COMPUTATION OF SYSTEMIC RISK MEASURES: A MIXED-INTEGER LINEAR PROGRAMMING APPROACH

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In the scope of finance, systemic risk is concerned with the instability of a financial system, where the members of the system are interdependent in the sense that the failure of some institutions may trigger defaults throughout the system. National and global economic crises are important examples of such system collapses. One of the factors that contribute to systemic risk is the existence of mutual liabilities that are met through a clearing procedure. In this study, two network models of systemic risk involving a clearing procedure, the Eisenberg-Noe network model and the Rogers-Veraart network model, are investigated and extended from the optimization point of view. The former one is extended to the case where operating cash flows in the system are unrestricted in sign. Two mixed integer linear programming (MILP) problems are introduced, which provide programming characterizations of clearing vectors in both the signed Eisenberg-Noe and Rogers-Veraart network models. The modifications made to these network models are financially interpretable. Based on these modifications, two MILP aggregation functions are introduced and used to define systemic risk measures. These systemic risk measures, which are not necessarily convex set-valued functions, are then approximated by a Benson type algorithm with respect to a user-defined error level and a user-defined upper-bound vector. This algorithm involves approximating the upper images of some associated non-convex vector optimization problems. A computational study is conducted on two-group and three-group systemic risk measures. In addition, sensitivity analyses are performed on two-group systemic risk measures.

Keywords: systemic risk measure, aggregation function, set-valued risk measure, systemic risk, Eisenberg-Noe model, Rogers-Veraart model, Benson's algorithm, non-convex vector optimization.

ÖZET

SİSTEMİK RİSK ÖLÇÜLERİNİN HESAPLANMASI: KARIŞIK TAMSAYILI DOĞRUSAL PROGRAMLAMA YAKLAŞIMI

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Finans kapsamında sistemik risk, üyeleri birbirine bağımlı olan bir finansal sistemin istikrarsızlığı ile ilgili bir olgudur. Ulusal ve küresel ekonomik krizler bu tür sistem çökmelerinin önemli örnekleridir. Sistemik riske katkısı olan etkenlerden biri de karşılıklı yükümlülüklerin varlığıdır. Bu yükümlülükler bir takas işlemi aracılığıyla yerine getirilmektedir. Bu çalışmada, takas işlemi içeren iki sistemik risk ağ modeli, Eisenberg-Noe ağ modeli ve Rogers-Veraart ağ modeli, araştırılmış ve eniyileme açısından genişletilmiştir. Ayrıca Eisenberg-Noe ağ modeli, sistemdeki işletme nakit akışlarının negatif olmama kısıtlaması kaldırılarak, akışlar işaretli olabilecek biçimde, genişletilmiştir. İşaretli Eisenberg-Noe ve Rogers-Veraart ağ modellerinde, takas vektörlerinin programlama açısından nitelenmesini sağlayan iki karmaşık tamsayılı doğrusal programlama problemi ortaya konulmuştur. Bu ağ modellerinde yapılan değişiklikler finansal olarak yorumlanabilir. Yapılan değişikliklere dayanarak iki birleştirme fonksiyonu tanıtılmıştır, ve bu fonksiyonlar sistemik risk ölçülerinde kullanılmıştır. Sistemik risk ölçüleri küme değerli fonksiyonlar olduğu için kullanıcı tarafından tanımlanan bir hata düzeyine ve üst sınır vektörüne göre Benson tipi bir vektör eniyileme algoritmasıyla yaklaşıklanmıştır. Bu işlem, bazı ilgili dışbükey olmayan vektör eniyileme problemlerinin üst görüntülerinin yaklaşıklanmasını içerir. İki gruplu ve üç gruplu sistemik risk ölçüleri üzerinde hesaplamalı çalışma gerçekleştirilmiştir. Ayrıca iki gruplu sistemik risk ölçüleri üzerinde duyarlılık analizleri yapılmıştır.

Anahtar sözcükler: sistemik risk ölçüsü, birleştirme fonksiyonu, küme-değerli risk ölçüsü, sistemik risk, Eisenberg-Noe modeli, Rogers-Veraart modeli, Benson algoritması, dışbükey olmayan vektör eniyilemesi.

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Chapter 1

Introduction

Financial contagion is usually associated with a quick and unpredictable chain of defaults in a financial system caused by high correlation between the members of the system and leading to disastrous, from an economic point of view, consequences such as high risk of national and global financial crises, necessity for bailout loans, long-lasting economic regression and rise in national debt. A good example is a so-called *bank run*, when a bank receives a lot of claims for deposits due to a panic or decrease in confidence in the bank, causing insolvency of the bank. In its turn, the bank probably calls its claims from the other banks, decreasing confidence in them and causing new bank runs. Being not able to meet their liabilities, some of the banks may become bankrupt and, thus, worsen the contagion even further. Unlike the usual notion of risk, when it is associated with a single entity, systemic risk is related to the strength of an entire financial system against financial contagions.

One of the factors that contribute to systemic risk is the existence of mutual liabilities between the members of a financial system. Clearing mechanisms of a financial system clear these mutual liabilities. It can be done by calculating a clearing vector of the system. There are many network models of systemic risk and the corresponding algorithms that can treat financial systems as network models and calculate their clearing vectors. This thesis extends two network models, the Eisenberg-Noe and Rogers-Veraart models, from the optimization point of view by developing optimization problems that provide clearing vectors in these models.

Systemic risk of a financial system can be decreased by imposing capital requirements to the financial institutions in the system, so that the system can overcome financial shocks. This can be accomplished by computing systemic risk measures. Furthermore, to make the computations of systemic risk measures easier, the members of a financial system can be grouped into two or more categories and the same capital requirement can be imposed to the financial institutions in the same group. In this thesis, set-valued systemic risk measures are considered and computed in the scope of the Eisenberg-Noe and Rogers-Veraart network models, which are introduced in subsequent chapters. For each model, the corresponding grouped systemic risk measure is established and approximated using a Benson type algorithm for non-convex problems introduced in [1].

The rest of the thesis is structured as follows. Chapter 2 reviews the literature on seminal and recent studies in network models of systemic risk and systemic risk measures. For the sake of completeness, seminal studies on scalar and set-valued risk measures are discussed as well.

The Eisenberg-Noe and Rogers-Veraart network models of systemic risk are studied in detail and the corresponding optimization characterizations of clearing vectors are provided in Chapter 3. The Eisenberg-Noe model is extended to the case where operating cash flows in a system are not restricted to be non-negative. Two approaches are applied to accomplish this task: the first one is applied naively, being a conjecture in Eisenberg and Noe [2], and an entirely novel one resulting from the analysis of the drawbacks of the first approach and imposing some seniority assumptions. Mixed-integer linear programming (MILP) characterizations of clearing vectors are established for both the Eisenberg-Noe and Rogers-Veraart network models. Proofs of some of the related results can be found in Appendix A. Moreover, two aggregation functions, one for each network model, are introduced in terms of these MILP formulations. These aggregation functions play a significant role in systemic risk measures and serve as intermediaries between the Eisenberg-Noe and Rogers-Veraart network models and the corresponding grouped systemic risk measures.

In Chapter 4, the grouping notion for systemic risk measures is introduced, which groups the members of a financial system and decreases the dimension of systemic risk measures, making it easier to approximate them. Two aggregation functions, introduced in the previous chapter, are applied to systemic risk measures and the Eisenberg-Noe and Rogers-Veraart grouped systemic risk measures are defined. Furthermore, these systemic risk measures are looked at from a vector optimization point of view and considered as the upper images of their associated vector optimization problems. Two types of optimization problems used in Benson's algorithm, one being a weighted-sum scalarization and the other being a minimum step-length function, are formulated as MILP problems for the Eisenberg-Noe and Rogers-Veraart grouped systemic risk measures. Some results on boundedness and feasibility of the corresponding MILP problems are provided and their proofs can be found in Appendix B.

The results of Chapter 4 allow one to approximate systemic risk measures with a Benson type algorithm for non-convex vector optimization problems, which is introduced in [1] and described in detail in Chapter 5. The assumptions and definitions made in [1], a modification of the algorithm for this study and the corresponding pseudo-codes to approximate inner and outer approximations of the Eisenberg-Noe and Rogers-Veraart grouped systemic risk measures are provided in detail.

In Chapter 6, computational results and approximations of the Eisenberg-Noe and Rogers-Veraart grouped systemic risk measures are presented. Two- and three-group financial networks are generated with mutual liabilities and random operating cash flows. In addition, sensitivity analyses are performed for two-group networks by changing various parameters of the generated networks and the corresponding grouped systemic risk measures.

In Chapter 7, an overview of this study and some suggestions for future research are provided.

1.1 Problem Definition

This study unites two research areas: network models of systemic risk and systemic risk measures. In the scope of network models of systemic risk, there are three main objectives of this thesis. The first one is to extend the Eisenberg-Noe network model by relaxing the non-negativity assumption for operating cash flows. Clearing vectors in the original Eisenberg-Noe network model have a nice mathematical programming characterization in terms of an optimization problem with linear constraints. Hence, the second objective in this area is to formulate mathematical programming characterizations of clearing vectors in the Eisenberg-Noe network model with signed operating cash flows as well as in the Rogers-Veraart network model. To unify the impacts of different members of a financial system on the economy, systemic risk measures use aggregation functions, which are described in Section 2.3. One can refer to [3] for a general framework of this approach. Hence, the third objective in this scope is to define aggregation functions in terms of the obtained mathematical programming characterizations of clearing vectors.

From the systemic risk measures point of view, the objective of this thesis is to apply the aggregation functions, obtained from the first part of the thesis, in systemic risk measures and attempt a computation of these systemic risk measures by applying a Benson type algorithm for non-convex problems, since the studied systemic risk measures are set-valued and not-necessarily convex. In addition, it is aimed to perform sensitivity analyses in the computation of systemic risk measures by generating two-group networks with different parameters.

Chapter 2

Literature Review

In this chapter, some significant works on network models of systemic risk and on risk measures relevant to this study are reviewed. In the first part, an overview of several network models is provided. The notions of monetary (scalar) risk measure, multivariate (set-valued) risk measure and acceptance set are presented in the second part. For monetary risk measures, some general definitions given in Föllmer and Schied [4] are provided. In the last part of this chapter, works done on the cross-section of network models and risk measures are reviewed and compared.

2.1 Network Models of Systemic Risk

In this part, network models of systemic risk that were proposed by different scholars in the period from 2001 up to the present are reviewed. The foundation of this approach is given in Eisenberg and Noe [2]. Suzuki [5] is known for developing a similar model, independent from Eisenberg and Noe [2], as well as for introducing cross shareholdings into the model. Cifuentes *et al.* [6] applies Eisenberg and Noe's approach in [2] to the systems with regulatory policies which force the defaulting institutions to sell their illiquid assets. Elsinger [7] investigates the seniorities of liabilities in the system. Rogers and Veraart [8] modifies the Eisenberg-Noe model by adding bankruptcy costs in terms of limited realization of assets by defaulting banks. Weber and Weske [9] proposes to investigate all these factors in a joint model. Kabanov *et al.* [10] presents a survey on the works that are devoted to network models of systemic risk.

Eisenberg and Noe [2] is known to be the first work to model the mutual liabilities in financial systems as a directed network, where each node corresponds to a member of a financial system (e.g. a bank, fund, company or any other financial institution) and directed arcs correspond to nominal liabilities between the members of the financial system. The paper introduces the notion of a clearing payment vector, defining it as a vector of payments that should be made by each member the system in order to clear mutual liabilities. It is insisted that the following three criteria should be satisfied by a clearing vector: (1) limited liability, which means that a node cannot pay more than what it owns, (2) the priority of debt claims over the equity values of nodes, meaning that all nodes should meet their obligations either in full or until they default, and (3) proportionality, that is, a defaulting node pays each creditor a portion of its assets that is proportional to the creditor's claim on the defaulting node's assets. Two characterizations of clearing vectors are provided in the work: a fixed point characterization and a mathematical programming characterization, which are summarized below.

A system of $n \in \mathbb{N}$ interconnected institutions is modeled as a quadruplet $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$, where $\boldsymbol{\pi} \in \mathbb{R}_+^{n \times n}$ is a relative liabilities matrix, $\bar{\boldsymbol{p}} \in \mathbb{R}_+^n$ is a total obligation vector of the system, $\boldsymbol{x} \in \mathbb{R}_+^n$ is an operating cash flow vector and $\mathcal{N} = \{1, \ldots, n\}$.

According to the fixed point characterization, a clearing vector of $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ is a fixed point of a mapping $\Phi^{\mathrm{EN}_+} : [\mathbf{0}, \bar{\boldsymbol{p}}] \to [\mathbf{0}, \bar{\boldsymbol{p}}]$, where $[\mathbf{0}, \bar{\boldsymbol{p}}] = [0, \bar{p}_1] \times \ldots \times [0, \bar{p}_n]$ is the Cartesian product of n intervals, $\mathbf{0} \in \mathbb{R}^n$ is the vector of zeros and

$$\Phi^{\mathrm{EN}_+}(\boldsymbol{p}) := \left(\boldsymbol{\Pi}^{\mathsf{T}} \boldsymbol{p} + \boldsymbol{x}\right) \wedge \bar{\boldsymbol{p}}, \tag{2.1.1}$$

or more explicitly, for each $i \in \mathcal{N}$,

$$\Phi_i^{\mathrm{EN}+}(\boldsymbol{p}) = \left(\sum_{j=1}^{n} \pi_{ji} p_j + x_i\right) \wedge \bar{p}_i, \tag{2.1.2}$$

where $a \wedge b = \min\{a, b\}$ for real numbers a, b.

On the other hand, according to the mathematical programming characterization, for every strictly increasing (with respect to the componentwise ordering in $[\mathbf{0}, \bar{\boldsymbol{p}}]$) function $f : [\mathbf{0}, \bar{\boldsymbol{p}}] \to \mathbb{R}$, an optimal solution of the optimization problem

$$\begin{aligned} \max \quad & f(\boldsymbol{p}) \\ \text{s.t.} \quad & \boldsymbol{p} \leq \boldsymbol{\Pi}^{\mathsf{T}} \boldsymbol{p} + \boldsymbol{x} \\ & \boldsymbol{p} \in [\mathbf{0}, \bar{\boldsymbol{p}}] \end{aligned} \tag{2.1.3}$$

is a clearing vector of the network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$. (The constraint is understood in a componentwise manner.)

Eisenberg and Noe [2] proposes a simple and easy-to-interpret algorithm, called the *fictitious default algorithm*, to find a clearing vector of the system. Starting from the assumption that initially all nodes meet their obligations, the aim of the algorithm is to find a vector of payments at each step. At the next step, using the current vector of payments and keeping in mind defaulting nodes, it updates the vector of payments. The algorithm stops either when there is no defaulting node at the current step or when all nodes default. It is proved that this sequence of vectors of payments converge to a clearing vector in finitely many steps.

Suzuki [5] introduces a similar approach to evaluate clearing vectors (*payoff functions* in the paper) as in Eisenberg and Noe [2]. The difference is that Eisenberg and Noe [2] studies interconnectedness only in terms of liabilities, whereas Suzuki [5] considers cross-holdings of stock among members of a financial system, as well. Similar to Eisenberg and Noe [2], Suzuki [5] points out that a clearing vector of a system satisfies a fixed-point property for a general function Φ, which covers the model in Eisenberg and Noe [2] as a special case, because the model in Suzuki [5] includes cross-holdings of stock, whereas the model in [2] does not.

The paper also proposes a version of the fictitious default algorithm, which it calls the *contraction principle*, to find a clearing vector. However, unlike [2], Suzuki [5] does not provide a mathematical programming characterization of clearing vectors.

Cifuentes *et al.* [6] investigates systemic risk in terms of liquidity of institutions in a financial system. In financial markets, externally imposed solvency regulations or institutions' internal risk regulations require the sale of assets whenever there is a shock on the economy, and usually the sale decreases the prices of the assets, because the supply increases while the demand does not. In its turn, this decrease in price induces even more sales by the institutions. The whole process has a disastrous effect on a falling market. Hence, liquidity requirements on the members are at least as important as capital requirements in preventing contagious failures.

Unlike earlier works, Cifuentes *et al.* [6] considers not only direct interconnectedness in balance sheet, but also unsteadiness of asset prices. The model in the paper is based on Eisenberg and Noe's framework in [2]. However, rather than assuming that all assets of the institutions are liquid, these assets are differentiated as illiquid and liquid. The "cash" introduced in Eisenberg and Noe [2] now becomes a market value of all assets of an institution, which is a function of the prices of illiquid assets. Since a clearing vector of the financial system depends on the equity values of the entities, it also becomes a function of the prices of illiquid assets. Hence, in order to find a clearing vector, the unsteadiness of these prices should be handled.

For the sake of simplicity, it is assumed in [6] that there is only one illiquid asset. The scholars introduce a function $\Phi_{\text{price}} : [q_{\text{eq}}, 1] \to [q_{\text{eq}}, 1]$ such that

$$\Phi_{\text{price}}(q) = d^{-1}\left(\sum_i s_i(q)\right)$$

where d^{-1} is a downward sloping inverse demand function, $s_i(q)$ the amount of the illiquid asset sold by an entity i at a price q, and q_{eq} is the equilibrium price of the system. $\Phi_{\text{price}}(q)$ is interpreted as a market-clearing price of the illiquid asset when this asset is initially evaluated at price q. Hence, a fixed point of Φ_{price} gives an equilibrium price of the illiquid asset in the system. The obtained equilibrium price can be used by a regulator of a financial system in order to decrease liquidity risks and prevent contagions in the system.

Elsinger [7] extends the work in Eisenberg and Noe [2] in several ways. Firstly, the so-called cross-holdings structure is added to model a financial system, similar to the one proposed by Suzuki [5], which is done by introducing a holding matrix of proportional ownership of each institution's equity by other institutions in the system. In the model, Elsinger [7] relaxes the non-negativity assumption on the operating cash flow of an institution, claiming that insisting on non-negativity of operating cash flows would mean that all liabilities of a node except the most junior ones are always paid in full. The paper also introduces liabilities outside the network, however this detail is insignificant in the structure of the model and proofs, because it is included in the total liability of a node.

As in Suzuki [5], since the cross-holdings play a significant role in the model in [7], equity values are brought to the forefront. Given a vector of payments, a vector of equity values of a financial system must be a fixed point of a certain map given explicitly in the paper. A clearing vector is then defined in terms of this vector of equity values. In addition, a fixed-point characterization of a clearing vector is provided, which is a generalization on the version in [2]. The paper provides existence and uniqueness proofs for a clearing payment vector, as well as a uniqueness proof for a vector of equity values. In addition, a modification of the fictitious default algorithm is provided in Elsinger [7] to calculate a clearing vector. It is asserted that both the modified algorithm and the original one in Eisenberg and Noe [2] have the same interpretation in terms of institutions defaulting in different rounds depending on their exposure to systemic risk.

Furthermore, Elsinger [7] introduces a seniority structure of liabilities by assuming different classes of seniorities and modifying the matrix of related liabilities accordingly. It is claimed that this modification does not affect the results on existence and uniqueness of a clearing vector. Two approaches are proposed to calculate a clearing vector under a seniority structure of liabilities. The first one is a modification of the fictitious default algorithm mentioned above and the second one is a sequential calculation of a clearing vector starting from the most junior liabilities and assuming that all other claims of higher seniority are satisfied in full. If any institution is not able to satisfy the current level of seniority, payments are reduced next to the most junior and so on. It is pointed out that if there are no bankruptcy costs, then it is not reasonable to bail out defaulting institutions. On the other hand, under strictly positive bankruptcy costs, bailing out defaulting institutions may become reasonable. In addition, it is asserted that introducing bankruptcy costs does not affect the existence of a clearing vector.

Rogers and Veraart [8] investigates contagion in a financial system, where institutions are interconnected in a way that is presented in Eisenberg and Noe [2]. It is claimed that, in reality, failing institutions are not able to realize their assets in full to meet their obligations and this condition depresses the system even further. Thus, the model in [8] includes default (or bankruptcy) costs.

The model in the paper is based on the Eisenberg-Noe network model in [2]. A system of interconnected financial institutions is modeled as a sextuple $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$, where $\boldsymbol{\pi} \in \mathbb{R}_+^{n\times n}$ is a relative liabilities matrix, $\bar{\boldsymbol{p}} \in \mathbb{R}_+^n$ is a total obligation vector of the system, $\boldsymbol{x} \in \mathbb{R}_+^n$ is an operating cash flow vector and $\mathcal{N} = \{1, \ldots, n\}$. These parameters are in line with the Eisenberg-Noe network model. The parameters $\alpha, \beta \in (0, 1]$ represent default costs. They are fractions of the values realized from the liquidation of the defaulting institution's assets and from the payments obtained from other entities, respectively.

As in Eisenberg and Noe [2], a clearing vector is defined as an n-dimensional vector of payments made by all members of the financial system, which is a fixed point of a mapping $\Phi^{\mathrm{RV}_+} : [\mathbf{0}, \bar{\boldsymbol{p}}] \to [\mathbf{0}, \bar{\boldsymbol{p}}]$, defined as follows: for each $i \in \{1, \ldots, n\}$,

$$\Phi_i^{\mathrm{RV}+}(\boldsymbol{p}) := \begin{cases} \bar{p}_i & \text{if } \bar{p}_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j, \\ \alpha x_i + \beta \sum_{j=1}^n \pi_{ji} p_j & \text{otherwise.} \end{cases} \tag{2.1.4}$$

Even though the uniqueness of a clearing vector in this model is not guaranteed, the existence can still be proved, which is done in [8]. In addition, a modification of the fictitious default algorithm proposed in Eisenberg and Noe [2], which is called the *greatest clearing vector algorithm*, is provided for the construction of clearing vectors.

The second main focus of the work lies on the issue of bailing out failing institutions. Rogers and Veraart [8] claim that in the absence of default costs, there is no reason for solvent institutions to rescue insolvent ones. However, if there are strictly positive default costs, then it might be beneficial for some subset of solvent institutions to take over insolvent institutions. This subset of solvent institutions is called a *rescue consortium* and is characterized by two conditions, an ability to rescue insolvent institutions and an incentive to do so.

Unlike Eisenberg and Noe [2], Rogers and Veraart [8] does not provide a mathematical programming characterization of clearing vector in the network model. Defining a mixed-integer linear programming characterization of clearing vectors in Rogers-Veraart network model and implementing it in an aggregation function of a systemic risk measure is one of the main contributions of this thesis which is discussed in detail in Chapters 3 and 4.

For a detailed review of network models on systemic risk one can refer to Kabanov *et al.* [10]. It is a survey of the main results on clearing systems. The common focus in these works is the existence and uniqueness of clearing vectors in the corresponding network models of systemic risk. The survey [10] consists of several network models considered in the literature, including the ones reviewed above, and discussions about the algorithms provided in the reviewed papers for calculating fixed point solutions. In particular, Kabanov *et al.* [10] considers the models proposed in Eisenberg and Noe [2], Rogers and Veraart [8], Suzuki [5], Elsinger [7], Fisher [11] and some other models with illiquid assets and price impact.

Among the papers reviewed in Kabanov *et al.* [10], Fisher [11] adds derivative liabilities with different seniorities to the usual debts in the seniority model proposed in Elsinger [7]. Thus, the model consists of two types of liabilities, represented by two sets of matrices with seniorities. The usual direct liabilities in the model are fixed, as input parameters to the network, whereas the derivative liabilities may depend on clearing vectors and, thus, are functions of clearing vectors. Fisher [11] provides some results on existence of clearing vectors in such models.

In addition to the above works, the survey [10] reviews two more network models, where the main assumption is that the nodes in a network may own not only cash, but also several types of illiquid assets. In the first model it is assumed that institutions sell illiquid assets in equal proportions. The pricing in these assets is modeled by some monotone decreasing and continuous inverse demand function. Thus, when the clearing is applied, any node either pays its debts with cash or sells its illiquid assets to generate more cash if its initial cash amount is not enough. Conditions for existence and uniqueness of clearing vectors in such systems are provided. The second model assumes that each institution sells its illiquid assets independently from other members of the system according to its individual strategy. In such a network, the main goal of each institution is to maximize the value of its illiquid assets given a clearing vector, market prices of the assets and a total sale of each illiquid asset by the other entities.

Weber and Weske [9] integrates many of the factors that contribute to systemic risk into one network model. These factors include cross-holdings introduced in Suzuki [5] and Elsinger [7], fire sales (or “forced” sale of assets) investigated in Cifuentes *et al.* [6] and bankruptcy costs that were viewed in Elsinger [7] and Rogers and Veraart [8]. Weber and Weske [9] takes the Eisenberg-Noe network model as a base and introduces all the above factors simultaneously. The notion of equilibrium in the paper consists of two parts: a clearing vector and a clearing price of a single representative illiquid asset (for the sake of simplicity). While uniqueness of an equilibrium is not discussed, a result for its existence is provided. The paper also provides two complex algorithms that calculate the greatest price-payment equilibrium and the least price-payment equilibrium. These algorithms are based on the fictitious default algorithm introduced in Eisenberg and Noe [2] and on the procedures of calculating clearing vectors in Rogers and Veraart [8] involving bankruptcy costs.

Weber and Weske [9] also provides a series of case studies, where systemic risk factors such as bankruptcy costs, forced sales of illiquid assets and cross-holdings are investigated both jointly and separately. Having investigated these factors separately, it is concluded that bankruptcy costs and fire sales increase the threat of systemic default. On the other hand, cross-holdings seem to be beneficial, under the condition that they can be exchanged for liquid assets. Under the joint model, bankruptcy costs prove to be more significant than other factors. The paper concludes that if these costs are not too large, then a higher integration of cross-shareholdings decreases the number of defaults. Hence, for regulatory institutions, a good policy is to stimulate cross shareholdings. However, this policy seems to be inefficient for high bankruptcy costs.

2.2 Risk Measures

In this part, the literature on risk measures is summarized, including the seminal work by Artzner *et al.* [12] and works on scalar and set-valued risk measures.

2.2.1 Scalar (Univariate) Risk Measures

Quantifying risk has become a popular subject of study in late 90's. The seminal paper Artzner *et al.* [12] introduces an axiomatic approach for measuring risk. First, a risky position is defined in terms of random future values. Second, the notions of risk measure and acceptance set are introduced. Artzner *et al.* [12] provides axioms on both acceptance sets and risk measures that reflect logical behavior in financial decision making. Hence, the risk measures that comply with these axioms are called *coherent risk measures*. Artzner *et al.* [12] provides several results that relate acceptance sets and coherent risk measures.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In [12] risky financial position is defined as a random variable $X : \Omega \to \mathbb{R}$. Let $L^\infty(\mathbb{R})$ be the linear space of all essentially bounded financial positions $X : \Omega \to \mathbb{R}$, where two random variables are considered identical if they are equal $\mathbb{P}$-almost surely, and, for any $X, Y \in L^\infty(\mathbb{R})$, we write $X \leq Y$ when $\mathbb{P}\{X > Y\} = 0$. Consider the following properties for a mapping $\rho : L^\infty(\mathbb{R}) \to \mathbb{R}$.

- *Monotonicity*: $X \leq Y$ implies $\rho(X) \geq \rho(Y)$, for every $X, Y \in L^\infty(\mathbb{R})$.
- *Translation property (cash additivity)*: $\rho(X + \mu) = \rho(X) - \mu$, for every $\mu \in \mathbb{R}$, $X \in L^\infty(\mathbb{R})$.
- *Convexity*: $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for every $\lambda \in [0, 1]$, $X, Y \in L^\infty(\mathbb{R})$.
- *Positive homogeneity*: $\rho(\eta X) = \eta\rho(X)$, for every $\eta \geq 0$, $X \in L^\infty(\mathbb{R})$.

A mapping ρ is called a *monetary risk measure* if it satisfies *monotonicity* and *translation property*. Monotonicity is interpreted as follows: between two financial positions, if the future value of one of them is greater than that of the other one under any scenario, then the former one is less risky. The translation property is motivated by the interpretation of $\rho(X)$ as a capital requirement for $X \in L^\infty(\mathbb{R})$. If some deterministic amount of cash is added to X, then its capital requirement will be reduced by the same amount.

If a monetary risk measure satisfies *convexity*, then it is called a *convex risk measure*. Convexity corresponds to a thesis: "Diversification reduces risk." If, in addition to convexity, a monetary risk measure satisfies *positive homogeneity*, then it is called a *coherent risk measure*.

In Föllmer and Schied [4], for a given mapping ρ, the corresponding *acceptance* *set* is defined as

$$\mathcal{A}_\rho := \{X \in L^\infty(\mathbb{R}) | \rho(X) \leq 0\}. \tag{2.2.1}$$

The following statements summarize the relationship between ρ and $\mathcal{A}_\rho$.

- If ρ is a risk measure, then $X \in \mathcal{A}_\rho, Y \geq X$ imply $Y \in \mathcal{A}_\rho$, for every $X, Y \in L^\infty(\mathbb{R})$.
- If ρ is a convex risk measure, then $\mathcal{A}_\rho$ is a convex subset of $L^\infty(\mathbb{R})$.
- If ρ is a coherent risk measure, then $\mathcal{A}_\rho$ is a convex cone.

Moreover, ρ can be recovered from $\mathcal{A}_\rho$ by

$$\rho(X) = \inf\{\mu \in \mathbb{R} | X + \mu \in \mathcal{A}_\rho\}. \tag{2.2.2}$$

The following examples of scalar risk measures are provided in [4].

Example 2.2.1 (Worst-case risk measure). The *worst-case risk measure* ρ_{max} is defined by

$$\rho_{max}(X) = \sup_{\mathbb{Q} \in \mathcal{M}_1} \mathbb{E}_{\mathbb{Q}}[-X], \tag{2.2.3}$$

where $\mathcal{M}_1$ is the class of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$. Note that ρ_{max} is a coherent risk measure.

Example 2.2.2 (Average value at risk). The *average value at risk (or the conditional value at risk, or expected shortfall) at level* $\lambda \in (0, 1]$ of a position $X \in L^\infty(\mathbb{R})$ is given by

$$AV@R_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda V@R_\theta(X)\, d\theta, \tag{2.2.4}$$

where

$$V@R_\lambda(X) = \inf\{\mu \in \mathbb{R} | \mathbb{P}[X + \mu < 0] \leq \lambda\} \tag{2.2.5}$$

is the value at risk at level $\lambda \in (0, 1]$ of X. The average value at risk is a coherent risk measure.

Example 2.2.3 (**Entropic risk measure**)**.** The *entropic risk measure* of a position $X \in L^\infty(\mathbb{R})$ is defined by

$$\rho_\eta(X) = \frac{1}{\eta} \log \mathbb{E}[e^{-\eta X}], \tag{2.2.6}$$

where $\eta > 0$ is a given constant. The entropic risk measure is a convex but not coherent risk measure.

2.2.2 Set-Valued (Multivariate) Risk Measures

Hamel *et al.* [13] gives a general representation of multivariate risk measures and corresponding acceptance sets as follows. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $L^\infty(\mathbb{R}^n)$ be the linear space of all essentially bounded n-dimensional random variables $\boldsymbol{X} : \Omega \to \mathbb{R}^n$, where two random variables are considered identical if they are equal $\mathbb{P}$-almost surely. Consider the following properties for a set-valued mapping $R : L^\infty(\mathbb{R}^n) \to 2^{\mathbb{R}^n}$.

- *Monotonicity*: $\boldsymbol{X} \geq \boldsymbol{Y}$ implies $R(\boldsymbol{X}) \supseteq R(\boldsymbol{Y})$ for every $\boldsymbol{X}, \boldsymbol{Y} \in L^\infty(\mathbb{R}^n)$.
- *Translation property*: $R(\boldsymbol{X} + \boldsymbol{z}) = R(\boldsymbol{X}) - \boldsymbol{z}$ for every $\boldsymbol{X} \in L^\infty(\mathbb{R}^n)$, $\boldsymbol{z} \in \mathbb{R}^n$.
- *Convexity*: $R(\lambda \boldsymbol{X} + (1-\lambda)\boldsymbol{Y}) \supseteq \lambda R(\boldsymbol{X}) + (1-\lambda) R(\boldsymbol{Y})$, for every $\lambda \in (0,1)$, $\boldsymbol{X}, \boldsymbol{Y} \in L^\infty(\mathbb{R}^n)$.
- *Positive homogeneity*: $R(\lambda \boldsymbol{X}) = \lambda R(\boldsymbol{X})$, for every $\lambda \in (0, +\infty)$, $\boldsymbol{X} \in L^\infty(\mathbb{R}^n)$.

A *set-valued risk measure* is a function $R : L^\infty(\mathbb{R}^n) \to 2^{\mathbb{R}^n}$ which satisfies *monotonicity*, *translation property* and $R(0) \neq \emptyset$. For a given financial position $\boldsymbol{X}$, the set $R(\boldsymbol{X})$ consists of all capital allocation vectors that, added to $\boldsymbol{X}$ (componentwisely), make it acceptable as in monetary risk measures.

Given a set-valued risk measure $R : L^\infty(\mathbb{R}^n) \rightarrow 2^{\mathbb{R}^n}$, the corresponding acceptance set is defined as

$$\mathcal{A}_R = \{\boldsymbol{X} \in L^\infty(\mathbb{R}^n) | 0 \in R(\boldsymbol{X})\}. \tag{2.2.7}$$

In other words, a financial position $\boldsymbol{X}$ is acceptable in terms of R, if it does not require an additional capital allocation.

Hamel *et al.* [13] provides the following results that relate set-valued risk measures and acceptance sets in $L^\infty(\mathbb{R}^n)$.

- If R is a set-valued risk measure, then $\mathcal{A}_R + L^\infty\left(\mathbb{R}^n_+\right) \subseteq \mathcal{A}_R$.
- If R is a convex set-valued risk measure, then $\mathcal{A}_R$ is a convex subset of $L^\infty(\mathbb{R}^n)$.
- If R is a positively homogeneous set-valued risk measure, then $\mathcal{A}_R$ is a cone.

Moreover, a set-valued risk measure R can be recovered from $\mathcal{A}_R$ by

$$R(\boldsymbol{X}) = \{\boldsymbol{z} \in \mathbb{R}^n | \boldsymbol{X} + \boldsymbol{z} \in \mathcal{A}_R\}. \tag{2.2.8}$$

2.3 Systemic Risk Measures

Chen *et al.* [14] applies an axiomatic approach to risk measures proposed by Artzner *et al.* [12] to systemic risk. For the sake of clarity, results are presented in a financial setting. However, it is argued that the notion of systemic risk measure can be applied to analyze the risk in any system that consists of individual parts that contribute to that risk. In Chen *et al.* [14], an economy (or financial market) is defined as a matrix of random profits of finite number of firms (let there be n firms in the economy) under scenarios from Ω, a finite set of states of nature, where each column of the matrix corresponds to the profits of the firms under a particular scenario. Thus, given any probability distribution, without loss of generality one can consider the economy as a random vector $\boldsymbol{X} : \Omega \to \mathbb{R}^n$ of income profiles of the firms. Here, negative entries of $\boldsymbol{X}(\omega)$ under some scenario $\omega \in \Omega$ would imply negative incomes of the corresponding firms. Assume $L^\infty(\mathbb{R}^n)$ be a vector space of all such random vectors. Let $\mathbb{1} \in L^\infty(\mathbb{R}^n)$ be a random vector whose entries are equal to one under any scenario.

In [14], the notions of systemic risk measure and aggregation function are introduced. A *systemic risk measure* is a function $\rho^{sys} : L^\infty(\mathbb{R}^n) \to \mathbb{R}$ that satisfies the following conditions.

- *Monotonicity*: $\boldsymbol{X} \leq \boldsymbol{Y}$ implies $\rho^{sys}(\boldsymbol{X}) \geq \rho^{sys}(\boldsymbol{Y})$, for every $\boldsymbol{X}, \boldsymbol{Y} \in L^\infty(\mathbb{R}^n)$.
- *Positive homogeneity*: $\rho^{sys}(\eta \boldsymbol{X}) = \eta \rho^{sys}(\boldsymbol{X})$, for every $\eta \in \mathbb{R}_+$, $\boldsymbol{X}, \boldsymbol{Y} \in L^\infty(\mathbb{R}^n)$.
- *Preference consistency*: if $\rho^{sys}(\boldsymbol{X}(\omega)\mathbb{1}) \geq \rho^{sys}(\boldsymbol{Y}(\omega)\mathbb{1})$ for every $\omega \in \Omega$, then $\rho^{sys}(\boldsymbol{X}) \geq \rho^{sys}(\boldsymbol{Y})$, for every $\boldsymbol{X}, \boldsymbol{Y} \in L^\infty(\mathbb{R}^n)$.
- *Outcome convexity*: $\rho^{sys}(\lambda \boldsymbol{X} + (1-\lambda)\boldsymbol{Y}) \leq \lambda \rho^{sys}(\boldsymbol{X}) + (1-\lambda)\rho^{sys}(\boldsymbol{Y})$, for every $\lambda \in [0,1]$, $\boldsymbol{X}, \boldsymbol{Y} \in L^\infty(\mathbb{R}^n)$.
- *Risk convexity*: if $\rho^{sys}(\boldsymbol{Z}(\omega)\mathbb{1}) = \lambda \rho^{sys}(\boldsymbol{X}(\omega)\mathbb{1}) + (1-\lambda)\rho^{sys}(\boldsymbol{Y}(\omega))$ for every $\omega \in \Omega$ and $\lambda \in [0,1]$, then $\rho^{sys}(\boldsymbol{Z}) \leq \lambda \rho^{sys}(\boldsymbol{X}) + (1-\lambda)\rho^{sys}(\boldsymbol{Y})$, for every $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in L^\infty(\mathbb{R}^n)$.

Chen *et al.* [14] defines an *aggregation function* as a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following properties.

- *Monotonicity*: if $\boldsymbol{x} \geq \boldsymbol{y}$, then $\Lambda(\boldsymbol{x}) \geq \Lambda(\boldsymbol{y})$, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.
- *Positive homogeneity*: $\Lambda(\eta \boldsymbol{x}) = \eta \Lambda(\boldsymbol{x})$, for every $\eta \in \mathbb{R}_+$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.
- *Convexity*: $\Lambda(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda \Lambda(\boldsymbol{x}) + (1-\lambda)\Lambda(\boldsymbol{y})$, for every $\lambda \in [0,1]$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

The main result of the paper is the following theorem, which allows to extend a one-dimensional risk measure to a systemic risk measure with help of an *aggregation function*, which summarizes an income profile of the economy under some scenario into a single number, thus, making it possible to measure systemic risk via a one-dimensional risk measure.

Theorem 2.3.1. *[14, Theorem 1] A function $\rho^{sys} : L^{\infty}(\mathbb{R}^n) \to \mathbb{R}$ is a systemic risk measure if and only if there exists an aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ and a coherent one-dimensional risk measure $\rho : L^{\infty}(\mathbb{R}) \to \mathbb{R}$ such that ρ^{sys} is the composition of ρ and Λ, that is, $\rho^{sys}(\boldsymbol{X}) = \rho(\Lambda(\boldsymbol{X}))$ for every $\boldsymbol{X} \in L^{\infty}(\mathbb{R}^n)$.*

It is emphasized that the main factor that makes this result possible is the preference consistency axiom mentioned above, which is a novel axiom and one of the main contributions of the paper [14]. In addition, it is claimed that the result can be modified to cases where either monotonicity, positive homogeneity or convexity does not hold, so long as the preference consistency holds. In particular, the last part of the paper is devoted to a detailed investigation of a matter when convexity does not hold, which yields a new class of systemic risk measures called *homogeneous systemic risk measures*.

Feinstein *et al.* [15] proposes a general approach to systemic risk. In the paper, it is maintained that systemic risk consists of two components: a cash-flow model, which captures the randomness of outcomes for the entities in the system, and an acceptability criterion, which is based on the notion of acceptance set in Artzner *et al.* [12].

A cash-flow model in the framework of [15] is described in terms of a non-decreasing random field $F : \mathbb{R}^n \to L^{\infty}(\mathbb{R}^n)$, where $L^{\infty}(\mathbb{R}^n)$ is a set of n-dimensional random vectors on some probability space and for each *cash-flow vector* $\boldsymbol{z} \in \mathbb{R}^n$, $F_{\boldsymbol{z}} \in L^{\infty}(\mathbb{R}^n)$ is a random variable representing some random outcome in the system, which then can be interpreted according to the assumed setting.

For a given random field F, a systemic risk measure R is defined as a set of additional capital allocations by

$$R(F) = \{\boldsymbol{z} \in \mathbb{R}^n | F_{\boldsymbol{z}} \in \mathcal{A}\}, \tag{2.3.1}$$

where $\mathcal{A}$ is some acceptance set. The resulting systemic risk measure is a set-valued risk measure discussed in Hamel *et al.* [13].

