2}{\mu}\dot{Y}^i\ddot{X}^j - m^2(Y^iY^j + X^iX^j)\right] \tag{4.78}
$$

here, a, μ, m are parameters [38]. L^{ST} is an example of second order degenerate Lagrangian since the determinant of extended Hessian matrix W_{ij} is zero.

For the Sarıoğlu-Tekin Lagrangian L^{ST}, the second order Euler-Lagrange equations (3.4) take the particular form

$$
m^2X^i + a\ddot{X}^i = \frac{1}{\mu}\dddot{Y}^i\,, \qquad m^2Y^i + a\ddot{Y}^i = -\frac{1}{\mu}\dddot{X}^i, \tag{4.79}
$$

whereas the Lagrangian one-form (3.11) becomes

$$
\theta_L = \delta_{ij}(a\dot{X}^i - \frac{1}{\mu}\ddot{Y}^i)dX^j + \delta_{ij}(a\dot{Y}^i + \frac{1}{\mu}\ddot{X}^i)dY^j + \frac{1}{\mu}\delta_{ij}\dot{Y}^i d\dot{X}^j.
$$

The exterior derivative of the one-form results in the pre-symplectic two-form

$$
\Omega_L = a\delta_{ij}(d\dot{X}^i \wedge dX^j + d\dot{Y}^i \wedge dY^j) + \frac{\delta_{ij}}{\mu}d\dot{Y} \wedge d\dot{X} + \frac{\delta_{ij}}{\mu}(d\ddot{X}^i \wedge dY^j - d\ddot{Y}^i \wedge dX^j).
$$

4.2.2. Jacobi-Ostrogradsky Method

On the Hamiltonian phase space T^*TN, we have the canonical Poisson bracket relations defined as

$$\{X^i, R^0_j\} = \{Y^i, S^0_j\} = \{\dot{X}^i, R^1_j\} = \{\dot{Y}^i, S^1_j\} = \delta^i_j \tag{4.80}$$

and all others are zero.

Proposition 4.2.1. *For the second order Lagrangian* L^{ST} *in* (4.78) *the Jacobi-Ostrogradsky momenta* (3.9) *and* (3.10) *are computed as*

$$R^0_i = \frac{\partial L^{ST}}{\partial \dot{X}^i} - \frac{d}{dt}\left(\frac{\partial L^{ST}}{\partial \ddot{X}^i}\right) = a\delta_{ij'}\dot{X}^{j'} - \frac{1}{\mu}\delta_{ij'}\ddot{Y}^{j'} \tag{4.81}$$

$$R^1_i = \frac{\partial L^{ST}}{\partial \ddot{X}^i} = \frac{1}{\mu}\delta_{ij'}\dot{Y}^{j'} \tag{4.82}$$

$$S^0_i = \frac{\partial L^{ST}}{\partial \dot{Y}^i} - \frac{d}{dt}\left(\frac{\partial L^{ST}}{\partial \ddot{Y}^i}\right) = a\delta_{ij'}\dot{Y}^{j'} + \frac{1}{\mu}\delta_{ij'}\ddot{X}^{j'} \tag{4.83}$$

$$S^1_i = \frac{\partial L^{ST}}{\partial \ddot{Y}^i} = 0 \tag{4.84}$$

respectively for $X^i, \dot{X}^i, Y^i$ *and* $\dot{Y}^i$.

Proposition 4.2.2. *Total Hamiltonian function for* L^{ST} *in* (4.78) *is given by*

$$\begin{aligned} H^{ST}_T = & -\frac{a}{2}\delta_{ij}(\dot{X}^i\dot{X}^j - \dot{Y}^i\dot{Y}^j) + \frac{m^2}{2}\delta_{ij}(Y^iY^j + X^iX^j) \\ & + R^0_j\dot{X}^j + a\mu(\dot{X}^jS^1_j - \dot{Y}^jR^1_j) + \mu\delta^{ij}(S^0_iR^1_j - R^0_iS^1_j). \end{aligned} \tag{4.85}$$

Proof. Since we cannot solve $\ddot{X}^i$ and $\ddot{Y}^i$ from equations (4.82) or (4.84), there exist primary constraints

$$\Gamma^x_i = R^1_i - \frac{1}{\mu}\delta_{ij}\dot{Y}^j = 0 \tag{4.86}$$

$$\Gamma^y_i = S^1_i = 0. \tag{4.87}$$

The canonical Hamiltonian function (3.15) for the second order Lagrangian L^{ST} takes the particular form

$$\begin{aligned} H_c^{ST} &= R_j^0 \dot{X}^j + R_j^1 \ddot{X}^j + S_j^0 \dot{Y}^j + S_j^1 \ddot{Y}^j - L^{ST} \\ &= R_j^0 \dot{X}^j + R_j^1 \ddot{X}^j + S_j^0 \dot{Y}^j - S_j^1 \ddot{Y}^j - \frac{\delta_{ij}}{2}\big[a(\dot{X}^i \dot{X}^j + \dot{Y}^i \dot{Y}^j)\big] \\ &- \frac{\delta_{ij}}{2}\big[\frac{2}{\mu}\dot{Y}^i \ddot{X}^j - m^2(Y^i Y^j + X^i X^j)\big]. \end{aligned} \tag{4.88}$$

The substitutions of $R_i^1 = \frac{\delta_{ij}}{\mu}\dot{Y}$ and $S_i^1 = 0$ result with the Hamiltonian function

$$H_c^{ST} = R_j^0 \dot{X}^j + S_j^0 \dot{Y}^j - \frac{\delta_{ij}}{2}\big[a(\dot{X}^i \dot{X}^j + \dot{Y}^i \dot{Y}^j) - m^2(Y^i Y^j + X^i X^j)\big]. \tag{4.89}$$

According to (3.19), define the total Hamiltonian function

$$H_T^{ST} = H_c^{ST} + U^j \Gamma_j^x + V^j \Gamma_j^y \tag{4.90}$$

by adding the primary constraints Γ_i^x and Γ_j^y with the Lagrange multipliers U^j and V^j. In order to guarantee the consistency of the primary constraints Γ_i^x, we compute

$$\begin{aligned} \dot{\Gamma}_i^x = \{\Gamma_i^x, H_T^{ST}\} &\approx \{\Gamma_i^x, H_c^{ST}\} + U^j\{\Gamma_i^x, \Gamma_j^x\} + V^j\{\Gamma_i^x, \Gamma_j^y\} \\ &\approx a\delta_{ij'}\dot{X}^{j'} - R_i^0 + V^j\big(-\frac{1}{\mu}\delta_{ij}\big) \end{aligned} \tag{4.91}$$

and for Γ_i^y we compute

$$\begin{aligned} \dot{\Gamma}_i^y = \{\Gamma_i^y, H_T^{ST}\} &\approx \{\Gamma_i^y, H_c^{ST}\} + U^j\{\Gamma_i^y, \Gamma_j^x\} + V^j\{\Gamma_i^y, \Gamma_j^y\} \\ &= a\delta_{ij'}\dot{Y}^{j'} - S_i^0 + U^j\big(\frac{1}{\mu}\delta_{ij}\big). \end{aligned} \tag{4.92}$$

From the consistency checks, no more constraint has been arisen, and the Lagrange multipliers are determined as

$$U^j \approx \mu(-a\delta_{j'}^j \dot{Y}^{j'} + \delta^{jj'} S_{j'}^0) \tag{4.93}$$

$$V^j \approx \mu(a\delta_{j'}^j \dot{X}^{j'} - \delta^{jj'} R_{j'}^0). \tag{4.94}$$

By substituting U^i and V^i into the total Hamiltonian function (4.90) the proof is completed.

□

Remark 4.2.3. *It is possible to solve* $\ddot{X}^i$ *and* $\ddot{Y}^i$ *from the equations* (4.81) *and* (4.83) *as*

$$\ddot{X}^i = \mu(\delta^{ij} S^0_j - a\dot{Y}^i), \quad \ddot{Y}^i = \mu(a\dot{X}^i - \delta^{ij} R^0_j) \tag{4.95}$$

and the substitution of the velocities into the canonical Hamiltonian (4.88) gives the total Hamiltonian (4.85) *directly without any constraint analysis.*

Proposition 4.2.4. *Hamilton equations generated by total Hamiltonian* (4.85) *are*

$$\begin{aligned}
\dot{X}^i &= \dot{X}^i - \mu\delta^{ij} S^1_j, \quad \ddot{X}^i = -a\mu\dot{Y}^i + \mu\delta^{ij} S^0_j \\
\dot{Y}^i &= mu\delta^{ij} R^1_j, \quad \ddot{Y}^i = a\mu\dot{X}^i - \mu\delta^{ij} R^0_j \\
\dot{R}^0_i &= -m^2\delta_{ij} X^j, \quad \dot{R}^1_i = a\delta_{ij}\dot{X}^j - R^0_i - a\mu S^1_i \\
\dot{S}^0_i &= -m^2\delta_{ij} Y^j, \quad \dot{S}^1_i = -a\delta_{ij}\dot{Y}^j + a\mu R^1_i.
\end{aligned}$$

Proof. We compute the Hamilton equations acoording to the total Hamiltonian function (4.85) and the canonical Poisson bracket relations (4.320). The first set of equations related with the base components

$$\dot{X}^i = \{X^i, H^{ST}_T\} = \dot{X}^i - \mu\delta^{ij} S^1_j \approx \dot{X}^i \tag{4.96}$$

$$\ddot{X}^i = \{\dot{X}^i, H^{ST}_T\} = -a\mu\dot{Y}^i + \mu\delta^{ij} S^0_j \tag{4.97}$$

$$\dot{Y}^i = \{Y^i, H^{ST}_T\} = \mu\delta^{ij} R^1_j \tag{4.98}$$

$$\ddot{Y}^i = \{\dot{Y}^i, H^{ST}_T\} = a\mu\dot{X}^i - \mu\delta^{ij} R^0_j \tag{4.99}$$

are satisfied identically using definitions of momenta $(4.81) - (4.84)$, whereas the set related with the momenta are

$$\dot{R}^0_i = \{R^0_i, H^{ST}_T\} = -m^2\delta_{ij} X^j \tag{4.100}$$

$$\dot{R}^1_i = \{R^1_i, H^{ST}_T\} = a\delta_{ij}\dot{X}^j - R^0_i - a\mu S^1_i \approx a\delta_{ij}\dot{X}^j - R^0_i \tag{4.101}$$

$$\dot{S}^0_i = \{S^0_i, H^{ST}_T\} = -m^2\delta_{ij} Y^j \tag{4.102}$$

$$\dot{S}^1_i = \{S^1_i, H^{ST}_T\} = -a\delta_{ij}\dot{Y}^j + a\mu R^1_i. \tag{4.103}$$

The equations defining $\dot{R}^0_i$ and $\dot{S}^0_i$ give the Euler-Lagrange equations (4.79). To see this we perform the following calculation

$$\begin{aligned}\dot{R}_i^0 &= -m^2\delta_{ij}X^j \\ a\delta_{ij'}\ddot{X}^{j'} - \frac{1}{\mu}\delta_{ij'}\dddot{Y}^{j'} &= -m^2\delta_{ij}X^j \\ \frac{1}{\mu}\delta_{ij'}\dddot{Y}^{j'} &= a\delta_{ij'}\ddot{X}^{j'} + m^2\delta_{ij}X^j \end{aligned} \tag{4.104}$$

using (4.81), and perform

$$\begin{aligned}\dot{S}_i^0 &= -m^2\delta_{ij}Y^j \\ a\delta_{ij'}\ddot{Y}^{j'} + \frac{1}{\mu}\delta_{ij'}\dddot{X}^{j'} &= -m^2\delta_{ij}Y^j \\ -\frac{1}{\mu}\delta_{ij'}\dddot{X}^{j'} &= a\delta_{ij'}\ddot{Y}^{j'} + m^2\delta_{ij}Y^j \end{aligned} \tag{4.105}$$

using (4.83). □

Dirac Bracket Formalism for L^{ST}: The set of constraints

$$\Gamma_i^x = R_i^1 - \frac{1}{\mu}\delta_{ij'}\dot{Y}^{j'}, \quad \Gamma_i^y = S_i^1 \tag{4.106}$$

consists of the second class constraints since their Poisson brackets are nonzero.

Proposition 4.2.5. *The Dirac bracket for the second order degenerate Sarıoğlu-Tekin Lagrangian L^{ST} is defined by*

$$\{X^i, R_j^0\}_{DB} = \delta_j^i \tag{4.107}$$

$$\{\dot{X}^i, \dot{Y}^i\}_{DB} = \mu\delta^{ij} \tag{4.108}$$

$$\{\dot{X}^i, R_j^1\}_{DB} = \delta_j^i \tag{4.109}$$

$$\{Y^i, S_j^0\}_{DB} = \delta_j^i \tag{4.110}$$

and, all the others are zero.

Proof. General form of the Dirac bracket is given in (2.39), using this we compute

$$\begin{aligned}\{F,G\}_{DB} &= \{F,G\} - \{F,\Gamma^x_m\}M^{mn}_{xy}\{\Gamma^y_n,G\} - \{F,\Gamma^y_m\}M^{mn}_{yx}\{\Gamma^x_n,G\} \\ &= \{F,G\} - \mu\delta^{mn}\big(\{F,\Gamma^x_m\}\{\Gamma^y_n,G\} + \{F,\Gamma^y_m\}\{\Gamma^x_n,G\}\big)\end{aligned} \tag{4.111}$$

by replacing inverse

$$M^{-1} = \begin{bmatrix} M^{mn}_{xx} & M^{mn}_{xy} \\ M^{mn}_{yx} & M^{mn}_{yy} \end{bmatrix} = \mu \begin{bmatrix} 0_{3\times3} & \delta^{mn} \\ -\delta^{mn} & 0_{3\times3} \end{bmatrix} \tag{4.112}$$

where M is

$$M = \begin{bmatrix} \{\Gamma^x_n,\Gamma^x_m\} & \{\Gamma^x_n,\Gamma^y_m\} \\ \{\Gamma^y_n,\Gamma^x_m\} & \{\Gamma^y_n,\Gamma^y_m\} \end{bmatrix} = \frac{1}{\mu}\begin{bmatrix} 0_{3\times3} & -\delta_{mn} \\ \delta_{mn} & 0_{3\times3} \end{bmatrix}. \tag{4.113}$$

To derive Dirac bracket relations of the coordinates, we also need to compute the Poisson brackets of the coordinates with the primary constraints

$$\{\dot{X}^i,\Gamma^x_m\} = \{\dot{X}^i, R^1_m - \frac{1}{\mu}\delta_{mj'}\dot{Y}^{j'}\} = \delta^i_m \tag{4.114}$$

$$\{\dot{Y}^i,\Gamma^y_m\} = \{\dot{Y}^i, S^1_m\} = \delta^i_m \tag{4.115}$$

$$\{S^1_i,\Gamma^x_m\} = \{S^1_i, R^1_m - \frac{1}{\mu}\delta_{mj'}\dot{Y}^{j'}\} = \frac{1}{\mu}\delta_{im} \tag{4.116}$$

and all others are zero. Let us derive some of Dirac brackets of the coordinates. For example Dirac bracket of X^i and Y^i

$$\begin{aligned}\{X^i,Y^j\}_{DB} &= \{X^i,Y^j\} - \mu\delta^{mn}\big(\{X^i,\Gamma^x_m\}\{\Gamma^y_n,Y^j\} + \{X^i,\Gamma^y_m\}\{\Gamma^x_n,Y^j\}\big) \\ &= 0\end{aligned} \tag{4.117}$$

since $\{X^i,Y^j\} = \{X^i,\Gamma^x_m\} = \{Y^i,\Gamma^y_n\} = 0$. Dirac bracket of X^i and $\dot{Y}^i$

$$\begin{aligned}\{\dot{X}^i,\dot{Y}^j\}_{DB} &= \{\dot{X}^i,\dot{Y}^j\} - \mu\delta^{mn}\big(\{\dot{X}^i,\Gamma^x_m\}\{\Gamma^y_n,\dot{Y}^j\} + \{\dot{X}^i,\Gamma^y_m\}\{\Gamma^x_n,\dot{Y}^j\}\big) \\ &= -\mu\delta^{mn}\delta^i_m(-\delta^j_n) = \mu\delta^{ij}\end{aligned} \tag{4.118}$$

since $\{\dot{X}^i,\dot{Y}^j\} = \{\dot{X}^i,\Gamma^y_m\} = 0$ and $\{\dot{X}^i,\Gamma^x_m\} = \{\dot{Y}^i,\Gamma^y_m\} = \delta^i_m$. Dirac brackets of the other coordinates can be proved in a similar way. □

Proposition 4.2.6. *Using the Dirac bracket defined in the Proposition* 4.2.5*, the Hamilton equations of motion for the canonical Hamiltonian function* (4.89) *are*

$$\dot{X}^i = \dot{X}^i, \quad \ddot{X}^i = \mu\delta^{ij}S_j^0 - a\mu\dot{Y}^i, \quad \dot{Y}^i = \dot{Y}^i, \quad \ddot{Y}^i = -\mu\delta^{ij}R_j^0 + a\mu\dot{X}^i$$
$$\dot{R}_i^0 = -m^2\delta_{ij}X^j, \quad \dot{R}_i^1 = -R_i^0 + a\delta_{ij}\dot{X}^j, \quad \dot{S}_i^0 = -m^2\delta_{ij}Y^j, \quad \dot{S}_i^1 = 0.$$

Proof. Let us derive these equations one by one. The equations of motion for X^i's are satisfied identically

$$\dot{X}^i = \{X^i, H_c^{ST}\}_{DB} = \{X^i, R_j^0\}_{DB}\dot{X}^j = \delta_j^i\dot{X}^j = \dot{X}^i \tag{4.119}$$

since Dirac bracket of X^i only with R_j^0 is nonzero. The equations of motion for $\dot{X}^i$'s

$$\begin{aligned}\ddot{X}^i &= \{\dot{X}^i, H_c^{ST}\}_{DB} = \{\dot{X}^i, \dot{Y}^j\}_{DB}S_j^0 - \frac{a}{2}\{\dot{X}^i, \delta_{jj'}\dot{Y}^j\dot{Y}^{j'}\}_{DB}\\ &= \mu\delta^{ij}S_j^0 - a\mu\dot{Y}^i = \mu\delta^{ij}\big(a\delta_{jj'}\dot{Y}^{j'} + \frac{1}{\mu}\delta_{jj'}\ddot{X}^{j'}\big) - a\mu\dot{Y}^i\\ &= \ddot{X}^i\end{aligned} \tag{4.120}$$

are identically satisfied replacing the definition of S_j^0. Equations of motion for Y^i's are satisfied identically

$$\dot{Y}^i = \{Y^i, H_c^{ST}\}_{DB} = \{Y^i, S_j^0\}_{DB}\dot{Y}^j = \delta_j^i\dot{Y}^j = \dot{Y}^i \tag{4.121}$$

since Dirac bracket of Y^i only with S_j^0 is nonzero. The equations of motion for $\dot{Y}^i$ are

$$\begin{aligned}\ddot{Y}^i &= \{\dot{Y}^i, H_c^{ST}\}_{DB} = \{\dot{Y}^i, \dot{X}^j\}_{DB}R_j^0 - \frac{a}{2}\{\dot{Y}^i, \delta_{jj'}\dot{X}^j\dot{X}^{j'}\}_{DB}\\ &= -\mu\delta^{ij}R_j^0 + a\mu\dot{X}^i = -\mu\delta^{ij}\big(a\delta_{jj'}\dot{X}^{j'} - \frac{1}{\mu}\delta_{jj'}\ddot{Y}^{j'}\big) + a\mu\dot{X}^i\\ &= \ddot{Y}^i\end{aligned} \tag{4.122}$$

using R_i^0. The equations of motion for R_i^0's are

$$\dot{R}_i^0 = \{R_i^0, H_c^{ST}\}_{DB} = \{R_i^0, \frac{m^2}{2}\delta_{jj'}X^jX^{j'}\}_{DB} = -m^2\delta_{ij'}X^{j'} \tag{4.123}$$

since Dirac bracket of R_i^0 only with X^j is nonzero. Using the definition of R_i^0, we compute

$$
\begin{aligned}
\dot{R}_i^0 &= -m^2\delta_{ij'}X^{j'} \\
a\delta_{ij'}\ddot{X}^{j'} - \frac{1}{\mu}\delta_{ij'}\dddot{Y}^{j'} &= -m^2\delta_{ij'}X^{j'} \\
\frac{1}{\mu}\dddot{Y}^{i} &= a\ddot{X}^i + m^2X^i
\end{aligned}
\tag{4.124}
$$

we get one of the Euler-Lagrange equations of motion (4.79). The equations of motion for R_i^1's are

$$
\begin{aligned}
\dot{R}_i^1 &= \{R_i^1, H_c^{ST}\}_{DB} = \{R_i^1, \dot{X}^j\}_{DB}R_j^0 + \{R_i^1, -\frac{a}{2}\delta_{jj'}\dot{X}^j\dot{X}^{j'}\}_{DB} \\
&= -R_i^0 + a\delta_{ij'}\dot{X}^{j'} = -\big(a\delta_{ij'}\dot{X}^{j'} - \frac{1}{\mu}\delta_{ij'}\ddot{Y}^{j'}\big) + a\delta_{ij'}\dot{X}^{j'} \\
&= \frac{1}{\mu}\delta_{ij'}\ddot{Y}^{j'} = \dot{R}_i^1
\end{aligned}
\tag{4.125}
$$

using the definition of R_i^0 and R_i^1. The equations of motion for S_i^0's are

$$
\dot{S}_i^0 = \{S_i^0, H_c^{ST}\}_{DB} = \{S_i^0, \frac{m^2}{2}\delta_{jj'}Y^jY^{j'}\}_{DB} = -m^2\delta_{ij}Y^j \tag{4.126}
$$

since Dirac bracket of S_i^0 is nonzero only with Y^j . Using definition of S_i^0

$$
\begin{aligned}
\dot{S}_i^0 &= -m^2\delta_{ij'}Y^{j'} \\
a\delta_{ij'}\ddot{Y}^{j'} + \frac{1}{\mu}\delta_{ij'}\dddot{X}^{j'} &= -m^2\delta_{ij'}Y^{j'} \\
-\frac{1}{\mu}\dddot{X}^{i} &= a\ddot{Y}^i + m^2Y^i
\end{aligned}
\tag{4.127}
$$

we get other one of the Euler- Lagrange equations of motion (4.79). Equations of motion for S_i^1

$$
\dot{S}_i^1 = \{S_i^1, H_c^{ST}\}_{DB} = 0. \tag{4.128}
$$

□

4.2.3. First Order Formalisms

We can write the second order Sarıoğlu-Tekin Lagrangian (4.78) as a degenerate first order Lagrangian functions (c.f. (3.23) and (3.24)) in two different ways given by

$$L^{ST}_{C_0} = \frac{\delta_{ij}}{2}\left[a(\dot{w}^i_0\dot{w}^j_0 + \dot{Y}^i\dot{Y}^j) + \frac{2}{\mu}\dot{Y}^i\dot{w}^j_1 - m^2(Y^iY^j + w^i_0w^j_0)\right] + \lambda^0_j(\dot{w}^j_0 - w^j_1) \quad (4.129)$$

$$L^{ST}_{C_1} = \frac{\delta_{ij}}{2}\left[a(w^i_1w^j_1 + \dot{Y}^i\dot{Y}^j) + \frac{2}{\mu}\dot{Y}^i\dot{w}^j_1 - m^2(Y^iY^j + w^i_0w^j_0)\right] + \lambda^1_j(\dot{w}^j_0 - w^j_1) \quad (4.130)$$

Here, we use the coordinate transformations $X^i = w^i_0, \dot{X}^i = \dot{w}^i_0 = w^i_1, \ddot{X}^i = \dot{w}^i_1$, with Lagrange multipliers λ^0_j and λ^1_i. These two first order Lagrangian give Euler-Lagrange equations of motion (4.79). The variations of $L^{ST}_{C_0}$ and $L^{ST}_{C_1}$ with respect to w^i_0 give one of the Euler-Lagrange equations of motion (4.79)

$$\frac{\delta L^{ST}_{C_0}}{\delta w^i_0} = -m^2w^i_0 - a\ddot{w}^i_0 - \delta^{ij}\dot{\lambda}^0_j = 0 \quad (4.131)$$

$$\frac{\delta L^{ST}_{C_1}}{\delta w^i_0} = -m^2w^i_0 - \delta^{ij}\dot{\lambda}^1_j = 0 \quad (4.132)$$

substituting λ^0_i and λ^1_i

$$\lambda^0_i = -\delta_{ij}\frac{1}{\mu}\ddot{Y}^j, \quad \lambda^1_i = \delta_{ij}(aw^j_1 - \frac{1}{\mu}\ddot{Y}^j) \quad (4.133)$$

obtained from variation of $L^{ST}_{C_0}$ and $L^{ST}_{C_1}$ with respect w^i_1. The other Euler-Lagrange equations of motion (4.79)

$$\frac{\delta L^{ST}_{C_0}}{\delta Y^i} = \frac{\delta L^{ST}_{C_1}}{\delta Y^i} = -m^2Y^i - a\ddot{Y}^i - \frac{1}{\mu}\ddot{w}^i_1 \quad (4.134)$$

are obtained from the variations of $L^{ST}_{C_0}$ and $L^{ST}_{C_1}$ with respect to Y^i.

