TATE'S RIGID ANALYTIC SPACES

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ABSTRACT

TATE'S RIGID ANALYTIC SPACES

During his work on reductions of p-adic elliptic curves, John Tate discovered that the existence of a non-archimedean analytic space with meaningful notions of connectedness and analytic continuation is important. In this thesis, we will inspect his construction of rigid analytic spaces in order to achieve this goal. In order to do his construction, we start with similar ideas to algebraic geometry; by considering the maximal (but not prime) spectrum of so called affinoid algebras which will replace coordinate rings of affine varieties from algebraic geometry. Unfortunately, the Zariski topology is too coarse to define a meaningful notion of connectedness. Tate uses so called G-topologies to overcome this problem and define a structure sheaf with properties analogue to archimedean geometry.

ÖZET

TATE'IN KATI ANALITIK UZAYLARI

p-sel eliptik eğrilerin indirgenmesi üzerine çalışmaları sırasında John Tate, Arşimetsel olmayan analitik uzaylarda mantıklı bir kümelerde bağlılık ve analitik süreklilik teorisinin bulunmasının önemli olduğunu farketti. Bu tezde, bu Tate'in bu amaca ulaşmak için yaptığı katı analitik uzayların inşaasını inceleyeceğiz. Başlangıçta, cebirsel geometrideki afin varyetelerin koordinat halkaları yerine, Tate'in tanımladığı afinoit denilen halkaların maksimal (klasik cebirsel geometrinin aksine asal değil) tayflarını incelemekle başlayacağız. Ne yazık ki Zariski topolojisinin açık kümeleri, mantıklı bir kümelerde bağlılık teorisi oluşturmak için çok büyükler. Tate bu sorunu G-topoloji denilen bir inşaa ile aşarak Arşimetsel duruma benzer özellikleri olan bir yapısal bağlam kurmayı başarıyor.

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LIST OF SYMBOLS/ABBREVIATIONS

k_a	Algebraic closure of k
R^o	Elements with $\|\cdot\| \leqslant 1$
R^{oo}	Elements with $\|\cdot\| < 1$
$\overline{R}$	Residue ring of the normed ring R
T_n	Tate algebra of order n
Γ	Galois group $Gal(k_a : k)$
e.g.	(exempli gratia) means "for example"
etc.	(et cetera) means "and so on"
f.g.	finitely generated
i.e.	(id est) means "that is"
T.F.A.E.	the following are equivalent
UFD	Unique Factorization Domain
w.l.o.g.	without loss of generality
w.r.t.	with respect to

1. INTRODUCTION

To understand p-adic numbers, one has to reconsider how we constructed our number system $\mathbb{C}$ in the first place:

$$\mathbb{N} \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$$

- To solve the equation $x + 1 = 0$, we added negative numbers to our collection of numbers.
- To solve the equation $5x = 1$, we added rational numbers to our collection.
- The existence of irrational numbers was known for a long time (famous proof of the irrationality of $\sqrt{2}$), but the operation of adding all of the irrationals to our collection was formalized by Cauchy Sequences.
- To solve the equation $x^2 + 1 = 0$, we added imaginary numbers to our collection.

Since $\mathbb{C}$ is Cauchy complete and algebraically closed, we were done.
Let's start with basic definitions:

Definition 1.0.1. A Cauchy sequence is defined to be a sequence c_n such that:

$$\forall \varepsilon > 0, \quad \exists N > 0, \quad \forall n, m > N \qquad \text{we have } |c_n - c_m| < \varepsilon \tag{1.1}$$

Hensel's discovery of other absolute values on $\mathbb{Q}$ led us to different completions of $\mathbb{Q}$:
Fix a prime number $p \in \mathbb{N}$ rational number $x \in \mathbb{Q}$ can obviously be represented as: $x = \frac{rp^a}{s}$ where $\gcd(s, p) = \gcd(r, p) = 1$,
then we can associate to x a number $|x|_p = p^{-a}$.
$|\cdot|_p$ is an absolute value. The completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$ is noted as $\mathbb{Q}_p$.
Then we need to compute the algebraic closure $\mathbb{Q}_{p_a}$ of $\mathbb{Q}_p$ and then its completion via $|\cdot|_p$ denoted $\mathbb{C}_p$ which is at last Cauchy complete and algebraically closed.
Then the Ostrowski's theorem gave us the complete picture:

Theorem 1.0.1. *Let $|\cdot|$ be a non-trivial absolute value defined on rational numbers $\mathbb{Q}$, $|\cdot|$ is either equivalent to the real absolute value or a p-adic absolute value.*

p-adic absolute values satisfy what is called the non-archimedean property, to be precise: $\forall x, y$, we have $|x+y|_p \leqslant max\left(|x|_p, |y|_p\right)$.
They're called non-archimedean absolute values.
Even though Marc Krasner was the first one to study p-adic analysis, John Tate's work has been the most influential on the field.
Tate's aim was to introduce a sheaf of analytic functions on a non-archimedean field k.
This aim proves to be difficult since if one tries to define an analytic function naively to be locally analytic functions, even Liouville's theorem (every bounded entire function is constant) fails to work.
In fact, in the non-archimedean case:

$$f(x) = \begin{cases} 0 & \text{if} \quad |z| \leqslant 1 \\ 1 & \text{if} \quad |z| > 1 \end{cases} \tag{1.2}$$

becomes analytic since $\{z : |z| \leqslant 1\}$ and $\{z : |z| > 1\}$ are open in k.
The problem here is that the non-archimedean field k is totally disconnected.
Actually Tate called these kind of constructions floppy, hence he called his construction rigid since it satisfied expected results.
Also, one has to know that Tate didn't inspect this subject by pure curiosity. He discovered that having meaningful complex analysis-like notions for p-adic analytic constructions helps with the theory on Elliptic Curves.

2. PRELIMINARIES

The aim of this chapter is to remind some basic definitions and propositions about analysis. The reader of course will need some knowledge on algebra, especially algebraic geometry in the following chapters. But we choose to remind those subjects when they will be needed.

2.1. ABSOLUTE VALUES, NORMS AND VALUATIONS

Definition 2.1.1. A norm on an abelian group A is defined to be a map:

$$\|\cdot\| : A \longrightarrow \mathbb{R} \tag{2.1}$$

satisfying following conditions:

$$\forall x, y \in A$$

$$\text{(i) } \big(\|x\| > 0 \text{ and } \|x\| = 0\big) \Leftrightarrow x = 0 \tag{2.2}$$

$$\text{(ii) } \|x + y\| \leqslant \|x\| + \|y\| \tag{2.3}$$

Then A is called a normed group. If A is a ring, we require the norm to satify another porperty:

$$\text{(iii)} \forall x, y \in A, \qquad \|xy\| \leqslant \|x\| \cdot \|y\| \tag{2.4}$$

Then A is called a normed ring.
Finally for a normed field k, a normed k-algebra A is a normed ring with a k-algebra structure.

Definition 2.1.2. A norm on an object (group, ring or algebra) A is said to be non-archimedean if it satisfies the following property:

$$\forall x, y \in A,\ \|x + y\| \leqslant max\,(\,\|x\|, \|y\|) \tag{2.5}$$

This property which is a stronger version of the triangular inequality is called the ultrametric property.

Definition 2.1.3. Let R be a normed ring, let $x \in R$, then if:

$$\forall y \in R, \quad \|xy\| = \|x\| \cdot \|y\| \tag{2.6}$$

then x is called a multiplicative element of R.

Definition 2.1.4. Let R be a normed ring, if every element of R is multiplicative, the norm on R is called a multiplicative.

Definition 2.1.5. A normed field k with a non-archimedean multiplicative norm is called a valued field and its norm is called a valuation.

Definition 2.1.6. Let A be a normed object, the set:

$$|A| = \{|x| : x \in A \text{ such that } x = |a|\} \subset \mathbb{R}_+ \tag{2.7}$$

is called the value set.
The value set can be bounded, finite etc.
For a valued field k, the value set $|k|$ becomes a multiplicative subgroup of $\mathbb{R}_+$. It is called the value group.

Definition 2.1.7. Let k be a normed field, and A a normed k-algebra. Then the norm defined on k is called an absolute value.
i.e. Absolute value is used to call the norm defined on the base field. Absolute values are denoted with one bar on each side of the element:

$$\text{For } x \in k,\ |x| = \text{ the absolute value of x} \tag{2.8}$$

Definition 2.1.8. A Cauchy sequence $(s_n)_{n\in\mathbb{N}}$ in a normed object (group, ring or algebra) A is a sequence in A such that:

$$\forall \varepsilon > 0,\ \exists M_\varepsilon > 0,\ \forall n, m > M,\ \text{we have } \|s_n - s_m\| < \varepsilon \tag{2.9}$$

Definition 2.1.9. A normed object (group, ring or algebra) A is said to be complete if every Cauchy sequence is convergent to an element in A.

Definition 2.1.10. For a normed object (group, ring or algebra) A, a completion $\hat{A}$ of A is a normed object of the same category such that:

(i) $\hat{A}$ is complete.

(ii) $i : A \to \hat{A}$ is an isometric homomorphism.

(iii) $i(A)$ is dense in $\hat{A}$.

Theorem 2.1.1. *For any normed object (group, ring or algebra) A, there exists a completion $\hat{A}$ of A. $\hat{A}$ is unique up to isometric isomorphisms.*

Proof. The construction of A is fairly algebraic. If one considers the object $\mathcal{C}(k)$ of Cauchy sequences on A and the object $c_0(A)$ of zero sequences on A, c_0 becomes a sub-object of $\mathcal{C}$ and the quotient object becomes a completion. Uniqueness is fairly straightforward. For details the reader should take a look at [1], section 1.1.7. □

3. TATE ALGEBRAS

In this chapter, we will construct our basic algebraic structure. They are called Tate algebras and affinoid algebras. They will replace in a way coordinate rings of the affine space and affine varieties.

The main theorem is called the Weierstrass Division and Preparation Theorem which implies that Tate algebras and affinoid algebras have very good algebraic properties. For example, they are noetherian.