As special cases, Feinstein *et al.* [15] provides the notions of insensitive and sensitive (to capital levels) random fields. Letting $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be an aggregation function, as defined in Chen *et al.* [14], a random field $F : \mathbb{R}^n \to L^\infty(\mathbb{R}^n)$ can be characterized with

$$F_{\boldsymbol{z}} := \Lambda(\boldsymbol{X}) + \sum_{i=1}^{n} z_i, \quad \boldsymbol{z} \in \mathbb{R}^n, \tag{2.3.2}$$

for the insensitive case, and with

$$F_{\boldsymbol{z}} := \Lambda(\boldsymbol{X} + \boldsymbol{z}), \quad \boldsymbol{z} \in \mathbb{R}^n, \tag{2.3.3}$$

for the sensitive case, where $\boldsymbol{X}$ is some n-dimensional random vector representing, for instance, the values or wealths of entities at some future date, and $\boldsymbol{z} \in \mathbb{R}^n$ is a capital level.

Axioms for risk measures, such as *translation property*, *monotonicity*, *convexity* proposed in previous works are adjusted to this framework and defined accordingly. In addition, the paper proposes a grid search algorithm to approximately solve set-valued systemic risk measures with a specified level of accuracy and provides numerical case studies.

Biagini *et al.* [3] independently proposes a general framework for systemic risk measures similar to the one in Feinstein *et al.* [15]. Unlike Feinstein *et al.* [15], where systemic risk measures are characterized as set-valued risk measures, Biagini *et al.* [3] generalize the axiomatic approach to systemic risk measures introduced in Chen *et al.* [14], where systemic risk measures are defined as compositions of an aggregation function and a monetary risk measure. In [3], systemic risk measures are interpreted as minimum capital allocations to make the corresponding systems acceptable. Here, acceptability notion is motivated by acceptance sets in Artzner *et al.* [12], but in terms of multidimensional acceptance sets, that is, subsets of $L^{\infty}(\mathbb{R}^n)$. Similar to Feinstein *et al.* [15], Biagini *et al.* [3] classifies systemic risk measures into two groups, insensitive and sensitive systemic risk measures.

Consider a system of n entities, where $\boldsymbol{X} \in L^{\infty}(\mathbb{R}^n)$ is a random vector representing random profits of entities at some fixed future date. Then, an insensitive systemic risk measure is defined as

$$\rho^{ins}(\boldsymbol{X}) := \inf \{\mu \in \mathbb{R} | \Lambda(\boldsymbol{X}) + \mu \in \mathcal{A}\}, \tag{2.3.4}$$

and interpreted as the minimum cost of recovering the system after a random shock, whereas a sensitive systemic risk measure is defined as,

$$\rho^{sen}(\boldsymbol{X}) := \inf \left\{ \sum_{i=1}^{n} z_i \middle| \boldsymbol{z} = (z_1, \ldots, z_n)^{\mathsf{T}} \in \mathbb{R}^n, \Lambda(\boldsymbol{X} + \boldsymbol{z}) \in \mathcal{A} \right\}, \tag{2.3.5}$$

and interpreted as a minimum capital allocation for each entity to avoid unacceptable consequences of a random shock. Here, $\mathcal{A} \in L^{\infty}(\mathbb{R})$ is some acceptance set which imposes some acceptability criterion, and $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function that calculates the effect that random shock $\boldsymbol{X}$ has on the economy.

Biagini *et al.* [3] generalizes systemic risk measures in multiple directions. Firstly, the notion of scenario-dependent (capital) allocations is introduced. Previously, in the context of systemic risk measures, only deterministic capital allocations were considered in the literature. Secondly, systemic risk measures are investigated under multi-dimensional acceptance sets, which makes it possible to analyze acceptability of random positions of the entities individually. The paper provides a theoretical framework, which represent a systemic risk measure and its properties under given generalizations. In addition, Biagini *et al.* [3] investigates previously studied families of systemic risk measures from this new perspective.

Ararat and Rudloff [16] studies representability of multivariate systemic risk measures from a convex duality perspective. Two types of multivariate systemic risk measures are considered in the paper, *insensitive* and *sensitive*. The definitions of set-valued systemic risk measures given in Feinstein *et al.* [15] are reformulated in the following fashion. An insensitive systemic risk measure is formulated as

$$R^{ins}\left(\boldsymbol{X}\right) := \left\{\boldsymbol{z} \in \mathbb{R}^n \middle| \Lambda\left(\boldsymbol{X}\right) + \sum_{i=1}^{n} z_i \in \mathcal{A}\right\}, \tag{2.3.6}$$

for every $\boldsymbol{X} \in L^\infty\left(\mathbb{R}^n\right)$, where $L^\infty\left(\mathbb{R}^n\right)$ is the space of n-dimensional essentially bounded random vectors. Here, $\mathcal{A} \subseteq L^\infty\left(\mathbb{R}\right)$ is an acceptance set that is in line with the notions from Artzner *et al.* [12]. It is remarked that an insensitive risk measure has its one-dimensional counter-part, a scalar systemic risk measure ρ^{ins}, formulated in Biagini *et al.* [3],

$$\rho^{ins}\left(\boldsymbol{X}\right) = \inf_{\boldsymbol{z} \in \mathbb{R}^n} \left\{\sum_{i=1}^{n} z_i \middle| \Lambda\left(\boldsymbol{X}\right) + \sum_{i=1}^{n} z_i \in \mathcal{A}\right\}, \tag{2.3.7}$$

in the sense that R^{ins} and ρ^{ins} can determine each other.

A sensitive systemic risk measure is formulated as

$$R^{sen}\left(\boldsymbol{X}\right) := \left\{\boldsymbol{z} \in \mathbb{R}^n \middle| \Lambda\left(\boldsymbol{X} + \boldsymbol{z}\right) \in \mathcal{A}\right\}. \tag{2.3.8}$$

The paper investigates the above formulations of R^{ins} and R^{sen} in the scope of the general framework for multivariate risk measures proposed by Hamel and Heyde [17]. It proves that R^{ins} is a set-valued convex risk measure that lacks translation and positive homogeneity properties in general. On the other hand, R^{sen} is proved to be a set-valued convex risk measure, which satisfies all the listed properties except positive homogeneity. However, the paper provides sufficient conditions for both R^{ins} and R^{sen} to satisfy positive homogeneity. Even though R^{sen} cannot be recovered from ρ^{ins}, due to its closedness and convexity, it can be scalarized as follows,

$$\rho^{sen}_{\boldsymbol{w}}\left(\boldsymbol{X}\right) := \inf_{\boldsymbol{z} \in \mathbb{R}^n} \left\{\boldsymbol{w}^\mathsf{T} \boldsymbol{z} \middle| \Lambda\left(\boldsymbol{X} + \boldsymbol{z}\right) \in \mathcal{A}\right\}, \tag{2.3.9}$$

where $\boldsymbol{w} \in \mathbb{R}^n_+ \backslash \left\{\boldsymbol{0}\right\}$.

The main contribution of [16] lies in dual representations for both insensitive and sensitive systemic risk measures, and for their scalarizations. These representations are formulated in terms of probability measures and weight vectors, and interpreted as the capital allocations of the entities in the presence of model uncertainty and weight ambiguity. The paper applies these representations to examples of systemic risk measures defined by some known and previously studied aggregation functions and monetary risk measures. These examples include total profit-loss, total loss, entropic, Eisenberg-Noe, resource allocation and network flow models.

Chapter 3

Network Models of Systemic Risk

In this chapter, the Eisenberg-Noe and the Rogers-Veraart network models are presented in detail. A modified model is introduced for the Eisenberg-Noe network model by assuming signed operating cash flows. For both the Eisenberg-Noe network model with signed operating cash flows and the Rogers-Veraart network model, novel mixed-integer linear programming formulations of clearing vectors are proposed. The related notation and assumptions are summarized below.

Let $n \in \{1, 2, \ldots\}$. For real numbers a, b and vectors $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathsf{T}}, \boldsymbol{b} = (b_1, \ldots, b_n)^{\mathsf{T}} \in \mathbb{R}^n$, the following operations are defined:

- $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.
- $\boldsymbol{a} \wedge \boldsymbol{b} = (a_1 \wedge b_1, \ldots, a_n \wedge b_n)^{\mathsf{T}}$ and $\boldsymbol{a} \vee \boldsymbol{b} = (a_1 \vee b_1, \ldots, a_n \vee b_n)^{\mathsf{T}}$.
- $a^+ = 0 \vee a = \max\{0, a\}$ and $a^- = 0 \vee (-a) = \max\{0, -a\}$.
- $\boldsymbol{a}^+ = \left(a_1^+, \ldots, a_n^+\right)^{\mathsf{T}}$ and $\boldsymbol{a}^- = \left(a_1^-, \ldots, a_n^-\right)^{\mathsf{T}}$.
- $\boldsymbol{a} \odot \boldsymbol{b} = (a_1 b_1, \ldots, a_n b_n)^{\mathsf{T}}$ is a Hadamard product (componentwise multiplication).
- $\mathbf{0} = (0, \ldots, 0)^{\mathsf{T}} \in \mathbb{R}^n$ is the vector of zeros.

- $\mathbb{1} = (1, \ldots, 1)^\mathsf{T} \in \mathbb{R}^n$ is the vector of ones.

- $\boldsymbol{a} \leq \boldsymbol{b}$ if and only if $a_i \leq b_i$ for each $i \in \{1, \ldots, n\}$.

- Assume $\boldsymbol{a} \leq \boldsymbol{b}$, then $[\boldsymbol{a}, \boldsymbol{b}] = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is a Cartesian product of n intervals.

- $\|\boldsymbol{a}\|_\infty = \max_{i \in \{1,\ldots,n\}} |a_i|$.

3.1 Eisenberg-Noe Network Model

In this section, the original Eisenberg-Noe network model in [2] and the corresponding aggregation function are provided for completeness.

Definition 3.1.1. *A quadruple $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ is called an Eisenberg-Noe network if $\mathcal{N} = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$, $\boldsymbol{\pi} = (\pi_{ij})_{i,j \in \mathcal{N}} \in \mathbb{R}_+^{n \times n}$ is a stochastic matrix with $\pi_{ii} = 0$ and $\sum_{j=1}^n \pi_{ji} < n$ for each $i \in \mathcal{N}$, $\bar{\boldsymbol{p}} = (\bar{p}_1, \ldots, \bar{p}_n)^\mathsf{T} \in \mathbb{R}_{++}^n$, and $\boldsymbol{x} = (x_1, \ldots, x_n)^\mathsf{T} \in \mathbb{R}_+^n$.*

In Definition 3.1.1, $\mathcal{N}$ is an index set of nodes in a network that represents a financial system of n institutions. For every $i \in \mathcal{N}$, $\bar{p}_i > 0$ is the total amount of liabilities of node i and the vector $\bar{\boldsymbol{p}}$ is called the *total obligation vector*.

For every $i, j \in \mathcal{N}$ such that $i \neq j$, $\pi_{ij} > 0$ is the fraction of the total liability of node i owed to node j and the stochastic matrix $\boldsymbol{\pi}$ is called the *matrix of relative liabilities*. For every $i \in \mathcal{N}$, the assumption $\pi_{ii} = 0$ implies that node i cannot have liabilities to itself. By $\sum_{j=1}^n \pi_{ji} < n$ for every $i \in \mathcal{N}$, it is assumed that no node owns all the claims in the network. Note that, given $\bar{\boldsymbol{p}}$ and $\boldsymbol{\pi}$, for every $i, j \in \mathcal{N}$, the nominal liability of node i to node j, l_{ij}, can be calculated as $l_{ij} = \pi_{ij}\bar{p}_i$.

For each $i \in \mathcal{N}$, $x_i \geq 0$ is the operating cash flow of node i and the vector $\boldsymbol{x}$ is called the *operating cash flow vector*.

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ be an Eisenberg-Noe network. For each $i \in \mathcal{N}$, let $p_i \geq 0$ be the sum of all payments made by node i to the other nodes in the network. Then $\boldsymbol{p} = (p_1, \ldots, p_n)^\mathsf{T} \in \mathbb{R}^n_+$ is called a *payment vector.*

Definition 3.1.2. *A vector $\boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ is called a* clearing vector *for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ if it satisfies the following properties:*

- Limited liability*: for each $i \in \mathcal{N}, p_i \leq \sum_{j=1}^n \pi_{ji} p_j + x_i$, which implies that node i cannot pay more than it has.*
- Absolute priority*: for each $i \in \mathcal{N}$, either $p_i = \bar{p}_i$ or $p_i = \sum_{j=1}^n \pi_{ji} p_j + x_i$, which implies that node i has to meet its obligations in full. Otherwise, it pays as much as it has.*

Definition 3.1.3. *Let $\Phi^{EN_+} : [\boldsymbol{0}, \bar{\boldsymbol{p}}] \to [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ be defined by*

$$\Phi^{EN_+}(\boldsymbol{p}) := \left(\boldsymbol{\pi}^\mathsf{T} \boldsymbol{p} + \boldsymbol{x}\right) \wedge \bar{\boldsymbol{p}}. \tag{3.1.1}$$

Remark 3.1.4. By a discussion in [2], a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ is a fixed point of the mapping Φ^{EN_+} in (3.1.1).

Recall, from (2.1.3), the relation between the optimization problem with linear constraints in Eisenberg and Noe [2] and the fixed point problem $\Phi^{\mathrm{EN}_+}(\boldsymbol{p}) = \boldsymbol{p}$. Its proof is given for completeness, as well as for its generalizations in the coming sections. Note that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called strictly increasing if and only if $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{a} \neq \boldsymbol{b}$ imply $f(\boldsymbol{a}) < f(\boldsymbol{b})$ for every $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$.

Proposition 3.1.5. *[2, Lemma 4] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly increasing function. Consider the following optimization problem with linear constraints:*

$$\begin{aligned} \max \quad & f(\boldsymbol{p}) \\ s.t. \quad & \boldsymbol{p} \leq \boldsymbol{\pi}^\mathsf{T} \boldsymbol{p} + \boldsymbol{x} \\ & \boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]. \end{aligned} \tag{3.1.2}$$

If $\boldsymbol{p} \in \mathbb{R}^n_+$ is an optimal solution to this optimization problem, then it is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$.

Proof. Let $\boldsymbol{p}$ be an optimal solution to (3.1.2). Then $\boldsymbol{p}$ satisfies *limited liability* by the feasibility of the constraints $\boldsymbol{p} \leq \boldsymbol{\pi}^{\mathsf{T}}\boldsymbol{p} + \boldsymbol{x}$.

Now assume $\boldsymbol{p}$ does not satisfy *absolute priority.* Then, there exists a node $i \in \mathcal{N}$ such that

$$p_i < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \quad \text{and} \quad p_i < \bar{p}_i.$$

Now let $\boldsymbol{p}^{\boldsymbol{\epsilon}} \in \mathbb{R}^n$ be equal to $\boldsymbol{p}$ in all components except the i^{th} one, and let

$$p_i^{\epsilon} = p_i + \epsilon,$$

where $\epsilon > 0$ is sufficiently small $\left(\text{for instance, } \epsilon = \min\left\{\frac{\bar{p}_i - p_i}{2}, \frac{\sum_{j=1}^{n} \pi_{ji} p_j + x_i - p_i}{2}\right\}\right)$ to ensure

$$p_i^{\epsilon} < \bar{p}_i \quad \text{and} \quad p_i^{\epsilon} < \sum_{j=1}^{n} \pi_{ji} p_j^{\epsilon} + x_i.$$

Now, for each $k \in \mathcal{N}$ such that $k \neq i$,

$$\sum_{j=1}^{n} \pi_{jk} p_j^{\epsilon} + x_k = \sum_{\substack{j \in \mathcal{N} \\ j \neq i}} \pi_{jk} p_j + \pi_{ik}\left(p_i + \epsilon\right) + x_k = \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} \geq p_k = p_k^{\epsilon},$$

by the feasibility of $\boldsymbol{p}$. Hence, $\boldsymbol{p}^{\boldsymbol{\epsilon}}$ is a feasible solution to (3.1.2).

Since $\boldsymbol{p}^{\boldsymbol{\epsilon}} \geq \boldsymbol{p}$ with $\boldsymbol{p}^{\boldsymbol{\epsilon}} \neq \boldsymbol{p}$ and f is a strictly increasing function, it holds $f\left(\boldsymbol{p}^{\boldsymbol{\epsilon}}\right) > f\left(\boldsymbol{p}\right)$, which is a contradiction to the optimality of $\boldsymbol{p}$. Hence, $\boldsymbol{p}$ satisfies *absolute priority* and is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$. □

Remark 3.1.6. Observe that the optimization problem in Proposition 3.1.5 can be reformulated as

$$\begin{aligned} \max \quad & f\left(\boldsymbol{p}\right) \\ \text{s.t.} \quad & A\boldsymbol{p} \leq b \\ & \boldsymbol{p} \geq \mathbf{0} \end{aligned} \tag{3.1.3}$$

where

$$A = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{\pi}^{\mathsf{T}} \\ \boldsymbol{I} \end{bmatrix} \in \mathbb{R}^{2n \times n}, \qquad b = \begin{bmatrix} \boldsymbol{x} \\ \bar{\boldsymbol{p}} \end{bmatrix} \in \mathbb{R}^{2n},$$

and $\boldsymbol{I}$ is the $n \times n$ identity matrix.

Each member in a network has its impact on economy. Aggregation functions summarize these individual effects and provide a total impact of the network on economy. They play a significant role in evaluating systemic risks and in the computation of systemic risk measures. The aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ for the Eisenberg-Noe network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ is defined as

$$\Lambda(\boldsymbol{x}) := \sup\left\{ f(\boldsymbol{p}) \,\middle|\, \boldsymbol{p} \leq \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p} + \boldsymbol{x}, \boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}] \right\}, \tag{3.1.4}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly increasing function.

3.2 Signed Eisenberg-Noe Network Model

In the original Eisenberg-Noe network model, it is assumed that the operating cash flow vector is nonnegative. In reality, however, it is not always the case. It may happen that an institution has liabilities to external entities not modeled as part of the network resulting in a negative *operating cash flow* or positive *operating costs.*

Definition 3.2.1. *A quadruple* $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ *is called a signed Eisenberg-Noe network if* $\mathcal{N}$, $\boldsymbol{\pi}$ *and* $\bar{\boldsymbol{p}}$ *are defined as in Definition 3.1.1, and* $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathsf{T}} \in \mathbb{R}^n$.

Note that Definition 3.2.1 removes the nonnegativity assumption on the operating cash flow vector $\boldsymbol{x}$. The objective is to modify the original Eisenberg-Noe network model and calculate systemic risk measures for this network model. Two approaches are considered to reach this objective.

The first approach, given in Section 3.2.1 below, is provided for completeness and motivated by a conjecture in Eisenberg and Noe [2], stating that, given $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ with a signed operating cash flow, negative operating cash flows in some nodes can be regarded as liabilities to some additional node, which itself has neither obligations nor operating cash flow, and that is why the operating cash flow vector $\boldsymbol{x}$ can be assumed to be nonnegative without loss of generality. Applying this conjecture directly without any seniority assumptions, a new network $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ of $n+1$ nodes is introduced, where the matrix of relative liabilities $\tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}$ and the total obligation vector $\tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}$ depend on the signed operating cash flow vector $\boldsymbol{x}$ from the initially given network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$. It turns out that the obtained network $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ lacks a solid interpretation in terms of the original network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$, even though this approach is intuitive and valid for the fictitious default algorithm described in Eisenberg and Noe [2], in the sense that this way a clearing vector for the original network can be found. Nevertheless, this approach is provided in detail to justify and give some insight for the second approach.

In the second approach, some seniority conditions on performing clearing are imposed, which results in modifying not the network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ itself, but the mapping Φ^{EN_+} in (3.1.1). The resulting network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ is still a network with n nodes, however, a clearing vector for the network is now determined by solving a fixed point problem of the new mapping Φ^{EN}, which is discussed in more detail in Section 3.2.2.

3.2.1 A Naive Approach

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ be a signed Eisenberg-Noe network. In this approach, Eisenberg and Noe's conjecture is applied directly, which states that any such network can be extended by an additional node, which may be seen as "society," to which each node with a negative operating cash flow owe the absolute value of that amount. Hence, $\mathcal{N}$ is replaced with a new index set $\tilde{\mathcal{N}} = \mathcal{N} \cup \{n+1\} = \{1, \ldots, n+1\}$. If the previous section is followed and an aggregation function in terms of an LP problem is formulated, then the resulting optimization problem appears to be non-linear in $\boldsymbol{x}$, as discussed below.

Consider negative entries of the *operating cash flow vector* $\boldsymbol{x}$ as liabilities of the corresponding nodes to the additional node, "society." Now the *total obligation vector*, the *matrix of relative liabilities* and the *operating cash flow vector* of the extended network can be constructed as follows.

For every $i \in \tilde{\mathcal{N}}$, let the total amount of liabilities of node i be defined as

$$\tilde{\bar{p}}_i := \begin{cases} \bar{p}_i + x_i^- & \text{if } i \in \mathcal{N}, \\ 0 & \text{if } i = n+1. \end{cases}$$

The vector $\tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}} = (\tilde{\bar{p}}_1, \ldots, \tilde{\bar{p}}_{n+1})^{\mathsf{T}} \in \mathbb{R}_+^n$ is called the *extended total obligation vector*. Observe that $\tilde{\bar{p}}_{n+1} = 0$ because the "society" does not have any obligations to the nodes.

For every $i \in \tilde{\mathcal{N}}, j \in \tilde{\mathcal{N}}$, let the fraction of the total liability of node i owed to node j be defined as

$$\tilde{\pi}_{ij} := \begin{cases} \frac{\pi_{ij}\bar{p}_i}{\bar{p}_i + x_i^-} & \text{if } i, j \in \mathcal{N}, \\ \frac{x_i^-}{\bar{p}_i + x_i^-} & \text{if } i \in \mathcal{N}, j = n+1, \\ 0 & \text{if } i = n+1. \end{cases} \tag{3.2.1}$$

The matrix $\tilde{\boldsymbol{\pi}}_{\boldsymbol{x}} = (\tilde{\pi}_{ij})_{i,j} \in \mathbb{R}_+^{(n+1)\times(n+1)}$ is called the *extended matrix of relative liabilities*. Then, for each $i, j \in \tilde{\mathcal{N}}$, a liability of node i to node j is defined by $\tilde{l}_{ij} := \tilde{\pi}_{ij}\tilde{\bar{p}}_i$. Hence, liabilities between nodes $i, j \in \mathcal{N}$ remain the same, any negative operating cash flow of node $i \in \mathcal{N}$ becomes a liability to node $n+1$, society, and society itself does not have any obligations to the other nodes. For each $i \in \tilde{\mathcal{N}}$, $\tilde{l}_{ii} = 0$ still holds. In other words, a node cannot have liabilities to itself.

Now, for every $i \in \tilde{\mathcal{N}}$, let the nonnegative operating cash flow of node i be defined as

$$\tilde{x}_i = \begin{cases} x_i^+ & \text{if } i \in \mathcal{N}, \\ 0 & \text{if } i = n+1. \end{cases}$$

The vector $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \ldots, \tilde{x}_{n+1})^{\mathsf{T}} \in \mathbb{R}_+^n$ is called the *extended operating cash flow* *vector.* Even though $\tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}$ is not a stochastic matrix and the modified network $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ is not an Eisenberg-Noe network in the sense of Definition 3.1.1, $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ still satisfies the original Eisenberg-Noe network definition with $n+1$ nodes described in [2] since $\tilde{\boldsymbol{x}}$ is nonnegative.

If $\boldsymbol{x}$ is nonnegative, then $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ reduces an Eisenberg-Noe network originally described in [2] with n nodes and one isolated node, which has no relationship with the other nodes in the sense of mutual liabilities.

Let $i \in \tilde{\mathcal{N}}$ and p_i the sum of all payments done by node i to all other nodes in the network. Then $\boldsymbol{p} = (p_1, \ldots, p_{n+1})^{\mathsf{T}} \in \mathbb{R}_+^{n+1}$ is a *payment vector.* A vector $\boldsymbol{p} \in \left[\boldsymbol{0}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}\right]$ is a *clearing vector* for $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ if it satisfies *limited liability* and *absolute priority* in Definition 3.1.2. For a clearing vector $\boldsymbol{p} = (p_1, \ldots, p_{n+1})^{\mathsf{T}}$ for $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$, it can be observed that $p_{n+1} = 0$ by *absolute priority*, because the society does not have any liabilities inside the network.

According to the fixed point characterization in Eisenberg and Noe [2], a clearing vector for $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ is a fixed point of a mapping $\tilde{\Phi}^{\mathrm{EN}} : \left[\boldsymbol{0}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}\right] \to \left[\boldsymbol{0}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}\right]$, where

$$\tilde{\Phi}^{\mathrm{EN}}(\boldsymbol{p}) := \left(\tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}^{\mathsf{T}} \boldsymbol{p} + \tilde{\boldsymbol{x}}\right) \wedge \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}. \tag{3.2.2}$$

By Eisenberg and Noe [2], $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ has a clearing vector, or, in other words, the fixed point problem $\tilde{\Phi}^{\mathrm{EN}_+}(\boldsymbol{p}) = \boldsymbol{p}$ has a solution in $\left[\boldsymbol{0}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}\right]$, where $\boldsymbol{0}, \boldsymbol{p} \in \mathbb{R}^{n+1}$.

Remark 3.2.2. Observe that if each node in $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ has a nonnegative operating cash flow then $\tilde{\Phi}^{\mathrm{EN}}$ becomes a simple extension of the function Φ^{EN_+} in the original Eisenberg-Noe network model from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$.

Recall Proposition 3.1.5, the relationship between the optimization problem in (3.1.2) and the fixed point problem $\Phi^{\mathrm{EN}_+}(\boldsymbol{p}) = \boldsymbol{p}$. A similar result is provided for $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$.

Corollary 3.2.3. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing function. Consider* *the following optimization problem:*

$$\begin{aligned} \max \quad & f(\boldsymbol{p}) \\ s.t. \quad & \boldsymbol{p} \leq \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}^{\mathsf{T}} \boldsymbol{p} + \boldsymbol{x}^{+} \\ & \boldsymbol{p} \in \left[\boldsymbol{0}, \bar{\boldsymbol{p}} + \boldsymbol{x}^{-}\right] \end{aligned} \tag{3.2.3}$$

If $\boldsymbol{p} \in \mathbb{R}_{+}^{n+1}$ *is an optimal solution to this optimization problem, then it is a clearing vector for* $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$.

The proof of Corollary 3.2.3 follows directly from Proposition 3.1.5 because $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ is a network of $n+1$ nodes described in the original network model in Eisenberg and Noe [2].

The aggregation function $\tilde{\Lambda} : \mathbb{R}^n \to \mathbb{R}$ for $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ is defined as

$$\tilde{\Lambda}(\boldsymbol{x}) := \sup\left\{ f(\boldsymbol{p}) \,\middle|\, \boldsymbol{p} \leq \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}^{\mathsf{T}} \boldsymbol{p} + \boldsymbol{x}^{+}, \boldsymbol{p} \in \left[\boldsymbol{0}, \bar{\boldsymbol{p}} + \boldsymbol{x}^{-}\right] \right\}, \tag{3.2.4}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a strictly increasing function. Observe that the constraint is not linear in $\boldsymbol{x}$ because $\tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}$ and $\boldsymbol{x}^{+}$ are not linear in $\boldsymbol{x}$.

Even though it is possible to find a clearing vector for $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ by applying the fictitious default algorithm in Eisenberg and Noe [2], it is not clear how to interpret it for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ and there is no guarantee that a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ exists, since the original proof in Eisenberg and Noe [2] on existence of a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ assumes $\boldsymbol{x}$ is nonnegative.

This approach has the following major drawback. Given a signed network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$, a network $\left(\tilde{\mathcal{N}}, \tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}, \tilde{\bar{\boldsymbol{p}}}_{\boldsymbol{x}}, \tilde{\boldsymbol{x}}_{\boldsymbol{x}}\right)$ defined as described above is a totally different network, due to the structure of $\tilde{\boldsymbol{\pi}}_{\boldsymbol{x}}$, and cannot be interpreted in terms of $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$. The main reason of this inconvenience is the absence of seniority between society and the other nodes in the network. Nevertheless, motivated by this observation, it is a good idea to impose some seniority between society and the other nodes, which is described in detail in the following section.

3.2.2 A Seniority-Based Approach

In this section, the second approach on modeling a signed Eisenberg-Noe network is described in detail. The definition of a clearing vector in Definition 3.1.2 and the mapping Φ^{EN_+} in (3.1.1) are modified accordingly. Both a fixed-point and a mathematical programming characterization of a clearing vector for such networks are provided. In addition, a mixed-integer linear programming aggregation function in the scope of the signed Eisenberg-Noe network model is introduced.

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ be a signed Eisenberg-Noe network. In this approach, it is assumed that the nodes having obligations outside the network, that is, the nodes having negative operating cash flows have to meet these obligations first, and if they do not default in the first round, then they should meet their obligations to the other nodes inside the network. At this step, as in the original Eisenberg-Noe network model, they either meet their obligations to the other nodes in full or pay as much as they have at hand and default. Hence, the following definition for clearing vectors is introduced.

Definition 3.2.4. *A vector $\boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ is called a* clearing vector *for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ if it satisfies the following properties:*

- Immediate default*: for each $i \in \mathcal{N}$, $\sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0$ implies $p_i = 0$.*
- Limited liability*: for each $i \in \mathcal{N}$, if $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0$, then $p_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i$, which implies that if node i has a strictly positive operating cash flow, then it cannot pay more than it has.*
- Absolute priority*: for each $i \in \mathcal{N}$, if $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0$, then either $p_i = \bar{p}_i$ or $p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i$, which implies that if node i has a strictly positive operating cash flow, then it has to meet its obligations in full. Otherwise, it pays as much as it has.*

Definition 3.2.5. *Let $\Phi^{EN} : [\boldsymbol{0}, \bar{\boldsymbol{p}}] \to [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ be defined by*

$$\Phi^{EN}(\boldsymbol{p}) := \left(\bar{\boldsymbol{p}} \wedge \left(\boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p} + \boldsymbol{x}\right)\right)^{+}, \tag{3.2.5}$$

or more explicitly, for each $i \in \mathcal{N}$,

$$\Phi_i^{EN}(\boldsymbol{p}) = \begin{cases} 0 & \text{if } \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0, \\ \sum_{j=1}^{n} \pi_{ji} p_j + x_i & \text{if } 0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i, \\ \bar{p}_i & \text{if } \sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i. \end{cases} \tag{3.2.6}$$

Proposition 3.2.6. *A vector $\boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$ if and only if it is a fixed point of the mapping Φ^{EN}.*

Proof. To prove the "only if" part, let $\boldsymbol{p} = (p_1, \ldots, p_n)^\mathsf{T} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ be a *clearing vector*. To show that $\boldsymbol{p}$ is a fixed point of the mapping Φ^{EN}, let $i \in \mathcal{N}$.

If $\sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0$, then $p_i = 0$, by *immediate default*, and $\Phi_i^{\text{EN}}(\boldsymbol{p}) = 0$, by Definition 3.2.5. Hence, $\Phi_i^{\text{EN}}(\boldsymbol{p}) = p_i$.

If $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0$, then, by *absolute priority*, either $p_i = \bar{p}_i$ or $p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i$. If $p_i = \bar{p}_i$, then, by *limited liability*, $\bar{p}_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i$ and, thus, by Definition 3.2.5, $\Phi_i^{\text{EN}}(\boldsymbol{p}) = \bar{p}_i$. Hence, $\Phi_i^{\text{EN}}(\boldsymbol{p}) = p_i$. On the other hand, if $p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i < \bar{p}_i$ then, by Definition 3.2.5, $\Phi_i^{\text{EN}}(\boldsymbol{p}) = \sum_{j=1}^{n} \pi_{ji} p_j + x_i$. Hence, again $\Phi_i^{\text{EN}}(\boldsymbol{p}) = p_i$. Thus, $\boldsymbol{p}$ is a fixed point of Φ^{EN}.

To prove the "if" part, let $\boldsymbol{p} = (p_1, \ldots, p_n)^\mathsf{T}$ be a fixed point of Φ^{EN}. In other words, for every $i \in \mathcal{N}$, $\Phi_i^{\text{EN}}(\boldsymbol{p}) = p_i$. To show that $\boldsymbol{p}$ is a *clearing vector*, let $i \in \mathcal{N}$.

If $\sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq 0$, then $\Phi_i^{\text{EN}}(\boldsymbol{p}) = p_i = 0$, by Definition 3.2.5. Hence, *immediate default* holds.

If $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0$, then $\Phi_i^{\text{EN}}(\boldsymbol{p}) = p_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i$, by Definition 3.2.5. Hence, *limited liability* holds.

Now assume $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0$. If $\sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i$, then $\Phi^{\text{EN}}(\boldsymbol{p}) = p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i$. If $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i$, then $\Phi^{\text{EN}}(\boldsymbol{p}) = p_i = \bar{p}_i$, by Definition 3.2.5. Hence, *absolute priority* holds as well. Hence, $\boldsymbol{p}$ is a clearing vector. $\square$

Remark 3.2.7. Observe that, if $\boldsymbol{x} \in \mathbb{R}^n_+$, then Φ^{EN} coincides with the function Φ^{EN_+} in (3.1.1) defined for the original Eisenberg-Noe network model.

For every $\boldsymbol{a} \in \mathbb{R}^{n_1}$, $\boldsymbol{b} \in \mathbb{R}^{n_2}$, let $(\boldsymbol{a}, \boldsymbol{b}) = (a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2})^{\mathsf{T}} \in \mathbb{R}^{n_1+n_2}$ be a vector concatenation.

Theorem 3.2.8. *Let $\Lambda^{EN} : \mathbb{R}^n \to \mathbb{R}$ be the following mixed-integer linear programming (MILP) aggregation function*

$$\begin{aligned}\Lambda^{EN}(\boldsymbol{y}) := \sup \Big\{ f(\boldsymbol{p}) \Big| \boldsymbol{p} \le \left[\boldsymbol{\pi}^{\mathsf{T}}\boldsymbol{p} + \boldsymbol{y} + M(\mathbb{1} - \boldsymbol{s})\right] \wedge (\bar{\boldsymbol{p}} \odot \boldsymbol{s}), \\ \boldsymbol{\pi}^{\mathsf{T}}\boldsymbol{p} + \boldsymbol{y} \le M\boldsymbol{s}, \boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s} \in \{0,1\}^n \Big\},\end{aligned} \tag{3.2.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly increasing linear function and $M = n\left\|\bar{\boldsymbol{p}}\right\|_\infty + \left\|\boldsymbol{y}\right\|_\infty$.

If $(\boldsymbol{p}, \boldsymbol{s})$ is an optimal solution to MILP for $\Lambda^{EN}(\boldsymbol{x})$, then $\boldsymbol{p}$ is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$.

Observe that $\Lambda^{\mathrm{EN}}(\boldsymbol{x})$ can be written more explicitly as

$$\text{maximize} \quad f(\boldsymbol{p}) \tag{3.2.8}$$

$$\text{subject to} \quad p_i \le \sum_{j=1}^{n} \pi_{ji} p_j + x_i + M(1 - s_i), \qquad i \in \mathcal{N}, \tag{3.2.9}$$

$$p_i \le \bar{p}_i s_i, \qquad i \in \mathcal{N}, \tag{3.2.10}$$

$$\sum_{j=1}^{n} \pi_{ji} p_j + x_i \le M s_i, \qquad i \in \mathcal{N}, \tag{3.2.11}$$

$$0 \le p_i \le \bar{p}_i, \qquad i \in \mathcal{N}, \tag{3.2.12}$$

$$s_i \in \{0,1\}, \qquad i \in \mathcal{N}. \tag{3.2.13}$$

Let $\boldsymbol{u} = (u_1, \ldots, u_n)^{\mathsf{T}} \in \{0,1\}^n$ be a binary vector, where $u_i = 0$ if $x_i < 0$, and $u_i = 1$ if $x_i \ge 0$, for each $i \in \mathcal{N}$. Then $(\boldsymbol{p}, \boldsymbol{s}) = (\boldsymbol{0}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{Z}^n$ is a feasible solution to the MILP in (3.2.8). Moreover, if f is a bounded function on the interval $[\mathbf{0}, \bar{\boldsymbol{p}}] \subseteq \mathbb{R}^n_+$, then by Meyer [18, Theorem 2.1], the MILP in (3.2.8) has an optimal solution. Observe that, by Theorem 3.2.8, the existence of an optimal solution to the MILP in (3.2.8) proves the existence of a clearing vector for the network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$.

Remark 3.2.9. In Theorem 3.2.8, $M = n \|\bar{\boldsymbol{p}}\|_\infty + \|\boldsymbol{x}\|_\infty$ is taken to ensure the feasibility in the constraint (3.2.11). In other words, it is enough to choose M such that $\sum_{j=1}^n \pi_{ji} p_j + x_i \leq M$, for each $i \in \mathcal{N}$ and for every $\boldsymbol{p} \in [\mathbf{0}, \bar{\boldsymbol{p}}]$. Furthermore, for each $i \in \mathcal{N}$ and for every $\boldsymbol{p} \in [\mathbf{0}, \bar{\boldsymbol{p}}]$, since $\sum_{j=1}^n \pi_{ji} < n$, it holds $\sum_{j=1}^n \pi_{ji} p_j < n \|\bar{\boldsymbol{p}}\|_\infty$. Hence, $\sum_{j=1}^n \pi_{ji} p_j + x_i \leq n \|\bar{\boldsymbol{p}}\|_\infty + \|\boldsymbol{x}\|_\infty = M$

Remark 3.2.10. Linearity of f is not a necessary condition for Theorem 3.2.8 to hold.

The proof of Theorem 3.2.8 is based on the following lemma.

Lemma 3.2.11. *Let $(\boldsymbol{p}, \boldsymbol{s})$ be an optimal solution to the MILP for $\Lambda^{EN}(\boldsymbol{x})$. Let $i \in \mathcal{N}$ such that $0 < \sum_{j=1}^n \pi_{ji} p_j + x_i$. Then, $p_i = \min\left\{\sum_{j=1}^n \pi_{ji} p_j + x_i, \bar{p}_i\right\}$.*

The proofs of Lemma 3.2.11 and Theorem 3.2.8 can be found in Appendices A.1 and A.2, respectively.

Remark 3.2.12. The aggregation function Λ^{EN} in (3.2.7) is applied in Chapter 4 to define the Eisenberg-Noe grouped systemic risk measures.

3.3 Rogers-Veraart Network Model

Rogers and Veraart [8] extends the original Eisenberg-Noe network model by introducing default costs. It is assumed that a defaulting node is not able to use all of its liquid assets to satisfy its creditors. Unlike Eisenberg and Noe [2], Rogers and Veraart [8] does not provide a programming formulation for clearing vectors in the network model. This gap is filled here by proposing an MILP whose optimal solution includes a clearing vector for the Rogers-Veraart network model. Hence, a mathematical characterization for a clearing vector is obtained for this model for the first time. In addition, an aggregation function based on this characterization is defined and its relationship to the network model is provided. Finally, inspired by Definition 3.1.2, a weak definition of a clearing vector for the Rogers-Veraart network model is proposed.

Definition 3.3.1. *A sextuple* $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$ *is called a Rogers-Veraart network if* $\mathcal{N} = \{1, \ldots, n\}$ *for some* $n \in \mathbb{N}$, $\boldsymbol{\pi} = (\pi_{ij})_{i,j\in\mathcal{N}} \in \mathbb{R}_+^{n\times n}$ *is a stochastic matrix with* $\pi_{ii} = 0$ *and* $\sum_{j=1}^{n} \pi_{ji} < n$ *for each* $i \in \mathcal{N}$, $\bar{\boldsymbol{p}} = (\bar{p}_1, \ldots, \bar{p}_n)^{\mathsf{T}} \in \mathbb{R}_{++}^n$, $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathsf{T}} \in \mathbb{R}_+^n$ *and* $\alpha, \beta \in (0, 1]$.

As in Definition 3.1.1, $\mathcal{N}$ is the index set of nodes in a network that represents a financial system of n institutions, $\bar{\boldsymbol{p}}$ is the *total obligation vector*, $\boldsymbol{\pi}$ is the *matrix of relative liabilities* and $\boldsymbol{x}$ is the *operating cash flow vector*. As part of the network model, it is assumed that a defaulting node may not be able to use all of its liquid assets to meet its obligations. Hence, α denotes the fraction of the operating cash flow and β denotes the fraction of cash inflow from other nodes that can be used by a defaulting node to meet its obligations.

