WHICH ALGEBRAIC K3 SURFACES COVER AN ENRIQUES SURFACE

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

WHICH ALGEBRAIC K3 SURFACES COVER AN ENRIQUES SURFACE

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2018

We partially determine the necessary and sufficient conditions on the entries of the intersection matrix of the transcendental lattice of algebraic $K3$ surface with Picard number $18 \leq \rho(X) \leq 19$ for the surface to doubly cover an Enriques surface.

Keywords: K3 Surface, Enriques Surface, Lattice.

ÖZET

HANGİ TEKİL K3 YÜZEYLERİ BİR ENRİQUES YÜZEYİNI ÖRTERLER

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Bu tezde, Picard sayısı $\rho(X)$, 18 ve 19 olan cebirsel bir $K3$ yüzeyinin aşkın örgüsü üzerinde, bu $K3$ yüzeyinin bir Enriques yüzeyinin iki kat örteni olması için gerek ve yeter şartlarını kısmi olarak tespit ediyoruz.

Anahtar sözcükler: K3 Yüzeyi, Enriques Yüzeyi, Örgü.

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Contents

Chapter 1

Introduction

1.1 Historical Background and Motivation

The primary essence of this thesis is to study the connection between K3 and Enriques surfaces from lattice theoretic point of view. The classification of algebraic surfaces culminated during 1900's as an analogous to Riemann's classification of complex curves with respect to their genus. Even though algebraic curves have only one invariant called genus, on the contrary, for algebraic surfaces, there exist several invariants. hence it causes the further problem of classification of algebraic surfaces. To achieve the goal, Enriques and Castelnuovo showed that for the characterization of the surfaces, it is enough to consider their three invariants namely, their Kodaira dimension $\kappa(X)$, irregularity $q(X)$, geometric genus $p_g(X)$. Further, Enriques divided all surfaces into four classes. In this classification regarding Kodaira dimension $\kappa(X)$, irregularity $q(X)$, geometric genus $p_g(X)$, Class II with $\kappa(X) = 0$ has divided into four subclasses such as:

- $q(X) = 0$, $p_g(X) = 0$. These surfaces are called Enriques surfaces
- $q(X) = 0$, $p_g(X) = 1$. These surfaces are called $K3$ surfaces inspired by

three mathematicians, Kummer, Kahler and Kodaira, and the mountain $K2$.

- $q(X) = 1$, $p_g(X) = 0$. These surfaces are called hyperelliptic surfaces.
- $q(X) = 2$, $p_g(X) = 1$. These surfaces are called the abelian surfaces.

Although $K3$ surfaces satisfy $K(X) = 0$, Enriques surfaces require the conditions $2K(Y) = 0$ and $K(Y) \neq 0$, where K is the canonical divisor. Moreover, these two kinds of surfaces are closely related to each other. It is well known result that for each Enriques surface Y there exists a $K3$ surface X and a fixed-point-free involution $\iota : X \hookrightarrow X$ such that the quotient surface X/ι is isomorphic to Y i.e., $Y \cong X/\iota$. Conversely, universal double covering of an Enriques surface is a $K3$ surface.

In 1990, Keum gave a criterion for an algebraic $K3$ surfaces over $\mathbb{C}$ to cover an Enriques surface, as the following arguments are equivalent:[12]

- X admits a fixed point free involution.
- There exists a primitive embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$ such that $\mathrm{Im}(T_X)^\perp$ doesn't contain any vector of self intersection -2, where U is the even unimodular lattice of signature $(1, 1)$ and E_8 is the even unimodular lattice of signature $(0, 8)$.

This characterization raised the question which algebraic $K3$ surfaces over $\mathbb{C}$ cover an Enriques surface and the solution of the problem leads us to lattice theory. This criterion above is indeed the most principal motivation for our present work.

Keum showed that for Picard number $17 \leq \rho(X) \leq 20$, every algebraic Kummer surfaces is a $K3$ surfaces which doublely covers an Enriques surface.

In 2005, Sertöz implemented this criterion to find explicit necessary and sufficient conditions on the entries of transcendental lattice T_X so that X covers an

Enriques surface when $\rho(X) = 20$, he resolved all difficulties arised in case of the $K3$ surface with $\rho(X) = 20$.[18]

In 2012, following the works of Keum and Sertöz, Lee attacked the problem of finding explicit necessary and sufficient conditions on the entries of transcendental lattice T_X so that X covers an Enriques surface when $\rho(X) = 19$.[14] But this problem still remains open for the cases which does not depend only on parity.

1.2 Strategy

When X is an algebraic K3 surface with Picard number $\rho(X) = 19$ over the field $\mathbb{C}$, the transcendental lattice T_X of X is denoted by its intersection matrix

$$\begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix} \tag{1.1}$$

with respect to some basis $\{x, y, z\}$, the transcendental lattice T_X of X has signature $(2; 20 - \rho(X)) = (2; 1)$. When X is an algebraic K3 surface with Picard number $\rho(X) = 18$ over the field $\mathbb{C}$, T_X is given by

$$\begin{pmatrix} 2a & e & f & g \\ e & 2b & h & i \\ f & h & 2c & j \\ g & i & j & 2d \end{pmatrix} \tag{1.2}$$

with respect to some basis $\{x, y, z, t\}$, the transcendental lattice T_X of X has signature $(2; 20 - \rho(X)) = (2; 2)$. Sometimes, it is possible to exhibit an embedding $\phi : T_X \to \Lambda^-$ such that

- it is possible to demonstrate that ϕ is primitive and that
- it is possible to show that the existence of a self intersection -2-vector in $\phi(T_X)^\perp$ leads to a contradiction.

It is hard work to demonstrate that for every primitive embedding the orthogonal complement of the image ϕ has a self intersection -2-vector, but we seek to find, in this case, particular primitive embedding such that there is no a self intersection -2-vector in $\phi(T_X)^{\perp}$.

In Lee' article [14], his strategy was the following:

- to find the particular embedding $\phi : T_X \to \Lambda^{-}$
- to demonstrate that ϕ is primitive using Sertöz Theorem[18]
- to show that for this particular primitive embedding, there does not exists a self intersection -2-vector in $\phi(T_X)^{\perp}$.