In the next part we will discuss Hamiltonian formalism for reduced first order Sarıoğlu-Tekin Lagrangians $L^{ST}_{C_0}$ and $L^{ST}_{C_1}$ in $(4.129) - (4.130)$ and unconstraint Sarıoğlu-Tekin Lagrangians obtained by substituting λ^0_i and λ^1_i in (4.133) into $(4.129) - (4.130)$

4.2.4. The First Order Formalism as $L_{C_0}^{ST}$

Hamiltonian Formalism for $L_{C_0}^{ST}$: In order to write the Hamiltonian formulation of the reduced first order systems, we introduce the conjugate momenta $(w_0^i, w_1^i, \lambda_i^0, Y^i, \pi_i^0, \pi_i^1, \pi_\lambda^i, \pi_j^Y)$ on the dual space. The canonical Poisson bracket relations are defined as

$$\{w_o^i, \pi_j^0\} = \{w_1^i, \pi_j^1\} = \{\dot{Y}^i, \pi_j^Y\} = \{\lambda_j^0, \pi_\lambda^i\} = \delta_j^i \tag{4.135}$$

and all the others are zero.

For the reduced first order Lagrangian $L_{C_0}^{ST}$ presented in the equation (4.129), the conjugate momenta corresponding to coordinates $(w_0^i, w_1^i, \lambda_i^0, Y^i)$ are defined by

$$\pi_i^0 = \frac{\partial L_{C_0}^{ST}}{\partial \dot{w}_0^i} = a\delta_{ij}\dot{w}_0^j + \lambda_i^0 \tag{4.136}$$

$$\pi_i^1 = \frac{\partial L_{C_0}^{ST}}{\partial \dot{w}_1^i} = \frac{1}{\mu}\delta_{ij}\dot{Y}^j \tag{4.137}$$

$$\pi_\lambda^i = \frac{\partial L_{C_0}^{ST}}{\partial \dot{\lambda}_i^0} = 0 \tag{4.138}$$

$$\pi_i^Y = \frac{\partial L_{C_0}^{ST}}{\partial \dot{Y}^i} = a\delta_{ij}\dot{Y}^j + \frac{1}{\mu}\delta_{ij}\dot{w}_1^j. \tag{4.139}$$

Proposition 4.2.7. *The total Hamiltonian for the first order Lagrangian* $L_{C_0}^{ST}$ *in* (4.129) *is given by*

$$\begin{aligned} H_{T1}^{ST} &= \mu\delta^{ij}\pi_i^1\big(\pi_j^Y - \frac{a\mu}{2}\pi_j^1\big) + \frac{m^2\delta_{ij}}{2}(Y^iY^j + w_0^iw_0^j) + \lambda_i^0 w_1^i \\ &+ \frac{1}{2a}\delta^{ij}(\pi_i^0 - \lambda_i^0)(\pi_j^0 - \lambda_j^0) - \big(m^2\delta_{ij}w_0^j - a^2\mu^2\pi_i^1 + a\mu\pi_i^Y\big)\pi_\lambda^i \\ &- \frac{1}{a}\delta^{ij}\big((\pi_i^0 - \lambda_i^0) - \delta_{ik}w_1^k\big)\big((\pi_j^0 - \lambda_j^0) - \delta_{kj}w_1^j\big). \end{aligned} \tag{4.140}$$

Proof. From the conjugate momenta π_i^0, π_i^1 and π_j^Y it is possible to solve velocities for

$$\dot{Y}^i = \mu\delta^{ij}\pi_j^1 \tag{4.141}$$

$$\dot{w}_0^i = \frac{1}{a}\delta^{ij}\left(\pi_j^0 - \lambda_j^0\right) \tag{4.142}$$

$$\dot{w}_1^i = \mu\delta^{ij}\left(\pi_j^Y - a\mu\pi_j^1\right) \tag{4.143}$$

and the other momenta π_λ^i leads to a primary constraint

$$\psi_\lambda^i = \pi_\lambda^i = 0. \tag{4.144}$$

For $L_{C_0}^{ST}$, after the substitution of $\dot{w}_0^i, \dot{w}_1^i, \dot{Y}^i$ and the primary constraint, the canonical Hamiltonian function (2.20) turns out to be

$$\begin{aligned} H_{c_0}^{ST} &= \pi_i^0 \dot{w}_0^i + \pi_i^1 \dot{w}_1^i + \pi_\lambda^i \dot{\lambda}_i^0 + \pi_i^Y \dot{Y}^i - L_{C_0}^{ST} \\ &= \mu \delta^{ij} \pi_i^1 \pi_j^Y - \frac{a\mu^2}{2} \delta^{ij} \pi_j^1 \pi_i^1 + \frac{m^2 \delta_{ij}}{2} (Y^i Y^j + w_0^i w_0^j) + \lambda_i^0 w_1^i \\ &+ \frac{1}{2a} \delta^{ij} (\pi_i^0 - \lambda_i^0)(\pi_j^0 - \lambda_j^0). \end{aligned} \tag{4.145}$$

The total Hamiltonian is defined as

$$H_T^{ST} = H_{c_0}^{ST} + U_i \psi_\lambda^i \tag{4.146}$$

by adding the primary constraint with a Lagrange multiplier U_j. Consistency of the primary constraint ψ_λ^i

$$\begin{aligned} \dot{\psi}_\lambda^i = \{\psi_\lambda^i, H_T^{ST}\} &\approx \{\psi_\lambda^i, H_{c_0}^{ST}\} + U_j \{\psi_\lambda^i, \psi_\lambda^j\} \\ &\approx \frac{1}{a} \delta^{ij} (\pi_j^0 - \lambda_j^0) - w_1^i \end{aligned} \tag{4.147}$$

leads us to a secondary constraint

$$\psi^i \approx \frac{1}{a} \delta^{ij} (\pi_j^0 - \lambda_j^0) - w_1^i. \tag{4.148}$$

Note that (4.147) will vanish weakly when we use $\dot{w}_0 = w_1$ in the definition of π_i^0. In this case, it is not possible to find the Lagrange multiplier U, thus equation of motion for λ_i^0 remains arbitrary. To solve this, we consider ψ^i as a secondary constraint. Revised total Hamiltonian is

$$H_{T1}^{ST} = H_{c_0}^{ST} + U_i \psi_\lambda^i + V_i \psi^i \tag{4.149}$$

by adding secondary constraint with arbitrary functions V_i's. The consistency of the secondary constraint ψ^i can be checked through

$$\begin{aligned}\dot{\psi}^i &= \{\psi^i, H^{ST}_{T1}\} \approx \{\psi^i, H^{ST}_{c_0}\} + U_j\{\psi^i, \psi^j_\lambda\} + V_j\{\psi^i, \psi^j\} \\ &\approx -\frac{m^2}{a}w^i_0 + a\mu^2\delta^{ij}\pi^1_j - \mu\delta^{ij}\pi^Y_j + U_j\big(-\frac{1}{a}\delta^{ij}\big)\end{aligned} \tag{4.150}$$

which leads us to determine the Lagrange multiplier U_j as

$$U_i \approx -m^2\delta_{ij}w^j_0 + a^2\mu^2\pi^1_i - a\mu\pi^Y_i. \tag{4.151}$$

On the other hand, the consistency of ψ^i_λ

$$\begin{aligned}\dot{\psi}^i_\lambda &= \{\psi^i_\lambda, H^{ST}_{T1}\} \approx \{\psi^i_\lambda, H^{ST}_{c_0}\} + U_j\{\psi^i_\lambda, \psi^j_\lambda\} + V_j\{\psi^i_\lambda, \psi^j\} \\ &\approx \psi^i + V_j\big(\frac{1}{a}\delta^{ij}\big)\end{aligned} \tag{4.152}$$

leads to us to determine V_j as $V_i \approx -a\delta_{ij}\psi^j$. Substitution of U_i and V_i into the total Hamiltonian function H^{ST}_T in (4.149) completes the proof. □

Proposition 4.2.8. *Hamilton equations of motion using the total Hamiltonian function in the proposition* 4.2.7 *are*

$$\dot{w}^i_0 = \frac{1}{a}\delta^{ij}(\pi^0_j - \lambda^0_j), \quad \dot{w}^i_1 \approx \mu\delta^{ij}(\pi^Y_j - a\mu^2\pi^1_j), \quad \dot{Y}^i \approx \mu\delta^{ij}\pi^1_j \tag{4.153}$$

$$\dot{\lambda}^0_i = -m^2\delta_{ij}w^j_0 + a^2\mu^2\pi^1_i - a\mu\pi^Y_i, \quad \dot{\pi}^0_i \approx -\delta_{ij}m^2w^j_0 \tag{4.154}$$

$$\dot{\pi}^1_i = -\lambda^0_i, \quad \dot{\pi}^i_\lambda = \frac{1}{a}\delta^{ij}(\pi^0_j - \lambda^0_j) - w^i_1, \quad \dot{\pi}^Y_i = -m^2\delta_{ij}Y^i \tag{4.155}$$

Proof. Using the total Hamiltonian in the proposition 4.2.7 and the Poisson brackets in (4.135), the equations

$$\dot{w}^i_0 = \{w^i_0, H^{ST}_T\} = \frac{1}{a}\delta^{ij}(\pi^0_j - \lambda^0_j) \tag{4.156}$$

$$\dot{w}^i_1 = \{w^i_1, H^{ST}_T\} = \delta^{ij}(\pi^Y_j - a\mu^2\pi^1_j) + a^2\mu^2\pi^i_\lambda \approx \mu\delta^{ij}(\pi^Y_j - a\mu^2\pi^1_j) \tag{4.157}$$

$$\dot{Y}^i = \{Y^i, H^{ST}_T\} = \mu\delta^{ij}\pi^1_j - a\mu\pi^i_\lambda \approx \mu\delta^{ij}\pi^1_j \tag{4.158}$$

$$\dot{\pi}^1_i = \{\pi^1_i, H^{ST}_T\} = -\lambda^0_i \tag{4.159}$$

$$\dot{\pi}^i_\lambda = \{\pi^i_\lambda, H^{ST}_T\} = \frac{1}{a}\delta^{ij}(\pi^0_j - \lambda^0_j) - w^i_1. \tag{4.160}$$

are satisfied identically from definition of momenta (4.136) and (4.139). The rest

$$\dot{\lambda}_i^0 = \{\lambda_i^0, H_T^{ST}\} = -m^2\delta_{ij}w_0^j + a^2\mu^2\pi_i^1 - a\mu\pi_i^Y \tag{4.161}$$

$$\dot{\pi}_i^0 = \{\pi_i^0, H_T^{ST}\} = \delta_{ij}(-m^2w_0^j + m^2\pi_\lambda^j) \approx -\delta_{ij}m^2w_0^j \tag{4.162}$$

$$\dot{\pi}_i^Y = \{\pi_i^Y, H_T^{ST}\} = -m^2\delta_{ij}Y^i \tag{4.163}$$

give the equations of motion using the definition of momenta (4.136) and (4.139). □

Dirac Bracket Formalism: The constraints

$$\psi^i = \frac{1}{a}\delta^{ij}(\pi_j^0 - \lambda_j^0) - w_1^i, \quad \psi_\lambda^i = \pi_\lambda^i \tag{4.164}$$

for the first order Lagrangian $L_{c_0}^{ST}$ are of second class since the Poisson brackets

$$\{\psi^i, \psi_\lambda^j\} = \frac{1}{a}\delta^{ij} \tag{4.165}$$

are nonzero.

Proposition 4.2.9. *Under the existence of the constraints (4.164), the Dirac bracket is defined by*

$$\{w_0^i, \lambda_j^0\}_{DB} = \delta_j^i \tag{4.166}$$

$$\{w_0^i, \pi_j^0\}_{DB} = \delta_j^i \tag{4.167}$$

$$\{w_1^i, \pi_j^1\}_{DB} = \delta_j^i \tag{4.168}$$

$$\{\lambda_i^0, \pi_j^1\}_{DB} = -a\delta_{ij} \tag{4.169}$$

$$\{Y^i, \pi_j^Y\}_{DB} = \delta_j^i \tag{4.170}$$

and all others are zero.

Proof. Recall the definition of the Dirac bracket presented in (2.39). In particular, for the constraints (4.164), we arrive

$$\{F, G\}_{DB} = \{F, G\} + a\{F, \psi^k\}\delta_{kn}\{\psi_\lambda^n, G\} - a\{F, \psi_\lambda^k\}\delta_{kn}\{\psi^n, G\} \tag{4.171}$$

after the substitution of the inverse matrix of

$$M = \begin{bmatrix} \{\psi^k, \psi^n\} & \{\psi^k, \psi^n_\lambda\} \\ \{\psi^n_\lambda, \psi^k\} & \{\psi^n_\lambda, \psi^k_\lambda\} \end{bmatrix} = \frac{1}{a} \begin{bmatrix} 0_{3\times 3} & \delta^{kn} \\ -\delta^{nk} & 0_{3\times 3} \end{bmatrix}. \tag{4.172}$$

Note that, the Poisson brackets of the coordinates and the constraints are

$$\{w^i_0, \psi^j\} = \frac{1}{a}\delta^i_j \tag{4.173}$$

$$\{\lambda^0_i, \psi^j_\lambda\} = \delta^j_i \tag{4.174}$$

$$\{\pi^1_i, \psi^j\} = \delta^j_i \tag{4.175}$$

$$\{\pi^i_\lambda, \psi^j\} = -\frac{1}{a}\delta^{ij} \tag{4.176}$$

and all others are zero. Using the Dirac bracket (4.171) and the equations $(4.173) - (4.176)$, we find

$$\begin{aligned} \{w^i_0, w^i_1\}_{DB} &= \{w^i_0, w^i_1\} + a\{w^i_0, \psi^k\}\delta_{kn}\{\psi^n_\lambda, w^i_1\} - a\{w^i_0, \psi^k_\lambda\}\delta_{kn}\{\psi^n, w^i_1\} \\ &= 0 \end{aligned} \tag{4.177}$$

since $\{w^i_0, w^i_1\} = \{w^i_0, \psi^k_\lambda\} = \{w^i_1, \psi^k_\lambda\}$. Dirac bracket of w^i_0 with λ^0_j is

$$\begin{aligned} \{w^i_0, \lambda^0_j\}_{DB} &= \{w^i_0, \lambda^0_j\} + a\{w^i_0, \psi^k\}\delta_{kn}\{\psi^n_\lambda, \lambda^0_j\} - a\{w^i_0, \psi^k_\lambda\}\delta_{kn}\{\psi^n, \lambda^0_j\} \\ &= \delta^i_j \end{aligned} \tag{4.178}$$

employing $\{w^i_0, \lambda^0_j\} = \{w^i_0, \psi^k_\lambda\} = 0$ and $\{w^i_0, \psi^j\} = \frac{1}{a}\delta^i_j$, $\{\lambda^0_i, \psi^j_\lambda\} = \delta^j_i$. One may continue to the proof in a similar manner. □

4.2.5. Unconstraint Variational Formalism for $L^{ST}_{C_0}$

To get the unconstraint Sarıoğlu-Tekin Lagrangians, we substitute the Lagrange multiplier λ^0_i obtained in (4.133) into the first order Lagrangian density $L^{ST}_{C_0}$

$$L^{ST}_{U_0} = \frac{\delta_{ij}}{2}\left[a(\dot{w}^i_0\dot{w}^j_0 + \dot{Y}^i\dot{Y}^j) + \frac{2}{\mu}\dot{Y}^i\dot{w}^j_1 - m^2(Y^iY^j + w^i_0w^j_0)\right] - \frac{\delta_{ij}}{\mu}(\dot{w}^i_0 - w^i_1)\ddot{Y}^j. \tag{4.179}$$

Note that, this is a second order Lagrangian. At this point it is possible to define the Jacobi-Ostrogradsky momenta for $L^{ST}_{U_0}$ and then apply Dirac analysis. Alternatively, one may reduce $L^{ST}_{U_0}$ into a first order Lagrangian by introducing a coordinate transformation

$$Y^i = q^i_0 \quad \dot{Y}^i = \dot{q}^i_0 = q^i_1, \quad \ddot{Y}^i = \dot{q}^i_1. \tag{4.180}$$

In this case, we arrive the following first order constraint Lagrangians

$$L^{ST}_{NC_0} = L^{ST}_{U_{00}}(q^i_0, \dot{q}^i_0, \dot{q}^i_1, w^i_0, \dot{w}^i_0, \dot{w}^i_1) + \chi^0_i(\dot{q}^i_0 - q^i_1) \tag{4.181}$$

$$L^{ST}_{NC_1} = L^{ST}_{U_{10}}(q^i_0, q^i_1, \dot{q}^i_1, w^i_0, \dot{w}^i_0, \dot{w}^i_1) + \chi^1_i(\dot{q}^i_0 - q^i_1) \tag{4.182}$$

where χ^0_i and χ^1_i are the Lagrange multipliers depending on $\dot{q}^i_0$ or q^i_1 in the first order Lagrangian $L^{ST}_{U_0}$ (4.179). In this case $L^{ST}_{U_{00}}$ and $L^{ST}_{U_{10}}$ become

$$L^{ST}_{U_{00}} = \frac{\delta_{ij}}{2}\big[a(\dot{q}^i_0\dot{q}^j_0 + \dot{w}^i_0\dot{w}^j_0) + \frac{2}{\mu}\dot{q}^i_0\dot{w}^j_1 - m^2(q^i_0q^j_0 + w^i_0w^j_0) - \frac{2}{\mu}(\dot{w}^i_0 - w^i_1)\dot{q}^j_1\big]$$
$$L^{ST}_{U_{10}} = \frac{\delta_{ij}}{2}\big[a(q^i_1q^j_1 + \dot{w}^i_0\dot{w}^j_0) + \frac{2}{\mu}q^i_1\dot{w}^j_1 - m^2(q^i_0q^j_0 + w^i_0w^j_0) - \frac{2}{\mu}(\dot{w}^i_0 - w^i_1)\dot{q}^j_1\big].$$

We can apply Dirac analysis directly for first order Lagrangians $L^{ST}_{NC_0}$ and $L^{ST}_{NC_1}$ in (4.181) and (4.182), or we write them as in unconstraint form substituting χ^0 and χ^1 into $(4.181) - (4.182)$ and then apply the Dirac analysis. Variations of $L^{ST}_{NC_0}$ and $L^{ST}_{NC_1}$ with respect to q^i_1 give χ^0_i and χ^1_i as

$$\chi^0_i = \frac{\delta_{ij}}{\mu}(\ddot{w}^j_0 - \dot{w}^j_1) = 0, \quad \chi^1_i = \delta_{ij}(aq^j_1 + \frac{1}{\mu}\dot{w}^j_1) \tag{4.183}$$

since $\dot{w}^i_0 = w^i_1$. Substitutions of these Lagrange multipliers χ^0_i and χ^0_i into (4.181) and (4.182) give first order unconstraint Lagrangians respectively

$$L^{ST}_{NU_0} = L^{ST}_{U_{00}} \tag{4.184}$$

$$L^{ST}_{NU_1} = L^{ST}_{U_{10}} + \delta_{ij}(aq^j_1 + \frac{1}{\mu}\dot{w}^j_1)(\dot{q}^i_0 - q^i_1). \tag{4.185}$$

Note that variations of these Lagrangians with respect to q^i_0 and w^i_0 give Euler-Lagrange equations (4.79). We will continue with Hamiltonian analysis of unconstraint Lagrangians $L^{ST}_{NU_0}$ and $L^{ST}_{NU_1}$ in $(4.184) - (4.185)$.

Hamiltonian Formalism for $L^{ST}_{NU_0}$: For the unconstraint Lagrangian $L^{ST}_{NU_0}$ in (4.184) the conjugate momenta are

$$R^0_i = \frac{\partial L^{ST}_{NU_0}}{\partial \dot{q}^i_0} = a\delta_{ij}\dot{q}^j_0 + \frac{1}{\mu}\delta_{ij}\dot{w}^j_1 \tag{4.186}$$

$$R^1_i = \frac{\partial L^{ST}_{NU_0}}{\partial \dot{q}^i_1} = \frac{1}{\mu}\delta_{ij}(w^j_1 - \dot{w}^j_0) \tag{4.187}$$

$$S^0_i = \frac{\partial L^{ST}_{NU_0}}{\partial \dot{w}^i_0} = \delta_{ij}(a\dot{w}^j_0 - \frac{1}{\mu}\dot{q}^j_1) \tag{4.188}$$

$$S^1_i = \frac{\partial L^{ST}_{NU_0}}{\partial \dot{w}^i_1} = \frac{1}{\mu}\delta_{ij}\dot{q}^j_0. \tag{4.189}$$

Proposition 4.2.10. *The canonical Hamiltonian function corresponding to the unconstraint Lagrangian $L^{ST}_{NU_0}$ is*

$$\begin{aligned} H^{ST}_{c_0} = &-\frac{a}{2}\delta^{ij}(\delta_{ik}w^k_1 - \mu R^1_i)(\delta_{jl}w^l_1 - \mu R^1_j) - \frac{a\mu^2}{2}\delta^{ij}S^1_iS^1_j + \mu\delta^{ij}R^0_iS^1_j \\ &+ S^0_i(w^i_1 - \mu\delta^{ij}R^1_j) + \frac{m^2}{2}\delta_{ij}(q^i_0q^j_0 + w^i_0w^j_0) \end{aligned} \tag{4.190}$$

Proof. From conjugate momenta R^0_i, R^1_i, S^0_i and S^1_i it is possible to solve the velocities as

$$\dot{q}^i_0 = \mu\delta^{ij}S^1_j \tag{4.191}$$

$$\dot{q}^i_1 = a\mu(w^i_1 - \mu\delta^{ij}R^1_j) - \mu\delta^{ij}S^0_j \tag{4.192}$$

$$\dot{w}^i_0 = w^i_1 - \mu\delta^{ij}R^1_j \tag{4.193}$$

$$\dot{w}^i_1 = \mu\delta^{ij}(R^0_j - a\mu S^1_j). \tag{4.194}$$

Using the definition of the canonical Hamiltonian function we have

$$\begin{aligned} H^{ST}_{c_0} &= R^0_i\dot{q}^i_0 + R^1_i\dot{q}^i_1 + S^0_i\dot{w}^i_0 + S^1_i\dot{w}^i_1 - L^{ST}_{NU_0} \\ &= R^0_i\dot{q}^i_0 + R^1_i\dot{q}^i_1 + S^0_i\dot{w}^i_0 + S^1_i\dot{w}^i_1 - L^{ST}_{C_0} \\ &= R^0_i\dot{q}^i_0 + R^1_i\dot{q}^i_1 + S^0_i\dot{w}^i_0 + S^1_i\dot{w}^i_1 - \frac{\delta_{ij}}{2}\big[a(\dot{q}^i_0\dot{q}^j_0 + \dot{w}^i_0\dot{w}^j_0) + \frac{2}{\mu}\dot{q}^i_0\dot{w}^j_1 \\ &- m^2(q^i_0q^j_0 + w^i_0w^j_0) - \frac{2}{\mu}(\dot{w}^i_0 - w^i_1)\dot{q}^j_1\big]. \end{aligned} \tag{4.195}$$

After the substitutions of the velocities, we arrive

$$
\begin{aligned}
H_{c_0}^{ST} &= -\frac{a}{2}\delta^{ij}(\delta_{ik}w_1^k - \mu R_i^1)(\delta_{jl}w_1^l - \mu R_j^1) - \frac{a\mu^2}{2}\delta^{ij}S_i^1 S_j^1 + \mu\delta^{ij}R_i^0 S_j^1 \\
&\quad + S_i^0(w_1^i - \mu\delta^{ij}R_j^1) + \frac{m^2}{2}\delta_{ij}(q_0^i q_0^j + w_0^i w_0^j).
\end{aligned} \tag{4.196}
$$