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$ be a Rogers-Veraart network. For each $i \in \mathcal{N}$, let $p_i \geq 0$ be the sum of all payments made by node i to the other nodes in the network. Then $\boldsymbol{p} = (p_1, \ldots, p_n)^{\mathsf{T}} \in \mathbb{R}_+^n$ is called a *payment vector*.

Motivated by Definition 3.1.2 of a clearing vector for an Eisenberg-Noe network, the following similar definition of a clearing vector for a Rogers-Veraart network is proposed.

Definition 3.3.2. *A vector* $\boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ *is called a* clearing vector *for* $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$ *if it satisfies the following properties:*

- Limited liability*: for each* $i \in \mathcal{N}, p_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j$*, which implies that node* i *cannot pay more than it has.*

- Absolute priority*: for each* $i \in \mathcal{N}$*, either* $p_i = \bar{p}_i$ *or* $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$*, which implies that node* i *has to meet its obligations in full. Otherwise, it has to pay as much as it can.*

Definition 3.3.3. *Let* $\Phi^{RV_+} : [\mathbf{0}, \bar{\boldsymbol{p}}] \to [\mathbf{0}, \bar{\boldsymbol{p}}]$ *be defined by*

$$\Phi_i^{RV_+}(\boldsymbol{p}) := \begin{cases} \bar{p}_i & \text{if } \bar{p}_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j, \\ \alpha x_i + \beta \sum_{j=1}^n \pi_{ji} p_j & \text{if } \bar{p}_i > x_i + \sum_{j=1}^n \pi_{ji} p_j, \end{cases} \tag{3.3.1}$$

for each $i \in \mathcal{N}$.

Remark 3.3.4. Observe that, if $\alpha = 1$ and $\beta = 1$, then the function Φ^{RV_+} becomes the usual Φ^{EN_+} in (3.1.1) from the original Eisenberg-Noe network model.

Proposition 3.3.5. *A fixed point* $\boldsymbol{p} \in [\mathbf{0}, \bar{\boldsymbol{p}}]$ *of* Φ^{RV_+} *is a clearing vector for* $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$.

Proof. Let $\boldsymbol{p} = (p_1, \ldots, p_n)^{\mathsf{T}}$ be a fixed point of the mapping Φ^{RV_+}. To show that $\boldsymbol{p}$ is a *clearing vector* for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$, let $i \in \mathcal{N}$.

If $\bar{p}_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j$, then $\Phi_i^{\mathrm{RV}_+}(\boldsymbol{p}) = \bar{p}_i = p_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j$, and if $\bar{p}_i > x_i + \sum_{j=1}^n \pi_{ji} p_j$, then $\Phi_i^{\mathrm{RV}_+}(\boldsymbol{p}) = \alpha x_i + \beta \sum_{j=1}^n \pi_{ji} p_j = p_i \leq x_i + \sum_{j=1}^n \pi_{ji} p_j$, by the definition of Φ^{RV_+} in (3.3.1) and since $\boldsymbol{p}$ is a fixed point of Φ^{RV_+}. Hence, both *limited liability* and *absolute priority* in Definition 3.3.2 hold. Hence, $\boldsymbol{p}$ is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$. $\square$

Remark 3.3.6. Unfortunately, the converse of Proposition 3.3.5 fails to hold in general. Here is a counterexample. Consider a Rogers-Veraart network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$, where $\mathcal{N} = \{1, 2\}$, $\boldsymbol{\pi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\bar{\boldsymbol{p}} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, $\alpha = 0.5$, $\beta = 0.5$, and let $\boldsymbol{p} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$. Observe that $\boldsymbol{p}$ is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$ since it satisfies *absolute priority* and *limited liability* in Definition 3.3.2. However, by Definition 3.3.3, $\Phi_2^{\mathrm{RV}_+}(\boldsymbol{p}) = 25 > p_2 = 15$. Hence, the fixed point property does not hold for $\boldsymbol{p}$.

Theorem 3.3.7. *Let $\Lambda^{RV_+} : \mathbb{R}^n \to \mathbb{R}$ be the following mixed-integer linear programming (MILP) aggregation function*

$$\Lambda^{RV_+}(\boldsymbol{y}) := \begin{cases} \sup\Big\{ f(\boldsymbol{p}) \Big| \boldsymbol{p} \le \alpha \boldsymbol{y} + \beta \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p} + \bar{\boldsymbol{p}} \odot \boldsymbol{s}, \\ \qquad \bar{\boldsymbol{p}} \odot \boldsymbol{s} \le \boldsymbol{y} + \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p}, \boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s} \in \{0,1\}^n \Big\}, & \text{if } \boldsymbol{y} \in \mathbb{R}^n_+, \\ -\infty, & \text{if } \boldsymbol{y} \notin \mathbb{R}^n_+, \end{cases} \tag{3.3.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly increasing linear function.

If $(\boldsymbol{p}, \boldsymbol{s})$ is an optimal solution to the MILP for $\Lambda^{RV_+}(\boldsymbol{x})$, then $\boldsymbol{p}$ is a clearing vector for the network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$.

Observe that $\Lambda^{\mathrm{RV}_+}(\boldsymbol{x})$ can be written more explicitly as

$$\begin{aligned} \text{maximize} \quad & f(\boldsymbol{p}) && && (3.3.3) \\ \text{subject to} \quad & p_i \le \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i, && i \in \mathcal{N}, && (3.3.4) \\ & \bar{p}_i s_i \le x_i + \sum_{j=1}^{n} \pi_{ji} p_j, && i \in \mathcal{N}, && (3.3.5) \\ & 0 \le p_i \le \bar{p}_i, && i \in \mathcal{N}, && (3.3.6) \\ & s_i \in \{0,1\}, && i \in \mathcal{N}. && (3.3.7) \end{aligned}$$

It is easy to check that $(\boldsymbol{p}, \boldsymbol{s}) = (\boldsymbol{0}, \boldsymbol{0}) \in \mathbb{R}^n \times \mathbb{Z}^n$ is a feasible solution to the MILP in (3.3.3). Moreover, if f is a bounded function on the interval $[\boldsymbol{0}, \bar{\boldsymbol{p}}] \subseteq \mathbb{R}^n_+$, then by Meyer [18, Theorem 2.1], the MILP in (3.3.3) has an optimal solution. Observe that, by Theorem 3.3.7, the existence of an optimal solution to the MILP in (3.3.3) proves the existence of a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$. Hence, if the function f is bounded on $[\boldsymbol{0}, \bar{\boldsymbol{p}}]$, then Theorem 3.3.7 provides an alternative argument for proof of the original theorem [8, Theorem 3.1] on existence of a clearing vectors in Rogers-Veraart network model.

Remark 3.3.8. Linearity of f is not a necessary condition for Theorem 3.3.7 to hold.

The proof of Theorem 3.3.7 relies on the following three lemmata.

Lemma 3.3.9. *Let* $(\boldsymbol{p}, \boldsymbol{s})$ *be an optimal solution to the MILP for* $\Lambda^{RV_+}(\boldsymbol{x})$*. Let* $i \in \mathcal{N}$ *such that*

$$\alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j < \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j.$$

Then, $s_i = 1$*.*

Lemma 3.3.10. *Let* $(\boldsymbol{p}, \boldsymbol{s})$ *be an optimal solution to the MILP for* $\Lambda^{RV_+}(\boldsymbol{x})$*. Let* $i \in \mathcal{N}$ *with* $\bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j$*. Then,* $p_i = \bar{p}_i$*.*

Lemma 3.3.11. *Let* $(\boldsymbol{p}, \boldsymbol{s})$ *be an optimal solution to the MILP for* $\Lambda^{RV_+}(\boldsymbol{x})$*. Let* $i \in \mathcal{N}$ *with* $\bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j$*. Then* $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$*.*

Proofs of Lemmata 3.3.9, 3.3.10, 3.3.11 and Theorem 3.3.7 can be found in Appendices A.3, A.4, A.5 and A.6, respectively.

Remark 3.3.12. The aggregation function Λ^{RV_+} in (3.3.2) is applied in Chapter 4 to define the Rogers-Veraart grouped systemic risk measures.

In the following theorem, an alternative mixed-integer linear programming aggregation function for the Rogers-Veraart network model is introduced. However, using this MILP aggregation function may be too costly since it has $2n$ more binary variables, n more continuous slack variables and $5n$ more constraints compared to the previously defined aggregation function Λ^{RV_+}. It is provided to show that there is no unique way of defining an aggregation function that represents a clearing vector in a Rogers-Veraart network. Furthermore, if $\alpha = \beta = 1$, then the MILP aggregation function Λ^{RV_+} can represent a clearing vector in the original Eisenberg-Noe network model with nonnegative $\boldsymbol{x}$.

Theorem 3.3.13. *Let $\Lambda_{alt}^{RV+} : \mathbb{R}_{+}^{n} \to \mathbb{R}$ be the following MILP aggregation function*

$$\begin{aligned}\Lambda_{alt}^{RV+}(\boldsymbol{x}) := \sup\Big\{ f(\boldsymbol{p}) \Big| & \boldsymbol{p} \leq \alpha \boldsymbol{x} + \beta \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p} + \bar{\boldsymbol{p}} \odot \left(\boldsymbol{s}^{1} + \boldsymbol{s}^{2}\right), \\ & \bar{\boldsymbol{p}} \odot \boldsymbol{s}^{1} \leq \boldsymbol{x} + \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p}, \\ & \bar{\boldsymbol{p}} + M \boldsymbol{s}^{1} - \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p}, \\ & \left(\mathbb{1} - \boldsymbol{s}^{2}\right) \odot \left(M \mathbb{1} + \bar{\boldsymbol{p}}\right) \geq \boldsymbol{y}, \\ & \mathbb{1} - \boldsymbol{s}^{2} \leq M \boldsymbol{y}, \\ & \boldsymbol{p} \leq \alpha \boldsymbol{x} + \beta \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p} + \bar{\boldsymbol{p}} \odot \left(\mathbb{1} - \boldsymbol{s}^{3}\right), \\ & \bar{\boldsymbol{p}} \odot \boldsymbol{s}^{3} \leq \alpha \boldsymbol{x} + \beta \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p}, \\ & \boldsymbol{y} \in \mathbb{R}_{+}^{n}, \boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^{1}, \boldsymbol{s}^{2}, \boldsymbol{s}^{3} \in \{0,1\}^{n} \Big\},\end{aligned} \tag{3.3.8}$$

where $f : \mathbb{R}^{n} \to \mathbb{R}$ is a strictly increasing linear function and $M = n \|\bar{\boldsymbol{p}}\|_{\infty} + \|\boldsymbol{x}\|_{\infty}$.

If $\left(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{s}^{1}, \boldsymbol{s}^{2}, \boldsymbol{s}^{3}\right)$ is an optimal solution to the MILP for $\Lambda_{alt}^{RV+}(\boldsymbol{x})$, then $\boldsymbol{p}$ is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x}, \alpha, \beta)$.

Observe that $\Lambda_{\text{alt}}^{\text{RV}+}(\boldsymbol{x})$ can be written more explicitly as

$$\max \quad f(\boldsymbol{p}) \tag{3.3.9}$$

$$\text{s.t.} \quad p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i \left(s_i^1 + s_i^2\right), \qquad i \in \mathcal{N}, \tag{3.3.10}$$

$$\bar{p}_i s_i^1 \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j, \qquad i \in \mathcal{N}, \tag{3.3.11}$$

$$\bar{p}_i + M s_i^1 - y_i = x_i + \sum_{j=1}^{n} \pi_{ji} p_j, \qquad i \in \mathcal{N}, \tag{3.3.12}$$

$$\left(1 - s_i^2\right)\left(M + \bar{p}_i\right) \geq y_i, \qquad i \in \mathcal{N}, \tag{3.3.13}$$

$$\left(1 - s_i^2\right) \leq M y_i, \qquad i \in \mathcal{N}, \tag{3.3.14}$$

$$p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i \left(1 - s_i^3\right), \qquad i \in \mathcal{N}, \tag{3.3.15}$$

$$\bar{p}_i s_i^3 \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j, \qquad i \in \mathcal{N}, \tag{3.3.16}$$

$$y_i \geq 0, \qquad i \in \mathcal{N}, \tag{3.3.17}$$

$$0 \leq p_i \leq \bar{p}_i, \qquad i \in \mathcal{N}, \tag{3.3.18}$$

$$s_i^1, s_i^2, s_i^3 \in \{0,1\}, \qquad i \in \mathcal{N}. \tag{3.3.19}$$

Assume $M \geq 1$ and $\min_{i \in \mathcal{N}} \{\bar{p}_i - x_i\} \geq 1$. In other words, assume that for each $i \in \mathcal{N}$, the operating cash flow of the node i plus one is less than its total debt. Let $\boldsymbol{s}^1 = (s_1^1, \ldots, s_n^1)^{\mathsf{T}} \in \{0,1\}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n)^{\mathsf{T}} \in \mathbb{R}^n$ be vectors, where, for every $i \in \mathcal{N}$, $s_i^1 = 0$ and $y_i = \bar{p}_i - x_i$ if $x_i \leq \bar{p}_i$, and $s_i^1 = 1$ and $y_i = \bar{p}_i + M - x_i$ if $x_i > \bar{p}_i$. Let $\boldsymbol{p}, \boldsymbol{s}^2, \boldsymbol{s}^3 = \boldsymbol{0}$. Then $(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{s}^1, \boldsymbol{s}^2, \boldsymbol{s}^3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^n$ is a feasible solution to the MILP in (3.3.9). Moreover, if f is a bounded function on the interval $[\boldsymbol{0}, \bar{\boldsymbol{p}}] \subseteq \mathbb{R}_+^n$, then by Meyer [18, Theorem 2.1], the MILP in (3.3.9) has an optimal solution. The proof of Theorem 3.3.13 can be found in Appendix A.7.

In the next chapter, an application of the MILP aggregation functions Λ^{EN} and Λ^{RV_+} to systemic risk measures is demonstrated, the resulting systemic risk measures are considered from a vector optimization point of view and some related results are provided.

Chapter 4

Grouping in Systemic Risk Measures

In this chapter, systemic risk measures are looked at from a vector optimization point of view. A notion of grouping in systemic risk measures is introduced. It allows one to categorize the members of a financial system into groups and makes it easier to compute systemic risk measures. To approximate systemic risk measures by a Benson type algorithm for non-convex problems, two scalarization problems are introduced as single objective optimization problems of a vector optimization problem in the scope of the signed Eisenberg-Noe and the Rogers-Veraart network models. Two more optimization problems are introduced as minimum step-length functions. Mixed-integer linear programming formulations of these problems are provided. Some results on the boundedness and feasibility of these problems are presented.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the set of scenarios Ω is finite. Assume $\Omega = \left\{\omega^1, \ldots, \omega^K\right\}$ for some integer $K \geq 1$. Let $\mathcal{K} = \{1, \ldots, K\}$ be an index set of Ω. Assume $q^k := \mathbb{P}\left\{\omega^k\right\} > 0$ for every $k \in \mathcal{K}$. Let $L(\mathbb{R}^n)$ be the linear space of all n-dimensional random variables $\boldsymbol{X} : \Omega \to \mathbb{R}^n$. For every $\boldsymbol{X} \in L(\mathbb{R}^n)$, let

$$\|\boldsymbol{X}\|_\infty := \max_{i \in \{1,\ldots,n\},\ \omega^k \in \Omega} \left|X_i\left(\omega^k\right)\right|.$$

Let $\boldsymbol{X} \in L(\mathbb{R}^n)$. In the scope of this thesis, the following sensitive systemic risk measures, studied in Feinstein *et al.* [15] and Ararat and Rudloff [16], are considered:

$$R^{sen}(\boldsymbol{X}) := \left\{ \boldsymbol{z} \in \mathbb{R}^n \middle| \Lambda(\boldsymbol{X} + \boldsymbol{z}) \in \mathcal{A} \right\}, \tag{4.0.1}$$

where $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function and $\mathcal{A}$ is an acceptance set.

When there are many institutions in the financial system, they can be grouped into two or three groups in order to simplify the computation of systemic risk measures by decreasing their dimensions. If a network of banks is considered, then this grouping can be interpreted as classifying the banks into small, medium and large ones and assigning the same risk level to all banks in a particular group.

Let $G \geq 1$ be an integer denoting the number of groups and $\mathcal{G} = \{1, \ldots, G\}$ the set of groups in the network. Let $(\mathcal{N}_\ell)_{\ell \in \mathcal{G}}$ be a partition on $\mathcal{N}$, where $\mathcal{N}_\ell$ denotes the set of all institutions that belong to group $\ell \in \mathcal{G}$. Hence, each institution in the network belongs to exactly one group. Let $B_1, \ldots, B_G$ be matrices, where B_ℓ, $\ell \in \mathcal{G}$, is the following matrix having 1's in the ℓ^{th} row and 0's elsewhere:

$$B_\ell := \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{G \times n_\ell},$$

where n_ℓ is the number of nodes in group ℓ. It is easy to observe that $n = \sum_{\ell \in \mathcal{G}} n_\ell$. Let $B \in \mathbb{R}^{G \times n}$ be the following grouping matrix:

$$B := \begin{bmatrix} B_1 & B_2 & \ldots & B_G \end{bmatrix}. \tag{4.0.2}$$

Then, by overwriting the definition in (4.0.1), the grouped sensitive systemic risk measures are defined as

$$R^{sen}(\boldsymbol{X}) := \left\{ \boldsymbol{z} \in \mathbb{R}^G \middle| \Lambda(\boldsymbol{X} + B^{\mathsf{T}} \boldsymbol{z}) \in \mathcal{A} \right\}, \tag{4.0.3}$$

where $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function and $\mathcal{A} \subseteq L(\mathbb{R})$ is an acceptance set.

For simplicity and computational reasons, from now on, let

$$\mathcal{A} = \left\{ Y \in L(\mathbb{R}) \middle| \mathbb{E}[Y] \geq \gamma \right\}, \tag{4.0.4}$$

where $\gamma \in \mathbb{R}$ is some suitable threshold.

For an arbitrary set A, let 2^A denote its power set. Consider a general optimization aggregation function $\Lambda^{\mathrm{OPT}} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ of the form

$$\Lambda^{\mathrm{OPT}}(\boldsymbol{x}) \coloneqq \sup \left\{ f(\boldsymbol{p}) \middle| (\boldsymbol{p}, \boldsymbol{s}) \in \mathcal{Y}(\boldsymbol{x}), \boldsymbol{p} \in \mathbb{R}^n, \boldsymbol{s} \in \mathbb{Z}^n \right\}, \tag{4.0.5}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly increasing and continuous function, and $\mathcal{Y} : \mathbb{R}^n \to 2^{\mathbb{R}^n \times \mathbb{Z}^n}$ is a set-valued constraint function such that $\mathcal{Y}(\boldsymbol{x})$ is a compact set for every $\boldsymbol{x} \in \mathbb{R}^n$. Then, the corresponding systemic risk measure $R^{sen} : L(\mathbb{R}^n) \to \mathbb{R}^n$ with respect to the aggregation function Λ^{OPT} becomes

$$R^{sen}(\boldsymbol{X}) = \left\{ \boldsymbol{z} \in \mathbb{R}^G \middle| \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq \gamma \right\}. \tag{4.0.6}$$

In this thesis, the special cases $\Lambda = \Lambda^{\mathrm{EN}}$ and $\Lambda = \Lambda^{\mathrm{RV}+}$ are considered in more detail. Let us define

$$R^{sen}_{\mathrm{EN}}(\boldsymbol{X}) \coloneqq \left\{ \boldsymbol{z} \in \mathbb{R}^G \middle| \mathbb{E}\left[\Lambda^{\mathrm{EN}}(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z})\right] \geq \gamma \right\}, \tag{4.0.7}$$

$$R^{sen}_{\mathrm{RV}}(\boldsymbol{X}) \coloneqq \left\{ \boldsymbol{z} \in \mathbb{R}^G \middle| \mathbb{E}\left[\Lambda^{\mathrm{RV}+}(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z})\right] \geq \gamma \right\}, \tag{4.0.8}$$

called the Eisenberg-Noe and Rogers-Veraart systemic risk measures, respectively.

Remark 4.0.1. In (4.0.8), the condition $\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z} \geq 0$ is implied by definition of $\Lambda^{\mathrm{RV}+}$ in (3.3.2).

Remark 4.0.2. The number of groups G can be at most n, since each node in a network must be assigned to exactly one group. If $G = n$, then the grouping matrix B becomes the identity matrix $I \in \mathbb{R}^{n \times n}$ and the grouped systemic risk measure in (4.0.3) reduces to the systemic risk measure in (4.0.1).

4.1 Weakly Minimal Elements of Systemic Risk Measures

In this section, a vector optimization problem for the systemic risk measure in (4.0.6) is introduced and some results on linearizing the corresponding weighted-sum scalarization problem are provided. The following definition of a weakly minimal element of a set is borrowed from Jahn [19].

Definition 4.1.1. *Let $A \subseteq \mathbb{R}^n$ be an arbitrary set. A point $\boldsymbol{z} \in A$ is a weakly minimal element of a set A if $\left(\{\boldsymbol{z}\} - int\left(\mathbb{R}^n_+\right)\right) \cap A = \emptyset$, or, in other words, if there is no other point $\boldsymbol{z}' \in A$ such that $z'_i < z_i$ for each $i \in \{1, \dots, n\}$.*

Consider the following vector optimization problem

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{z} \in \mathbb{R}^G \text{ with respect to } \leq \\ \text{subject to} \quad & \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq \gamma, \end{aligned} \tag{4.1.1}$$

where "$\leq$" is the usual componentwise ordering in $\mathbb{R}^n$. Note that $R^{sen}(\boldsymbol{X})$ coincides with the so-called *upper image* of this vector optimization problem in the sense that

$$R^{sen}(\boldsymbol{X}) = \left\{\boldsymbol{z} + \mathbb{R}^G_+ \middle| \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq \gamma\right\}. \tag{4.1.2}$$

An approximation algorithm presented in Löhne *et al.* [20] works for convex upper images by calculating finitely many weakly minimal elements. Since $R^{sen}(\boldsymbol{X})$ is not necessarily convex in the scope of this thesis, the Benson type algorithm proposed in Nobakhtian and Shafiei [1] is applied instead, which works for non-convex upper images.

For $\boldsymbol{w} \in \mathbb{R}^G_+ \setminus \{\boldsymbol{0}\}$, let $\mathcal{P}_1(\boldsymbol{w})$ be given by

$$\mathcal{P}_1(\boldsymbol{w}) := \inf_{\boldsymbol{z} \in \mathbb{R}^G} \left\{ \boldsymbol{w}^\mathsf{T} \boldsymbol{z} | \Lambda^{\mathrm{OPT}} \left(\boldsymbol{X} + B^\mathsf{T} \boldsymbol{z} \right) \in \mathcal{A} \right\} \tag{4.1.3}$$

as the optimal value of a weighted-sum scalarization problem.

Theorem 4.1.2. *Let $\boldsymbol{w} \in \mathbb{R}^G_+ \setminus \{\boldsymbol{0}\}$. Consider a weighted-sum scalarization problem*

$$\mathcal{P}_1(\boldsymbol{w}) = \inf_{\boldsymbol{z} \in \mathbb{R}^G} \left\{ \boldsymbol{w}^\mathsf{T} \boldsymbol{z} \middle| \mathbb{E}\left[\Lambda^{OPT} \left(\boldsymbol{X} + B^\mathsf{T} \boldsymbol{z} \right) \right] \geq \gamma \right\}, \tag{4.1.4}$$

and let

$$\mathcal{Z}_1(\boldsymbol{w}) := \inf_{\boldsymbol{z} \in \mathbb{R}^G} \left\{ \boldsymbol{w}^\mathsf{T} \boldsymbol{z} \middle| \sum_{k \in \mathcal{K}} q^k f\left(\boldsymbol{p}^k \right) \geq \gamma, \right. \\ \left. \left(\boldsymbol{p}^k, \boldsymbol{s}^k \right) \in \mathcal{Y}\left(\boldsymbol{X}\left(\omega^k \right) + B^\mathsf{T} \boldsymbol{z} \right), \boldsymbol{p}^k \in \mathbb{R}^{n_1}, \boldsymbol{s}^k \in \mathbb{Z}^{n_2}, k \in \mathcal{K} \right\}. \tag{4.1.5}$$

If $\mathcal{P}_1(\boldsymbol{w})$ and $\mathcal{Z}_1(\boldsymbol{w})$ have finite optimal values, then $\mathcal{P}_1(\boldsymbol{w}) = \mathcal{Z}_1(\boldsymbol{w})$.

Proof. Let $\mathring{\boldsymbol{z}}$ be an optimal solution to $\mathcal{P}_1(\boldsymbol{w})$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k \right)_{k \in \mathcal{K}} \right)$ is an optimal solution to $\mathcal{Z}_1(\boldsymbol{w})$. Let $\mathcal{P}_1(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T} \mathring{\boldsymbol{z}} = \mathring{\mu}$ and $\mathcal{Z}_1(\boldsymbol{w}) = \boldsymbol{w}^\mathsf{T} \boldsymbol{z} = \mu$. Hence, the aim is to prove $\mathring{\mu} = \mu$.

First, $\mathring{\mu} \leq \mu$ is shown. For an arbitrary $k \in \mathcal{K}$, $\left(\boldsymbol{p}^k, \boldsymbol{s}^k \right)$ is a feasible solution to $\Lambda^{\mathrm{OPT}} \left(\boldsymbol{X}\left(\omega^k \right) + B^\mathsf{T} \boldsymbol{z} \right)$ in (4.0.5) because the optimization problem in (4.1.5) includes the constraints of (4.0.5). Hence,

$$\Lambda^{\mathrm{OPT}} \left(\boldsymbol{X}\left(\omega^k \right) + B^\mathsf{T} \boldsymbol{z} \right) \geq f\left(\boldsymbol{p}^k \right), \quad \text{for every } k \in \mathcal{K},$$

which implies

$$\mathbb{E}\left[\Lambda^{\mathrm{OPT}} \left(\boldsymbol{X} + B^\mathsf{T} \boldsymbol{z} \right) \right] \geq \sum_{k=1}^{K} q^k f\left(\boldsymbol{p}^k \right) \geq \gamma,$$

where the second inequality holds by feasibility of $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k \right)_{k \in \mathcal{K}} \right)$. Hence, $\boldsymbol{z}$ is a feasible solution to $\mathcal{P}_1(\boldsymbol{w})$. Hence, $\mathring{\mu} \leq \mu$.

Now, $\dot{\mu} \geq \mu$ is shown. For each $k \in \mathcal{K}$, let $\left(\dot{\boldsymbol{p}}^k, \dot{\boldsymbol{s}}^k\right)$ be an optimal solution to

$$\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\dot{\boldsymbol{z}}\right).$$

Then,

$$\sum_{k=1}^{K} q^k f\left(\dot{\boldsymbol{p}}^k\right) = \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\dot{\boldsymbol{z}}\right)\right] \geq \gamma,$$

by the definition of $\mathcal{P}_1(\boldsymbol{w})$. Hence, $\left(\dot{\boldsymbol{z}}, \left(\dot{\boldsymbol{p}}^k, \dot{\boldsymbol{s}}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1(\boldsymbol{w})$, which implies $\dot{\mu} \geq \mu$. Hence, $\mathcal{Z}_1(\boldsymbol{w}) = \mathcal{P}_1(\boldsymbol{w})$. $\square$

Remark 4.1.3. Let $\ell \in \mathcal{G}$ and $\boldsymbol{e}^\ell$ the corresponding standard unit vector in $\mathbb{R}^G$. Observe that the following weighted-sum scalarization problem

$$\mathcal{P}_1\left(\boldsymbol{e}^\ell\right) = \inf_{\boldsymbol{z}\in\mathbb{R}^G}\left\{z_\ell \,\middle|\, \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq \gamma\right\} \tag{4.1.6}$$

is a single objective optimization problem of the vector optimization problem in (4.1.1). By Theorem 4.1.2, if $\mathcal{P}_1\left(\boldsymbol{e}^\ell\right)$ and $\mathcal{Z}_1\left(\boldsymbol{e}^\ell\right)$ have finite optimal values, then $\mathcal{P}_1\left(\boldsymbol{e}^\ell\right) = \mathcal{Z}_1\left(\boldsymbol{e}^\ell\right)$.

Remark 4.1.4. Let $\boldsymbol{z}^{\mathrm{ideal}} \in \mathbb{R}^G$ be the ideal point of the vector optimization problem in (4.1.1) in the sense that the entries of $\boldsymbol{z}^{\mathrm{ideal}}$ minimize each of the objective functions of the vector optimization problem. In other words, one can define

$$\boldsymbol{z}^{\mathrm{ideal}} := \left(\mathcal{P}_1\left(\boldsymbol{e}^1\right), \ldots, \mathcal{P}_1\left(\boldsymbol{e}^G\right)\right)^{\mathsf{T}} \in \mathbb{R}^G. \tag{4.1.7}$$

However, the single objective optimization problems $\left(\mathcal{P}_1\left(\boldsymbol{e}^\ell\right)\right)_{\ell\in\mathcal{G}}$ are not linear. Theorem 4.1.2 allows one to solve G optimization problems with compact feasible sets $\left(\mathcal{Z}_1\left(\boldsymbol{e}^\ell\right)\right)_{\ell\in\mathcal{G}}$ to obtain the ideal point of the vector optimization problem in (4.1.1). In other words, one can calculate $\boldsymbol{z}^{\mathrm{ideal}} = \left(\mathcal{Z}_1\left(\boldsymbol{e}^1\right), \ldots, \mathcal{Z}_1\left(\boldsymbol{e}^G\right)\right)^{\mathsf{T}}$.

The following two sections apply the results from Theorem 4.1.2 to specific cases $\Lambda^{\mathrm{OPT}} = \Lambda^{\mathrm{EN}}$ and $\Lambda^{\mathrm{OPT}} = \Lambda^{\mathrm{RV}_+}$, respectively.

In the subsequent sections, the function $f : \mathbb{R}^n \to \mathbb{R}$ in the objective functions of the MILP aggregation functions Λ^{EN} in (3.2.7), and $\Lambda^{\mathrm{RV}+}$ in (3.3.2), is fixed as

$$f\left(\boldsymbol{p}\right) := \mathbb{1}^{\mathsf{T}}\boldsymbol{p}.$$

It is easy to check that f is a strictly increasing continuous linear function bounded on the interval $[\mathbf{0}, \bar{\boldsymbol{p}}] \subseteq \mathbb{R}^n$.

4.1.1 $\mathcal{P}_1$ Problem for Eisenberg-Noe Systemic Risk Measures

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X})$ be a signed Eisenberg-Noe network.

Corollary 4.1.5. *Let $\ell \in \mathcal{G}$. Consider the single objective optimization problem*

$$\mathcal{P}_1^{EN}\left(\boldsymbol{e}^{\ell}\right) := \inf_{\boldsymbol{z}\in\mathbb{R}^G} \left\{ z_\ell \middle| \mathbb{E}\left[\Lambda^{EN}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq \gamma \right\}, \tag{4.1.8}$$

and let

$$\begin{aligned}\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right) := \inf_{\boldsymbol{z}\in\mathbb{R}^G} \Bigg\{ z_\ell \Bigg| & \sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^{\mathsf{T}}\boldsymbol{p}^k\right] \geq \gamma, \\ & \boldsymbol{p}^k \leq \left[\boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{p}^k + \left[\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\boldsymbol{z}\right] + M\left(\mathbb{1} - \boldsymbol{s}^k\right)\right] \wedge \left(\bar{\boldsymbol{p}} \odot \boldsymbol{s}^k\right), \\ & \boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{p}^k + \left[\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\boldsymbol{z}\right] \leq M\boldsymbol{s}^k, \\ & \boldsymbol{p}^k \in [\mathbf{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^k \in \{0,1\}^n, \forall k \in \mathcal{K} \Bigg\},\end{aligned} \tag{4.1.9}$$

where $M = 2\left\|\boldsymbol{X}\right\|_\infty + (n+1)\left\|\bar{\boldsymbol{p}}\right\|_\infty$.

If $\mathcal{P}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ and $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ have finite optimal values, then $\mathcal{P}_1^{EN}\left(\boldsymbol{e}^{\ell}\right) = \mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$.

The MILP problem $\mathcal{Z}_1^{\mathrm{EN}}\left(\boldsymbol{e}^{\ell}\right)$ in (4.1.9) can be written more explicitly as

$$\text{minimize} \quad z_\ell \tag{4.1.10}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\mathsf{T} \boldsymbol{p}^k\right] \geq \gamma, \tag{4.1.11}$$

$$p_i^k \leq \sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z}\right)_i\right] + M\left(1 - s_i^k\right), \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.12}$$

$$p_i^k \leq \bar{p}_i s_i^k, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.13}$$

$$\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z}\right)_i\right] \leq M s_i^k, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.14}$$

$$0 \leq p_i^k \leq \bar{p}_i, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.15}$$

$$s_i^k \in \{0, 1\}, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.16}$$

$$\boldsymbol{z} \in \mathbb{R}^G. \tag{4.1.17}$$

Proof of Corollary 4.1.5. Assume that $\mathring{\boldsymbol{z}}$ is an optimal solution to $\mathcal{P}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is an optimal solution to $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$.

Let $\mathcal{Y}_{\text{EN}} : \mathbb{R}^n \to 2^{\mathbb{R}^n \times \mathbb{Z}^n}$ be the following set-valued function

$$\mathcal{Y}_{\text{EN}}(\boldsymbol{x}) := \left\{ \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right) \in \mathbb{R}^n \times \mathbb{Z}^n \middle| \begin{array}{l} \boldsymbol{p}^k \leq \left[\boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \boldsymbol{x} + M\left(\mathbb{1} - \boldsymbol{s}^k\right)\right] \wedge \left(\bar{\boldsymbol{p}} \odot \boldsymbol{s}^k\right), \\ \boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \boldsymbol{x} \leq M\boldsymbol{s}^k, \\ \boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^k \in \{0,1\}^n, \forall k \in \mathcal{K} \end{array} \right\}. \tag{4.1.18}$$

Take $\mathcal{Y} = \mathcal{Y}_{\text{EN}}$ in Theorem 4.1.2 and let $\left(\mathring{\boldsymbol{p}}^k, \mathring{\boldsymbol{s}}^k\right)$ be an optimal solution to $\Lambda^{\text{EN}}\left(\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\mathring{\boldsymbol{z}}\right)$, for each $k \in \mathcal{K}$. Then by Theorem 4.1.2, $\mathcal{P}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right) = \mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$. □

The next three propositions present some boundedness and feasibility results for the MILP problem $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^{\ell}\right)$, $\ell \in \mathcal{G}$, in Corollary 4.1.5.

Proposition 4.1.6. *Let* $\ell \in \mathcal{G}$. *Consider* $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ *in Corollary 4.1.5. If* $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ *is an optimal solution to* $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$, *then* $z_\ell \leq z^{\max}$, *where* $z^{\max} = \|\boldsymbol{X}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty$.

Proposition 4.1.7. *Let* $\ell \in \mathcal{G}$. *Consider* $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ *in Corollary 4.1.5. Let the value of* M *in* $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ *MILP problem be taken as* $M = 2\|\boldsymbol{X}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty$. *If* $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ *is feasible, then it is bounded.*

Proposition 4.1.8. *Let* $\ell \in \mathcal{G}$. *Then,* $\mathcal{Z}_1^{EN}\left(\boldsymbol{e}^{\ell}\right)$ *in Corollary 4.1.5 is feasible if and only if* $\gamma \leq \mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}$.

The proofs of Propositions 4.1.6, 4.1.7 and 4.1.8 can be found in Appendices B.1, B.2 and B.3, respectively.

Remark 4.1.9. Let $\ell \in \mathcal{G}$. Consider an optimal solution $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ to the MILP problem $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^{\ell}\right)$. By the structure of the matrix B, for each $i \in \mathcal{N}$, it holds $\left(B^{\mathsf{T}}\boldsymbol{z}\right)_i = z_t$ for some $t \in \mathcal{G}$. Hence, by Proposition 4.1.7, $\left(B^{\mathsf{T}}\boldsymbol{z}\right)_i \leq \|\boldsymbol{X}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty$ holds for each $i \in \mathcal{N}$. In addition, for every $i \in \mathcal{N}$, $k \in \mathcal{K}$, and $\boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$, it holds $\sum_{j=1}^{n} \pi_{ji} p_j^k < n\|\bar{\boldsymbol{p}}\|_\infty$ and $X_i\left(\omega^k\right) \leq \|\boldsymbol{X}\|_\infty$. Hence, the choice of $M = 2\|\boldsymbol{X}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty$ in Corollary 4.1.5 is justified, since, to ensure the feasibility in constraint (4.1.14) of the explicit formulation of $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^{\ell}\right)$ in (4.1.10), it is enough to choose M such that

$$\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^{\mathsf{T}}\boldsymbol{z}\right)_i\right] \leq M$$

for every $i \in \mathcal{N}$, $k \in \mathcal{K}$ and $\boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$.

4.1.2 $\mathcal{P}_1$ Problem for Rogers-Veraart Systemic Risk Measures

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ be a Rogers-Veraart network, where $\boldsymbol{X} \in L\left(\mathbb{R}_+^n\right)$.

Corollary 4.1.10. *Let $\ell \in \mathcal{G}$. Consider the single objective optimization problem*

$$\mathcal{P}_1^{RV}\left(\boldsymbol{e}^\ell\right) := \inf_{\boldsymbol{z}\in\mathbb{R}^G} \left\{ z_\ell \middle| \mathbb{E}\left[\Lambda^{RV_+}\left(\boldsymbol{X} + B^\mathsf{T}\boldsymbol{z}\right)\right] \geq \gamma \right\}, \tag{4.1.19}$$

and let

$$\begin{aligned}\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right) := \inf_{\boldsymbol{z}\in\mathbb{R}^G} \Bigg\{ z_\ell \Bigg| & \sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \geq \gamma, \\ & \boldsymbol{p}^k \leq \alpha\left[\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{z}\right] + \beta\boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \bar{\boldsymbol{p}} \odot \boldsymbol{s}^k, \\ & \bar{\boldsymbol{p}} \odot \boldsymbol{s}^k \leq \left[\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{z}\right] + \boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k, \\ & \boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{z} \geq 0, \\ & \boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^k \in \{0,1\}^n, \forall k \in \mathcal{K} \Bigg\}.\end{aligned} \tag{4.1.20}$$

If $\mathcal{P}_1^{RV}\left(\boldsymbol{e}^\ell\right)$ and $\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$ have finite optimal values, then $\mathcal{P}_1^{RV}\left(\boldsymbol{e}^\ell\right) = \mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$.

The MILP problem $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$ in (4.1.20) can be written more explicitly as

$$\text{minimize} \quad z_\ell \tag{4.1.21}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \geq \gamma, \tag{4.1.22}$$

$$p_i^k \leq \alpha\left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z}\right)_i\right] + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.23}$$

$$\bar{p}_i s_i^k \leq \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z}\right)_i\right] + \sum_{j=1}^n \pi_{ji}p_j^k, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.24}$$

$$X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z}\right)_i \geq 0, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.25}$$

$$0 \leq p_i^k \leq \bar{p}_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.26}$$

$$s_i^k \in \{0,1\}, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.1.27}$$

$$\boldsymbol{z} \in \mathbb{R}^G. \tag{4.1.28}$$

Here, constraint (4.1.25) ensures $\boldsymbol{X} + B^\mathsf{T}\boldsymbol{z} \geq 0$ so that $\Lambda^{\mathrm{RV}+}\left(\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{z}\right) \neq -\infty$ for every $k \in \mathcal{K}$.

Proof of Corollary 4.1.10. Assume that $\dot{\boldsymbol{z}}$ is an optimal solution to $\mathcal{P}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is an optimal solution to $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$.