His techniques is inconclusive in the cases which do not depend only on parity. Even if we find the particular primitive embedding ϕ, it is impossible to show the non-existence of a self intersection -2-vector in $\phi(T_X)^{\perp}$ just using $\mod 2$ argument.

Indeed, the cases can be classified into two groups,

- the cases in which the solutions depend not only parity, but also on the determinant of the intersection matrix and integral quadratic forms, these cases are exactly the ones that at least one of the diagonal entries of the intersection matrix is odd, and remaining entries in the intersection matrix are even. We will call these cases Type 1,
- the cases in which the solutions depend only on parity, these cases are the remaining cases which are not in Type 1. We will call these cases Type 2.

When X is an algebraic K3 surface with Picard number $\rho(X) = 19$ over the field $\mathbb{C}$, all the cases in Type 2 are resolved, but the cases in Type 1 remained open, but it is known that all the cases in Type 1 is equivalent. Thus we obtained the results for the one case in Type 1 under some conditions.

When X is an algebraic K3 surface with Picard number $\rho(X) = 18$ over the field $\mathbb{C}$, all the cases in Type 2 can be resolved fully. We established theorems for the cases in Type 2 whether X covers an Enriques surface. All the cases in Type 1 is also equivalent, hence we deduced the theorem for the one case in Type 1 under some conditions.

Our theorems are heavily based on the integral quadratic form and Sertöz' approach in his article[18]. The most diffucult part of the problem that whether there exist any primitive embedding which does not have a self intersection -2-vector in $\phi(T_X)^{\perp}$, can be overcome by looking every primitive embedding and using lattice theoretic perspective under some conditions. Now our techniques based on Sertöz's approach fulfilled in his article [18] are as the following:

- to find the particular embedding $\phi : T_X \to \Lambda^-$
- to demonstrate that ϕ is primitive using Sertöz Theorem [18]
- to look for any vector in $\phi(T_X)^{\perp}$, solving integral quadratic equation derived from orthogonality of a primitive embedding whether it has the non-existence of a self intersection -2-vector in $\phi(T_X)^{\perp}$.

Using this techniques, we conclude the following theorems:

Theorem 1.1. *If X is an algebraic K3 surface with Picard number $\rho(X) = 19$ and transcendental lattice given as*

$$\begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix}, \tag{1.3}$$

then X covers an Enriques surface if the following conditions hold:

- *d, e, f are even; a and b are odd; c is negative even.*
- *The form $aX^2 + dXY + bY^2$ is positive definite and does not represent 1.*

In the case when $aX^2 + dXY + bY^2$ represents 1, by using theorem 3.6 we proved in the following chapters, the Gram matrix of Transcendental lattice T_X given in (1.3) can be transformed into the following matrix form :

$$\begin{pmatrix} 2 & d & 0 \\ d & 2b & f \\ 0 & f & 2c \end{pmatrix}. \tag{1.4}$$

We prove the following theorem which states:

Theorem 1.2. *If X is a algebraic K3 surface with Picard number $\rho(X) = 19$ and transcendental lattice given as in (1.4), then X covers an Enriques surface if the following conditions hold:*

- *$d = 0$, f is even. c is a negative integer.*
- *The form $X^2 + bY^2$ is positive definite and represents 1, and $b \neq 1, 2, 4$.*

In case of Picard number $\rho(X) = 18$, by using Jacobi Theorem, the transcendental lattice T_X of X denoted by its intersection matrix given in 1.2 can be reduced to its triple-diagonal form:

$$\begin{pmatrix} 2a & e & 0 & 0 \\ e & 2b & f & 0 \\ 0 & f & 2c & g \\ 0 & 0 & g & 2d \end{pmatrix}. \tag{1.5}$$

We establish the following theorem;

Theorem 1.3. *If X is an algebraic K3 surface with Picard number $\rho(X) = 18$ and transcendental lattice given as in (1.5), then X covers an Enriques surface if the following conditions hold:*

- *a is a positive integer; c is a positive even integer; b, d are negative even integers; e, f, g are even integers.*

- *The form $aX^2 + cY^2$ is positive definite and does not represent* 1.

In case of Picard number $\rho(X) = 18$, to determine which $K3$ surface does not cover an Enriques surface is fairly an easy problem in which the conditions that the parity is only determiner. We proved these cases also.

Chapter 2

K3 Surfaces and Lattices

2.1 The Fundamental Definitions and Theorems on Lattice Theory

The main references for this section are [7], [13], [11], [5].

Before we begin to give the definition of a lattice and its algebraic structures, we recall the basic facts about a module over a principal ideal domain. Every module over a commutative ring R with a finite generating set does not have a basis generally. But in the case of a commutative ring R being a principal ideal domain, we have the similar assertions as in the case of a vector space over a field $\mathbb{F}$:

1. Every finitely generated torsion-free R-module M is free, i.e, it has a finite basis and M is isomorphic to R^n for a unique $n > 0$.
2. Every finitely generated R-module M is isomorphic to $R^k \oplus T$, where $k \geq 0$ and T is a finitely generated torsion module.

Now we are ready to give the definition of a lattice:

Definition 2.1. Let R be a principal ideal domain and F be its fractional field.

A lattice is a finitely generated torsion-free R-module M endowed with a bilinear form

$$<,>: M \times M \longrightarrow F$$

Since the lattice is equipped with a bilinear form, we define it as follow:

Definition 2.2 (Bilinear Form)**.** Let R be a principal ideal domain and F be its fractional field and M be a finitely generated torsion-free R-module. A mapping $\beta =<,>: M \otimes M \longrightarrow F$ is a bilinear form on M if it is linear with respect to each variables, that is:

- $\beta(\alpha x + y, v) = \alpha\beta(x, v) + \beta(y, v)$
- $\beta(v, \alpha x + y) = \alpha\beta(v, x) + \beta(v, y)$

  for all $v, x, y \in M$, where $\alpha \in R$.

1. The bilinear form β is called symmetric if $< x, y >=< y, x >$ for all $x, y \in M$; or
2. The bilinear form β skew-symmetric if$< x, y >= - < y, x >$ for all $x, y \in M$.

Definition 2.3 (Quadratic Form)**.** Let R be a principal ideal domain and F be its fractional field and M be a finitely generated torsion-free R-module. The function $f : M \longrightarrow F$ given by $f(x) = \beta(x, x)$ is called the quadratic form associated with the bilinear form β.