□

Proposition 4.2.11. *Hamilton equations of motion using the Hamiltonian function in proposition* 4.2.10 *are*

$$
\begin{aligned}
&\dot{q}_0^i = \mu\delta^{ij}S_j^1, \quad \dot{q}_1^i = -a\mu w_1^i + \mu\delta^{ij}(aR_j^1 - S_j^0), \quad \dot{w}_0^i = w_1^i - \mu\delta^{ij}R_j^1 \\
&\dot{w}_1^i = \mu R^0 - a\mu^2 S_1, \quad \dot{R}_i^0 = -m^2\delta_{ij}q_0^j, \quad \dot{R}_i^1 = 0 \\
&\dot{S}_i^0 = -m^2\delta_{ij}w_0^j, \quad \dot{S}_i^1 = -S_i^0 + a\delta_{ij}w_1^j.
\end{aligned} \tag{4.197}
$$

Proof. Governing by the canonical Hamiltonian, the equations

$$\dot{q}_0^i = \{q_0^i, H_{c_0}^{ST}\} = \mu\delta^{ij}S_j^1 \tag{4.198}$$

$$\dot{q}_1^i = \{q_1^i, H_{c_0}^{ST}\} = -a\mu\left(w_1 - R^1\right) - \mu S^0 \tag{4.199}$$

$$\dot{w}_0^i = \{w_0^i, H_{c_0}^{ST}\} = w_1^i - \mu\delta^{ij}R_j^1 \tag{4.200}$$

$$\dot{w}_1^i = \{w_1^i, H_{c_0}^{ST}\} = \mu\delta^{ij}(R_j^0 - a\mu S_j^1) \tag{4.201}$$

are satisfied identically. Equations of motion for momenta are

$$\dot{R}_i^0 = \{R_i^0, H_{c_0}^{ST}\} = -m^2\delta_{ij}q_0^j \tag{4.202}$$

$$\dot{R}_i^1 = \{R_i^1, H_{c_0}^{ST}\} = 0 \tag{4.203}$$

$$\dot{S}_i^0 = \{S^0, H_{c_0}^{ST}\} = -m^2\delta_{ij}w_0^j \tag{4.204}$$

$$\dot{S}_i^1 = \{S_i^1, H_{c_0}^{ST}\} = -S_i^0 + a\delta_{ij}w_1. \tag{4.205}$$

The first and third of these equations give the Euler-Lagrange equations, and the rest two are satisfied identically. □

Hamiltonian Formalism for $L_{NU_1}^{ST}$: For the Lagrangian $L_{NU_1}^{ST}$ in (4.185), the conjugate momenta are

$$r_i^0 = \frac{\partial L_{NU_1}^{ST}}{\partial \dot{q}_0^i} = aq_1^i + \frac{1}{\mu}\dot{w}_1^i \tag{4.206}$$

$$r_i^1 = \frac{\partial L_{NU_1}^{ST}}{\partial \dot{q}_1^i} = \frac{1}{\mu}(w_1^i - \dot{w}_0^i) \tag{4.207}$$

$$s_i^0 = \frac{\partial L_{NU_1}^{ST}}{\partial \dot{w}_0^i} = a\dot{w}_0^i - \frac{1}{\mu}\dot{q}_1^i \tag{4.208}$$

$$s_i^1 = \frac{\partial L_{NU_1}^{ST}}{\partial \dot{w}_1^i} = \frac{1}{\mu}\dot{q}_0^i. \tag{4.209}$$

Proposition 4.2.12. *The canonical Hamiltonian function corresponding to the Lagrangian $L_{NC_1}^{ST}$ is*

$$\begin{aligned} H_{c_1}^{ST} &= s_i^0(w_1^i - \mu\delta^{ij}r_j^1) + \mu s_i^1(\delta^{ij}r_j^0 - aq_1^i) + \frac{a}{2}\delta^{ij}q_1^iq_1^j + \frac{m^2}{2}\delta^{ij}(q_0^iq_0^j + w_0^iw_0^j) \\ &- \frac{a}{2}\delta_{ij}(w_1^i - \mu\delta^{ik}r_k^1)(w_1^j - \mu\delta^{jl}r_l^1). \end{aligned} \tag{4.210}$$

Proof. From the definitions of the conjugate momenta, it is possible to solve velocities as follows

$$\dot{q}_0^i = \mu\delta^{ij}s_j^1 \tag{4.211}$$

$$\dot{q}_1^i = a\mu w_1^i - \mu\delta^{ij}(s_j^0 - a\mu r_j^1) \tag{4.212}$$

$$\dot{w}_0^i = w_1^i - \delta^{ij}\mu r_j^1 \tag{4.213}$$

$$\dot{w}_1^i = \mu\delta_j^{ij}r^0 - a\mu q_1^i. \tag{4.214}$$

Using these, we write the canonical Hamiltonian function as

$$\begin{aligned} H_{c_1}^{ST} &= r_i^0\dot{q}_0^i + r_i^1\dot{q}_1^i + s_i^0\dot{w}_0^i + s_i^1\dot{w}_0^i - L_{NU_1}^{ST} \\ &= r_i^0\dot{q}_0^i + r_i^1\dot{q}_1^i + s_i^0\dot{w}_0^i + s_i^1\dot{w}_0^i - (L_{C_1}^{ST} + \delta_{ij}(aq_1^j + \frac{1}{\mu}\dot{w}_1^j)(\dot{q}_0^i - q_1^i)) \\ &= r_i^0\dot{q}_0^i + r_i^1\dot{q}_1^i + s_i^0\dot{w}_0^i + s_i^1\dot{w}_0^i - \frac{\delta_{ij}}{2}\big[a(q_1^iq_1^j + \dot{w}_0^i\dot{w}_0^j) + \frac{2}{\mu}q_1^i\dot{w}_1^j \\ &- m^2(q_0^iq_0^j + w_0^iw_0^j) - \frac{2}{\mu}(\dot{w}_0^i - w_1^i)\dot{q}_1^j\big] \\ &= s_i^0(w_1^i - \mu\delta^{ij}r_j^1) + \mu s_i^1(\delta^{ij}r_j^0 - aq_1^i) + \frac{a}{2}\delta^{ij}q_1^iq_1^j + \frac{m^2}{2}\delta^{ij}(q_0^iq_0^j + w_0^iw_0^j) \\ &- \frac{a}{2}\delta_{ij}(w_1^i - \mu\delta^{ik}r_k^1)(w_1^j - \mu\delta^{jl}r_l^1). \end{aligned} \tag{4.215}$$

□

Proposition 4.2.13. *Using the Hamiltonian function given in the proposition* 4.2.12*, the Hamilton equations are*

$$\dot{q}_0^i = \mu\delta^{ij}s_j^1, \quad \dot{q}_1^i = -\mu\delta^{ij}s_j^0 + a\mu(w_1^i - \mu\delta^{ij}r_j^1), \quad \dot{w}_0^i = w_1^i - \mu\delta^{ij}r_j^1$$
$$\dot{w}_1^i = \mu\delta^{ij}r_j^0 - a\mu q_1^i, \quad \dot{r}_i^0 = -m^2\delta_{ij}q_0^j, \quad \dot{r}_i^1 = -a\delta_{ij}q_1^j + a\mu s_i^1 \tag{4.216}$$
$$\dot{s}_i^0 = -m^2\delta^{ij}w_0^j, \quad \dot{s}_i^1 = -s_i^0 + a\delta_{ij}w_1^j - \mu r_i^1.$$

Proof. The Hamilton's equations for the variables q_0^i, q_1^i, w_0^i and w_1^i generated by $H_{c_1}^{ST}$ given in the proposition 4.2.12 are

$$\dot{q}_0^i = \{q_0^i, H_{c_1}^{ST}\} = \mu\delta^{ij}s_j^1 \tag{4.217}$$
$$\dot{q}_1^i = \{q_1^i, H_{c_1}^{ST}\} = -\mu\delta^{ij}s_j^0 + a\mu(w_1^i - \mu\delta^{ij}r_j^1) \tag{4.218}$$
$$\dot{w}_0^i = \{w_0^i, H_{c_1}^{ST}\} = w_1^i - \mu\delta^{ij}r_j^1 \tag{4.219}$$
$$\dot{w}_1^i = \{w_1^i, H_{c_1}^{ST}\} = \mu\delta^{ij}r_j^0 - a\mu q_1^i \tag{4.220}$$

which are identically satisfied after the substitutions of momenta. Equations of motion for momenta are

$$\dot{r}_i^0 = \{r_i^0, H_{c_1}^{ST}\} = -m^2\delta_{ij}q_0^j \tag{4.221}$$
$$\dot{r}_i^1 = \{r_i^1, H_{c_1}^{ST}\} = -a\delta_{ij}q_1^j + a\mu s_i^1 \tag{4.222}$$
$$\dot{s}_i^0 = \{s_i^0, H_{c_1}^{ST}\} = -m^2\delta^{ij}w_0^j \tag{4.223}$$
$$\dot{s}_i^1 = \{s_i^1, H_{c_1}^{ST}\} = -s_i^0 + a\delta_{ij}w_1^j - \mu r_i^1. \tag{4.224}$$

From these equations, the ones in (4.221) and (4.223) give the Euler-Lagrange equations of motion whereas the other two are identically satisfied. □

4.2.6. An Alternative Reduction to the First Order Formalism

Hamiltonian Formalism for $L^{ST}_{C_1}$: Consider the momentum phase space with coordinates $(w^i_0, w^i_1, \lambda^0_i, Y^i, \Pi^0_i, \Pi^1_i, \Pi^i_\lambda, \Pi^Y_j)$ and the canonical Poisson bracket defined as

$$\{w^i_o, \Pi^0_j\} = \{w^i_1, \Pi^1_j\} = \{\dot{Y}^i, \Pi^Y_i\} = \delta^i_j, \quad \{\lambda^0_i, \Pi^j_\lambda\} = \delta^j_i \tag{4.225}$$

and all others are zero. The fiber derivatives of $L^{ST}_{C_1}$ establish the relationship between the velocities and the momenta as follows

$$\Pi^0_i = \frac{\partial L^{ST}_{C_1}}{\partial \dot{w}^i_0} = \lambda^1_i \tag{4.226}$$

$$\Pi^1_i = \frac{\partial L^{ST}_{C_1}}{\partial \dot{w}^i_1} = \frac{1}{\mu}\delta_{ij}\dot{Y}^j \tag{4.227}$$

$$\Pi^i_\lambda = \frac{\partial L^{ST}_{C_1}}{\partial \dot{\lambda}^1_i} = 0 \tag{4.228}$$

$$\Pi^Y_i = \frac{\partial L^{ST}_{C_1}}{\partial \dot{Y}^i} = a\delta_{ij}\dot{Y}^j + \frac{1}{\mu}\delta_{ij}\dot{w}^j_1. \tag{4.229}$$

Proposition 4.2.14. *The total Hamiltonian function corresponding to the Lagrangian density* $L^{ST}_{C_1}$ *in* (4.130) *is given by*

$$\begin{aligned} H^{ST}_T &= \mu\delta^{ij}(\Pi^1_i\Pi^Y_j - \frac{a\mu}{2}\Pi^1_i\Pi^1_j) - \frac{a}{2}\delta_{ij}w^i_1w^j_1 + \Pi^0_iw^i_1 + \frac{m^2}{2}\delta_{ij}(Y^iY^j + w^i_0w^i_0) \\ &- \delta_{ij}m^2w^i_0\Pi^j_\lambda. \end{aligned} \tag{4.230}$$

Proof. From the conjugate momenta in equations (4.227) and (4.229), it is possible to solve $\dot{Y}$ and $\dot{w}_1$ as functions of coordinates and momenta given by

$$\dot{Y}^i = \mu\delta^{ij}\Pi^1_j \tag{4.231}$$

$$\dot{w}^i_1 = \mu\delta^{ij}\left(\Pi^Y_j - a\mu\Pi^1_j\right), \tag{4.232}$$

but, unfortunately, the others lead to primary constraints

$$\Psi^0_i = \Pi^0_i - \lambda^1_i \tag{4.233}$$

$$\Psi^i_\lambda = \Pi^i_\lambda. \tag{4.234}$$

The canonical Hamiltonian function (3.15) turns out to be

$$
\begin{aligned}
H_{c_1}^{ST} &= \Pi_i^0 \dot{w}_0^i + \Pi_i^1 \dot{w}_1^i + \Pi_\lambda^i \dot{\lambda}_i^1 + \Pi_i^Y \dot{Y}^i - L_{C_1}^{ST} \\
&= \Pi_i^0 \dot{w}_0^i + \Pi_i^1 \dot{w}_1^i + \Pi_\lambda^i \dot{\lambda}_i^1 + \Pi_i^Y \dot{Y}^i - \frac{\delta_{ij}}{2}\big[a(w_1^i w_1^j + \dot{Y}^i \dot{Y}^j) + \frac{2}{\mu}\dot{Y}^i \dot{w}_1^j \\
&- m^2(Y^i Y^j + w_0^i w_0^j)\big] - \lambda_j^1(\dot{w}_0^j - w_1^j).
\end{aligned} \tag{4.235}
$$

After the substitution of $\dot{Y}^i, \dot{w}_1^i$ and the primary constraints, the Hamiltonian function becomes

$$
H_{c_1}^{ST} = \delta^{ij}(\mu \Pi_i^1 \Pi_j^Y - \frac{a\mu^2}{2}\Pi_i^1 \Pi_i^1) - \frac{a}{2}\delta_{ij} w_1^i w_1^j + \Pi_i^0 w_1^i + \frac{m^2}{2}\delta_{ij}(Y^i Y^j + w_0^i w_0^j). \tag{4.236}
$$

We define the total Hamiltonian as

$$
H_T^{ST} = H_{c_1}^{ST} + u_0^i \Psi_i^0 + u_i^\lambda \Psi_\lambda^i \tag{4.237}
$$

where u_0^i, u_i^λ arbitrary function. The consistency checks for Ψ_i^0

$$
\begin{aligned}
\dot{\Psi}_i^0 = \{\Psi_i^0, H_T\} &\approx \{\Psi_i^0, H_{c_1}^{ST}\} + u_0^j \{\Psi_i^0, \Psi_j^0\} + u_j^\lambda \{\Psi_i^0, \Psi_\lambda^j\} \\
&\approx -m^2 \delta_{ij} w_0^j - u_i^\lambda
\end{aligned} \tag{4.238}
$$

and for Ψ_λ^i

$$
\dot{\Psi}_\lambda^i = \{\Psi_\lambda^i, H_T\} \approx \{\Psi_\lambda^i, H_{c_1}^{ST}\} + u_0^j \{\Psi_\lambda^i, \Psi_j^0\} + u_j^\lambda \{\Psi_\lambda^i, \Psi_\lambda^j\} \tag{4.239}
$$

$$
\approx u_0^i \tag{4.240}
$$

allow us to determine u_0^i and u_i^λ. The substitutions of u_0^i and u_i^λ lead to write the total Hamiltonian H_T^{ST} in an explicit form. □

Proposition 4.2.15. *Hamilton equations of motion using the total Hamiltonian function in* *proposition* 4.2.14 *are*

$$\dot{w}_0^i = w_1^i, \quad \dot{w}_1^i = \delta^{ij}(\mu\Pi_j^Y - a\mu^2\Pi_j^1), \quad \dot{Y}^i = \mu\delta^{ij}\Pi_j^1$$
$$\dot{\Pi}_i^1 = a\delta_{ij}w_1^j - \Pi_i^0, \quad \dot{\Pi}_i^\lambda = 0, \quad \dot{\Pi}_i^0 \approx -\delta_{ij}m^2 w_0^j, \quad \dot{\lambda}_i^1 = -m^2\delta_{ij}w_0^j \tag{4.241}$$
$$\dot{\Pi}_i^0 = -\delta_{ij}m^2 w_0^j, \quad \dot{\lambda}_i^1 = -\delta_{ij}m^2 w_0^j, \quad \dot{\Pi}_i^Y = -m^2\delta_{ij}Y^j.$$

Proof. The Hamilton's equations for w_0 determine one of the constraints

$$\dot{w}_0^i = \{w_0^i, H_T^{ST}\} = w_1^i. \tag{4.242}$$

The equations governing w_1^i, Y^i, Π_i^1 and Π_i^λ

$$\dot{w}_1^i = \{w_1^i, H_T^{ST}\} = \delta^{ij}(\mu\Pi_j^Y - a\mu^2\Pi_j^1) \tag{4.243}$$
$$\dot{Y}^i = \{Y^i, H_T^{ST}\} = \mu\delta^{ij}\Pi_j^1 \tag{4.244}$$
$$\dot{\Pi}_i^1 = \{\Pi_i^1, H_T^{ST}\} = a\delta_{ij}w_1^j - \Pi_i^0 \tag{4.245}$$
$$\dot{\Pi}_i^\lambda = \{\Pi_i^\lambda, H_T^{ST}\} = 0 \tag{4.246}$$

are satisfied identically using the definitions of momenta. The equations governing Π_i^0 and λ_i^1

$$\dot{\Pi}_i^0 = \{\Pi_i^0, H_T^{ST}\} = \delta_{ij} - m^2 w_0^j + m^2\Pi_\lambda^j \approx -\delta_{ij}m^2 w_0^j \tag{4.247}$$
$$\dot{\lambda}_i^1 = \{\lambda_i^1, H_T^{ST}\} = -m^2\delta_{ij}w_0^j \tag{4.248}$$

give one half of the Euler-Lagrange equations (4.79). To see this, we compute

$$\dot{\Pi}_i^0 = -\delta_{ij}m^2 w_0^j \tag{4.249}$$
$$\dot{\lambda}_i^1 = -\delta_{ij}m^2 w_0^j \tag{4.250}$$
$$a\dot{w}_1^i - \frac{1}{\mu}\ddot{Y}^i = -\delta_{ij}m^2 w_0^j \tag{4.251}$$
$$a\ddot{X}^i + m^2 X^i = \frac{1}{\mu}\ddot{Y}^i \tag{4.252}$$

using λ^1. The equations governing the other momenta Π_i^Y

$$\dot{\Pi}_i^Y = \{\Pi_i^Y, H_T^{ST}\} = -m^2 \delta_{ij} Y^j. \tag{4.253}$$

give the rest half of the Euler-Lagrange equations. See that,

$$\begin{aligned} \dot{\Pi}_i^Y &= -m^2 \delta_{ij} Y^j \\ a\ddot{Y}^i + \frac{1}{\mu}\ddot{w}_1^i &= -m^2 Y^i \\ a\ddot{Y}^i + m^2 Y^i &= -\frac{1}{\mu}\dddot{X}^i \end{aligned} \tag{4.254}$$

using definition of Π_i^Y. □

Dirac Bracket Formalism: Now we are going to arrive the Hamilton's equations by defining the Dirac bracket for the constraint space given by

$$\Psi_i^0 = \Pi_i^0 - \lambda_i^1 \tag{4.255}$$

$$\Psi_\lambda^i = \Pi_\lambda^i. \tag{4.256}$$

See that, Poisson brackets of constraints

$$\{\Psi_k^0, \Psi_n^0\} = \{\Psi_k^\lambda, \Psi_n^\lambda\} = 0 \tag{4.257}$$

$$\{\Psi_k^0, \Psi_\lambda^n\} = -\delta_k^n \tag{4.258}$$

are non vanishing. So that they are of the second class.

Proposition 4.2.16. *For the constraint space defined by* (4.255) *and* (4.256)*, Dirac brackets of the coordinates are*

$$\{w_0^i, \lambda_j^1\}_{DB} = \delta_j^i \tag{4.259}$$

$$\{w_0^i, \Pi_j^0\}_{DB} = \delta_j^i \tag{4.260}$$

$$\{w_1^i, \Pi_j^1\}_{DB} = \delta_j^i \tag{4.261}$$

$$\{Y^i, \Pi_j^Y\}_{DB} = \delta_j^i, \tag{4.262}$$

and the rest is zero.

Proof. The Dirac bracket presented in (2.39) turns out to be

$$\{F,G\}_{DB}=\{F,G\}-\{F,\Psi_k^0\}\delta_n^k\{\Psi_\lambda^n,G\}+\{F,\Psi_\lambda^k\}\delta_k^n\{\Psi_n^0,G\}. \tag{4.263}$$

where we substitute the inverse of

$$M=\begin{pmatrix}\{\Psi_k^0,\Psi_n^0\} & \{\Psi_k^0,\Psi_\lambda^n\}\\ \{\Psi_\lambda^k,\Psi_n^0\} & \{\Psi_\lambda^k,\Psi_\lambda^n\}\end{pmatrix}=\begin{pmatrix}0_{3\times3} & -\delta_k^n\\ \delta_n^k & 0_{3\times3}\end{pmatrix}. \tag{4.264}$$

To derive the Dirac bracket of the coordinates, we also need to the Poisson brackets of the coordinates with constraints i.e.

$$\{w_0^i,\Psi_k^0\}=\{\lambda^i,\Psi_k^\lambda\}=\delta_k^i,\quad \{\Pi_i^\lambda,\Psi_k^0\}=\delta_{ik} \tag{4.265}$$

and all the others are zero. The rest simply results of a direct calculation. To demonstrate this, let us prove some of the Dirac brackets

$$\{w_0^i,w_1^j\}_{DB}=\{w_0^i,w_1^j\}-\{w_0^i,\Psi_k^0\}\delta_n^k\{\Psi_\lambda^n,w_1^j\}+\{w_0^i,\Psi_\lambda^k\}\delta_k^n\{\Psi_n^0,w_1^j\}=0$$

since $\{w_0^i,w_1^j\}=\{\Psi_\lambda^n,w_1^j\}=\{\Psi_n^0,w_1^j\}=0$. See also that,

$$\{w_0^i,\lambda_j^1\}_{DB}=\{w_0^i,\lambda_j^1\}-\{w_0^i,\Psi_k^0\}\delta_n^k\{\Psi_\lambda^n,\lambda_j^1\}+\{w_0^i,\Psi_\lambda^k\}\delta_k^n\{\Psi_n^0,\lambda_j^1\}=\delta_j^i$$

using $\{w_0^i,\lambda_j^1\}=0$ and $\{w_0^i,\Psi_j^0\}=\{\lambda_i^1,\Psi_\lambda^j\}=\delta_i^j$. Similarly one can calculate the remaining Dirac brackets. □

Proposition 4.2.17. *Hamilton's equation of motion using the canonical Hamiltonian in* (4.236) *are*

$$\begin{aligned}&\dot{w}_0^i=w_1^i,\quad \dot{w}_1^i=\mu\delta^{ij}\Pi_j^Y-a\mu^2\delta^{ij}\Pi_j^1,\quad \dot{Y}^i=\mu\delta^{ij}\Pi_j^1\\ &\dot{\lambda}_i^1=-m^2\delta_{ij}w_0^j,\quad \dot{\Pi}_i^0=-m^2\delta_{ij}w_0^j,\quad \dot{\Pi}_i^1=a\delta_{ij}w_1^j-\Pi_i^0\\ &\dot{\Pi}_i^Y=-m^2\delta_{ij}Y^j,\quad \dot{\Pi}_\lambda^i=0.\end{aligned} \tag{4.266}$$

Proof. Recall the canonical Hamiltonian in (4.236), and the Dirac bracket presented in Proposition 4.2.16, the Hamilton's equation of motion are

$$
\begin{aligned}
\dot{w}_0^i &= \{w_0^i, H_c^{ST}\}_{DB} = \{w_0^i, \Pi_j^0\}_{DB} w_1^j = \delta_j^i w_1^j = w_1^i \\
\dot{w}_1^i &= \{w_1^i, H_c^{ST}\}_{DB} = \mu\delta^{jj'}\Pi_{j'}^Y\{w_1^i, \Pi_j^1\}_{DB} - \frac{a\mu^2}{2}\delta^{jj'}\{w_1^i, \Pi_j^1\Pi_{j'}^1\} \\
&= \mu\delta^{ij}\Pi_j^Y - a\mu^2\delta^{ij}\Pi_j^1 \\
\dot{Y}^i &= \{Y^i, H_c^{ST}\}_{DB} = \{Y^i, \mu\delta^{jj'}\Pi_j^1\Pi_{j'}^Y\}_{DB} = \mu\delta^{ij}\Pi_j^1 \\
\dot{\lambda}_i^1 &= \{\lambda_i^1, H_c^{ST}\}_{DB} = \{\lambda_i^1, \frac{m^2}{2}\delta_{jj'}w_0^j w_0^{j'}\}_{DB} = -m^2\delta_{ij}w_0^j \\
\dot{\Pi}_i^0 &= \{\Pi_i^0, H_c^{ST}\}_{DB} = \{\Pi_i^0, \frac{m^2}{2}\delta_{jj'}w_0^j w_0^{j'}\}_{DB} = -m^2\delta_{ij}w_0^j \\
\dot{\Pi}_i^1 &= \{\Pi_i^1, H_c^{ST}\}_{DB} = \{\Pi_i^1, \frac{a\mu^2}{2}\delta^{jj'}w_1^j w_1^{j'}\}_{DB} + \{\Pi_i^1, w_1^j\}\Pi_j^0 = a\delta_{ij}w_1^j - \Pi_i^0 \\
\dot{\Pi}_i^Y &= \{\Pi_i^Y, H_c^{ST}\}_{DB} = \{\Pi_i^Y, \frac{m^2}{2}\delta_{jj'}Y^j Y^{j'}\}_{DB} = -m^2\delta_{ij}Y^j \\
\dot{\Pi}_\lambda^i &= \{\Pi_\lambda^i, H_c^{ST}\}_{DB} = 0.
\end{aligned}
$$

