ON THE BLOW-UP SOLUTIONS TO SOME QUASI-LINEAR BI-HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION UNDER DYNAMICAL BOUNDARY CONDITIONS

by

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ABSTRACT

ON THE BLOW-UP SOLUTIONS TO SOME QUASI-LINEAR BI-HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION UNDER DYNAMICAL BOUNDARY CONDITIONS

In this thesis, we show the non-existence of the global solutions as a class of initial boundary value problem by considering various dissipative terms in the boundary conditions for quasi-linear bi-hyperbolic equations. In particular, we obtain blow-up solutions for the positive initial energy. This work is inspired by the paper of V.Bayrak-M.Can in [11] in which they studied the same problem for the non-positive initial energy. While their result is achieved by applying O.Ladyzhenskaya-V.K. Kalantarov lemma, called generalized convexity method, our approach is based on the blow-up lemma by M. O. Korpusov.

ÖZET

DİNAMİK SINIR KOŞULLARI ALTINDA BAZI QUASI-LİNEER BI-HİPERBOLİK KISMİ DİFERANSİYEL DENKLEMLER İÇİN ÇÖZÜMLERİN PATLAMASI ÜZERİNE

Bu tezde, başlangıç sınır değer problemlerinin bir sınıfı olarak, sınır koşullarında bazı yayılım terimlerinin düşünülmesiyle, quasi-lineer çift-hiperbolik denklemlerin küresel çözümlerinin olmadığı gösteriliyor. Bilhassa, pozitif başlangıç enerjisi için patlayan çözümler elde ediyoruz. Bu tez [11]'de aynı problemi negatif başlangıç enerjisi için çalışan V. Bayrak ve M. Can'ın makalesinden esinlenmiştir. Onların sonucu, genelleştirilmiş dış bükeylik metodu olarak adlandırılan O. Ladyzhenskaya ve V.K. Kalantarov'un lemması kullanılarak elde edilirken, bizim yaklaşımımız M.O. Korpusov'un patlama lemmasına dayanıyor.

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1. INTRODUCTION

Initial-boundary value problems for quasi-linear bi-hyperbolic partial equations provide powerful and flexible tools for modelling problems in physics, engineering, and other fields. Many researchers have worked on the problems of such equations. There are great number of articles on the global non-existence of solutions to quasi-linear hyperbolic type equations of order two, see [1, 2, 3].

The non-existence of solutions of quasi-linear hyperbolic equations for lack of dissipative terms on the boundary conditions have been studied by many authors among which we refer to and the references therein, [1, 4, 5, 6, 7] .

We must mention that many books on blow-up of solutions of non-linear partial differential equations have been published for the last decades: The books of Samarskii, Galaktinov, Kurdyumov and Mikhailov [8] and Hu [9] are devoted to the study of blow-up of solutions of non-linear parabolic equations and systems, and the book of Pokhozhaev and Mitidieri [10] is devoted to problems of solutions of non-linear parabolic and hyperbolic equations and inequalities.

In the last decades, the studies of dynamical boundary conditions of hyperbolic equations have been appeared in many articles. Majority of them are devoted the second order equations. In addition, hyperbolic equations of fourth-order have been studied by M. Can-V. Bayrak [11] , I. Lasiecka [12], F. Maksudov- F. Aliev [13]. The mathematical tool that M. Kirane [14] and M. Can [11] used in their work is the blow-up lemma which is so called O. A. Ladyzhenskaya-V. K. Kalantarov lemma [15]. The important point in applying of the above argument is to find appropriate function that includes dissipation of boundary in the proof of the hypotheses of this lemma. The goal of this thesis is to work on blow-up solutions for bi-hyperbolic quasi-linear equations under different type of boundary conditions when the initial energy is positive.

The following is an organization of this thesis: Chapter 2 is devoted to literature; in Chapter 3 notations and auxiliary propositions are given; in Chapter 4 our main results and their proofs are given; The final chapter is devoted to an initial boundary value problem of a second order hyperbolic partial differential equation and its blow-up solutions under the positive initial energy.

2. PREVIOUS RESULT

We recall a blow-up lemma which is so called O. A. Ladyzhenskaya-V. K. Kalantarov lemma [15].

Lemma 1 *Suppose that* $\Phi(t) \in C^2([0,T])$, $\Phi(t) \geq 0$ *satisfies the inequality*

$$\Phi''(t)\,\Phi(t) - (1+\gamma)\,(\Phi'(t))^2 \geq -2\hat{C}_1\Phi(t)\,\Phi'(t) - \hat{C}_2\Phi(t)^2 \tag{2.1}$$

where $\gamma > 0$, $\hat{C}_1$, $\hat{C}_2 \geq 0$, *then the following hold*

***a**) If*

$$\Phi(0) > 0, \quad \Phi'(0) > -\gamma_2\gamma^{-1}\Phi(0) \quad and \quad \hat{C}_1 + \hat{C}_2 > 0 \tag{2.2}$$

then,
$\Phi(t) \to +\infty$ *as* $t{\to}t_1 \leq t_2 = \frac{1}{\sqrt[2]{\hat{C}_1^2+\hat{C}_2^2}} \ln\left(\frac{\gamma_1\Phi(0)+\gamma\Phi'(0)}{\gamma_2\Phi(0)+\gamma\Phi'(0)}\right)$,

where
$\gamma_1 = -\hat{C}_1 + \sqrt[2]{\hat{C}_1^2 + \gamma\hat{C}_2}$, $\gamma_2 = -\hat{C}_1 - \sqrt[2]{\hat{C}_1^2 + \gamma\hat{C}_2}$.

***b**) If*

$$\Phi(0) > 0, \quad \Phi'(0) > 0 \quad and \quad \hat{C}_1 = \hat{C}_2 = 0\ , \tag{2.3}$$

then $\Phi(t) \to +\infty$ *as* $t{\to}t_2 \leq t_2 = \frac{\Phi(0)}{\gamma\Phi'(0)}$.

The below two initial and boundary value problems were studied by M. Kirane, S. Kouachi and N. Tatar [14] .

The first problem:

$$\begin{cases} w_{tt} + \varDelta^2 w = f(w), \quad x \in \varOmega,\ t \in (0,T), \\ \dfrac{\partial \varDelta w}{\partial \vartheta} = 0, \quad \varDelta w = -\alpha(x) \dfrac{\partial w_t}{\partial \vartheta}, \quad x \in \partial\varOmega, \quad t \in (0,T), \\ w(x,0) = w_0(x), \quad \dfrac{\partial w}{\partial t}(x,0) = w_1(x), \quad x \in \varOmega. \end{cases}$$

where $\varOmega \subset \mathbb{R}^n$ is a bounded domain with regular boundary $\partial\varOmega := \varGamma$, $T > 0$ is any real number, the function $\alpha(x) \geq 0$ is smooth on $\varGamma$; $\vartheta = (\vartheta_1, ..., \vartheta_n)$ is the unit outward normal to $\varGamma$ and $\dfrac{\partial w}{\partial \vartheta} = \sum_{i=1}^{i=n} \dfrac{\partial w}{\partial x_i}\vartheta_i$ is the normal derivative of w on $\varGamma$.
Assume that the function $f(w)$ and its primitive $F(w) = \int_0^w f(\xi)\, d\xi$ have the following property:

$$w f(w) \geq 2(2\gamma + 2) F(w) - C_0. \tag{2.4}$$

Here $\gamma > 0$, $C_0 > 0$ and w are real numbers .
Their result for this problem is given below:

Theorem 1 *Let $w_0(x)$, $w_1(x)$ enjoy the initial functions with properties below*

$$A := \int_\varOmega w_0^2(x)\, dx + \int_\varGamma \alpha(x) \left(\frac{\partial w_0}{\partial \vartheta}(x)\right)^2 d\sigma > 0 ,$$

$$B := \int_\varOmega (2w_0(x) w_1(x) + \gamma_2 \gamma^{-1} w_0^2(x))\, dx \geq 0 ,$$

$$(2\gamma + 1) \int_\varOmega (w_1^2(x) + |\varDelta w_0(x)|^2)\, dx + C_0\, measure(\varOmega) \leq 2(2\gamma+1) \int_\varOmega F(w_0(x))\, dx .$$

Then

$$\lim_{t\to t_1}\left(\int_{\Omega} w^2(x)\,dx+\int_0^t\int_{\Gamma}\alpha(x)\left(\frac{\partial w}{\partial\vartheta}(x)\right)^2 d\sigma\,ds\right)=+\infty,\quad for\ some$$

$$t_1\le t_2:=\frac{1}{2\sqrt{\hat{C}_1^2+\hat{C}_2^2}}\ln\left(\frac{\gamma_1 A+B}{\gamma_2 A+B}\right)$$

In the proof of this result they use the following function:

$$\Phi(t)=\int_{\Omega} w^2(x)\,dx+\int_0^t\int_{\Gamma}\alpha(x)\left(\frac{\partial w}{\partial\vartheta}(x)\right)^2 d\sigma\,ds+\int_{\Gamma}\alpha(x)\left(\frac{\partial w_0}{\partial\vartheta}(x)\right)^2 d\sigma. \quad (2.5)$$

and then, they obtained a lower estimate for the function

$$\chi(t):=\Phi''\Phi-(1+\gamma)(\Phi')^2 \quad (2.6)$$

which they proved

$$\chi(t)\ge-(\gamma+2)\Phi^2-4(1+\gamma)\Phi\Phi'\,. \quad (2.7)$$

Hence, from the Lemma 1 for $\hat{C}_1\equiv 2(1+\gamma)$, $\hat{C}_2\equiv 2+\gamma$ gives the result.

The second problem:

$$\begin{cases} w_{tt}+\Delta^2 w=f(w), \quad x\in\Omega,\ t\in(0,T), \\ \Delta w=0, \quad \alpha(x)\,w_t-\dfrac{\partial\Delta w}{\partial\vartheta}=0, \quad x\in\partial\Omega,\ t\in(0,T), \\ w(x,0)=w_0(x), \quad \dfrac{\partial w}{\partial t}(x,0)=w_1(x), \quad x\in\Omega. \end{cases}$$

Here $\Omega\subset\mathbb{R}^n$ is a bounded domain with regular boundary $\partial\Omega:=\Gamma$. $T>0$ is any real number, ϑ is the outward normal to Γ and the function $\alpha(x)\ge 0$ is smooth on Γ.