Let $\mathcal{Y}_{\mathrm{RV}} : \mathbb{R}^n \to 2^{\mathbb{R}^n \times \mathbb{Z}^n}$ be the following set-valued function

$$\mathcal{Y}_{\mathrm{RV}}(\boldsymbol{x}) := \Bigg\{ \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right) \in \mathbb{R}^n \times \mathbb{Z}^n \,\Bigg|\, \boldsymbol{p}^k \leq \alpha\boldsymbol{x} + \beta\boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \bar{\boldsymbol{p}} \odot \boldsymbol{s}^k, \\ \bar{\boldsymbol{p}} \odot \boldsymbol{s}^k \leq \boldsymbol{x} + \boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k, \boldsymbol{x} \geq 0, \\ \boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^k \in \{0,1\}^n, \forall k \in \mathcal{K} \Bigg\}. \tag{4.1.29}$$

Take $\mathcal{Y} = \mathcal{Y}_{\mathrm{RV}}$ in Theorem 4.1.2 and let $\left(\dot{\boldsymbol{p}}^k, \dot{\boldsymbol{s}}^k\right)$ be an optimal solution to $\Lambda^{\mathrm{RV}+}\left(\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\dot{\boldsymbol{z}}\right)$, for each $k \in \mathcal{K}$. Then by Theorem 4.1.2, $\mathcal{P}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right) = \mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$. □

The next three propositions present some boundedness and feasibility results for the MILP problem $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$, $\ell \in \mathcal{G}$, in Corollary 4.1.10.

Proposition 4.1.11. *Let $\ell \in \mathcal{G}$. Consider $\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$ in Corollary 4.1.10. If $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is an optimal solution to $\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$, then $z_\ell \leq z^{\max}$, where $z^{\max} = \|\boldsymbol{X}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty$.*

Proposition 4.1.12. *Let $\ell \in \mathcal{G}$. Consider $\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$ in Corollary 4.1.10. If $\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$ is feasible, then it is bounded.*

Proposition 4.1.13. *Let $\ell \in \mathcal{G}$. Then, $\mathcal{Z}_1^{RV}\left(\boldsymbol{e}^\ell\right)$ in Corollary 4.1.10 is feasible if and only if $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$.*

The proofs of Propositions 4.1.11, 4.1.12 and 4.1.13 can be found in Appendices B.4, B.5 and B.6, respectively.

4.2 Minimum Step-Length Function

In this section, minimum step-length functions, also referred to as $\mathcal{P}_2$ problems (see Gerstewitz and Iwanow [21], Göpfert *et al.* [22] for details), are defined in the scope of the Eisenberg-Noe and Rogers-Veraart systemic risk measures in (4.0.7) and (4.0.8). Some results on MILP formulations of these $\mathcal{P}_2$ problems are provided.

Let $\boldsymbol{v} \in \mathbb{R}^G$ and $\mathcal{P}_2(\boldsymbol{v})$ the following minimum step-length function

$$\mathcal{P}_2(\boldsymbol{v}) := \inf\left\{\mu \in \mathbb{R} \middle| \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{v} + \mu\mathbb{1}\right)\right] \geq \gamma\right\}. \tag{4.2.1}$$

$\mathcal{P}_2(\boldsymbol{v})$ can be interpreted as a minimum step-length in the direction $\mathbb{1}$ from the point $\boldsymbol{v}$ to the boundary of the systemic risk measure $R^{sen}(\boldsymbol{X})$.

The following theorem provides an alternative formulation for $\mathcal{P}_2(\boldsymbol{v})$.

Theorem 4.2.1. *Let $\boldsymbol{v} \in \mathbb{R}^G$ and let*

$$\begin{aligned}\mathcal{Z}_2(\boldsymbol{v}) := \inf\Bigg\{\mu \in \mathbb{R} \Bigg| & \sum_{k\in\mathcal{K}} q^k f\left(\boldsymbol{p}^k\right) \geq \gamma, \\ & \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right) \in \mathcal{Y}\left(\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\boldsymbol{v} + \mu\mathbb{1}\right), \\ & \boldsymbol{p}^k \in \mathbb{R}^n, \boldsymbol{s}^k \in \mathbb{Z}^n, k \in \mathcal{K}\Bigg\}.\end{aligned} \tag{4.2.2}$$

If $\mathcal{P}_2(\boldsymbol{v})$ and $\mathcal{Z}_2(\boldsymbol{v})$ have finite optimal values, then $\mathcal{P}_2(\boldsymbol{v}) = \mathcal{Z}_2(\boldsymbol{v})$.

Proof. Let $\mathring{\mu} \in \mathbb{R}$ be an optimal solution to $\mathcal{P}_2(\boldsymbol{v})$ and $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ be an optimal solution to $\mathcal{Z}_2(\boldsymbol{v})$. Then, by definitions, $\mathcal{P}_2(\boldsymbol{v}) = \mathring{\mu}$ and $\mathcal{Z}_2(\boldsymbol{v}) = \mu$. Hence, the aim is to prove $\mathring{\mu} = \mu$.

First, $\mathring{\mu} \leq \mu$ is shown. For an arbitrary $k \in \mathcal{K}$, $\left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)$ is a feasible solution to $\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\boldsymbol{v} + \mu\mathbb{1}\right)$ in (4.0.5) because $\mathcal{Z}_2\left(\boldsymbol{v}\right)$ in (4.2.2) includes the constraints of (4.0.5). Hence,

$$\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\boldsymbol{v} + \mu\mathbb{1}\right) \geq f\left(\boldsymbol{p}^k\right), \quad \text{for every } k \in \mathcal{K},$$

which implies

$$\mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{v} + \mu\mathbb{1}\right)\right] \geq \sum_{k=1}^{K} q^k\left[f\left(\boldsymbol{p}^k\right)\right] \geq \gamma,$$

where the second inequality holds by feasibility of $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$. Then, μ is a feasible solution to $\mathcal{P}_2\left(\boldsymbol{v}\right)$. Hence, $\mathring{\mu} \leq \mu$.

Now, $\mathring{\mu} \geq \mu$ is shown. For each $k \in \mathcal{K}$, let $\left(\mathring{\boldsymbol{p}}^k, \mathring{\boldsymbol{s}}^k\right)$ be an optimal solution to

$$\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X}\left(\omega^k\right) + B^{\mathsf{T}}\boldsymbol{v} + \mathring{\mu}\mathbb{1}\right).$$

Then

$$\sum_{k=1}^{K} q^k\left[f\left(\mathring{\boldsymbol{p}}^k\right)\right] = \mathbb{E}\left[\Lambda^{\mathrm{OPT}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{v} + \mathring{\mu}\mathbb{1}\right)\right] \geq \gamma,$$

by definition of $\mathcal{P}_2\left(\boldsymbol{v}\right)$. Hence $\left(\mathring{\mu}, \left(\mathring{\boldsymbol{p}}^k, \mathring{\boldsymbol{s}}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2\left(\boldsymbol{v}\right)$, which implies $\mathring{\mu} \geq \mu$. Hence, $\mathcal{P}_2\left(\boldsymbol{v}\right) = \mathcal{Z}_2\left(\boldsymbol{v}\right)$. □

The following two sections apply the result from Theorem 4.2.1 to specific cases $\Lambda^{\mathrm{OPT}} = \Lambda^{\mathrm{EN}}$ and $\Lambda^{\mathrm{OPT}} = \Lambda^{\mathrm{RV}_+}$, respectively.

4.2.1 $\mathcal{P}_2$ Problem for Eisenberg-Noe Systemic Risk Measures

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X})$ be an Eisenberg-Noe network.

Corollary 4.2.2. *Let* $\boldsymbol{v} \in \mathbb{R}^G$ *and let*

$$\mathcal{P}_2^{EN}(\boldsymbol{v}) \coloneqq \inf\left\{\mu \in \mathbb{R} \middle| \mathbb{E}\left[\Lambda^{EN}\left(\boldsymbol{X} + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right)\right] \geq \gamma\right\}, \tag{4.2.3}$$

and

$$\begin{aligned}\mathcal{Z}_2^{EN}(\boldsymbol{v}) \coloneqq \inf\Bigg\{\mu \in \mathbb{R}\Bigg| & \sum_{k\in\mathcal{K}} q^k\left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \geq \gamma, \\ & \boldsymbol{p}^k \leq \Big[\boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \left[\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right] \\ & \qquad + M\left(\mathbb{1} - \boldsymbol{s}^k\right)\Big] \wedge \left(\bar{\boldsymbol{p}} \odot \boldsymbol{s}^k\right), \qquad (4.2.4) \\ & \boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \left[\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right] \leq M\boldsymbol{s}^k, \\ & \boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^k \in \{0,1\}^n, \forall k \in \mathcal{K}\Bigg\},\end{aligned}$$

where $M = 2\|\boldsymbol{X}\|_\infty + 2\|\boldsymbol{v}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty$.

If $\mathcal{P}_2^{EN}(\boldsymbol{v})$ *and* $\mathcal{Z}_2^{EN}(\boldsymbol{v})$ *have finite optimal values, then* $\mathcal{P}_2^{EN}(\boldsymbol{v}) = \mathcal{Z}_2^{EN}(\boldsymbol{v})$.

The MILP problem $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$ in (4.2.4) can be written more explicitly as

$$\text{minimize} \quad \mu \tag{4.2.5}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} q^k\left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \geq \gamma, \tag{4.2.6}$$

$$p_i^k \leq \sum_{j=1}^n \pi_{ji}p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu\right] \tag{4.2.7}$$

$$+ M\left(1 - s_i^k\right), \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.2.8}$$

$$p_i^k \leq \bar{p}_i s_i^k, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.2.9}$$

$$\sum_{j=1}^n \pi_{ji}p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu\right] \leq M s_i^k, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.2.10}$$

$$0 \leq p_i^k \leq \bar{p}_i, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, \tag{4.2.11}$$

$$s_i^k \in \{0,1\}, \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}. \tag{4.2.12}$$

Proof of Corollary 4.2.2. Let $\mathring{\mu} \in \mathbb{R}$ be an optimal solution to $\mathcal{P}_2^{\text{EN}}(\boldsymbol{v})$ and let $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ be an optimal solution to $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$.

Take $\mathcal{Y} = \mathcal{Y}_{\text{EN}}$ in Theorem 4.2.1 and let $\left(\mathring{\boldsymbol{p}}^k, \mathring{\boldsymbol{s}}^k\right)$ be an optimal solution to $\Lambda^{\text{EN}}\left(\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mathring{\mu}\mathbb{1}\right)$, for each $k \in \mathcal{K}$. Then by Theorem 4.2.1, $\mathcal{P}_2^{\text{EN}}(\boldsymbol{v}) = \mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$. □

The next three propositions present some boundedness and feasibility results for the MILP problem $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$, $\boldsymbol{v} \in \mathbb{R}^G$, in Corollary 4.2.2.

Proposition 4.2.3. *Let $\boldsymbol{v} \in \mathbb{R}^G$ and consider $\mathcal{Z}_2^{EN}(\boldsymbol{v})$ in Corollary 4.2.2. If $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is an optimal solution to $\mathcal{Z}_2^{EN}(\boldsymbol{v})$, then $\mu \leq \mu^{\max}$, where $\mu^{\max} = \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty$.*

Proposition 4.2.4. *Let $\boldsymbol{v} \in \mathbb{R}^G$ and consider $\mathcal{Z}_2^{EN}(\boldsymbol{v})$ in Corollary 4.2.2. Let the value of M in $\mathcal{Z}_2^{EN}(\boldsymbol{v})$ MILP problem be taken as $M = 2\|\boldsymbol{X}\|_\infty + 2\|\boldsymbol{v}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty$. If $\mathcal{Z}_2^{EN}(\boldsymbol{v})$ is feasible, then it is bounded.*

Proposition 4.2.5. *Let $\boldsymbol{v} \in \mathbb{R}^G$. Then, $\mathcal{Z}_2^{EN}(\boldsymbol{v})$ in Corollary 4.2.2 is feasible if and only if $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$.*

The proofs of Propositions 4.2.3, 4.2.4 and 4.2.5 can be found in Appendices B.7, B.8 and B.9.

Remark 4.2.6. Let $\boldsymbol{v} \in \mathbb{R}^G$ and consider an optimal solution $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ to the MILP problem $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$. By Proposition (4.2.3), $\mu \leq \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty$. By the structure of the matrix B, for each $i \in \mathcal{N}$, it holds $\left(B^\mathsf{T}\boldsymbol{v}\right)_i = \boldsymbol{v}_t$ for some $t \in \mathcal{G}$. Hence, for every $\boldsymbol{v} \in \mathbb{R}^G$, $\left(B^\mathsf{T}\boldsymbol{v}\right)_i \leq \|\boldsymbol{v}\|_\infty$. In addition, for every $i \in \mathcal{N}$, $k \in \mathcal{K}$, and $\boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$, it holds $\sum_{j=1}^n \pi_{ji} p_j^k < n\|\bar{\boldsymbol{p}}\|_\infty$ and $X_i\left(\omega^k\right) \leq \|\boldsymbol{X}\|_\infty$. Hence, the choice of $M = 2\|\boldsymbol{X}\|_\infty + 2\|\boldsymbol{v}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty$ in Corollary 4.2.2

is justified, since, to ensure the feasibility in constraint (4.2.10) of the explicit formulation of $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$ in (4.2.5), it is enough to choose M such that

$$\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu\right] \leq M$$

for every $i \in \mathcal{N}$, $k \in \mathcal{K}$, $\boldsymbol{v} \in \mathbb{R}^G$ and $\boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$.

4.2.2 $\mathcal{P}_2$ Problem for Rogers-Veraart Systemic Risk Measures

Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ be a Rogers-Veraart network, where $\boldsymbol{X} \in L\left(\mathbb{R}_+^n\right)$.

Corollary 4.2.7. *Let* $\boldsymbol{v} \in \mathbb{R}^G$ *and let*

$$\mathcal{P}_2^{RV}(\boldsymbol{v}) := \inf\left\{\mu \in \mathbb{R} \middle| \mathbb{E}\left[\Lambda^{RV+}\left(\boldsymbol{X} + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right)\right] \geq \gamma\right\}, \tag{4.2.13}$$

and

$$\begin{aligned} \mathcal{Z}_2^{RV}(\boldsymbol{v}) := \inf\Bigg\{\mu \in \mathbb{R} \Bigg| & \sum_{k\in\mathcal{K}} q^k\left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \geq \gamma, \\ & \boldsymbol{p}^k \leq \alpha\left[\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right] + \beta\boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k + \bar{\boldsymbol{p}} \odot \boldsymbol{s}^k, \\ & \bar{\boldsymbol{p}} \odot \boldsymbol{s}^k \leq \left[\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right] + \boldsymbol{\Pi}^\mathsf{T}\boldsymbol{p}^k, \\ & \boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1} \geq 0, \\ & \boldsymbol{p}^k \in [\boldsymbol{0}, \bar{\boldsymbol{p}}], \boldsymbol{s}^k \in \{0,1\}^n, \forall k \in \mathcal{K}\Bigg\}. \end{aligned} \tag{4.2.14}$$

If $\mathcal{P}_2^{RV}(\boldsymbol{v})$ *and* $\mathcal{Z}_2^{RV}(\boldsymbol{v})$ *have finite optimal values, then* $\mathcal{P}_2^{RV}(\boldsymbol{v}) = \mathcal{Z}_2^{RV}(\boldsymbol{v})$.

The MILP problem $\mathcal{Z}_2^{\text{RV}}(\boldsymbol{v})$ in (4.2.14) can be written more explicitly as

$$\text{minimize} \quad \mu \tag{4.2.15}$$

$$
\begin{aligned}
\text{subject to} \quad & \sum_{k \in \mathcal{K}} q^k \left[\mathbb{1}^\mathsf{T} \boldsymbol{p}^k \right] \geq \gamma, && (4.2.16) \\
& p_i^k \leq \alpha \left[X_i \left(\omega^k \right) + \left(B^\mathsf{T} \boldsymbol{v} \right)_i + \mu \right] + \beta \sum_{j=1}^{n} \pi_{ji} p_j^k + \bar{p}_i s_i^k, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, && (4.2.17) \\
& \bar{p}_i s_i^k \leq \left[X_i \left(\omega^k \right) + \left(B^\mathsf{T} \boldsymbol{v} \right)_i + \mu \right] + \sum_{j=1}^{n} \pi_{ji} p_j^k, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, && (4.2.18) \\
& X_i \left(\omega^k \right) + \left(B^\mathsf{T} \boldsymbol{v} \right)_i + \mu \geq 0, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, && (4.2.19) \\
& 0 \leq p_i^k \leq \bar{p}_i, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, && (4.2.20) \\
& s_i^k \in \{0, 1\}, \quad \forall i \in \mathcal{N}, k \in \mathcal{K}. && (4.2.21)
\end{aligned}
$$

Here, constraint (4.2.19) ensures $\boldsymbol{X} + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1} \geq 0$, so that for every $k \in \mathcal{K}$ it holds that $\Lambda^{\mathrm{RV}+}\left(\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right) \neq -\infty$.

Proof of Corollary 4.2.7. Let $\dot{\mu} \in \mathbb{R}$ be an optimal solution to $\mathcal{P}_2^{\mathrm{RV}}(\boldsymbol{v})$ and let $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k \in \mathcal{K}}\right)$ be an optimal solution to $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$.

Take $\mathcal{Y} = \mathcal{Y}_{\mathrm{RV}}$ in Theorem 4.2.1 and let $\left(\dot{\boldsymbol{p}}^k, \dot{\boldsymbol{s}}^k\right)$ be an optimal solution to $\Lambda^{\mathrm{RV}+}\left(\boldsymbol{X}\left(\omega^k\right) + B^\mathsf{T}\boldsymbol{v} + \mu\mathbb{1}\right)$, for each $k \in \mathcal{K}$. Then by Theorem 4.2.1, $\mathcal{P}_2^{\mathrm{RV}}(\boldsymbol{v}) = \mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$. □

The next three propositions present some boundedness and feasibility results for the MILP problem $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$, $\boldsymbol{v} \in \mathbb{R}^G$, in Corollary 4.2.7.

Proposition 4.2.8. *Let $\boldsymbol{v} \in \mathbb{R}^G$ and consider $\mathcal{Z}_2^{RV}(\boldsymbol{v})$ in Corollary 4.2.7. If $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k \in \mathcal{K}}\right)$ is an optimal solution to $\mathcal{Z}_2^{RV}(\boldsymbol{v})$, then $\mu \leq \mu^{\max}$, where $\mu^{\max} = \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty$.*

Proposition 4.2.9. *Let $\boldsymbol{v} \in \mathbb{R}^G$ and consider $\mathcal{Z}_2^{RV}(\boldsymbol{v})$ in Corollary 4.2.7. If $\mathcal{Z}_2^{RV}(\boldsymbol{v})$ is feasible, then it is bounded.*

Proposition 4.2.10. *Let $\boldsymbol{v} \in \mathbb{R}^G$. Then, $\mathcal{Z}_2^{RV}(\boldsymbol{v})$ in Corollary 4.2.7 is feasible if and only if $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$.*

The proofs of Propositions 4.2.8, 4.2.9 and 4.2.10 can be found in Appendices B.10, B.11 and B.12.

Remark 4.2.11. For $\ell \in \mathcal{G}$ and $\boldsymbol{v} \in \mathbb{R}^G$, a threshold γ in the MILP problems $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$, $\mathcal{Z}_1^{\text{RV}}\left(\boldsymbol{e}^\ell\right)$, $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$, $\mathcal{Z}_2^{\text{RV}}(\boldsymbol{v})$ in Corollaries 4.1.5, 4.1.10, 4.2.2 and 4.2.7, respectively, can be taken as some percentage of $\mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$, sum of the debts of all nodes in the network. Then this threshold ensures that the expected total amount of payments exceeds this fraction of the total debt in the system. Indeed, Corollaries 4.1.8, 4.1.13, 4.2.5 and 4.2.10 show that the MILP problems $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$, $\mathcal{Z}_1^{\text{RV}}\left(\boldsymbol{e}^\ell\right)$, $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$, $\mathcal{Z}_2^{\text{RV}}(\boldsymbol{v})$ are feasible if and only if $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$. Hence, this choice of γ threshold is justified.

Proposition 4.1.7 shows that if the MILP problem $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$ in (4.1.9) is feasible for every $\ell \in \mathcal{G}$, then the ideal point $z^{\text{ideal}} \in \mathbb{R}^n$ exists for the vector optimization problem in (4.1.1) with $\Lambda^{\text{OPT}} = \Lambda^{\text{EN}}$. Proposition 4.1.12 provides the same result for the vector optimization problem in (4.1.1) with $\Lambda^{\text{OPT}} = \Lambda^{\text{RV}+}$. In addition, the results of Propositions 4.1.6, 4.1.7, 4.2.3 and 4.2.4 allow one to choose the exact value for the upper bound M in the corresponding MILP problems instead of assuming some vague heuristic values.

The next chapter outlines the Benson type algorithm for non-convex problems introduced in Nobakhtian and Shafiei [1]. It is described in detail how this algorithm approximates the Eisenberg-Noe and Rogers-Veraart systemic risk measures, which are not necessarily convex. The related pseudo-codes are provided.

Chapter 5

A Benson Type Algorithm to Approximate the Eisenberg-Noe and Rogers-Veraart Systemic Risk Measures

In this chapter, an algorithm that approximates the Eisenberg-Noe and Rogers-Veraart systemic risk measures is presented. The systemic risk measures are approximated with respect to a user-defined approximation error $\epsilon > 0$ and an upper bound point $\boldsymbol{z}^{\text{UB}} \in \mathbb{R}^G$. The algorithm is based on the Benson type algorithm for non-convex multi-objective programming problems described in Nobakhtian and Shafiei [1]. The following definitions are borrowed from [1].

Let $\mathcal{L} \subseteq \mathbb{R}^G$. A point $\boldsymbol{v} \in \mathcal{L}$ is called a *vertex* of $\mathcal{L}$ if it cannot be expressed as a strict convex combination of two distinct points of $\mathcal{L} \cap N$, where N is a neighborhood of $\boldsymbol{v}$. A set of all vertices of $\mathcal{L}$ is denoted by vert$\mathcal{L}$. The notation int$\mathcal{L}$ denotes the interior of $\mathcal{L}$. Given a point $\boldsymbol{z} \in \mathbb{R}^G$ and $\mathcal{L} \subseteq \mathbb{R}^G$, $\mathcal{L}|_{\boldsymbol{z}} := \{\boldsymbol{v} \in \mathcal{L} | \boldsymbol{v} \leq \boldsymbol{z}\}$ denotes the set of all points in $\mathcal{L}$ which are less than or equal to $\boldsymbol{z}$ in all components.

Let $R, \mathcal{L}, \mathcal{U} \subseteq \mathbb{R}^G$, $\boldsymbol{z} \in \mathbb{R}^G$ and $\epsilon > 0$ be a positive real number. The set $\mathcal{L}$ is called an *outer approximation* for R with respect to ϵ and $\boldsymbol{z}$, if $R \subseteq \mathcal{L}$ and $\mathcal{L}|_{\boldsymbol{z}} \subseteq R + B(\mathbf{0}, \epsilon)$, where $B(\mathbf{0}, \epsilon)$ is the closed ball in $\mathbb{R}^G$ centered at $\mathbf{0}$ and with radius ϵ. The set $\mathcal{U}$ is called an *inner approximation* for R with respect to ϵ and $\boldsymbol{z}$ if R is an outer approximation for $\mathcal{U}$ with respect to ϵ and $\boldsymbol{z}$.

The algorithm that calculates inner and outer approximations of a systemic risk measure works as follows. It is provided in detail only for the Eisenberg-Noe systemic risk measures, since it works similarly for the Rogers-Veraart systemic risk measures. Let $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X})$ be a signed Eisenberg-Noe network. Let G be the number of groups in the network and $\mathcal{G} = \{1, \ldots, G\}$. Consider the corresponding Eisenberg-Noe systemic risk measure $R^{sen}_{\text{EN}}(\boldsymbol{X})$ in (4.0.7). Let $\boldsymbol{z}^{\text{ideal}} \in \mathbb{R}^G$ be the ideal point of the vector optimization problem in (4.1.1) with $\Lambda^{\text{OPT}} = \Lambda^{\text{EN}}$, in the sense that the entries of $\boldsymbol{z}^{\text{ideal}}$ minimize each of the objective functions of the vector optimization problem. One can calculate $\boldsymbol{z}^{\text{ideal}} = \left(\mathcal{Z}^{\text{EN}}_1(\boldsymbol{e}^1), \ldots, \mathcal{Z}^{\text{EN}}_1(\boldsymbol{e}^G)\right)^{\mathsf{T}}$ by Corollary 4.1.5. In addition, for $\boldsymbol{v} \in \mathbb{R}^G$, the minimum step-length $\mathcal{P}^{\text{EN}}_2(\boldsymbol{v})$ can be obtained by solving the MILP problem $\mathcal{Z}^{\text{EN}}_2(\boldsymbol{v})$ in (4.2.4), by Corollary 4.2.2.

The algorithm starts with an initial inner approximation $\mathcal{U}^0 := \boldsymbol{z}^{\text{UB}} + \mathbb{R}^G_+$ and an initial outer approximation $\mathcal{L}^0 := \boldsymbol{z}^{\text{ideal}} + \mathbb{R}^G_+$, which satisfy $\mathcal{U}^0 \subseteq R^{sen}_{\text{EN}}(\boldsymbol{X}) \subseteq \mathcal{L}^0$. Let $\varepsilon = \epsilon\mathbb{1}$ and initially set $t \leftarrow 0$. At the t^{th} iteration, for a vertex $\boldsymbol{v}^t \in \text{vert}\mathcal{L}^t|_{\boldsymbol{z}^{\text{UB}}}$ such that $\boldsymbol{v}^t + \varepsilon \notin \text{int}\,\mathcal{U}^t$, the algorithm solves $\mathcal{Z}^{\text{EN}}_2(\boldsymbol{v}^t)$ to obtain the minimum step-length μ^t from the point $\boldsymbol{v}^t$ to the boundary of $R^{sen}_{\text{EN}}(\boldsymbol{X})$ in the direction $\mathbb{1} \in \mathbb{R}^G$. In other words, $\boldsymbol{y}^t = \boldsymbol{v}^t + \mu^t\mathbb{1}$ is a boundary point of the set $R^{sen}(\boldsymbol{X})$. Then the algorithm excludes the cone $\boldsymbol{y}^t - \mathbb{R}^G_+$ from $\mathcal{L}^t$ to obtain $\mathcal{L}^{t+1}$ by $\mathcal{L}^{t+1} := \mathcal{L}^t \setminus \left(\boldsymbol{y}^t - \mathbb{R}^G_+\right)$, and adds the cone $\boldsymbol{y}^t + \mathbb{R}^G_+$ to $\mathcal{U}^t$ to obtain $\mathcal{U}^{t+1}$ as follows: $\mathcal{U}^{t+1} := \mathcal{U}^t \cup \left(\boldsymbol{y}^t + \mathbb{R}^G_+\right)$. Therefore, at each step of the algorithm we have $\mathcal{U}^t \subseteq \mathcal{U}^{t+1} \subseteq R^{sen}_{\text{EN}}(\boldsymbol{X}) \subseteq \mathcal{L}^{t+1} \subseteq \mathcal{L}^t$. At the end of the t^{th} iteration, $\text{vert}\mathcal{L}^{t+1}$ is computed. The computation of $\text{vert}\mathcal{L}^{t+1}$ is described in detail in Gourion and Luc [23]. The above process repeats for $t \leftarrow t+1$. The algorithm stops at T^{th} iteration, when $\text{vert}\mathcal{L}^T|_{\boldsymbol{z}^{\text{UB}}} + \varepsilon \subseteq \text{int}\,\mathcal{U}^T$. The sets $\mathcal{U}^T$ and $\mathcal{L}^T$ are the inner and outer approximations for $R^{sen}_{\text{EN}}(\boldsymbol{X})$ with respect to $\epsilon > 0$ and $\boldsymbol{z}^{\text{UB}} \in \mathbb{R}^G$. Note that $\boldsymbol{z}^{\text{UB}}$ have to be chosen such that $\boldsymbol{z}^{\text{UB}} \in R^{sen}_{\text{EN}}(\boldsymbol{X})$ to get non-empty approximations. The pseudo-codes of the algorithm for the Eisenberg-Noe and Rogers-Veraart systemic risk measures are provided in Algorithm 1 and Algorithm 2, respectively.

Algorithm 1. Inner and outer approximation algorithm for the Eisenberg-Noe systemic risk measures

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Initialization.

(i1) Let $\boldsymbol{z}^{\mathrm{UB}} \in R_{\mathrm{EN}}^{sen}(\boldsymbol{X})$ be an upper bound, $\mathcal{L}^0 = \boldsymbol{z}^{\mathrm{ideal}} + \mathbb{R}^G_+$, $\mathcal{U}^0 = \boldsymbol{z}^{\mathrm{UB}} + \mathbb{R}^G_+$ and $\epsilon > 0$ be an approximation error.

(i2) Put $\varepsilon = \epsilon \mathbb{1}$ and $t \leftarrow 0$.

(i3) Let S be an empty set.

Iteration steps.

(k1) If $\mathrm{vert}\mathcal{L}^t|_{\boldsymbol{z}^{\mathrm{UB}}} \subseteq S$ set $T = t$ and stop. Otherwise, choose $\boldsymbol{v}^t \in \mathrm{vert}\mathcal{L}^t|_{\boldsymbol{z}^{\mathrm{UB}}} \backslash S$.

(k2) If $\boldsymbol{v}^t + \varepsilon \in \mathrm{int}\,\mathcal{U}^t$, add $\boldsymbol{v}^t$ to S and go to (k1).

(k3) Suppose that $\mu^t = \mathcal{P}_2^{\mathrm{EN}}(\boldsymbol{v}^t)$. Define $\boldsymbol{y}^t = \boldsymbol{v}^t + \mu^t \mathbb{1}$.

(k4) Define $\mathcal{L}^{t+1} := \mathcal{L}^t \backslash \left(\boldsymbol{y}^t - \mathbb{R}^G_+\right)$ and $\mathcal{U}^{t+1} := \mathcal{U}^t \cup \left(\boldsymbol{y}^t + \mathbb{R}^G_+\right)$.

(k5) Determine $\mathrm{vert}\mathcal{L}^{t+1}$ and set $t \leftarrow t+1$. Go to (k1).

Results.

(r1) $\mathcal{L}^T$ is an outer approximation and $\mathcal{U}^T$ is an inner approximation for $R_{\mathrm{EN}}^{sen}(\boldsymbol{X})$ with respect to $\epsilon > 0$ and $\boldsymbol{z}^{\mathrm{UB}} \in \mathbb{R}^G$.

Algorithm 2. Inner and outer approximation algorithm for the Rogers-Veraart systemic risk measures

Initialization.

(i1) Let $\boldsymbol{z}^{\text{UB}} \in R^{sen}_{\text{RV}}(\boldsymbol{X})$ be an upper bound, $\mathcal{L}^0 = \boldsymbol{z}^{\text{ideal}} + \mathbb{R}^G_+$, $\mathcal{U}^0 = \boldsymbol{z}^{\text{UB}} + \mathbb{R}^G_+$ and $\epsilon > 0$ be an approximation error.

(i2) Put $\varepsilon = \epsilon \mathbb{1}$ and $t \leftarrow 0$.

(i3) Let S be an empty set.

Iteration steps.

(k1) If $\text{vert}\mathcal{L}^t|_{\boldsymbol{z}^{\text{UB}}} \subseteq S$ set $T = t$ and stop. Otherwise, choose $\boldsymbol{v}^t \in \text{vert}\mathcal{L}^t|_{\boldsymbol{z}^{\text{UB}}} \backslash S$.

(k2) If $\boldsymbol{v}^t + \varepsilon \in \text{int}\,\mathcal{U}^t$, add $\boldsymbol{v}^t$ to S and go to (k1).

(k3) Suppose that $\mu^t = \mathcal{P}^{\text{RV}}_2(\boldsymbol{v}^t)$. Define $\boldsymbol{y}^t = \boldsymbol{v}^t + \mu^t \mathbb{1}$.

(k4) Define $\mathcal{L}^{t+1} := \mathcal{L}^t \backslash \left(\boldsymbol{y}^t - \mathbb{R}^G_+\right)$ and $\mathcal{U}^{t+1} := \mathcal{U}^t \cup \left(\boldsymbol{y}^t + \mathbb{R}^G_+\right)$.

(k5) Determine $\text{vert}\mathcal{L}^{t+1}$ and set $t \leftarrow t+1$. Go to (k1).

Results.

(r1) $\mathcal{L}^T$ is an outer approximation and $\mathcal{U}^T$ is an inner approximation for $R^{sen}_{\text{RV}}(\boldsymbol{X})$ with respect to $\epsilon > 0$ and $\boldsymbol{z}^{\text{UB}} \in \mathbb{R}^G$.

Chapter 6

Computational Results and Analysis

In this chapter, some computational results are presented to illustrate the approximation of the Eisenberg-Noe and Rogers-Veraart systemic risk measures by the Benson type algorithm described in Chapter 5. The algorithm is implemented on Java Photon (Release 4.8.0) calling Gurobi Interactive Shell (Version 7.5.2) and run on an Intel(R) Core(TM) i7-4790 processor with 3.60 GHz and 4 GB RAM. First, the Eisenberg-Noe and Rogers-Veraart systemic risk measures are approximated within two-group frameworks. Then, sensitivity analyses are performed in the scope of these frameworks. Finally, the Eisenberg-Noe and Rogers-Veraart systemic risk measures are approximated within three-group frameworks. The corresponding computational results and approximations are presented.

Recall that n is the number of institutions in a financial system, n_ℓ is the number of nodes in a group $\ell \in \mathcal{G}$, K is the number of scenarios, ϵ is a user-defined approximation error and $\boldsymbol{z}^{\mathrm{UB}}$ is a user-defined upper-bound vector that limits the approximated region of a systemic risk measure. Throughout the computation of systemic risk measures, except for the Rogers-Veraart case in a three-group framework, $\boldsymbol{z}^{\mathrm{UB}}$ is taken as

$$\boldsymbol{z}^{\mathrm{UB}} = \hat{\boldsymbol{z}}^{\mathrm{ideal}} + 2\left\|\bar{\boldsymbol{p}}\right\|_{\infty},$$

where $\hat{\boldsymbol{z}}^{\mathrm{ideal}}$ is the ideal point of the corresponding systemic risk measure with $\gamma = \mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}$, that is, when it is required that the expected total value of payments is at least as much as the total amount of liabilities in the network.

Throughout this chapter, γ is taken as some percentage of $\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}$, the total debt in a network with the total obligation vector $\bar{\boldsymbol{p}}$. Hence, the choice of γ threshold has a nice and intuitive financial interpretation as the minimum amount of liabilities that should be met on average in the network. As it was already mentioned in Chapter 4, this choice of γ is justified by Corollaries 4.1.8, 4.1.13, 4.2.5 and 4.2.10. For the simplicity of notation in the subsequent sections, let γ^{p} denote the fraction of the total debt that should be met on average in the network, that is, $\gamma = \gamma^{\mathrm{p}}\left(\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}\right)$ and $\gamma^{\mathrm{p}} \in [0, 1]$.

6.1 Data Generation

From now on, nodes in any particular network are considered as banks. In the scope of this computational study, a network of n banks is grouped into two or three categories. These correspond to cases when the number of groups G is 2 or 3, respectively, $\mathcal{G} = \{1, \ldots, G\}$, $\mathcal{N} = \bigcup_{\ell\in\mathcal{G}} \mathcal{N}_\ell = \{1, \ldots, n\}$, and $n_\ell = |\mathcal{N}_\ell|$ corresponds to the number of banks in group $\ell \in \mathcal{G}$. In the case of two groups, $\ell = 1$ and $\ell = 2$ correspond to big and small banks, respectively. In the case of three groups, $\ell = 1$, $\ell = 2$ and $\ell = 3$ correspond to big, medium and small banks, respectively. For a signed Eisenberg-Noe network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X})$ and a Rogers-Veraart network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$, the corresponding interbank liabilities matrix $\boldsymbol{l} := (l_{ij})_{i,j\in\mathcal{N}} \in \mathbb{R}_{+}^{n\times n}$ and the random operating cash flow vector $\boldsymbol{X}$ are generated in the following fashion.

Recall that a financial system is considered as a network with nodes corresponding to the members of the system and directed arcs corresponding to the liabilities between the members. To generate these connections in terms of the interbank liabilities matrix $\boldsymbol{l}$ we use the idea of random graphs, also referred as the Erdös-Rényi model, presented by Erdös and Rényi [24] and, independently, by Gilbert [25]. First, we fix a connectivity probabilities matrix $\boldsymbol{q}^{\mathrm{con}} := \left(q^{\mathrm{con}}_{\ell,\hat{\ell}}\right)_{\ell,\hat{\ell}\in\mathcal{G}} \in \mathbb{R}^{G\times G}$ and an intergroup liabilities matrix $\boldsymbol{l}^{\mathrm{gr}} := \left(l^{\mathrm{gr}}_{\ell,\hat{\ell}}\right)_{\ell,\hat{\ell}\in\mathcal{G}} \in \mathbb{R}^{G\times G}$. These matrices are interpreted as follows. For any two banks $i, j \in \mathcal{N}$ with $i \in \mathcal{N}_\ell$, $j \in \mathcal{N}_{\hat{\ell}}$ and $\ell, \hat{\ell} \in \mathcal{G}$, $q^{\mathrm{con}}_{\ell,\hat{\ell}}$ is a probability that the bank i owes $l^{\mathrm{gr}}_{\ell,\hat{\ell}}$ amount to the bank j. Then the liability l_{ij} is generated by the Bernoulli trial

$$l_{ij} = \begin{cases} l^{\mathrm{gr}}_{\ell,\hat{\ell}}, & \text{if } U_{ij} < q^{\mathrm{con}}_{\ell,\hat{\ell}}, \\ 0, & \text{otherwise,} \end{cases}$$

where U_{ij} is the realization of a continuous random variable with a standard uniform distribution on a separate probability space. Once the interbank liabilities matrix $\boldsymbol{l}$ is generated, the relative liabilities matrix $\boldsymbol{\pi}$ and the total obligation vector $\bar{\boldsymbol{p}}$ are derived from it by the following relations, described originally in Eisenberg and Noe [2],

$$\bar{p}_i = \sum_{j\in\mathcal{N}} l_{ij}, \quad i \in \mathcal{N}; \qquad \pi_{ij} = \frac{l_{ij}}{\bar{p}_i}, \quad i, j \in \mathcal{N}.$$

Recall that the operating cash flow vector $\boldsymbol{X} = (X_1, \ldots, X_n) \in L(\mathbb{R}^n)$ is a multivariate random vector and Ω is a finite set of K scenarios. It is assumed that all scenarios are equally likely to happen, the operating cash flows have a common standard deviation σ, and there is a common correlation ϱ between any two operating cash flows. Then, each entry X_i, $i \in \mathcal{N}$, is generated as a random sample of size K as described below.

For the Eisenberg-Noe network, the mean values of operating cash flows in each group, $\boldsymbol{\nu} := (\nu_\ell)_{\ell\in\mathcal{G}}$, are fixed and the random vector $\boldsymbol{X}$ is generated in a way that its joint cumulative distribution function is stated in terms of a Gaussian copula and its marginal distributions are Gaussian distributions. Hence, for each $i \in \mathcal{N}$ with $i \in \mathcal{N}_\ell$ for some $\ell \in \mathcal{G}$, X_i has a Gaussian distribution with mean ν_ℓ and common standard deviation σ.

On the other hand, for the Rogers-Veraart network, first, shape parameters $\boldsymbol{\kappa} := (\kappa_\ell)_{\ell \in \mathcal{G}}$ and scale parameters $\boldsymbol{\theta} := (\theta_\ell)_{\ell \in \mathcal{G}}$ are fixed and then, due to the assumption that operating cash flows are nonnegative, $\boldsymbol{X}$ is generated in a way that its joint cumulative distribution function is stated in terms of a Gaussian copula and its marginal distributions are gamma distributions. That is, for each $i \in \mathcal{N}$ with $i \in \mathcal{N}_\ell$ for some $\ell \in \mathcal{G}$, X_i has a gamma distribution with shape parameter κ_i and scale parameter θ_i. Then, $\nu_\ell = \kappa_\ell \theta_\ell$ for each $\ell \in \mathcal{G}$. Shape and scale parameters of the gamma distributions are chosen in a way that the random operating cash flows have a common standard deviation $\sigma = \sqrt{\kappa_\ell}\theta_\ell$, for each $\ell \in \mathcal{G}$. For both Eisenberg-Noe and Rogers-Veraart networks, it is assumed that X_i and X_j have a common correlation coefficient $\varrho \in [-1, 1]$, for every $i, j \in \mathcal{N}$ such that $i \neq j$.