3.1. FIRST DEFINITIONS

Definition 3.1.1. Let k be a complete field w.r.t. a non-archimedean multiplicative norm, then the Tate algebra on k of degree n is defined to be:

$$\begin{aligned} T_n(k) &= k\langle X_1, X_2, \dots, X_n\rangle \\ &= \left\{ \sum_{\nu_1,\dots,\nu_n \geqslant 0} a_{\nu_1,\dots,\nu_n} X_1^{\nu_1} \cdots X_n^{\nu_n} \ : \ a_{\nu_1,\dots,\nu_n} \in k \text{ and } |a_{\nu_1,\dots,\nu_n}| \to 0 \right. \\ &\qquad\qquad \left. \text{when } |\nu_1 + \cdots + \nu_n| \to \infty \right\} \end{aligned} \tag{3.1}$$

If k is known from the context, we will write just T_n.
Also T_0 is defined to be the base field k.
Elements of T_n are called strictly convergent power series.
For notation purposes, we will put from now on:

$$X = (X_1, \dots, X_n) \tag{3.2}$$

$$\nu = (\nu_1, \dots, \nu_n) \tag{3.3}$$

$$X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n} \tag{3.4}$$

$$|\nu| = |\nu_1 + \cdots + \nu_n| \tag{3.5}$$

Definition 3.1.2. Let k be a complete field w.r.t. a non-archimedean multiplicative norm, consider the map:

$$\|\cdot\| : \quad T_n \longrightarrow \mathbb{R}$$
$$f = \sum_{\nu \geqslant 0} a_\nu X^\nu \longmapsto \max_\nu \ |a_\nu| \tag{3.6}$$

$\|\cdot\|$ is called the Gauss norm on T_n.

Note that the Gauss norm is well defined since for every $f = \sum_{\nu \geqslant 0} a_\nu X^\nu \in T_n$, we have the property: $|a_\nu| \to 0$ when $|\nu| \to \infty$ Thus the maximum exists.

Definition 3.1.3. For a normed ring R, let:

$$R^o = \{x \in A : \|x\| \leqslant 1\} \tag{3.7}$$

$$R^{oo} = \{x \in A : \|x\| < 1\} \tag{3.8}$$

Then by the definition of ring norm, R^o is a subring of R and R^{oo} is an ideal of R^o.

The quotient ring $\overline{R} = R^o / R^{oo}$ is called the residue ring of R.

If $R = k$ a valued field, the residue ring $\overline{k}$ is a field (due to the fact that k is a field and the norm on k is a valuation) and is called the residue field of k.

3.2. SOME BASIC PROPERTIES OF T_n AND GAUSS NORM

In this section k will always be a complete field w.r.t. a non-archimedean multiplicative norm unless specified otherwise.

Lemma 3.2.1. *T_n with the Gauss norm is isometric to $c_0(k)$ the space of all zero sequences of k.*

Proof. For $f = \sum_{\nu \geqslant 0} a_\nu X^\nu \in T_n$

Using Cantor's diagonal procedure, we can reorder a_ν's as a sequence of k.

This transformation trivially gives us an isometry between T_n and $c_0(k)$. □

Proposition 3.2.2. *T_n is indeed a k-subalgebra of $k[[X]]$ the algebra of formal power series on k.*

Also $(T_n, \|\cdot\|)$ *is a k-Banach algebra containing* $k[X]$ *as a dense k-subalgebra.*
The Gauss norm on T_n *is a non-archimedean norm extending the norm defined on* k.

Proof. It is obvious that we have:

$$k \subset k[X] \subset T_n \subset k[[X]] \quad \text{as sets.} \tag{3.9}$$

Now let $f, g \in T_n$ and
denote:

$$f = \sum_{\nu \geqslant 0} a_\nu X^\nu \tag{3.10}$$

$$g = \sum_{\mu \geqslant 0} b_\mu X^\mu \tag{3.11}$$

we have:

$$f + g = \sum_{\nu \geqslant 0} (a_\nu + b_\nu) X^\nu \tag{3.12}$$

but

$$a_\nu \xrightarrow[|\nu| \to \infty]{} 0 \qquad \text{since } f \in T_n \tag{3.13}$$

$$b_\nu \xrightarrow[|\nu| \to \infty]{} 0 \qquad \text{since } f \in T_n \tag{3.14}$$

thus we have (since k is a normed field, thus Hausdorff):

$$a_\nu \pm b_\nu \xrightarrow[|\nu| \to \infty]{} 0 \tag{3.15}$$

it means that $f \pm g \in T_n$ which implies that T_n is closed under addition, and thus $T_n \subset k[[X]]$ as an abelian group.
We now need to prove that it is closed under multiplication.
We have

$$fg = \sum_{\nu+\mu=\lambda} a_\nu b_\mu X^\lambda \tag{3.16}$$

but

$$\forall \nu, \mu; \quad |a_\nu b_\mu| \leqslant \max_{\nu+\mu=\lambda} \left(|a_\nu| \cdot |b_\mu|\right) \tag{3.17}$$

thus

$$a_\nu b_\mu \xrightarrow[\lambda \to \infty]{} 0 \tag{3.18}$$

so T_n is closed under multiplication. It is indeed a k-subalgebra.
Also:

$$\begin{aligned}\|f \pm g\| &= \max_{\nu} |a_\nu \pm b_\nu| \\ &\leqslant \max_{\nu}\big(\{a_\nu\}_\nu \cup \{b_\nu\}_\nu\big) \\ &= \max\,(|f|, |g|)\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|fg\| &\leqslant \max_{\nu+\mu=\lambda}\big(|a_\nu| \cdot |b_\mu|\big) \\ &\leqslant \max\,(|f|\,, |g|)\end{aligned} \tag{3.20}$$

Thus the Gauss norm is indeed a non-archimedean norm and since obviously for $a \in k$, $\max\,(|a|, 0, 0, \ldots) = |a|$, it extends the norm defined on k.
We also can deduce that T_n is complete w.r.t. the Gauss norm since it is isometric to $c_0(k)$ and since we have the fact that $c_0(k)$ is complete when k is complete.
Finally, for $f = \sum_{\nu \geqslant 0} a_\nu X^\nu$

$$\left\| f - \sum_{\nu=0}^{m} a_\nu X^\nu \right\| \xrightarrow[m \to \infty]{} 0 \tag{3.21}$$

thus $k[X]$ is dense in T_n. □

Lemma 3.2.3. *T_n is a field if and only if $n = 0$.*

Proof. $T_0 = k$ is a field by definition.
If $n > 0$, then $X_1 \in T_n$ and X_1 is not a unit. □

Lemma 3.2.4. *For all non-zero f in T_n, there exists c in k such that $\|cf\| = 1$.*

Proof. Let

$$|T_n| = \{\|f\| \in \mathbb{R} : \quad f \in T_n\} \tag{3.22}$$

$$|k| = \{|x| \in \mathbb{R} : \quad x \in k\} \tag{3.23}$$

Clearly we have $|T_n| = |k|$.

By definition of the Gauss norm:

$$f = \sum_{\nu \geqslant 0} a_\nu X^\nu \tag{3.24}$$

$$\|f\| = \max_{\nu \geqslant 0}(|a_\nu|) \tag{3.25}$$

there exists $\nu \in k$ so that $\|f\| = |a_\nu|$

but k being a field, ν has an inverse in k. Put $c = \frac{1}{a_\nu}$

The norm on k being multiplicative, we get $\|cf\| = 1$. □

Lemma 3.2.5. *(Gauss lemma)* *Let R be a normed ring such that:*

(i) $\forall x \in R,\ \exists m \in R - \{0\},\ \exists s \in \mathbb{N}, \quad \|mx^s\| = \|m\| \cdot \|x\|^s = 1$

(ii) $R^o/R^{oo} = \overline{R}$ *is an integral domain.*

then $\|\cdot\|$ *is multiplicative on R.*

Proof. Suppose that $\exists x_1, x_2 \in R$ such that:

$$\|x_1 x_2\| < \|x_1\| \cdot \|x_2\| \tag{3.26}$$

so $x_1 \neq 0$ and $x_2 \neq 0$, since if else we would get $0 < 0$. but (i) implies:

$$\exists m_1, m_2 \in R, \exists s_1, s_2 \in \mathbb{N},\ s \geqslant 1 \text{ such that:}$$

$$\|m_1 x_1^{s_1}\| = \|m_1\| \cdot \|x_1\|^{s_1} = 1 \tag{3.27}$$

$$\|m_2 x_2^{s_2}\| = \|m_2\| \cdot \|x_2\|^{s_2} = 1 \tag{3.28}$$

Assume now w.l.o.g. that $s_2 \geqslant s_1$.

Since m_1 and m_2 are multiplicative, we have:

$$\begin{aligned} \|(m_1 x_1^{s_1})(m_2 x_2^{s_2})\| &= \|m_1\| \cdot \|m_2\| \cdot \|x_1^{s_1} x_2^{s_2}\| \\ &\leqslant \|m_1\| \cdot \|m_2\| \cdot \|x_1 x_2\|^{s_1} \cdot \|x_2\|^{s_2 - s_1} \\ &< \|m_1\| \cdot \|m_2\| \cdot (\|x_1\| \cdot \|x_2\|)^{s_1} \cdot \|x_2\|^{s_2 - s_1} \\ &= \|m_1\| \cdot \|x_1\|^{s_1} \cdot \|m_2\| \cdot \|x_2\|^{s_2} = 1 \end{aligned} \tag{3.29}$$

So $\left(m_1 x_1^{S_1}\right)\left(m_2 x_2^{S_2}\right) \in R^{oo}$ which is a contradiction since by (ii).

Indeed R^{oo} is a prime ideal in R^{o}, but neither $m_1 x_1^{S_1}$ nor $m_1 x_1^{S_1}$ is in R^{oo}.

Thus $\|\cdot\|$ is multiplicative. □

Theorem 3.2.6. *The Gauss norm on T_n is multiplicative, thus a valuation on T_n and $\overline{T_n} = \overline{k}[X]$.*

Proof. Consider the morphism:

$$\begin{array}{rccc} \Phi: & T_n^o & \longrightarrow & \overline{k}[X] \\ & f = \sum\limits_{\nu \geqslant 0} a_\nu X^\nu & \longmapsto & \sum\limits_{\nu \geqslant 0} \overline{a_\nu} X^\nu \end{array} \tag{3.30}$$

This is a well defined surjective map since $\lim\ a_\nu \to 0$,

thus $\overline{a_\nu} = 0$ after some finite number of coefficients,

and so $\Phi(f)$ is indeed an element of $\overline{k}[X]$.