Assume that $f\,(w)$ satisfies (2.4). For the second problem, their conclusion is:

Theorem 2 *Let* $w_0\,(x)\,,\,w_1\,(x)$ *enjoy the initial functions with properties below*

$$\int_\Omega w_0^2\,(x)\,dx + \int_\Gamma \alpha\,(x)\,w_0^2\,(x)\,d\sigma > 0,$$

$$\int_\Omega (2w_0\,(x)\,w_1\,(x) + \gamma_2\gamma^{-1}w_0^2\,(x))\,dx \geq 0,$$

$$(1+2\gamma)\int_\Omega (w_1^2\,(x) + |\varDelta w_0\,(x)\,|^2)\,dx + C_0\,measure\,(\Omega) \leq 2(1+2\gamma)\int_\Omega F(w_0\,(x))\,dx.$$

$Then$

$$\lim_{t\to t_1}\left(\int_\Omega w^2\,(x)\;dx + \int_0^t\int_\Gamma \alpha\,(x)\,w^2\,(x)\,d\sigma\,ds\right) = +\infty, \quad for\ some$$

$$t_1 \leq t_2 := \frac{1}{2\sqrt{\hat{C}_1^2+\hat{C}_2^2}}\ln\left(\frac{\gamma_1 A + B}{\gamma_2 A + B}\right)$$

As in the previous proof they used the functional,

$$\varPhi\,(t) = \int_\Omega w^2\,(x)\,dx + \int_0^t\int_\Gamma \alpha\,(x)\,w^2\,(x)\,d\sigma\,ds + \int_\Gamma \alpha\,(x)\,w_0^2\,(x)\;d\sigma \qquad (2.8)$$

and obtain similar estimates as (2.6) and (2.7).

Hence, from the Lemma 1 for $\hat{C}_1 \equiv 2(1+\gamma)\,,\,\hat{C}_2 \equiv 2+\gamma$ gives the conclusion.

M. Can and V. Bayrak [11] studied some variations of the following problem:

$$\begin{cases} u_{tt} + \mathit{\Delta}^2 u = \mathit{\Delta} u + f\left(-\mathit{\Delta} u\right), & t \in (0,T)\,,\ x \in (\Omega \cup \partial\Omega)\,, \\ -\mathit{\Delta} u = 0, \qquad \dfrac{\partial u_t}{\partial \vartheta} = \mathit{\Delta}^2 u, & t \in (0,T)\,,\ x \in \partial\Omega, \\ u\left(x,0\right) = u_0\left(x\right), \quad \dfrac{\partial u}{\partial t}\left(x,0\right) = u_1\left(x\right), & x \in \Omega. \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and its boundary $\partial\Omega := \Gamma$ is regular, $T > 0$ is any real number, and ϑ is the outward normal to Γ.

They used the function,

$$\Phi\left(t\right) = \left\|\nabla u\right\|^2 + \int_0^t \int_\Gamma \alpha\left(x\right)\left(\frac{\partial u}{\partial \vartheta}\right)^2 d\sigma\, ds + \int_\Gamma \alpha\left(x\right)\left(\frac{\partial u_0}{\partial \vartheta}\right)^2 d\sigma. \tag{2.9}$$

And, use the Lemma 1, to prove:

Theorem 3 *Asumme that $f\left(u\right)$ and its primitive $F\left(u\right) = \int_0^u f\left(\xi\right) d\xi$ enjoy the restrictions:*

$$f\left(0\right) = 0, \qquad uf\left(u\right) \geq 2\left(2\gamma + 1\right) F\left(u\right), \qquad \textit{for all}\, u \in \mathbb{R} \tag{2.10}$$

with some real number $\gamma > 0$. Let $u_0\left(x\right)$, $u_1\left(x\right)$ are functions satisfying

- *Φ and its derivative Φ' satisfy (2.2) of Lemma 1.*
- *The energy at $t = 0$*

$$E\left(0\right) = \left\|\nabla u_1\right\|^2 + \int_\Omega \left|\nabla \mathit{\Delta} u_0\right|^2 dx + \int_\Omega \left|\mathit{\Delta} u_0\right|^2 dx - 2\int_\Omega F\left(\mathit{\Delta} u_0\right) dx \leq 0. \tag{2.11}$$

For $t_2 > 0$ is given in Lemma 1, then there exists $0 < t_1 < t_2$ such that
$\lim_{t \longrightarrow t_1 \uparrow} \Phi(t) = +\infty$.

Our main tool in this work is the blow-up lemma by M. O. Korpusov [26] :

Lemma 2 *Assume that*

$$(a_1) \quad \varphi\,(t) \in C^2([0,\hat{T}]),\ \hat{T} > 0, \quad \varphi\,(t) \geq 0, \quad \varphi\,(0) > 0, \quad \varphi'\,(0) > 0.$$

This implies the existence of $t_0 > 0$ *with* $\varphi'\,(t) > 0$ *on* $[0, t_0)$ *and* $\varphi\,(t) > \varphi\,(0) \geq 0$ *on* $[0, t_0)$

$$\alpha > 1, \quad \beta > 0.$$

(a_2) $\quad Assume\,(\varphi'(0))^2 > \frac{2\beta}{2\alpha-1}\varphi(0)$ *holds for the solution of the differential inequality*

$$\varphi''(t)\varphi(t) - \alpha(\psi'(t))^2 + \beta\varphi(t) \geq 0. \tag{2.12}$$

Then

$$\varphi(t) \geq \left(\varphi^{1-\alpha}(0) - At\right)^{\frac{-1}{\alpha-1}} \quad \textit{and} \quad \lim_{t\uparrow T_\infty} \varphi(t) = +\infty.$$

3. NOTATIONS AND AUXILIARY PROPOSITIONS

Throughout the thesis we are using the following notations:

• $L^2(\Omega)$ is a usual Lebesgue space with the inner product $\langle .,. \rangle$ and the norm $\| \, . \, \|$.

• $H^1(\Omega)$ is a Sobolev space of functions $v \in L^2(\Omega)$ whose weak derivatives also belong to $L^2(\Omega)$. This space is a Hilbert space with the inner product

$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega \Big(u(x) \, v(x) + \nabla u(x) \, \nabla v(x) \Big) dx \tag{3.1}$$

and the norm

$$\| \, v \, \|_{H^1(\Omega)} = \left(\| \, v \, \|^2 + \| \, \nabla v \, \|^2 \right)^{\frac{1}{2}}. \tag{3.2}$$

• $H_0^1(\Omega)$ is the Sobolev space obtained by completion of $C_0^\infty(\Omega)$ with respect to the norm of $H^1(\Omega)$. The inner product and the norm in this space are defined as follows

$$\langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u(x) \, \nabla v(x) \, dx \tag{3.3}$$

and

$$\| \, v \, \|_{H_0^1(\Omega)} = \| \, \nabla v \, \|. \tag{3.4}$$

We will need the following inequalities:

• Cauchy-Schwartz Inequality:

Let Ω is a region in $\mathbb{R}^n$ and $u\left(x\right)$ and $v\left(x\right)$ be two integrable vectors in Ω

$$\left|\int_{\Omega} u\left(x\right) v\left(x\right) dx\right| \leq \Big(\int_{\Omega} |u\left(x\right)|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega} |v\left(x\right)|^2 dx\Big)^{\frac{1}{2}}. \tag{3.5}$$

• Hölder's Inequality:

$$\int_{\Omega} |f\left(x\right) g\left(x\right) dx| \leq \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^{p'}(\Omega)} \tag{3.6}$$

which holds for each $f \in L^p\left(\Omega\right)$ and $g \in L^{p'}\left(\Omega\right)$ the inequality and $1/p + 1/p' = 1$.

• Young's Inequality:

For $a, b \geq 0$ and $p, q > 0$ with $1/p + 1/q = 1$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{3.7}$$

• Green's Formula:

Let u , $\vartheta \in C^2\left(\bar{\Omega}\right)$. Then

$$\int_{\Omega} \nabla\vartheta\nabla u\, dx = -\int_{\Omega} u\Delta\vartheta\, dx + \int_{\partial\Omega} \frac{\partial\vartheta}{\partial\eta} u\, d\sigma \tag{3.8}$$

where η is the outward normal vector to the boundary of Ω.

• Poincare- Friedrichs Inequality:

$$\|w\| \le \lambda_1^{\frac{1}{2}} \|\nabla w\| \tag{3.9}$$

which holds for each $w \in H_0^1(\Omega)$. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, λ_1 is the first eigenvalue of the problem

$$\begin{cases} -\Delta\psi = \lambda\psi, \quad x \in \Omega \\ \psi = 0, \quad x \in \partial\Omega \end{cases}$$

If $w \in H^2(\Omega) \bigcap H_0^1(\Omega)$, then Poincare- Friedrichs Inequality implies that

$$\|\nabla w\| \le \lambda_1^{-\frac{1}{2}} \|\Delta w\|\,. \tag{3.10}$$

3.1. THE INITIAL AND BOUNDARY VALUE PROBLEM UNDER VARIOUS DYNAMICAL BOUNDARY CONDITIONS

In this case and the following cases Ω is a bounded domain in $\mathbb{R}^n$ and its boundary $\partial\Omega := \Gamma$ is regular, $T > 0$ is any real number, and η is the outward normal of Γ and the function $\alpha(x) \geq 0$ is smooth on Γ.

3.1.1. CASE: 1

We consider the following initial and boundary value problem as our first problem:

$$v_{tt} + \Delta^2 v = \Delta v + bf(-\Delta v), \qquad x \in \bar{\Omega},\ t \in (0,T), \tag{3.11}$$

$$\Delta v = 0, \qquad \alpha(x)\frac{\partial v_t}{\partial \eta} = \Delta^2 v, \qquad x \in \Gamma,\ t \in (0,T), \tag{3.12}$$

$$v(x,0) = v_0(x), \qquad \frac{\partial v}{\partial t}(x,0) = v_1(x), \qquad x \in \Omega. \tag{3.13}$$

To prove the existence of blow-up solution we use the Lemma 2 for the solution $v(x,t)$ of $(3.11) - (3.13)$ as our first result.
Assume that $f(v)$ and its primitive $F(v) = \int_0^v f(\xi)\, d\xi$ enjoy the following restrictions:

$$f(0) = 0, \qquad vf(v) \geq 2(2\gamma + 1) F(v), \qquad \text{for all } v \in \mathbb{R} \tag{3.14}$$

for reals $\gamma > 0$.
Firstly, we start by obtaining an estimate for the energy function $E(t)$ defined by

$$E(t) := \|\nabla v_t\|^2 + \|\Delta v\|^2 + \|\nabla \Delta v\|^2 - 2b\langle F(-\Delta v), 1\rangle. \tag{3.15}$$

Multiplying the equation (3.11) in $L^2(\Omega)$ by $-2\Delta v_t$ gives us the equality:

$$\underbrace{-2\int_\Omega v_{tt}\Delta v_t\,dx}_{\text{I}}+\underbrace{2\int_\Omega \Delta v\Delta v_t\,dx}_{\text{II}}\underbrace{-2\int_\Omega \Delta^2 v\Delta v_t\,dx}_{\text{III}}=\underbrace{-2b\int_\Omega f(-\Delta v)\,\Delta v_t\,dx}_{\text{IV}} \quad (3.16)$$