In the following sections, computational results and approximations of some Eisenberg-Noe and Rogers-Veraart systemic risk measures are presented and analyzed from different points of view. First, general computational results for a two-group Eisenberg-Noe systemic risk measure are presented. Then sensitivity analysis results are provided for two-group Eisenberg-Noe and Rogers-Veraart systemic risk measures by changing various parameters of the generated networks, namely, connectivity probabilities, number of scenarios, γ threshold, numbers of banks in groups, α and β parameters of Rogers-Veraart networks, mean values of operating cash flows and a common correlation ϱ between operating cash flows.

6.2 A Two-Group Signed Eisenberg-Noe Network with 50 Nodes and 100 Scenarios

In this part, a two-group Eisenberg-Noe network of banks is generated and the corresponding systemic risk measure is approximated by the Benson type algorithm with different levels of approximation error. The corresponding results, namely, computation times, inner and outer approximations are provided below.

The network is generated with the following parameters: the number of banks in the network $n = 50$, the number of big banks $n_1 = 15$, the number of small banks $n_2 = 35$, the number of scenarios $K = 100$, the common standard deviation $\sigma = 100$, and the common correlation $\varrho = 0.05$. In addition, the following connectivity probabilities matrix, the intergroup liabilities matrix and the mean values of the operating cash flows are assumed:

$$\boldsymbol{q}^{\text{con}} = \begin{bmatrix} 0.9 & 0.3 \\ 0.7 & 0.5 \end{bmatrix}, \quad \boldsymbol{l}^{\text{gr}} = \begin{bmatrix} 10 & 5 \\ 8 & 5 \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} -50 & -100 \end{bmatrix}.$$

We take $\gamma^p = 0.7$. Hence,

$$R^{sen}_{\text{EN}}(\boldsymbol{X}) = \left\{ \boldsymbol{z} \in \mathbb{R}^2 \middle| \mathbb{E}\left[\Lambda^{\text{EN}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq 0.7\left(\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}\right) \right\}, \tag{6.2.1}$$

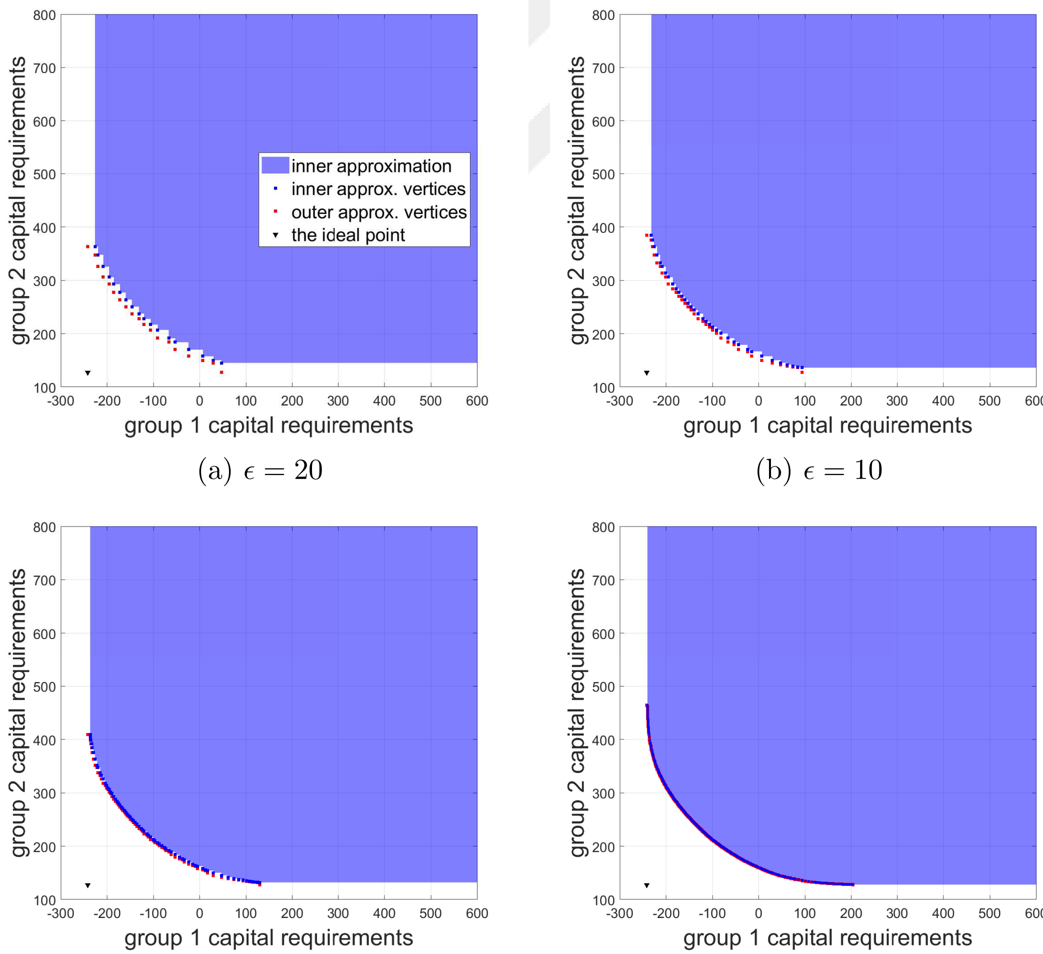
where the aggregation function Λ^{EN} is defined as in (3.2.7).

The Benson type algorithm is run with four different approximation errors ϵ to demonstrate different approximation levels both for inner and outer approximations. Table 6.1 presents the computational performance of the algorithm for $\epsilon \in \{1, 5, 10, 20\}$.

Figure 6.1 shows the inner approximations of the Eisenberg-Noe systemic risk measure in (6.2.1) for $\epsilon \in \{1, 5, 10, 20\}$. Figure 6.2 consists of zoomed portions of the inner approximations in Figure 6.1. Figure 6.3 shows the corresponding outer approximations. Figure 6.4 consists of zoomed portions of the outer approximations in Figure 6.3.

ϵ	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
20	18	19	18	663.546	11944	3.318
10	35	36	35	541.419	18950	5.264
5	73	74	73	512.998	37449	10.403
1	394	395	394	492.597	194083	53.912

Table 6.1: Computational performance of the algorithm for a network of 15 big and 35 small banks, 100 scenarios and approximation errors $\epsilon \in \{1, 5, 10, 20\}$.



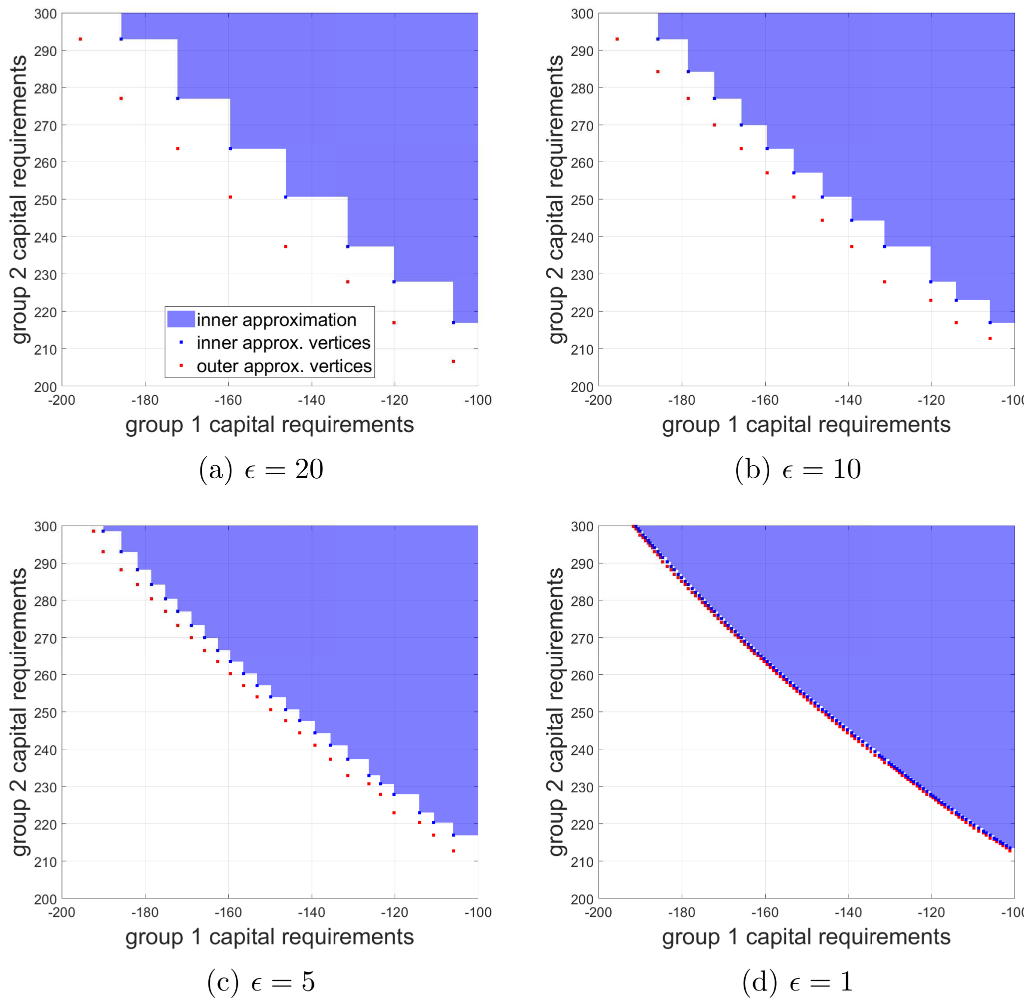
(a) $\epsilon = 20$

(b) $\epsilon = 10$

(c) $\epsilon = 5$

(d) $\epsilon = 1$

Figure 6.1: Inner approximations of the Eisenberg-Noe systemic risk measure for $\epsilon \in \{1, 5, 10, 20\}$.



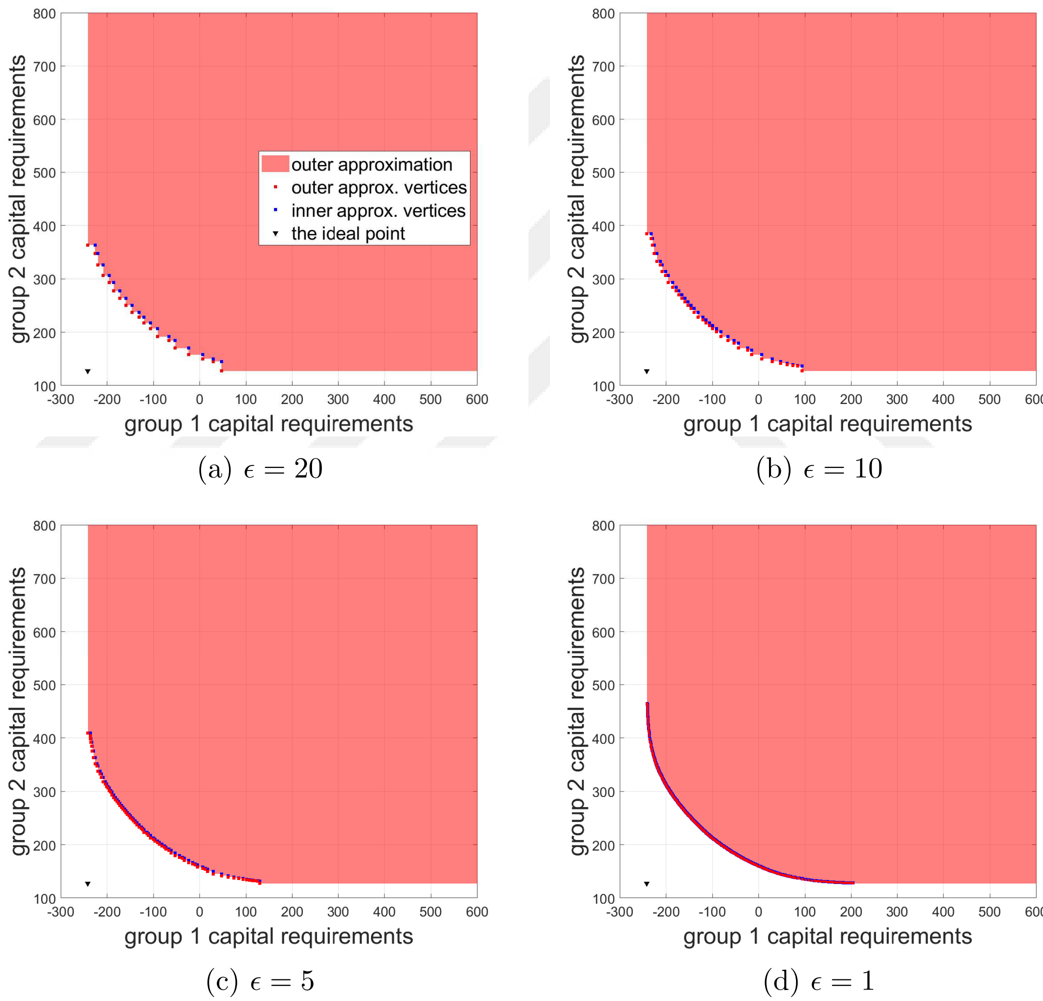
(a) $\epsilon = 20$

(b) $\epsilon = 10$

(c) $\epsilon = 5$

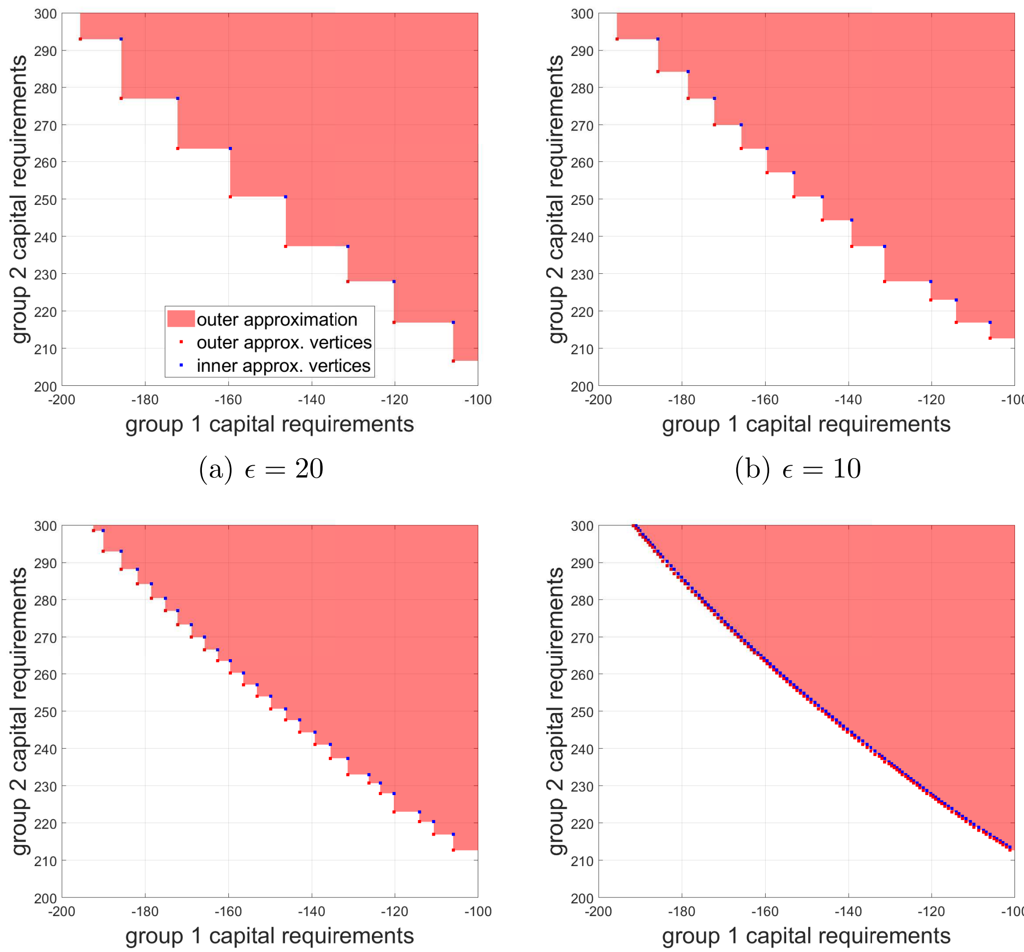
(d) $\epsilon = 1$

Figure 6.2: Zoomed portions of the inner approximations in Figure 6.1.



(a) $\epsilon = 20$ (b) $\epsilon = 10$ (c) $\epsilon = 5$ (d) $\epsilon = 1$

Figure 6.3: Outer approximations of the Eisenberg-Noe systemic risk measure for $\epsilon \in \{1, 5, 10, 20\}$.



(a) $\epsilon = 20$ (b) $\epsilon = 10$

(c) $\epsilon = 5$ (d) $\epsilon = 1$

Figure 6.4: Zoomed portions of the outer approximations in Figure 6.3.

One can easily observe from Figure 6.2 and Figure 6.4 that as ϵ decreases the algorithm gives more precise approximations of the systemic risk measure. In addition, as the number of $\mathcal{P}_2$ problems increases, the average computation time per $\mathcal{P}_2$ problem decreases. This may be attributed to the warm start feature of the Gurobi solver. When a sequence of mixed-integer programming problems are solved, the solver constructs an initial solution out of the previously obtained optimal solution. This feature is explained in detail in Gurobi Optimizer Reference Manual [26, Chapter 10.2, pp. 594-595].

In the next two sections, sensitivity analyses on several parameters are performed and the corresponding computation times and inner approximations of Eisenberg-Noe systemic risk measures are compared. The parameters investigated are connectivity probabilities between big and small banks, number of scenarios, γ threshold and distribution of nodes among groups. These are performed on two different two-group Eisenberg-Noe networks, one with 50 banks and 100 scenarios and the other one with 70 banks and 50 scenarios.

6.3 Sensitivity Analyses on Two-Group Signed Eisenberg-Noe Networks with 50 Nodes and 100 Scenarios

In this section, sensitivity analyses are performed on connectivity probabilities between big and small banks and on the number of scenarios. We take the two-group signed Eisenberg-Noe network investigated in the previous section as a base, where

$$\boldsymbol{q}^{\text{con}} = \begin{bmatrix} 0.9 & 0.3 \\ 0.7 & 0.5 \end{bmatrix}, \quad \boldsymbol{l}^{\text{gr}} = \begin{bmatrix} 10 & 5 \\ 8 & 5 \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} -50 & -100 \end{bmatrix},$$

$n = 50$, $n_1 = 15$, $n_2 = 35$, $K = 100$, $\sigma = 100$, and the $\varrho = 0.05$. The corresponding Eisenberg-Noe systemic risk measure requires $\gamma^p = 0.7$ and the approximation error in the Benson type algorithm is taken as $\epsilon = 1$.

6.3.1 Connectivity Probabilities

In this part, a sensitivity analysis on connectivity probabilities between the groups of big banks and small banks is performed and the corresponding changes in Eisenberg-Noe systemic risk measures are reported and analyzed. Connectivity probabilities play a major role in generating networks of banks for the computational purposes in this study because they define the existence of liabilities between the banks. Hence, changing them changes the structure of the generated network. First, the connectivity probability corresponding to liabilities of big banks to small banks is changed, keeping all the other parameters fixed. Then, the one corresponding to liabilities of small banks to big banks is considered in a similar way.

6.3.1.1 Connectivity Probability that a Big Bank is Liable to a Small Bank

Here we present the results of the sensitivity analysis on $q^{\text{con}}_{1,2}$, the probability that a big bank is liable to a small bank. Originally, this connectivity probability is taken as $q^{\text{con}}_{1,2} = 0.3$. Table 6.2 shows the computational performance of the algorithm for $q^{\text{con}}_{1,2} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Figure 6.5 consists of the corresponding inner approximations.

Observe from Table 6.2 that the average time per $\mathcal{P}_2$ problem increases with $q^{\text{con}}_{1,2}$. This is the case because as $q^{\text{con}}_{1,2}$ increases, big and small banks in the network become more connected in terms of liabilities. Hence, the corresponding MILP formulations of $\mathcal{P}_2$ problems need more time to be solved. This seems to be the only factor behind the increase because most of the algorithm runtime is devoted to solving $\mathcal{P}_2$ problems and the number of $\mathcal{P}_2$ problems in each case does not change much.

$q_{1,2}^{\text{con}}$	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.1	279	280	358	294.07	105 277	29.244
0.3	394	395	394	492.597	194 083	53.912
0.5	360	361	360	556.795	200 447	55.680
0.7	364	365	364	633.644	230 647	64.069
0.9	377	378	377	772.76	291 331	80.925

Table 6.2: Computational performance of the algorithm for $q_{1,2}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

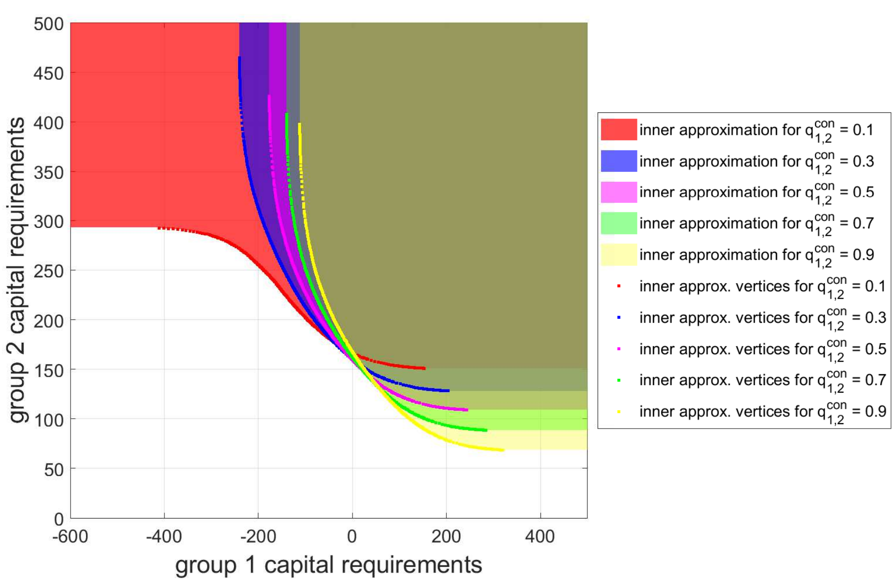


Figure 6.5: Inner approximations of the Eisenberg-Noe systemic risk measure for $q_{1,2}^{\text{con}} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

It can be observed that, as $q^{\text{con}}_{1,2}$ increases, the corresponding inner approximations of systemic risk measures in Figure 6.5 shift from the top left corner towards the bottom right corner. It can be interpreted as follows: as $q^{\text{con}}_{1,2}$ increases, the first group, the group of big banks, loses capital allocation options, while the second group, the group of small banks, gains a wider range of capital allocation options. It can also be observed from Figure 6.5 that generating a network with $q^{\text{con}}_{1,2} = 0.1$ results in a nonconvex Eisenberg-Noe systemic risk measure. However, for the values $q^{\text{con}}_{1,2} \in \{0.3, 0.5, 0.7, 0.9\}$, the corresponding Eisenberg-Noe systemic risk measures seem to be convex sets. Probably, for these cases, there is some breakpoint between 0.1 and 0.3 that switches these Eisenberg-Noe systemic risk measures from a nonconvex shape to a convex one, meaning that, whenever the probability $q^{\text{con}}_{1,2}$ is less than this breakpoint, big banks are less likely to be liable to small banks and have even more capital allocation options than they have in the other cases.

6.3.1.2 Connectivity Probability that a Small Bank is Liable to a Big Bank

Here we present the results of the sensitivity analysis on $q^{\text{con}}_{2,1}$, the probability that a big bank is liable to a small bank. Originally, this connectivity probability is taken as $q^{\text{con}}_{2,1} = 0.7$. Table 6.3 shows the computational performance of the algorithm for $q^{\text{con}}_{2,1} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Figure 6.6 consists of the corresponding inner approximations.

$q^{\text{con}}_{2,1}$	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.1	257	258	257	233.243	59943	16.651
0.3	294	295	294	319.511	93936	26.093
0.5	328	329	328	377.398	123787	34.385
0.7	394	395	394	492.597	194083	53.912
0.9	435	436	512	487.547	249624	69.340

Table 6.3: Computational performance of the algorithm for $q^{\text{con}}_{2,1} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

As in the previous sensitivity analysis, observe from Table 6.3 that the average time per $\mathcal{P}_2$ problem increases with $q^{\text{con}}_{2,1}$. Hence, it is another justification of the presumption that this happens because with higher connectivity probabilities the network becomes more connected in terms of liabilities and the corresponding MILP formulations of $\mathcal{P}_2$ problems need more time to be solved.

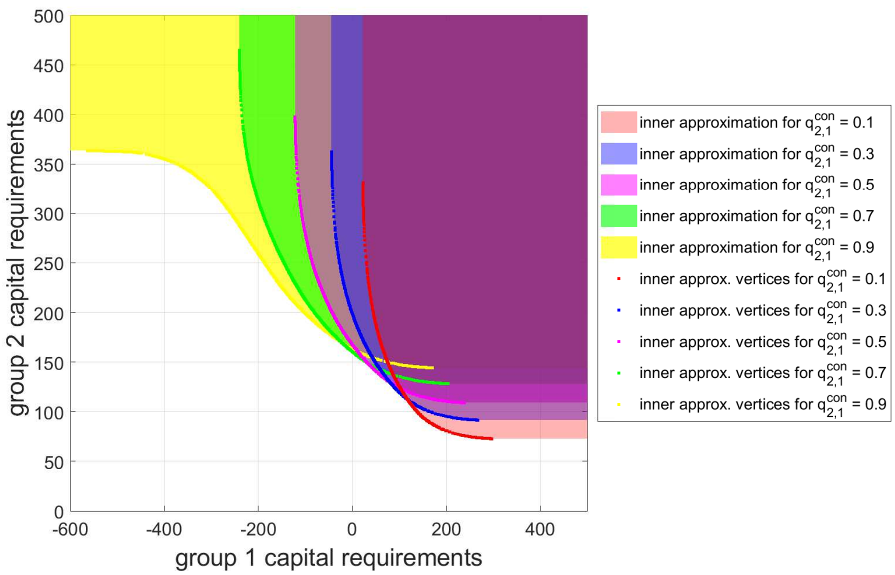


Figure 6.6: Inner approximations of the Eisenberg-Noe systemic risk measure for $q^{\text{con}}_{2,1} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

Note that as $q^{\text{con}}_{2,1}$ increases, the inner approximations of the corresponding Eisenberg-Noe systemic risk measures in Figure 6.6 shift from the bottom right corner towards the top left corner. Conversely to the previous sensitivity analysis, it can be interpreted as follows: as $q^{\text{con}}_{2,1}$ increases, the first group gains a wider range of capital allocation options, while the second group loses capital allocation options. It can also be observed from Figure 6.6 that generating a network with $q^{\text{con}}_{2,1} = 0.9$ results in a nonconvex Eisenberg-Noe systemic risk measure. However, for the values $q^{\text{con}}_{2,1} \in \{0.1, 0.3, 0.5, 0.7\}$, the corresponding Eisenberg-Noe systemic risk measures seem to be convex sets. As in the previous sensitivity analysis, it can be presumed that for these cases there is some breakpoint between 0.7 and 0.9 that switches these Eisenberg-Noe systemic risk measures from a convex shape to a nonconvex one, meaning that, whenever the probability $q^{\text{con}}_{2,1}$ is higher than this breakpoint, small banks are more likely to be liable to big banks and the latter have even more capital allocation options than they have in the other cases.

6.3.2 Number of Scenarios

In this part, a sensitivity analysis is performed by changing the number K of scenarios in the set $\{10, 20, \ldots, 100\}$. The main purpose is to analyze how computation times and the corresponding systemic risk measures change. Since the network structure remains the same all the time, it is expected that there will be no major changes in Eisenberg-Noe systemic risk measures. However, since each scenario adds n continuous and n binary variables to the corresponding $\mathcal{P}_2$ problem and its MILP formulation $\mathcal{Z}_2^{\mathrm{EN}}$, defined in (4.2.5), one would expect major changes in computation times.

Table 6.4 shows the computational performance of the algorithm for $K \in \{10, 20, \ldots, 100\}$ and Figure 6.7 provides the inner approximations of the corresponding Eisenberg-Noe systemic risk measures.

K	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
10	376	377	376	3.088	1 161	0.323
20	380	381	380	11.977	4 551	1.264
30	389	390	389	28.134	10 944	3.040
40	381	382	381	56.685	21 597	5.999
50	373	374	373	96.488	35 990	9.997
60	381	382	381	151.635	57 773	16.048
70	385	386	385	206.924	79 666	22.129
80	390	391	390	293.155	114 330	31.758
90	381	382	381	378.346	144 150	40.042
100	394	395	394	492.597	194 083	53.912

Table 6.4: Computational performance of the algorithm for $K \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$.

Finally, Figure 6.8 and Figure 6.9 suggest that the average time per $\mathcal{P}_2$ problem and the total algorithm time increase faster than linearly with K. At the

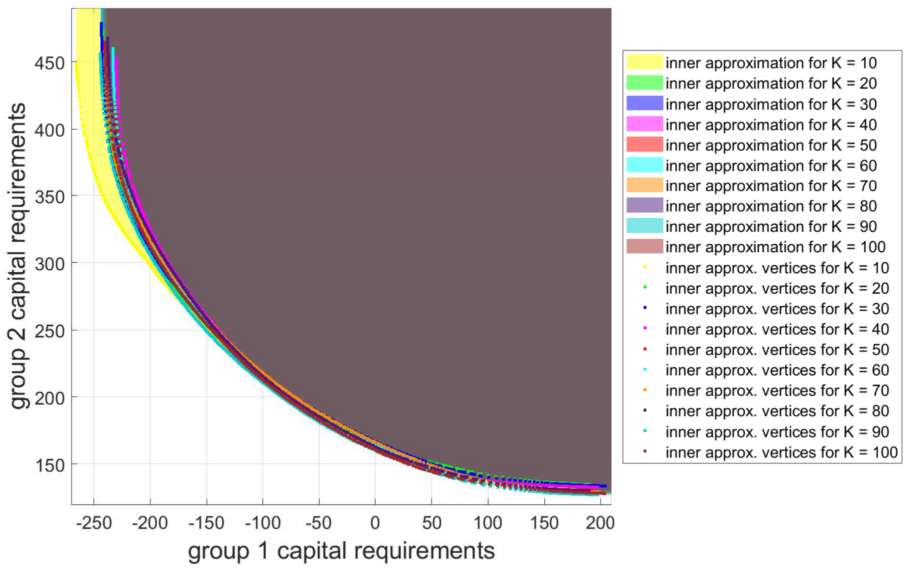


Figure 6.7: Inner approximations of the Eisenberg-Noe systemic risk measure for $K \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$.

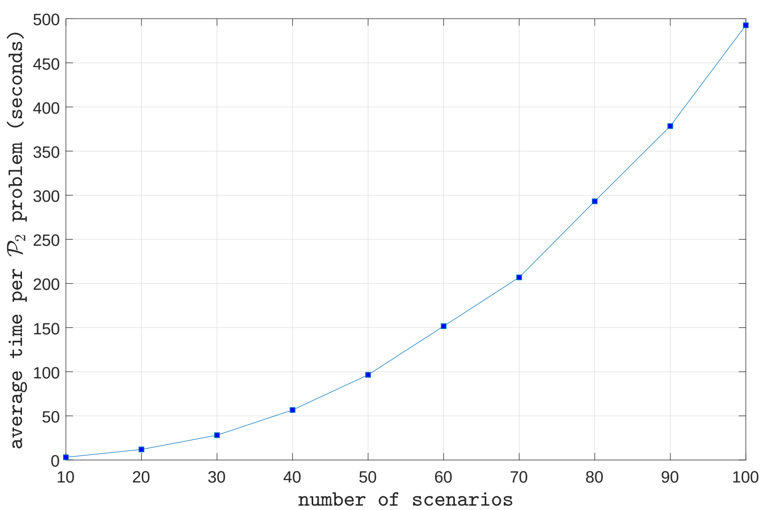


Figure 6.8: Scenarios-average time per $\mathcal{P}_2$ problem plot for the signed Eisenberg-Noe network of 50 banks.

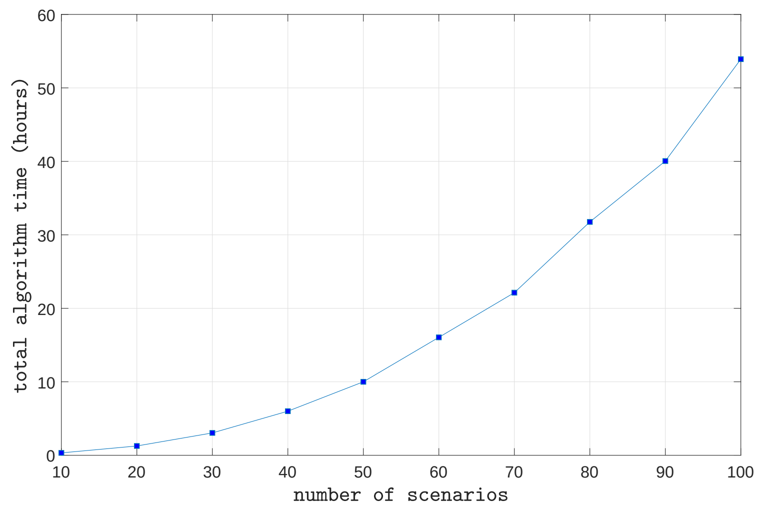


Figure 6.9: Scenarios-total algorithm time plot for the signed Eisenberg-Noe network of 50 banks.

same time, it can be observed from Figure 6.7 that the corresponding inner approximations of the Eisenberg-Noe systemic risk measures do not change much. Hence, the results obtained justify the expectations.

6.4 Sensitivity Analyses on Two-Group Signed Eisenberg-Noe Networks with 70 Nodes and 50 Scenarios

In this section, sensitivity analyses are performed on γ threshold, the distribution of nodes among groups, and the number of scenarios. An Eisenberg-Noe network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X})$ is generated with the following parameters:

$$\boldsymbol{q}^{\text{con}} = \begin{bmatrix} 0.7 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}, \quad \boldsymbol{l}^{\text{gr}} = \begin{bmatrix} 10 & 5 \\ 8 & 5 \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} -50 & -100 \end{bmatrix},$$

$n = 70$, $n_1 = 10$, $n_2 = 60$, $K = 50$, $\sigma = 100$, and $\varrho = 0.05$. The corresponding Eisenberg-Noe systemic risk measure requires $\gamma^p = 0.9$. Hence,

$$R^{sen}_{\mathrm{EN}}(\boldsymbol{X}) = \left\{ \boldsymbol{z} \in \mathbb{R}^2 \middle| \mathbb{E}\left[\Lambda^{\mathrm{EN}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq 0.9\left(\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}\right)\right\}, \qquad (6.4.1)$$

where the aggregation function Λ^{EN} is defined as in (3.2.7). The approximation error in the Benson type algorithm is taken as $\epsilon = 1$.

6.4.1 Threshold Level

In this part, different γ levels are compared and analyzed to investigate how the corresponding Eisenberg-Noe systemic risk measures and computational times change when the requirement that some fraction of the total amount of liabilities in the network should be met on average gets more strict. The values $\gamma^{\mathrm{p}} \in \{0.01, 0.1, 0.2, \ldots, 0.9, 0.95, 0.99, 1\}$ are considered. Recall that $\gamma = \gamma^{\mathrm{p}}\left(\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}\right)$.

Table 6.5 illustrates the computational performance of the algorithm for different values of γ^{p} and Figure 6.10 represents the corresponding inner approximations of the Eisenberg-Noe systemic risk measures.

It can be noted from Table 6.5 that the average times per $\mathcal{P}_2$ problem are high for the values of γ^{p} around 0.3, and the number of $\mathcal{P}_2$ problems are high for the values of γ^{p} around 0.5. These two factors result in high total algorithm times for the values of γ^{p} around 0.4. In addition, it can be observed that the difference between the number of inner and outer approximation vertices and the number of $\mathcal{P}_2$ problems increases drastically for the values of γ^{p} around 0.5. This happens because the boundaries (weakly minimal elements) of the corresponding Eisenberg-Noe systemic risk measures in Figure 6.10 contain "flat" regions, which makes the algorithm solve more $\mathcal{P}_2$ problems without actually improving the approximation. Observe from Figure 6.10 that as γ^{p} increases, each subsequent Eisenberg-Noe systemic risk measure is contained in the previous one. This result is fully consistent with the corresponding Eisenberg-Noe systemic risk measure in (6.4.1) since capital allocations that satisfy a particular γ threshold can at

γ^{p}	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.01	376	377	376	3.088	1 161	0.323
0.1	210	210	437	305.389	133 455	37.071
0.2	145	146	727	492.418	357 988	99.441
0.3	90	91	893	560.268	500 320	138.978
0.4	87	88	1037	494.65	512 952	142.487
0.5	91	95	1099	448.063	492 421	136.784
0.6	94	95	1065	240.982	256 646	71.291
0.7	96	97	927	97.501	90 383	25.106
0.8	141	142	719	45.546	32 748	9.097
0.9	234	235	461	15.285	7 047	1.957
0.95	217	218	217	11.622	2 522	0.701
0.99	136	137	136	2.504	341	0.095
1.00	1	1	1	0.203	0.204	0

Table 6.5: Computational performance of the algorithm for $\gamma^{\mathrm{p}} \in \{0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 1\}$.

the same time satisfy any smaller threshold. Hence, these capital allocations are included in any Eisenberg-Noe systemic risk measure with a smaller threshold level.

6.4.2 Distribution of Nodes among Groups

In this part, a sensitivity analysis is performed by changing the distribution of nodes among the groups for a fixed total number of nodes $n = 70$ and the number of big banks n_1 takes values in $\{5, 10, 20, \ldots, 60, 65\}$. Then the number of small banks is $n_2 = n - n_1$. As previously, the main purpose is to analyze how computation times and the corresponding systemic risk measures change. The generated random operating cash flows remain the same all the time, while the network structure changes at each run. Hence, the corresponding Eisenberg-Noe systemic risk measures are expected to vary a lot.

Table 6.6 shows the computational performance of the algorithm for $n_1 \in$

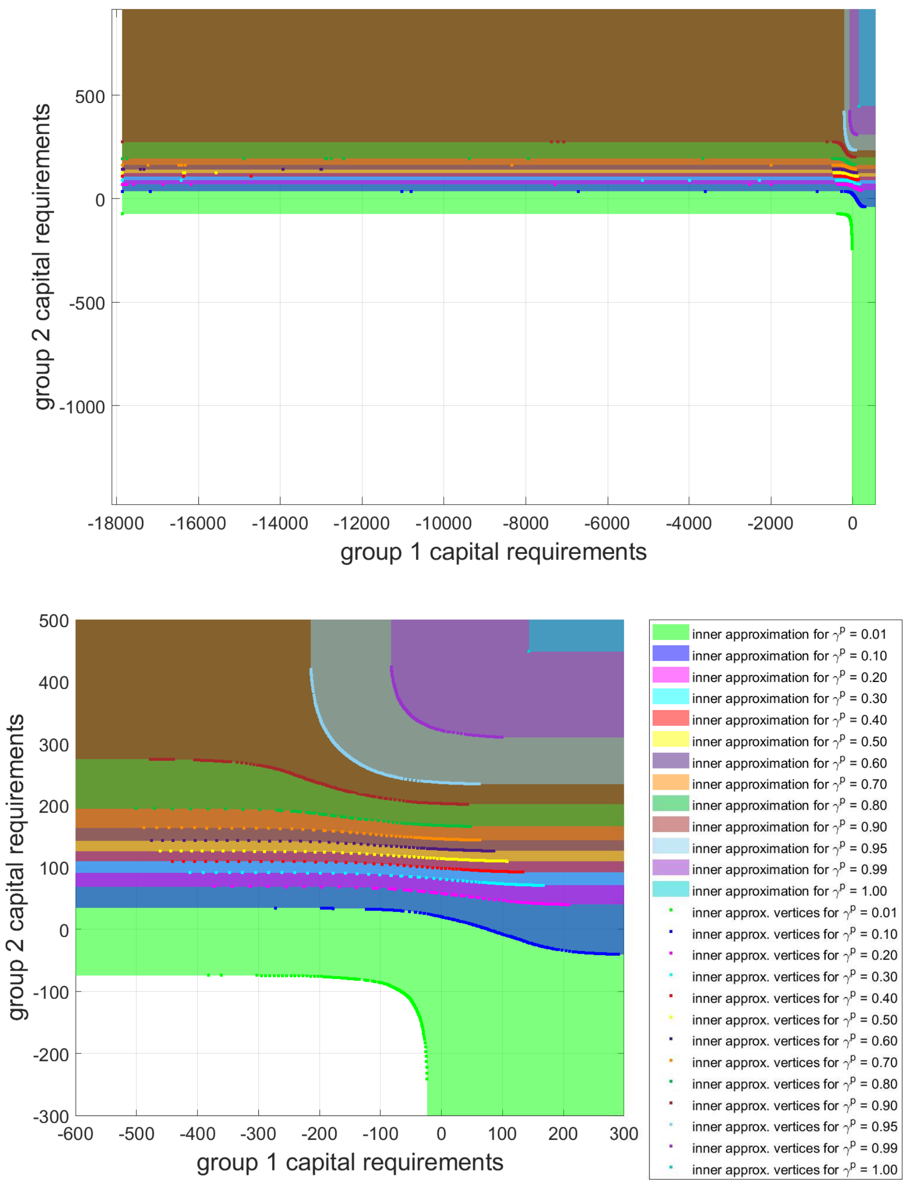


Figure 6.10: Inner approximations of the Eisenberg-Noe systemic risk measure for $\gamma^{\mathrm{p}} \in \{0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 1\}$.