□

4.2.7. Unconstraint Variational Formalism for $L_{C_1}^{ST}$

By substituting the Lagrange multiplier λ_i^1 in (4.133) into the Lagrangian function $L_{C_1}^{ST}$ in (4.130), we define an unconstraint Lagrangian

$$
\begin{aligned}
L_{U_1}^{ST} = \frac{\delta_{ij}}{2}\big[a(\dot{Y}^i\dot{Y}^j + w_1^i w_1^j) + \frac{2}{\mu}\dot{Y}^i\dot{w}_1^j - m^2(Y^iY^j + w_0^i w_0^j)\big] \\
+ \delta_{ij}(\dot{w}_0^i - w_1^i)(aw_1^j - \frac{1}{\mu}\ddot{Y}^j).
\end{aligned} \tag{4.267}
$$

This is a second order Lagrangian with respect to Y^i. We will apply similar analysis as it has been done for $L_{C_0}^{ST}$ in Section $4.2.5$, first reduce it into first order Lagrangian and express it in an unconstraint form. To reduce $L_{U_1}^{ST}$ we define the following transformations

$$
Y^i = q_0^i \quad \dot{Y}^i = \dot{q}_0^i = q_1^i, \quad \ddot{Y}^i = \dot{q}_1^i. \tag{4.268}
$$

As a result, we define two first order Lagrangian functions

$$L^{ST}_{NC^0} = L^{ST}_{U^{00}}(q^i_0, \dot{q}^i_0, \dot{q}^i_1, w^i_0, \dot{w}^i_0, w^i_1, \dot{w}^i_1) + \nu^0_j(\dot{q}^j_0 - q^j_1) \tag{4.269}$$

$$L^{ST}_{NC^1} = L^{ST}_{U^{10}}(q^i_0, q^i_1, \dot{q}^i_1, w^i_0, \dot{w}^i_0, w^i_1, \dot{w}^i_1) + \nu^1_j(\dot{q}^j_0 - q^j_1) \tag{4.270}$$

where ν^0_j and ν^1_j are Lagrange multipliers depending on $\dot{q}^0_i$ or q^i_1 used in $L^{ST}_{U_1}$. Thus, $L^{ST}_{U^{00}}$ and $L^{ST}_{U^{10}}$ are

$$\begin{aligned}
L^{ST}_{U^{00}} &= \frac{\delta_{ij}}{2}\left[a(\dot{q}^i_0\dot{q}^j_0 + w^i_1 w^j_1) + \frac{2}{\mu}\dot{q}^i_0\dot{w}^j_1 - m^2(q^i_0 q^j_0 + w^i_0 w^j_0)\right] \\
&\quad + \delta_{ij}(\dot{w}^i_0 - w^i_1)(a w^j_1 - \frac{1}{\mu}\dot{q}^j_1) \\
L^{ST}_{U^{10}} &= \frac{\delta_{ij}}{2}\left[a(q^i_1 q^j_1 + w^i_1 w^j_1) + \frac{2}{\mu}q^i_1\dot{w}^j_1 - m^2(q^i_0 q^j_0 + w^i_0 w^j_0)\right] \\
&\quad + \delta_{ij}(\dot{w}^i_0 - w^i_1)(a w^j_1 - \frac{1}{\mu}\dot{q}^j_1).
\end{aligned} \tag{4.271}$$

Now we have to write $L^{ST}_{NC^0}$ and $L^{ST}_{NC^1}$ given in (4.269) and (4.270) in an unconstraint form. The variation of $L^{ST}_{NC^0}$ and $L^{ST}_{NC^1}$ with respect to q^i_1 give the Lagrange multipliers respectively

$$\nu^0_i = -\delta_{ij}(\frac{1}{\mu}\dot{w}^j_1 - \ddot{w}^j_0) = 0 \tag{4.272}$$

$$\nu^1_i = \delta_{ij}(a q^j_1 + \frac{1}{\mu}\dot{w}^j_1). \tag{4.273}$$

Substitutions of ν^0_i and ν^1_i into $L^{ST}_{NC^0}$ and $L^{ST}_{NC^1}$ lead to first order unconstraint Lagrangians

$$L^{ST}_{NU^0} = L^{ST}_{U^{00}} \tag{4.274}$$

$$L^{ST}_{NU^1} = L^{ST}_{U^{10}} + \delta_{ij}(a q^j_1 + \frac{1}{\mu}\dot{w}^j_1)(\dot{q}^j_0 - q^j_1), \tag{4.275}$$

respectively.

Hamiltonian Formalism for $L^{ST}_{NU^0}$: Recalling the Lagrangian density $L^{ST}_{NU^0}$ in (4.274), the conjugate momenta $q_0^i, q_1^i, w_0^i, w_1^i$ are defined as

$$R_i^0 = \frac{\partial L_{NU^0}^{ST}}{\partial \dot{q}_0^i} = a\dot{q}_0^i + \frac{1}{\mu}\dot{w}_1^i \tag{4.276}$$

$$R_i^1 = \frac{\partial L_{NU_0}^{ST}}{\partial \dot{q}_1^i} = \frac{1}{\mu}(w_1^i - \dot{w}_0^i) \tag{4.277}$$

$$S_i^0 = \frac{\partial L_{NU_0}^{ST}}{\partial \dot{w}_0^i} = aw_1^i - \frac{1}{\mu}\dot{q}_1^i \tag{4.278}$$

$$S_i^1 = \frac{\partial L_{NU_0}^{ST}}{\partial \dot{w}_1^i} = \frac{1}{\mu}\dot{q}_0^i. \tag{4.279}$$

The only difference between conjugate momenta for $L_{NC^0}^{ST}$ and $L_{NC^1}^{ST}$ is the presence of w_1^i in S_i^0 instead of $\dot{w}_0^i$ as expected.

Proposition 4.2.18. *The canonical Hamiltonian function for the unconstraint Lagrangian $L_{NU^0}^{ST}$ is*

$$\begin{aligned} H_{c^0}^{ST} = &-\frac{a}{2}\delta^{ij}(\mu^2 S_i^1 S_j^1 + w_1^i w_1^j) + \mu\delta^{ij} R_i^0 S_j^1 + S_i^0(w_1^i - \mu\delta^{ij} R_j^1) \\ &+ \frac{m^2}{2}\delta_{ij}(q_0^i q_0^j + w_0^i w_0^j) + a\mu R_i^1 w_1^i. \end{aligned} \tag{4.280}$$

Proof. From the definitions of the conjugate momenta R_i^0, R_i^1, S_i^0 and S_i^1, it is possible to solve velocities as

$$\dot{q}_0^i = \mu\delta^{ij} S_j^1 \tag{4.281}$$

$$\dot{q}_1^i = \mu(aw_1^i - \delta^{ij} S_j^0) \tag{4.282}$$

$$\dot{w}_0^i = w_1^i - \mu\delta^{ij} R_j^1 \tag{4.283}$$

$$\dot{w}_1^i = \mu\delta^{ij}(R_j^0 - a\mu S_j^1). \tag{4.284}$$

The canonical Hamiltonian function is computed by

$$\begin{aligned} H_{c^0}^{ST} &= R_i^0\dot{q}_0^i + R_i^1\dot{q}_1^i + S_i^0\dot{w}_0^i + S_i^1\dot{w}_1^i - L_{NU^0}^{ST} \\ &= R_i^0\dot{q}_0^i + R_i^1\dot{q}_1^i + S_i^0\dot{w}_0^i + S_i^1\dot{w}_1^i - L_{C^0}^{ST} \\ &= R_i^0\dot{q}_0^i + R_i^1\dot{q}_1^i + S_i^0\dot{w}_0^i + S_i^1\dot{w}_1^i - \frac{\delta_{ij}}{2}\big[a(\dot{q}_0^i\dot{q}_0^j + w_1^i w_1^j) + \frac{2}{\mu}\dot{q}_0^i\dot{w}_1^j \\ &\quad - m^2(q_0^i q_0^j + w_0^i w_0^j)\big] - \delta_{ij}(\dot{w}_0^i - w_1^i)(aw_1^j - \frac{1}{\mu}\dot{q}_1^j). \end{aligned} \tag{4.285}$$

By substituting the velocities, we have

$$\begin{aligned} H^{ST}_{c^0} &= -\frac{a}{2}\delta^{ij}(\mu^2 S^1_i S^1_j + w^i_1 w^j_1) + \mu\delta^{ij} R^0_i S^1_j + S^0_i(w^i_1 - \mu\delta^{ij} R^1_j) \\ &\quad + \frac{m^2}{2}\delta_{ij}(q^i_0 q^j_0 + w^i_0 w^j_0) + a\mu R^1_i w^i_1. \end{aligned} \tag{4.286}$$

□

Proposition 4.2.19. *Hamilton's equation of motion using the canonical Hamiltonian in the proposition* 4.2.18 *are*

$$\dot{q}^i_0 = \mu S^1 \quad \dot{q}^i_1 = a\mu w^i_1 - \mu\delta^{ij} S^0_j, \quad \dot{w}_0 = w_1 - \mu R^1, \quad \dot{w}_1 = \mu R^0 - a\mu^2 S_1 \tag{4.287}$$

$$\dot{R}^0 = -m^2 q_0, \quad \dot{R}^1 = 0, \quad \dot{S}^0 = -m^2 w_0, \quad \dot{S}^1 = -S^0 + a w_1 - a\mu R^1_i. \tag{4.288}$$

Proof. The Hamilton's equations are

$$\dot{q}^i_0 = \{q_0, H^{ST}_{c^0}\} = \mu S^1 \tag{4.289}$$

$$\dot{q}^i_1 = \{q_1, H^{ST}_{c^0}\} = a\mu w^i_1 - \mu\delta^{ij} S^0_j \tag{4.290}$$

$$\dot{w}_0 = \{w_0, H^{ST}_{c^0}\} = w_1 - \mu R^1 \tag{4.291}$$

$$\dot{w}_1 = \{w_1, H^{ST}_{c^0}\} = \mu R^0 - a\mu^2 S_1 \tag{4.292}$$

$$\dot{R}^0 = \{r^0, H^{ST}_{c_0}\} = -m^2 q_0 \tag{4.293}$$

$$\dot{R}^1 = \{r^1, H^{ST}_{c_0}\} = 0 \tag{4.294}$$

$$\dot{S}^0 = \{s^0, H^{ST}_{c_0}\} = -m^2 w_0 \tag{4.295}$$

$$\dot{S}^1 = \{s^1, H^{ST}_{c_0}\} = -S^0 + a w_1 - a\mu R^1_i. \tag{4.296}$$

generated by the canonical Hamiltonian $H^{ST}_{c^0}$ in the proposition 4.2.18. The equations (4.293) and (4.295) give the Euler-Lagrange equations whereas the others are satisfied identically.

□

Hamiltonian Formalism for $L^{ST}_{NU^1}$: For the Lagrangian $L^{ST}_{NU^1}$ given in (4.275), canonical momenta are defined as

$$r_i^0 = \frac{\partial L^{ST}_{NU^1}}{\partial \dot{q}_0^i} = \delta_{ij}(aq_1^j + \frac{1}{\mu}\dot{w}_1^j) \tag{4.297}$$

$$r_i^1 = \frac{\partial L^{ST}_{NU^1}}{\partial \dot{q}_1^i} = \frac{1}{\mu}\delta_{ij}(w_1^j - \dot{w}_0^j) \tag{4.298}$$

$$s_i^0 = \frac{\partial L^{ST}_{NU^1}}{\partial \dot{w}_0^i} = \delta_{ij}(aw_1^j - \frac{1}{\mu}\dot{q}_1^j) \tag{4.299}$$

$$s_i^1 = \frac{\partial L^{ST}_{NU^1}}{\partial \dot{w}_1^i} = \frac{1}{\mu}\delta_{ij}\dot{q}_0^j. \tag{4.300}$$

Proposition 4.2.20. *Canonical Hamiltonian function for $L^{ST}_{NU^1}$ is*

$$\begin{aligned} H^{ST}_{C^1} &= s_i^0(w_1^i - \mu\delta^{ij}r_j^1) - \frac{1}{2}\delta_{ij}\left[a(-q_1^iq_1^j + w_1^iw_1^j) - m^2(q_0^iq_0^j + w_0^iw_0^j)\right] \\ &\quad + a\mu(r_i^1w_1^i - s_i^1q_1^i) + \mu\delta^{ij}s_i^1r_j^0 \end{aligned} \tag{4.301}$$

Proof. From conjugate momenta it is possible to solve the velocities as

$$\dot{q}_0^i = \delta^{ij}\mu s_j^1 \tag{4.302}$$

$$\dot{q}_1^i = \mu(aw_1^i - \delta^{ij}s_j^0) \tag{4.303}$$

$$\dot{w}_0^i = w_1^i - \delta^{ij}\mu r_j^1 \tag{4.304}$$

$$\dot{w}_1^i = \mu(\delta^{ij}r_j^0 - aq_1^i) \tag{4.305}$$

and the canonical Hamiltonian function (3.15) is

$$\begin{aligned} H^{ST}_{c^1} &= r_i^0\dot{q}_0^i + r_i^1\dot{q}_1^i + s_i^0\dot{w}_0^i + s_i^1\dot{w}_1^i - L^{ST}_{NU^1} \\ &= r_i^0\dot{q}_0^i + r_i^1\dot{q}_1^i + s_i^0\dot{w}_0^i + s_i^1\dot{w}_1^i - L^{ST}_{C^1} - \delta_{ij}(aq_1^j + \frac{1}{\mu}\dot{w}_1^j)(\dot{q}_0^j - q_1^j). \end{aligned} \tag{4.306}$$

Substitution of the velocities give canonical Hamiltonian function

$$\begin{aligned} H^{ST}_{C^1} &= s_i^0(w_1^i - \mu\delta^{ij}r_j^1) - \frac{1}{2}\delta_{ij}\left[a(-q_1^iq_1^j + w_1^iw_1^j) - m^2(q_0^iq_0^j + w_0^iw_0^j)\right] \\ &\quad + a\mu(r_i^1w_1^i - s_i^1q_1^i) + \mu\delta^{ij}s_i^1r_j^0. \end{aligned} \tag{4.307}$$

□

Proposition 4.2.21. *Hamilton equations of motion using the Hamiltonian function given in* *the proposition* 4.2.20 *are*

$$\dot{q}_0^i = \mu\delta^{ij}s_j^1, \quad \dot{q}_1^i = -\mu\delta^{ij}s_j^0 + a\mu w_1^i, \quad \dot{w}_0^i = w_1^i - \mu\delta^{ij}r_j^1$$
$$\dot{w}_1^i = \mu\delta^{ij}r_j^0 - a\mu q_1^i, \quad \dot{r}_i^0 = -m^2\delta_{ij}q_0^j, \quad \dot{r}_i^1 = -a\delta_{ij}q_1^j + a\mu s_i^1 \tag{4.308}$$
$$\dot{s}_i^0 = -m^2\delta_{ij}w_0^j, \quad \dot{s}_i^1 = -s_i^0 + a\delta_{ij}w_1^j - a\mu r_i^1 \tag{4.309}$$

Proof. Using Hamiltonian function in the proposition 4.2.20, equations of motion for canonical coordinates are

$$\dot{q}_0^i = \{q_0^i, H_{C^1}^{ST}\} = \mu\delta^{ij}s_j^1 \tag{4.310}$$
$$\dot{q}_1^i = \{q_1^i, H_{C^1}^{ST}\} = -\mu\delta^{ij}s_j^0 + a\mu w_1^i \tag{4.311}$$
$$\dot{w}_0^i = \{w_0^i, H_{C^1}^{ST}\} = w_1^i - \mu\delta^{ij}r_j^1 \tag{4.312}$$
$$\dot{w}_1^i = \{w_1^i, H_{C^1}^{ST}\} = \mu\delta^{ij}r_j^0 - a\mu q_1^i \tag{4.313}$$

which are satisfied identically using definitions of momenta. Equations of motion for momenta are

$$\dot{r}_i^0 = \{r_i^0, H_{C^1}^{ST}\} = -m^2\delta_{ij}q_0^j \tag{4.314}$$
$$\dot{r}_i^1 = \{r_i^1, H_{C^1}^{ST}\} = -a\delta_{ij}q_1^j + a\mu s_i^1 \tag{4.315}$$
$$\dot{s}_i^0 = \{s_i^0, H_{C^1}^{ST}\} = -m^2\delta_{ij}w_0^j \tag{4.316}$$
$$\dot{s}_i^1 = \{s_i^1, H_{C^1}^{ST}\} = -s_i^0 + a\delta_{ij}w_1^j - a\mu r_i^1. \tag{4.317}$$

The equations (4.314) and (4.316) give the Euler-Lagrange equations (4.79) and the remaining are satisfied identically using the definition of momenta. □

4.3. CLÈMENT LAGRANGIAN

4.3.1. General Setting

Consider a three dimensional space M with local coordinates $X = (X^i)$. We introduce the following coordinates on the bundles

$$\begin{gathered} X \in M, \\ (X, \dot{X}) \in TM \\ (X, \dot{X}, \ddot{X}) \in T^2M \\ (X, \dot{X}, \ddot{X}, \dddot{X}) \in T^3M \\ (X, \dot{X}, P^0, P^1) \in T^*TM. \end{gathered}$$

Let us consider the following second order degenerate Lagrangian

$$L^C[X^i] = -\frac{m\zeta}{2}\delta_{ij}\dot{X}^i\dot{X}^j - 2m\zeta^{-1}\Lambda + \frac{\zeta^2}{2\mu m}\epsilon_{ijk}X^i\dot{X}^j\ddot{X}^k, \tag{4.318}$$

introduced by Clèment [37]. Here, the inner product $X^2 = T^2 - X^2 - Y^2$ is defined by the Lorentzian metric and the triple product is $\epsilon_{ijk}X^i\dot{X}^j\ddot{X}^k$. $\zeta = \zeta(t)$ is a function which allows arbitrary reparametrization of the variable t whereas Λ and $1/2m$ are the cosmological and Einstein gravitational constants, respectively. The second order Euler-Lagrange equations (3.4) governed by the Clèment Lagrangian are computed to be

$$-m\zeta\ddot{X}^i + \frac{\zeta^2}{\mu m}\delta^{ij}\epsilon_{jkl}\dddot{X}^k X^l + \frac{3\zeta^2}{2\mu m}\delta^{ij}\epsilon_{jkl}\ddot{X}^k\dot{X}^l = 0. \tag{4.319}$$

4.3.2. Jacobi-Ostrogradsky Method

On the Hamiltonian phase space T^*TM, the canonical Poisson bracket relations are defined as

$$\{X^i, P_j^0\} = \{\dot{X}^i, P_j^1\} = \delta_j^i \tag{4.320}$$

and all others are zero. Legendre transformations (3.9) and (3.10) are introduced through the definition of Jacobi-Ostrogradsky momenta as

$$P_i^0 = \frac{\partial L^C}{\partial \dot{X}^i} - \frac{d}{dt}\left(\frac{\partial L^C}{\partial \ddot{X}^i}\right) = -m\zeta\delta_{ij'}\dot{X}^{j'} + \frac{\zeta^2}{\mu m}\epsilon_{ink}\ddot{X}^n X^k \tag{4.321}$$

$$P_i^1 = \frac{\partial L^C}{\partial \ddot{X}^i} = \frac{\zeta^2}{2\mu m}\epsilon_{ikl}X^k\dot{X}^l \tag{4.322}$$

conjugated respectively to X^i and $\dot{X}^i$.

Proposition 4.3.1. *Total Hamiltonian function for the Clèment Lagrangian is*

$$\begin{aligned} H_{T^1}^C = \frac{1}{2}\dot{X}^i P_i^0 - \frac{3}{2m\zeta X^2}(XP_i^1)(\dot{X}^i B_i) - \frac{\mu m}{\zeta^2 X^2}\epsilon^{ijk}P_i^1 B_j \delta_{kl} X^l \\ + \frac{1}{2X^2}(B_i X^i)(\delta_{ij}X^i\dot{X}^j) - \frac{1}{m\zeta X^2}(X^i B_i)^2. \end{aligned} \tag{4.323}$$

Here, we used the abbreviations $B_i = m\zeta\delta_{ij}\dot{X}^j + P_i^0$ *and* $X^2 = \delta_{ij}X^iX^j$.

Proof. It is not possible to solve any component of $\ddot{X}^i$ from (4.322), instead we have to define 3 primary constraints

$$\Phi_i = P_i^1 - \frac{\zeta^2}{2\mu m}\epsilon_{ikl}X^k\dot{X}^l$$

from equation (4.322). The canonical Hamiltonian function (3.15) turns out to be

$$\begin{aligned} H^C &= P_j^0\dot{X}^j + P_j^1\ddot{X}^j - L^C \\ &= \frac{m}{2}\zeta\delta_{jj'}\dot{X}^j\dot{X}^{j'} + 2m\zeta^{-1}\Lambda + \dot{X}^j P_j^0 \end{aligned} \tag{4.324}$$

using (4.321) and (4.322) whereas the total Hamiltonian (3.19) becomes

$$\begin{aligned} H_T^C &= H^C + U^j\Phi_j \\ &= \frac{m}{2}\zeta\delta_{jj'}\dot{X}^j\dot{X}^{j'} + 2m\zeta^{-1}\Lambda + \dot{X}^j P_j^0 + U^j\big(P_j^1 - \frac{\zeta^2}{2\mu m}\epsilon_{jkl}X^k\dot{X}^l\big). \end{aligned} \tag{4.325}$$

Here, U^j's are arbitrary functions of the canonical variables. Consistency of the primary constraint Φ_i

$$\dot{\Phi}_i = \{\Phi_i, H^C_T\} \approx \{\Phi_i, H^C\} + U^j\{\Phi_i, \Phi_j\}$$
$$\approx -m\zeta\delta_{ij}\dot{X}^j - P^0_i + \frac{\zeta^2}{\mu m}\epsilon_{ijk}U^jX^k = 0 \tag{4.326}$$

leads to a secondary constraint

$$\Phi = X^i(m\zeta\delta_{ij}\dot{X}^j + P^0_i). \tag{4.327}$$

Note that this secondary constraint Φ also follows from equation (4.321). This is not suprising as we have explained in section $(3.2.1)$. We revise the total Hamiltonian presented in (4.325) as

$$H^C_{T^1} = H^C_T + U\Phi$$
$$= H^C + U^j\Phi_j + U\Phi \tag{4.328}$$

with the introduction of a Lagrange multiplier U. The consistency condition of secondary constraint Φ gives that

$$\dot{\Phi} = \{\Phi, H^C_{T^1}\} \approx \{\Phi, H^C\} + U^j\{\Phi, \Phi_j\} + U\{\Phi, \Phi\}$$
$$\approx (m\zeta\delta_{ij}\dot{X}^j + P^0_i)\dot{X}^i + U^j\big[\delta_{jk}m\zeta X^k + \frac{\zeta^2}{2\mu m}\epsilon_{jkl}X^k\dot{X}^l\big] = 0. \tag{4.329}$$

whereas the consistencies of the primary constraints Φ give

$$\dot{\Phi}_i = \{\Phi_i, H^C_{T^1}\} \approx \{\Phi_i, H^C\} + U^j\{\Phi_i, \Phi_j\} + U\{\Phi_i, \Phi\}$$
$$\approx -m\zeta\delta_{ij}\dot{X}^j - P^0_i + \frac{\zeta^2}{\mu m}\epsilon_{ijk}U^jX^k + U\big[-m\zeta\delta_{ij}X^j - \frac{\zeta^2}{2\mu m}\epsilon_{ijk}X^j\dot{X}^k\big] = 0. \tag{4.330}$$

No further constraint arises, instead we can solve the Lagrange multipliers U^i and U using

(4.329) and (4.330). Accordingly, we have

$$U^i \approx \frac{\mu m}{\zeta^2 X^2}\epsilon^{ijk}\delta_{jl}X^l B_k - \frac{3}{2}\left(\frac{B_j \dot{X}^j}{m\zeta X}2\right)X^i \tag{4.331}$$

$$U \approx -\frac{1}{m\zeta X^2}X^i B_i = -\frac{1}{m\zeta X^2}\Phi. \tag{4.332}$$