By using Green's Formula and the boundary conditions we get,

$$\begin{aligned}
\text{I} &= -2\int_\Omega v_{tt}\Delta v_t\,dx \\
&= 2\int_\Omega \nabla v_{tt}\nabla v_t\,dx - 2\int_\Gamma \frac{\partial v_t}{\partial \eta}v_{tt}\,d\sigma \\
&= \frac{d}{dt}\int_\Omega |\nabla v_t|^2\,dx - 2\int_\Gamma \frac{\partial v_t}{\partial \eta}v_{tt}\,d\sigma \\
\text{II} &= 2\int_\Omega \Delta v\Delta v_t\,dx \\
&= \frac{d}{dt}\int_\Omega |\Delta v|^2\,dx \\
\text{III} &= -2\int_\Omega \Delta v_t\Delta^2 v\,dx \\
&= 2\int_\Omega \nabla(\Delta v_t)\nabla(\Delta v)\,dx - 2\int_\Gamma \frac{\partial \Delta v}{\partial \eta}\Delta v_t\,d\sigma \\
&= \frac{d}{dt}\int_\Omega |\nabla\Delta v|^2\,dx - 2\int_\Gamma \frac{\partial \Delta v}{\partial \eta}\Delta v_t\,d\sigma \\
\text{IV} &= -2b\int_\Omega f(-\Delta v)\,\Delta v_t\,dx \\
&= \frac{d}{dt}\,2b\int_\Omega F(-\Delta v)\,dx.
\end{aligned}$$

Plugging them into (3.16) we have,

$$\frac{d}{dt}\left[\|\nabla v_t\|^2 + \|\varDelta v\|^2 + \|\nabla \varDelta v\|^2 - 2b\left\langle F\left(-\varDelta v\right), 1\right\rangle\right]$$

$$= 2\int_{\varGamma} \frac{\partial v_t}{\partial \eta} v_{tt}\, d\sigma + 2\int_{\varGamma} \frac{\partial \Delta v}{\partial \eta} \Delta v_t\, d\sigma.$$

Restricting the differential equation to $\partial\varOmega$ we get,

$$v_{tt}\Big|_{\partial\varOmega} = 0 - \varDelta^2 v + 0 = -\varDelta^2 v \qquad \text{and} \qquad \alpha\left(x\right)\frac{\partial v_t}{\partial \eta} = \varDelta^2 v$$

for all $(x,t) \in \partial\varOmega \times (0,T)$. Hence,

$$\frac{d}{dt}E\left(t\right) = 2\int_{\varGamma} \frac{\partial v_t}{\partial \eta} v_{tt}\, d\sigma + 2\int_{\varGamma} \frac{\partial\left(\Delta v\right)}{\partial \eta} \Delta v_t\, d\sigma$$

$$= -2\int_{\varGamma} \frac{\partial v_t}{\partial \eta} \varDelta^2 v\, d\sigma$$

$$= -2\int_{\varGamma} \alpha\left(x\right)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma. \tag{3.17}$$

It is obvious from (3.17) that $E(t) \le E(0)$ for all $t \ge 0$.

Define the following function

$$\varPhi\left(t\right) = \|\nabla v\|^2 + \int_0^t \int_{\varGamma} \alpha\left(x\right)\left(\frac{\partial v}{\partial \eta}\right)^2 d\sigma\, ds + \int_{\varGamma} \alpha\left(x\right)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma. \tag{3.18}$$

Differentiating the equation (3.18) with respect to t we obtain,

$$\varPhi'\left(t\right) = 2\left\langle \nabla v, \nabla v_t\right\rangle + 2\int_0^t \int_{\varGamma} \alpha\left(x\right)\frac{\partial v}{\partial \eta}\frac{\partial v_t}{\partial \eta} d\sigma\, ds + \int_{\varGamma} \alpha\left(x\right)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma. \tag{3.19}$$

Differentiating once more,

$$\Phi''(t) = 2\|\nabla v_t\|^2 + 2\langle \nabla v, \nabla v_{tt}\rangle + 2\int_\Gamma \alpha(x)\frac{\partial v}{\partial \eta}\frac{\partial v_t}{\partial \eta}d\sigma.$$

By the help of Green's Formula and the boundary conditions on the partial differential equation (3.11), we conclude the following from integrals second and third

$$\begin{aligned}\Phi''(t) &= 2\|\nabla v_t\|^2 + \langle v_{tt}, -2\Delta v\rangle + 2\int_\Gamma \alpha(x)\frac{\partial v}{\partial \eta}\frac{\partial v_t}{\partial \eta}d\sigma + 2\int_\Gamma \frac{\partial v}{\partial \eta} v_{tt}\, d\sigma \\ &= 2\|\nabla v_t\|^2 + \langle v_{tt}, -2\Delta v\rangle + \int_\Gamma \frac{\partial v}{\partial \eta}\left(2v_{tt} + 2\Delta^2 v\right) d\sigma \\ &= 2\|\nabla v_t\|^2 + \underbrace{\langle v_{tt}, -2\Delta v\rangle}_{(*)}.\end{aligned}$$

since

$$v_{tt} + \Delta^2 v = \Delta v + bf(-\Delta v)\Big|_{\partial\Omega} = v_{tt} - 0 + \alpha(x)\frac{\partial v_t}{\partial \eta} = 0$$

implies that

$$v_{tt} = -\alpha(x)\frac{\partial v_t}{\partial \eta} \;\Rightarrow\; 2v_{tt} = -2\Delta^2 v.$$

By substituting v_{tt} as in the equation (3.11) and using the inequality (3.14) we get,

$$\begin{aligned}(*) = \langle v_{tt}, -2\Delta v\rangle &= -2\|\Delta v\|^2 + 2\left\langle \Delta^2 v, \Delta v\right\rangle + 2b\langle f(-\Delta v), -\Delta v\rangle \\ &= -2\|\Delta v\|^2 - 2\|\nabla\Delta v\|^2 + 2b\langle f(-\Delta v), -\Delta v\rangle.\end{aligned}$$

Therefore,

$$\Phi''(t) \geq 2\|\nabla v_t\|^2 - 2\|\Delta v\|^2 - 2\|\nabla\Delta v\|^2 + 4b(2\gamma+1)\langle F(-\Delta v), 1\rangle$$

$$= -2(2\gamma+1)E(t) + 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\Delta v\|^2 + 4\gamma\|\nabla\Delta v\|^2. \tag{3.20}$$

Thanks to (3.17) we have,

$$E(t) = E(0) - 2\int_0^t \int_\Gamma \alpha(x)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma\, ds \tag{3.21}$$

Thus, we obtain from the inequality (3.20) that

$$\Phi''(t) \geq -2(2\gamma+1)E(0) + 4(2\gamma+1)\int_0^t \int_\Gamma \alpha(x)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma\, ds$$

$$+ 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\Delta v\|^2 + 4\gamma\|\nabla\Delta v\|^2$$

$$\geq 4(\gamma+1)\left[\|\nabla v_t\|^2 + \int_0^t \int_\Gamma \alpha(x)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma\, ds + \frac{1}{2}\int_\Gamma \alpha(x)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma\right] - d_0$$

where

$$d_0 := 2(2\gamma+1)E(0) + 2(\gamma+1)\int_\Gamma \alpha(x)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma.$$

Theorem 4 *For any solution* $v(x,t)$ *problem* $(3.11) - (3.13)$ *we obtain the following inequality*

$$\Phi''(t)\Phi(t) - (\gamma+1)[\Phi'(t)]^2 \geq -d_0\Phi(t).$$

• **Proof**: Multiplying both sides of

$$\Phi''(t) \geq 4(\gamma+1)\underbrace{\left[\|\nabla v_t\|^2 + \int_0^t \int_\Gamma \alpha(x)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma\, ds + \frac{1}{2}\int_\Gamma \alpha(x)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma\right]}_{\text{A}} - d_0$$

by $\Phi(t)$ we attain,

$$\Phi''(t)\Phi(t) \geq 4(1+\gamma)A\Phi(t) - d_0\Phi(t). \tag{3.22}$$

From (3.19) we have,

$$(1+\gamma)\left[\Phi'(t)\right]^2 =$$

$$4(1+\gamma)\left[\langle \nabla v, \nabla v_t\rangle + \int_0^t \int_\Gamma \alpha(x)\frac{\partial v}{\partial \eta}\frac{\partial v_t}{\partial \eta} d\sigma\, ds + \frac{1}{2}\int_\Gamma \alpha(x)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma\right]^2. \tag{3.23}$$

By Schwartz's inequality

$$\int_\Omega \nabla v \nabla v_t \leq \left(\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v_t|^2\right)^{\frac{1}{2}}$$

and

$$\int_0^t \int_\Gamma \alpha(x)\frac{\partial v}{\partial \eta}\frac{\partial v_t}{\partial \eta} d\sigma\, ds \leq \left\{\int_0^t \left[\int_\Gamma \alpha(x)\left(\frac{\partial v}{\partial \eta}\right)^2 d\sigma\right] ds\right\}^{\frac{1}{2}}\left\{\int_0^t \left[\int_\Gamma \alpha(x)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma\right] ds\right\}^{\frac{1}{2}}$$

Let us write into (3.23)

$$(1+\gamma)\left[\Phi'(t)\right]^2 \leq 4(1+\gamma)\left[\|\nabla v\|\|\nabla v_t\| + \left\{\int_0^t \left[\int_\Gamma \alpha(x)\left(\frac{\partial v}{\partial \eta}\right)^2 d\sigma\right] ds\right\}^{\frac{1}{2}}\cdot\right.$$

$$\left.\left\{\int_0^t \left[\int_\Gamma \alpha(x)\left(\frac{\partial v_t}{\partial \eta}\right)^2 d\sigma\right] ds\right\}^{\frac{1}{2}} + \frac{1}{2}\int_\Gamma \alpha(x)\left(\frac{\partial v_0}{\partial \eta}\right)^2 d\sigma\right]^2 \tag{3.24}$$

Let us introduce the following notations:

$$X_1 := \|\nabla v\| \ , \quad X_2 := \left\{ \int_0^t \left[\int_\Gamma \alpha(x) \left(\frac{\partial v}{\partial \eta} \right)^2 d\sigma \right] ds \right\}^{\frac{1}{2}} \ ,$$

$$Y_1 := \|\nabla v_t\| \ , \quad Y_2 := \left\{ \int_0^t \left[\int_\Gamma \alpha(x) \left(\frac{\partial v_t}{\partial \eta} \right)^2 d\sigma \right] ds \right\}^{\frac{1}{2}} , \quad Z := \int_\Gamma \alpha(x) \left(\frac{\partial v_0}{\partial \eta} \right)^2 d\sigma.$$