Also it is easy to see that $ker(\Phi) = T_n^{oo}$.

Thus we have $\overline{T_n} = \overline{k}[X]$.

But since $\overline{k}$ is a field, $\overline{k}[X]$ and thus $\overline{T_n}$ is an integral domain.

Also, we know that we can "normalize" every $f \in T_n$ i.e.

$\forall f \in T_n, \exists c \in k$ such that $\|cf\| = 1$.

So we can use the Gauss lemma which proves that the Gauss norm on T_n is multiplicative. □

Proposition 3.2.7. *For $f \in T_n$ such that $\|f\| = 1$, T.F.A.E.:*

(i) *f is a unit in T_n^o.*

(ii) *f is a unit in T_n.*

(iii) *$\overline{f}$ is constant in $\overline{k}[X] - \{0\}$.*

(iv) *$|f(0)| = 1$ and $\|f - f(0)\| < 1$.*

Proof. Obviously, (i) and (ii) are equivalent since the Gauss norm is multiplicative.

Also we know that $(i) \Rightarrow (iii)$ since

$\overline{T_n} = T_n^o / T_n^{oo} = \overline{k}[X]$ $\quad$ ($\Rightarrow \overline{f}$ is a unit in $\overline{k}[X]$ thus a non-zero constant).

(iii) and (iv) are equivalent by the definition of the residue field $\overline{k}$.

Finally suppose that (iv) is true. Let $g = 1/f(0)$.

Then there exist some u, such that $fg = 1 + u$ and $\|u\| < 1$.

Thus $\sum_{\nu \geqslant 0} (-u)^\nu$ is convergent and is an inverse for $1 + u$.

So

$$fg \sum_{\nu \geqslant 0} (-u)^\nu = (1 + u) \sum_{\nu \geqslant 0} (-u)^\nu = 1 \tag{3.31}$$

Which concludes that:

$$f^{-1} = g \sum_{\nu \geqslant 0} (-u)^\nu \tag{3.32}$$

f is then unit in T_n^o $\left(g \sum_{\nu \geqslant 0} (-u)^\nu \in T_n^o \text{ since } \|u\| < 1 \text{ and } |g| = 1 \right)$. □

Proposition 3.2.8. *Let $f \in T_n$ such that $\|f\| = 1$.*

There exists $c \in k$ with $|c| = 1$ so that $f + c$ is not a unit.

Proof. We are going to use Proposition 3.2.7.(iv).

There are two cases:

(i) If $|f(0)| < 1$,

since $\|f\| = 1$, f has a coefficient a_ν other than the first one, such that $|a_\nu| = 1$

Put $c = 1$.

Then

$$|(f + c)(0)| = 1 \tag{3.33}$$

but,

$$\|(f + c) - (f + c)(0)\| = \|f - f(0)\| = 1 \tag{3.34}$$

Thus $f + c$ is not a unit.

(ii) If $|f(0)| = 1$,

Then putting $c = f(0)$, we get $|(f + c)(0)| = 0$

which cannot be a unit in T_n. □

We have to prove a little lemma from abstract algebra before the next proposition:

Lemma 3.2.9. *Let R be a ring:*

(i) *There exists a maximal ideal* $\mathfrak{M}$ *in* R.

(ii) *If* $I \neq R$ *is an ideal of* R, I *is contained in a maximal ideal in* R.

(iii) *Every non unit in R is contained in a maximal ideal.*

Proof. (i) Let $\Sigma = \{I : I \text{ ideal in } R\}$.

Σ is not empty since $(0) \in \Sigma$

and Σ can be ordered by the inclusion of ideals, so that Σ is partially ordered.

To use Zorn's lemma we'll just need to prove that every chain of ideals in Σ has an upper bound.

But for a chain of ideals $(I_n)_{n\in\Lambda}$, we have that

$$\bigcup_{n\in\Lambda} I_n \text{ is an ideal of } R. \tag{3.35}$$

Thus the union of every ideal in the chain is in Σ.

Zorn's lemma implies that Σ has at least one maximal element.

(ii) We can define $\Sigma_I = \{ J : I \subset J \text{ and } J \text{ is an ideal of } R \}$ and imitate the same proof as (i).

Alternatively, we can consider the quotient ring R/I and use the (order-preserving) bijection between ideals of the quotient ring R/I and ideals of R which contain I.

(iii) Let $x \in R$ a non unit element.

By (ii), (x) is contained in a maximal ideal $\mathfrak{M}$. Then $x \in \mathfrak{M}$. □

Proposition 3.2.10.

$$\bigcap_{\mathfrak{M} \text{ maximal ideal of } T_n} \mathfrak{M} = (0) \tag{3.36}$$

Proof. Suppose that the intersection is bigger than the zero ideal.

Then there exists an f in the intersection.

We can assume that $\|f\| = 1$ because of Lemma 3.2.4.

But by Proposition 3.2.8, there exists a $c \in k$ with $|c| = 1$ such that $f + c$ is not a unit.

Since $f + c$ is not a unit, it has to be contained in a maximal ideal $\mathfrak{M}$ by Lemma 3.2.9.

But f being in the intersection of all maximal ideals, $f \in \mathfrak{M}$, thus $f + c - f = c \in \mathfrak{M}$.

Of course c being in the base field k, is a unit in T_n.

Which conclude $\mathfrak{M} = T_n$ which is a contradiction. □

3.3. WEIERSTRASS DIVISION AND PREPARATION THEOREMS

Definition 3.3.1. Let $f \in T_n$, f is called normalized if $\|f\| = 1$.

Recall that, we can always normalize an element in T_n to have its norm equal to 1 because of Lemma 3.2.4.

For the next definition, remember that $\overline{T_n} = \overline{k}[X]$ (Theorem 3.2.6).

Definition 3.3.2. Let $f \in T_n$, $f = \sum_{i=0}^{\infty} a_i(X_1, \dots, X_{n-1})X_n^i$ is called X_n-distinguished of degree d if:

(i) a_d is a unit in T_{n-1}

(ii) $\|a_d\| = \|f\|$ and $\|a_d\| > \|a_v\|$ for all $i > d$.

Note that, if $\|f\| = 1$, f is X_n-distinguished of degree d if and only if $\overline{f} \in \overline{T_n}$ is a unitary polynomial ($i.e.$ its leading coefficient is a unit) of degree d in the polynomial ring $\overline{k}[X_1, \dots, X_{n-1}][X_n]$ (using Proposition 3.2.7 (iv) on $a_i's$).

Theorem 3.3.1. *(Weierstrass)*

(i) *(Division) Suppose that $f \in T_n^o$ is normalized distinguished in X_n of degree d.*

Then every $g \in T_n$ can be uniquely represented as:

$$g = fq + r \tag{3.37}$$

such that $q \in T_n$, $r \in T_{n-1}[X_n]$ and degree of r in $T_{n-1}[X_n]$ is less than d.

Also, $\|g\| = \max(\|q\|, \|r\|)$.

(ii) *(Preparation) Suppose that $f \in T_n^o$ is normalized distinguished in X_n of degree d.*

Then there is a unique way to write $f = gh$ *such that:*

$$g \in T_{n-1}^{o} \text{ and unitary of degree } d$$
$$h \in (T_n^o)^*$$

(iii) *(Distinction) If* $f_1, \dots, f_m \in T_n^o$ *all normalized,*
then there exists an automorphism σ *of* T_n *preserving Gauss norms and such that* $\sigma(f_1), \dots, \sigma(f_n)$ *are normalized distinguished in* X_n.

Proof. (i) (Division)
(Uniqueness) Let's first prove the uniqueness:
Let $g \in T_n$ and let $g = qf + r = q'f + r'$ two different divisions of g by f.
Then $(q - q')f = r' - r$ and it implies $\|q - q'\| \cdot \|f\| = \|r' - r\|$ since the the Gauss norm is multiplicative.
But $\|f\| = 1$, so we have $\|q - q'\| = \|r' - r\|$.
Since $|T_n| = |k|$, there exists $c \in k$ such that $|c| = \|q - q'\|^{-1}$.
Then $c(q - q')f = c(r' - r)$ and also $\|c(q - q')\| \cdot \|f\| = \|c(r' - r)\| = 1$.
Thus $\bar{c}(\overline{q - q'})\bar{f} = \bar{c}(\overline{r' - r}) \neq 0$ in $\overline{T_n}$.
This translates to $\overline{fcq} + \overline{cr} = \overline{fcq'} + \overline{cr'}$
But this contradicts the uniqueness of Euclidean division on
$\bar{k}[X_1, \dots, X_{n-1}][X_n]$.
Thus the Weierstrass division is unique.
Here $\bar{k}[X_1, \dots, X_{n-1}][X_n]$ is not an Euclidean domain, but it is an integral domain and f is unitary. So, we're using the Euclidean division by a unitary polynomial.
Now let's prove the property of the division on the norms:
If we have:

$$g = fq + r \tag{3.38}$$

then:

$$\|g\| \leqslant \max(\|qf\|, \|r\|) = \max(\|q\|, \|r\|) \tag{3.39}$$

If $\|q\| \neq \|r\|$ we have the equality.

But, if $\|q\| = \|r\| > g$, then we can choose $c \in k$ such that $|c| = \|q\|^{-1}$, and then

$$cg = fcq + cr \tag{3.40}$$

Here $1 = \|cq\| > \|cg\|$, thus $\overline{cg} = 0$.

The equation with bars turns out to be the division 0 by f on $\bar{k}[X_1, \dots, X_{n-1}][X_n]$:

$$0 = \overline{cg} = \overline{fcq} + \overline{cr} \tag{3.41}$$

From the uniqueness of the division, we deduce:

$$\overline{cq} = 0 \tag{3.42}$$

$$\overline{cr} = 0 \tag{3.43}$$

which is a contradiction to $\|cq\| = 1$.