Here, we used the abbreviations $B_i = m\zeta\delta_{ij}\dot{X}^j + P_i^0$ and $X^2 = \delta_{ij}X^i X^j$. Substitutions of the constraints Φ_i, Φ and Lagrange multipliers U^i, U in (4.331), (4.332) into the total Hamiltonian function $H_{T^1}^C$ in (4.328) prove the proposition. □

Proposition 4.3.2. *Hamilton equations of motion for the total Hamiltonian function H_{T^1} in the proposition 4.3.1 are*

$$\dot{X}^i \approx \frac{1}{2}\dot{X}^i + \frac{\mu m}{\zeta^2 X^2}\epsilon^{ijk}\delta_{kl}P_j^1 X^l + \frac{1}{2X^2}X^i(\delta_{kj}X^k\dot{X}^j) \tag{4.333}$$

$$\ddot{X}^i \approx \frac{\mu m}{\zeta^2 X^2}\epsilon^{ijk}\delta_{jl}X^l B_k - \frac{3}{2\zeta X^2}(B_j\dot{X}^j)X^i \tag{4.334}$$

$$\begin{aligned}\dot{P}_i^0 \approx{}& \frac{\mu m}{\zeta^2 X^2}\delta_{ij}\epsilon^{jkl}P_k^1 B_l + \frac{3}{2m\zeta X^2}(B_j\dot{X}^j)P_i^1 - \frac{1}{2X^2}(\delta_{jk}\dot{X}^j X^k)B_i\\ &- \frac{2}{X^4}\frac{\mu m}{\zeta^2}(\epsilon^{rsk}\delta_{kl}P_r^1 B_s X^l)\delta_{ij}X^j\end{aligned} \tag{4.335}$$

$$\dot{P}_i^1 \approx -\frac{1}{2}P_i^0 - \frac{\mu m^2}{\zeta X^2}\epsilon_{ijk}\delta^{jl}P_l^1 X^k - \frac{m\zeta}{2X^2}X^i(\delta_{jk}X^j\dot{X}^k). \tag{4.336}$$

Proof. Using the total Hamiltonian $H_{T^1}^C$ in the proposition 4.3.1, equation of motion for X^i is identically satisfied. To have this, consider the following calculation

$$\begin{aligned}\dot{X}^i = \{X^i, H_{T^1}^C\} &\approx \frac{1}{2}\dot{X}^i + \frac{\mu m}{\zeta^2 X^2}\epsilon^{ijk}\delta_{lk}P_j^1 X^l + \frac{1}{2X^2}(\delta_{kj}X^k\dot{X}^j)X^i\\ &\approx \frac{1}{2}\dot{X}^i - \frac{1}{2X^2}\left[X^i(\delta_{jk}X^j\dot{X}^k) - \dot{X}^i(\delta_{jk}X^j X^k)\right] + \frac{1}{2X^2}(\delta_{jk}X^j\dot{X}^k)X^i\\ &\approx \dot{X}^i.\end{aligned} \tag{4.337}$$

Likewise equation of motion for $\dot{X}^i$ is identically satisfied

$$\begin{aligned}
\ddot{X}^i = \{\dot{X}^i, H^C_{T^1}\} &\approx \frac{\mu m}{\zeta^2 X^2}\epsilon^{ijk}\delta_{jl}X^l B_k - \frac{3}{2m\zeta X^2}(B_j\dot{X}^j)X^i \\
&\approx \frac{1}{X^2}\big[\ddot{X}^i(\delta_{jk}X^j X^k) - X^i(\delta_{jk}X^j\ddot{X}^k)\big] - \frac{3\zeta}{2\mu m^2}(\epsilon_{jkl}\ddot{X}^j X^k \dot{X}^l)X^i \\
&\approx \ddot{X}^i - \frac{1}{m\zeta X^2}\big[m\zeta\delta_{jk}X^j\ddot{X}^k + \frac{3\zeta^2}{2\mu m}\epsilon_{jkl}\ddot{X}^j X^k\dot{X}^l\big]X^i \qquad (4.338)
\end{aligned}$$

since the term in the parenthesis is zero, it is the dot product of Euler-Lagrange equations of motion (4.319) with X^i.

Equation of motion for P^0_i gives Euler-Lagrange equations of motion (4.319). To show this we perform the following computation

$$\begin{aligned}
\dot{P}^0_i &= \{P^0_i, H^C_{T^1}\} \\
&\approx \frac{\mu m}{\zeta^2 X^2}\delta_{ij}\epsilon^{jkl}P^1_k B_l + \frac{3}{2m\zeta X^2}(B_j\dot{X}^j)P^1_i - \frac{1}{2X^2}(\delta_{jk}\dot{X}^j X^k)B_i \\
&- \frac{2}{X^4}\frac{\mu m}{\zeta^2}(\epsilon^{rsk}\delta_{kl}P^1_r B_s X^l)\delta_{ij}X^j \\
&\approx \frac{-1}{2X^2}(\delta_{ij}X^j(B_k\dot{X}^k) - \delta_{ij}\dot{X}^j(B_k X^k)) + \frac{3\zeta}{4\mu m^2 X^2}(B_l\dot{X}^l)\epsilon_{ijk}X^j\dot{X}^k \\
&- \frac{1}{2X^2}(\delta_{jk}\dot{X}^j X^k)B_i + \frac{1}{X^4}\big(X^2(B_k\dot{X}^k) - (X^k B_k)(\delta_{lr}X^l\dot{X}^r)\big)\delta_{ij}X^j
\end{aligned}$$

using definition of P^1_i. After Substituting the constraint $\Phi = B_i X^i = 0$ if we use definition of B_i

$$\begin{aligned}
\dot{P}^0_i &\approx \frac{-1}{2X^2}\delta_{ij}X^j(B_k\dot{X}^k) + \frac{3\zeta}{4\mu m^2 X^2}(B_l\dot{X}^l)\epsilon_{ijk}X^j\dot{X}^k - \frac{1}{2X^2}(\delta_{jk}\dot{X}^j X^k)B_i \\
&\approx \frac{-1}{2X^2}\epsilon_{ijk}\dot{X}^j(\epsilon^{klr}\delta_{rs}B_l\dot{X}^s) + \frac{3\zeta}{4\mu m^2 X^2}(B_l\dot{X}^l)\epsilon_{ijk}X^j\dot{X}^k \\
&\approx \frac{\zeta^2}{2\mu m}\epsilon_{ijk}\dot{X}^j\ddot{X}^k - \frac{\zeta^2}{2\mu m X^2}\epsilon_{ijk}\dot{X}^J X^k\big[\delta_{rl}X^r\ddot{X}^l + \frac{3\zeta}{2\mu m^2}\epsilon_{lrs}\dot{X}^l\ddot{X}^r X^s\big]
\end{aligned} \qquad (4.339)$$

the term in the parenthesis is zero since it is the dot product of Euler-Lagrange equations of motion (4.319) with X^i and the remaining term leads to Euler-Lagrange equations of motion (4.319) using the definition of P^0_i.

The equation of motion for P_i^1 is identically satisfied

$$\begin{aligned}\dot{P}_i^1 = \{P_i^1, H_{T^1}^C\} &\approx -\frac{1}{2}P_i^0 - \frac{\mu m^2}{\zeta X^2}\epsilon_{ijk}\delta^{jl}P_l^1 X^k - \frac{m\zeta}{2X^2}(\delta_{jk}X^j\dot{X}^k)X^i \\ &\approx -\frac{1}{2}P_i^0 + \frac{m\zeta}{2X^2}\left[X^i(\delta_{jk}X^j\dot{X}^k) - \dot{X}^i(\delta_{jk}X^jX^k)\right] - \frac{m\zeta}{2X^2}X^i(\delta_{jk}X^j\dot{X}^k) \\ &\approx -\frac{1}{2}P_i^0 - \frac{m\zeta}{2}\delta_{ij}\dot{X}^j \approx \dot{P}_i^1 \end{aligned} \tag{4.340}$$

using the definition of P_i^0 and P_i^1. □

4.3.3. Dirac Bracket Formalism

In this part, we will construct the Dirac brackets using the constraints

$$\Phi_i = P_i^1 - \frac{\zeta^2}{2\mu m}\epsilon_{ikk'}X^k\dot{X}^{k'} \tag{4.341}$$

$$\chi = X^l\left(m\zeta\delta_{ll'}\dot{X}^{l'} + P_l^0\right) \tag{4.342}$$

which are of second class.

Proposition 4.3.3. *Dirac brackets of canonical coordinates are*

$$\begin{aligned}
\{X^i, X^j\}_{DB} &= 0 \\
\{\dot{X}^i, \dot{X}^j\}_{DB} &= \frac{-\mu}{\zeta^3X^2}\epsilon^{ijk}A_k \\
\{X^i, \dot{X}^j\}_{DB} &= \frac{-1}{m\zeta X^2}X^iX^j \\
\{X^i, P_j^0\}_{DB} &= \delta_j^i - \frac{\zeta}{2m^2\mu X^2}X^i\epsilon_{jkn}\dot{X}^kX^n \\
\{X^i, P_j^1\}_{DB} &= 0 \\
\{\dot{X}^i, P_j^0\}_{DB} &= -\frac{1}{2m\zeta X^2}A_j\dot{X}^i + \frac{1}{2m\zeta X^2}(A_k\dot{X}^k)\delta_j^i - \frac{1}{m\zeta X^2}X^iB_j \\
\{\dot{X}^i, P_j^1\}_{DB} &= \delta_j^i - \frac{1}{2m\zeta X^2}(A_kX^k)\delta_j^i + \frac{1}{2m\zeta X^2}A_jX^i - \frac{1}{X^2}X^i\delta_{jl}X^l \\
\{P_i^0, P_j^0\}_{DB} &= \frac{\zeta}{4m^2\mu X^2}\epsilon_{jil}\dot{X}^l(A_k\dot{X}^k) - \frac{\zeta}{2m^2\mu X^2}(\epsilon_{ikn}\dot{X}^kX^nB_j - \epsilon_{jkn}\dot{X}^kX^nB_i) \\
\{P_i^0, P_j^1\}_{DB} &= \frac{-\zeta}{4m^2\mu X^2}\epsilon_{jik}\dot{X}^k(A_lX^l) + \frac{\zeta}{4m^2\mu X^2}A_j\epsilon_{ikl}\dot{X}^kX^l \\
\{P_i^1, P_j^1\}_{DB} &= \frac{-\zeta}{4m^2\mu X^2}\epsilon_{jki}X^k(A_lX^l).
\end{aligned}$$

where we used the abbreviations $A_i = m\zeta\delta_{ij}X^j + P_i^1, B_i = m\zeta\delta_{ij}\dot{X}^j + P_i^0$ *and* $X^2 = \delta_{ij}X^iX^j$.

Proof. To derive these relations, we use the Dirac bracket (2.39)

$$\begin{aligned}\{F,G\}_{DB} &= \{F,G\} - \{F,\Phi_n\}M^{nn'}\{\Phi_{n'},G\} - \{F,\Phi_n\}M^{n1}\{\chi,G\} \\ &\quad - \{F,\chi\}M^{1n'}\{\Phi_{n'},G\}\end{aligned} \tag{4.343}$$

with the substitution of the inverse matrix

$$M^{-1} = \begin{pmatrix} M^{nn'} & M^{n1} \\ M^{1n'} & M^{11} \end{pmatrix} = \begin{pmatrix} \frac{-\mu\epsilon^{nn'k}(m\zeta\delta_{sk}X^s + P_k^1)}{\zeta^3X^2} & \frac{X^n}{m\zeta X^2} \\ -\frac{X^{n'}}{m\zeta X^2} & 0 \end{pmatrix}$$

of the matrix

$$\begin{aligned} M &= \begin{pmatrix} \{\Phi_n,\Phi_{n'}\} & \{\Phi_n,\chi\} \\ \{\chi,\Phi_{n'}\} & \{\chi,\chi\} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\zeta^2}{\mu m}\epsilon_{nn'k}X^k & -m\zeta\delta_{nl}X^l - \frac{\zeta^2}{2\mu m}\epsilon_{nlk'}X^l\dot{X}^{k'} \\ m\zeta\delta_{n'l}X^l + \frac{\zeta^2}{2\mu m}\epsilon_{n'lk'}X^l\dot{X}^{k'} & 0 \end{pmatrix} \end{aligned}$$

First, we define the Poisson brackets of coordinates $X^i, \dot{X}^i$ and P_i^0, P_i^1 with the set of constraints as follows

$$\{X^i,\chi\} = \{X^i, X^l(m\zeta\delta_{ll'}\dot{X}^{l'} + P_l^0)\} = X^i \tag{4.344}$$

$$\{\dot{X}^i,\Phi_n\} = \{\dot{X}^i, P_n^1 - \frac{\zeta^2}{2\mu m}\epsilon_{nkk'}X^k\dot{X}^{k'}\} = \delta_n^i \tag{4.345}$$

$$\{P_i^0,\Phi_n\} = \{P_i^0, P_n^1 - \frac{\zeta^2}{2\mu m}\epsilon_{nkk'}X^k\dot{X}^{k'}\} = \frac{\zeta^2}{2\mu m}\epsilon_{nik'}\dot{X}^{k'} \tag{4.346}$$

$$\{P_i^0,\chi\} = \{P_i^0, X^l(m\zeta\delta_{ll'}\dot{X}^{l'} + P_l^0)\} = -m\zeta\delta_{il'}\dot{X}^{l'} - P_i^0 \tag{4.347}$$

$$\{P_i^1,\Phi_n\} = \{P_i^1, P_n^1 - \frac{\zeta^2}{2\mu m}\epsilon_{nkk'}X^k\dot{X}^{k'}\} = \frac{\zeta^2}{2\mu m}\epsilon_{nki}X^k \tag{4.348}$$

$$\{P_i^1,\chi\} = \{P_i^1, X^l(m\zeta\delta_{ll'}\dot{X}^{l'} + P_l^0)\} = -m\zeta\delta_{il}X^l \tag{4.349}$$

and all the others are zero. Substitution of these into the bracket (4.343) lead to the proof of the theorem. For example, the Dirac bracket of X^i and $\dot{X}^i$ is computed by

$$\begin{aligned}\{X^i,\dot{X}^j\}_{DB} &= \{X^i,\dot{X}^j\} - \{X^i,\Phi_n\}M^{nn'}\{\Phi_{n'},\dot{X}^j\} - \{X^i,\Phi_n\}M^{n1}\{\chi,\dot{X}^j\} \\ &- \{X^i,\chi\}M^{1n'}\{\Phi_{n'},\dot{X}^j\} \\ &= -(X^i)(-\frac{X^{n'}}{m\zeta X^2})(-\delta^j_{n'}) \\ &= \frac{-1}{m\zeta X^2}X^iX^j \end{aligned} \tag{4.350}$$

since $\{X^i,\dot{X}^j\} = \{X^i,\Phi_n\} = 0$. Dirac brackets for the other coordinates can be derived in a similar way. □

Proposition 4.3.4. *Hamilton equations using the Dirac algebra of the constraints in the proposition* 4.3.3 *are*

$$\dot{X}^i = \frac{-1}{m\zeta X^2}X^iX^jB_j + \dot{X}^i \tag{4.351}$$

$$\ddot{X}^i = \frac{-\mu}{\zeta^3X^2}\epsilon^{ijk}B_jA_k - \frac{X^i}{m\zeta X^2}B_j\dot{X}^j \tag{4.352}$$

$$\begin{aligned}\dot{P}^0_i &= \frac{-1}{2m\zeta X^2}B_i(A_j\dot{X}^j) + \frac{1}{2m\zeta X^2}A_i(B_k\dot{X}^k) + \frac{1}{m\zeta X^2}B_i(X^jB_j) \\ &- \frac{\zeta}{2\mu m^2X^2}\epsilon_{ikl}\dot{X}^kX^l(B_j\dot{X}^j)\end{aligned} \tag{4.353}$$

$$\dot{P}^1_i = -B_i + \frac{1}{2m\zeta X^2}B_i(A_kX^k) - \frac{1}{2m\zeta X^2}A_i(B_jX^j) + \frac{1}{X^2}\delta_{ik}X^k(X^jB_j) \tag{4.354}$$

where we used the abbreviations $A_i = m\zeta\delta_{ij}X^j + P^1_i, B_i = m\zeta\delta_{ij}\dot{X}^j + P^0_i$ *and* $X^2 = \delta_{ij}X^iX^j$.

Proof. Using the Dirac brackets of the coordinates in the proposition 4.3.3 and the Hamiltonian function (4.324), equations of motion for X^i's

$$\begin{aligned}\dot{X}^i &= \{X^i, H^C\}_{DB} \\ &= \{X^i, \dot{X}^j\}_{DB}(m\zeta\delta_{jj'}\dot{X}^{j'} + P^0_j) + \{X^i, P^0_j\}_{DB}\dot{X}^j \\ &= \big(\frac{-1}{m\zeta X^2}X^iX^j\big)(m\zeta\delta_{jj'}\dot{X}^{j'} + P^0_j) + \big(\delta^i_j - \frac{\zeta}{2m^2\mu X^2}X^i\epsilon_{jk'n'}\dot{X}^{k'}X^{n'}\big)\dot{X}^j \\ &= \frac{-1}{m\zeta X^2}X^iX^j(m\zeta\delta_{jj'}\dot{X}^{j'} + P^0_j) + \dot{X}^i \\ &= \frac{-1}{m\zeta X^2}X^iX^j\big(\frac{\zeta^2}{\mu m}\epsilon_{jkl}\ddot{X}^kX^l\big) + \dot{X}^i \end{aligned} \tag{4.355}$$

after cancellation of $\dot{X}^i$, the remaining gives secondary constraint Φ since X^i is nonzero.

The equation of motion for $\dot{X}^i$'s are

$$\begin{aligned}\ddot{X}^i &= \{\dot{X}^i, H^C\}_{DB} \\ &= \{\dot{X}^i, \dot{X}^j\}_{DB}(m\zeta\delta_{jj'}\dot{X}^{j'} + P^0_j) + \{\dot{X}^i, P^0_j\}_{DB}\dot{X}^j \\ &= \frac{-\mu}{\zeta^3X^2}\epsilon^{ijk}A_kB_j - \frac{X^i}{m\zeta X^2}B_j\dot{X}^j. \end{aligned} \tag{4.356}$$

Substitutions of $A_k = m\zeta\delta_{kl}X^l + P^1_k$ and $B_k = m\zeta\delta_{kl}\dot{X}^l + P^0_k$ into equation (4.356) give

$$\begin{aligned}\ddot{X}^i = &\frac{-\mu}{\zeta^3X^2}\epsilon^{ijk}(m\zeta\delta_{sk}X^s + P^1_k)(m\zeta\delta_{jj'}\dot{X}^{j'} + P^0_j) \\ &- \frac{X^i}{m\zeta X^2}(m\zeta\delta_{jl'}\dot{X}^{l'} + P^0_j)\dot{X}^j \end{aligned} \tag{4.357}$$

and with the help of P^0_i and P^1_i the last equation can be written as

$$\ddot{X}^i = \frac{-1}{X^2}(\delta_{sn}X^s\ddot{X}^n)X^i + \ddot{X}^i - \frac{3\zeta}{2\mu m^2X^2}\big(\epsilon_{jnk}\dot{X}^j\ddot{X}^nX^k\big)X^i. \tag{4.358}$$

After cancellation we get

$$\frac{-X^i}{X^2}\left(\delta_{sn}X^s\ddot{X}^n + \frac{3\zeta}{2\mu m^2}\epsilon_{jnk}\dot{X}^j\ddot{X}^nX^k\right) = 0 \tag{4.359}$$

since X^i is nonzero the term in the parenthesis must be zero. This is not a contradiction since it is the dot product of the Euler-Lagrange equation (4.319) with X^i.

Hamilton equations of motion for P_i^0's are

$$
\begin{aligned}
\dot{P}_i^0 &= \{P_i^0, H^C\}_{DB} \\
&= \{P_i^0, \dot{X}^j\}_{DB}(m\zeta\delta_{jj'}\dot{X}^{j'} + P_j^0) + \{P_i^0, P_j^0\}_{DB}\dot{X}^j \\
&= \frac{-1}{2m\zeta X^2}B_i(A_j\dot{X}^j) + \frac{1}{2m\zeta X^2}A_i(B_k\dot{X}^k) + \frac{1}{m\zeta X^2}B_i(X^jB_j) \\
&- \frac{\zeta}{2\mu m^2X^2}\epsilon_{ikl}\dot{X}^kX^l(B_j\dot{X}^j)
\end{aligned} \tag{4.360}
$$

First line in the equation (4.360) can be written explicitly as

$$
\begin{aligned}
&\frac{-1}{2m\zeta X^2}B_i(A_j\dot{X}^j) + \frac{1}{2m\zeta X^2}A_i(B_k\dot{X}^k) + \frac{1}{m\zeta X^2}B_i(X^jB_j) \\
&= \Bigg(\frac{-1}{2m\zeta X^2}(m\zeta\delta_{si}X^s + P_i^1)\dot{X}^j + \frac{1}{2m\zeta X^2}(m\zeta\delta_{sk}X^s + P_k^1)\dot{X}^k\delta_i^j \\
&- \frac{X^j}{m\zeta X^2}(m\zeta\delta_{il'}\dot{X}^{l'} + P_i^0)\Bigg)(m\zeta\delta_{jj'}\dot{X}^{j'} + P_j^0) \\
&= \frac{-\zeta^2}{2\mu m^2X^2}(\delta_{sk}X^s\dot{X}^k)\epsilon_{inn'}\ddot{X}^nX^{n'} \\
&+ \frac{\zeta}{2\mu m^2X^2}\epsilon_{jnn'}\dot{X}^j\ddot{X}^nX^{n'}\big(m\zeta_{is}X^s + \frac{\zeta^2}{2\mu m}\epsilon_{ill'}X^l\dot{X}^{l'}\big)
\end{aligned} \tag{4.361}
$$

by substituting the A_k and B_k in terms of P_i^0 and P_i^1 in X coordinates. Second line in the equation (4.360) can be written as

$$
\begin{aligned}
&- \frac{\zeta}{2\mu m^2X^2}\epsilon_{ikl}\dot{X}^kX^l(B_j\dot{X}^j) \\
&= -\frac{\zeta}{2m^2\mu X^2}\epsilon_{ik'n}\dot{X}^{k'}X^n(m\zeta\delta_{jl'}\dot{X}^{l'} + P_j^0)\dot{X}^j \\
&= -\frac{\zeta^3}{2m^3\mu^2X^2}\epsilon_{ik'n}\dot{X}^{k'}X^n\epsilon_{jll'}\dot{X}^j\ddot{X}^lX^{l'}
\end{aligned} \tag{4.362}
$$

using definition of P_i^0. Hence summation of (4.361) and (4.362) in equation (4.360) give

Hamilton equations of motion for P_i^0,

$$
\begin{aligned}
\dot{P}^0 &= \frac{-\zeta^2}{2\mu m^2 X^2}(\delta_{sk}X^s\dot{X}^k)\epsilon_{inn'}\ddot{X}^n X^{n'} + \frac{\zeta^2}{2\mu m X^2}(\epsilon_{jnn'}\dot{X}^j\ddot{X}^n X^{n'})\delta_{is}X^s \\
&+ \frac{3\zeta^3}{4\mu^2 m^3 X^2}\epsilon_{ill'}X^l\dot{X}^{l'}(\epsilon_{jnn'}\dot{X}^j\ddot{X}^n X^{n'}) \\
&= \frac{-\zeta^2}{2\mu m X^2}\epsilon_{ijk}\dot{X}^j\epsilon^{klr}(\epsilon_{ll'm}\ddot{X}^{l'}X^m)X^{r'}\delta_{rr'} \\
&+ \frac{3\zeta^3}{4\mu^2 m^3 X^2}\epsilon_{ill'}X^l\dot{X}^{l'}(\epsilon_{jnn'}\dot{X}^j\ddot{X}^n X^{n'}) \\
&= \frac{\zeta^2}{2\mu m X^2}\epsilon_{ijk}\dot{X}^j\big(\ddot{X}^k(\delta_{lr}X^l X^r) - X^k(\delta_{lr}X^l\ddot{X}^r)\big) \\
&+ \frac{3\zeta^3}{4\mu^2 m^3 X^2}\epsilon_{ill'}X^l\dot{X}^{l'}(\epsilon_{jnn'}\dot{X}^j\ddot{X}^n X^{n'}) \\
&= \frac{\zeta^2}{2\mu m}\epsilon_{ijk}\dot{X}^j\ddot{X}^k - \frac{\zeta^2}{2\mu m}\epsilon_{ijk}\dot{X}^j X^k\big(\delta_{lr}X^l\ddot{X}^r + \frac{3\zeta}{2\mu m^2}(\epsilon_{jnn'}\dot{X}^j\ddot{X}^n X^{n'})\big).
\end{aligned}
\tag{4.363}
$$

The term in the parenthesis vanish since it is the dot product of the Euler-Lagrange with X^i and the remaining term gives the Euler-Lagrange equations (4.319) substituting the definition of P_i^0.