Hence, from (3.24) we have,

$$4(1+\gamma) \left[X_1 \cdot Y_1 + X_2 \cdot Y_2 + \frac{Z}{2} \right]^2$$

$$= 4(1+\gamma) \left[\left(X_1^2 \cdot Y_1^2 + X_2^2 \cdot Y_2^2 + \frac{Z^2}{4} \right) + 2 \Big(X_1 \cdot Y_1 \cdot X_2 \cdot Y_2 + X_1 \cdot Y_1 \cdot \frac{Z}{2} + X_2 \cdot Y_2 \cdot \frac{Z}{2} \Big) \right].$$

By Cauchy's inequality

- $Z \cdot X_1 \cdot Y_1 \leq Z \cdot \left(\frac{X_1^2}{2} + \frac{Y_1^2}{2} \right)$
- $Z \cdot X_2 \cdot Y_2 \leq Z \cdot \left(\frac{X_2^2}{2} + \frac{Y_2^2}{2} \right)$

On the other hand,

$$4(1+\gamma) A\Phi(t) =$$

$$4(1+\gamma) \left[X_1^2 \cdot Y_1^2 + X_1^2 \cdot Y_2^2 + X_1^2 \cdot \frac{Z}{2} + X_2^2 \cdot Y_1^2 + X_2^2 \cdot Y_2^2 + X_2^2 \cdot \frac{Z}{2} + Z \cdot Y_1^2 + Z \cdot Y_2^2 + \frac{Z^2}{2} \right]$$

and we also have,

$$X_1^2 \cdot Y_2^2 + X_2^2 \cdot Y_1^2 = \Big(X_1 \cdot Y_2 - X_2 \cdot Y_1\Big)^2 + 2 \cdot X_1 \cdot Y_2 \cdot X_2 \cdot Y_1$$

so, we get

$$(1+\gamma)\left[\Phi'(t)\right]^2 \leq 4(1+\gamma)A\Phi(t). \tag{3.25}$$

As a result, by subtracting (3.25) from (3.22) we find,

$$\Phi''(t)\Phi(t) - (1+\gamma)\left[\Phi'(t)\right]^2 \geq -d_0\Phi(t)$$

as we desired.

3.1.2. CASE: 2

In this part, we shall study the initial and boundary value problem below as our second problem:

$$v_{tt} + \Delta^2 v = \Delta v + bf(-\Delta v), \qquad x \in \bar{\Omega},\ t \in (0,T), \tag{3.26}$$

$$\frac{\partial v}{\partial \eta} = 0, \qquad \frac{\partial \Delta v}{\partial \eta} = -\alpha(x)\,\Delta v_t, \qquad x \in \Gamma,\ t \in (0,T), \tag{3.27}$$

$$v(x,0) = v_0(x), \qquad \frac{\partial v}{\partial t}(x,0) = v_1(x), \qquad x \in \Omega. \tag{3.28}$$

To obtain the blow-up solution we will use the Lemma 2 for the solution $v(x,t)$ of $(3.26) - (3.28)$ as our second result.
Assume that (3.14) is satisfied.

First of all, we start by obtaining an estimate for the energy function $E(t)$ defined by

$$E(t) := \|\nabla v_t\|^2 + \|\Delta v\|^2 + \|\nabla \Delta v\|^2 - 2b\,\langle F(-\Delta v), 1\rangle. \tag{3.29}$$

Multiplying the equation (3.26) in $L^2(\Omega)$ by $-2\Delta v_t$ gives us the equality:

$$\underbrace{-2\int_\Omega v_{tt}\Delta v_t dx}_{\text{I}} + \underbrace{2\int_\Omega \Delta v \Delta v_t\, dx}_{\text{II}} \underbrace{-2\int_\Omega \Delta^2 v \Delta v_t\, dx}_{\text{III}} = \underbrace{-2b\int_\Omega f(-\Delta v)\,\Delta v_t\, dx}_{\text{IV}} \tag{3.30}$$

By using Green's Formula and the boundary conditions, we get

$$\mathrm{I} = -2\int_{\Omega} v_{tt}\Delta v_t\,dx$$

$$= 2\int_{\Omega} \nabla v_{tt}\nabla v_t\,dx - 2\int_{\Gamma}\frac{\partial v_t}{\partial\eta}v_{tt}\,d\sigma$$

$$= \frac{d}{dt}\int_{\Omega}|\nabla v_t|^2\,dx - 2\int_{\Gamma}\frac{\partial v_t}{\partial\eta}v_{tt}\,d\sigma$$

$$\mathrm{II} = 2\int_{\Omega}\Delta v\Delta v_t\,dx$$

$$= \frac{d}{dt}\int_{\Omega}|\Delta v|^2\,dx$$

$$\mathrm{III} = -2\int_{\Omega}\Delta v_t\Delta^2 v\,dx$$

$$= 2\int_{\Omega}\nabla(\Delta v_t)\nabla(\Delta v)\,dx - 2\int_{\Gamma}\frac{\partial\Delta v}{\partial\eta}\Delta v_t\,d\sigma$$

$$= \frac{d}{dt}\int_{\Omega}|\nabla\Delta v|^2\,dx - 2\int_{\Gamma}\frac{\partial\Delta v}{\partial\eta}\Delta v_t\,d\sigma$$

$$\mathrm{IV} = -2b\int_{\Omega} f(-\Delta v)\,\Delta v_t\,dx$$

$$= \frac{d}{dt}\,2b\int_{\Omega} F(-\Delta v)\,dx.$$

Plugging them into (3.30) we have,

$$\frac{d}{dt}\left[\|\nabla v_t\|^2 + \|\Delta v\|^2 + \|\nabla \Delta v\|^2 - 2b\left\langle F\left(-\Delta v\right), 1\right\rangle\right]$$

$$= 2\int_{\Gamma} \underbrace{\frac{\partial v_t}{\partial \eta}}_{*} v_{tt}\, d\sigma + 2\int_{\Gamma} \underbrace{\frac{\partial \Delta v}{\partial \eta}}_{**} \Delta v_t\, d\sigma.$$

Restricting the differential equation to $\partial\Omega$ we obtain,

$$\frac{\partial v}{\partial \eta} = 0, \qquad \frac{\partial \Delta v}{\partial \eta} + \alpha\left(x\right)\Delta v_t = 0$$

$$2\int_{\Gamma} \underbrace{\frac{\partial v_t}{\partial \eta}}_{*} v_{tt}\, d\sigma = 0 \quad and \quad \underbrace{\frac{\partial \Delta v}{\partial \eta}}_{**} = -\alpha\left(x\right)\Delta v_t$$

we have,

$$\frac{d}{dt}E\left(t\right) = 2\int_{\Gamma} \frac{\partial v_t}{\partial \eta} v_{tt}\, d\sigma + 2\int_{\Gamma} \frac{\partial\left(\Delta v\right)}{\partial \eta}\Delta v_t\, d\sigma$$

$$= -2\int_{\Gamma} \alpha\left(x\right)\left(\Delta v_t\right)\Delta v_t\, d\sigma$$

$$= -2\int_{\Gamma} \alpha\left(x\right)\left(\Delta v_t\right)^2 d\sigma$$

so, we get the following equality

$$\frac{d}{dt}E\left(t\right) = -2\int_{\Gamma} \alpha\left(x\right)\left(\Delta v_t\right)^2 d\sigma. \tag{3.31}$$

It is obvious from (3.31) that $E(t) \leq E(0)$ for all $t \geq 0$.

Define the following function

$$\Phi(t) = \|\nabla v\|^2 + \int_0^t \int_\Gamma \alpha(x)(\Delta v)^2 \, d\sigma \, ds + \int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma. \tag{3.32}$$

Differentiating the function Φ defined in the equation (3.32) for t, we obtain

$$\Phi'(t) = 2\langle \nabla v, \nabla v_t \rangle + 2\int_0^t \int_\Gamma \alpha(x)\, \Delta v \Delta v_t d\sigma \, ds + \int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma. \tag{3.33}$$

Differentiating once more with respect to t gives

$$\Phi''(t) = 2\|\nabla v_t\|^2 + 2\langle \nabla v, \nabla v_{tt} \rangle + 2\int_\Gamma \alpha(x)\, \Delta v \Delta v_t \, d\sigma.$$

By the help of Green's Formula and the boundary conditions on the partial differential equation (3.26), we have

$$\Phi''(t) = 2\|\nabla v_t\|^2 \, dx + \underbrace{2\int_\Omega \nabla v \nabla v_{tt} \, dx}_{*} + 2\int_\Gamma \alpha(x)\, \Delta v \Delta v_t \, d\sigma.$$

By Green's Formula

$$\int_\Omega u \Delta v \; dx = -\int_\Omega \nabla u \nabla v \; dx + \int_\Gamma \frac{\partial v}{\partial \eta} u \, d\sigma$$

$$\int_\Omega \nabla u \nabla v \; dx = -\int_\Omega u \Delta v \; dx + \int_\Gamma \frac{\partial v}{\partial \eta} u \, d\sigma$$

$(*)$ turns out to be

$$2\int_\Omega \nabla v \nabla v_{tt} \; dx = -2\int_\Omega v_{tt} \Delta v \; dx + 2\int_\Gamma \frac{\partial v}{\partial \eta} v_{tt} \, d\sigma$$

$$\Phi''(t) = 2\|\nabla v_t\|^2 + \langle v_{tt}, -2\Delta v\rangle + 2\int_\Gamma \underbrace{\frac{\partial v}{\partial \eta}}_{=0} v_{tt}\, d\sigma + 2\int_\Gamma \alpha(x)\, \Delta v \Delta v_t\, d\sigma$$

$$= 2\|\nabla v_t\|^2 + \underbrace{\langle v_{tt}, -2\Delta v\rangle}_{\mathrm{I}} + 2\int_\Gamma \alpha(x)\, \Delta v \Delta v_t\, d\sigma.$$

By substituting v_{tt} as in the equation (3.26) we obtain,

$$\mathrm{I} = \langle v_{tt}, -2\Delta v\rangle = -2\|\Delta v\|^2 + \underbrace{2\left\langle \Delta^2 v, \Delta v\right\rangle}_{\mathrm{II}} + 2b\langle f(-\Delta v), -\Delta v\rangle$$