Throughout all the thesis, instead of studying on a general principal ideal domain R, we will assume $R = \mathbb{Z}$, the ring of integer, its fractional field $F = \mathbb{Q}$. Also we restrict our scope on an integral symmetric bilinear form and an integral quadratic form associated with this symmetric bilinear form. In this context, we study a lattice as a $\mathbb{Z}$-module equipped with an integral symmetric bilinear form, $M \cong (\mathbb{Z}^n, <,>)$.

We are going to give the fundamental definitions about the lattice to allow us classifying them with respect to their parity, rank and signature;

Definition 2.4 (Parity)**.** The lattice M is even or of Type-II if $< x, x >\equiv 0 \mod 2$ for all $x \in M$, and is odd or of Type-I otherwise.

Definition 2.5 (Rank)**.** The rank of the lattice $r(M)$ is the $rank_{\mathbb{Z}} M$, i.e., $r(M) = rank_{\mathbb{Z}} M$ and denoted by $r(M)$

Since M is the $\mathbb{Z}$-lattice endowed with symmetric bilinear form, we can extend and diagonalize a bilinear form over M tensoring by the fractional field $\mathbb{Q}$, we obtain $V = M \otimes_{\mathbb{Z}} \mathbb{Q}$, therefore the signature of the lattice is sign$(M) = (b^+, b^-, b^0)$ where b^+, b^-, b^0 represent the number of $+1$'s,-1's and 0's respectively in the diagonalized form.

Definition 2.6 (Index)**.** The index of the lattice is determined by the difference $b^+ - b^-$, and is denoted by $\sigma(M)$

Definition 2.7 (Definiteness)**.** The lattice M is a positive definite if $b^- = b^0 = 0$, that is $r(M) = \sigma(M)$, and is a negative definite if $b^+ = b^0 = 0$, i.e., $r(M) = -\sigma(M)$ and is called indefinite if $b^- > 0$, $b^+ > 0$.

Definition 2.8 (Non-degeneracy)**.** The lattice M is non-degenerate if it satisfies the following condition:

$$ker(M) := \{x \in M| < x, y >= 0 \text{ for all y} \in M\} = 0$$

Definition 2.9 (Duality)**.** Let M be an integral lattice, then the dual lattice of M is

$$M^* := Hom(M, \mathbb{Z}) = \{x \in M \otimes_{\mathbb{Z}} \mathbb{Q}| < x, y >\in \mathbb{Z} \text{ for all y} \in M\}$$

In the following corollary, we are going to give the equivalent definition of the non-degeneracy of the lattice. Since the integral lattice carries $\mathbb{Z}$-module structure, by the definition, it has an abelian group structure.

Corollary 2.10. *The lattice M is non-degenerate if and only if the canonical group homomorphism*

$$\iota : M \longrightarrow M^* := Hom(M, \mathbb{Z}) = x \mapsto < x, . >$$

is a monomorphism.

In the following definition, we will connect the integral lattice with its associated matrix.

Let $B = \{e_1, \cdots, e_n\}$ be a basis of $\mathbb{Z}$-module M and let $\beta =<,>$ be a symmetric bilinear form on M;

Definition 2.11 (Gram Matrix)**.** The symmetric matrix $G = (a_{ij}) \in \mathcal{M}_n(\mathbb{Z})$ given by $a_{ij} = \beta(e_i, e_j)$ is the Gram matrix of M with respect to this basis B; it can be written as $M \cong G$.

Conversely, the Gram matrix also determines the symmetric bilinear form β, the reason is that given a basis $B = \{e_1, \cdots, e_n\}$, for $x, y \in M$, $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$ for some $x_i, y_i \in \mathbb{Z}$, then $\beta(x, y) = \beta(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i) = \sum_{1 \leq i,j \leq n} x_i \beta(e_i, e_j) y_j = (x_1, \cdots, x_n)\, G\, (y_1, \cdots, y_n)^{tr}$.

Let M be an integral lattice and $B = \{e_1, \cdots, e_n\}$ be a basis of M, G be the Gram matrix $a_{ij} = \beta(e_i, e_j)$. The determinant of G, $det(G) = det(a_{ij})$ is independent of the choice of the basis, i.e., it is an invariant property of the integral lattice. The determinant of G is called the determinant of M, denoted by $det(M)$. and $|det(G)|$ is called the discriminant of M.

A bilinear form and a Gram matrix can be defined in the following:

Definition 2.12. Let M, N be $\mathbb{Z}$-modules and $\beta : M \times N \to \mathbb{Q}$ be a bilinear form. Suppose $B = \{e_1, \cdots, e_m\}$, $B' = \{e'_1, \cdots, e'_n\}$ are the bases of M, N respectively. The Gram matrix with respect to these bases is $G := (\beta(e_i, e'_j))$ and denoted $G_{B,B'}$ with respect to B, B'.

If we replace B with an another basis $B'' = \{e''_1, \cdots, e''_m\}$ of M, then $G_{B'',B'} = A\, G_{B,B'}$ and the matrix $A = (a_{ji})$ that expresses B'' in terms of B where $e''_j = \sum_{i=1}^{n} a_{ji} e_i$.

In the similar fashion, if we replace B' with an another basis $B''' = \{e'''_1, \cdots, e'''_n\}$ of N, then $G_{B,B'''} = G_{B,B'}\, C^{tr}$ and the matrix $C = (c_{ji})$ that expresses B''' in terms of B' where $e'''_j = \sum_{i=1}^{n} c_{ji} e'_i$.

Let M be $\mathbb{Z}$-lattice and N be its dual lattice. Suppose $B = \{e_1, \cdots, e_n\}$, $B' = \{e'_1, \cdots, e'_n\}$ the bases of M, N respectively. Then $\beta(e_i, e'_j) = \delta_{ij}$, so its Gram matrix is the identity matrix $G_{B,B'} = I_{n\times n}$.

Let the matrix $A = (a_{ji})$ that expresses B' in terms of B where $e'_j = \sum_{i=1}^{n} a_{ji} e_i$. Similarly, let the matrix $C = (c_{ji})$ that expresses B in terms of B' where $e_j = \sum_{i=1}^{n} c_{ji} e'_i$. We can conclude that $G_{B,B'} = A\, C = I_{n\times n}$, and $G_{B,B} = G_{B,B'}\, C^{tr} = C^{tr}$.