So if $\|q\| = \|r\|$, then $\|q\| = \|r\| = \|g\|$.

(Existence) Finally, we shall prove the existence:

Since f is normalized distinguished in X_n, we can decompose f as:

$$\begin{aligned} f &= f_0 - D \\ &\text{where } f_0 = \sum_{i=0}^{d-1} a_i X_n^i + \lambda X_n^d \text{ with } \forall a_i \in T_{n-1}^o \\ &\text{and } \|D\| < 1 \end{aligned} \tag{3.44}$$

Then f_0 is also normalized distinguished in X_n of degree d.

First, suppose that $D = 0$.

Thus, $f = f_0$.

But f_0 being a unitary polynomial, we can use the Euclidean division by f_0 in $T_{n-1}[X_n]$.

Let $g = \sum_\nu b^\nu X^\nu$.

We can compute $\forall \nu$, the Euclidean division:

$$\begin{aligned} X^\nu &= q_\nu f_0 + r_\nu \\ &\text{with } q_\nu, r_\nu \in T_{n-1}[X_n] \\ &\text{and } d = \deg_{X_n} f_0 > \deg_{X_n} r_\nu \end{aligned} \tag{3.45}$$

But this is also a Weierstrass division and we proved that:

$$\max(\|q_\nu\|, \|r_\nu\|) = \|X^\nu\| = 1 \tag{3.46}$$

Since $g \in T_n$, $|b_\nu| \to 0$ by definition and thus:

$$q = \sum_\nu b_\nu q_\nu \text{ and } r = \sum_\nu b_\nu r_\nu \tag{3.47}$$

are convergent in T_n.

Also since $\forall \nu$, r_ν has degree less than d, we have:

$r \in T_{n-1}[X_n]$ with a degree less than d. Then by definition, we get:

$$g = fq + r \tag{3.48}$$

This proves the existence of the division when $f = f_0$.

Now, if $D \neq 0$, we use an induction argument to get to the case "$f = f_0$":

Remember that $f = f_0 - D$ and $\|D\| < 1$.

Put $g_0 = g$ and define g_{i+1} as the following:

$$g_i = f_0 q_i + r_i = f q_i + r_i + D q_i \tag{3.49}$$

$$\text{then } g_{i+1} = D q_i \tag{3.50}$$

So $\forall i$,

$$g_i - g_{i+1} = f q_i + r_i$$

$$\text{where } r_i \in T_{n-1}[X_n] \text{ with degree } < d \text{ in } X_n \tag{3.51}$$

Summing the last equation $\forall i$, we get:

$$g = g_0 = f \sum_i q_i + \sum_i r_i \tag{3.52}$$

But $g_1 = D q_0$ and since $\|D\| \leqslant D$ we have:

$$\|g_1\| \leqslant \|D\| \cdot \|g_0\|$$

$$\|g_2\| \leqslant \|D\| \cdot \|g_1\| \leqslant \|D\|^2 \cdot \|g_0\|$$

$$\|g_3\| \leqslant \|D\| \cdot \|g_2\| \leqslant \|D\|^3 \cdot \|g_0\| \tag{3.53}$$

and so on.

Since we proved the property of the division on norms, we have:

$$\lim \|g_i\| = \lim \|q_i\| = \lim \|r_i\| = 0 \tag{3.54}$$

proving that the series

$$\sum_i q_i \quad , \quad \sum_i r_i \quad \text{are Cauchy.} \tag{3.55}$$

But T_n is complete.

Hence defining:

$$q = \sum_i q_i \quad , \quad r = \sum_i r_i \in T_n \tag{3.56}$$

But by the definition of r implies that $r \in T_{n-1}[X_n]$ of degree less than d in X_n (since $\forall i, r_i \in T_{n-1}[X_n]$ and of degree less than d).

So we have:

$$g = fq + r \tag{3.57}$$

satisfying claims of the theorem, concluding the proof for the existence of the Weierstrass division.

(ii)(Preparation)

(Existence) Let $f \in T_n^o$ be normalized distinguished in X_n of degree d.

Divide X_n^d by f via Weierstrass division to obtain:

$$X_n^d = fq' + r' \tag{3.58}$$

Putting $q = X_n^d - r'$,

we get $q \in T_{n-1}^o[X_n]$ since $\|X_n^d\| = 1$

And q is unitary of degree d in X_n, since r' has degree less than d in X_n.

Also $q = fq'$.

Reducing the equality, we get:

$$\overline{q} = \overline{fq'} \qquad \text{where } \deg_{X_n} \overline{q} = \deg_{X_n} \overline{f} \tag{3.59}$$

Hence $\overline{q'}$ can only be a unit and by Lemma 3.2.7, q' is also a unit.

So we can put:

$$f = gh \quad \text{with } g = q \text{ and } h = (q')^{-1} \tag{3.60}$$

which satisfies claims of the part (ii) of the theorem.

(Uniqueness) Let's now prove that this decomposition is unique.

If $f = gh$ is such a decomposition, we have:

$$X_n^d = fh^{-1} + (X_n^d - g) \tag{3.61}$$

which is a Weierstrass division of X_n by f. Thus is uniquely determined.
Uniqueness of the decomposition follows, since this equation uniquely determines g, h (otherwise we would have a contradiction to the uniqueness of the Weierstrass division).
(iii) Recall that $\nu = (\nu_1, \dots, \nu_n)$.
Now write $f_i = \sum\limits_{\nu} a_{\nu,i} X^\nu$ for $i = 1, \dots, m$.
Since $f_i \in T_n^o$, there exists finite amount of tuples ν such that $|a_\nu| = 1$.
Choose now $e_1, \dots, e_{n-1} \geqslant 0$ such that:

$$\begin{aligned} &\text{Whenever} \quad \nu \neq \mu \quad \text{and} \quad |a_{\nu,i}| = |a_{\mu,i}| = 1 \\ &\text{we have} \quad e_1\nu_1 + \cdots + e_{n-1}\nu_{n-1} + \nu_n \neq e_1\mu_1 + \cdots + e_{n-1}\mu_{n-1} + \mu_n \end{aligned} \tag{3.62}$$

Now let σ be the automorphism defined by substituting X_i's with $X_i + X_n^{e_i}$ for $i = 1, \dots, n-1$ and X_n with X_n.
Then:

$$\overline{\sigma(f_i)} = \sum \overline{a_{\nu,i}} \left(X_1 + X_n^{e_1}\right)^{\nu_1} \cdots \left(X_{n-1} + X_n^{e_{n-1}}\right)^{\nu_{n-1}} X_n^{\nu_n} \tag{3.63}$$

There is a unique tuple ν maximizing $e = e_1\nu_1 + \cdots + e_{n-1}\nu_{n-1} + \nu_n$.
So the term of the highest degree of the reduction of $\sigma(f_i)$ is X_n^e.
Thus $\sigma(f_i)$'s are distinguished in X_n. □

3.4. SOME ALGEBRAIC PROPERTIES OF TATE ALGEBRAS AND AFFINOID ALGEBRAS

We start by proving an analogue of Hilbert Basis Theorem from algebraic geometry.

Proposition 3.4.1. *The ring T_n is noetherian.*

Proof. We will use induction on n.
Let I be a non-zero ideal of T_n.

We know that $T_0 = k$ is noetherian and suppose that T_{n-1} is noetherian.

Given $f \in I$, we can choose an automorphism σ such as described in the Weierstrass theorem (Theorem 3.3.1 (iii)) so that $\sigma(f)$ is distinguished in X_n of some degree d.

Now, using the Weierstrass division (Theorem 3.3.1(i)), we can see that $\sigma(I)$ can be generated by $\sigma(f)$ and $J = \sigma(I) \cap T_{n-1}[X_n]$.

We supposed that T_{n-1} is noetherian and thus $T_{n-1}[X_n]$ is noetherian by the Hilbert Basis Theorem.

So J is finitely generated. It implies that $\sigma(I)$ is finitely generated.

σ being an automorphism, I is also finitely generated. □

Proposition 3.4.2. *The ring T_n is a UFD.*

Proof. We'll use again an induction on n.

We know that k is a UFD and suppose that T_{n-1} is also a UFD.

Let f be a non-zero element of T_n.

By the Weierstrass theorem (Theorem 3.3.1(iii)), we can find an automorphism of T_n, σ such that $\sigma\,(f)$ is normalized distinguished in X_n of some degree d.

Consider the Weierstrass division:

$$X_n^d = \sigma(f)q + r \tag{3.64}$$

So $X_n^d - r$ is also normalized distinguished in X_n of degree d (because $\deg_{X_n} r < d$).

Thus we have another Weierstrass division:

$$\sigma(f) = (X_n^d - r)q' + r' \tag{3.65}$$

Substituing the first equality in the second, we get:

$$\sigma(f) = \sigma(f)qq' + r' \tag{3.66}$$

But the uniqueness of the Weierstrass division implies:

$$\begin{aligned} qq' &= 1 \\ r' &= 0 \end{aligned} \tag{3.67}$$

Thus q' is a unit in T_n and:

$$\sigma(f) = q'(X_n^d - r) \tag{3.68}$$

We supposed that T_{n-1} is a UFD and thus $T_{n-1}[X_n]$ is a UFD.

This means that we can write:

$X_n^d - r = f_1 \cdots f_s$ uniquely upto multiplication by units in $T_{n-1}[X_n]$
and where f_i's are unitary and irreducible in $T_{n-1}[X_n]$

Now we claim that f_i's are irreducible in T_n:

Let $g \in T_{n-1}[X_n]$ be unitary, irreducible and normalized.

Let $g = g_1 g_2$ be a decomposition of g in T_n such that g_1 is not a unit in T_n.

We may suppose that g_1, g_2 are normalized (if not, we can easily normalize the decomposition by dividing everything by their norm).

Then $\overline{g} = \overline{g_1 g_2}$.

This implies that g_1 is unitary normalized distinguished in X_n.

But then the Weierstrass division of g by g_1 can be done and the unique division is:

$$g = g_1 g_2 \tag{3.69}$$

However g, g_1 being in $T_{n-1}[X_n]$ and g_1 being unitary we also have the unique Euclidean division:

$$g = g_1 h + r \tag{3.70}$$

which is also by definition a Weierstrass division.