$\{5, 10, 20, \ldots, 60, 65\}$ and Figure 6.11 represents the corresponding inner approximations of the Eisenberg-Noe systemic risk measures.

n_1	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
5	93	94	1096	16.88	18 501	5.139
10	234	235	461	15.285	7 047	1.957
20	209	210	209	38.512	8 049	2.236
30	201	202	201	45.225	9 090	2.525
40	213	214	213	55.444	11 809	3.280
50	250	251	250	61.329	15 332	4.259
60	403	404	639	79.577	50 850	14.125
65	205	206	1092	131.431	143 523	39.867

Table 6.6: Computational performance of the algorithm for $n_1 \in \{5, 10, 20, 30, 40, 50, 60, 65\}$.

Note that the average time per $\mathcal{P}_2$ problem in Table 6.6 tends to increase as the number of big banks increases. This happens because the highest connectivity probability, $q^{\text{con}}_{1,1} = 0.7$, is the probability that one big bank is liable to another big bank. Hence, as the number of big banks increases, the nodes in the network become more connected with liabilities and it takes more time to solve a $\mathcal{P}_2$ problem because the MILP formulations of $\mathcal{P}_2$ problems get more complex in terms of constraints. In addition, it can be observed that the difference between the numbers of inner and outer approximation vertices and the number of $\mathcal{P}_2$ problems increases as the distribution of nodes changes toward the two extreme cases: 5 big banks and 65 big banks. As in the previous sensitivity analysis, this happens because the boundaries (weakly minimal elements) of the Eisenberg-Noe systemic risk measures around these extreme cases in Figure 6.11 contain "flat" regions, which makes the algorithm solve more $\mathcal{P}_2$ problems without actually improving the approximation.

Observe from Figure 6.11 that as the number of big banks increases and the number of small banks decreases, the small banks get a wider range of capital allocation options, as opposed to the big banks. This happens because the total number of banks is fixed and the group with less number of banks has a wider range of capital allocation options since it has more claims to the other group's

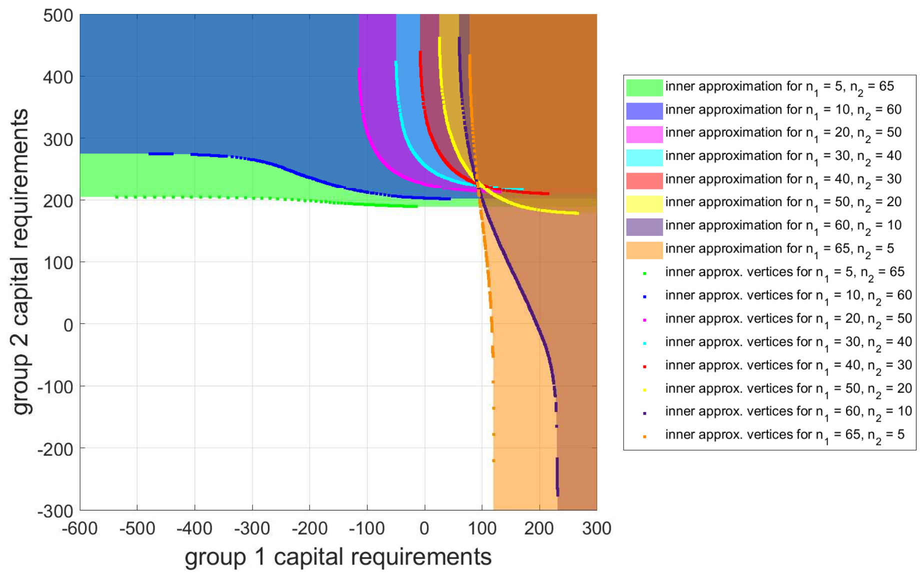


Figure 6.11: Inner approximations of the Eisenberg-Noe systemic risk measure for $n_1 \in \{5, 10, 20, 30, 40, 50, 60, 65\}$.

banks. When the number of banks in each group is evenly distributed, the group of big banks has a wider range of capital allocation options. The reason lies behind connectivity probabilities. Recall that for this set-up it is assumed that the connectivity probability from big banks to small banks is $q_{12}^{\text{con}} = 0.1$, while the connectivity probability from small banks to big banks is $q_{21}^{\text{con}} = 0.5$. It means that small banks are more likely to be liable to big banks and, since big banks have more claims compared to small banks, they have a wider range of capital allocation options.

6.4.3 Number of Scenarios

In this part, a sensitivity analysis is performed by changing the number K of scenarios in the set $\{10, 20, \dots, 100\}$. As in the previous sensitivity analysis on the number of scenarios, since the network structure remains the same all the time, it is expected that there will be no major changes in the corresponding Eisenberg-Noe systemic risk measures. The main purpose is to compare computation times. Since each scenario adds n continuous and n binary variables to the corresponding $\mathcal{P}_2$ problem and its MILP formulation $\mathcal{Z}_2^{\text{EN}}$, defined in (4.2.5), one would expect major changes in computation times. Table 6.7 shows the computational performance of the algorithm for $K \in \{10, 20, \ldots, 100\}$ and Figure 6.12 provides the inner approximations of the corresponding Eisenberg-Noe systemic risk measures.

K	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
10	232	233	453	0.817	370	0.103
20	220	221	446	2.282	1 018	0.283
30	217	218	455	4.912	2 235	0.621
40	220	221	451	9.293	4 191	1.164
50	234	235	461	15.285	7 047	1.957
60	219	220	439	23.642	10 379	2.883
70	213	214	436	31.066	13 545	3.762
80	222	223	451	38.392	17 315	4.810
90	230	231	462	62.928	29 073	8.076
100	216	217	444	77.758	34 524	9.590

Table 6.7: Computational performance of the algorithm for $K \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$.

Finally, Figure 6.13 and Figure 6.14 suggest that the average time per $\mathcal{P}_2$ problem and the total algorithm time increase faster than linearly with K. At the same time, it can be observed from Figure 6.12 that the corresponding inner approximations of the Eisenberg-Noe systemic risk measures do not change much. Hence, the results obtained justify the expectations.

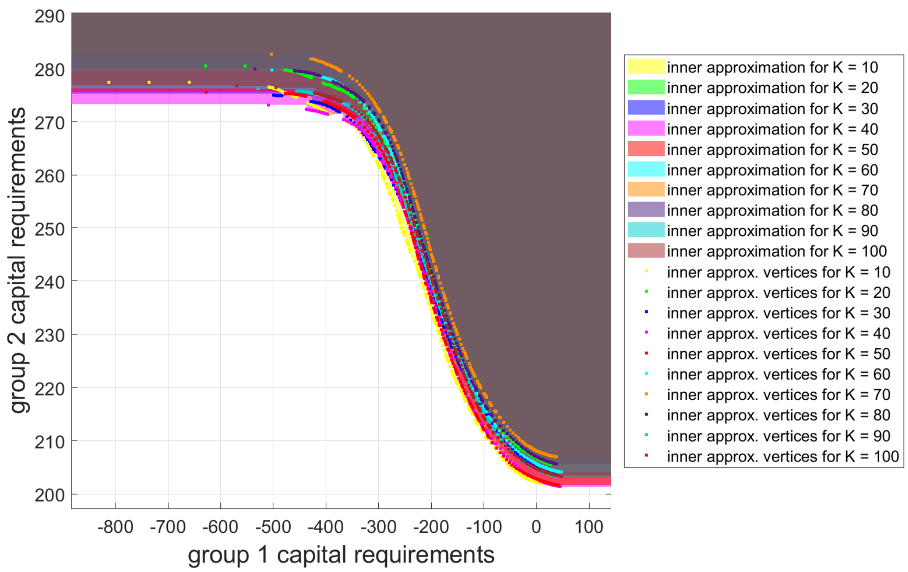


Figure 6.12: Inner approximations of the Eisenberg-Noe systemic risk measure for $K \in \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$.

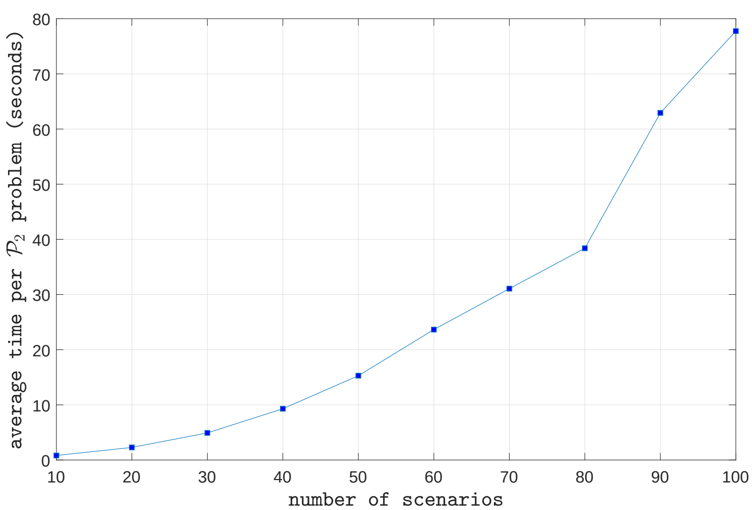


Figure 6.13: Scenarios-average time per $\mathcal{P}_2$ problem plot for the signed Eisenberg-Noe network of 70 banks.

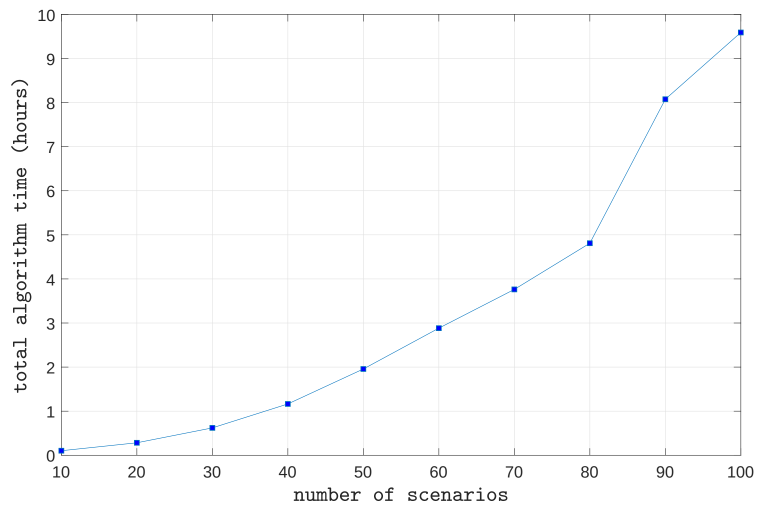


Figure 6.14: Scenarios-total algorithm time plot for the signed Eisenberg-Noe network of 70 banks.

6.5 Sensitivity Analyses on Two-Group Rogers-Veraart Networks with 45 Nodes and 50 Scenarios

In this section, sensitivity analyses are performed in computation of Rogers-Veraart systemic risk measures by changing the α and β parameters of Rogers-Veraart networks, γ threshold, distributions of nodes among groups, mean values of the random operating cash flows for a fixed expected total value of the operating cash flows in Rogers-Veraart networks, and the common correlation between the random operating cash flows. A Rogers-Veraart network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ is generated with the following parameters:

$$\boldsymbol{q}^{\text{con}} = \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & 0.5 \end{bmatrix}, \quad \boldsymbol{l}^{\text{gr}} = \begin{bmatrix} 200 & 100 \\ 50 & 50 \end{bmatrix},$$

$n = 45$, $n_1 = 15$, $n_2 = 30$, $K = 50$, and $\varrho = 0.05$. In addition, the liquid fraction of the random operating cash flows available to a defaulting node is fixed as $\alpha = 0.7$, and the liquid fraction of the realized claims available to a defaulting node is fixed as $\beta = 0.9$. The shape and scale parameters of gamma distributions of the random operating cash flows X_i, $i \in \mathcal{N}_\ell$, $\ell \in \mathcal{G}$, are chosen as

$$\kappa = \begin{bmatrix} 100 & 64 \end{bmatrix}, \quad \theta = \begin{bmatrix} 1 & 1.25 \end{bmatrix}.$$

Then the mean values of the random operating cash flows in the corresponding groups are

$$\boldsymbol{\nu} = \begin{bmatrix} 100 & 80 \end{bmatrix}$$

and the common standard deviation is $\sigma = 10$.

In the Rogers-Veraart systemic risk measure $\gamma^p = 0.9$ is taken, that is,

$$R^{sen}_{\text{RV}}(\boldsymbol{X}) = \left\{ \boldsymbol{z} \in \mathbb{R}^2 \middle| \mathbb{E}\left[\Lambda^{\text{RV}+}\left(\boldsymbol{X} + B^\mathsf{T}\boldsymbol{z}\right)\right] \geq 0.9\left(\mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}\right) \right\}, \tag{6.5.1}$$

where the aggregation function $\Lambda^{\text{RV}+}$ is defined as in (3.3.2). The approximation error in the Benson type algorithm is taken as $\epsilon = 1$.

6.5.1 Rogers-Veraart α Parameter

In this part, a sensitivity analysis is performed by changing α, the liquid fraction of the operating cash flow that can be used by a defaulting node to meet its obligations. It is analyzed how computation times and the corresponding systemic risk measures change. The generated network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ remains the same in all cases.

Table 6.8 illustrates the computational performance of the algorithm for $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and Figure 6.15 consists of the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

Note from Table 6.8 that the average time per $\mathcal{P}_2$ problem decreases with α. It can be presumed that this happens because of the following observation: as α parameter increases, the discontinuity in the fixed-point characterization of clearing vectors in the Rogers-Veraart model in (3.3.1) decreases and it gets easier

α	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.1	273	274	333	12.165	4 051	1.125
0.3	461	462	484	10.572	5 117	1.421
0.5	592	593	602	5.231	3 149	0.875
0.7	583	584	584	3.876	2 264	0.629
0.9	589	590	589	3.395	2 000	0.555

Table 6.8: Computational performance of the algorithm for $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

to solve the corresponding MILP formulation of a $\mathcal{P}_2$ problem because it contains the constraints of (3.3.2), the MILP characterization of clearing vectors in the Rogers-Veraart model.

Observe from Figure 6.15 that the Rogers-Veraart systemic risk measures expand significantly as α increases. It means that both big and small banks get less strict capital requirements as default costs decrease. One can also observe that in each case allocating zero capital requirement to the groups is not an available option. In addition, in each case big banks can be allocated a negative amount of capital requirement given that the capital requirements for small banks are high enough. On the other hand, small banks do not have this privilege.

6.5.2 Rogers-Veraart β Parameter

In this part, a sensitivity analysis is performed by changing β, the liquid fraction of the realized claims from the other nodes that can be used by a defaulting node to meet its obligations. The generated network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ remains the same in all cases.

Table 6.9 shows the computational performance of the algorithm for $\beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and Figure 6.16 provides the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

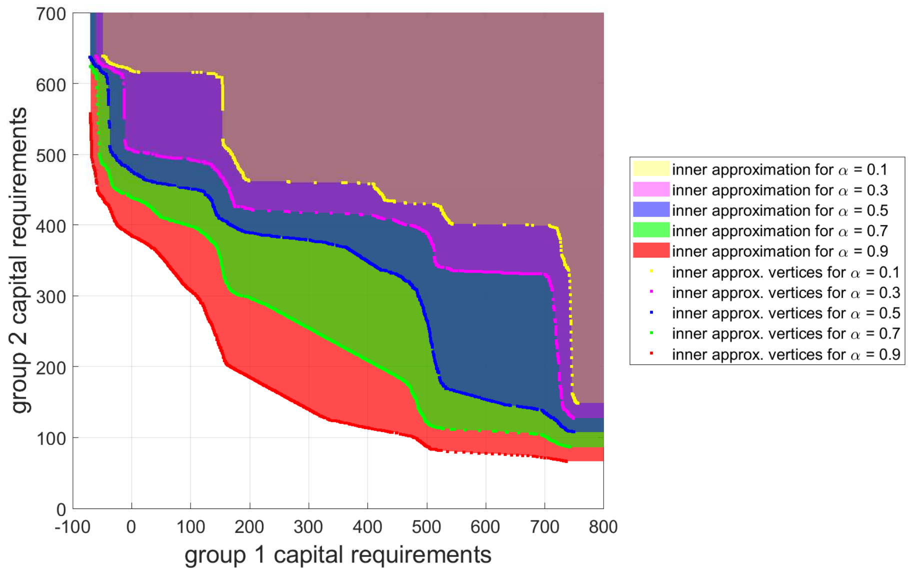


Figure 6.15: Inner approximations of the Rogers-Veraart systemic risk measures for $\alpha \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

β	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.1	187	189	214	5.014	1 073	0.298
0.3	223	225	270	5.561	1 502	0.417
0.5	323	324	350	3.733	1 307	0.363
0.7	394	395	401	3.710	1 488	0.413
0.9	583	584	584	3.876	2 264	0.629

Table 6.9: Computational performance of the algorithm for $\beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

Note from Table 6.9 that the total number of $\mathcal{P}_2$ problems increases with β. We can observe smaller average times per $\mathcal{P}_2$ problem for higher values of β. As in the case of the α parameter, it can be presumed that this happens because of the following observation: as β parameter increases, the discontinuity in the fixed-point characterization of clearing vectors in the Rogers-Veraart model in (3.3.1) decreases, which makes it easier to solve the MILP formulation of a $\mathcal{P}_2$ problem.

Observe from Figure 6.16 that the Rogers-Veraart systemic risk measures expand significantly as β increases. It means that both big and small banks get less strict capital requirements if defaulting banks are able to use larger fractions of realized claims. It can also be observed that in each case allocating zero capital requirement to the groups is not an available option. In addition, if $\beta = 0.9$ then big banks can be allocated a negative amount of capital requirement given that the capital requirements for small banks are high enough. On the other hand, small banks do not have this privilege.

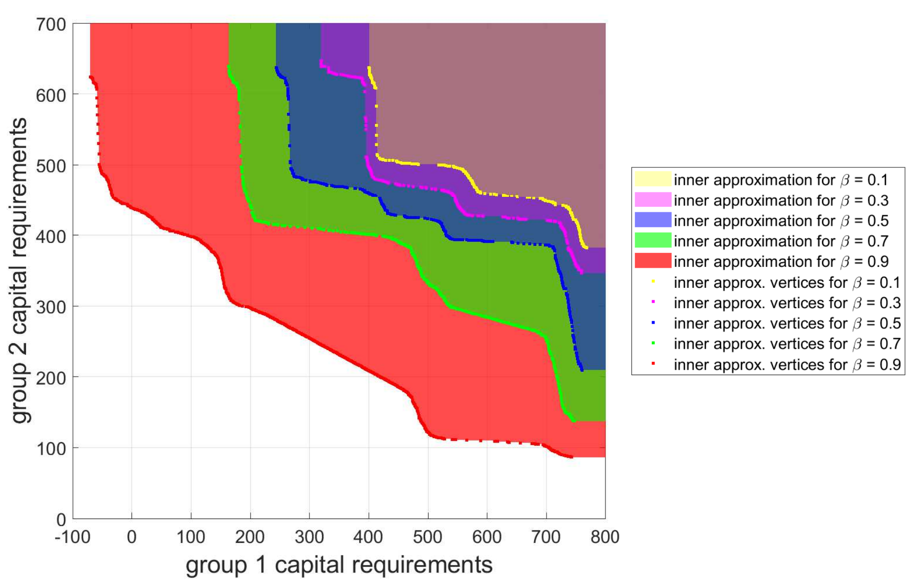


Figure 6.16: Inner approximations of the Rogers-Veraart systemic risk measures for $\beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

6.5.3 Rogers-Veraart α and β Parameters

Now, Rogers-Veraart systemic risk measures and the corresponding computation times are analyzed by changing α and β parameters simultaneously and assuming that $\alpha = \beta$. Again, the generated network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ remains the same in all cases. Table 6.10 shows the computational performance of the algorithm for $\alpha, \beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and Figure 6.17 represents the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

α, β	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.1	130	134	142	6.585	935	0.260
0.3	168	169	214	4.440	950	0.264
0.5	263	265	287	5.169	1 484	0.412
0.7	394	395	401	3.710	1 488	0.413
0.9	589	590	589	3.395	2 000	0.555

Table 6.10: Computational performance of the algorithm for $\alpha, \beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

In Table 6.10, note that the average time per $\mathcal{P}_2$ problem decreases and the number of $\mathcal{P}_2$ problems increases as α and β increase, which results in increase in the total algorithm time. As in the previous two sensitivity analyses, it can be presumed that this happens because the discontinuity in the fixed-point characterization of clearing vectors in the Rogers-Veraart model in (3.3.1) decreases as α and β increase.

Observe from Figure 6.17 that the Rogers-Veraart systemic risk measures expand significantly as α and β increase. It means that both big and small banks get less strict capital requirements if defaulting nodes can use larger fractions of their assets and realized claims to meet their obligations. One can also observe that in each case allocating zero capital requirement to the groups is not an available option. In addition, if $\alpha = 0.9$ and $\beta = 0.9$ then big banks can be allocated a negative amount of capital requirement given that the capital requirements for small banks are high enough. On the other hand, small banks do not have this privilege.

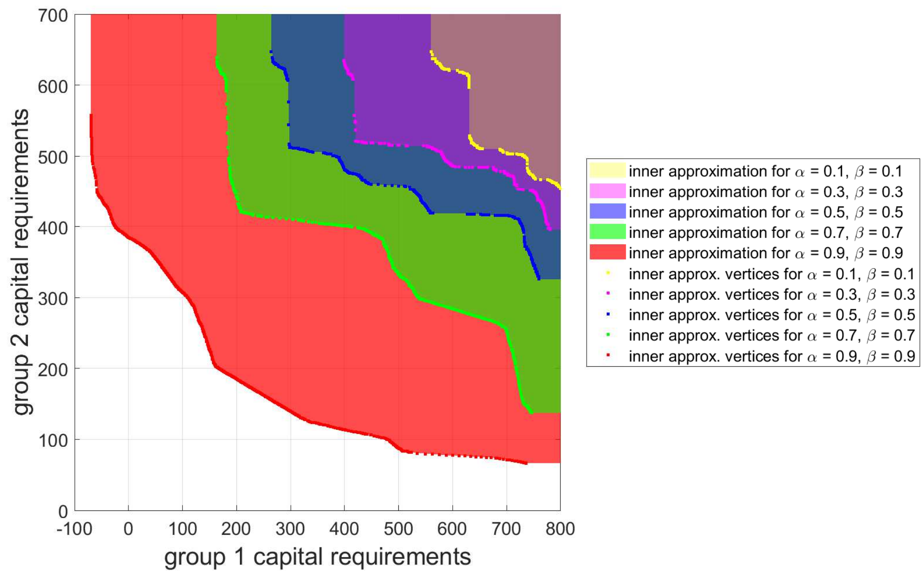


Figure 6.17: Inner approximations of the Rogers-Veraart systemic risk measures for $\alpha, \beta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

6.5.4 Threshold Level

In this part, different γ levels are compared and analyzed, where γ^{p} takes values in $\{0.1, 0.2, \dots, 0.9, 0.95, 0.99, 1\}$ and $\gamma = \gamma^{\mathrm{p}} \left(\mathbb{1}^{\mathsf{T}} \bar{\boldsymbol{p}}\right)$. Table 6.11 shows the computational performance of the algorithm for different values of γ^{p} and Figure 6.18 consists of the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

It can be noted from Table 6.11 that the average time per $\mathcal{P}_2$ problem and the total algorithm time are high for γ^{p} values around 0.7. In addition, the number of $\mathcal{P}_2$ problems increases up to $\gamma^{\mathrm{p}} = 0.9$ and then decreases. Observe that in Figure 6.18 the Rogers-Veraart systemic risk measures with smaller γ^{p} values contain the ones that have higher γ^{p} values. This result is fully consistent with the corresponding Eisenberg-Noe systemic risk measure in (6.5.1) since capital allocations that satisfy high values of γ threshold can at the same time satisfy smaller values of γ threshold. Hence, these capital allocations are included in any Rogers-Veraart systemic risk measures with a smaller γ threshold.

γ^{p}	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0.1	1	1	1	0.384	0.384	0
0.2	13	14	13	13.809	180	0.050
0.3	51	52	51	30.273	1 544	0.429
0.4	94	95	94	36.645	3 445	0.957
0.5	165	166	165	98.625	16 273	4.520
0.6	223	224	223	138.532	30 893	8.581
0.7	389	390	389	204.288	79 468	22.075
0.8	395	396	395	91.600	36 182	10.051
0.9	583	584	584	3.876	2 264	0.629
0.95	418	419	431	2.946	1 270	0.353
0.99	66	67	74	1.639	121	0.034
1.00	1	1	1	0.132	0.132	0

Table 6.11: Computational performance of the algorithm for $\gamma^{\mathrm{p}} \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 1\}$.

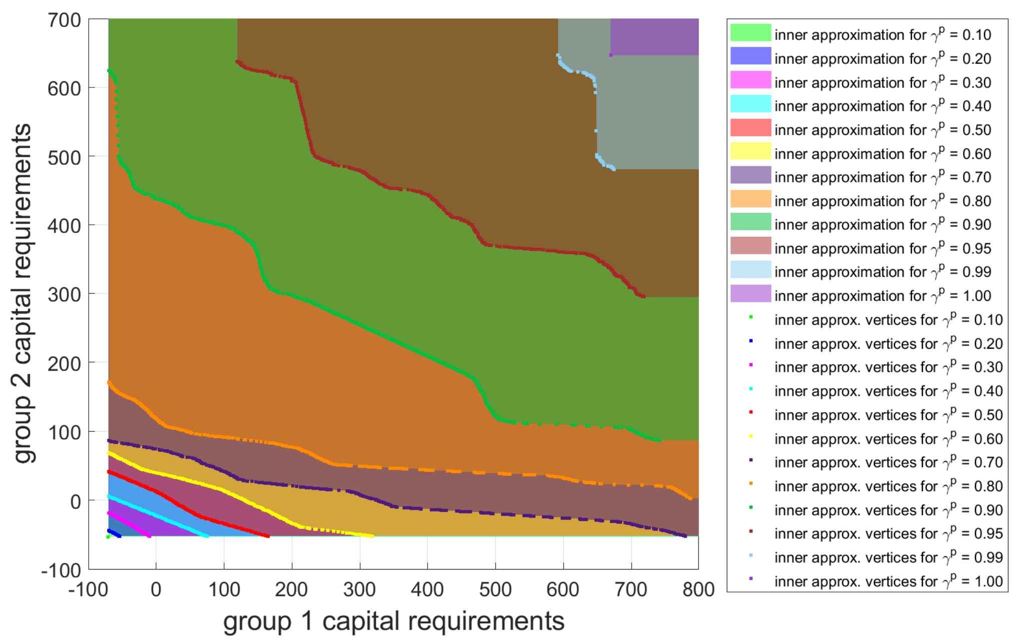


Figure 6.18: Inner approximations of the Rogers-Veraart systemic risk measure for $\gamma^{\mathrm{p}} \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 1\}$.

6.5.5 Distribution of Nodes among Groups

In this part, a sensitivity analysis is performed by changing the distribution of nodes among the groups for a fixed total number of nodes $n = 45$ and the number of big banks n_1 takes values in $\{5, 10, 15, 20, 25, 30, 35, 40\}$. Then the number of small banks is $n_2 = n - n_1$. As previously, the main purpose of is to analyze how computation times and the corresponding systemic risk measures change. The generated random operating cash flows remain the same all the time, while the network structure changes at each run. Hence, the corresponding Rogers-Veraart systemic risk measures are expected to vary a lot. Table 6.12 shows the computational performance of the algorithm for $n_1 \in \{5, 10, 15, \ldots, 40\}$ and Figure 6.19 provides the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

n_1	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
5	6	7	6	1.006	6	0.002
10	436	437	436	3.994	1 742	0.484
15	583	584	584	3.876	2 264	0.629
20	516	517	517	7.887	4 078	1.133
25	557	558	557	6.118	3 408	0.947
30	371	372	371	5.786	2 147	0.596
35	187	188	187	6.100	1 141	0.317
40	106	107	108	5.196	561	0.156

Table 6.12: Computational performance of the algorithm for $n_1 \in \{5, 10, 15, 20, 25, 30, 35, 40\}$.

Note that the average time per $\mathcal{P}_2$ problem in Table 6.12 is relatively high for the values $n_1 \in \{20, 25, 30, 35, 40\}$. In addition, the number of $\mathcal{P}_2$ problems is greater for the values around $n_1 = 20$. Observe from Figure 6.19 that as the number of big banks increases and the number of small banks decreases, the small banks get a wider range of capital allocation options, as opposed to the big banks. This happens because the total number of banks is fixed and the group with less number of banks has a wider range of capital allocation options since it has more claims to the other group's banks in the scope of this set-up.

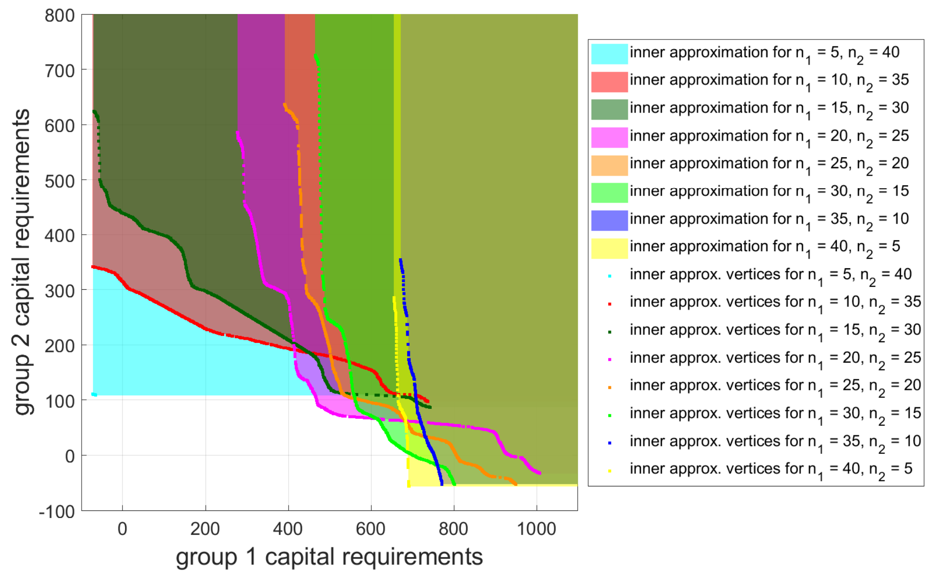


Figure 6.19: Inner approximations of the Rogers-Veraart systemic risk measure for $n_1 \in \{5, 10, 15, 25, 30, 35, 40\}$.

6.5.6 Mean Values of Random Operating Cash Flows

In this part, a sensitivity analysis is performed by changing the mean values of random operating cash flows X_i, $i \in \mathcal{N}$, for a fixed mean value of total operating cash flow in the network and fixed common standard deviation $\sigma = 10$. The mean total operation cash flow in $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ is

$$\mathbb{E}\left[\sum_{i\in\mathcal{N}} X_i\right] = \sum_{i\in\mathcal{N}} \mathbb{E}\left[X_i\right] = \sum_{i\in\mathcal{N}_\ell}\sum_{\ell\in\mathcal{G}} \mathbb{E}\left[X_i\right] = n_1\nu_1 + n_2\nu_2 = 15\cdot 100 + 30\cdot 80 = 3900.$$

Let ν_1 take values in $\{10, 30, 50, 80, 100, 120, 150, 180, 200, 240\}$. Then, for the fixed mean total operation cash flow, the corresponding ν_2 values are $\{125, 115, 105, 90, 80, 70, 55, 40, 30, 10\}$. Tables 6.13 and 6.14 represent the corresponding shape and scale parameters of the gamma distributions of generated operating cash flows X_i, $i \in \mathcal{N}_\ell, \ell \in \mathcal{G}$. Table 6.15 illustrates the computational performance of the algorithm for these values of ν_1 and ν_2, and Figure 6.20 represents the inner approximations of some of the corresponding Rogers-Veraart systemic risk measures.

ν_1	10	30	50	80	100	120	150	180	200	240
κ_1	1	9	25	64	100	144	225	324	400	576
θ_1	10	$\frac{10}{3}$	2	1.25	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{5}{9}$	0.5	$\frac{5}{12}$

Table 6.13: Shape and scale parameter values of gamma distributions of generated operation cash flows for big banks with mean values $\nu_1 \in \{10, 30, 50, 80, 100, 120, 150, 180, 200, 240\}$ and standard deviation $\sigma = 10$.

ν_2	125	115	105	90	80	70	55	40	30	10
κ_2	156.25	132.25	110.25	81	64	49	30.25	16	9	1
θ_2	0.8	$\frac{20}{23}$	$\frac{20}{21}$	$\frac{10}{9}$	1.25	$\frac{10}{7}$	$\frac{20}{11}$	2.5	$\frac{10}{3}$	10

Table 6.14: Shape and scale parameter values of gamma distributions of generated operation cash flows for small banks with mean values $\nu_2 \in \{125, 115, 105, 90, 80, 70, 55, 40, 30, 10\}$ and standard deviation $\sigma = 10$.

Note that the computational results in Table 6.15 do not change much with mean values of operating cash flows. The reason behind this observation lies in Figure 6.20, where the corresponding Rogers-Veraart systemic risk measures do not change much in shape, but only shift as the mean values of operating cash flows change. One can presume that this happens because the mean total value of operating cash flows in the network is fixed. Interpreting it from the financial perspective, as the mean operating cash flow increases for one group of banks and decreases for the other one, the former one gets less strict capital requirements while the latter gets more strict capital requirements and vice versa, but it does not change the picture as a whole. However, one should keep in mind that these results do not imply a general behavior and depend on generated networks.

6.5.7 Common Correlation

In this part, a sensitivity analysis is performed by changing the common correlation, ϱ, between random operating cash flows X_i, $i \in \mathcal{N}$. Table 6.16 shows the computational performance of the algorithm for $\varrho \in$

ν_1	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
10	599	600	600	3.828	2 297	0.638
30	586	587	586	3.878	2 273	0.631
50	580	581	580	3.912	2 269	0.630
80	598	599	598	3.803	2 274	0.632
100	583	584	584	3.876	2 264	0.629
120	596	597	596	3.812	2 272	0.631
150	603	604	603	3.826	2 307	0.641
180	589	590	589	3.829	2 256	0.627
200	589	590	589	3.824	2 252	0.626
240	579	580	579	4.206	2 435	0.676

Table 6.15: Computational performance of the algorithm for $\nu_1 \in \{10, 30, 50, 80, 100, 120, 150, 180, 200, 240\}$.

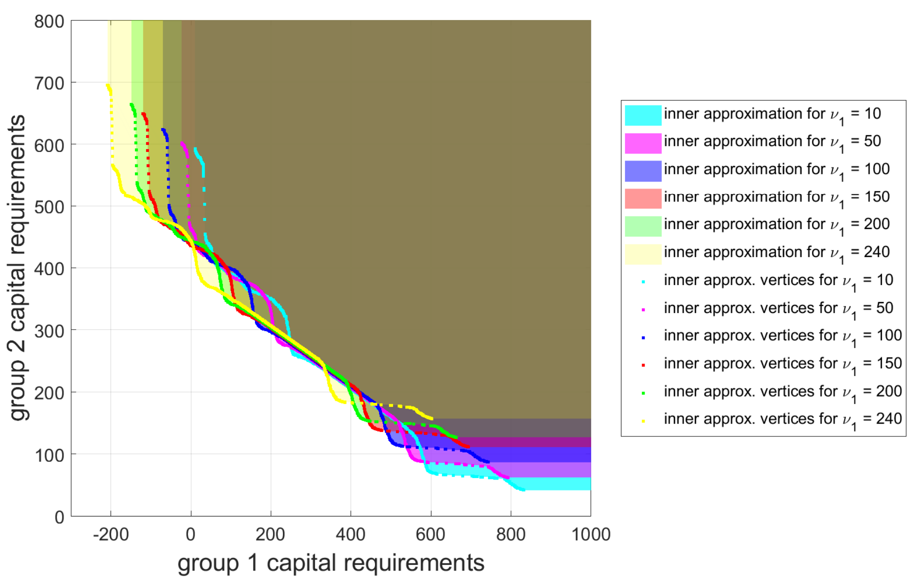


Figure 6.20: Inner approximations of the Rogers-Veraart systemic risk measure for $\nu_1 \in \{10, 50, 100, 150, 200, 240\}$.

$\{0, 0.01, 0.05, 0.1, 0.2, \ldots, 0.9\}$. Figure 6.21 provides the inner approximations of the corresponding Rogers-Veraart systemic risk measures.

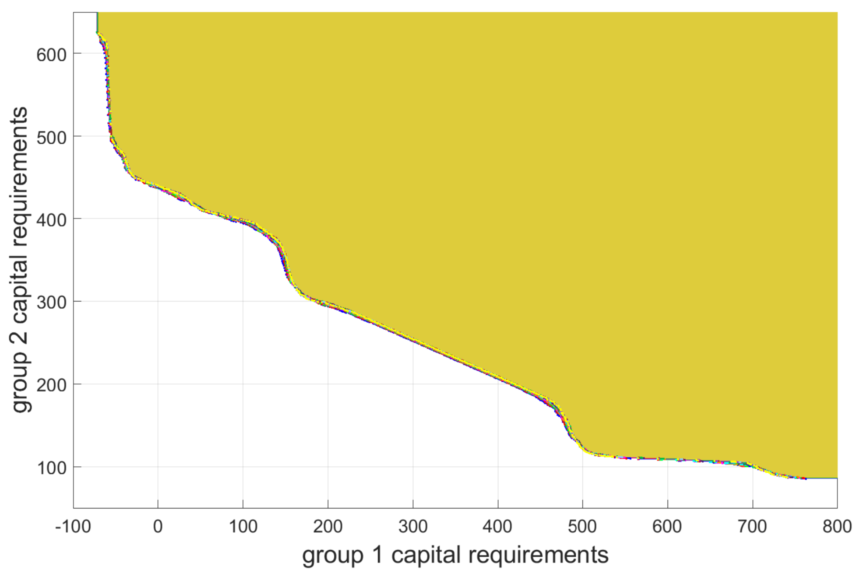
ϱ	Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
0	575	576	577	3.887	2 243	0.623
0.01	582	583	582	3.788	2 205	0.612
0.05	583	584	584	3.876	2 264	0.629
0.1	591	592	591	3.866	2 285	0.635
0.2	588	589	590	3.852	2 273	0.631
0.3	600	601	602	3.792	2 283	0.634
0.4	601	602	603	3.819	2 303	0.640
0.5	616	617	619	3.751	2 322	0.645
0.6	604	605	607	3.850	2 337	0.649
0.7	616	617	619	3.821	2 365	0.657
0.8	618	619	618	3.847	2 378	0.660
0.9	614	615	614	3.896	2 392	0.664

Table 6.16: Computational performance of the algorithm for $\varrho \in \{0, 0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

Note that the computational results in Table 6.16 do not change much with the common correlation ϱ. It can also be observed from Figure 6.21 that the Rogers-Veraart systemic risk measures are almost the same for all values of ϱ. One can attribute it to the possibility that in this network operating cash flows do not have much impact on the network.

6.6 A Three-Group Signed Eisenberg-Noe Network with 60 Nodes and 50 Scenarios

In this section, a three-group signed Eisenberg-Noe network of banks is generated and the corresponding Eisenberg-Noe systemic risk measure is approximated by the Benson type algorithm. The computation times and the inner approximations are presented below.