Remark 2.13. In the thesis, given a integral lattice and its associated Gram matrix with respect to some bases, we express $M \cong G$ instead of writing $M \cong G_B$.

The Gram matrix determines the symmetric bilinear form β and also the quadratic form associated to the symmetric bilinear form β.

Two symmetric bilinear forms α, β are equivalent if there exists $C \in GL_n(\mathbb{Z})$ such that $G' = C^{tr}\, G\, C$, where G and G' are their associated Gram matrices respectively.

Definition 2.14 (Unimodularity)**.** The lattice is unimodular if its associated Gram matrix is unimodular,i.e., $det(G) = det(a_{ij}) = \pm 1$.

Remark 2.15.

- For any integer n we denote by $\langle n \rangle$ the lattice $\mathbb{Z}e$ of rank 1 with $< e, e >= n$.
- For any integer n, and $M \cong G = (a_{ij})$ we denote by $M(n)$ the lattice which is isomorphic to $(n a_{ij})$

Over the course of the thesis, the most primary examples of integral unimodular lattice that we will use are the following:

Examples 2.16.

- The lattice of rank 1 of the signature $(1,0)$ and its associated Gram matrix $< 1 >$ denoted by I_+.
- The lattice of rank 1 of the signature $(0,1)$ and its associated Gram matrix $< -1 >$ denoted by I_-.
- The lattice of rank 2 with the signature (1,1) and its associated Gram matrix denoted by $U \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- E_8 denotes the even unimodular negative definite lattice of signature (0,8). Its associated Gram matrix given by

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}. \qquad (2.1)$$

The rank $r(M)$, the parity, the index $\sigma(M)$, the determinant $det(M)$ are the invariants of the unimodular lattice. All these invariants determine indefinite unimodular lattices. We will give the classification of indefinite unimodular lattices. Before this, we need to clarify the concept of the lattice isomorphism:

Definition 2.17 (Lattice Morphism)**.** Suppose M and N are two lattices, β and β' be bilinear form on M, N respectively. A group homomorphism $\alpha : M \to N$ is called a lattice morphism if it satisfies the isometry relation: $\beta(x, y) = \beta'(\alpha(x), \alpha(y))$ for all $x, y \in M$.

Definition 2.18 (Lattice Embedding)**.** Suppose M and N are two lattices. A lattice morphism $\alpha : M \to N$ is called a lattice embedding if a group homomorphism is a group monomorphism.

Definition 2.19 (Lattice Isomorphism)**.** Suppose M and N are two lattices. A lattice morphism $\alpha : M \to N$ is called a lattice isomorphism if a group homomorphism is a group isomorphism and denoted by $M \cong N$.

Hence we are ready to state classification of indefinite unimodular lattices:

In the case of Type-I or oddness, lattice isomorphism determined by the rank, and the index of lattice.

Theorem 2.20. *Let M be an indefinite odd unimodular $\mathbb{Z}$-lattice, let sign $(M) = (b^+, b^-)$ be its signature, then M is isomorphic to $M \cong < +1 >^{\oplus b^+} \oplus < -1 >^{\oplus b^-}$*

Proof. We refer to [7, p.189]. □

In the case of Type-II or evenness, lattice isomorphism also determined by the rank, and the index of lattice.

Theorem 2.21. *Let M be an indefinite even unimodular $\mathbb{Z}$-lattice, let sign $(M) = (b^+, b^-)$ be its signature. Then*

- *if $b^+ = b^-$, $M \cong U^{\oplus b^+}$*
- *if $b^+ > b^-$, $M \cong U^{\oplus b^-} \oplus E_8(-1)^{\oplus (b^+ - b^-)/8}$*
- *if $b^- > b^+$, $M \cong U^{\oplus b^+} \oplus E_8^{\oplus (b^- - b^+)/8}$*

Proof. We refer to [7, p.191]. □

Thus by using the theorem 2.21 and the theorem 2.20, we can state that indefinite unimodular lattices are isomorphic if and only if they have the same rank, the same index and the same parity and similarly, they are uniquely determined by the rank, the index and the parity.

The classification of indefinite even unimodular lattices and the following corollary are essentially the significant to determine the $K3$ lattice and the Enriques lattice. This corollary is one of the characterizations of the unimodular lattice by the group isomorphism.

Corollary 2.22. *The lattice M is unimodular if and only if the canonical group homomorphism*

$$\iota : M \longrightarrow M^* := Hom(M, \mathbb{Z})$$

is an isomorphism.

Proof. Let $B = \{e_1, \cdots, e_n\}$ be a basis of M and $B^* = \{e_1^*, \cdots, e_n^*\}$ be the dual basis of M^* given by $e_i^*(e_j) = \delta_{ij}$.

Suppose the lattice is unimodular, hence $M \cong G$ and $det(G) = det(a_{ij}) = \pm 1$, and $G = (a_{ij})$ is invertible hence it defines the isomorphism $\iota : M \longrightarrow M^*$ given by $\iota(e_i) = \sum\limits_{k=1}^{n} a_{ij} e_k^*$.

Conversely, suppose $\iota : M \longrightarrow M^*$ given by $\iota(e_i) = \sum\limits_{k=1}^{n} a_{ij} e_k^*$ is the isomorphism, hence (a_{ij}) is invertible, so only an invertible matrix over $\mathbb{Z}$ is unimodular, i.e., $det(G) = det(a_{ij}) = \pm 1$.

$\square$

The primary content of the thesis evolves around the concept of the embedding of lattices. Before giving one of the substantial concepts that is primitivity which plays a central role in the thesis, we will bring forth the definition of sublattice as follows:

Definition 2.23 (Sublattice)**.** A $\mathbb{Z}$-submodule of a $\mathbb{Z}$-lattice M is called a $\mathbb{Z}$-sublattice of M.

Definition 2.24 (Primitivity)**.** A $\mathbb{Z}$-sublattice L of a $\mathbb{Z}$-lattice M is called primitive if M/L is a free module,i.e., it has a basis.