Since

$$\mathrm{II} = 2\int_\Omega \Delta^2 v \Delta v\, dx$$

$$= -2\int_\Omega \nabla(\Delta v)\nabla(\Delta v)\; dx + 2\int_\Gamma \frac{\partial(\Delta v)}{\partial \eta}\Delta v\, d\sigma$$

$$= -2\int_\Omega |\nabla \Delta v|^2\; dx + 2\int_\Gamma \underbrace{\frac{\partial(\Delta v)}{\partial \eta}}_{=-\alpha(x)\Delta v_t} \Delta v\, d\sigma$$

$$= -2\int_\Omega |\nabla \Delta v|^2\; dx - 2\int_\Gamma \alpha(x)\, \Delta v_t \Delta v\, d\sigma.$$

we have,

$$\mathrm{I} = -2\|\Delta v\|^2 - 2\|\nabla \Delta v\|^2 + 2b\langle f(-\Delta v), -\Delta v\rangle.$$

thus, we get

$$\Phi''(t) = 2\|\nabla v_t\|^2 - 2\|\Delta v\|^2 - 2\|\nabla \Delta v\|^2 + 2b\langle f(-\Delta v), -\Delta v\rangle.$$

By using the inequality (3.14) we obtain,

$$\Phi''(t) \geq 2\|\nabla v_t\|^2 - 2\|\Delta v\|^2 - 2\|\nabla\Delta v\|^2 + 4b(2\gamma+1)\langle F(-\Delta v), 1\rangle$$

$$= -2(2\gamma+1)E(t) + 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\Delta v\|^2 + 4\gamma\|\nabla\Delta v\|^2. \quad (3.34)$$

Thanks to (3.31) we attain,

$$E(t) = E(0) - 2\int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds \quad (3.35)$$

Thus, we obtain from the inequality (3.34) that

$$\Phi''(t) \geq -2(2\gamma+1)E(0) + 4(2\gamma+1)\int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds$$

$$+ 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\Delta v\|^2 + 4\gamma\|\nabla\Delta v\|^2$$

$$\geq 4(\gamma+1)\left[\|\nabla v_t\|^2 + \int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds + \frac{1}{2}\int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma\right] - d_0$$

where

$$d_0 := 2(2\gamma+1)E(0) + 2(\gamma+1)\int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma.$$

Theorem 5 *Under the assumptions on the parameter of second problem we have,*

$$\Phi''(t)\Phi(t) - (\gamma+1)[\Phi'(t)]^2 \geq -d_0\Phi(t).$$

• **Proof**: Multiplying both sides of

$$\Phi''(t) \geq 4(\gamma+1)\underbrace{\left[\|\nabla v_t\|^2 + \int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds + \frac{1}{2}\int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma\right]}_{\mathrm{B}} - d_0$$

by $\Phi(t)$ we get,

$$\Phi''(t)\Phi(t) \geq 4(1+\gamma) B\Phi(t) - d_0\Phi(t). \tag{3.36}$$

From (3.33) we have,

$$(1+\gamma)[\Phi'(t)]^2 =$$

$$4(1+\gamma)\left[\langle\nabla v, \nabla v_t\rangle + \int_0^t \int_\Gamma \alpha(x)\Delta v \Delta v_t \, d\sigma \, ds + \frac{1}{2}\int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma\right]^2. \tag{3.37}$$

By Schwartz's inequality

$$\int_\Omega \nabla v \nabla v_t \leq \left(\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v_t|^2\right)^{\frac{1}{2}}$$

and

$$\int_0^t \int_\Gamma \alpha(x)\Delta v \Delta v_t d\sigma \, ds$$

$$\leq \left\{\int_0^t \left[\int_\Gamma \alpha(x)(\Delta v)^2 \, d\sigma\right] ds\right\}^{\frac{1}{2}} \left\{\int_0^t \left[\int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma\right] ds\right\}^{\frac{1}{2}}$$

Let us write into (3.37) and we make similar calculations on inequality like the first problem, we obtain

$$(1+\gamma)\left[\Phi'(t)\right]^2 \leq 4(1+\gamma)B\Phi(t). \tag{3.38}$$

As a result, by subtracting (3.38) from (3.36) we get,

$$\Phi''(t)\Phi(t) - (1+\gamma)\left[\Phi'(t)\right]^2 \geq -d_0\Phi(t)$$

as we desired.

3.1.3. CASE: 3

In this case, we consider the below problem:

$$v_{tt} + \Delta^2 v = \Delta v + bf\left(-\Delta v\right), \qquad x \in \bar{\Omega},\ t \in (0,T)\ , \tag{3.39}$$

$$\frac{\partial \Delta v}{\partial \eta} = 0, \qquad \frac{\partial v}{\partial \eta} = -\alpha\left(x\right) v_t, \qquad x \in \Gamma,\ t \in (0,T)\ , \tag{3.40}$$

$$v\left(x,0\right) = v_0\left(x\right), \qquad \frac{\partial v}{\partial t}\left(x,0\right) = v_1\left(x\right), \qquad x \in \Omega. \tag{3.41}$$

To prove the existence of blow-up solution we apply the Lemma 2 for the solution $v\left(x,t\right)$ of $(3.39) - (3.41)$ as our third result.
Assume that (3.14) is satisfied.

To begin with, we start by obtaining an estimate for the energy function $E\left(t\right)$ defined by

$$E\left(t\right) := \left\|\nabla v_t\right\|^2 + \left\|\Delta v\right\|^2 + \left\|\nabla \Delta v\right\|^2 - 2b\left\langle F\left(-\Delta v\right), 1\right\rangle. \tag{3.42}$$

Multiplying the equation (3.39) in $L^2\left(\Omega\right)$ by $-2\Delta v_t$ gives us the equality:

$$\underbrace{-2\int_{\Omega} v_{tt}\Delta v_t\,dx}_{\text{I}} + \underbrace{2\int_{\Omega} \Delta v \Delta v_t\,dx}_{\text{II}} \underbrace{-2\int_{\Omega} \Delta^2 v \Delta v_t\,dx}_{\text{III}} = \underbrace{-2b\int_{\Omega} f\left(-\Delta v\right)\Delta v_t\,dx}_{\text{IV}} \tag{3.43}$$

By using Green's Formula and the boundary conditions we find,

$$\text{I} = -2\int_{\Omega} v_{tt}\Delta v_t \, dx$$

$$= 2\int_{\Omega} \nabla v_{tt}\nabla v_t \, dx - 2\int_{\Gamma} \frac{\partial v_t}{\partial \eta} v_{tt} \, d\sigma$$

$$= \frac{d}{dt}\int_{\Omega} |\nabla v_t|^2 \, dx - 2\int_{\Gamma} \frac{\partial v_t}{\partial \eta} v_{tt} \, d\sigma$$

$$\text{II} = 2\int_{\Omega} \Delta v \Delta v_t \, dx$$

$$= \frac{d}{dt}\int_{\Omega} |\Delta v|^2 \, dx$$

$$\text{III} = -2\int_{\Omega} \Delta v_t \Delta^2 v \, dx$$

$$= 2\int_{\Omega} \nabla\left(\Delta v_t\right)\nabla\left(\Delta v\right) \, dx - 2\int_{\Gamma} \frac{\partial \Delta v}{\partial \eta} \Delta v_t \, d\sigma$$

$$= \frac{d}{dt}\int_{\Omega} |\nabla \Delta v|^2 \, dx - 2\int_{\Gamma} \underbrace{\frac{\partial \Delta v}{\partial \eta}}_{=0} \Delta v_t \, d\sigma$$

$$= \frac{d}{dt}\int_{\Omega} |\nabla \Delta v|^2 \, dx$$

$$\text{IV} = -2b\int_{\Omega} f\left(-\Delta v\right)\Delta v_t \, dx$$

$$= \frac{d}{dt}\, 2b\int_{\Omega} F\left(-\Delta v\right) \, dx.$$

Plugging them into (3.43) we have,

$$\frac{d}{dt}\left[\|\nabla v_t\|^2+\|\Delta v\|^2+\|\nabla\Delta v\|^2-2b\left\langle F\left(-\Delta v\right),1\right\rangle\right]=2\int_{\Gamma}\underbrace{\frac{\partial v_t}{\partial\eta}}_{*}v_{tt}\,d\sigma.$$

Since

$$\frac{\partial v}{\partial\eta}+\alpha\left(x\right)v_t=0\quad\Rightarrow\quad\frac{d}{dt}\left(\frac{\partial v}{\partial\eta}+\alpha\left(x\right)v_t\right)=0\quad\Rightarrow\quad(*)\quad\frac{\partial v_t}{\partial\eta}=-\alpha\left(x\right)v_{tt}$$

we get,

$$\frac{d}{dt}\left[\|\nabla v_t\|^2+\|\Delta v\|^2+\|\nabla\Delta v\|^2-2b\left\langle F\left(-\Delta v\right),1\right\rangle\right]=-2\int_{\Gamma}\alpha\left(x\right)\left(v_{tt}\right)^2d\sigma.$$

$$\frac{d}{dt}E\left(t\right)=-2\int_{\Gamma}\alpha\left(x\right)\left(v_{tt}\right)^2d\sigma. \tag{3.44}$$

It is obvious from (3.44) that $E(t)\leq E(0)$ for all $t\geq 0$.

Define the following function

$$\Phi\left(t\right)=\|\nabla v\|^2+\int_0^t\int_{\Gamma}\alpha\left(x\right)v_t^2\,d\sigma\,ds+\int_{\Gamma}\alpha\left(x\right)v_1^2\,d\sigma. \tag{3.45}$$

Differentiating the function Φ defined in the equation (3.45) for t, we obtain

$$\Phi'\left(t\right)=2\left\langle\nabla v,\nabla v_t\right\rangle+2\int_0^t\int_{\Gamma}\alpha\left(x\right)v_tv_{tt}\,d\sigma\,ds+\int_{\Gamma}\alpha\left(x\right)v_1^2\,d\sigma. \tag{3.46}$$

Differentiating once more,

$$\Phi''\left(t\right)=2\|\nabla v_t\|^2+2\left\langle\nabla v,\nabla v_{tt}\right\rangle+2\int_{\Gamma}\alpha\left(x\right)v_tv_{tt}\,d\sigma.$$

By the help of Green's Formula and the boundary conditions on the partial differential equation (3.39) we have,

$$2\int_{\Omega}\nabla v\nabla v_{tt}\,dx=-2\int_{\Omega}v_{tt}\Delta v\,dx+2\int_{\Gamma}\frac{\partial v}{\partial\eta}v_{tt}\,d\sigma$$

$$\Phi''(t)=2\left\|\nabla v_t\right\|^2-2\int_{\Omega}v_{tt}\Delta v+\underbrace{2\int_{\Gamma}\frac{\partial v}{\partial\eta}v_{tt}\,d\sigma+2\int_{\Gamma}\alpha(x)\,v_t v_{tt}\,d\sigma}_{\text{A}}.$$

$$\text{A}=2\int_{\Gamma}v_{tt}\left(\underbrace{\frac{\partial v}{\partial\eta}}_{-\alpha(x)v_t}+\alpha(x)\,v_t\right)d\sigma=0.$$