Finally Hamilton equation of motion for P_i^1's are identically satisfied

$$
\begin{aligned}
\dot{P}_i^1 &= \{P_i^1, H^C\}_{DB} \\
&= \{P_i^1, \dot{X}^j\}_{DB}(m\zeta\delta_{jj'}\dot{X}^{j'} + P_j^0) + \{P_i^1, P_j^0\}_{DB}\dot{X}^j \\
&= -B_i + \frac{1}{2m\zeta X^2}B_i(A_k X^k) - \frac{1}{2m\zeta X^2}A_i(B_j X^j) + \frac{1}{X^2}\delta_{ik}X^k(X^j B_j)
\end{aligned}
\tag{4.364}
$$

substituting A_i and B_i

$$
\dot{P}_i^1 = \frac{-\zeta^2}{2\mu m}\epsilon_{ink}\ddot{X}^n X^k. \tag{4.365}
$$

□

4.3.4. The First Order Formalism

In this section, we will analyze first order constraint Clèment Lagrangians

$$L^C_{C_0} = -\frac{m\zeta}{2}\delta_{ij}\dot{Q}^i_0\dot{Q}^j_0 + \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^i_0\dot{Q}^j_0\dot{Q}^k_1 + \lambda^0_j(\dot{Q}^j_0 - Q^j_1) \tag{4.366}$$

$$L^C_{C_1} = -\frac{m\zeta}{2}\delta_{ij}Q^i_1Q^j_1 + \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^i_0Q^j_1\dot{Q}^k_1 + \lambda^1_j(\dot{Q}^j_0 - Q^j_1) \tag{4.367}$$

using coordinate transformations $X^i = Q^i_0$, $\dot{X}^i = Q^i_1$, $\ddot{X}^i = \dot{Q}^i_1$ and Lagrange multipliers λ^0_i and λ^1_i.

Both of the variations of $L^C_{C_0}$ and $L^C_{C_1}$ with respect to λ^0_i and λ^1_i give the constraint equation $\dot{Q}^i_0 - Q^i_1 = 0$. Variation with respect to Q^i_1 implies the expression

$$\lambda^0_i = -\frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^j_0\ddot{Q}^k_0 \tag{4.368}$$

$$\lambda^1_i = -m\zeta\delta_{ij}Q^j_1 - \frac{\zeta^2}{\mu m}\epsilon_{ijk}Q^j_0\dot{Q}^k_1 + \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^j_1\dot{Q}^k_0 \tag{4.369}$$

for the Lagrange multipliers respectively. Finally, the equation from the variation of Q^i_0 is

$$\dot{\lambda}^0_i = -m\zeta\ddot{Q}^i_0 + \frac{\zeta^2}{\mu m}\epsilon_{ijk}\dot{Q}^j_0\dot{Q}^k_1 - \frac{\zeta^2}{2\mu m}\epsilon_{ijk}\ddot{Q}^j_1Q^k_0 \tag{4.370}$$

$$\dot{\lambda}^1_i = \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^j_1\dot{Q}^k_1 \tag{4.371}$$

which give the Euler-Lagrange equations (4.319) for X^i when the identification (4.368) and (4.369) of λ^0_i, λ^1_i and the constraints $\dot{Q}^i_0 - Q^i_1 = 0$ are used.

4.3.5. The First Order Formalism as $L^C_{C_1}$

Hamiltonian Formalism for $L^C_{C_1}$: To pass the Hamiltonian formalism for the $L^C_{C_1}$ in (4.367), the conjugate momenta are defined by

$$\Pi^0_i \equiv \frac{\partial L^C_{C_1}}{\partial \dot{Q}^i_0} = \lambda^1_i \tag{4.372}$$

$$\Pi^1_i \equiv \frac{\partial L^C_{C_1}}{\partial \dot{Q}^i_1} = \frac{\zeta^2}{2\mu m}\epsilon_{ikl}Q^k_0 Q^l_1 \tag{4.373}$$

$$\Pi^i_\lambda \equiv \frac{\partial L^C_{C_1}}{\partial \dot{\lambda}^1_i} = 0. \tag{4.374}$$

Proposition 4.3.5. *Total Hamiltonian function is*

$$\begin{aligned} H_{T1} = {} & \frac{1}{2}\Pi^0_j Q^j_1 + \frac{1}{2Q^2_0}(\delta_{ij}Q^i_0 Q^j_1)(D_i Q^i_0) + \frac{\mu m}{\zeta^2 {Q_0}^2}\epsilon^{ijk}\delta_{jl}\Pi^1_i Q^l_0 D_k \\ & - \frac{3}{2m\zeta Q^2_0}(D_j Q^j_1)(Q^i_0 \Pi^1_i) + \frac{1}{2Q^2_0}\epsilon_{ijk}\Pi^i_\lambda Q^j_1 \epsilon^{klr}\delta_{ls}Q^s_0 D_r \\ & - \frac{3\zeta}{4\mu m^2 Q^2_0}\epsilon_{ijk}\Pi^i_\lambda Q^j_1 Q^k_0 (D_l Q^l_1) - \frac{(D_i Q^i_0)^2}{m\zeta Q^2_0} \end{aligned} \tag{4.375}$$

where we used abbreviations $D_i = m\zeta\delta_{ij}Q^j_1 + \Pi^0_i$ *and* $\delta_{ij}Q^i_0 Q^j_0 = {Q_0}^2$.

Proof. Recall the definition of momenta $(4.372) - (4.374)$. Neither of the momenta can be written as explicit functions of the $\dot{Q}^i_0$, $\dot{Q}^i_1$ and $\dot{\lambda}^1_i$, instead we introduce the primary constraints

$$\phi^0_i \equiv \Pi^0_i - \lambda^1_i \tag{4.376}$$

$$\phi^1_i \equiv \Pi^1_i - \frac{\zeta^2}{2\mu m}\epsilon_{ikl}Q^k_0 Q^l_1 \tag{4.377}$$

$$\phi^i_\lambda \equiv \Pi^i_\lambda. \tag{4.378}$$

For the Lagrangian $L_{C_1}^C$, the canonical Hamiltonian function is defined by

$$\begin{aligned} H_c &= \Pi_j^0 \dot{Q}_0^j + \Pi_j^1 \dot{Q}_1^j + \Pi_\lambda^j \dot{\lambda}_j^1 - L_{C_1}^C \\ &= \Pi_j^0 \dot{Q}_0^j + \big(\frac{\zeta^2}{2\mu m}\epsilon_{jkl} Q_0^k Q_1^l\big)\dot{Q}_1^j + \Pi_\lambda^j \dot{\lambda}_j^1 + \frac{m\zeta}{2}\delta_{jj'} Q_1^j Q_1^{j'} \\ &- \frac{\zeta^2}{2\mu m}\epsilon_{kln} Q_0^k Q_1^l \dot{Q}_1^n - \lambda_j^1(\dot{Q}_0^j - Q_1^j) \\ &= \frac{m\zeta}{2}\delta_{jj'} Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j \end{aligned} \tag{4.379}$$

using the primary constraints. Then, the total Hamiltonian becomes

$$\begin{aligned} H_T &= H_c + V_0^j \phi_j^0 + V_1^j \phi_j^1 + V_j^\lambda \phi_\lambda^j \\ &= \frac{m\zeta}{2}\delta_{jj'} Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j + V_0^j(\Pi_j^0 - \lambda_j^1) \\ &+ V_1^j(\Pi_j^1 - \frac{\zeta^2}{2\mu m}\epsilon_{jkl} Q_0^k Q_1^l) + V_j^\lambda \Pi_\lambda^j \end{aligned} \tag{4.380}$$

where V_0^j, V_1^j and V_j^λ are arbitrary functions of canonical variables.

Now we check the consistency condition for each of the primary constraint ϕ_i^0, ϕ_i^1 and ϕ_λ^i. The Poisson brackets of constraints are

$$\{\phi_i^0, \phi_j^1\} = \{\Pi_i^0 - \lambda_i^1, \Pi_j^1 - \frac{\zeta^2}{2\mu m}\epsilon_{jkl} Q_0^k Q_1^l\} = \frac{\zeta^2}{2\mu m}\epsilon_{jil} Q_1^l \tag{4.381}$$

$$\{\phi_i^0, \phi_\lambda^j\} = \{\Pi_i^0 - \lambda_i^1, \Pi_\lambda^j\} = -\delta_i^j \tag{4.382}$$

$$\begin{aligned} \{\phi_i^1, \phi_j^1\} &= \{\Pi_i^1 - \frac{\zeta^2}{2\mu m}\epsilon_{ikl} Q_0^k Q_1^l, \Pi_j^1 - \frac{\zeta^2}{2\mu m}\epsilon_{jk'l'} Q_0^{k'} Q_1^{l'}\} \\ &= \frac{\zeta^2}{\mu m}\epsilon_{ijk} Q_0^k \end{aligned} \tag{4.383}$$

and all the others are zero. The brackets of constraints and the canonical Hamiltonian are

$$\{\phi_i^0, H_c\} = \{\Pi_i^0 - \lambda_i^1, \frac{m\zeta}{2}\delta_{jj'} Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j\} = 0 \tag{4.384}$$

$$\begin{aligned} \{\phi_i^1, H_c\} &= \{\Pi_i^1 - \frac{\zeta^2}{2\mu m}\epsilon_{ikl} Q_0^k Q_1^l, \frac{m\zeta}{2}\delta_{jj'} Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j\} \\ &= -m\zeta\delta_{ij} Q_1^j - \Pi_i^0 \end{aligned} \tag{4.385}$$

$$\{\phi_\lambda^i, H_c\} = \{\Pi_\lambda^i, \frac{m\zeta}{2}\delta_{jj'} Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j\} = 0. \tag{4.386}$$

Using these, we write the consistency conditions for each the primary constraints as follows, for ϕ_i^0's,

$$\begin{aligned} \dot{\phi}_i^0 = \{\phi_i^0, H_T\} &\approx \{\phi_i^0, H_c\} + V_0^j\{\phi_i^0, \phi_j^0\} + V_1^j\{\phi_i^0, \phi_j^1\} + V_j^\lambda\{\phi_i^0, \phi_\lambda^j\} \\ &\approx \frac{\zeta^2}{2\mu m}\epsilon_{ilj}Q_1^l V_1^j - V_i^\lambda, \end{aligned} \tag{4.387}$$

for ϕ_i^1's

$$\begin{aligned} \dot{\phi}_i^1 = \{\phi_i^1, H_T\} &\approx \{\phi_i^1, H_c\} + V_0^j\{\phi_i^1, \phi_j^0\} + V_1^j\{\phi_i^1, \phi_j^1\} + V_j^\lambda\{\phi_i^1, \phi_\lambda^j\} \\ &\approx m\zeta\delta_{ij}Q_1^j - \Pi_i^0 - \frac{\zeta^2}{2\mu m}\epsilon_{ijl}V_0^j Q_1^l + \frac{\zeta^2}{\mu m}\epsilon_{ijk}V_1^j Q_0^k, \end{aligned} \tag{4.388}$$

and for ϕ_λ^i's

$$\begin{aligned} \dot{\phi}_\lambda^i = \{\phi_\lambda^i, H_T\} &\approx \{\phi_\lambda^i, H_c\} + V_0^j\{\phi_\lambda^i, \phi_j^0\} + V_1^j\{\phi_\lambda^i, \phi_j^1\} + V_j^\lambda\{\phi_\lambda^i, \phi_\lambda^j\} \\ &\approx V_0^i. \end{aligned} \tag{4.389}$$

Since $V_0^i \approx 0$, from the equation (4.387) we arrive

$$V_i^\lambda \approx \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q_1^j V_1^k. \tag{4.390}$$

By taking the dot product of Q_0^i and Eq.(4.388), we have a new constraint

$$\psi = \big(m\zeta\delta_{ii'}Q_1^{i'} + \Pi_i^0\big)Q_0^i. \tag{4.391}$$

We revise the total Hamiltonian as

$$H_{T1} = H_c + V_0^j\phi_j^0 + V_1^j\phi_j^1 + V_j^\lambda\phi_\lambda^j + V\psi \tag{4.392}$$

by adding the secondary constraint ψ by multiplying it with a Lagrange multiplier V.

The Poisson bracket of the secondary constraint ψ with the canonical Hamilton function is

$$\begin{aligned} \{\psi, H_c\} &= \{(m\zeta\delta_{ii'}Q_1^{i'} + \Pi_i^0)Q_0^i, \frac{m\zeta}{2}\delta_{jj'}Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j\} \\ &= (m\zeta\delta_{ii'}Q_1^{i'} + \Pi_i^0)Q_1^i. \end{aligned} \tag{4.393}$$

The Poisson bracket of the secondary constraint ψ with the primary constraints are

$$\{\psi,\phi^0_j\} = \{(m\zeta\delta_{ii'}Q^{i'}_1 + \Pi^0_i)Q^i_0, \Pi^0_j - \lambda_j\} = (m\zeta\delta_{ji'}Q^{i'}_1 + \Pi^0_j) \tag{4.394}$$

$$\begin{aligned}\{\psi,\phi^1_j\} &= \{(m\zeta\delta_{ii'}Q^{i'}_1 + \Pi^0_i)Q^i_0, \Pi^1_j - \frac{\zeta^2}{2\mu m}\epsilon_{jkl}Q^k_0 Q^l_1\} \\ &= m\zeta\delta_{ij}Q^i_0 + \frac{\zeta^2}{2\mu m}\epsilon_{jil}Q^l_1 Q^i_0\end{aligned} \tag{4.395}$$

$$\{\psi,\phi^j_\lambda\} = \{(m\zeta\delta_{ii'}Q^{i'}_1 + \Pi^0_i)Q^i_0, \Pi^j_\lambda\} = 0 \tag{4.396}$$

The consistency condition for ψ is

$$\begin{aligned}\dot{\psi} &= \{\psi, H_{T1}\} \\ &= \{\psi, H_c\} + V^j_0\{\psi,\phi^0_j\} + V^j_1\{\psi,\phi^1_j\} + V^\lambda_j\{\psi,\phi^j_\lambda\} + V\{\psi,\psi\} \\ &= (m\zeta\delta_{ii'}Q^{i'}_1 + \Pi^0_i)Q^i_1 + V^j_1\big(m\zeta\delta_{ij}Q^i_0 + \frac{\zeta^2}{2\mu m}\epsilon_{jil}Q^l_1 Q^i_0\big).\end{aligned} \tag{4.397}$$

The consistency of the secondary constraint ψ is not giving a tertiary constraint but it defines V^3_1 using $V^1_1(V^3_1)$ and $V^2_1(V^3_1)$ from the equations (4.388). Hence, we have

$$V^i_1 = \frac{\mu m}{\zeta^2 {Q_0}^2}\epsilon^{ijk}\delta_{jl}Q^l_0 D_k - \frac{3}{2m\zeta Q^2_0}(D_j Q^j_1)Q^i_0 \tag{4.398}$$

$$V^\lambda_i = \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^j_1\big(\frac{\mu m}{\zeta^2 {Q_0}^2}\epsilon^{klr}\delta_{ls}Q^s_0 D_r - \frac{3}{2m\zeta Q^2_0}(D_j Q^j_1)Q^k_0\big) \tag{4.399}$$

where we used abbreviations $D_i = m\zeta\delta_{ij}Q^j_1 + \Pi^0_i$ and $\delta_{ij}Q^i_0 Q^j_0 = {Q_0}^2$. On the other hand, consistency of the primary constraints ϕ^0_i and ϕ^i_λ are

$$\dot{\phi}^0_i = \{\phi^0_i, H_{T1}\} = \frac{\zeta^2}{2\mu m}\epsilon_{ilj}Q^l_1 V^j_1 - V^\lambda_i - V(m\zeta\delta_{ij}Q^j_1 + \Pi^0_i) \tag{4.400}$$

$$\dot{\phi}^i_\lambda = \{\phi^i_\lambda, H_{T1}\} = 0 \tag{4.401}$$

whereas consistency of the primary constraints ϕ^1_i are

$$\dot{\phi}^1_i = \{\phi^1_i, H_{T1}\} = m\zeta\delta_{ij}Q^j_1 - \Pi^0_i + \frac{\zeta^2}{\mu m}\epsilon_{ijk}V^j_1 Q^k_0 - V(m\zeta\delta_{ij}Q^j_0 + \frac{\zeta^2}{2\mu m}\epsilon_{ijl}Q^j_1 Q^l_0). \tag{4.402}$$

These leads to determine $V = -\frac{\psi}{m\zeta Q^2_0}$. The substitutions of $V^i_0, V^i_1, V^\lambda_i$ and V into the total Hamiltonian (4.392) complete the proof. □

Proposition 4.3.6. *The Hamilton's equations generated by the total Hamiltonian in the proposition* 4.3.5 *are*

$$\dot{Q}_0^i \approx \frac{1}{2}Q_1^i + \frac{1}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^k)Q_0^i + \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{kl}\Pi_j^1 Q_0^l \tag{4.403}$$

$$\dot{Q}_1^i \approx \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{jl}Q_0^l D_k - \frac{3}{2m\zeta Q_0^2}(D_jQ_1^j)Q_0^i \tag{4.404}$$

$$\dot{\lambda}_i^1 \approx \frac{1}{2Q_0^2}\epsilon_{ijk}Q_1^j\epsilon^{krs}\delta_{lr}Q_0^l D_s - \frac{3\zeta}{4\mu m^2 Q_0^2}\epsilon_{ijk}Q_1^jQ_0^k(D_lQ_1^l) \tag{4.405}$$

$$\dot{\Pi}_i^0 \approx \frac{-1}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^k)D_i - \frac{\mu m}{\zeta^2 Q_0^2}\delta_{il}\epsilon^{ljk}D_j\Pi_k^1 + \frac{3}{2m\zeta Q_0^2}\Pi_i^1(D_jQ_1^j)$$
$$+ \frac{2\mu m}{\zeta^2 Q_0^4}(\epsilon^{jkl}\delta_{kk'}\Pi_j^1 Q_0^{k'}D_l)Q_0^i \tag{4.406}$$

$$\dot{\Pi}_i^1 \approx -\frac{1}{2}\Pi_i^0 - \frac{m\zeta}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^k)Q_0^i - \frac{\mu m^2}{\zeta Q_0^2}\epsilon_{ijk}\delta^{jl}\Pi_l^1 Q_0^k \tag{4.407}$$

$$\dot{\Pi}_\lambda^i \approx 0 \tag{4.408}$$

where we used abbreviations $D_i = m\zeta\delta_{ij}Q_1^j + \Pi_i^0$ *and* $\delta_{ij}Q_0^iQ_0^j = {Q_0}^2$.

Proof. The equations of motion for Q_0^i's are

$$\dot{Q}_0^i = \{Q_0^i, H_{T1}\}$$
$$\approx \frac{1}{2}Q_1^i + \frac{1}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^k)Q_0^i + \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{kl}\Pi_j^1 Q_0^l \tag{4.409}$$

identically satisfied

$$\dot{Q}_0^i \approx \frac{1}{2}Q_1^i + \frac{1}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^k)Q_0^i + \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{kl}(\frac{\zeta^2}{2\mu m}\epsilon_{jrs}Q_0^rQ_1^s)Q_0^l$$
$$\approx \frac{1}{2}Q_1^i + \frac{1}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^k)Q_0^i - \frac{1}{2Q_0^2}(Q_0^i(\delta_{jk}Q_0^jQ_1^k) - Q_1^i(\delta_{jk}Q_0^jQ_0^k))$$
$$\approx Q_1^i. \tag{4.410}$$

The Hamilton's equations governing Q_1^i generated by the total Hamiltonian function are

$$\dot{Q}_1^i = \{Q_1^i, H_{T1}\} \approx \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{jl}Q_0^l D_k - \frac{3}{2m\zeta Q_0^2}(D_jQ_1^j)Q_0^i \tag{4.411}$$

using D_i, definition of Π^0_i, we compute

$$\begin{aligned}\dot{Q}^i_1 &\approx \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{jl}Q^l_0(\Pi^0_k + m\zeta\delta_{kl}Q^l_1) - \frac{3}{2m\zeta Q_0^2}((\Pi^0_j + m\zeta\delta_{jl}Q^l_1)Q^j_1)Q^i_0 \\ &\approx \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{jl}Q^l_0(-\frac{\zeta^2}{\mu m}\epsilon_{krs}Q^r_0\dot{Q}^s_1) - \frac{3}{2m\zeta Q_0^2}((-\frac{\zeta^2}{\mu m}\epsilon_{jrs}Q^r_0\dot{Q}^s_1)Q^j_1)Q^i_0 \\ &\approx -\frac{1}{Q_0^2}\big(Q^i_0(\delta_{jk}Q^j_0\dot{Q}^k_1) - \dot{Q}^i_1(\delta_{jk}Q^j_0Q^k_0)\big) + \frac{3\zeta}{2\mu m^2 Q_0^2}((\epsilon_{jrs}Q^r_0\dot{Q}^s_1)Q^j_1)Q^i_0 \\ &\approx \dot{Q}^i_1 + \frac{1}{Q_0^2}\big(-\delta_{kl}Q^k_0\dot{Q}^l_1 + \frac{3\zeta}{2\mu m^2 Q_0^2}\epsilon_{jrs}Q^j_1Q^r_0\dot{Q}^s_1\big)Q^i_0. \end{aligned} \tag{4.412}$$

Note that, the term in the parenthesis is the dot product of Euler-Lagrange equation with Q^i_0 so that it is zero.

The equations of motion for λ^1_i's

$$\begin{aligned}\dot{\lambda}^1_i &= \{\lambda^1_i, H_{T1}\} \\ &\approx \frac{1}{2Q_0^2}\epsilon_{ijk}Q^j_1\epsilon^{krs}\delta_{lr}Q^l_0 D_s - \frac{3\zeta}{4\mu m^2 Q_0^2}\epsilon_{ijk}Q^j_1Q^k_0(D_lQ^l_1)\end{aligned} \tag{4.413}$$

substitution of Π^0 in D_i, and some cross product property

$$\dot{\lambda}^1_i \approx \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^j_1\dot{Q}^k_1 + \frac{\zeta^2}{2\mu m}\big(-\delta_{kl}Q^k_0\dot{Q}^l_1 + \frac{3\zeta}{2\mu m^2 Q_0^2}\epsilon_{jrs}Q^j_1Q^r_0\dot{Q}^s_1\big) \tag{4.414}$$

which is exactly true since the term in the parenthesis is zero since it is the dot product of the Euler-Lagrange equation with Q^i_0 .

The equations of motion for Π^0_i's are

$$\begin{aligned}\dot{\Pi}^0_i &= \{\Pi^0_i, H_{T1}\} \\ &\approx \frac{-1}{2Q_0^2}(\delta_{jk}Q^j_0Q^k_1) - \frac{\mu m}{\zeta^2 Q_0^2}\delta_{il}\epsilon^{ljk}D_k\Pi^1_k + \frac{3}{2m\zeta Q_0^2}\Pi^1_i(D_jQ^j_1) \\ &+ \frac{2\mu m}{\zeta^2 Q_0^4}(\epsilon^{jkl}\Pi^1_jQ^k_0D_l)Q^i_0.\end{aligned} \tag{4.415}$$

If we impose $D_iQ^i_0 = \Pi^i_\lambda = 0$ and use the definition of Π^1_i, and we arrive

$$\dot{\Pi}^0_i \approx \frac{1}{2Q_0^2}\epsilon_{ijk}Q^j_1\left(\epsilon^{klr}\delta_{ll'}Q^{l'}_0D_r - \frac{3\zeta}{2\mu m^2}Q^k_0(D_lQ^l_1)\right). \tag{4.416}$$

After imposing the definition of D_i in terms of Q_0^i and Q_1^i, we get

$$\dot{\Pi}_i^0 \approx \frac{\zeta^2}{\mu m Q_0^2}\epsilon_{ijk}Q_1^j\dot{Q}_1^k + \frac{\zeta^2}{\mu m Q_0^2}\epsilon_{ijk}Q_1^jQ_0^k\left(-\delta_{kl}Q_0^k\dot{Q}_1^l - \frac{3\zeta}{2\mu m^2}\epsilon_{lkr}Q_1^lQ_0^k\dot{Q}_1^r\right). \quad (4.417)$$

These give the Euler-Lagrange equations (4.319).