Definition 2.25 (Primitive Embedding)**.** Suppose M and N be two lattices. Let $\alpha : M \to N$ be a lattice embedding. The embedding is called a primitive if $N/\alpha(M)$ is a free module,i.e., it has a basis.

The theorem we are about to give characterizes the primitive embedding of the lattices.

Theorem 2.26 (Sertoz)**.** *A lattice embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of basis is 1.*

Proof. We refer to [18]. □

The discrimant group of the lattice has the crucial role for the primitive embedding of lattices, hence we bring forth its definition and algebraic structure.

Definition 2.27 (Discriminant Group)**.** Let M be a nondegenerate $\mathbb{Z}$-lattice and M^* be its dual lattice. The abelian group M^*/M is called the discriminant group of lattice M.

To understand the structure of the discriminant group, we digress here and state the some of the important theorems and definitions in the module theory over principal ideal domain.

If A is a diagonal matrix with nonnegative integers d_i as diagonal entries such that $d_i|d_{i+1}$ for all $i < n$ and $d_i \geq 0$, then the matrix A is called in Smith normal form.

Theorem 2.28. *Suppose A be $n \times n$ nonsingular matrix such that $A \in \mathcal{M}_{n,m}(\mathbb{Z})$. Then there exist a unique matrix B in the Smith normal form such that $B = UAV$ where $U, V \in GL_n(\mathbb{Z})$.*

Proof. We refer to [11, p.187]. $\square$

Remark 2.29. We will denote the index of a sublattice M inside a lattice N by $[N/M]$.

Theorem 2.30 (Structure Theorem)**.** *Suppose M is a $\mathbb{Z}$-submodule of a finitely generated free module N such that $r(M) = r(N)$ and $B = \{e_1, \cdots, e_n\}$ is a basis of N. Then there are unique positive integers $d_1, \cdots, d_n$ satisfying $d_i | d_{i+1}$ such that*

1. *N/M isomorphic to $\bigoplus_{1 \leq i \leq n} \mathbb{Z}/d_i\mathbb{Z} \cong \bigoplus_{1 \leq i \leq n} \mathbb{Z}_{d_i}$, particularly, $[N/M] = d_1 \cdots d_n$.*
2. *There exists a basis $B' = \{b_1, \cdots, b_n\}$ of M such that $b_i = d_i e_i$ for all $1 \leq i \leq n$.*

Proof. We refer to [11, p.187]. $\square$

We can deduce the following theorem from the structure theorem:

Theorem 2.31. *Let M be a $\mathbb{Z}$-sublattice of a $\mathbb{Z}$-lattice N such that $r(M) = r(N)$. Then the following equality is satisfied*

$$det(M) = det(N)[N/M]^2$$

Proof. Suppose $B = \{e_1, \cdots, e_n\}$ is a basis of N. By the structure theorem, there exists a basis $B' = \{b_1, \cdots, b_n\}$ of M such that $b_i = d_i e_i$ for all $1 \leq i \leq n$. $det(M) = det(G) = det(\beta(b_i, b_j)) = det(\beta(d_i e_i, d_j e_j))$, then

$$det(G) = \begin{pmatrix} d_1 d_1 < e_1, e_1 > & . & . \\ . & d_i d_j < e_i, e_j > & . \\ . & . & d_n d_n < e_n, e_n > \end{pmatrix} \tag{2.2}$$

$$= det \left(\begin{pmatrix} d_1 & \cdots & 0 \\ . & \cdots & . \\ 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} < e_1, e_1 > & . & . \\ . & < e_i, e_j > & . \\ . & . & < e_n, e_n > \end{pmatrix} \begin{pmatrix} d_1 & \cdots & 0 \\ . & \cdots & . \\ 0 & \cdots & d_n \end{pmatrix} \right) \tag{2.3}$$

Hence we are done.

$\square$

Theorem 2.32. *Let M be a nondegenerate $\mathbb{Z}$-lattice. Then the index of its discriminant group $disc(M) = M^*/M$ is equal to $det(M)$.*

Proof. Suppose $B = \{e_1, \cdots, e_m\}$, $B' = \{e'_1, \cdots, e'_n\}$ are the bases of M, M^* respectively. By the Structure theorem, there exists a basis $B' = \{b_1, \cdots, b_n\}$ of M such that $e_i = d_i e'_i$ for all $1 \leq i \leq n$. Hence $G_{B,B'} \in GL_n(\mathbb{Z})$ as we showed above. $G_{B,B} = G_{B,B'} C^{tr}$ where C expresses B in terms of B'. So C is in the Smith normal form. Thus $det(M) = det(G_{B,B}) = \pm d_1 \cdots d_n$. $disc(M) = M^*/M \cong \bigoplus_{1 \leq i \leq n} \mathbb{Z}/d_i\mathbb{Z}$ and its index is equal to $[M^*/M] = d_1 \cdots d_n$. Therefore $[M^*/M] = |det(M)|$.

$\square$

Definition 2.33 (Orthogonal Complement)**.** Suppose L is a $\mathbb{Z}$-submodule of a $\mathbb{Z}$-lattice M, $L^\perp = \{x \in M | < x, y >= 0 \text{ for all } y \in L\}$ is called the sublattice orthogonal to L.

Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a primitive element in the lattice M, that is, $\gcd(\alpha_1, \ldots, \alpha_n) = 1$. Denote the orthogonal complement of α in M by $\alpha^\perp$.

Lemma 2.34. *The index of $\alpha \oplus \alpha^\perp$ in M divides $< \alpha, \alpha >_M$.*

Proof. We refer to [18]. $\square$

The theorem we are about to give informs us whether there exists a primitive embedding between two lattices under some conditions.

Theorem 2.35 (Nikulin)**.** *Let M be an even non-degenerate lattice of signature (m^+, m^-) and N be an even unimodular lattice of signature (n^+, n^-), then there exists a primitive embedding of M into N if the following three inequalities are satisfied simultaneously:*

- $m^+ \leq n^+$
- $m^- \leq n^-$
- $l(M) + 1 < rank(N) - rank(M)$, *where $l(M)$ denotes is the minimum number of generators of $disc(M)$*

Proof. We refer to [16, p.122]. $\square$

Using Nikulin's theorem, we will provide the corollary which we will use in the next chapters for our purposes.