Therefore,

$$\Phi''(t)=2\left\|\nabla v_t\right\|^2-2\int_{\Omega}v_{tt}\Delta v\,dx.$$

By substituting v_{tt} as in the equation (3.39) we obtain,

$$\Phi''(t)=2\left\|\nabla v_t\right\|^2-2\int_{\Omega}\left(\Delta v-\Delta^2 v+bf(-\Delta v)\right)\Delta v\,dx$$

$$=2\left\|\nabla v_t\right\|^2-2\left\|\Delta v\right\|^2+\underbrace{2\left\langle\Delta^2 v,\Delta v\right\rangle}_{\text{II}}-2b\int_{\Omega}f(-\Delta v)\,\Delta v\,dx.$$

Since

$$\text{II}=2\int_{\Omega}\Delta^2 v\Delta v\,dx=-2\int_{\Omega}\nabla(\Delta v)\,\nabla(\Delta v)\,dx+2\int_{\Gamma}\underbrace{\frac{\partial(\Delta v)}{\partial\eta}}_{=0}\Delta v\,d\sigma=-2\int_{\Omega}\left|\nabla\Delta v\right|^2\,dx.$$

we have,

$$\varPhi''(t) = 2\|\nabla v_t\|^2 - 2\|\varDelta v\|^2 - 2\|\nabla\varDelta v\|^2 + 2b\langle f(-\varDelta v), -\varDelta v\rangle.$$

By using the inequality (3.14) we get,

$$\varPhi''(t) \geq 2\|\nabla v_t\|^2 - 2\|\varDelta v\|^2 - 2\|\nabla\varDelta v\|^2 + 4b(2\gamma+1)\langle F(-\varDelta v), 1\rangle$$

$$= -2(2\gamma+1)E(t) + 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\varDelta v\|^2 + 4\gamma\|\nabla\varDelta v\|^2. \tag{3.47}$$

Thanks to (3.44) we find,

$$E(t) = E(0) - 2\int_0^t\int_\varGamma \alpha(x)(v_{tt})^2\, d\sigma\, ds \tag{3.48}$$

Thus, we get from the inequality (3.47) that

$$\varPhi'' \geq -2(2\gamma+1)E(0) + 4(2\gamma+1)\int_0^t\int_\varGamma \alpha(x)(v_{tt})^2\, d\sigma\, ds$$

$$+ 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\varDelta v\|^2 + 4\gamma\|\nabla\varDelta v\|^2.$$

$$\varPhi'' \geq 4(\gamma+1)\left[\|\nabla v_t\|^2 + \int_0^t\int_\varGamma \alpha(x)(v_{tt})^2\, d\sigma\, ds + \frac{1}{2}\int_\varGamma \alpha(x)v_1^2\, d\sigma\right] - d_0$$

where

$$d_0 := 2(2\gamma+1)E(0) + 2(\gamma+1)\int_\varGamma \alpha(x)v_1^2\, d\sigma.$$

Theorem 6 *Under the assumption on the parameter of third problem we have,*

$$\varPhi''(t)\varPhi(t) - (\gamma+1)[\varPhi'(t)]^2 \geq -d_0\varPhi(t).$$

• **Proof**: Multiplying both sides of

$$\varPhi'' \geq 4\left(\gamma+1\right) \underbrace{\left[\left\|\nabla v_t\right\|^2 + \int_0^t \int_{\varGamma} \alpha\left(x\right)\left(v_{tt}\right)^2 d\sigma\, ds + \frac{1}{2}\int_{\varGamma} \alpha\left(x\right) v_1^2\, d\sigma\right]}_{\mathrm{C}} - d_0$$

by $\varPhi\left(t\right)$ we attain,

$$\varPhi''\left(t\right)\varPhi\left(t\right) \geq 4\left(1+\gamma\right) C\varPhi\left(t\right) - d_0\varPhi\left(t\right). \tag{3.49}$$

From (3.46) we have,

$$\left(1+\gamma\right)\left[\varPhi'\left(t\right)\right]^2 =$$

$$4\left(1+\gamma\right)\left[\left\langle \nabla v, \nabla v_t\right\rangle + \int_0^t \int_{\varGamma} \alpha\left(x\right) v_t v_{tt}\, d\sigma\, ds + \frac{1}{2}\int_{\varGamma} \alpha\left(x\right) v_1^2\, d\sigma\right]^2 \tag{3.50}$$

By Schwartz's inequality

$$\int_{\varOmega} \nabla v \nabla v_t \leq \left(\int_{\varOmega} \left|\nabla v\right|^2\right)^{\frac{1}{2}} \left(\int_{\varOmega} \left|\nabla v_t\right|^2\right)^{\frac{1}{2}}$$

and

$$\int_0^t \int_{\varGamma} \alpha\left(x\right) v_t v_{tt}\, d\sigma\, ds$$

$$\leq \left\{\int_0^t \left[\int_{\varGamma} \alpha\left(x\right)\left(v_t\right)^2 d\sigma\right] ds\right\}^{\frac{1}{2}} \left\{\int_0^t \left[\int_{\varGamma} \alpha\left(x\right)\left(v_{tt}\right)^2 d\sigma\right] ds\right\}^{\frac{1}{2}}$$

Let us write into (3.50) and we make similar calculations on inequality like the first problem, we get

$$(1+\gamma)\left[\Phi'(t)\right]^2 \leq 4(1+\gamma)\,C\Phi(t)\,. \tag{3.51}$$

As a result, by subtracting (3.51) from (3.49) we find,

$$\Phi''(t)\,\Phi(t)-(1+\gamma)\left[\Phi'(t)\right]^2 \geq -d_0\Phi(t)$$

as we desired.

3.1.4. CASE: 4

Finally, we consider the following initial and boundary value problem:

$$v_{tt} + \varDelta^2 v = \varDelta v + bf\left(-\varDelta v\right), \qquad x \in \bar{\varOmega},\ t \in (0,T), \tag{3.52}$$

$$v = 0, \qquad \frac{\partial \varDelta v}{\partial \eta} = -\alpha\left(x\right)\varDelta v_t, \qquad x \in \varGamma,\ t \in (0,T), \tag{3.53}$$

$$v\left(x,0\right) = v_0\left(x\right), \qquad \frac{\partial v}{\partial t}\left(x,0\right) = v_1\left(x\right), \qquad x \in \varOmega. \tag{3.54}$$

To obtain the blow-up solution we use the Lemma 2 for the solution $v\left(x,t\right)$ of $(3.52)-(3.54)$ as our final result.
Assume that (3.14) is satisfied.

We start by obtaining an estimate for the energy function $E\left(t\right)$ defined by

$$E\left(t\right) := \left\|\nabla v_t\right\|^2 + \left\|\varDelta v\right\|^2 + \left\|\nabla \varDelta v\right\|^2 - 2b\left\langle F\left(-\varDelta v\right), 1\right\rangle. \tag{3.55}$$

Multiplying the equation (3.52) in $L^2\left(\varOmega\right)$ by $-2\varDelta v_t$ gives us the equality:

$$\underbrace{-2\int_{\varOmega} v_{tt}\varDelta v_t\,dx}_{\text{I}} + \underbrace{2\int_{\varOmega} \varDelta v\varDelta v_t\,dx}_{\text{II}} \underbrace{-2\int_{\varOmega} \varDelta^2 v\varDelta v_t\,dx}_{\text{III}} = \underbrace{-2b\int_{\varOmega} f\left(-\varDelta v\right)\varDelta v_t\,dx}_{\text{IV}} \tag{3.56}$$

By using Green's Formula and the boundary conditions we get,

$$\begin{aligned}
\mathrm{I} &= -2\int_{\Omega} v_{tt}\Delta v_t\,dx \\
&= 2\int_{\Omega} \nabla v_{tt}\nabla v_t\,dx - 2\int_{\Gamma}\frac{\partial v_t}{\partial \eta}v_{tt}\,d\sigma \\
&= \frac{d}{dt}\int_{\Omega}\left|\nabla v_t\right|^2\,dx - 2\int_{\Gamma}\frac{\partial v_t}{\partial \nu}v_{tt}\,d\sigma
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &= 2\int_{\Omega}\Delta v\Delta v_t dx \\
&= \frac{d}{dt}\int_{\Omega}\left|\Delta v\right|^2\,dx
\end{aligned}$$

$$\begin{aligned}
\mathrm{III} &= -2\int_{\Omega}\Delta^2 v\Delta v_t\,dx \\
&= 2\int_{\Omega}\nabla\left(\Delta v_t\right)\nabla\left(\Delta v\right)\,dx - 2\int_{\Gamma}\underbrace{\frac{\partial\left(\Delta v\right)}{\partial\eta}}_{-\alpha(x)\Delta v_t}\Delta v_t\,d\sigma \\
&= \frac{d}{dt}\int_{\Omega}\left|\nabla\Delta v\right|^2\,dx + 2\int_{\Gamma}\alpha\left(x\right)\left(\Delta v_t\right)^2\,d\sigma
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV} &= -2b\int_{\Omega} f\left(-\Delta v\right)\Delta v_t\,dx \\
&= \frac{d}{dt}2b\int_{\Omega}F\left(-\Delta v\right)\,dx.
\end{aligned}$$

Plugging them into (3.56) we have,

$$\frac{d}{dt}\left[\|\nabla v_t\|^2+\|\Delta v\|^2+\|\nabla\Delta v\|^2-2b\left\langle F\left(-\Delta v\right),1\right\rangle\right]=2\int_{\Gamma}\underbrace{\frac{\partial v_t}{\partial\eta}}_{=0}v_{tt}\,d\sigma-2\int_{\Gamma}\alpha\left(x\right)\left(\Delta v_t\right)^2d\sigma.$$

$$\frac{d}{dt}E\left(t\right)=-2\int_{\Gamma}\alpha\left(x\right)\left(\Delta v_t\right)^2d\sigma \tag{3.57}$$

It is obvious from (3.57) that $E(t)\leq E(0)$ for all $t\geq 0$.