The equations of motion for Π_i^1's are

$$\dot{\Pi}_i^1 = \{\Pi_i^1, H_{T1}\} \approx \frac{-1}{2}\Pi_i^0 - \frac{m\zeta}{2Q_0^2}(\delta_{jk}Q_0^jQ_1^l)Q_0^i - \frac{\mu m^2}{\zeta Q_0^2}\epsilon_{ijk}\delta^{jl}\Pi_l^1Q_0^k. \quad (4.418)$$

After substitution of the definitions Π_i^0 and Π_i^1, we get

$$\dot{\Pi}_i^1 \approx \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q_0^j\dot{Q}_1^k. \quad (4.419)$$

Finally, the equations of motion for Π_λ^i's are

$$\dot{\Pi}_\lambda^i = \{\Pi_\lambda^i, H_{T1}\} = 0 \quad (4.420)$$

since $\Pi_\lambda^i = 0$. □

As it is pointed out in [30] that, the constraints $\phi_\lambda^i = \Pi_\lambda^i$ effect only the equation of motion for λ_i. So that, we may omit to add them to the total Hamiltonian function H_T. In this case, $V_0^i = 0$ from the consistency condition hence we may additionally omit the constraint ϕ_i^0. So that, the total Hamiltonian function reduces to

$$H_T = H_C + V_1^j\phi_j^1 \quad (4.421)$$

with the constraints

$$\phi_i^1 = \Pi_i^1 - \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q_0^jQ_1^k \quad (4.422)$$

$$\psi = \left(m\zeta\delta_{ii'}Q_1^{i'} + \Pi_i^0\right)Q_0^i \quad (4.423)$$

identical to Φ and χ given in terms of the X coordinates (4.341) and (4.342).

4.3.6. Dirac Bracket Formalism for First Order Lagrangian

We shall write the Hamilton's equations using the Dirac bracket. To this end, we record here the set of second class constraints

$$\phi_i^0 \equiv \Pi_i^0 - \lambda_i \tag{4.424}$$

$$\phi_i^1 \equiv \Pi_i^1 - \frac{\zeta^2}{2\mu m}\epsilon_{ikk'}Q_0^k Q_1^{k'} \tag{4.425}$$

$$\phi_\lambda^i \equiv \Pi_\lambda^i \tag{4.426}$$

$$\psi \equiv (m\zeta\delta_{ll'}Q_1^{l'} + \Pi_l^0)Q_0^l. \tag{4.427}$$

Proposition 4.3.7. *The Dirac brackets of the coordinates are*

$$\{Q_0^i, Q_1^j\}_{DB} = -\frac{1}{m\zeta Q_0^2}Q_0^i Q_0^j \tag{4.428}$$

$$\{Q_0^i, \lambda_j^1\}_{DB} = \delta_j^i - \frac{1}{m\zeta Q_0^2}Q_0^i\delta_{nj}Q_0^n \tag{4.429}$$

$$\{Q_0^i, \Pi_j^0\}_{DB} = \delta_j^i - \frac{\zeta}{2m^2\mu Q_0^2}\epsilon_{jk'n'}Q_0^{n'}Q_1^{k'} \tag{4.430}$$

$$\{Q_1^i, Q_1^j\}_{DB} = -\frac{\mu}{\zeta^3 Q_0^2}\epsilon^{ijk}E_k \tag{4.431}$$

$$\{Q_1^i, \lambda_j^1\}_{DB} = A_j^i \tag{4.432}$$

$$\{Q_1^i, \Pi_j^0\}_{DB} = -\frac{1}{2m\zeta Q_0^2}\epsilon^{irk}\epsilon_{jrl}Q_1^l E_k - \frac{Q_0^i}{m\zeta Q_0^2}D_j$$

$$\{Q_1^i, \Pi_j^1\}_{DB} = \delta_j^i - \frac{1}{2m\zeta Q_0^2}\epsilon^{in'k}\epsilon_{jn'k'}Q_0^{k'}E_k - \frac{Q_0^i}{Q_0^2}\delta_{jl}Q_0^l$$

$$\{\lambda_i^1, \lambda_j^1\}_{DB} = \delta_{in}\delta_{jn'}C^{nn'} \tag{4.433}$$

$$\{\lambda_i^1, \Pi_j^0\}_{DB} = \frac{\zeta^2}{2\mu m}\delta_{in}B^{nr}\epsilon_{jkrQ_1^k} + \frac{\zeta}{2m^2\mu Q_0^2}\epsilon_{irs}Q_0^r Q_1^s D_j \tag{4.434}$$

$$\{\lambda_i^1, \Pi_j^1\}_{DB} = \frac{\zeta^2}{2\mu m}\delta_{in}B^{nr}\epsilon_{jrk}Q_0^k + \frac{\zeta^2}{2m\mu Q_0^2}\epsilon_{irs}Q_0^r Q_1^s(\delta_{jl}Q_0^l) \tag{4.435}$$

$$\{\Pi_i^0, \Pi_j^0\}_{DB} = \frac{\zeta\epsilon_{ik'j}Q_1^{k'}E_k Q_1^k}{4\mu m^2 Q_0^2} - \frac{\zeta\epsilon_{ikn}Q_1^k Q_0^n D_j}{2\mu m^2 Q_0^2} + \frac{\zeta\epsilon_{jk'n'}Q_1^{k'}Q_0^{n'}D_i}{2\mu m^2 Q_0^2} \tag{4.436}$$

$$\{\Pi_i^0, \Pi_j^1\}_{DB} = -\frac{\zeta\epsilon_{ik'j}Q_1^{k'}D_i Q_0^k}{4\mu m^2 Q_0^2} + \frac{\zeta\epsilon_{ik'n}Q_1^{k'}Q_0^n E_j}{4\mu m^2 Q_0^2} \tag{4.437}$$

$$\{\Pi_i^1, \Pi_j^1\}_{DB} = -\frac{\zeta}{4\mu m^3 Q_0^2}\epsilon_{ink'}Q_0^{k'}\epsilon^{nrk}E_k\epsilon_{jrl}Q_0^l \tag{4.438}$$

and all the others are zero. Here we used abbreviations $D_i = m\zeta\delta_{ij}Q_1^j + \Pi_i^0$, $E_i = m\zeta\delta_{ij}Q_0^j + \Pi_i^1$ *and* $\delta_{ij}Q_0^iQ_0^j = {Q_0}^2$.

Proof. Recall that the Dirac bracket (2.39) can be written as

$$\begin{aligned}\{F,G\}_{DB} = &\{F,G\} - \{F,\phi_n^0\}M_{0\lambda}^{nn'}\{\delta_{n'l}\phi_\lambda^l,G\} - \{F,\phi_n^1\}M_{11}^{nn'}\{\phi_{n'}^1,G\}\\ &- \{F,\phi_n^1\}M_{1\lambda}^{nn'}\{\delta_{n'l}\phi_\lambda^l,G\} - \{F,\phi_n^1\}M_{1\psi}^{n}\{\psi,G\}\\ &- \{F,\delta_{nl}\phi_\lambda^l\}M_{\lambda 0}^{nn'}\{\phi_{n'}^0,G\} - \{F,\delta_{nl}\phi_\lambda^l\}M_{\lambda 1}^{nn'}\{\phi_{n'}^1,G\}\\ &- \{F,\delta_{nl}\phi_\lambda^l\}M_{\lambda\lambda}^{nn'}\{\delta_{n'r}\phi_n^r,G\} - \{F,\delta_{nl}\phi_\lambda^l\}M_{\lambda\psi}^{n}\{\psi,G\}\\ &- \{F,\psi\}M_{\psi 1}^{n'}\{\phi_{n'}^1,G\} - \{F,\psi\}M_{\psi 1}^{n'}\{\delta_{n'l}\psi_\lambda^l,G\}\end{aligned} \tag{4.439}$$

substituting the inverse of M given by

$$\begin{aligned}M &= \begin{pmatrix} \{\phi_n^0,\phi_{n'}^0\} & \{\phi_n^0,\phi_{n'}^1\} & \{\phi_n^0,\phi_\lambda^{n'}\} & \{\phi_n^0,\psi\}\\ \{\phi_n^1,\phi_{n'}^0\} & \{\phi_n^1,\phi_{n'}^1\} & \{\phi_n^1,\phi_\lambda^{n'}\} & \{\phi_n^1,\psi\}\\ \{\phi_\lambda^n,\phi_{n'}^0\} & \{\phi_\lambda^n,\phi_{n'}^1\} & \{\phi_\lambda^n,\phi_\lambda^{n'}\} & \{\phi_\lambda^n,\psi\}\\ \{\psi,\phi_{n'}^0\} & \{\psi,\phi_{n'}^1\} & \{\psi,\phi_\lambda^{n'}\} & \{\psi,\psi\}\end{pmatrix}\\ &= \begin{pmatrix} 0 & \frac{\zeta^2}{2\mu m}\epsilon_{n'nk'}Q_1^{k'} & -\delta_n^{n'} & -D_n\\ -\frac{\zeta^2}{2\mu m}\epsilon_{nn'k'}Q_1^{k'} & \frac{\zeta^2}{\mu m}\epsilon_{nn'k}Q_0^k & 0 & -E_n\\ \delta_{n'}^n & 0 & 0 & 0\\ D_{n'} & E_{n'} & 0 & 0\end{pmatrix}\end{aligned} \tag{4.440}$$

whose determinant is $\frac{\zeta^6}{\mu^2}Q_0^2$. The inverse of M is

$$\begin{aligned}M^{-1} &= \begin{pmatrix} M_{00}^{nn'} & M_{01}^{nn'} & M_{0\lambda}^{nn'} & M_{0\psi}^{n}\\ M_{10}^{nn'} & M_{11}^{nn'} & M_{1\lambda}^{nn'} & M_{1\psi}^{n}\\ M_{\lambda 0}^{nn'} & M_{\lambda 1}^{nn'} & M_{\lambda\lambda}^{nn'} & M_{\lambda\psi}^{n}\\ M_{\psi 0}^{n'} & M_{\psi 1}^{n'} & M_{\psi\lambda}^{n'} & M_{\psi\psi}\end{pmatrix}\\ &= \begin{pmatrix} 0 & 0 & \delta^{nn'} & 0\\ 0 & \frac{-\mu\epsilon^{nn'k}E_k}{\zeta^3Q_0^2} & A^{nn'} & \frac{Q_0^n}{m\zeta Q_0^2}\\ -\delta^{nn'} & B^{nn'} & C^{nn'} & -\frac{\zeta\epsilon_{krs}\delta^{nk}Q_0^rQ_1^s}{2m^2\mu Q_0^2}\\ 0 & -\frac{Q_0^{n'}}{m\zeta Q_0^2} & \frac{\zeta\epsilon_{krs}\delta^{n'k}Q_0^rQ_1^s}{2m^2\mu Q_0^2} & 0\end{pmatrix}\end{aligned} \tag{4.441}$$

where

$$A^n_{n'} = \frac{1}{2m\zeta Q_0^2}\epsilon^{nrk}\epsilon_{n'rl}Q_1^l E_k + \frac{1}{m\zeta Q_0^2}Q_0^n D_{n'}, \qquad B^{nn'} = -\frac{\epsilon_{rkp}\delta^{nk}Q_1^p\epsilon^{rn's}E_s}{2m\zeta Q_0^2} + \frac{D^n Q_0^{n'}}{Q_0^2}.$$

To arrive at the Hamilton's equations using the Dirac bracket, we first evaluate the Poisson brackets of the coordinates and the constraints as follows

$$\{Q_0^i, \phi_n^0\} = \{Q_0^i, \Pi_n^0 - \lambda_n^1\} = \delta_n^i \tag{4.442}$$

$$\{Q_0^i, \psi\} = \{Q_0^i, (m\zeta\delta_{ll'}Q_1^{l'} + \Pi_l^0)Q_0^l\} = Q_0^i \tag{4.443}$$

$$\{Q_1^i, \phi_n^1\} = \{Q_1^i, \Pi_n^1 - \frac{\zeta^2}{2\mu m}\epsilon_{nkk'}Q_0^k Q_1^{k'}\} = \delta_n^i, \tag{4.444}$$

$$\{\lambda_i^1, \phi_\lambda^n\} = \{\lambda_i^1, \Pi_\lambda^n\} = \delta_i^n \tag{4.445}$$

$$\{\Pi_i^0, \phi_n^1\} = \{\Pi_i^0, \Pi_n^1 - \frac{\zeta^2}{2\mu m}\epsilon_{nkk'}Q_0^k Q_1^{k'}\} = \frac{1}{2\mu m}\zeta^2\epsilon_{nik'}Q_1^{k'} \tag{4.446}$$

$$\{\Pi_i^0, \psi\} = \{\Pi_i^0, (m\zeta\delta_{ll'}Q_1^{l'} + \Pi_l^0)Q_0^l\} = -m\zeta\delta_{il'}Q_1^{l'} - \Pi_i^0 \tag{4.447}$$

$$\{\Pi_i^1, \phi_n^1\} = \{\Pi_i^1, \Pi_n^1 - \frac{\zeta^2}{2\mu m}\epsilon_{nkk'}Q_0^k Q_1^{k'}\} = \frac{1}{2\mu m}\zeta^2\epsilon_{nki}Q_0^k \tag{4.448}$$

$$\{\Pi_i^1, \psi\} = \{\Pi_i^1, (m\zeta\delta_{ll'}Q_1^{l'} + \Pi_l^0)Q_0^l\} = -m\zeta\delta_{li}Q_0^l \tag{4.449}$$

and all others are zero.

Dirac bracket of Q_0^i and Q_0^j is

$$\{Q_0^i, Q_0^j\}_{DB} = 0 \tag{4.450}$$

since $\{Q_0^i, Q_0^j\} = \{Q_0^i, \phi_\lambda^j\} = \{Q_0^i, \phi_j^1\} = 0$. Dirac bracket of Q_0^i with Q_1^j is

$$\begin{aligned}\{Q_0^i, Q_1^j\}_{DB} &= -\{Q_0^i, \psi\}M^{n'}_{\psi 1}\{\phi^1_{n'}, Q_1^j\} \\ &= -Q_0^i\big(-\frac{Q_0^{n'}}{m\zeta Q^2}\big)\big(-\delta^j_{n'}\big) = -\frac{1}{m\zeta Q^2}Q_0^i Q_0^j\end{aligned} \tag{4.451}$$

since $\{Q_0^i, Q_1^j\} = \{Q_0^i, \phi_\lambda^j\} = \{Q_0^i, \phi_j^1\} = \{Q_1^i, \phi_\lambda^j\} = \{Q_1^i, \phi_j^1\} = \{Q_1^i, \psi\} = 0$. Dirac bracket of Q_0^i with λ_j^1 is

$$\begin{aligned}\{Q_0^i, \lambda_j^1\}_{DB} &= -\{Q_0^i, \phi_n^0\} M_{0\lambda^1}^{nn'} \{\delta_{n'l}\phi_\lambda^l, \lambda_j^1\} - \{Q_0^i, \psi\} M_{\psi 1}^{n'} \{\delta_{n'l}\phi_\lambda^l, \lambda_j^1\} \\ &= -(\delta_n^i)(\delta^{nn'})(-\delta_{n'j}) - Q_0^i(-\frac{Q_0^{n'}}{m\zeta Q^2})(-\delta_{n'j}) \\ &= \delta_j^i - \frac{1}{m\zeta Q^2} Q_0^i \delta_{n'j} Q_0^{n'} \end{aligned} \tag{4.452}$$

since $\{Q_0^i, \lambda_j^1\} = \{Q_0^i, \phi_\lambda^j\} = \{Q_0^i, \phi_j^1\} = \{\lambda_i^1, \phi_j^0\} = \{Q_1^i, \phi_j^1\} = \{\lambda_i^1, \phi_j^1\} = 0$. In a similar way one can derive the remaining Dirac brackets of coordinates. □

Proposition 4.3.8. *Hamilton equations of motion using the Dirac bracket of coordinates in the Proposition* 4.3.7 *and the Hamiltonian function in* (4.379) *are*

$$\dot{Q}_0^i = -\frac{1}{m\zeta Q_0^2}(D_j Q_0^j) Q_0^i + Q_1^i \tag{4.453}$$

$$\dot{Q}_1^i = -\frac{\mu}{\zeta^3 Q_0^2}\epsilon^{ijk} D_j E_k - Q_0^i \frac{1}{m\zeta Q_0^2}(Q_1^j D_j) \tag{4.454}$$

$$\dot{\lambda}_i^1 = A_i^j D_j + \frac{\zeta}{2m\zeta Q_0^2}(D_j Q_1^j)\epsilon_{irs} Q_0^r Q_1^s \tag{4.455}$$

$$\dot{\Pi}_i^0 = \frac{1}{2m\zeta Q_0^2}\epsilon_{ijk}\epsilon^{jlr} D_l E_r Q_1^k + \frac{1}{m\zeta Q_0^2} D_i (D_j Q_0^j) - \frac{\zeta}{2\mu m^2 Q_0^2}\epsilon_{ijk} Q_1^j Q_0^k (D_l Q_1^l) \tag{4.456}$$

$$\dot{\Pi}_i^1 = -D_i + \frac{1}{2m\zeta Q_0^2}\epsilon_{ijk} Q_0^j \epsilon^{klr} D_l E_r \tag{4.457}$$

$$\dot{\Pi}_\lambda^i = 0 \tag{4.458}$$

here $D_i = m\zeta\delta_{ij}Q_1^j + \Pi_i^0$, $E_i = m\zeta\delta_{ij}Q_0^j + \Pi_i^1$, $A_{n'}^n = \frac{1}{2m\zeta Q_0^2}\epsilon^{nrk}\epsilon_{n'rl}Q_1^l E_k + \frac{1}{m\zeta Q_0^2}Q_0^n D_{n'}$ *and* $\delta_{ij}Q_0^i Q_0^j = {Q_0}^2$.

Proof. The equations of motion for Q_0^i's

$$\begin{aligned}\dot{Q}_0^i = \{Q_0^i, H_c\}_{DB} &= (m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)\{Q_0^i, Q_1^j\}_{DB} + Q_1^j\{Q_0^i, \Pi_j^0\}_{DB} \\ &= (m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)(-\frac{Q_0^i Q_0^j}{m\zeta Q^2}) + Q_1^j(\delta_j^i - \frac{\zeta\epsilon_{jk'n'}Q_0^{n'}Q_1^{k'}}{2m^2\mu Q^2}) \end{aligned} \tag{4.459}$$

give the secondary constraint

$$Q_0^i Q_0^j (m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0) = 0 \tag{4.460}$$

since $\dot{Q}_0^i = Q_1^i$. The equations of motion for Q_1^i's are

$$\begin{aligned}\dot{Q}_1^i &= \{Q_1^i, H_c\}_{DB} = (m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)\{Q_1^i, Q_1^j\}_{DB} + Q_1^j\{Q_1^i, \Pi_j^0\}_{DB} \\ &= -\frac{\mu}{\zeta^3 Q_0^2}\epsilon^{ijk}(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)E_k - Q_1^j\frac{1}{m\zeta Q_0^2}(D_j Q_0^i) \\ &= (m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)(-\frac{\mu}{\zeta^3 Q_0^2}\epsilon^{ijk}(m\zeta\delta_{sk}Q_0^s + \Pi_k^1)) \\ &- Q_1^j(\frac{Q_0^i(m\zeta\delta_{jl'}Q_1^{l'} + \Pi_j^0)}{m\zeta Q_0^2})\end{aligned} \tag{4.461}$$

and using the definition of Π_i^1 and combining similar terms in the last equation, we compute

$$\begin{aligned}\dot{Q}_1^i &= -\frac{\mu m}{\zeta^3 Q_0^2}\epsilon^{ijk}\delta_{sk}Q_0^s(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0) - \frac{3}{2m\zeta Q_0^2}Q_0^i Q_1^j(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0) \\ &+ \frac{1}{2m\zeta Q_0^2}Q_0^j Q_1^i(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0).\end{aligned} \tag{4.462}$$

This is the same with the first line of (4.412) since $Q_0^j(m\zeta\delta_{jk}Q_1^{jk} + \Pi_j^0) = 0$. So the equation of motion for Q_0^i are identically satisfied. The equations of motion for λ_i^1's are

$$\begin{aligned}\dot{\lambda}_i^1 &= \{\lambda_i^1, H_c\}_{DB} = \left(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0\right)\{\lambda_i^1, Q_1^j\}_{DB} + Q_1^j\{\lambda_i^1, \Pi_j^0\}_{DB} \\ &= -(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)A_i^j + \frac{\zeta}{2\mu m^2}\epsilon_{irs}Q_0^r Q_1^s(m\zeta\delta_{jl'}Q_1^{l'} + \Pi_j^0)Q_1^j.\end{aligned} \tag{4.463}$$

Substitution of $A_{n'}^n = \frac{1}{2m\zeta Q_0^2}\epsilon^{nrk}\epsilon_{n'rl}Q_1^l E_k + \frac{1}{m\zeta Q_0^2}Q_0^n D_{n'}$ results with

$$\begin{aligned}\dot{\lambda}_i^1 &= \frac{-1}{2m\zeta Q_0^2}D_i(E_j Q_1^j) + \frac{1}{2m\zeta Q_0^2}E_i(D_k Q_1^k) + \frac{1}{m\zeta Q_0^2}D_i(Q_0^j D_j) \\ &- \frac{\zeta}{2\mu m^2 Q_0^2}\epsilon_{ikl}Q_1^k Q_0^l(D_j Q_1^j)\end{aligned} \tag{4.464}$$

which is the same with the equation presented in (4.360) derived for P_i^0 with $D_i = B_i, E_i = A_i, Q_0^i = X^i$.

The equations of motion for Π_i^0's are

$$\begin{aligned}\dot{\Pi}_i^0 &= \{\Pi_i^0, H_c\}_{DB} = \{\Pi_i^0, \frac{m\zeta}{2}\delta_{jj'}Q_1^j Q_1^{j'} + \Pi_j^0 Q_1^j\}_{DB} \\ &= \left(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0\right)\{\Pi_i^0, Q_1^j\}_{DB} + Q_1^j\{\Pi_i^0, \Pi_j^0\}_{DB}.\end{aligned} \tag{4.465}$$

After substituting the Dirac brackets $\{\Pi_i^0, Q_1^j\}_{DB}$ and $\{\Pi_i^0, \Pi_j^0\}_{DB}$, using the definition of

Π_i^0, Π_i^1, and applying some cross product properties, we write

$$\dot{\Pi}_i^0 = \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q_1^j\dot{Q}_1^k - \frac{\zeta^2}{\mu m Q_0^2}\epsilon_{ijk}Q_1^jQ_0^k\left(\delta_{kl}Q_0^k\dot{Q}_1^l - \frac{3\zeta}{2\mu m^2}\epsilon_{lkr}Q_1^lQ_0^k\dot{Q}_1^r\right) \tag{4.466}$$

which equivalent to the Euler-Lagrange equations (4.319).

The equation of motion for Π_i^1's are

$$\dot{\Pi}_i^1 = \{\Pi_i^1, H_c\}_{DB} = \left(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0\right)\{\Pi_i^1, Q_1^j\}_{DB} + Q_1^j\{\Pi_i^1, \Pi_j^0\}_{DB}. \tag{4.467}$$

Substitution of the Dirac brackets $\{\Pi_i^1, Q_1^j\}_{DB}$ and $\{\Pi_i^1, \Pi_j^0\}_{DB}$ lead to simplified expression

$$\dot{\Pi}_i^1 = -m\zeta\delta_{ij}Q_1^j - \Pi_i^0 + \frac{1}{2m\zeta Q_0^2}\epsilon_{ijk}Q_0^j\epsilon^{klr}(m\zeta\delta_{ll'}Q_1^{l'} + \Pi_l^0)(m\zeta\delta_{rr'}Q_0^{r'} + \Pi_r^1). \tag{4.468}$$

Imposing the definition of Π_i^1 and applying some cross product properties, we get

$$\begin{aligned}\dot{\Pi}_i^1 &= -\frac{1}{2}(m\zeta\delta_{ij'}Q_1^{j'} + \Pi_i^0) + \frac{1}{2Q_0^2}(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)Q_0^jQ_0^i \\ &\quad - \frac{\zeta}{2\mu m^2}\epsilon_{ill'}Q_0^lQ_1^{l'}(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0)Q_0^j.\end{aligned} \tag{4.469}$$

Finally, the equation of motion for Π_λ^i's are

$$\begin{aligned}\dot{\Pi}_\lambda^i &= \{\Pi_\lambda^i, H_c\}_{DB} \\ &= \left(m\zeta\delta_{jj'}Q_1^{j'} + \Pi_j^0\right)\{\Pi_\lambda^i, Q_1^j\}_{DB} + Q_1^j\{\Pi_\lambda^i, \Pi_j^0\}_{DB} \\ &= 0\end{aligned} \tag{4.470}$$

since $\{\Pi_\lambda^i, Q_1^j\}_{DB} = 0$ and $\{\Pi_\lambda^i, \Pi_j^0\}_{DB} = 0$. □

4.3.7. Unconstrained Variational Formalism

Unconstraint variational formalism corresponding to the first order Clèment Lagrangian (4.367) is

$$L_{U_1}^C = \frac{m\zeta}{2}\delta_{ij}Q_1^iQ_1^j - m\zeta\delta_{ij}Q_1^i\dot{Q}_0^j - \frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q_0^iQ_1^j\dot{Q}_1^k + \frac{\zeta^2}{\mu m}\epsilon_{ijk}Q_0^i\dot{Q}_0^j\dot{Q}_1^k \tag{4.471}$$

by substituting λ^1 in (4.369) into $L^C_{C_1}$ in (4.367).