By the uniqueness of the Weierstrass division, we get that $h = g_2$ and $r = 0$.

But then $g_2 \in T_{n-1}[X_n]$.

Since g is irreducible in $T_{n-1}[X_n]$, g_2 must be a unit in $T_{n-1}[X_n]$ and thus in T_n.

The uniqueness of the decomposition can be proven in a similar manner.

So for any f in T_n, $\sigma(f)$ and thus f itself can be uniquely written as the product of irreducibles, which concludes the proof. □

We can now prove an analogue for the Noether Normalization Lemma.

First, we start some definitions and a lemma which will help us prove the Noether Normalization.

Definition 3.4.1. Let R and S be ring and $\Phi : R \to S$ be a ring homomorphism.
Every S-module M can be seen as an R-module defining the scalar product in the R-module M as $rx = \Phi(r)x$.
In this case, we say that Φ makes the S-module M and R-module.
Notice that, this definition can be used in the case where S is an S-module over itself.
So essentially, a ring homomorphism $R \to S$ defines an R-module structure on S.

Definition 3.4.2. Let $\Phi : R \to S$ be a ring homomorphism. If Φ makes S into a f.g. R-module, we say that Φ is a finite homomorphism.
If there exists a finite, injective homomorphism $R \to S$, we say that S is a finite extension of R; or sometimes if the context is clear, we can also say that S is finite over R.

Definition 3.4.3. The Krull dimension of a ring R is the supremum of the length of all chains of prime ideals in R.

The next proposition is an analogue of the Noether Normalization lemma.
For the proof, we will use the following lemma from algebra.

Lemma 3.4.3. *If R and S are noetherian and if there exists $\Phi : R \to S$, a finite injective ring homomorphism, then R and S have the same Krull dimension.*

Proposition 3.4.4. *The Krull dimension of T_n is n.*
For every ideal $I \subset T_n$, there exists an integer d and an injective, finite k-algebra homomorphism $T_d \hookrightarrow T_n/I$ such that the Krull dimension of T_n/I is d.

Proof. Let $f \in T_n$ be normalized distinguished in X_n of degree d.
The Weierstrass division theorem is equivalent to the following:
The natural k-algebra homomorphism: $T_{n-1} \to T_n/(f)$ (obtained by composing the natural injection of T_{n-1} into T_n and the canonical projection from T_n into T_n/I) makes $T_n/(f)$ into a free T_{n-1}-module with the free basis $1, X_n, \ldots, X_n^{d-1}$.
Indeed, for any g in T_n by Weierstrass division by f, we can mod out a multiple of f and the

rest is a polynomial with degree less than d. This translates to the fact that by modding out f from any $g \in T_n$, we obtain a linear combination of powers (less than d) of X_n with elements from T_{n-1} as coefficients.

Now with this fact, let's prove that the Krull dimension of T_n is n.

Since in T_n, we have the obvious chain of prime ideals:

$$(0) \subset (X_1) \subset \dots \subset (X_1, \dots, X_n) \quad (3.71)$$

which has a length n. The Krull dimension is at least n.

Now, let f be irreducible in T_n. Then (f) is a prime ideal and the Krull dimension of T_n is less or equal than $1+$ the Krull dimension of $T_n/(f)$. That is because we have the bijection between the collection of prime ideals of T_n containing (f) and the collection of prime ideals of $T_n/(f)$.

Also, by the distinction part of Weierstrass theorems, we can suppose that f is normalized distinguished in X_n.

But from what we have done (the reformulation of the Weierstrass division), we conclude that $T_n/(f)$ is finite over T_{n-1}.

By the lemma 3.4.3, the Krull dimension of T_{n-1} is the same as the Krull dimension of $T_n/(f)$. The latter being greater or equal than the Krull dimension of $T_n - 1$, we get:

$$\text{the Krull dimension of } T_n \leqslant \text{ the Krull dimension of } T_{n-1} + 1 \quad (3.72)$$

Since the Krull dimension of $T_0 = k$ is obviously 0, a simple induction on n shows us that the Krull dimension of $T_n \leqslant n$.

Thus the Krull dimension of $T_n = n$.

Now lets prove the second part of the theorem:

Let I be a non-zero ideal of T_n.

Again, by the distinction part of Weierstrass theorems, we can suppose that there exists an element f in I such that f is normalized distinguished in X^n.

From what we have done, we know that $T_n/(f)$ is a f.g. T_{n-1}-module.

Put now, $J = I \cap T_{n-1}$.

Then, $T_{n-1}/J \to T_n/I$ is injective and finite.

Everytime, we do this operation, the degree of the module gets smaller, thus there must exist (by induction) an integer d such that:

$$T_d \to T_{n-1}/J \tag{3.73}$$

is finite and injective.

Then, clearly $T_d \to T_{n-1}/J \to T_n/I$ is finite and injective.

We conclude from the lemma 3.4.3 that the Krull dimension of T_n/I is d. □

Corollary 3.4.4.1. *For any maximal ideal $\mathfrak{M}$ and any integer n, $T_n/\mathfrak{M}$ is a finite field extension of k.*

Proof. Since $\mathfrak{M}$ is a maximal ideal, $T_n/\mathfrak{M}$ is obviously a field.

Consider now the finite injective morphism $T_d \to T_n/\mathfrak{M}$ defined by the Proposition 3.4.4.

But, this is an injective morphism into a field, thus T_d has to be a field.

So $d = 0$ and $T_d = k$.

We conclude that the morphism defined in the Proposition 3.4.4 is a finite injection of fields, which implies the result. □

Definition 3.4.4. A k-algebra A which is also a finite extension of T_n for some $n \geqslant 0$ is called an affinoid algebra.

Affinoid algebras are the basic building block of Rigid Analytic Spaces and we just proved (Proposition 3.4.4) some interesting fact about them.

Let's now focus on their other properties.

Proposition 3.4.5. *Every affinoid algebra is noetherian.*

Proof. From the Proposition 3.4.1, we know that T_n is noetherian for any $n \geqslant 0$.

But by the definition of affinoid algebras, for an affinoid algebra A, we have a injective, finite homomorphism:

$$\Phi : T_n \to A \tag{3.74}$$

for some n.

So A is a f.g. T_n-module via this homomorphism.

But then, T_n being a noetherian ring, the T_n-module A becomes a noetherian module.

Thus if we let I be an ideal of A,

every element x of I can be written as:

$$x = a_1\Phi(t_1) + \cdots + a_d\Phi(t_d) \tag{3.75}$$

where $a_i \in A$ and $t_i \in T_n$.

So $\Phi(t_i)$'s generate the ideal I. □

Before proving the next proposition, we need a lemma from p-adic functional analysis.

Lemma 3.4.6. *Let A be a Banach algebra over k and let M be a Banach A-module. If A is a noetherian ring and M is f.g., then every A-submodule of M is closed.*

Proof. Let N be a A-submodule of M and let $\overline{N}$ be its closure.

Since A is noetherian, $\overline{N}$ is a f.g. module over A.

Choose now generators $e_1, \dots, e_n$ of $\overline{N}$ and consider the A-module homomorphism:

$$\begin{aligned} \Phi : \quad & A^n \longrightarrow \overline{N} \\ & (a_1, \dots, a_n) \longmapsto \textstyle\sum a_i e_i \end{aligned} \tag{3.76}$$

Here A^n has the topology provided by the norm $\|(a_i)\| = \max(\{\|a_i\|\})$.

By the open mapping theorem, there exists $c \in (0, 1)$ such that each $x \in \overline{N}$ can be written as $\sum a_i e_i$ such that $c \cdot \max(\|a_i\|) \leqslant \|x\|$.

Choose $f_1, \dots, f_n \in N$ with $\|e_i - f_i\| \leqslant c^2$.

Given $x \in \overline{N}$, choose f_i's as above and define:

$$x_0 = x \tag{3.77}$$

and writing each x_j as:

$$x_j = \sum_i a_{j,i} e_i \quad \text{with } c \cdot \max\left(\{\|a_{j,i}\|\}\right) \leqslant \|x_j\| \tag{3.78}$$

Put:

$$x_{j+1} = \sum_i a_{j,i}(e_i - f_i) \tag{3.79}$$

so that $\|x_{j+1}\| \leqslant c\|x_j\|$.

But this means that $x_j \to 0$ and for each i, the series $\sum_j a_{j,i}$ converges to a limit a_i such that $x = \sum_i a_i f_i$. Thus f_i's generate $\overline{N}$ which implies $N = \overline{N}$. □

Proposition 3.4.7. *Let A be an affinoid algebra and let $\|\cdot\|$ be a norm w.r.t. which A is a Banach algebra over k, then every ideal I of A is closed w.r.t. $\|\cdot\|$.*

Proof. We already proved that affinoid algebras are noetherian.

Thus the lemma above proves the proposition since every ideal of A is by definition an A-submodule. □

Proposition 3.4.8. *Any affinoid algebra A can be written as T_n/I for some $n \geqslant 0$ and for some ideal I of T_n.*

The Gauss norm on T_n induces a norm on A w.r.t. which A is a Banach algebra over k.

Proof. Let A be an affinoid algebra.

Thus by definition, we have a finite, injective homomorphism $\tau : T_d \to A$ for some $d \geqslant 0$.

Here, we name T_d's indeterminates as Z_i's. So $T_d = k\langle Z_1, \dots, Z_d\rangle$.

But then, every element e of A is integral over T_d and we can find an $a \in k^*$ such that ae is integral over T_d^o.

Thus the T_n-module A can be generated by elements $e_1, \dots, e_s$, such that $e_1, \dots, e_s$ are integral over T_d^o.

Fix polynomials:

$$P_i(X_i) = X_i^{d_i} + \sum_{j=0}^{d_i - 1} a_{i,j} X_i^j \tag{3.80}$$

such that $P_i(e_i) = 0$ where $i = 1, \dots, s$.

Now T_{d+s} can be written as:

$$T_{d+s} = k\langle Z_1, \dots, Z_n, X_1, \dots, X_s\rangle \tag{3.81}$$

But the Weierstrass division implies that the canonical morphism:

$$\Phi : \frac{k\langle Z_1, \dots, Z_d\rangle [X_1, \dots, X_s]}{(P_1, \dots, P_s)} \longrightarrow \frac{k\langle Z_1, \dots, Z_d, X_1, \dots, X_s\rangle}{(P_1, \dots, P_s)} \tag{3.82}$$

is an isomorphism.