Define the following function

$$\varPhi\left(t\right)=\|\nabla v\|^2+\int_0^t\int_{\Gamma}\alpha\left(x\right)\left(\Delta v\right)^2d\sigma\,ds+\int_{\Gamma}\alpha\left(x\right)\left(\Delta v_0\right)^2d\sigma. \tag{3.58}$$

Differentiating the function $\varPhi$ defined in the equation (3.58) for t, we obtain

$$\varPhi'\left(t\right)=2\left\langle\nabla v,\nabla v_t\right\rangle+2\int_0^t\int_{\Gamma}\alpha\left(x\right)\Delta v\Delta v_t\,d\sigma\,ds+\int_{\Gamma}\alpha\left(x\right)\left(\Delta v_0\right)^2\,d\sigma. \tag{3.59}$$

Differentiating once more with respect to t gives

$$\varPhi''\left(t\right)=2\left\|\nabla v_t\right\|^2+2\left\langle\nabla v,\nabla v_{tt}\right\rangle+2\int_{\Gamma}\alpha\left(x\right)\Delta v\Delta v_t\,d\sigma.$$

By the help of Green's Formula and the boundary conditions on the partial differential equation (3.52), we conclude the following from second and third integrals:
By Green's Formula:

$$\int_{\Omega}u\Delta v\,dx=-\int_{\Omega}\nabla u\nabla v\,dx+\int_{\Gamma}\frac{\partial v}{\partial\eta}u\,d\sigma$$

$$\int_{\Omega}\nabla u\nabla v\,dx=-\int_{\Omega}u\Delta v\,dx+\int_{\Gamma}\frac{\partial v}{\partial\eta}u\,d\sigma$$

we have,

$$2\int_{\Omega}\nabla v\nabla v_{tt}\,dx=-2\int_{\Omega}v_{tt}\Delta v\,dx+2\int_{\Gamma}\frac{\partial v}{\partial\eta}v_{tt}\,d\sigma$$

Therefore, we get the following equality

$$\Phi''(t)=2\left\|\nabla v_t\right\|^2-2\int_{\Omega}v_{tt}\Delta v\,dx+2\int_{\Gamma}\underbrace{\frac{\partial v}{\partial\eta}}_{=0}v_{tt}\,d\sigma+2\int_{\Gamma}\alpha(x)\,\Delta v\Delta v_t\,d\sigma.$$

By substituting v_{tt} as in the equation (3.52) and using the inequality (3.14), we obtain

$$\begin{aligned}\Phi''(t)&=2\left\|\nabla v_t\right\|^2-2\int_{\Omega}\left(\Delta v-\Delta^2v+bf(-\Delta v)\right)\Delta v\,dx+2\int_{\Gamma}\alpha(x)\,\Delta v\Delta v_t\,d\sigma\\&=2\left\|\nabla v_t\right\|^2-2\int_{\Omega}\left|\Delta v\right|^2\,dx+\underbrace{2\int_{\Omega}\Delta^2v\Delta v\,dx}_{*}+2b\left\langle f(-\Delta v),-\Delta v\right\rangle+2\int_{\Gamma}\alpha(x)\,\Delta v\Delta v_t\,d\sigma\\&=2\left\|\nabla v_t\right\|^2-2\left\|\Delta v\right\|^2-2\left\|\nabla\Delta v\right\|^2+2b\left\langle f(-\Delta v),-\Delta v\right\rangle\\&\quad\underbrace{+2\int_{\Gamma}\frac{\partial\Delta v}{\partial\eta}\Delta v\,d\sigma+2\int_{\Gamma}\alpha(x)\,\Delta v\Delta v_t\,d\sigma}_{=0}.\end{aligned}$$

where

$$(*)\qquad 2\int_{\Omega}\Delta^2v\Delta v\,dx=-2\int_{\Omega}\nabla(\Delta v)\,\nabla(\Delta v)\,dx+2\int_{\Gamma}\frac{\partial\Delta v}{\partial\eta}\Delta v\,d\sigma$$

$$\begin{aligned}\Phi''(t)&\geq2\left\|\nabla v_t\right\|^2-2\left\|\Delta v\right\|^2-2\left\|\nabla\Delta v\right\|^2+4b(2\gamma+1)\left\langle F(-\Delta v),1\right\rangle\\&=-2(2\gamma+1)\,E(t)+4(\gamma+1)\left\|\nabla v_t\right\|^2+4\gamma\left\|\Delta v\right\|^2+4\gamma\left\|\nabla\Delta v\right\|^2.\end{aligned}\qquad(3.60)$$

Thanks to (3.57) we obtain,

$$E(t) = E(0) - 2\int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds \tag{3.61}$$

Thus, we obtain from the inequality (3.60) that

$$\Phi'' \geq -2(2\gamma+1)E(0) + 4(2\gamma+1)\int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds$$

$$+ 4(\gamma+1)\|\nabla v_t\|^2 + 4\gamma\|\Delta v\|^2 + 4\gamma\|\nabla\Delta v\|^2$$

$$\geq 4(\gamma+1)\left[\|\nabla v_t\|^2 + \int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2 \, d\sigma \, ds + \frac{1}{2}\int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma\right] - d_0$$

where

$$d_0 := 2(2\gamma+1)E(0) + 2(\gamma+1)\int_\Gamma \alpha(x)(\Delta v_0)^2 \, d\sigma.$$

Theorem 7 *Under the assumptions on the parameter of finally problem we have,*

$$\Phi''(t)\Phi(t) - (\gamma+1)[\Phi'(t)]^2 \geq -d_0\Phi(t).$$

• **Proof**: Multiplying both sides of

$$\Phi'' \geq 4(\gamma+1)\underbrace{\left[\|\nabla v_t\|^2 + \int_0^t \int_\Gamma \alpha(x)(\Delta v_t)^2\, d\sigma\, ds + \frac{1}{2}\int_\Gamma \alpha(x)(\Delta v_0)^2\, d\sigma\right]}_{\text{D}} - d_0$$

by $\Phi(t)$ we attain,

$$\Phi''(t)\,\Phi(t) \geq 4(1+\gamma)\,D\Phi(t) - d_0\Phi(t)\,. \tag{3.62}$$

From (3.59) we have,

$$(1+\gamma)\left[\Phi'(t)\right]^2 =$$

$$4(1+\gamma)\left[\langle\nabla v, \nabla v_t\rangle + \int_0^t \int_\Gamma \alpha(x)\,\Delta v \Delta v_t\, d\sigma\, ds + \frac{1}{2}\int_\Gamma \alpha(x)(\Delta v_0)^2\; d\sigma\right]^2. \tag{3.63}$$

By Schwartz's inequality

$$\int_\Omega \nabla v \nabla v_t \leq \left(\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v_t|^2\right)^{\frac{1}{2}}$$

and

$$\int_0^t \int_\Gamma \alpha(x)\,\Delta v \Delta v_t\, d\sigma\, ds$$

$$\leq \left\{\int_0^t \left[\int_\Gamma \alpha(x)(\Delta v)^2\, d\sigma\right] ds\right\}^{\frac{1}{2}}\left\{\int_0^t \left[\int_\Gamma \alpha(x)(\Delta v_t)^2\, d\sigma\right] ds\right\}^{\frac{1}{2}}$$

Let us write into (3.63) and we make similar calculations on inequality like the first problem, we get

$$(1+\gamma)\left[\Phi'(t)\right]^2 \leq 4(1+\gamma)\, D\Phi(t)\,. \tag{3.64}$$

As a result, by subtracting (3.64) from (3.62) we find,

$$\Phi''(t)\,\Phi(t)-(1+\gamma)\left[\Phi'(t)\right]^2 \geq -d_0\Phi(t)$$

as we desired.

3.2. BLOW-UP SOLUTION OF A SECOND ORDER WAVE EQUATION WITH INITIAL AND BOUNDARY CONDITIONS

In this section, we consider the following initial and boundary value problem. Models of the following type comes from various areas of mathematical physics [28, 29, 30].

$$v_{tt} - (a(x)\, v_x)_x + bv_t = kv_{xx} + f(v)\,, \qquad x \in [0,1]\,,\ t \in (0,T)\,, \tag{3.65}$$

$$v(0,t) = v_x(0,t) = v(1,t) = v_x(1,t) = 0, \qquad t \in (0,T)\,, \tag{3.66}$$

$$v(x,0) = v_0(x)\,, \qquad \frac{\partial v}{\partial t}(x,0) = v_1(x)\,, \qquad x \in [0,1]\,. \tag{3.67}$$

where $a(x) \in C^1[0,1]$ and $a(x) > 0$, k and b nonnegative constant, $f \in C^n$.

To obtain the blow-up solution we will use the Lemma 2 for the solution $v(x,t)$ of (3.65)-(3.67).

Let the function $f(v)$ with its primitive $F(v) = \int_0^v f(\xi)\, d\xi$ satisfy the following inequality

$$vf(v) \geq 2(2\alpha+1)\, F(v)\,, \qquad \forall v \in \mathbb{R} \tag{3.68}$$

for some real number $\alpha > 0$.

As first step, we start by obtaining an estimate for the energy function.

Multiplying the equation (3.65) by $2v_t$ and integrating over $(0,1)$ gives us the energy equality:

$$\frac{d}{dt}\left[\|v_t\|_2^2 + k\,\|v_x\|_2^2 + \int_0^1 a(x)\, v_x^2\, dx - 2\int_0^1 F(v)\ dx\right] = -2b\,\|v_t\|_2^2\,. \tag{3.69}$$

where,

$$F(v) = \int_0^v f(\xi)\, d\xi$$

So, the energy equation of the initial boundary problem (3.65) is defined by

$$E(t) := \|v_t\|_2^2 + k\|v_x\|_2^2 + \int_0^1 a(x)\, v_x^2\, dx - 2\int_0^1 F(v)\ dx. \tag{3.70}$$

From the equation (3.69) we obtain,

$$\frac{d}{dt}E(t) = -2b\|v_t\|_2^2\,. \tag{3.71}$$

It is obvious from (3.71) that $E(t) \leq E(0)$ for all $t \geq 0$.

Thanks to (3.71) we find,

$$E(t) = E(0) - 2b\int_0^t \|v_t\|_2^2\,. \tag{3.72}$$

Define the following function

$$\psi(t) = \|v\|_2^2 + \gamma(t+\tau)^2 \tag{3.73}$$

where γ , $\tau > 0$.