Corollary 2.36. *There exists a primitive embedding of $M \cong < 2a >$ into the even unimodular lattice E_8 for any negative integer a.*

There exists a primitive embedding of $M \cong < 2a > \oplus < 2b >$ into the even unimodular lattice E_8 for any negative integers a, b.

There exists a primitive embedding of $M \cong < 2a > \oplus < 2b > \oplus < 2c >$ into the even unimodular lattice E_8 for any negative integer a, b and c.

Proof. Since the signature of the lattice M of rank 1 of the signature $sign(M) = (0, 1)$, and $l(M) = 1$, the signature of E_8 that is $sign(E_8) = (0, 8)$, all three conditions in Nikulin's theorem satisfied by these numbers, hence we are done.

Since the signature of the lattice M of rank 2 of the signature $sign(M) = (0, 2)$, and $l(M) = 2$, the signature of E_8 that is $sign(E_8) = (0, 8)$, all three conditions in Nikulin's theorem satisfied by these numbers, hence we are done.

Since the signature of the lattice M of rank 3 of the signature $sign(M) = (0, 3)$, and $l(M) = 3$, the signature of E_8 that is $sign(E_8) = (0, 8)$, all three conditions in Nikulin's theorem satisfied by these numbers, hence we are done.

$\square$

2.2 The Fundamental Definitons and Theorems on The Complex 4-Manifold

For this section, our main references are [17], [9], [1], [8], [6].

In this section, we will study a complex 4-manifold, hence it always endows a canonical orientation inherited by its complex structure. Before we deal with the algebraic and analytics invariants of complex manifold, we will recall the homology and cohomology of the general complex manifold.

2.2.1 Homology and Cohomology of The Complex Manifold

Utilizing the universal coefficients theorems and by Poincare duality, the homology and cohomology of a manifold are determined. The universal coefficients theorems organize the relationship between homology and cohomology theories.

Theorem 2.37 (The universal Coefficients Theorem for Homology)**.** *Suppose G is an abelian group. For any $k > 0$, there exists the short exact sequence;*

$$0 \to H_k(M, \mathbb{Z}) \otimes G \to H_k(M; G) \to Tor(H_{k-1}(M; \mathbb{Z}), G) \to 0.$$

The sequence splits, but not necessarily canonical.

Proof. We refer to [9, p.195]. $\square$

Theorem 2.38 (The universal Coefficients Theorem for Cohomology)**.** *Suppose G is an abelian group. For any $k > 0$, there exists the short exact sequence;*

$$0 \to Ext((H_{k-1}(M;\mathbb{Z}), G)) \to H^k(M;G) \to Hom(H_k(M;\mathbb{Z})) \to 0.$$

The sequence splits, but not necessarily canonical.

Proof. We refer to [9, p.195]. $\square$

So $H^k(M;G) \cong Hom(H_k(M;\mathbb{Z})) \oplus Ext((H_{k-1}(M;\mathbb{Z}), G))$, this isomorphism is not canonical.

In the case of $G = \mathbb{Z}$, we assert that:

$H^k(M;\mathbb{Z}) \cong \mathrm{Hom}(H_k(M;\mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_{k-1}(G;\mathbb{Z}), \mathbb{Z})$, suppose $H_k(M;\mathbb{Z}) \cong \mathbb{Z}^{\omega_k} \oplus T_k$ written as the free and torsion part for some finite group T_k and $H_{p-1}(G;\mathbb{Z}) \cong \mathbb{Z}^{\omega_{k-1}} \oplus T_{k-1}$ for some finite group T_{k-1}. Then we have:

$H^k(M;\mathbb{Z}) \cong \mathbb{Z}^{\omega_k} \oplus \mathrm{Hom}(T_k, \mathbb{Z}) \oplus \mathrm{Ext}(T_{k-1}, \mathbb{Z})$. Using the main properties of Hom and Ext, let $G \cong \mathbb{Z}^w \oplus \mathbb{Z}/c_1\mathbb{Z} \oplus \ldots \mathbb{Z}/c_u\mathbb{Z}$:

- $\mathrm{Hom}(T_k, G) \cong \bigoplus_{1 \le i \le s} \mathrm{Ann}_G(a_i)$ where $\mathrm{Ann}_G(a_i) = \{x \in G \mid a_i x = 0\}$, i.e., the a_i-torsion of G. Hence in our case $G = \mathbb{Z}$, $\mathrm{Hom}(T_k, \mathbb{Z})$ is trivial.
- $\mathrm{Ext}(T_{k-1}, G) \cong T_{k-1}^w \oplus \bigoplus_{1 \le i \le t, 1 \le j \le u} \mathbb{Z}/\gcd(b_i, c_j)\mathbb{Z}$. Hence in our case $G = \mathbb{Z}$, so $\mathrm{Ext}(T_{k-1}, \mathbb{Z}) \cong T_{k-1}^{\omega}$,

Finally, $H^k(M;\mathbb{Z}) \cong \mathbb{Z}^{\omega_k} \oplus T_{k-1}$ and $H_k(M;\mathbb{Z}) \cong \mathbb{Z}^{\omega_k} \oplus T_k$ for $k > 0$

Further by using Poincare Duality Theorem, which states:

Theorem 2.39. *Let M be an oriented n-manifold. Then there exists canonical isomorphism:*

$$H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$$

Proof. We refer to [9, p.241]. □

We establish the following isomorphisms between the homology and cohomology of an oriented connected 4-manifold,

- $F_k \cong F_{n-k}$ and $T_k \cong T_{n-k-1}$, where F_k is the free part of $H_k(M;\mathbb{Z})$.
- $H^0(M,\mathbb{Z}) \cong \mathbb{Z}$ since M is connected, and $H^4(M,\mathbb{Z}) \cong \mathbb{Z}$ since M is oriented.

k	0	1	2	3	4
$H^k(M,\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{\omega_1}$	$\mathbb{Z}^{\omega_2} \oplus T_1$	$\mathbb{Z}^{\omega_1} \oplus T_1$	$\mathbb{Z}$
$H_k(M,\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{\omega_1} \oplus T_1$	$\mathbb{Z}^{\omega_2} \oplus T_1$	$\mathbb{Z}^{\omega_1}$	$\mathbb{Z}$

As noticed, all cohomology and homology are governed by $H_1(M,\mathbb{Z})$ and $H_2(M,\mathbb{Z})$, since $H_1(M,\mathbb{Z}) = \pi_1(M)/[\pi_1(M),\pi_1(M)]$ as the abelianization of the fundamental group $\pi_1(M)$, so $\pi_1(M)$ and $H_2(M,\mathbb{Z})$ determine all the groups. If the 4-manifold is simply connected, $H_1(M,\mathbb{Z})$ vanishes.