We also can construct a surjective k-algebra homomorphism γ from $\frac{k\langle Z_1,\dots,Z_d\rangle[X_1,\dots,X_s]}{(P_1,\dots,P_s)}$ to A extending the defining homomorphism τ of A by mapping each X_i to e_i.

Now consider the following diagram:

$$T_{d+s} \xrightarrow[\pi]{} \frac{k\langle Z_1,\dots,Z_d,X_1,\dots,X_s\rangle}{(P_1,\dots,P_s)} \xrightarrow[\Phi^{-1}]{} \frac{k\langle Z_1,\dots,Z_d\rangle[X_1,\dots,X_s]}{(P_1,\dots,P_s)} \xrightarrow[\gamma]{} A \tag{3.83}$$

π being the canonical morphism is surjective.

Also Φ is an isomorphism, so Φ^{-1} is surjective.

And we constructed ϕ to be surjective.

So by computing their composition, we conclude that there exists a surjective k-algebra homomorphism $\rho : T_n \to A$ with $n = d + s$.

So $A = T_n/I$ where $I = ker\rho$.

Additionally since the kernel is an ideal, by the Proposition 3.4.7, the kernel is closed.

So the induced semi-norm:

$$\|f\|_A = \inf\{\|g\| : g \in T_n \text{ and } \rho(g) = f\} \tag{3.84}$$

is a norm and A is a Banach space w.r.t. it.

The inequality $\|f_1 f_2\|_A \leqslant \|f_1\|_A \cdot \|f_2\|_A$ comes from the same inequality for elements of T_n.

Finally, we need to prove that $\|1\|_A = 1$.

Trivially, $\|1\|_A \leqslant 1$.

Now suppose that $\|1\|_A < 1$,

then there is an element $g \in T_n$ with $\rho(g) = 1$ and $\|g\| < 1$.

Thus $\rho(1 - g) = 0$.

However $1 - g$ has an inverse $\sum_j g^j$ in T_n which is a contradiction.

So $\|1\|_A = 1$ and thus A is a Banach k-algebra. □

The last proposition we gave, with the Proposition 3.4.4 gives us another definition of affinoid algebras.
They are k-algebras isomorphic to a quotient of a Tate algebra by an ideal.
We also proved that they are Banach algebras with the norm induced from the Gauss norm.
We will now prove an important proposition about affinoid spaces, their algebra structure dictates their topology.

Proposition 3.4.9. *Let two affinoid algebras A_1 and A_2 are assigned with norms which make them Banach algebras.*
If $u : A_1 \to A_2$ is a k-algebra homomorphism, then u is continuous w.r.t. the norms assigned.

Proof. We will use the closed graph theorem. Let $u : A_1 \to A_2$ a morphism.
Since we're dealing with algebras, we can check the requirements of the closed graph theorem at zero.
So the closed graph theorem translates to:

$$\text{Let } x_n \text{ be a sequence in } A_1 \text{ with } \lim x_n = 0 \text{ and } \lim u(x_n) = y \in A_2, \text{ then } y = 0 \tag{3.85}$$

Take an ideal $I \subset A_2$ with finite co-dimension (i.e. A_2/I has finite dimension over k) and consider:

$$v_I : A_1 \xrightarrow[u]{} A_2 \longrightarrow A_2/I \tag{3.86}$$

Then $J = ker\ v$ is an ideal of finite co-dimension (since v sends an injective image of A_1/J into A_2/I which has finite dimension over k).
The map v of course factors as:

$$v_I : A_1 \xrightarrow[\pi_1]{} A_1/J \xrightarrow[v']{} A_2/I \tag{3.87}$$

Lets induce the norms defined on A_1 and A_2 onto A_1/J and A_2/I. Of course, we can do this because I and J are closed (Proposition 3.4.7).

But any linear mapping between finite dimensional topological vector spaces is continuous, and thus v_I is continuous (it is the composition of π_1 and v').

Now, v_I being continuous, it is compatible with limits and thus $v(y) = 0$ in A_2/I.

We proved that the image of y for any ideal with co-finite dimension of A_2 is zero, thus y is in I.

The trick here is to show that the intersection $\bigcap I$ of the ideals with co-finite dimension is (0), so that y being in every co-finite dimensional ideal of A_2, will be equal to zero.

Before, we proved that any maximal ideal $\mathfrak{M} \subset A$ where A is an affinoid algebra, $\mathfrak{M}$ has finite co-dimension (Corollary 3.4.4.1).

Then, for any $n \geqslant 1$, $\mathfrak{M}^n$ has also finite co-dimension.

Take $y \in A$, such that $\forall n \geq 1$, $y \in \mathfrak{M}^n$.

$J := \{a \in A : ay = 0\}$ is an ideal of A.

If $y \neq 0, J \neq A$ and thus J lies in a maximal ideal of A (Lemma 3.2.9).

The image z of y in the localization $A_{\mathfrak{M}}$ of A by $\mathfrak{M}$ is not zero since, $J \subset \mathfrak{M}$ and also since $z \in \mathfrak{M}^n A_{\mathfrak{M}} = (\mathfrak{M}A_{\mathfrak{M}})^n$, $\forall n \geq 1$.

However, $A_{\mathfrak{M}}$ being a noetherian local ring, the intersection of the powers of its maximal ideal is (0).

But we just said that z, the image of y in $A_{\mathfrak{M}}$ is not zero.

It's a contradiction.

Thus $y = 0$, which proves that:

$$\bigcap I \subseteq \bigcap_n \mathfrak{M}^n = (0) \tag{3.88}$$

Here remember that all the powers of maximal ideals had finite co-dimension.

But, this proves the result we wanted. □

Corollary 3.4.9.1. *Let A be an affinoid algebra, let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two norms w.r.t. which A is a Banach algebra over k,*

then $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent.*

Proof. Consider the identity morphism:

$$\begin{aligned} id : A &\longrightarrow A \\ f &\longmapsto f \end{aligned} \tag{3.89}$$

It is obviously isomorphic, and id and id^{-1} are continuous.
Thus id is a homeomorphism. □

Corollary 3.4.9.2. *Every norm which makes T_n a Banach algebra is equivalent to the Gauss norm.*

Proof. The proof follows effortlessly from the fact that the Gauss norm makes T_n a Banach algebra (Proposition 3.2.2). □

4. SPECTRAL THEORY OF AFFINOID ALGEBRAS

In a fashion much like Algebraic Geometry, we will treat A as a function algebra and elements of A functions defined on the maximal spectrum $Max\ A$ of A. We'll call the maximal spectrum of an affinoid algebra the affinoid space associated to A.

We'll see that those functions satisfy a very familiar property: The Maximum Modulus Principle.

The spectral theory works almost exactly the same way (obviously we have a norm here, so one has to investigate its behaviour). So in this chapter, we'll be seeing some expected results on the spectrum.

Before everything, let's illustrate with an example, how the spectrum machinery works consistently on a very specific affinoid algebra: T_n.

Example. Let $Max\ T_n$ be the maximal spectrum of $T_n = T_n(k)$, i.e.

$$Max\ T_n = \{\mathfrak{M} : \mathfrak{M} \text{ is a maximal ideal of } T_n\} \tag{4.1}$$

and where K is a valued field, let $B^n(K)$ to be the closed unit disc w.r.t. the valuation on K, i.e.

$$B^n(K) = \{(x_1, \dots, x_n) \in K^n : \max\ |x_\nu| \leqslant 1\} \tag{4.2}$$

There is a unique valuation on the algebraic closure k_a of k extending the valuation defined on k (for details one should read [1] (3.2.4)).

The elements of $T_n(k)$ and any element $f = \sum a_\nu X^\nu$ of $T_n(k)$ defines a map

$$f : B^n(k_a) \longrightarrow k_a \tag{4.3}$$

That's what we expected, $T_n(k)$ is supposed to be the analogue of analytic function on the closed unit polydisc. To see how an element $f = \sum a_\nu X^\nu$ behaves on the polydisc $B^n(k_a)$, take $x \in B^n(k_a)$ so $x = (x_1, \dots, x_n)$.

Now take a finite field extension over k which includes $x_1, \dots, x_n$ as elements. Then the field extension we took is complete and the series converges to a limit in the field extension. Thus we see that $f(x)$ is well defined in k_a. Actually $T_n(k)$ gives rise exactly to functions:

$$f : B^n(k_a) \longrightarrow k_a$$

$(i) f$ has a power series expansion over k_a on the whole unit polydisc $B^n(k_a)$

$(ii) f$ maps $B^n(k)$ to k

For the detailed proof of this fact, the reader should read [1] (5.1.4). Now, there is actually another way of making $f = \sum a^\nu X^\nu$ into a map, but this time defined on the space $Max\, T_n$ i.e.

$$f : Max\, T_n \longrightarrow k_a/\Gamma \qquad \text{where } \Gamma = Gal(k_a/k) \tag{4.4}$$

The construction of this map is rather easy to understand:
We proved that $T_n/\mathfrak{M}$ is a finite extension of k for every maximal ideal $\mathfrak{M}$ of T_n (Corollary 3.4.4.1). So we can consider the image of f in the quotient field $T_n/\mathfrak{M}$ for every maximal ideal $\mathfrak{M}$.
We want to assign to every maximal ideal, the image of f in the their respective quotient field. This image is also in the algebraic closure k_a of k since $T_n/\mathfrak{M}$ is a finite extension. But the embedding of $T_n/\mathfrak{M}$ into k_a is generally not unique. So to construct a well defined map, we have to quotient out the action of the Galois group to provide uniqueness of the image.
Now we have two ways of seeing the same element $f = \sum a_\nu X^\nu$ as a map defined on two different definition spaces: $B^n(k_a)$ and $Max\, T_n$.
The interesting thing about this situation is that they're compatible in a very sturdy way: Firstly, we want to build a map:

$$\tau : B^n(k_a) \rightarrow Max\, T_n \tag{4.5}$$

This map should be the analogue of a map used in Algebraic Geometry which assigns maximal ideals to points on for example the affine space.
Indeed, we use here exactly the map from Algebraic Geometry:

$$\tau(x) = \{f \in T_n : f(x) = 0\} \tag{4.6}$$

The set $\tau(x)$ is a maximal ideal, thus the map τ is well defined. Finally we can illustrate why two ways of seeing $f \in T_n$ as a function are consistent:

It turns out that the following diagram is commutative:

$$\begin{array}{ccc} B^n(k_a) & \xrightarrow[f]{} & k_a \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi} \\ Max\ T_n & \xrightarrow[f]{} & k_a/\Gamma \end{array} \tag{4.7}$$

Moreover, there is a bijection between $Max\ T_n$ and $B^n(k_a)/\Gamma$.