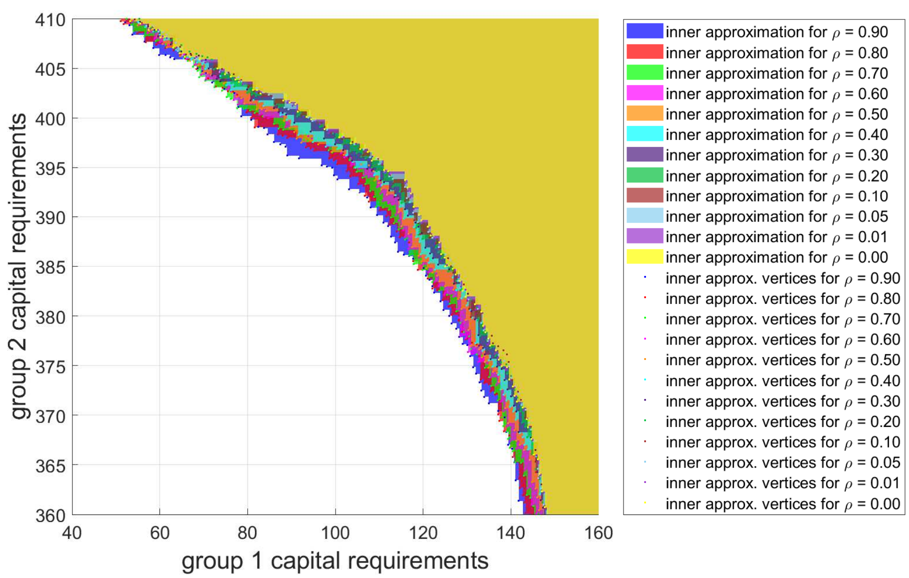


Figure 6.21: Inner approximations of the Rogers-Veraart systemic risk measure in (6.5.1) for $\varrho \in \{0, 0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

A signed Eisenberg-Noe network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X})$ is generated with $n = 60$, $n_1 = 10$, $n_2 = 20$, $n_3 = 30$, $K = 50$, $\sigma = 100$, and $\varrho = 0.05$. In addition,

$$\boldsymbol{q}^{\text{con}} = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}, \quad \boldsymbol{l}^{\text{gr}} = \begin{bmatrix} 20 & 15 & 8 \\ 15 & 10 & 6 \\ 8 & 6 & 5 \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} -50 & -100 & -150 \end{bmatrix}.$$

It is taken $\gamma^p = 0.95$ so that

$$R^{sen}_{\text{EN}}(\boldsymbol{X}) = \left\{ \boldsymbol{z} \in \mathbb{R}^3 \middle| \mathbb{E}\left[\Lambda^{\text{EN}}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq 0.95\left(\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}\right) \right\}, \tag{6.6.1}$$

where the aggregation function Λ^{EN} is defined as in (3.2.7).

Table 6.17 shows the computational performance of the algorithm for $\epsilon = 20$.

Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
413	516	1250	2.904	3631	1.009

Table 6.17: Computational performance of the algorithm for a signed Eisenberg-Noe network with 10 big, 20 medium and 30 small banks, 50 scenarios and approximation error $\epsilon = 20$.

Figure 6.22 represents the inner approximation of the corresponding three-group Eisenberg-Noe systemic risk measure. It can be presumed that this Eisenberg-Noe systemic risk measure is convex.

6.7 A Three-Group Rogers-Veraart Network with 60 Nodes and 50 Scenarios

In this section, a three-group network of banks is generated and the corresponding Rogers-Veraart systemic risk measure is approximated by the Benson type algorithm. The computation times and the inner approximations are presented below.

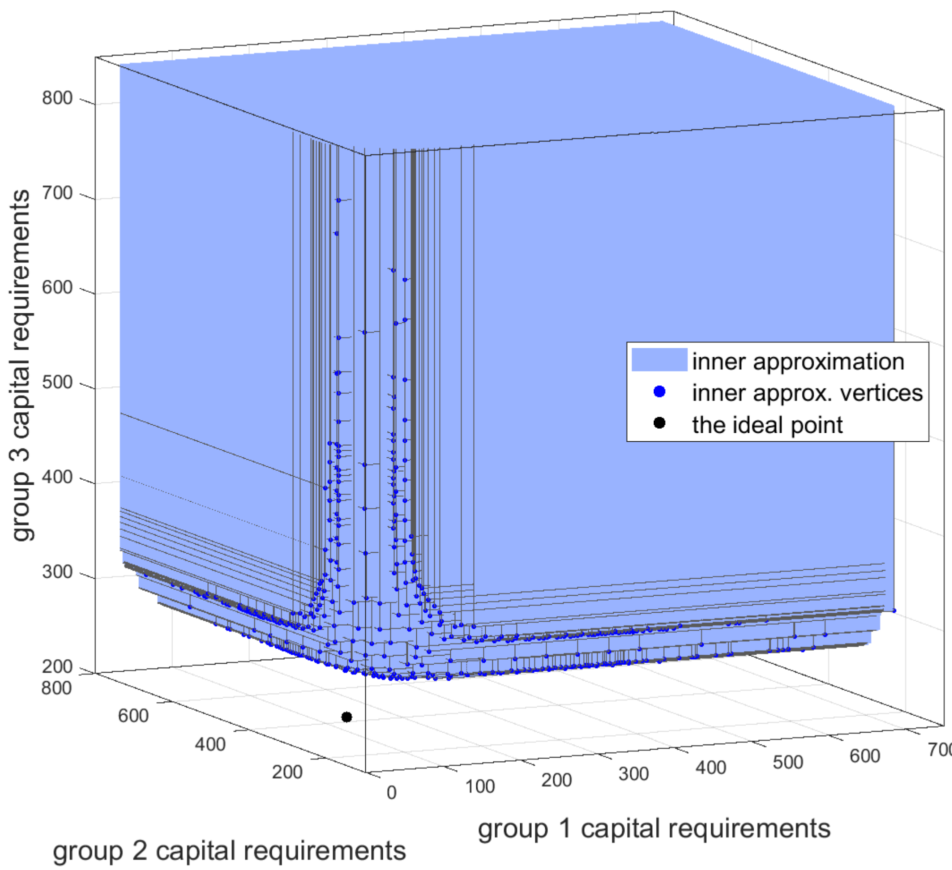


Figure 6.22: Inner approximation of the three-group Eisenberg-Noe systemic risk measure with 60 nodes, 50 scenarios and approximation error $\epsilon = 20$.

A Rogers-Veraart network $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{X}, \alpha, \beta)$ is generated with $n = 60$, $n_1 = 10$, $n_2 = 20$, $n_3 = 30$, $K = 50$, $\varrho = 0.05$,

$$\boldsymbol{q}^{\mathrm{con}} = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}, \quad \boldsymbol{l}^{\mathrm{gr}} = \begin{bmatrix} 200 & 190 & 180 \\ 190 & 190 & 180 \\ 180 & 180 & 170 \end{bmatrix}.$$

In addition, the liquid fraction of the random operating cash flows and the liquid fraction of the realized claims available to defaulting banks are fixed as $\alpha = \beta = 0.9$. The shape and scale parameters of gamma distributions of X_i, $i \in \mathcal{N}_\ell$, $\ell \in \mathcal{G}$, are chosen as

$$\kappa = \begin{bmatrix} 100 & 81 & 64 \end{bmatrix}, \quad \theta = \begin{bmatrix} 1 & \frac{10}{9} & 1.25 \end{bmatrix}.$$

Then, the corresponding mean values of the random operating cash flows in the groups are

$$\boldsymbol{\nu} = \begin{bmatrix} 100 & 90 & 80 \end{bmatrix}$$

and the common standard deviation is $\sigma = 10$.

It is taken $\gamma^p = 0.99$ so that

$$R^{sen}_{\mathrm{RV}}(\boldsymbol{X}) = \left\{ \boldsymbol{z} \in \mathbb{R}^3 \middle| \mathbb{E}\left[\Lambda^{\mathrm{RV}+}\left(\boldsymbol{X} + B^{\mathsf{T}}\boldsymbol{z}\right)\right] \geq 0.99\left(\mathbb{1}^{\mathsf{T}}\bar{\boldsymbol{p}}\right) \right\}, \tag{6.7.1}$$

where $\Lambda^{\mathrm{RV}+}$ is defined as in (3.3.2). The upper bound point in the approximation is taken as

$$\boldsymbol{z}^{\mathrm{UB}} = \hat{\boldsymbol{z}}^{\mathrm{ideal}} + \frac{1}{5}\left\|\bar{\boldsymbol{p}}\right\|_\infty.$$

Table 6.18 shows the computational performance of the algorithm for $\epsilon = 40$.

Inner approx. vertices	Outer approx. vertices	$\mathcal{P}_2$ problems	Avg. time per $\mathcal{P}_2$ prob. (seconds)	Total algorithm time (seconds)	Total algorithm time (hours)
975	1323	19382	0.427	8284	2.301

Table 6.18: Computational performance of the algorithm for a Rogers-Veraart network with 10 big, 20 medium and 30 small banks, 50 scenarios and approximation error $\epsilon = 40$.

Figure 6.23 provides the inner approximation of the corresponding three-group Rogers-Veraart systemic risk measure. It can be observed that this Rogers-Veraart systemic risk measure is not convex.

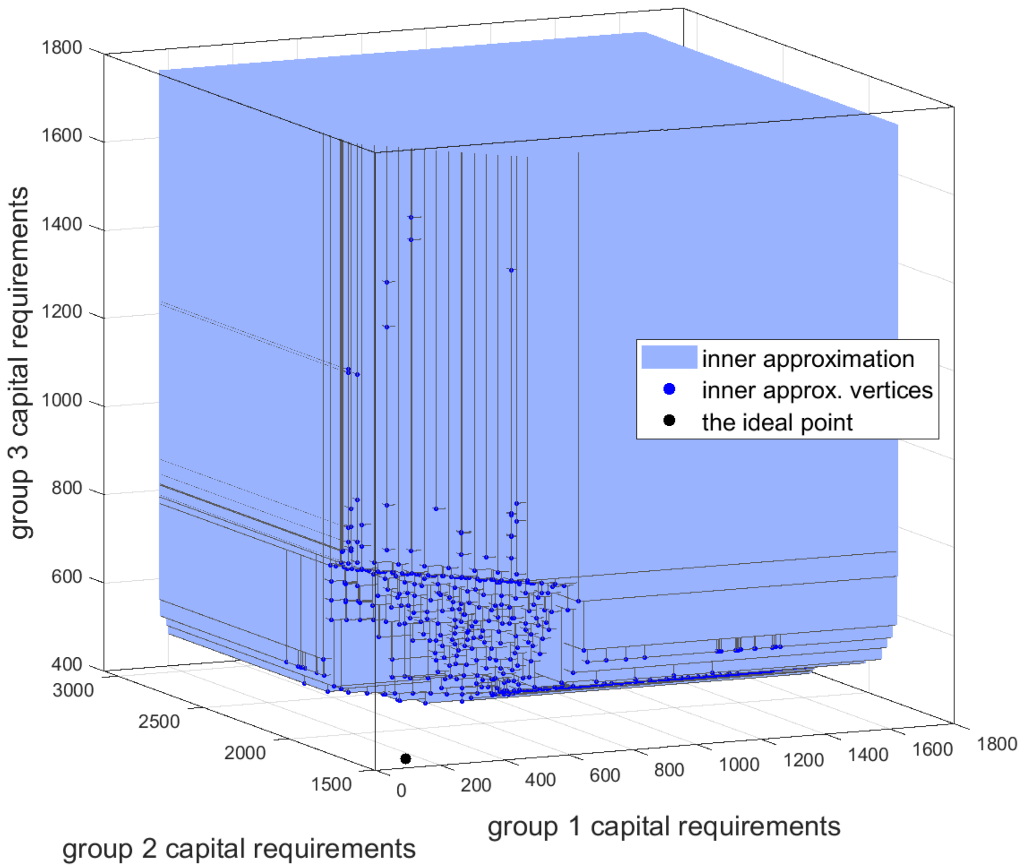


Figure 6.23: Inner approximation of the three-group Rogers-Veraart systemic risk measure with 60 nodes, 50 scenarios and approximation error $\epsilon = 40$.

Chapter 7

Conclusion and Future Research

Systemic risk is an important aspect of a financial system with tight interdependencies because it helps evaluate the risk of financial contagions and decrease the risks of financial crises and their consequences. One of the aspects studied by the scholars in systemic risk area is clearing, which reduces interdependency in financial systems by eliminating mutual liabilities. The study in this thesis is conducted at the interface of network models of systemic risk and systemic risk measures. To the best of our knowledge, it is the first attempt to compute systemic risk measures by implementing mixed-integer linear programming aggregation functions and applying a Benson type algorithm for non-convex problems. Computation of systemic risk measures is considered from the vector optimization point of view.

The Eisenberg-Noe and Rogers-Veraart network models of systemic risk are investigated and extended from the optimization point of view in Chapter 3. MILP characterizations of clearing vectors in these network models are presented. The nonnegative operating cash flow assumption in the Eisenberg-Noe network model is relaxed and two modification approaches are described: without and with seniority assumption. It is shown that the former approach, a naive attempt originated from a conjecture made in Eisenberg and Noe [2], leads to a formulation of a totally different problem and makes it difficult to interpret the new model in terms of the original one due to absence of seniority in the structure of the model. Analyzing this approach leads to the second approach with a seniority assumption and proves to be more applicable both in terms of financial interpretation and from optimization point of view. In addition, two MILP aggregation functions are presented, which are then implemented in systemic risk measures in Chapter 4.

A grouping notion for the members of a financial system is employed in Chapter 4 in terms of a grouping matrix applied in the structure of systemic risk measures. MILP characterizations of $\mathcal{P}_1$, the weighted-sum scalarization problem, and $\mathcal{P}_2$, the problem of finding a minimum step-length, are formulated for both the Eisenberg-Noe and Rogers-Veraart systemic risk measures. Important results on boundedness and feasibility of these MILP formulations are provided.

In Chapter 5, the Benson type algorithm for non-convex problems described in Nobakhtian and Shafiei [1] and the corresponding pseudo-codes for approximation of the Eisenberg-Noe and Rogers-Veraart systemic risk measures are presented. The roles of the MILP formulations of $\mathcal{P}_1$ and $\mathcal{P}_2$ problems in the algorithm are explained.

A computational study is performed in Chapter 6 by generating two-group and three-group signed Eisenberg-Noe and Rogers-Veraart networks and approximating the corresponding systemic risk measures. The computational performance of the algorithm and results of the sensitivity analyses performed on various parameters of the networks are provided. The obtained inner approximations of the corresponding systemic risk measures are compared and interpreted from the financial point of view as capital allocations (capital requirements) for the members of the financial system. It is observed that the most of the computation times are devoted to solving the optimization problems in terms of MILP formulations of the corresponding $\mathcal{P}_2$ problems.

For future research, this study can be extended in several ways. One option is to try different acceptance sets in systemic risk measures. For the sake of simplicity, the acceptance set of the negative expectation risk measure is applied in this study. One can try using acceptance sets related to the average value at risk (also known as the conditional value at risk or expected shortfall) or the entropic risk measure. As a slight extension for this research, the nonnegativity assumption on operating cash flows can be relaxed for the Rogers-Veraart network model using the ideas developed in Chapter 3. In the scope of computation of systemic risk measures, different aggregation functions can be formulated as optimization problems in various research areas and applied in systemic risk measures, which then can be approximated using the Benson type algorithm. A challenging attempt would be to try to formulate and compute a systemic risk measure with a network model (together with its aggregation function) that reflects as much as possible the real-world conditions and includes many aspects of financial systems, such as the network model developed in Weber and Weske [9]. One more option is to try to develop a heuristic algorithm that improves computational times in approximating systemic risk measures.

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Appendix A

Proofs of the Results in Chapter 3

A.1 Proof of Lemma 3.2.11

Proof. If $s_i = 0$, then constraint (3.2.11) is infeasible by assumption. Hence, $s_i = 1$, and this yields

$$p_i \leq \sum_{j=1}^{n} \pi_{ji} p_j + x_i \quad \text{and} \quad p_i \leq \bar{p}_i,$$

by constraints (3.2.9) and (3.2.10), respectively. Hence,

$$p_i \leq \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\}.$$

To get a contradiction, suppose that $p_i < \min\left\{\sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i\right\}$. Now let $\boldsymbol{p}^{\boldsymbol{\epsilon}} \in \mathbb{R}^n_+$ be equal to $\boldsymbol{p}$ in all components except the i^{th} one, and let $p_i^{\epsilon} = p_i + \epsilon$, where

$$\epsilon := \min\left\{\min\left\{\sum_{j=1}^{n}\pi_{ji}p_j + x_i, \bar{p}_i\right\} - p_i, \min_{l\in\mathcal{N}}\left\{M - \left(\sum_{j=1}^{n}\pi_{jl}p_j + x_l\right)\right\}, \epsilon'\right\} > 0,$$

where $\epsilon' = \min_{l\in\mathcal{N}:\sum_{j=1}^{n}\pi_{jl}p_j + x_l < 0}\left|\sum_{j=1}^{n}\pi_{jl}p_j + x_l\right|$ $\Big(\epsilon' = +\infty$ if there is no $l \in \mathcal{N}$ such that $\sum_{j=1}^{n}\pi_{jl}p_j + x_l < 0$ holds$\Big)$. This choice of ϵ ensures

$$p_i^{\epsilon} \leq \bar{p}_i \quad \text{and} \quad p_i^{\epsilon} \leq \sum_{j=1}^{n}\pi_{ji}p_j^{\epsilon} + x_i,$$

and will also be justified by other technical details later in this proof.

Let $\boldsymbol{s^\epsilon} \in \{0,1\}^n$ be a vector of binaries, where $s_l^\epsilon = 0$ if $\sum_{j=1}^{n}\pi_{jl}p_j^\epsilon + x_l < 0$ and $s_l^\epsilon = 1$ if $\sum_{j=1}^{n}\pi_{jl}p_j^\epsilon + x_l \geq 0$, for each $l \in \mathcal{N}$. It is shown that $(\boldsymbol{p^\epsilon}, \boldsymbol{s^\epsilon})$ is a feasible solution to $\Lambda^{\text{EN}}(\boldsymbol{x})$ by showing that all constraints in (3.2.8) are satisfied. First, for fixed $k \in \mathcal{N}\setminus\{i\}$, the k^{th} constraints in (3.2.8) are verified for $(\boldsymbol{p^\epsilon}, \boldsymbol{s^\epsilon})$. Three cases are considered:

$$(1)\ \sum_{j=1}^{n}\pi_{jk}p_j + x_k < 0, \quad (2)\ \sum_{j=1}^{n}\pi_{jk}p_j + x_k = 0, \quad (3)\ 0 < \sum_{j=1}^{n}\pi_{jk}p_j + x_k.$$

(1) Assume that $\sum_{j=1}^{n}\pi_{jk}p_j + x_k < 0$. If $s_k = 1$, then, by constraint (3.2.9),

$$p_k \leq \sum_{j=1}^{n}\pi_{jk}p_j + x_k + M(1-1) = \sum_{j=1}^{n}\pi_{jk}p_j + x_k < 0,$$

which is a contradiction to the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$ in constraint (3.2.12). Hence, $s_k = 0$, which in its turn implies $p_k = 0$ by (3.2.10) and (3.2.12).

By the definitions of $\boldsymbol{p^\epsilon}$ and $\boldsymbol{s^\epsilon}$, it holds that $p_k^\epsilon = p_k = 0$ since $k \neq i$, and $s_k^\epsilon = 0$. Constraint (3.2.9) holds as

$$p_k^\epsilon = p_k = 0 \leq \sum_{j=1}^{n}\pi_{jk}p_j^\epsilon + x_k + M(1 - s_k^\epsilon) = \sum_{j=1}^{n}\pi_{jk}p_j + x_k + M + \epsilon\pi_{ik}$$

by the feasibility of $p_k = 0$ and $s_k = 0$, and since $\epsilon > 0$ and $\pi_{ik} \geq 0$.

Constraint (3.2.11) holds as

$$\sum_{j=1}^{n} \pi_{jk} p_j^{\epsilon} + x_k = \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \leq 0 = M s_k^{\epsilon}$$

since $\sum_{j=1}^{n} \pi_{jk} p_j + x_k < 0$, $\pi_{ik} \leq 1$ and since a small enough $\epsilon > 0$ is taken to ensure $\sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \leq 0$. Constraints (3.2.10), (3.2.12), and (3.2.13) for node k hold trivially by the feasibility of $p_k = 0$ and $s_k = 0$. Hence, $p_k^{\epsilon} = 0$ and $s_k^{\epsilon} = 0$ satisfy the corresponding constraints in (3.2.8).

(2) Assume that $\sum_{j=1}^{n} \pi_{jk} p_j + x_k = 0$. Now, either $s_k = 0$ or $s_k = 1$ holds. If $s_k = 0$, then $p_k = 0$ by constraints (3.2.10) and (3.2.12). If $s_k = 1$, then, by the assumption of this case and (3.2.9), $p_k \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k + M(1-1) = 0$, which, together with (3.2.12), implies $p_k = 0$.

Also, $p_k^{\epsilon} = p_k = 0$ and $s_k^{\epsilon} = 1$, by the definitions of $\boldsymbol{p}^{\epsilon}$ and $\boldsymbol{s}^{\epsilon}$. Constraint (3.2.9) holds as

$$\begin{aligned} p_k^{\epsilon} = p_k = 0 &\leq \sum_{j=1}^{n} \pi_{jk} p_j^{\epsilon} + x_k + M\left(1 - s_k^{\epsilon}\right) \\ &= \sum_{j=1}^{n} \pi_{jk} p_j + x_k + M(1-1) + \epsilon \pi_{ik} = \epsilon \pi_{ik}, \end{aligned}$$

since $\sum_{j=1}^{n} \pi_{jk} p_j + x_k = 0$, $\epsilon > 0$ and $\pi_{ik} \geq 0$. Constraint (3.2.11) holds as

$$\sum_{j=1}^{n} \pi_{jk} p_j^{\epsilon} + x_k = \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} = \epsilon \pi_{ik} \leq M s_k^{\epsilon} = M$$

since $\sum_{j=1}^{n} \pi_{jk} p_j + x_k = 0$, $\epsilon \leq \min\limits_{l \in \mathcal{N}} \left\{ M - \left(\sum_{j=1}^{n} \pi_{jl} p_j + x_l \right) \right\} \leq M$ by the definition of ϵ, and $0 \leq \pi_{ik} \leq 1$. It is easy to observe that all other constraints in (3.2.8) for node k are satisfied trivially by $p_k^{\epsilon} = 0$ and $s_k^{\epsilon} = 1$.

(3) Assume that $0 < \sum_{j=1}^{n} \pi_{jk} p_j + x_k$. If $s_k = 0$, then, by constraint (3.2.11),

$$\sum_{j=1}^{n} \pi_{jk} p_j + x_k \leq M s_k = 0,$$

which is a contradiction to the assumption. Hence, $s_k = 1$. Also, $s^\epsilon_k = 1$, by the definition of $\boldsymbol{s}^\epsilon$.

Since $s_k = 1$, (3.2.10) and (3.2.12) hold by the feasibility of p_k since $p^\epsilon_k = p_k$ for $k \neq i$. Also, (3.2.11) holds since a small enough $\epsilon > 0$ is taken to ensure

$$\sum_{j=1}^{n} \pi_{jk} p^\epsilon_j + x_k = \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} \leq M. \tag{A.1.1}$$

Indeed, recall the assumption $\sum_{j=1}^{n} \pi_{jl} < n$, for each $l \in \mathcal{N}$. Hence, for each $l \in \mathcal{N}$ and for every $\boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]$, $\sum_{j=1}^{n} \pi_{jl} p_j + x_l < M$, where $M = n \|\bar{\boldsymbol{p}}\|_\infty + \|\boldsymbol{x}\|_\infty$. So, (A.1.1) is guaranteed by the choice of ϵ. (That is the reason behind including the term $\min\limits_{l \in \mathcal{N}} \left\{ M - \left(\sum_{j=1}^{n} \pi_{jl} p_j + x_l \right) \right\}$ in the definition of ϵ.)

Recall that, since $s_k = 1$, $p_k \leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k$ holds. Then constraint (3.2.9) is satisfied since

$$\begin{aligned} p^\epsilon_k = p_k &\leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k \\ &\leq \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} \\ &= \sum_{\substack{j \in \mathcal{N} \\ j \neq i}} \pi_{jk} p_j + \pi_{ik} (p_i + \epsilon) + x_k = \sum_{j=1}^{n} \pi_{jk} p^\epsilon_j + x_k. \end{aligned}$$

Constraint (3.2.13) is satisfied trivially. Hence, p^ϵ_k and s^ϵ_k satisfy the corresponding constraints in (3.2.8).

Next, it is shown that p^ϵ_i and s^ϵ_i satisfy the constraints in (3.2.8) for i. It holds $s^\epsilon_i = 1$, since $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > 0$ by the assumption of Lemma 3.2.11. Then, constraints (3.2.10) and (3.2.12) hold since $p^\epsilon_i = p_i + \epsilon > 0$ and $p^\epsilon_i = p_i + \epsilon \leq p_i + \bar{p}_i - p_i \leq \bar{p}_i$, where $\epsilon \leq \bar{p}_i - p_i$ holds since $\epsilon \leq \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} - \bar{p}_i$.

Constraint (3.2.9) holds as

$$\begin{aligned} p_i^\epsilon = p_i + \epsilon \le p_i + \sum_{j=1}^{n} \pi_{ji} p_j + x_i - p_i &= \sum_{j=1}^{n} \pi_{ji} p_j + x_i \\ \le \sum_{j=1}^{n} \pi_{jk} p_j + x_k + \epsilon \pi_{ik} = \sum_{\substack{j \in \mathcal{N} \\ j \ne i}} \pi_{jk} p_j + \pi_{ik} \left(p_i + \epsilon\right) + x_k &= \sum_{j=1}^{n} \pi_{jk} p_j^\epsilon + x_k, \end{aligned}$$

where $\epsilon \le \sum_{j=1}^{n} \pi_{ji} p_j + x_i - p_i$ holds since $\epsilon \le \min\left\{\sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i\right\} - \bar{p}_i$. Constraint (3.2.11) holds as

$$\sum_{j=1}^{n} \pi_{ji} p_j^\epsilon + x_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i + \epsilon \pi_{ii} = \sum_{j=1}^{n} \pi_{ji} p_j + x_i \le M$$

by the feasibility of $\boldsymbol{p}$ and since $\pi_{ll} = 0$, for each $l \in \mathcal{N}$. Constraint (3.2.13) is satisfied trivially. Hence, p_i^ϵ and s_i^ϵ satisfy the corresponding constraints in (3.2.8).

Hence, $(\boldsymbol{p}^\epsilon, \boldsymbol{s}^\epsilon)$ is a feasible solution to $\Lambda^{\mathrm{EN}}(\boldsymbol{x})$. However, since $\boldsymbol{p}^\epsilon \ge \boldsymbol{p}$ with $\boldsymbol{p}^\epsilon \ne \boldsymbol{p}$ and f is a strictly increasing function, it holds that $f(\boldsymbol{p}^\epsilon) > f(\boldsymbol{p})$, which is a contradiction to the optimality of $\boldsymbol{p}$. Hence, $p_i = \min\left\{\sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i\right\}$. □

A.2 Proof of Theorem 3.2.8

Proof. Let $(\boldsymbol{p}, \boldsymbol{s})$ be an optimal solution to the MILP for $\Lambda^{\mathrm{EN}}(\boldsymbol{x})$. To prove that $\boldsymbol{p}$ is a clearing vector, by Proposition 3.2.6, one can equivalently show that $\Phi^{\mathrm{EN}}(\boldsymbol{p}) = \boldsymbol{p}$. Let $i \in \mathcal{N}$. Recalling (3.2.6), three cases are considered:

$$(1)\ \sum_{j=1}^{n} \pi_{ji} p_j + x_i \le 0, \quad (2)\ 0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \le \bar{p}_i, \quad (3)\ \sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i.$$

(1) Assume that $\sum_{j=1}^{n} \pi_{ji} p_j + x_i \le 0$. Then, by Definition 3.2.5, $\Phi_i^{\mathrm{EN}}(\boldsymbol{p}) = 0$. By the arguments from the proof of Lemma 3.2.11 for this case, $p_i = 0$. Hence, $p_i = 0 = \Phi_i^{\mathrm{EN}}(\boldsymbol{p})$.

(2) Assume that $0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i \leq \bar{p}_i$. Then, by Definition 3.2.5, $\Phi_i^{\mathrm{EN}}(\boldsymbol{p}) = \sum_{j=1}^{n} \pi_{ji} p_j + x_i$. Since $0 < \sum_{j=1}^{n} \pi_{ji} p_j + x_i$, by Lemma 3.2.11,

$$p_i = \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} = \sum_{j=1}^{n} \pi_{ji} p_j + x_i.$$

Hence, $p_i = \sum_{j=1}^{n} \pi_{ji} p_j + x_i = \Phi_i^{\mathrm{EN}}(\boldsymbol{p})$.

(3) Assume that $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i$. Then, by Definition 3.2.5, $\Phi_i^{\mathrm{EN}}(\boldsymbol{p}) = \bar{p}_i$. Since $\sum_{j=1}^{n} \pi_{ji} p_j + x_i > \bar{p}_i > 0$, again by Lemma 3.2.11,

$$p_i = \min \left\{ \sum_{j=1}^{n} \pi_{ji} p_j + x_i, \bar{p}_i \right\} = \bar{p}_i.$$

Hence, $p_i = \bar{p}_i = \Phi_i^{\mathrm{EN}}(\boldsymbol{p})$.

Therefore, $\boldsymbol{p}$ is a clearing vector for $(\mathcal{N}, \boldsymbol{\pi}, \bar{\boldsymbol{p}}, \boldsymbol{x})$. □

A.3 Proof of Lemma 3.3.9

Proof. To get a contradiction, suppose that $s_i = 0$. Then $p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j < \bar{p}_i$ by constraint (3.3.4) and the assumption. Let $\boldsymbol{p'} \in \mathbb{R}_+^n$ be equal to $\boldsymbol{p}$ in all components except the i^{th} one, and let $p'_i = \bar{p}_i$. Also, let $\boldsymbol{s'} \in \mathbb{R}_+^n$ be equal to $\boldsymbol{s}$ in all components except the i^{th} one, and let $s'_i = 1$.

It is shown that $(\boldsymbol{p'}, \boldsymbol{s'})$ is a feasible solution to $\Lambda^{\mathrm{RV}+}(\boldsymbol{x})$ by showing that all constraints in (3.3.3) are satisfied. First, for fixed $k \in \mathcal{N} \setminus \{i\}$, the k^{th} constraints in (3.3.3) are verified for $(\boldsymbol{p'}, \boldsymbol{s'})$. Constraints (3.3.4) and (3.3.5) hold as

$$\begin{aligned} p'_k = p_k &\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k \\ &\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k + \pi_{ik} (\bar{p}_i - p_i) = \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p'_j + \bar{p}_k s'_k, \end{aligned}$$

and

$$\bar{p}_k s'_k = \bar{p}_k s_k \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j + \pi_{ik} \left(\bar{p}_i - p_i\right) = x_k + \sum_{j=1}^{n} \pi_{jk} p'_j,$$

since $p'_k = p_k$, $s'_k = s_k$ for every $k \in \mathcal{K}$ such that $k \neq i$, $\bar{p}_i - p_i > 0$, $\pi_{ik} \geq 0$, and by the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$. Constraints (3.3.6) and (3.3.7) hold trivially by the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$.

Next, the i^{th} constraints in (3.3.3) are verified for $p'_i = \bar{p}_i$, $s'_i = 1$. Constraints (3.3.4) and (3.3.5) hold as

$$\begin{aligned} p'_i = \bar{p}_i &\leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s'_i \\ &= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i + \pi_{ii} \left(\bar{p}_i - p_i\right) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j + \bar{p}_i, \end{aligned}$$

and

$$\bar{p}_i s'_i = \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} \left(\bar{p}_i - p_i\right) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,$$

since $\alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j \geq 0$, $\pi_{ii} = 0$ and by the assumption of Lemma 3.3.9. Constraints (3.3.6) and (3.3.7) are satisfied trivially.

Hence, $(\boldsymbol{p'}, \boldsymbol{s'})$ is a feasible solution to $\Lambda^{\text{RV}+}(\boldsymbol{x})$. However, since $\boldsymbol{p'} \geq \boldsymbol{p}$ with $\boldsymbol{p'} \neq \boldsymbol{p}$ and f is a strictly increasing function, it holds that $f(\boldsymbol{p'}) > f(\boldsymbol{p})$, which is a contradiction to the optimality of $\boldsymbol{p}$. Hence, $s_i = 1$. □

A.4 Proof of Lemma 3.3.10

Proof. To get a contradiction, suppose that $p_i < \bar{p}_i$. Let $\boldsymbol{p'} \in \mathbb{R}^n_+$ be equal to $\boldsymbol{p}$ in all components except the i^{th} one, and let $p'_i = \bar{p}_i$

It is shown that $(\boldsymbol{p'}, \boldsymbol{s})$ is a feasible solution to $\Lambda^{\text{RV+}}(\boldsymbol{x})$ by showing that all constraints in (3.3.3) are satisfied. First, for fixed $k \in \mathcal{N} \setminus \{i\}$, the k^{th} constraints in (3.3.3) are verified for $(\boldsymbol{p'}, \boldsymbol{s})$. Constraints (3.3.4) and (3.3.5) hold as

$$\begin{aligned}p'_k = p_k &\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k \\ &\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k + \pi_{ik}(\bar{p}_i - p_i) = \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p'_j + \bar{p}_k s_k,\end{aligned}$$

and

$$\bar{p}_k s_k \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j + \pi_{ik}(\bar{p}_i - p_i) = x_k + \sum_{j=1}^{n} \pi_{jk} p'_j,$$

since $p'_k = p_k$ for every $k \in \mathcal{K}$ such that $k \neq i$, $\bar{p}_i - p_i > 0$, $\pi_{ik} \geq 0$ and by the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$. Constraints (3.3.6) and (3.3.7) hold trivially by the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$.

Next, the i^{th} constraints in (3.3.3) are verified for $p'_i = \bar{p}_i$, s_i. Two cases are considered:

$$(1)\ \ \bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j, \quad (2)\ \ \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j < \bar{p}_i.$$

(1) If the first case holds, then constraints (3.3.4) and (3.3.5) hold for both $s_i = 0$ and $s_i = 1$ as

$$\begin{aligned}p'_i = \bar{p}_i &\leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i \\ &= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i + \pi_{ii}(\bar{p}_i - p_i) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j + \bar{p}_i s_i,\end{aligned}$$

and

$$\bar{p}_i s_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii}(\bar{p}_i - p_i) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,$$

since $\pi_{ii} = 0$ and by the assumption of Lemma 3.3.10. Constraints (3.3.6) and (3.3.7) are satisfied trivially.

(2) If the second case holds, then, by Lemma 3.3.9, $s_i = 1$. Then constraints (3.3.4) and (3.3.5) hold as

$$p'_i = \bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i$$
$$= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i + \pi_{ii} \left(\bar{p}_i - p_i\right) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j + \bar{p}_i,$$

and

$$\bar{p}_i s_i = \bar{p}_i \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} \left(\bar{p}_i - p_i\right) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,$$

since $\pi_{ii} = 0$ and by the assumption of Lemma 3.3.10. Constraints (3.3.6) and (3.3.7) are satisfied trivially.

Hence, $(\boldsymbol{p'}, \boldsymbol{s})$ is a feasible solution to $\Lambda^{\mathrm{RV}+}(\boldsymbol{x})$. However, since $\boldsymbol{p'} \geq \boldsymbol{p}$ with $\boldsymbol{p'} \neq \boldsymbol{p}$ and f is a strictly increasing function, it holds that $f(\boldsymbol{p'}) > f(\boldsymbol{p})$, which is a contradiction to the optimality of $\boldsymbol{p}$. Hence, $p_i = \bar{p}_i$. □

A.5 Proof of Lemma 3.3.11

Proof. To get a contradiction, suppose that $p_i \neq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. If $s_i = 1$, then constraint (3.3.5) is not satisfied by assumption. Hence, $s_i = 0$ and $p_i < \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$ by constraint (3.3.4). Let $\boldsymbol{p'} \in \mathbb{R}_+^n$ be equal to $\boldsymbol{p}$ in all components except the i^{th} one, and let $p'_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$.

It is shown that $(\boldsymbol{p'}, \boldsymbol{s})$ is a feasible solution to $\Lambda^{\mathrm{RV}+}(\boldsymbol{x})$ by showing that all constraints in (3.3.3) are satisfied. First, for fixed $k \in \mathcal{N} \setminus \{i\}$, the k^{th} constraints in (3.3.3) are verified for $(\boldsymbol{p'}, \boldsymbol{s})$. Constraints (3.3.4) and (3.3.5) hold as

$$\begin{aligned} p'_k = p_k &\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k \\ &\leq \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p_j + \bar{p}_k s_k + \pi_{ik} \left(\bar{p}_i - p_i\right) = \alpha x_k + \beta \sum_{j=1}^{n} \pi_{jk} p'_j + \bar{p}_k s_k, \end{aligned}$$

and

$$\bar{p}_k s_k \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j \leq x_k + \sum_{j=1}^{n} \pi_{jk} p_j + \pi_{ik} \left(\bar{p}_i - p_i\right) = x_k + \sum_{j=1}^{n} \pi_{jk} p'_j,$$

since $p'_k = p_k$ for every $k \in \mathcal{K}$ such that $k \neq i$, $\bar{p}_i - p_i > 0$, $\pi_{ik} \geq 0$ and by the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$. Constraints (3.3.6) and (3.3.7) hold trivially by the feasibility of $(\boldsymbol{p}, \boldsymbol{s})$.

Next, the i^{th} constraints in (3.3.3) are verified for $p'_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$, $s_i = 0$. Constraints (3.3.4) and (3.3.5) hold as

$$\begin{aligned} p'_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j &\leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i s_i \\ &= \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} \left(\bar{p}_i - p_i\right) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p'_j, \end{aligned}$$

and

$$\bar{p}_i s_i = 0 \leq x_i + \sum_{j=1}^{n} \pi_{ji} p_j = x_i + \sum_{j=1}^{n} \pi_{ji} p_j + \pi_{ii} \left(\bar{p}_i - p_i\right) = x_i + \sum_{j=1}^{n} \pi_{ji} p'_j,$$

since $\pi_{ii} = 0$ and $x_i + \sum_{j=1}^{n} \pi_{ji} p_j \geq 0$. Constraints (3.3.6) and (3.3.7) are satisfied trivially.

Hence, $(\boldsymbol{p'}, \boldsymbol{s})$ is a feasible solution to $\Lambda^{\text{RV}+}(\boldsymbol{x})$. However, since $\boldsymbol{p'} \geq \boldsymbol{p}$ with $\boldsymbol{p'} \neq \boldsymbol{p}$ and f is a strictly increasing function, it holds that $f(\boldsymbol{p'}) > f(\boldsymbol{p})$, which is a contradiction to the optimality of $\boldsymbol{p}$. Hence, $p_i = \bar{p}_i$. $\square$

A.6 Proof of Theorem 3.3.7

Proof. Let $(\boldsymbol{p}, \boldsymbol{s})$ be an optimal solution to the MILP for $\Lambda^{\mathrm{RV}+}(\boldsymbol{x})$. To prove that $\boldsymbol{p}$ is a clearing vector, thanks to Proposition 3.3.5, it suffices to show $\Phi^{\mathrm{RV}+}(\boldsymbol{p}) = \boldsymbol{p}$. Fix $i \in \mathcal{N}$. Recalling (3.3.1), two cases are considered:

$$(1)\ \bar{p}_i \le x_i + \sum_{j=1}^{n} \pi_{ji} p_j, \quad (2)\ \bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j.$$

(1) Assume that $\bar{p}_i \le x_i + \sum_{j=1}^{n} \pi_{ji} p_j$. Then, by Definition 3.3.3, $\Phi_i^{\mathrm{RV}+}(\boldsymbol{p}) = \bar{p}_i$. By Lemma 3.3.10, $p_i = \bar{p}_i$. Hence, $p_i = \bar{p}_i = \Phi_i^{\mathrm{RV}+}(\boldsymbol{p})$.

(2) Assume that $\bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j$. Then, by Definition 3.3.3, $\Phi_i^{\mathrm{RV}+}(\boldsymbol{p}) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. By Lemma 3.3.11, $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. Hence, $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j = \Phi_i^{\mathrm{RV}+}(\boldsymbol{p})$.