To arrive at Hamiltonian formalism for unconstraint Lagrangian (4.471), we introduce the momenta

$$\pi^0_i \equiv \frac{\partial L^C_{U_1}}{\partial \dot{Q}^i_0} = -m\zeta\delta_{ij}Q^j_1 - \frac{\zeta^2}{\mu m}\epsilon_{ijk}Q^j_0\dot{Q}^k_1 \quad (4.472)$$

$$\pi^1_i \equiv \frac{\partial L^C_{U_1}}{\partial \dot{Q}^i_1} = -\frac{\zeta^2}{2\mu m}\epsilon_{ijk}Q^j_0 Q^k_1 + \frac{\zeta^2}{\mu m}\epsilon_{ijk}Q^j_0\dot{Q}^k_0. \quad (4.473)$$

conjugated to Q^i_0 and Q^i_1.

Proposition 4.3.9. *Total Hamiltonian function for the first order unconstraint Lagrangian* (4.471) *is*

$$\begin{aligned} H_T = &-\frac{\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}D_i\delta_{jl}Q^l_0\Pi^1_k + \frac{1}{2Q^2_0}(D_iQ^i_0)(\delta_{jk}Q^j_0Q^k_1) + \frac{1}{2}D_iQ^i_1 \\ &- \frac{m\zeta}{2}\delta_{ij}Q^i_1Q^j_1 + \Big(\frac{\frac{2\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{il}Q^l_0\pi^1_jD_k - \frac{1}{2}D_iQ^i_1}{m\zeta Q^2_0}\Big)(\pi^1_rQ^r_0) \end{aligned} \quad (4.474)$$

where $D_i = \Pi^0_i + m\zeta\delta_{ij}Q^j_1$.

Proof. Equations (4.472) and (4.473) lead to solve two components of $\dot{Q}^i_0$ and $\dot{Q}^i_1$

$$\dot{Q}^i_0 = -\frac{\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{jl}Q^l_0(\pi^1_k + \frac{\zeta^2}{2\mu m}\epsilon_{klr}Q^l_0Q^r_1) + \frac{1}{Q^2_0}Q^i_0(\delta_{jk}Q^j_0\dot{Q}^k_0) \quad (4.475)$$

$$\dot{Q}^i_1 = -\frac{\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{jl}Q^l_0(\pi^0_k + m\zeta\delta_{kr}Q^r_1) + \frac{1}{Q^2_0}Q^i_0(\delta_{jk}Q^j_0\dot{Q}^k_1) \quad (4.476)$$

and for the remaining one, there exist primary constraints

$$\Phi = \pi^1_iQ^i_0, \quad \phi = (\pi^0_i + m\zeta\delta_{ij}Q^j_1)Q^i_0. \quad (4.477)$$

Then the canonical Hamiltonian function for the first order unconstraint Lagrangian (4.471) is

$$\begin{aligned} H_{U_1} &= \dot{Q}_{0^i}\pi^0_i + \dot{Q}^i_1\pi^1_i - L^C_{U_1} \\ &= -\frac{\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}D_i\delta_{jl}Q^l_0\Pi^1_k - \frac{1}{2Q^2_0}(D_iQ^i_0)(\delta_{jk}Q^j_0Q^k_1) + \frac{1}{2}D_iQ^i_1 - \frac{m\zeta}{2}\delta_{ij}Q^i_1Q^j_1 \end{aligned} \quad (4.478)$$

using $\dot{Q}^i_0$ and $\dot{Q}^i_1$, where $D_i = \pi^0_i + m\zeta\delta_{ij}Q^j_0$, $Q^2_0 = \delta_{ij}Q^i_0Q^j_j$. The total Hamiltonian function is

$$H_{T_1} = H_{U_1} + U\Phi + V\phi \tag{4.479}$$

where U and V arbitrary function. Consistency of the primary constraints Φ and ϕ

$$\begin{aligned}\dot{\Phi} &= \{\Phi, H_{T_1}\} \approx \{\Phi, H_{U_1}\} + U\{\Phi,\Phi\} + V\{\Phi,\phi\}\\ &\approx m\zeta\delta_{ij}Q^i_0Q^j_1 + V(-m\zeta\delta_{ij}Q^i_0Q^j_0)\end{aligned} \tag{4.480}$$

and

$$\begin{aligned}\dot{\phi} &= \{\Phi, H_{T_1}\} \approx \{\phi, H_{U_1}\} + U\{\phi,\Phi\} + V\{\phi,\phi\}\\ &\approx -\frac{2\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{il}Q^l_0\pi^1_jD_k + \frac{1}{2}D_iQ^i_1 + U(m\zeta\delta_{ij}Q^i_0Q^j_0)\end{aligned} \tag{4.481}$$

give no more constraint, instead we can solve U and V

$$U \approx \frac{\frac{2\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{il}Q^l_0\pi^1_jD_k - \frac{1}{2}D_iQ^i_1}{m\zeta Q^2_0} \tag{4.482}$$

$$V \approx \frac{\delta_{ij}Q^i_0Q^j_1}{Q^2_0}. \tag{4.483}$$

Substitutions of U and V into total Hamiltonian function (4.479) complete the proof. □

Proposition 4.3.10. *Hamilton equations of motion for the total Hamiltonian function in* (4.474) *are*

$$\dot{Q}^i_0 \approx -\frac{\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{jl}Q^l_0\pi^1_k + \frac{1}{2Q^2_0}Q^i_0(\delta_{jk}Q^j_0Q^k_1) + \frac{1}{2}Q^i_1 \tag{4.484}$$

$$\dot{Q}^i_1 \approx -\frac{\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}D_j\delta_{kl}Q^l_0 + \frac{1}{m\zeta Q^2_0}\Big(\frac{2\mu m}{\zeta^2 Q^2_0}\epsilon^{rjk}\delta_{rl}Q^l_0\pi^1_jD_k - \frac{1}{2}D_jQ^j_1\Big)Q^i_0 \tag{4.485}$$

$$\begin{aligned}\dot{\pi}^0_i &\approx \frac{\mu m}{\zeta^2 Q^2_0}\epsilon_{ijk}\delta^{jl}\Pi^1_lD_k - \frac{1}{2Q^2_0}D_i(\delta_{jk}Q^j_0Q^k_1)\\ &\quad - \Big(\frac{\frac{2\mu m}{\zeta^2 Q^2_0}\epsilon^{ijk}\delta_{il}Q^l_0\pi^1_jD_k - \frac{1}{2}D_iQ^i_1}{m\zeta Q^2_0}\Big)\pi^1_i - \frac{2\mu m}{\zeta^2 Q^4_0}\delta_{ij}Q^j_0(\epsilon^{ijk}\delta_{il}Q^l_0\pi^1_jD_k)\end{aligned} \tag{4.486}$$

$$\dot{\pi}^1_i \approx \frac{\mu m^2}{\zeta Q^2_0}\epsilon_{ijk}Q^j_0\delta^{kl}\pi^1_l - \frac{m\zeta}{2Q^2_0}\delta_{ij}Q^j_0(\delta_{kl}Q^k_0Q^l_1) - \frac{1}{2}\pi^0_i. \tag{4.487}$$

Proof. $\dot{\pi}_i^0$ gives the Euler-Lagrange equations and the remaining satisfied identically. The Hamilton equation of motion for Q_0^i's are satisfied identically

$$\begin{aligned}\dot{Q}_0^i &= \{Q_0^i, H_{T_1}\}\\ &\approx -\frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{jl}Q_0^l\pi_k^1 + \frac{1}{2Q_0^2}Q_0^i(\delta_{jk}Q_0^jQ_1^k) + \frac{1}{2}Q_1^i\\ &\approx -\frac{1}{2Q_0^2}\big(Q_0^i(\delta_{jk}Q_0^jQ_1^k) + Q_1^i(Q_0^2)\big) + \frac{1}{2Q_0^2}Q_0^i(\delta_{jk}Q_0^jQ_1^k) + \frac{1}{2}Q_1^i\\ &\approx Q_1^i \end{aligned} \tag{4.488}$$

using definition of π_i^1. The Hamilton equation of motion for Q_1^i's are satisfied identically

$$\begin{aligned}\dot{Q}_1^i &= \{Q_1^i, H_{T_1}\}\\ &\approx -\frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}D_j\delta_{kl}Q_0^l + \frac{1}{m\zeta Q_0^2}\big(\frac{2\mu m}{\zeta^2 Q_0^2}\epsilon^{rjk}\delta_{rl}Q_0^l\pi_j^1D_k - \frac{1}{2}D_jQ_1^j\big)Q_0^i\\ &\approx \dot{Q}_1^i - \frac{2}{Q_0^2}\big(\delta_{jk}Q_0^j\dot{Q}_1^k - \frac{3\zeta}{\mu m^2}\epsilon_{jkl}Q_0^j\dot{Q}_1^kQ_1^l\big)Q_0^i \end{aligned} \tag{4.489}$$

using definition of π_i^0 and π_i^1. Equation of motion for π_i^0's are

$$\begin{aligned}\dot{\pi}_i^0 &= \{\pi_i^0, H_{T_1}\}\\ &\approx \frac{\mu m}{\zeta^2 Q_0^2}\epsilon_{ijk}\delta^{jl}\Pi_l^1D_k - \frac{1}{2Q_0^2}D_i(\delta_{jk}Q_0^jQ_1^k)\\ &\quad - \Big(\frac{\frac{2\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{il}Q_0^l\pi_j^1D_k - \frac{1}{2}D_iQ_1^i}{m\zeta Q_0^2}\Big)\pi_i^1 - \frac{2\mu m}{\zeta^2 Q_0^4}\delta_{ij}Q_0^j(\epsilon^{ijk}\delta_{il}Q_0^l\pi_j^1D_k). \end{aligned} \tag{4.490}$$

Substitutions of D_i, π_i^0 and π_i^1 give same equation with first line of (4.339) with $Q_0^i = X^i, Q_1 = \dot{X}^i$. Equation of motion for π_i^1's are

$$\begin{aligned}\dot{\pi}_i^1 &= \{\pi_i^1, H_{T_1}\}\\ &\approx \frac{\mu m^2}{\zeta Q_0^2}\epsilon_{ijk}Q_0^j\delta^{kl}\pi_l^1 - \frac{m\zeta}{2Q_0^2}\delta_{ij}Q_0^j(\delta_{kl}Q_0^kQ_1^l) - \frac{1}{2}\pi_i^0 \end{aligned} \tag{4.491}$$

this is also same with the first line of the equation (4.340) with $Q_0^i = X^i, Q_1 = \dot{X}^i$. □

Dirac Bracket Formalism: Let us find the Hamilton equations of motion using the Dirac algebra. All constraints are second class since their Poisson bracket

$$\{\Phi, \phi\} = -m\zeta(\delta_{ij}Q_0^iQ_0^j) \tag{4.492}$$

is nonzero when $\delta_{ij}Q_0^iQ_0^j \neq 0$.

Proposition 4.3.11. *Dirac brackets of the coordinates are*

$$\{Q_0^i, Q_1^j\}_{DB} = -\frac{1}{m\zeta Q_0^2}Q_0^iQ_1^j \tag{4.493}$$
$$\{Q_0^i, \pi_j^0\}_{DB} = \frac{1}{m\zeta Q_0^2}Q_0^i\pi_j^1 + \delta_j^i \tag{4.494}$$
$$\{Q_1^i, \pi_j^0\}_{DB} = -\frac{1}{m\zeta Q_0^2}Q_0^i(\pi_j^0 + m\zeta\delta_{jk}Q_1^k) \tag{4.495}$$
$$\{Q_1^i, \pi_j^1\}_{DB} = \delta_j^i - \frac{1}{Q_0^2}Q_0^iQ_0^j \tag{4.496}$$
$$\{\pi_i^0, \pi_j^0\}_{DB} = \frac{1}{m\zeta Q_0^2}[\pi_i^1(\pi_j^0 + m\zeta\delta_{jk}Q_1^k) - (\pi_i^0 + m\zeta\delta_{ik}Q_1^k)\pi_j^1] \tag{4.497}$$
$$\{\pi_i^0, \pi_j^1\}_{DB} = \frac{1}{Q_0^2}\pi_i^1\delta_{jk}Q_1^k \tag{4.498}$$

and all the others are zero.

Proof. Recall that the Dirac bracket is given in (2.39), using this we compute

$$\{F, G\}_{DB} = \{F, G\} - \frac{1}{m\zeta Q_0^2}\big(\{F, \Phi\}\{\phi, G\} - \{F, \phi\}\{\Phi, G\}\big) \tag{4.499}$$

by replacing inverse of M

$$M = \begin{bmatrix} \{\Phi, \Phi\} & \{\Phi, \phi\} \\ \{\phi, \Phi\} & \{\phi, \phi\} \end{bmatrix} = m\zeta Q_0^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.500}$$

To prove Dirac brackets of coordinates we also need to compute Poisson brackets of the coordinates with the constraints, they are

$$\{Q_0^i, \phi\} = \{Q_0^i, (\pi_j^0 + m\zeta\delta_{jk}Q_1^k)Q_0^j\} = Q_0^i \tag{4.501}$$

$$\{Q_1^i, \Phi\} = \{Q_1^i, \pi_j^1 Q_0^j\} = Q_0^i \tag{4.502}$$

$$\{\pi_i^0, \Phi\} = \{\pi_i^0, \pi_j^1 Q_0^j\} = -\pi_i^1 \tag{4.503}$$

$$\{\pi_i^0, \phi\} = \{\pi_i^0, (\pi_j^0 + m\zeta\delta_{jk}Q_1^k)Q_0^j\} = -\pi_i^0 - m\zeta\delta_{ik}Q_1^k \tag{4.504}$$

$$\{\pi_i^1, \phi\} = \{\pi_i^1, (\pi_j^0 + m\zeta\delta_{jk}Q_1^k)Q_0^j\} = -m\zeta\delta_{ij}Q_0^j \tag{4.505}$$

and all others are zero. Substitution of these relations in Dirac bracket (4.499) complete the proof. For instance Dirac bracket of Q_0^i with Q_1^j is

$$\begin{aligned}\{Q_0^i, Q_1^j\}_{DB} &= \{Q_0^i, Q_1^j\} - \frac{1}{m\zeta Q_0^2}\big(\{Q_0^i, \Phi\}\{\phi, Q_1^j\} - \{Q_0^i, \phi\}\{\Phi, Q_1^j\}\big)\\ &= -\frac{1}{\zeta Q_0^2}Q_0^i Q_0^j\end{aligned} \tag{4.506}$$

since $\{Q_0^i, \phi\}$ and $\{Q_1^i, \Phi\}$ is nonzero. One can derive the other Dirac brackets of the coordinates similar to this. □

Proposition 4.3.12. *The Hamilton equations using the Dirac brackets of the coordinates in the proposition* $4.3.11$ *are*

$$\dot{Q}_0^i \approx -\frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{jsk}\delta_{sl}Q_0^l\pi_k^1 + \frac{1}{2}Q_1^i \tag{4.507}$$

$$\begin{aligned}\dot{Q}_1^i \approx\ & \frac{2\mu}{m\zeta^3 Q_0^4}\big(\epsilon^{rsk}\delta_{sl}D_r\pi_k^1 Q_0^l\big)Q_0^i - \frac{Q_0^i}{2m\zeta Q_0^2}Q_1^j(\pi_j^0 + m\zeta\delta_{jk}Q_1^k)\\ & - \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{irs}\delta_{sl}D_r Q_0^l\end{aligned} \tag{4.508}$$

$$\begin{aligned}\dot{\pi}_i^0 \approx\ & \frac{\mu m}{\zeta^2 Q_0^2}\epsilon_{ijk}\delta^{jl}\Pi_l^1 D_k - \frac{1}{2Q_0^2}D_i(\delta_{jk}Q_0^j Q_1^k)\\ & - \Big(\frac{\frac{2\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{il}Q_0^l\pi_j^1 D_k - \frac{1}{2}D_i Q_1^i}{m\zeta Q_0^2}\Big)\pi_i^1 - \frac{2\mu m}{\zeta^2 Q_0^4}\delta_{ij}Q_0^j(\epsilon^{ijk}\delta_{il}Q_0^l\pi_j^1 D_k)\end{aligned} \tag{4.509}$$

$$\dot{\pi}_i^1 \approx \frac{\mu m^2}{\zeta Q_0^2}\epsilon_{ilk}Q_0^l\delta^{sk}\pi_s^1 - \frac{1}{2}\pi_i^0 + \frac{1}{2Q_0^2}(\pi_j^0 Q_0^j)\delta_{ik}Q_0^k. \tag{4.510}$$

Proof. Using the Dirac brackets of the coordinates in the proposition $4.3.11$ and the Hamiltonian function 4.478, equations of motion for Q_0^i are satisfied identically

$$\begin{aligned} Q_0^i &= \{Q_0^i, H\}_{DB} \\ &= \big(-\frac{\mu m^2}{\zeta Q_0^2}\epsilon_{jsk}Q_0^s\pi_k^1 + \frac{1}{2}\pi_j^0\big)\{Q_0^i, Q_1^j\}_{DB} \\ &+ (-\frac{\mu m^2}{\zeta Q_0^2}\epsilon^{jsk}\delta_{sl}Q_0^l\pi_k^1 + \frac{1}{2}Q_1^j)\{Q_0^i, \Pi_j^0\}_{DB} \\ &\approx -\frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{isk}\delta_{sl}Q_0^l\pi_k^1 + \frac{1}{2}Q_1^i \end{aligned} \tag{4.511}$$

using definition of π_i^1. Equations of motion for Q_1^i are

$$\begin{aligned} Q_1^i &= \{Q_1^i, H\}_{DB} \\ &= \big(\frac{-\mu m}{\zeta^2 Q_0^2}\epsilon^{rsk}\delta_{sj}D_r\pi_k^1 + \frac{1}{2}\pi_j^0 + \frac{2\mu m}{\zeta^2 Q_0^4}\delta_{jj'}Q_0^{j'}\epsilon^{rsk}\delta_{sl}D_r\pi_k^1 Q_0^l\big)\{Q_0^i, Q_0^j\}_{DB} \\ &+ (\frac{-\mu m}{\zeta^2 Q_0^2}\epsilon^{jsk}\delta_{sl}Q_0^l\pi_k^1 + \frac{1}{2}Q_1^j)\{Q_0^i, \Pi_j^0\}_{DB} \\ &+ (\frac{-\mu m}{\zeta^2 Q_0^2}\epsilon^{rsj}\delta_{sl}Q_0^l D_r\big)\{Q_1^i, \Pi_j^1\}_{DB} \\ &\approx \frac{2\mu}{m\zeta^3 Q_0^4}\big(\epsilon^{rsk}\delta_{sl}D_r\pi_k^1 Q_0^l\big)Q_0^i - \frac{Q_0^i}{2m\zeta Q_0^2}Q_1^j(\pi_j^0 + m\zeta\delta_{jk}Q_1^k) \\ &- \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{irs}\delta_{sl}D_r Q_0^l. \end{aligned} \tag{4.512}$$

Substituting definitions of D_i, π_i^0 and π_i^1 the last equation

$$\approx \dot{Q}_1^i - \frac{2}{Q_0^2}\big(\delta_{jk}Q_0^j\dot{Q}_1^k - \frac{3\zeta}{\mu m^2}\epsilon_{jkl}Q_0^j\dot{Q}_1^k Q_1^l\big)Q_0^i \tag{4.513}$$

gives $\dot{Q}_1^i$ since the term in the parenthesis is zero. Equation of motion for π_i^0's are

$$\begin{aligned} \dot{\pi}_i^0 &= \{\pi_i^0, H\}_{DB} \\ &= \big(\frac{2\mu m}{\zeta^2 Q_0^4}Q_0^j\epsilon_{lkr}Q_0^l\delta^{sk}\pi_s^1 D_r - \frac{\mu m}{\zeta^2 Q_0^2}\delta_{js}\epsilon^{skr}D_r\Pi_k^1\big)\{\pi_i^0, Q_0^j\}_{DB} \\ &+ \big(-\frac{\mu m^2}{\zeta Q_0^2}\epsilon_{jlk}Q_0^l\delta^{sk}\pi_s^1 + \frac{1}{2}\pi_j^0\big)\{\pi_i^0, Q_1^j\}_{DB} \\ &+ (-\frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{jl}Q_0^l\pi_l^k + \frac{1}{2}Q_1^j)\{\pi_i^0, \pi_j^0\}_{DB} \end{aligned} \tag{4.514}$$

$$- \frac{\mu m}{\zeta^2 Q_0^2}\epsilon^{rsj}\delta_{sl}D_r Q_0^l\{\pi_i^0, \pi_j^1\}_{DB}. \tag{4.515}$$

Using the Dirac brackets in the Proposition $4.3.11$ we get

$$\approx \frac{\mu m}{\zeta^2 Q_0^2}\epsilon_{ijk}\delta^{jl}\Pi_l^1 D_k - \frac{1}{2Q_0^2}D_i(\delta_{jk}Q_0^j Q_1^k)$$
$$-\,(\frac{\frac{2\mu m}{\zeta^2 Q_0^2}\epsilon^{ijk}\delta_{il}Q_0^l\pi_j^1 D_k - \frac{1}{2}D_i Q_1^i}{m\zeta Q_0^2})\pi_i^1 - \frac{2\mu m}{\zeta^2 Q_0^4}\delta_{ij}Q_0^j(\epsilon^{ijk}\delta_{il}Q_0^l\pi_j^1 D_k). \qquad (4.516)$$

which is the same with equation (4.490) where $D_i = \Pi_i^0 + m\zeta\delta_{ij}Q_1^j$.

Equation of motion for π_i^1's are identically satisfied

$$\begin{aligned}
\dot{\pi}_i^1 &= \{\pi_i^1, H\}_{DB} \\
&= (\frac{-\mu m^2}{\zeta Q_0^2}\epsilon_{jlk}Q_0^l\delta^{sk}\pi_s^1 + \frac{1}{2}\pi_j^0)\{\pi_i^1, Q_1^j\}_{DB} \\
&+ (\frac{-\mu m^2}{\zeta Q_0^2}\epsilon_{jlk}Q_0^l\delta^{sk}\pi_s^1 + \frac{1}{2}Q_1^j)\{\pi_i^1, \pi_j^0\}_{DB} \\
&\approx \frac{\mu m^2}{\zeta Q_0^2}\epsilon_{ilk}Q_0^l\delta^{sk}\pi_s^1 - \frac{1}{2}\pi_i^0 + \frac{1}{2Q_0^2}(\pi_j^0 Q_0^j)\delta_{ik}Q_0^k \\
&\approx -\frac{1}{2}(\pi_i^0 + m\zeta_{ij}Q_1^j) = \dot{\pi}_i^1
\end{aligned} \qquad (4.517)$$

using the definitions of π_i^0 and π_i^1. □

5. CONCLUSIONS

In this thesis, the Hamiltonian formulations of the second order Pais-Uhlenbeck [39], Sarıoğlu-Tekin [38] and Clèment [37] Lagrangians have been presented. We note that, Pais-Uhlenbeck Lagrangian is non-degenerate whereas Sarıoğlu-Tekin and Clèment Lagrangians are degenerate. For the nondegenerate cases, the Legendre transformation is immediate after the introduction of the Jacobi-Ostrogradsky momenta. For degenerate ones, one needs to employ the Dirac-Bergmann algorithm.

In each of these cases, first we have studied directly the second order Lagrangians. We defined related Jacobi-Ostrogradsky momenta, and the canonical Hamiltonian functions. At this step, the Legendre transforation has been achieved for Pais-Uhlenbeck Lagrangian. For Sarıoğlu-Tekin and Clèment Lagrangians, further investigations have been needed. The Dirac-Bergmann constraint algorithm have been run and the Dirac-Poisson brackets have been constructed for these degenerate theories.

The reductions of Pais-Uhlenbeck, Sarıoğlu-Tekin and Clèment Lagrangians to the first order theories have been presented. The fiber derivatives have been computed and the Dirac-Bergmann algorithms have been performed for the reduced Lagrangians. It has been shown the possibility of arriving a reduced Lagrangian in a variational free form. Dirac analysis for this is quite similar to the reduced first order Lagrangian.

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