Differentiating the equation (3.73) with respect to t gives us

$$\psi'(t) = 2\langle v_t, v\rangle + 2\gamma(t+\tau) \tag{3.74}$$

and

$$(\psi'(t))^2 \leq 4\psi\left(\|v_t\|_2^2 + \gamma\right). \tag{3.75}$$

Differentiating once more,

$$\psi''(t) = \langle v_t, v_t \rangle + 2\gamma + 2\langle v_{tt}, v \rangle = 2\|v_t\|_2^2 + 2\gamma + 2\langle v_{tt}, v \rangle . \tag{3.76}$$

By substituting v_{tt} as in the equation (3.65) and integrating over $[0,1]$, using integration by parts and boundary conditions when necessary, we get

$$v_{tt} = kv_{xx} + (a(x) v_x)_x - bv_t + f(v),$$

$$2\langle v_{tt}, v \rangle = 2k \int_0^1 v_{xx} v \, dx + 2 \int_0^1 (a(x) v_x)_x v \, dx - 2b \int_0^1 v_t v \, dx + 2 \int_0^1 f(v) v \, dx$$

$$= -2k \|v_x\|_2^2 - 2 \int_0^1 a(x) v_x^2 \, dx - 2b \int_0^1 v v_t \, dx + 2 \int_0^1 v f(v) \, dx.$$

we obtain,

$$\psi''(t) = 2\|v_t\|_2^2 + 2\gamma - 2k\|v_x\|_2^2 - 2\int_0^1 a(x) v_x^2 \, dx - 2b\int_0^1 v v_t \, dx + 2\int_0^1 v f(v) \, dx. \tag{3.77}$$

Now,

$$\psi(t)\psi''(t) - \left(1 + \frac{\eta}{4}\right)(\psi'(t))^2 \geq$$

$$\psi \underbrace{\left(\psi'' - (4+\eta)(\|v_t\|)^2 - (4+\eta)\gamma\right)}_{(*)} \geq -\psi\zeta \tag{3.78}$$

$$(*) = \psi \left[2\|v_t\|_2^2 + 2\gamma - 2k\|v_x\|_2^2 - 2\int_0^1 a(x)\, v_x^2 \, dx \right.$$

$$\left. - 2b\int_0^1 vv_t \, dx + 2\int_0^1 vf(v) \, dx - (4+\eta)\left(\|v_t\|_2^2 + \gamma\right) \right] \tag{3.79}$$

$$= -\psi \left[(2+\eta)\|v_t\|_2^2 + (2+\eta)\gamma + 2k\|v_x\|_2^2 \right.$$

$$\left. + 2\int_0^1 a(x)\, v_x^2 \, dx + 2b\int_0^1 vv_t \, dx - 2\int_0^1 vf(v) \, dx \right] \tag{3.80}$$

where

$$\zeta = (2+\eta)\|v_t\|_2^2 + (2+\eta)\gamma + 2k\|v_x\|_2^2$$

$$+ 2\int_0^1 a(x)\, v_x^2 \, dx + 2b\int_0^1 vv_t \, dx - 2\int_0^1 vf(v) \, dx. \tag{3.81}$$

By the help of Cauchy-Schwartz's inequality and Young's inequality, respectively we have

$$\int_0^1 vv_t \, dx \le \int_0^1 |v||v_t| \, dx \le \|v\|\|v_t\| \le \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|v_t\|_2^2,$$

$$2b\int_0^1 vv_t \, dx \le 2b\int_0^1 |v||v_t| \, dx \le 2b\left(\|v\|\|v_t\|\right) \le b\|v\|_2^2 + b\|v_t\|_2^2. \tag{3.82}$$

By Poincare's inequality

$$\int_0^1 |v|^2 \, dx \leq \frac{1}{\kappa} \int_0^1 |v_x|^2 \, dx$$

$$b\|v\|_2^2 \leq b\kappa^{-1}\|v_x\|_2^2 \tag{3.83}$$

we have,

$$-\psi\Big[\quad (2+b+\eta)\,\|v_t\|_2^2 + \left(2k + b\kappa^{-1}\right)\|v_x\|_2^2 + (2+\eta)\,\gamma$$

$$+ 2\int_0^1 a\,(x)\, v_x^2 \, dx - 4\,(1+\alpha) \int_0^1 F\,(v) \; dx \quad \Big] \geq -\psi\,\zeta \tag{3.84}$$

Thus, we get the following inequality

$$\zeta \leq \; (2+b+\eta)\,\|v_t\|_2^2 + \left(2k + b\kappa^{-1}\right)\|v_x\|_2^2 + (2+\eta)\,\gamma$$

$$+ 2\int_0^1 a\,(x)\, v_x^2 \, dx - 4\,(1+\alpha) \int_0^1 F\,(v) \; dx \tag{3.85}$$

which shows that,

$$\zeta \leq \ell E\,(t)\,. \tag{3.86}$$

Multiplying both sides of (3.86) by $\psi\,(t)$ we find,

$$\psi\,(t)\;\zeta \leq \psi\,(t)\;\ell\,E\,(t) \tag{3.87}$$

Now, multiplying both sides of (3.87) by negative one, we attain

$$-\psi\ (t)\ \zeta\ \geq\ -\psi\,(t)\ \ell\,E\,(t)\ \geq\ -\psi\,(t)\ \ell\,E\,(0) \tag{3.88}$$

Now let ,

$$\ell = \max\{(2+\eta+b)\,,\left(2k+b\kappa^{-1}\right),2,4\,(1+\alpha)\} \tag{3.89}$$

Hence, we get

$\zeta \geq \ell E\,(t)$ and $-\psi\ (t)\ \zeta\ \geq\ -\psi\,(t)\ \ell\,E\,(0)$

Thus, (3.78) turns out to be

$$\psi\,(t)\,\psi''\,(t) - \left(1+\frac{\eta}{4}\right)(\psi'\,(t))^2 + \ell E\,(0)\,\psi\,(t) \geq 0 \tag{3.90}$$

as we desired.

REFERENCES

1. Glassey RT. Blow-up theorems for nonlinear wave equations. *Mathematische Zeitschrift*. 1973;132(3):183-203.

2. Kalantarov VK, Ladyzhenskaya OA. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types. *Journal of Soviet Mathematics*. 1978;10(1):53-70.

3. Keller JB. On solutions of nonlinear wave equations. *Communications on Pure and Applied Mathematics*. 1957;10(4):523-30.

4. Rühs F, Lions JL. Equations Differentielles Operationnelles et Problemes aux Limites. IX+ 292 S. Berlin/Göttingen/Heidelberg 1961. Springer-Verlag. Preis geb. 64,-. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*. 1962;42(7-8):363-4.

5. Levine HA. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM Journal on Mathematical Analysis*. 1974;5(1):138-46.

6. Levine HA. A note on a nonexistence theorem for nonlinear wave equations. *SIAM Journal on Mathematical Analysis*. 1974;5(4):644-8.

7. Levine HA. Instability and nonexistence of global solutions to nonlinear wave equations of the form $u_{tt} = -Au + F(u)$. *Transactions of The American Mathematical Society*. 1974;192:1-21.

8. Samarskii AA, Galaktionov VA, Kurdyumov SP, Mikhailov AP. *Blow-up in Quasilinear Parabolic Equations*. Berlin: De Gruyter; 1995.

9. Hu B. *Blow-up theories for semilinear parabolic equations*. New York: Springer; 2011.

10. Mitidieri E, Pokhozhaev SI. A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. *Trudy Matematicheskogo Instituta Imeni VA Steklova*. 2001;234:3-83.

11. Bayrak V, Can M. Nonexistense of global solutions of a quasilinear bi-hyperbolic equation with dynamical boundary conditions. *Electronic Journal of Qualitative Theory of Differential Equations*. 1999;1999(3):1-10.

12. Lasiecka I. Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary. *Journal of Differential Equations*. 1989;79(2):340-81.

13. Maksudov FG, Aliev FA. On a problem for a nonlinear hyperbolic equation of higher order with dissipation on the boundary of the domain. *In Soviet Math*. Dokl 1992;(44): 771-774.

14. Kirane M, Kouachi S, Tatar N. Nonexistence of global solutions of some quasilinear hyperbolic equations with dynamic boundary conditions. *Mathematische Nachrichten*. 1995;176(1):139-47.

15. Ladyzhenskaya OA, Kalantarov V. Blow up theorems for quasilinear parabolic and hyperbolic equations. *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta Imeni V. A. Steklova.* 1977;69:77-102.

16. Knops RJ, Levine HA, Payne LE. Non-existence, instability, and growth theorems for solutions of a class of abstract nonlinear equations with applications to nonlinear elastodynamics. *Archive for Rational Mechanics and Analysis*. 1974;55(1):52-72.

17. Levine HA. The role of critical exponents in blowup theorems. *Siam Review*. 1990;32(2):262-88.

18. Straughan B. *Instability, nonexistence and weighted energy methods in fluid dynamics and related theories.* Boston: Pitman Advanced Pub. Program; 1982.

19. Can M, Park SR, Aliyev F. Nonexistence of global solutions of some quasilinear hyperbolic equations. *Journal of Mathematical Analysis and Applications*. 1997;213(2):540-53.

20. Bayrak V, Can M, Aliyev FA. Nonexistence of global solutions of a quasilinear hyperbolic equation. *Math. Inequalities and Appl*. 1998;1:45-52.

21. Straughan B. Further global nonexistence theorems for abstract nonlinear wave equations. *Proceedings of the American Mathematical Society*. 1975;48(2):381-90.

22. Sattinger DH. Stability of nonlinear hyperbolic equations. *Archive for Rational Mechanics and Analysis*. 1968;28(3):226-44.

23. Sattinger DH. On global solution of nonlinear hyperbolic equations. *Archive for Rational Mechanics and Analysis*. 1968;30(2):148-72.

24. Kato T. Blow-up of solutions of some nonlinear hyperbolic equations. *Communications on Pure and Applied Mathematics*. 1980;33(4):501-5.

25. Pokhozhaev SI. Investigation of hyperbolic systems of quasilinear equations by the method of continuation, *Proceedings of Moscow Energy Institute*. 1975;250:74-88.

26. Korpusov MO. Blow-up of the solution of a nonlinear system of equations with positive energy. *Theoretical and Mathematical Physics*. 2012;171(3):725-38.

27. Levine HA. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM Journal on Mathematical Analysis*. 1974;5(1):138-46.

28. Bayrak V, Can M. Global nonexistence of solutions of the quasilinear hyperbolic equation of the vibrations of a riser. *Math. and Comp. Appl*. 1998;2:45-52.

29. Hao J, Li S, Zhang Y. Blow up and global solutions for a quasilinear riser problem. *Nonlinear Analysis: Theory, Methods & Applications*. 2007;67(3):974-80.

30. Wu JQ, Li SJ. Global solution and blow-up solution for a nonlinear damped beam with source term. *Applied Mathematics-A Journal of Chinese Universities*. 2010;25(4):447-53.

31. Evans LC. *Partial differential equations.* Berkeley: American Mathematical Society;1997.

32. Dinlemez Ü, Nabdel S. Blow-up and global solutions of a wave equation with the initial-boundary conditions. *Gazi University Journal of Science*. 2015;28(2):245-51.

33. Kalantarova J. *Nonlinear Second Order Parabolic and Hyperbolic Equations Blow-up and Asymptotic Behavior of Solutions*, Phd Thesis, Yeditepe University, 2015.