Therefore, $\boldsymbol{p}$ is a clearing vector. □

A.7 Proof of Theorem 3.3.13

Proof. Let $(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{s}^1, \boldsymbol{s}^2, \boldsymbol{s}^3)$ be an optimal solution to the MILP for $\Lambda_{\mathrm{alt}}^{\mathrm{RV}+}(\boldsymbol{x})$. To prove that $\boldsymbol{p}$ is a clearing vector, thanks to Proposition 3.3.5, it suffices to show $\Phi^{\mathrm{RV}+}(\boldsymbol{p}) = \boldsymbol{p}$. Fix $i \in \mathcal{N}$. Four cases are considered:

$$(1)\quad \bar{p}_i \le \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j \le x_i + \sum_{j=1}^{n} \pi_{ji} p_j, \quad \text{and} \quad \bar{p}_i < x_i + \sum_{j=1}^{n} \pi_{ji} p_j;$$

$$(2)\quad \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j < \bar{p}_i < x_i + \sum_{j=1}^{n} \pi_{ji} p_j;$$

$$(3)\quad \bar{p}_i = x_i + \sum_{j=1}^{n} \pi_{ji} p_j;$$

$$(4)\quad \bar{p}_i > x_i + \sum_{j=1}^{n} \pi_{ji} p_j \ge \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j.$$

(1) Assume the first case. Then, by Definition 3.3.3, $\Phi_i^{\text{RV+}}(\boldsymbol{p}) = \bar{p}_i$. By constraint (3.3.12), $s_i^1 = 1$. Otherwise, (3.3.12) is not satisfied by assumptions. If $y_i = 0$, then $s_i^2 = 1$ by (3.3.14). If $y_i > 0$, then $s_i^2 = 0$ by (3.3.13). For both $s_i^3 = 0$ and $s_i^3 = 1$, the constraint $p_i \leq \bar{p}_i$ in (3.3.18) is the tightest one on p_i by the assumption $\bar{p}_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. Hence, $p_i = \bar{p}_i = \Phi_i^{\text{RV+}}(\boldsymbol{p})$.

(2) Assume the second case. By Definition 3.3.3, $\Phi_i^{\text{RV+}}(\boldsymbol{p}) = \bar{p}_i$. By constraint (3.3.12), $s_i^1 = 1$. Otherwise, (3.3.12) is not satisfied by assumption. If $y_i = 0$, then $s_i^2 = 1$ by (3.3.14). If $y_i > 0$, then $s_i^2 = 0$ by (3.3.13). By constraint (3.3.16), $s_i^3 = 0$. Otherwise, (3.3.16) is not satisfied by assumption. Then, since $s_i^3 = 0$ and $s_i^1 = 1$ it holds that

$$p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i (1 - 0) \quad \text{and} \quad p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i \left(1 + s_i^2\right),$$

by constraints (3.3.15) and (3.3.10), respectively. Hence, the constraint $p_i \leq \bar{p}_i$ in (3.3.18) is the tightest one on p_i, despite the assumption $\bar{p}_i > \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. Hence, $p_i = \bar{p}_i = \Phi_i^{\text{RV+}}(\boldsymbol{p})$.

(3) Assume the third case. By Definition 3.3.3, $\Phi_i^{\text{RV+}}(\boldsymbol{p}) = \bar{p}_i$. If $s_i^1 = 0$ then $y_i = 0$ by (3.3.12). Then, $s_i^2 = 1$ by (3.3.14). If $s_i^1 = 1$ then $y_i = M > 0$ by (3.3.12). Then, $s_i^2 = 0$ by (3.3.14). So, either $s_i^1 = 1$ or $s_i^2 = 1$. Here, two more possible cases are considered: either $\bar{p}_i = x_i + \sum_{j=1}^{n} \pi_{ji} p_j = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$ or $\bar{p}_i = x_i + \sum_{j=1}^{n} \pi_{ji} p_j > \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. In the former case, for both $s_i^3 = 0$ and $s_i^3 = 1$, the constraint $p_i \leq \bar{p}_i$ in (3.3.18) is the tightest one for p_i. Hence, $p_i = \bar{p}_i = \Phi_i^{\text{RV+}}(\boldsymbol{p})$. In the latter case, $s_i^3 = 0$ by (3.3.16). Then, since $s_i^3 = 0$ and $s_i^1 + s_i^2 = 1$ it holds that

$$p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i (1 - 0) \quad \text{and} \quad p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j + \bar{p}_i \left(s_i^1 + s_i^2\right),$$

by constraints (3.3.15) and (3.3.10), respectively. Hence, the constraint $p_i \leq \bar{p}_i$ in (3.3.18) is the tightest one on p_i. Hence, $p_i = \bar{p}_i = \Phi_i^{\text{RV+}}(\boldsymbol{p})$. A detailed proof of the claim $p_i = \bar{p}_i$ can be given similar to the one in the proof of Theorem 3.2.8.

(4) Assume the fourth case. By Definition 3.3.3, $\Phi_i^{\text{RV}+}(\boldsymbol{p}) = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$. It holds that $s_i^1 = 0$ and $s_i^3 = 0$ by constraints (3.3.16) and (3.3.11), respectively. Otherwise, these constraints are not satisfied by assumption. Since $s_i^1 = 0$ it holds that $y_i > 0$ by constraint (3.3.12). Then, $s_i^2 = 0$ by constraint (3.3.13). Then, $s_i^1 = s_i^2 = s_i^3 = 0$ yields

$$p_i \leq \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$$

by both (3.3.15) and (3.3.10), and it is the tightest constraint on p_i by assumption. Hence, $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j = \Phi_i^{\text{RV}+}(\boldsymbol{p})$. A detailed proof of the claim $p_i = \alpha x_i + \beta \sum_{j=1}^{n} \pi_{ji} p_j$ can be given similar to the one in the proof of Theorem 3.2.8

Therefore, $\boldsymbol{p}$ is a clearing vector. □

Appendix B

Proofs of the Results in Chapter 4

B.1 Proof of Proposition 4.1.6

Proof. Let $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ be an optimal solution to $\mathcal{Z}_1^{\mathrm{EN}}\left(\boldsymbol{e}^\ell\right)$. The aim is to show $z_\ell \leq z^{\max}$.

To get a contradiction, suppose that $z_\ell > z^{\max}$. Let $\boldsymbol{z'} \in \mathbb{R}^G$ be the vector such that $z'_\ell = z^{\max}$ and $z'_{\hat{\ell}} = z_{\hat{\ell}}$, for each $\hat{\ell} \in \mathcal{G}$ such that $\hat{\ell} \neq \ell$. It is shown that $\left(\boldsymbol{z'}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\mathrm{EN}}\left(\boldsymbol{e}^\ell\right)$ by showing that all constraints of $\mathcal{Z}_1^{\mathrm{EN}}\left(\boldsymbol{e}^\ell\right)$ in (4.1.10) are satisfied. For each $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^\mathsf{T}\boldsymbol{z'}\right)_i = z^{\max}$, constraint (4.1.12) holds as

$$\begin{aligned} p_i^k &\leq \sum_{j=1}^n \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z'}\right)_i\right] + M\left(1 - s_i^k\right) \\ &= \sum_{j=1}^n \pi_{ji} p_j^k + X_i\left(\omega^k\right) + z^{\max} + M\left(1 - s_i^k\right) \\ &= \sum_{j=1}^n \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty + M\left(1 - s_i^k\right), \end{aligned}$$

since

$$\sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad p_i^k \leq \bar{p}_i \leq \|\bar{\boldsymbol{p}}\|_\infty, \quad \text{and} \quad M\left(1 - s_i^k\right) \geq 0.$$

Also, for each $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^\top \boldsymbol{z'}\right)_i = z^{\max}$, constraint (4.1.14) holds as

$$\begin{aligned}\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{z'}\right)_i\right] &= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + z^{\max} \\ &< \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + z_t \leq M s_i^k,\end{aligned}$$

by the assumption $z^{\max} < z_\ell$ and the feasibility of $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$. All the other constraints in (4.1.10) hold by the feasibility of $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$, since they are free of $z^{\max}$. Hence, $\left(\boldsymbol{z'}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$, which contradicts to the optimality of $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ by the assumption $z_\ell > z^{\max}$. Hence, $z_\ell \leq z^{\max} = \|\boldsymbol{X}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty$. □

B.2 Proof of Proposition 4.1.7

Proof. To get a contradiction, suppose that $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$ is feasible but unbounded. Then for any $\mu \in \mathbb{R}$ there exist $\epsilon > 0$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$, where $\boldsymbol{z} \in \mathbb{R}^n$ and $\left(\boldsymbol{p}^k, \boldsymbol{s}^k\right) \in \mathbb{R}^n \times \mathbb{Z}^n$ for each $k \in \mathcal{K}$, such that ${\boldsymbol{e}^\ell}^\top \boldsymbol{z} = z_\ell = \mu$ and $\left(\boldsymbol{z} - \epsilon \boldsymbol{e}^\ell, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$.

Let $\mu = -2M$. Then there exist $\epsilon > 0$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ such that ${\boldsymbol{e}^\ell}^\top \boldsymbol{z} = z_\ell = \mu = -2M$ and $\left(\boldsymbol{z} - \epsilon \boldsymbol{e}^\ell, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$. Fix $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^\top \boldsymbol{z}\right)_i = z_\ell = \mu = -2M$. Then constraint (4.1.12) violates constraint (4.1.15) as

$$
\begin{aligned}
p_i^k &\leq \sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\intercal \boldsymbol{z} - \epsilon \boldsymbol{e}^\ell\right)_i\right] + M\left(1 - s_i^k\right) \\
&\leq \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) - 2M - \epsilon + M \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) - \epsilon - 2\left\|\boldsymbol{X}\right\|_\infty - (n+1)\left\|\bar{\boldsymbol{p}}\right\|_\infty \\
&= \left(\sum_{j=1}^{n} \pi_{ji} p_j^k - n\left\|\bar{\boldsymbol{p}}\right\|_\infty\right) + \left(X_i\left(\omega^k\right) - 2\left\|\boldsymbol{X}\right\|_\infty\right) - \left\|\bar{\boldsymbol{p}}\right\|_\infty - \epsilon < 0,
\end{aligned}
$$

since

$$
\sum_{j=1}^{n} \pi_{ji} p_j^k < n\left\|\bar{\boldsymbol{p}}\right\|_\infty, \quad X_i\left(\omega^k\right) \leq 2\left\|\boldsymbol{X}\right\|_\infty, \quad -\left\|\bar{\boldsymbol{p}}\right\|_\infty < 0 \quad \text{and} \quad -\epsilon < 0.
$$

Hence, $\left(\boldsymbol{z} - \epsilon \boldsymbol{e}_\ell, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k \in \mathcal{K}}\right)$ is infeasible, which is a contradiction to the assumption. Hence, $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}_\ell\right)$ is bounded from below. In addition, by Proposition 4.1.6, $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}_\ell\right)$ is bounded from above. □

B.3 Proof of Proposition 4.1.8

Proof. Assume $\gamma \leq \mathbb{1}^\intercal \bar{\boldsymbol{p}}$. Let $\boldsymbol{z} = z^{\max}\mathbb{1}$, $\boldsymbol{p}^k = \bar{\boldsymbol{p}}$, $\boldsymbol{s}^k = \mathbb{1}$ for each $k \in \mathcal{K}$, where $z^{\max} = \left\|\boldsymbol{X}\right\|_\infty + \left\|\bar{\boldsymbol{p}}\right\|_\infty$. It is shown that $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k \in \mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$. Since $\boldsymbol{p}^k = \bar{\boldsymbol{p}}$ for each $k \in \mathcal{K}$, it holds that $\sum_{k \in \mathcal{K}} q^k \left[\mathbb{1}^\intercal \boldsymbol{p}^k\right] = \mathbb{1}^\intercal \bar{\boldsymbol{p}} \geq \gamma$. Hence, constraint (4.1.11) holds by hypothesis.

Now fix $i \in \mathcal{N}$, $k \in \mathcal{K}$. Constraint (4.1.12) holds as

$$\begin{aligned}
&\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T} \boldsymbol{z}\right)_i\right] + M\left(1 - s_i^k\right) \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + z^{\max}\left(B^\mathsf{T} \mathbb{1}\right)_i + M\left(1 - 1\right) \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + z^{\max} = \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty \geq \bar{p}_i = p_i^k,
\end{aligned}$$

since

$$\sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0,$$

$\left(B^\mathsf{T} \mathbb{1}\right)_i = 1$, by definition of a grouping matrix B, and $s_i^k = 1$, by the choice of $\boldsymbol{s}^k$. Constraint (4.1.14) holds as

$$\begin{aligned}
&\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T} \boldsymbol{z}\right)_i\right] = \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty \\
&\leq 2\|\boldsymbol{X}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty = M = M s_i^k,
\end{aligned}$$

since $\sum_{j=1}^{n} \pi_{ji} p_j^k \leq n\|\bar{\boldsymbol{p}}\|_\infty$. All the other constraints in (4.1.10) hold trivially by the choice of $\boldsymbol{z}$, $\boldsymbol{p}^k$ and $\boldsymbol{s}^k$, for each $k \in \mathcal{K}$. Hence, $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$.

Now, if $\gamma > \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$, then constraint (4.1.11) is infeasible, since $\sum_{k\in\mathcal{K}} q^k\left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}} < \gamma$, by constraint (4.1.15). Hence, $\mathcal{Z}_1^{\text{EN}}\left(\boldsymbol{e}^\ell\right)$ in Corollary 4.1.5 is feasible if and only if $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$. □

B.4 Proof of Proposition 4.1.11

Proof. Let $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ be an optimal solution to $\mathcal{Z}_1^{\text{RV}}\left(\boldsymbol{e}^\ell\right)$. The aim is to show $z_\ell \leq z^{\max}$.

To get a contradiction, suppose that $z_\ell > z^{\max}$. Let $\boldsymbol{z'} \in \mathbb{R}^n$ be the vector such that $z'_\ell = z^{\max}$ and $z'_{\hat{\ell}} = z_{\hat{\ell}}$, for each $\hat{\ell} \in \mathcal{G}$ such that $\hat{\ell} \neq \ell$. It is shown that $\left(\boldsymbol{z'}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$ by showing that all constraints of $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$ in (4.1.21) are satisfied. For each $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^\mathsf{T}\boldsymbol{z'}\right)_i = z^{\max}$, constraint (4.1.23) holds as

$$\begin{aligned} p_i^k &\leq \alpha\left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z'}\right)_i\right] + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\ &= \alpha X_i\left(\omega^k\right) + \alpha z^{\max} + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\ &= \alpha\left(X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty\right) + \|\bar{\boldsymbol{p}}\|_\infty + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k, \end{aligned}$$

since

$$0 < \alpha, \beta \leq 1, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad p_i^k \leq \|\bar{\boldsymbol{p}}\|_\infty, \quad \sum_{j=1}^n \pi_{ji}p_j^k \geq 0, \quad \bar{p}_i s_i^k \geq 0.$$

Also, for each $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^\mathsf{T}\boldsymbol{z'}\right)_i = z^{\max}$, constraint (4.1.24) holds as

$$\begin{aligned} \bar{p}_i s_i^k &\leq \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z'}\right)_i\right] + \sum_{j=1}^n \pi_{ji}p_j^k = X_i\left(\omega^k\right) + z^{\max} + \sum_{j=1}^n \pi_{ji}p_j^k \\ &= X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty + \sum_{j=1}^n \pi_{ji}p_j^k, \end{aligned}$$

since

$$0 < \alpha \leq 1, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \bar{p}_i s_i^k \leq \|\bar{\boldsymbol{p}}\|_\infty \leq \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty, \quad \sum_{j=1}^n \pi_{ji}p_j^k \geq 0.$$

In addition, for each $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^\mathsf{T}\boldsymbol{z'}\right)_i = z^{\max}$, constraint (4.1.25) holds as

$$X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{z'}\right)_i = X_i\left(\omega^k\right) + z^{\max} = X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty \geq 0.$$

All the other constraints in (4.1.21) hold by the feasibility of $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$, since they are free of $z^{\max}$. Hence, $\left(\boldsymbol{z'}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$, which contradicts to the optimality of $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ by the assumption $z_\ell > z^{\max}$. Hence, $z_\ell \leq z^{\max} = \|\boldsymbol{X}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty$. □

B.5 Proof of Proposition 4.1.12

Proof. To get a contradiction, suppose that $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$ is feasible but unbounded. Then for any $\mu \in \mathbb{R}$ there exist $\epsilon > 0$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$, where $\boldsymbol{z} \in \mathbb{R}^n$ and $\left(\boldsymbol{p}^k, \boldsymbol{s}^k\right) \in \mathbb{R}^n \times \mathbb{Z}^n$ for each $k \in \mathcal{K}$, such that ${\boldsymbol{e}^\ell}^{\mathsf{T}}\boldsymbol{z} = z_\ell = \mu$ and $\left(\boldsymbol{z} - \epsilon\boldsymbol{e}^\ell, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$.

Let $\mu = -M$, where $M = \|\boldsymbol{X}\|_\infty + \frac{1}{\alpha}(n+1)\|\bar{\boldsymbol{p}}\|_\infty$. Then there exist $\epsilon > 0$ and $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ such that ${\boldsymbol{e}^\ell}^{\mathsf{T}}\boldsymbol{z} = z_\ell = \mu = -M$ and $\left(\boldsymbol{z} - \epsilon\boldsymbol{e}^\ell, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$. Fix $i \in \mathcal{N}$, $k \in \mathcal{K}$ such that $\left(B^{\mathsf{T}}\boldsymbol{z}\right)_i = z_\ell = \mu = -M$. Then constraint (4.1.23) violates constraint (4.1.26) as

$$\begin{aligned} p_i^k &\leq \alpha\left[X_i\left(\omega^k\right) + \left(B^{\mathsf{T}}\left(\boldsymbol{z} - \epsilon\boldsymbol{e}^\ell\right)\right)\right] + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\ &= \alpha X_i\left(\omega^k\right) + \alpha\left(-M - \epsilon\right) + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\ &= \alpha\left(X_i\left(\omega^k\right) - \|\boldsymbol{X}\|_\infty\right) - (n+1)\|\bar{\boldsymbol{p}}\|_\infty - \alpha\epsilon + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\ &= \alpha\left(X_i\left(\omega^k\right) - \|\boldsymbol{X}\|_\infty\right) \\ &\qquad + \left(\beta\sum_{j=1}^n \pi_{ji}p_j^k - n\|\bar{\boldsymbol{p}}\|_\infty\right) + \left(\bar{p}_i s_i^k - \|\bar{\boldsymbol{p}}\|_\infty\right) - \alpha\epsilon < 0, \end{aligned}$$

since

$$0 < \alpha, \beta \leq 1,\ X_i\left(\omega^k\right) \leq \|\boldsymbol{X}\|_\infty,\ \sum_{j=1}^n \pi_{ji}p_j^k < n\|\bar{\boldsymbol{p}}\|_\infty,\ \bar{p}_i s_i^k \leq \|\bar{\boldsymbol{p}}\|_\infty,\ -\epsilon < 0.$$

Hence, $\left(\boldsymbol{z}-\epsilon \boldsymbol{e}_{\ell},\left(\boldsymbol{p}^{k}, \boldsymbol{s}^{k}\right)_{k \in \mathcal{K}}\right)$ is infeasible, which is a contradiction to the assumption. Hence, $\mathcal{Z}_{1}^{\mathrm{RV}}\left(\boldsymbol{e}_{\ell}\right)$ is bounded from below. In addition, by Proposition 4.1.11, $\mathcal{Z}_{1}^{\mathrm{RV}}\left(\boldsymbol{e}_{\ell}\right)$ is bounded from above. □

B.6 Proof of Proposition 4.1.13

Proof. Assume $\gamma \leq \mathbb{1}^{\mathsf{T}} \bar{\boldsymbol{p}}$. Let $\boldsymbol{z}=z^{\max } \mathbb{1}$, $\boldsymbol{p}^{k}=\bar{\boldsymbol{p}}$, $\boldsymbol{s}^{k}=\mathbb{1}$ for each $k \in \mathcal{K}$, where $z^{\max }=\|\boldsymbol{X}\|_{\infty}+\frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_{\infty}$. It is shown that $\left(\boldsymbol{z},\left(\boldsymbol{p}^{k}, \boldsymbol{s}^{k}\right)_{k \in \mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_{1}^{\mathrm{RV}}\left(\boldsymbol{e}^{\ell}\right)$. Since $\boldsymbol{p}^{k}=\bar{\boldsymbol{p}}$ for each $k \in \mathcal{K}$, it holds that $\sum_{k \in \mathcal{K}} q^{k}\left[\mathbb{1}^{\mathsf{T}} \boldsymbol{p}^{k}\right]=\mathbb{1}^{\mathsf{T}} \bar{\boldsymbol{p}} \geq \gamma$. Hence, constraint (4.1.22) holds by hypothesis.

Now fix $i \in \mathcal{N}$, $k \in \mathcal{K}$. Constraint (4.1.23) holds as

$$
\begin{aligned}
& \alpha\left[X_{i}\left(\omega^{k}\right)+\left(B^{\mathsf{T}} \boldsymbol{z}\right)_{i}\right]+\beta \sum_{j=1}^{n} \pi_{j i} p_{j}^{k}+\bar{p}_{i} s_{i}^{k} \\
& =\alpha\left[X_{i}\left(\omega^{k}\right)+z^{\max }\left(B^{\mathsf{T}} \mathbb{1}\right)_{i}\right]+\beta \sum_{j=1}^{n} \pi_{j i} p_{j}^{k}+\bar{p}_{i} \\
& =\alpha\left[X_{i}\left(\omega^{k}\right)+\|\boldsymbol{X}\|_{\infty}\right]+\|\bar{\boldsymbol{p}}\|_{\infty}+\beta \sum_{j=1}^{n} \pi_{j i} p_{j}^{k}+\bar{p}_{i} \geq \bar{p}_{i}=p_{i}^{k},
\end{aligned}
$$

since

$$
0<\alpha, \beta \leq 1, \quad X_{i}\left(\omega^{k}\right)+\|\boldsymbol{X}\|_{\infty} \geq 0, \quad\|\bar{\boldsymbol{p}}\|_{\infty} \geq 0, \quad \sum_{j=1}^{n} \pi_{j i} p_{j}^{k} \geq 0,
$$

and $s_{i}^{k}=1$, by the choice of $\boldsymbol{s}^{k}$. Constraint (4.1.24) holds as

$$
\begin{aligned}
\left[X_{i}\left(\omega^{k}\right)+\left(B^{\mathsf{T}} \boldsymbol{z}\right)_{i}\right]+\sum_{j=1}^{n} \pi_{j i} p_{j}^{k} & =X_{i}\left(\omega^{k}\right)+\|\boldsymbol{X}\|_{\infty}+\frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_{\infty}+\sum_{j=1}^{n} \pi_{j i} p_{j}^{k} \\
& \geq X_{i}\left(\omega^{k}\right)+\|\boldsymbol{X}\|_{\infty}+\|\bar{\boldsymbol{p}}\|_{\infty}+\sum_{j=1}^{n} \pi_{j i} p_{j}^{k} \\
& \geq \bar{p}_{i}=\bar{p}_{i} s_{i}^{k},
\end{aligned}
$$

since

$$0 < \alpha \leq 1, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0, \quad \|\bar{\boldsymbol{p}}\|_\infty \geq \bar{p}_i$$

and $s_i^k = 1$, by the choice of $\boldsymbol{s}^k$. In addition, constraint (4.1.25) holds as

$$X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{z}\right)_i = X_i\left(\omega^k\right) + z^{\max} = X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty \geq 0.$$

All the other constraints in (4.1.21) hold trivially by the choice of $\boldsymbol{z}$, $\boldsymbol{p}^k$ and $\boldsymbol{s}^k$, for each $k \in \mathcal{K}$. Hence, $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$.

Now, if $\gamma > \mathbb{1}^\top \bar{\boldsymbol{p}}$, then constraint (4.1.22) is infeasible, since $\sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\top \boldsymbol{p}^k\right] \leq \mathbb{1}^\top \bar{\boldsymbol{p}} < \gamma$, by constraint (4.1.26). Hence, $\mathcal{Z}_1^{\mathrm{RV}}\left(\boldsymbol{e}^\ell\right)$ in Corollary 4.1.10 is feasible if and only if $\gamma \leq \mathbb{1}^\top \bar{\boldsymbol{p}}$. □

B.7 Proof of Proposition 4.2.3

Proof. To get a contradiction, suppose that $\mu > \mu^{\max}$. It is shown that $\left(\mu^{\max}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{EN}}(\boldsymbol{v})$ by showing that all constraints of $\mathcal{Z}_2^{\mathrm{EN}}(\boldsymbol{v})$ in (4.2.5) are satisfied. Fix $i \in \mathcal{N}$, $k \in \mathcal{K}$. Constraint (4.2.8) holds as

$$\begin{aligned} p_i^k &\leq \sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \mu^{\max}\right] + M\left(1 - s_i^k\right) \\ &= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty + M\left(1 - s_i^k\right) \\ &= \sum_{j=1}^{n} \pi_{ji} p_j^k + \left(X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty\right) + \left(\left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty\right) + \|\bar{\boldsymbol{p}}\|_\infty + M\left(1 - s_i^k\right), \end{aligned}$$

since

$$\sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty \geq 0,$$
$$p_i^k \leq \|\bar{\boldsymbol{p}}\|_\infty, \quad M\left(1 - s_i^k\right) \geq 0.$$

Constraint (4.2.10) holds as

$$\begin{aligned}
&\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \mu^{\max}\right] \\
&< \sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \mu\right] \leq M s_i^k = M
\end{aligned}$$

by the assumption $\mu^{\max} < \mu$ and the feasibility of $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$. All the other constraints in (4.2.5) hold by the feasibility of $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$, since they are free of $\mu^{\max}$. Hence, $\left(\mu^{\max}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$, which contradicts to the optimality of $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ by the assumption $\mu > \mu^{\max}$. Hence, $\mu \leq \mu^{\max} = \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty$. □

B.8 Proof of Proposition 4.2.4

Proof. To get a contradiction, suppose that $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$ is feasible but unbounded. Then for any $\mu \in \mathbb{R}$ there exist $\epsilon > 0$ and $\left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}$, where $\left(\boldsymbol{p}^k, \boldsymbol{s}^k\right) \in \mathbb{R}^n \times \mathbb{Z}^n$ for each $k \in \mathcal{K}$, such that $\left(\mu - \epsilon, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$.

Let $\mu = -2M$. Then there exist $\epsilon > 0$ and $\left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}$ such that $\left(-2M - \epsilon, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\text{EN}}(\boldsymbol{v})$. Fix $i \in \mathcal{N}$, $k \in \mathcal{K}$.

Then constraint (4.2.8) violates constraint (4.2.11) as

$$
\begin{aligned}
p_i^k &\leq \sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T} \boldsymbol{v}\right)_i - 2M - \epsilon\right] + M\left(1 - s_i^k\right) \\
&\leq \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \left(B^\mathsf{T} \boldsymbol{v}\right)_i - \epsilon - M \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \left(B^\mathsf{T} \boldsymbol{v}\right)_i - \epsilon - 2\left\|\boldsymbol{X}\right\|_\infty - 2\left\|\boldsymbol{v}\right\|_\infty - (n+1)\left\|\bar{\boldsymbol{p}}\right\|_\infty \\
&= \left(\sum_{j=1}^{n} \pi_{ji} p_j^k - (n+1)\left\|\bar{\boldsymbol{p}}\right\|_\infty\right) \\
&\qquad + \left(X_i\left(\omega^k\right) - 2\left\|\boldsymbol{X}\right\|_\infty\right) + \left(\left(B^\mathsf{T} \boldsymbol{v}\right)_i - 2\left\|\boldsymbol{v}\right\|_\infty\right) - \epsilon < 0,
\end{aligned}
$$

since

$$
\sum_{j=1}^{n} \pi_{ji} p_j^k < (n+1)\left\|\bar{\boldsymbol{p}}\right\|_\infty, \quad X_i\left(\omega^k\right) \leq 2\left\|\boldsymbol{X}\right\|_\infty, \quad \left(B^\mathsf{T} \boldsymbol{v}\right)_i \leq 2\left\|\boldsymbol{v}\right\|_\infty, \quad -\epsilon < 0.
$$

Hence, $\left(-2M - \epsilon, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is infeasible, which is a contradiction to the assumption. Hence, $\mathcal{Z}_2^{\mathrm{EN}}\left(\boldsymbol{v}\right)$ is bounded from below. In addition, by Proposition 4.2.3, $\mathcal{Z}_2^{\mathrm{EN}}\left(\boldsymbol{v}\right)$ is bounded from above. □

B.9 Proof of Proposition 4.2.5

Proof. Assume $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$. Let $\mu = \mu^{\max}$, $\boldsymbol{p}^k = \bar{\boldsymbol{p}}$, $\boldsymbol{s}^k = \mathbb{1}$ for each $k \in \mathcal{K}$, where $\mu^{\max} = \left\|\boldsymbol{X}\right\|_\infty + \left\|\boldsymbol{v}\right\|_\infty + \left\|\bar{\boldsymbol{p}}\right\|_\infty$. It is shown that $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{EN}}\left(\boldsymbol{v}\right)$. Since $\boldsymbol{p}^k = \bar{\boldsymbol{p}}$ for each $k \in \mathcal{K}$, it holds that $\sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] = \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}} \geq \gamma$. Hence, constraint (4.2.6) holds by hypothesis.

Now fix $i \in \mathcal{N}$, $k \in \mathcal{K}$. Constraint (4.2.8) holds as

$$\begin{aligned}
&\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu\right] + M\left(1 - s_i^k\right) \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu + M\left(1 - 1\right) \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty \geq \bar{p}_i = p_i^k,
\end{aligned}$$

since

$$\sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty \geq 0,$$

and $s_i^k = 1$, by the choice of $\boldsymbol{s}^k$. Constraint (4.2.10) holds as

$$\begin{aligned}
&\sum_{j=1}^{n} \pi_{ji} p_j^k + \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu\right] \\
&= \sum_{j=1}^{n} \pi_{ji} p_j^k + X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty \\
&\leq 2\|\boldsymbol{X}\|_\infty + 2\|\boldsymbol{v}\|_\infty + (n+1)\|\bar{\boldsymbol{p}}\|_\infty = M = M s_i^k,
\end{aligned}$$

since $\sum_{j=1}^{n} \pi_{ji} p_j^k \leq n\|\bar{\boldsymbol{p}}\|_\infty$. All the other constraints hold trivially by the choice of μ, $\boldsymbol{p}^k$ and $\boldsymbol{s}^k$, for each $k \in \mathcal{K}$. Hence, $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{EN}}(\boldsymbol{v})$.

Now, if $\gamma > \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$, then constraint (4.2.6) is infeasible, since $\sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}} < \gamma$, by constraint (4.2.11). Hence, $\mathcal{Z}_2^{\mathrm{EN}}(\boldsymbol{v})$ in Corollary 4.2.2 is feasible if and only if $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$. □

B.10 Proof of Proposition 4.2.8

Proof. To get a contradiction, suppose that $\mu > \mu^{\max}$. It is shown that $\left(\mu^{\max}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$ by showing that all constraints of $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$ in (4.2.15) are satisfied. Fix $i \in \mathcal{N}$, $k \in \mathcal{K}$. Constraint (4.2.17) holds as

$$\begin{aligned}
p_i^k &\leq \alpha\left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu^{\max}\right] + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\
&= \alpha X_i\left(\omega^k\right) + \alpha\left(B^\mathsf{T}\boldsymbol{v}\right)_i + \alpha\mu^{\max} + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k \\
&= \alpha\left(X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty\right) + \alpha\left(\left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty\right) + \|\bar{\boldsymbol{p}}\|_\infty + \beta\sum_{j=1}^n \pi_{ji}p_j^k + \bar{p}_i s_i^k,
\end{aligned}$$

since

$$\begin{aligned}
&0 < \alpha, \beta \leq 1,\ X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0,\ \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty \geq 0, \\
&p_i^k \leq \bar{p}_i \leq \|\bar{\boldsymbol{p}}\|_\infty, \quad \sum_{j=1}^n \pi_{ji}p_j^k \geq 0, \quad \text{and} \quad \bar{p}_i s_i^k \geq 0.
\end{aligned}$$

Constraint (4.2.18) holds as

$$\begin{aligned}
\bar{p}_i s_i^k &\leq \left[X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu^{\max}\right] + \sum_{j=1}^n \pi_{ji}p_j^k \\
&= \left(X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty\right) + \left(\left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty\right) + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty + \sum_{j=1}^n \pi_{ji}p_j^k,
\end{aligned}$$

since

$$\begin{aligned}
&X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty \geq 0, \\
&0 < \alpha \leq 1, \quad \bar{p}_i s_i^k \leq \|\bar{\boldsymbol{p}}\|_\infty \leq \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty, \quad \text{and} \quad \sum_{j=1}^n \pi_{ji}p_j^k \geq 0.
\end{aligned}$$

In addition, constraint (4.2.19) holds as

$$X_i\left(\omega^k\right)+\left(B^{\mathsf{T}}\boldsymbol{v}\right)_i+\mu^{\max}=X_i\left(\omega^k\right)+\left(B^{\mathsf{T}}\boldsymbol{v}\right)_i+\|\boldsymbol{X}\|_\infty+\|\boldsymbol{v}\|_\infty+\frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty\geq 0.$$

All the other constraints hold by the feasibility of $\left(\mu,\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$, since they are free of $\mu^{\max}$. Hence, $\left(\mu^{\max},\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$, which contradicts to the optimality of $\left(\mu,\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ by the assumption $\mu>\mu^{\max}$. Hence, $\mu\leq\mu^{\max}=\|\boldsymbol{X}\|_\infty+\|\boldsymbol{v}\|_\infty+\frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty$. □

B.11 Proof of Proposition 4.2.9

Proof. To get a contradiction, suppose that $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$ is feasible but unbounded. Then for any $\mu\in\mathbb{R}$ there exist $\epsilon>0$ and $\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}$, where $\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)\in\mathbb{R}^n\times\mathbb{Z}^n$ for each $k\in\mathcal{K}$, such that $\left(\mu-\epsilon,\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$.

Let $\mu=-M$, where $M=\|\boldsymbol{X}\|_\infty+\|\boldsymbol{v}\|_\infty+\frac{1}{\alpha}(n+1)\|\bar{\boldsymbol{p}}\|_\infty$. Then there exist $\epsilon>0$ and $\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}$ such that $\left(-M-\epsilon,\left(\boldsymbol{p}^k,\boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$. Fix $i\in\mathcal{N}$, $k\in\mathcal{K}$. Then constraint (4.2.17) violates constraint (4.2.20) as

$$\begin{aligned}
p_i^k &\leq \alpha\left[X_i\left(\omega^k\right)+\left(B^{\mathsf{T}}\boldsymbol{v}\right)_i-M-\epsilon\right]+\beta\sum_{j=1}^n\pi_{ji}p_j^k+\bar{p}_i s_i^k\\
&=\alpha X_i\left(\omega^k\right)+\alpha\left(B^{\mathsf{T}}\boldsymbol{v}\right)_i+\alpha(-M-\epsilon)+\beta\sum_{j=1}^n\pi_{ji}p_j^k+\bar{p}_i s_i^k\\
&=\alpha\left(X_i\left(\omega^k\right)-\|\boldsymbol{X}\|_\infty\right)+\alpha\left(\left(B^{\mathsf{T}}\boldsymbol{v}\right)_i-\|\boldsymbol{v}\|_\infty\right)\\
&\quad+\left(\beta\sum_{j=1}^n\pi_{ji}p_j^k-n\|\bar{\boldsymbol{p}}\|_\infty\right)+\left(\bar{p}_i s_i^k-\|\bar{\boldsymbol{p}}\|_\infty\right)-\alpha\epsilon<0,
\end{aligned}$$

since

$$0 < \alpha, \beta \leq 1, \quad X_i\left(\omega^k\right) \leq \|\boldsymbol{X}\|_\infty, \quad \left(B^\mathsf{T}\boldsymbol{v}\right)_i \leq \|\boldsymbol{v}\|_\infty,$$
$$\beta \sum_{j=1}^{n} \pi_{ji} p_j^k < n\|\bar{\boldsymbol{p}}\|_\infty, \quad \bar{p}_i s_i^k \leq \|\bar{\boldsymbol{p}}\|_\infty, \quad \text{and} \quad -\epsilon < 0.$$

Hence, $\left(-M-\epsilon, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is infeasible, which is a contradiction to the assumption. Hence, $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$ is bounded from below. In addition, by Proposition 4.2.8, $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$ is bounded from above. □

B.12 Proof of Proposition 4.2.10

Proof. Assume $\gamma \leq \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}}$. Let $\mu = \mu^{\max}$, $\boldsymbol{p}^k = \bar{\boldsymbol{p}}$, $\boldsymbol{s}^k = \mathbb{1}$ for each $k \in \mathcal{K}$, where $\mu^{\max} = \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty$. It is shown that $\left(\mu, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{RV}}(\boldsymbol{v})$. Since $\boldsymbol{p}^k = \bar{\boldsymbol{p}}$ for each $k \in \mathcal{K}$, it holds that $\sum_{k\in\mathcal{K}} q^k\left[\mathbb{1}^\mathsf{T}\boldsymbol{p}^k\right] = \mathbb{1}^\mathsf{T}\bar{\boldsymbol{p}} \geq \gamma$. Hence, constraint (4.2.16) holds by hypothesis.

Now fix $i \in \mathcal{N}$, $k \in \mathcal{K}$. Constraint (4.2.17) holds as

$$\begin{aligned}
&\alpha\left(X_i\left(\omega^k\right) + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \mu\right) + \beta\sum_{j=1}^{n}\pi_{ji}p_j^k + \bar{p}_i s_i^k \\
&= \alpha\left(X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty + \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty\right) + \|\bar{\boldsymbol{p}}\|_\infty + \beta\sum_{j=1}^{n}\pi_{ji}p_j^k + \bar{p}_i \geq \bar{p}_i = p_i^k,
\end{aligned}$$

since

$$0 < \alpha, \beta \leq 1, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \left(B^\mathsf{T}\boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty \geq 0, \quad \sum_{j=1}^{n}\pi_{ji}p_j^k \geq 0,$$

and $s_i^k = 1$, by the choice of $\boldsymbol{s}^k$. Constraint (4.2.18) holds as

$$\begin{aligned}
&\left[X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \mu\right] + \sum_{j=1}^{n} \pi_{ji} p_j^k \\
&= X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty + \sum_{j=1}^{n} \pi_{ji} p_j^k \\
&\geq X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \|\bar{\boldsymbol{p}}\|_\infty + \sum_{j=1}^{n} \pi_{ji} p_j^k \geq \bar{p}_i = \bar{p}_i s_i^k,
\end{aligned}$$

since

$$0 < \alpha \leq 1, \quad X_i\left(\omega^k\right) + \|\boldsymbol{X}\|_\infty \geq 0, \quad \left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{v}\|_\infty \geq 0, \quad \sum_{j=1}^{n} \pi_{ji} p_j^k \geq 0,$$

and $s_i^k = 1$, by the choice of $\boldsymbol{s}^k$. In addition, constraint (4.2.19) holds as

$$X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \mu = X_i\left(\omega^k\right) + \left(B^\top \boldsymbol{v}\right)_i + \|\boldsymbol{X}\|_\infty + \|\boldsymbol{v}\|_\infty + \frac{1}{\alpha}\|\bar{\boldsymbol{p}}\|_\infty \geq 0.$$

All the other constraints hold trivially by the choice of $\boldsymbol{z}$, $\boldsymbol{p}^k$ and $\boldsymbol{s}^k$ for each $k \in \mathcal{K}$. Hence, $\left(\boldsymbol{z}, \left(\boldsymbol{p}^k, \boldsymbol{s}^k\right)_{k\in\mathcal{K}}\right)$ is a feasible solution to $\mathcal{Z}_2^{\mathrm{RV}}\left(\boldsymbol{v}\right)$.

Now, if $\gamma > \mathbb{1}^\top \bar{\boldsymbol{p}}$, then constraint (4.2.16) is infeasible, since $\sum_{k\in\mathcal{K}} q^k \left[\mathbb{1}^\top \boldsymbol{p}^k\right] \leq \mathbb{1}^\top \bar{\boldsymbol{p}} < \gamma$, by constraint (4.2.20). Hence, $\mathcal{Z}_2^{\mathrm{RV}}\left(\boldsymbol{v}\right)$ in Corollary 4.2.7 is feasible if and only if $\gamma \leq \mathbb{1}^\top \bar{\boldsymbol{p}}$. □