So basically, there is no essential difference between seeing f as a function over $B^n(k_a)$ and as a function over $Max\ T_n$.

4.1. SPECTRAL SEMINORM

Now let's move to affinoid algebras.

By the Corollary 3.4.4.1, for any maximal ideal $\mathfrak{M}$ of an affinoid algebra A, the quotient field $A/\mathfrak{M}$ is finite over k.

Definition 4.1.1. Like in the example above, for $f \in A$ and $x \in Max\ A$, we define $f(x)$ to be the image of f in $A/\mathfrak{M}_x$ where $\mathfrak{M}_x = x$.

For the sake of clarity, we write the maximal ideal $\mathfrak{M}_x$ as x when we want to see it as a point.

The spectral seminorm on A is defined to be:

$$\|f\|_{spec} = \sup_{x \in Max\ A} |f(x)| \tag{4.8}$$

Remember here that $A/\mathfrak{M}$ is finite over k, then there is a unique extension of the norm on k to $A/\mathfrak{M}_x$ so the definition make sense (also $Gal(k_a : k)$ acts on k isometrically, so we can use the image of f in k like we did in the example).

Let's prove now that the Maximum Modulus Principle holds for the spectral seminorm.

Proposition 4.1.1. *Let A be an affinoid algebra and $f \in A$, there exists $x \in Max(A)$ so that $\|f\|_{spec} = |f(x)|$.*

Also, if $\|f\|_{spec} = 0$*,* f *is nilpotent in* A*.*

Proof. The proof uses the theory of Newton Polygons, we will give here a sketch of the proof. The reader is encouraged to look into [3], notes from Kiran Kedlaya. If A is not reduced (no nilpotent elements), consider the quotient of A by its nilradical (the set of nilpotents; it forms an ideal, also turns out to be the intersection of prime ideals). Since the nilradical is the intersection of prime ideals, the spectral seminorm of an element doesn't change in the quotient. So we can restrict ourselves into the reduced affinoid algebras case.
We can also restrict our case to integral (domain) affinoid algebras case, since if else we can check the claim on componenents of the affinoid algebra. So, let A be an integral affinoid algebra.
From the Proposition 3.4.4, A is a finite extension of T_d for some $d \geqslant 0$. So we take an irreducible polynomial from $T_d[X]$ and use Newton Polygon methods on it to reduce the case to T_d where both assertions is easy to prove. □

Corollary 4.1.1.1. *The Jacobson radical (i.e. the intersection of maximal ideals) of an affinoid algebra A is equal to the nilradical of A*
i.e. A is a Jacobson ring.
In particular, the spectral seminorm is a norm if and only if A is reduced.

The main proposition we can give on the spectral seminorms is the following (it should remind the reader the Gelfand formula and it is another way of defining the spectral seminorm):

Proposition 4.1.2. *Let A be an affinoid algebra, under some norm* $\|\cdot\|$*,*
then for $f \in A$ *we have:*

$$\|f\|_{spec} = \lim_{n\to+\infty} \|f^n\|^{1/n} \tag{4.9}$$

Proof. Again, Kedlaya proves it rather thoroughly. We will give a sketch and encourage the reader to take a look at [3].
Like the proof of the last proposition, we try to use the finite morphism $T_d \to A$ to prove an

equivalence:

$$\|f\|_{spec} \leqslant 1 \Leftrightarrow \{\|f^n\|\}_{n=1}^{\infty} \text{ is bounded.} \tag{4.10}$$

Of course, here the norm we chose on A is arbitrary. If $\{\|f^n\|\}_{n=1}^{\infty}$ is bounded, then $|f(x)^n|$ is bounded for each $x \in Max\ A$, so $|f(x)|$ cannot be greater than 1. So $\|f\|_{spec} \leqslant 1$.
Also if the sequence is bounded, first we prove that f is integral over the subring of T_d consisting of elements with spectral seminorm less than 1. Then we write a polynomial admitting f as a root and bound the set $\{\|f^n\|\}_{n=1}^{\infty}$ on T_d.
The proof is finished by considering the continuity of the map $T_d \to A$. It should send a bounded set to a bounded set. □

We should also give these propositions about spectral seminorms in order to better understand the structure they define:
They are minimal norms in the following sense:

Proposition 4.1.3. *Let A be an affinoid algebra with norm* $\|\cdot\|$, *then for all* $f \in A$, $\|f\|_{spec} \leqslant \|\cdot\|$.

Proof. From the last proposition, we have:

$$\|f\|_{spec} = \lim_{n \to +\infty} \|f^n\|^{1/n} \tag{4.11}$$

but $\|\cdot\|$ is a k-algebra norm, thus:

$$\|f^n\| \leqslant \|f\|^n \tag{4.12}$$

The result follows (by taking the $1/n$'th power of both sides). □

Let's just state two last convenient properties of the seminorm without proving them ([3] does it, the proof of the first one is quite technical) We stated that spectral seminorms are norms if the affinoid algebra is reduced. Moreover they turn out to be Banach norms (they make their affinoid algebra complete).
Also, the spectral norm coincide with the Gauss norm on T_n.

So ultimately, the spectral seminorm gives us a way of comparing elements of an affinoid algebra as functions and satisfy convenient properties like the Maximum Modulus Principle (which is really important since we want to somehow imitate complex analytic spaces).

4.2. SPECTRUM OF AN AFFINOID ALGEBRA

Let's now move to the algebraic properties of the maximal spectrum. Some parts are almost identical to the theory of affine schemes.

Let A be an affinoid algebra, so it is T_n/I for an ideal I and an integer n. We proved that T_n is noetherian, so I is finitely generated, say $I = (f_1, \ldots, f_n)$. One should think of $Max\ A$ (similar to the case of affine schemes) as the space cut out by setting $f_1 = \cdots = f_n$. Also as in the case of affine schemes, a morphism of affinoid algebras induce a map on their spectra in the opposite direction, so the case is the following (A and B are affinoid algebras):

$$\forall \phi : A \longrightarrow B$$
$$\exists^* \phi : Max\ B \longrightarrow Max\ A \tag{4.13}$$

A reader with some knowledge on algebraic geometry would expect almost every definition and proposition from now on. Their proofs are not very interesting (very similar to the affine case), so we will not give complete proofs for following propositions. If the reader is interested on full proofs, they can be found in [1] Chapter 7.

The theory starts with the definition of affinoid subsets.

Definition 4.2.1. Let $F \subset Max\ T_n$ be a subset,

the set:

$$V(F) = \left\{ \mathfrak{M} \in Max\ T_n : f(\mathfrak{M}) = 0 \text{ for all } f \in F \right\} \tag{4.14}$$

is called an affinoid subset.

As expected, all affinoid subsets can be "generated" from a finite number of elements, it is of course an ideal of T_n:

Proposition 4.2.1. *Let $F \subset Max\ T_n$ be an affinoid subset, then there are finitely many func-*

tions $f_1, \dots, f_r \in T_n$ *such that* $F = V(f_1, \dots, f_n)$.
Also for an ideal $I \in T_n$,
we have:

$$V(I) = \{x \in Max\, T_n : I \subset \mathfrak{M}\} \tag{4.15}$$

First one is because T_n is noetherian and the second one turns out to be true because of our definition of affinoid subsets. We also have the ideal associated to an affine subset:

$$id(F) = \left\{ f \in T_n : f_{|_F} = 0 \right\} = \bigcap_{x \in F} \mathfrak{M}_F \tag{4.16}$$

As in algebraic geometry, we can obtain an affinoid subset from the ideal associated to itself:

Proposition 4.2.2. *Let* $F \subset Max\, T_n$ *be an affinoid subset, then:*

$$V(id(F)) = F \tag{4.17}$$

For the converse equality however, the situation is the same:
Hilbert Nullstellensatz holds for affinoid subsets:

Proposition 4.2.3. *For an ideal* $I \in T_n$,
we have:

$$id(V(I)) = rad(I) \tag{4.18}$$

This is really the result of T_n being Jacobson. In fact, the general version of Hilbert Nullstellensatz is formulated on Jacobson rings. At this point there is a mild difference between affinoid subsets and affine sets:

Proposition 4.2.4. V *and* id *define a bijection between reduced ideals of* T_n *and affinoid subsets of* $Max\, T_n$.

We can endow $Max\, T_n$ with the Zariski topology and we get expected results:

Proposition 4.2.5. *Any non-empty affinoid subset* $F \subset Max\ T_n$ *is irreducible (w.r.t. Zariski topology) if and only if* $id(F)$ *is a prime ideal.*

We can also consider those definition in the affinoid algebra case (on the maximum spectrum of an affinoid algebra). A morphism between two affinoid algebras induce a map between their spectra:

Definition 4.2.2. Let A be an affinoid algebra, the pair $Sp\ A = (MaxA, A)$ is called an affinoid variety.
Affinoid varieties form a category where morphisms are pairs $({}^*\phi, \phi)$ where:
${}^*\phi : Max\ B \longrightarrow Max\ A$ is induced from $\phi : A \longrightarrow B$.