2.2.2 Invariants of 4-Manifold

In this subsection, we give the topological and the analytic invariants of a compact, connected 4-manifold.

2.2.2.1 Topological and Analytic Invariants

The main topological and analytic invariants of the 4-manifold M are the fundamental group $\pi_1(M)$, the Betti numbers, the Chern numbers, the Hodge numbers, the intersection form.

The Betti numbers are defined by the dimension of simplicial or singular cohomology groups.

$$b_i = \dim_{\mathbb{Z}} H^i(X, \mathbb{Z}) = \dim_{\mathbb{R}} H^i(X, \mathbb{R}) = \dim_{\mathbb{C}} H^i(X, \mathbb{C}).$$

Hence by Poincare duality, $b_i = b_{4-i}$, $b_0 = b_4 = 1, b_1 = b_3$.

The Hodge numbers are defined by the dimensions of the cohomology groups of the sheaves of p-forms on M.

$$h^{p,q} = \dim H^q(M, \Omega^p)$$

By Serre duality and Hodge theory, $h^{p,q} = h^{q,p} = h^{2-p,2-q} = h^{2-q,2-p}$.

The irregularity $q = h^{1,0}$ as the special Hodge number is defined by the dimensions of the cohomology groups of the sheaves of global holomorphic 1-forms on M:

$$q = \dim H^0(M, \Omega^1)$$

The geometric genus $p_g = h^{2,0}$ as the special Hodge number defined by the dimensions of the cohomology groups of the sheaves of global holomorphic 2-forms on M:

$$p_g = \dim H^0(M, \Omega^2)$$

The Chern classes c_1 and c_2 are also the invariants depending on the almost complex structure. We have the relation between top Chern number and the Euler number:

$$c_2[M] = \chi(M) = \sum_{i=0}^{4}(-1)^i b_i = 2 - 2b_1 + b_2 \tag{2.4}$$

To express c_1, we need Hirzebruch signature theorem [15] which states :

$$b_2^+(M) - b_2^-(M) = \frac{1}{3}(c_1^2(M) - 2c_2(M)), \tag{2.5}$$

where b_2^+ and b_2^- represent the signature (b_2^+, b_2^-) of the intersection form. Hence $c_1^2(M)$ and $c_2(M)$ become the holomorphic invariants because of the fact that the intersection form and the Betti numbers are the invariants.

We have also the holomorphic Euler characteristic as follow;

$$\chi(\mathcal{O}(M)) = h^0(\mathcal{O}(M)) - h^1(\mathcal{O}(M)) + h^2(\mathcal{O}(M)) = 1 - q(M) + p_g(M) \tag{2.6}$$

where h^i represents the dimension of $H^i(M, \mathcal{O}(M))$.

Noether's formula[10] establishes the relations between the invariants the irregularity q and geometric genus p_g and states the following.

$$\chi(\mathcal{O}(M)) = \frac{1}{12}(c_1^2(M) + c_2(M)) \tag{2.7}$$

By eliminating $c_1^2(M)$ in the equations (2.5) and (2.7), we find that

$$(b_2^+ - 2p_g) + (2q - b_1) = 1 \tag{2.8}$$

Since $(b_2^+ - 2p_g \geq 0)$ and $(2q - b_1 \geq 0)$, we have

- In case of b_1 is even, then

$$2q = b_1,\ b_2^+ = 2p_g + 1,\ h^{1,1} = b_2^- + 1 \tag{2.9}$$

- In case of b_1 is odd, then

$$b_1 = 2q - 1,\ b_2^+ = 2p_g,\ h^{1,1} = b_2^- \tag{2.10}$$

Since $b_1 \geq 2h^{1,0}$ and $h^{0,1} + h^{1,0} \geq b_1$, from the equation (2.6), $b_1 = h^{1,0}$ and from the equation (2.7), $b_1 = 2h^{1,0} + 1$, in the both cases, we have $b_1 = h^{1,0} + h^{0,1}$.

By Serre Duality, we have $h^{i,j} = h^{2-i,2-j}$, and

$$\chi(M) = \sum_{1 \leq i,j \leq 2} (-1)^{i+j} h^{i,j} = 2 - 2b_1 + (h^{2,0} + h^{1,1} + h^{0,2}). \tag{2.11}$$

We conclude that $b_2 = h^{2,0} + h^{1,1} + h^{0,2}$ and $h^{1,1} = b_2 - 2h^{0,2} = b_2 - 2p_g$

2.2.2.2 Intersection Form

In this section, we deal with the intersection form on the closed oriented 4-manifold. Every homology class of a closed oriented 4-manifold can be represented by embedded submanifolds. Since M is closed and oriented, let S_a, S_b be two surfaces represented by the classes $a, b \in H_2(M, \mathbb{Z})$ such that their intersections are all transverse. We will assign a sign ± 1 to every intersection point of S_a and S_b by concatenating positive bases of the tangent spaces T_pS_a and T_pS_b at a point $p \in S_a \cap S_b$, so we obtain a basis of T_pM. We will call that the sign of the intersection at p is positive if this basis is positive, and negative otherwise. For $a, b \in H_2(M, \mathbb{Z})$, $\beta(a, b)$ is the number of points in $S_a \cap S_b$ counted with sign. Hence we are ready to give the intersection pairing as follow:

Let M be a closed oriented 4-manifold and S_a, S_b be two surfaces represented by the classes $a, b \in H_2(M, \mathbb{Z})$. Its intersection form defined by

$$\beta : H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z}$$

$$\beta(a,b) = S_a \cdot S_b$$

In the case of M being simply-connected, $H_2(M,\mathbb{Z})$ will be a free $\mathbb{Z}$-module, hence $H_2(M,\mathbb{Z}) \cong \mathbb{Z}^n$ where $n = b_2(M) = dim(H_2(M,\mathbb{R}))$. In the case of M being not simply-connected, $H_2(M,\mathbb{Z})$ have a torsion part which inherits the torsion of $H_1(M,\mathbb{Z})$ essentially coming from $\pi_1(M)$. Since the intersection form is linear, it vanishes on the torsion parts; hence, we can always assume that $H_2(M;Z)$ is a free $\mathbb{Z}$-module for the intersection form.