It turns out the Zariski topoology is too coarse for rigid spaces.
We want a topology for which notions like connectedness will have good meaning. There is one topology to consider. It is called the canonical topology and it is the topology $Max\ A$ inherits from k (remember the morphism between $B^n(k)$ and $Max\ T_n$). Before getting into the analysis of this topology, we should examine some very particular domains:

Definition 4.2.3. Let A be an affinoid algebra, $a = (a_1, \dots, a_n) \in A^n$, $b = (b_1, \dots, b_m) \in A^m$ and define:

$$A < a, b^{-1} >= A < X, Y > /(X_1 - a_1, \dots, X_n - a_n, b_1Y_1 - 1, \dots, b_mY_m - 1)$$
$$\text{where } X = (X_1, \dots, X_n) \text{ and } Y = Y_1, \dots, Y_m \qquad (4.19)$$

then $A < a, b^{-1} >$ will become the coordinate ring (remember from algebraic geometry) of what is called a Laurent domain (it is of course a subset of $Max\ A$).
If $m = 0$, the domain is called a Weierstrass domain.

Proposition 4.2.6. *There is a bijection between* $Max\ (A < a, b^{-1} >)$ *and*
$\{x \in Max\ A : |a_i(x)| \leqslant 1, |b_j(x)| \geqslant 1 \text{ for all } i, j\}$

It turns out that Laurent domains are a basis for the canonical topology. It gives us an idea

about the difference between Zariski topology and the canonical topology: Instead of considering the zero loci of functions (sections), we can consider loci of some comparisons. The problem is that the canonical topology is too fine. In fact, Laurent domains being a basis forces the topology to be totally disconnected.
So we need to ”restrict” the canonical topology in some way. Now, let us also introduce another class of important domains:

Definition 4.2.4. Let A be an affinoid algebra, $a_1, \dots, a_n, b \in A$ and consider:

$$A\left\langle \frac{a_1}{b}, \dots, \frac{a_n}{b} \right\rangle = A < X_1, \dots, A_n > /(bX_1 - a_1, \dots, bX_n - a_n) \tag{4.20}$$

Same as before, the subset of $Max\ A$ such that the algebra above is the coordinate ring is called a rational domain.

As for Laurent domains, the next proposition should give the reader some intuition about rational domains.

Proposition 4.2.7. *There is a bijection between* $A\left\langle \frac{a_1}{b}, \dots, \frac{a_n}{b} \right\rangle$ *and*
$\{x \in Max\ A : |a_j(x)| \leqslant |b(x)|\}$.

Before getting into the next chapter, we should explain why we consider the maximum spectrum instead of prime.
Well firstly, it is of course possible to do algebraic geometry via the maximum spectrum, but the existence of generic points gives us some flexibility on the topic.
One of the important aspects of this problem is to see that we are able to compute $|f(x)|$ uniquely when x is a maximal ideal, but there may be some prime ideals for which the result is not unique.

5. G-TOPOLOGIES AND TATE'S ACYCLICITY

We stated that the canonical topology is totally disconnected. Thus we have to restrict the topology in some way.
Tate's method was to introduce a so called Grothendieck topology on the maximum spectrum of an affinoid algebra.
A Grothendieck topology (or a G-topology) is a notion that Alexander Grothendieck came up with, hence the name. It is essentially an algebraic structure generalizing topology which should be the natural environment of sheaves.
Lets first recall the definition of a sheaf. First, the definition of a presheaf:

Definition 5.0.1. Let X be a topological space and $\mathcal{C}$ a category,
A presheaf of objects in $\mathcal{C}$ on X is a functor $\mathcal{O}$ which associates an object from $\mathcal{C}$ to every open set in X such that:

- For each open set U there exists an object $\mathcal{O}(U)$
- For each inclusion of open sets $U \subseteq V$, there exists a restriction morphism $\text{res}_{V,U} : \mathcal{O}(V) \longrightarrow \mathcal{O}(U)$ and restriction morphisms satisfy:
  - For each open set U, $\text{res}_{U,U} : \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$ is the identity morphism on $\mathcal{O}(U)$.
  - For each tower of open sets $U \subseteq V \subseteq W$, we have: $\text{res}_{V,U} \circ \text{res}_{W,V} = \text{res}_{W,U}$

For U, V open such that $U \subseteq V$ and for $s \in \mathcal{O}(V)$, we usually write $s|_U := \text{res}_{V,U}$ if V is not relevant in our situation.

Presheaves are algebraic structures mainly used to model function spaces. The restriction morphisms are generally restrictions of functions in to a smaller subset and etc (of course presheaves can be used to model some other algebraic structures such as vector bundles).
An element of $\mathcal{O}(U)$ is called a section of U, in this context one view it as a function defined on the open set U. A sheaf is a presheaf such that one can recover bigger sections from smaller ones.

Definition 5.0.2. A presheaf $\mathcal{O}$ of objects in $\mathcal{C}$ on a topological space X is called a sheaf if it satisfies two properties:

- (local property) If $(U_i)_{i\in\Lambda}$ is an open covering of an open set U and if $s, t \in \mathcal{O}(U)$ such that $\forall i \in \Lambda$, we have $s|_{U_i} = t|_{U_i}$, then $s = t$.
- (gluing property) If $(U_i)_{i\in\Lambda}$ is an open covering of an open set U and if $\forall i \in \Lambda$ we have $s_i \in \mathcal{O}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then
  $\exists s \in \mathcal{O}(U)$ such that $s|_{U_i} = s_i$

5.1. G-TOPOLOGIES AND AFFINOID SUBDOMAINS

A G-topology is defined as the following.

Definition 5.1.1. A G-topology $\mathfrak{T}$ on a set X consists of:

- a system S of subsets of X, called admissible opens ($\mathfrak{T}$-opens)
- a family $\{Cov\ U\}_{U\in S}$ of systems of coverings ($\mathfrak{T}$-coverings) for every $U \in S$ where each $Cov\ U$ is itself a collection which has elements of the form $\{U_i\}$ such that $U_i \in S$. S and $Cov\ U$ are subject to some properties:
  - $U, V \in S \Rightarrow U \cap V \in S$
  - $U \in S \Rightarrow \{U\} \in Cov\ U$
  - If $U \in S$, $\{U_i\}_{i\in I} \in Cov\ U$ and $\forall i \in I$, $\{V_{ij}\}_{j\in J} \in Cov\ U_i$ then $\{V_{ij}\}_{i\in I, j\in J} \in Cov\ U$
  - If $U, V \in S, V \subseteq U$ and if $\{U_i\}_{i\in I} \in Cov\ U$, then $\{V \cap U_i\} \in Cov\ V$

This is a logical definition, one has more power when defining a G-topology instead of an ordinary topology. For example, we can restrict ourselves to finite coverings only. This is actually exactly what we are going to do.
Now, let's define the basic building block of our affinoid space.

Definition 5.1.2. Let $A = T_n(k)/I$ an affinoid algebra and $X = \mathrm{Max}(A)$.
A subset $U \subseteq X$ is called an affinoid subdomain if it satisfies:

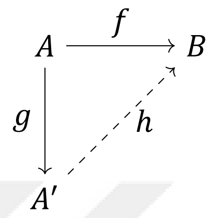
If $\exists g : A \to A'$ such that, $\forall B$ affinoid algebra:

$\forall f : A \to B$,

$f(\mathrm{Max}(B)) \subseteq U$

if and only if $f = h \circ g$ for some $h : A' \to B$.

This means that the following diagram is commutative if and only if ${}^*f(\mathrm{Max}(B)) \subseteq U$:



In this case, we put $\mathcal{O}(U) = A'$

5.2. TATE'S ACYCLICITY THEOREM

Affinoid subdomains have some good behaviour. For example their finite intersections are also affinoid subdomains.

This definition will be the basis for our so called admissible opens.

Definition 5.2.1. Let A be an affinoid algebra and $X = \mathrm{Max}(A)$.

A subset $U \subseteq X$ is called an admissible open if it has a covering(set-theoretic) by affinoid subdomains $U_i \subseteq X$ such that for any $f : A \to B$ with ${}^*f(\mathrm{Max}(B)) \subseteq U$ (remember the induced map on maximum spectra), then

the covering ${}^*f^{-1}(U_i)$ of $\mathrm{Max}(B)$ admits a finite subcovering.

Definition 5.2.2. Let U_i be a collection of admissible open subsets of X, $U = \bigcup_{i \in I} U_i$ as sets. U_i is called an admissible covering of U if for any $f : A \to B$ with ${}^*f(\mathrm{Max}(B)) \subseteq U$, the covering ${}^*f^{-1}(U_i)$ of $\mathrm{Max}(B)$ admits a covering which is finite by open affinoid subdomains of B.

The admissible opens with admissible coverings define a G-topology (sometimes called Tate topology).

Also of course with these definitions, we are also defining $\mathcal{O}(U)$ for an admissible open. We're just gluing coordinate rings of affinoid subdomains to obtain the coordinate ring of U. At last, what Tate's Acyclicity Theorem says actually is that the coordinate rings $\mathcal{O}(U)$ with the G-topology defined by admissible opens is an actual sheaf:

Theorem 5.2.1. *For any finite covering of $X = \text{Max}(A)$ where A is an affinoid algebra by affinoid subdomains U_i, the following sequence is exact:*

$$0 \longrightarrow \mathcal{O}(X) \longrightarrow \prod \mathcal{O}(U_i) \longrightarrow \prod \mathcal{O}(U_i \cap U_j) \tag{5.1}$$

The proof reduce the general case to a very specific case: Rational domains.

In order to do that, one uses a very important theorem: Gerritzen-Grauert theorem which states that any open affinoid is a finite union of rational domains.

Proving both theorems use some quite serious Čech theoretic computations. There is also an advanced version of the theorem telling that every Čech cohomology group of this sheaf vanishes, so the sheaf is ”acyclic” (hence the name of the theorem).

In the end, as in algebraic geometry we just put $Sp(A) = (Max\ A, \mathcal{O}_A)$. This becomes a locally ringed space. Rigid Analytic Spaces are spaces locally isomorphic to $Sp(A)$'s for some A's.

One can use these constructions to prove the so called GAGA (Géométrie Algébrique Géométrie Analytique) principle in the rigid analytic setting. Also one can examine coherent sheaves over the rigid analytic geometry. The Rigid Analytic Space also admits meaningful notions such as analytic continuation and connectedness. Or one can try expand the subject by considering another type of spectrum. That's what Vladimir Berkovich did with his theory of Berkovich Spaces or Roland Huber with his Adic Spaces.

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