Theorem 2.40. *The intersection form β of a 4-manifold is unimodular.*

Proof. As we proved in the theorem (2.22), the bilinear form β is unimodular if and only if $M = M^*$, if we apply this theorem to the $H_2(M;\mathbb{Z})$, this implies that let the basis $B = \{e_1, \cdots, e_m\}$ of $H_2(M;\mathbb{Z})$, then there exist a unique dual basis $B = \{e'_1, \cdots, e'_m\}$ of its dual space $Hom(H_2(M;\mathbb{Z}),\mathbb{Z})$, it means that taking the basis $B = \{e_1, \cdots, e_m\}$ in $H_2(M;\mathbb{Z})$, choose canonical dual basis in the $Hom(H_2(M;\mathbb{Z}),\mathbb{Z})$, then by using Poincare duality, taking it back to $H_2(M;\mathbb{Z})$, thus we obtain the desired basis $B''' = \{e'''_1, \cdots, e'''_m\}$ such that $\beta(e_i, e'''_j) = \delta_{i,j}$. $\square$

Tensoring by $\mathbb{R}$, the intersection form has the signature of $H_2(M,\mathbb{Z})$ denoted by $sign(H_2(M,\mathbb{Z})) = (b_2^+, b_2^-)$ where $b_2(M) = dim(H_2(M,\mathbb{R})) = b_2^+ + b_2^-$.

2.3 K3 Surface

In this section, we will give some properties about $K3$ surfaces and $K3$ lattice. The main references for this section are [1], [6], [2], [15], [4].

2.3.1 Basic Definitions and Invariants of K3 Surface

In the classication of algebraic surfaces, $K3$ surfaces encompass one of four types of minimal surfaces of Kodaira dimension 0. All algebraic K3 surfaces over $\mathbb{C}$ are the complex $K3$ surface, the most of complex $K3$ surfaces are not algebraic.

In this thesis, we restrict ourselves to study algebraic complex $K3$ surfaces.

Definition 2.41 (K3 Surface)**.** A complex K3 surface is a compact connected 2-dimensional complex manifold X such that

- the canonical line bundle is trivial; $\omega_X = \mathcal{O}_X$ and
- the cohomology group $H^1(X, \mathcal{O}_X)$ is trivial.

We can derive the basic informations from the definition of $K3$ surfaces.

- The irregularity $q = h^{0,1} = 0$ and the first Betti number $b_1 = 0$ and
- We have that $\Lambda^2 T^* X$ is trivial bundle and so there is the holomorphic 2-form ω_X which is nowhere zero, where T^*X is the holomorphic cotangent bundle.
- This holomorphic form which is non-zero everywhere is unique up to a multiplication, so $h^{0,2} = h^{2,0} = 1$.

Examples 2.42. Suppose X is a smooth complete intersection of type $(d_1, ..., d_r)$ in $\mathbb{P}^n$,i.e., X has codimension r and $X = H_1 \cap \cdots \cap H_r$, where H_i is a hypersurface of degree $d_i \geq 1$ for $i = 1, ..., r$. If $r = n - 2$ and $\sum d_i = n + 1$, by using the adjunction formula which states that if X is complete intersection of type $(d_1, ..., d_r)$ in $\mathbb{P}^n$, then $K_X = (\sum d_i - n - 1)H$, then X is a $K3$ surface. We can also suppose $d_i > 1$ since we do not want to drop the dimension $\mathbb{P}^n$, hence if we compute the possible values for d_i:

- in case of $n = 3$ so $d_1 = 4$, so X is a smooth quartic surface in $\mathbb{P}^3$.
- in case of $n = 4$ and $(d_1, d_2) = (2, 3)$, so X is a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$.
- in case of $n = 5$ and $(d_1, d_2, d_3) = (2, 2, 2)$, i.e., X is a smooth complete intersection of three quadrics in $\mathbb{P}^5$.

Next, we will compute the fundamental invariants of $K3$ surface X to obtain the lattice structure of X which we deal with later.

By using Noether's formula for the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ is equal to,

$$\chi(\mathcal{O}(X)) = \frac{1}{12}(c_1^2(X)) + c_2(X)). \tag{2.12}$$

Since the canonical divisor K_X is trivial, we have $c_1^2(X) = K_X.K_X = 0$ and so $\chi(\mathcal{O}_X) = \frac{1}{12}c_2(X)$, we know that

$$\chi(\mathcal{O}(X)) = h^0(\mathcal{O}(X)) - h^1(\mathcal{O}(X)) + h^2(\mathcal{O}(X)) \tag{2.13}$$

$h^0 = h^{0,0}$, $h^1 = h^{0,1}$ and $h^2 = h^{0,2}$, since X is connected, $h^{0,0} = 1$, by the definition of $K3$ surface, $\omega = \Omega^2 = \mathcal{O}_X$, $h^{0,2} = 1$, The irregularity $q = h^{0,1} = 0$, so we obtain

$$\chi(\mathcal{O}(X)) = 2, c_2(X) = 24 = \chi(X) = 24 \tag{2.14}$$

Hence, c_2 is the Euler characteristic,

$$c_2(X) = \chi(X) = \sum_{i=0}^{4} = (-1)^i b_i = 2 - 2b_1 + b_2 \tag{2.15}$$

we show that

$$b_2 = b_2^+ + b_2^- = 22 \tag{2.16}$$

The holomorphic Euler characteristic is also equal to

$$\chi(\mathcal{O}(X)) = 1 - q(X) + p_g(X) \tag{2.17}$$

from this equation, we get

$$p_g(X) = 1 \tag{2.18}$$

We know the following from the previous results

$$b_2^+ = 2p_g + 1 \tag{2.19}$$

so

$$b_2^+ = 3,\ b_2^- = 19 \tag{2.20}$$

2.3.2 Cohomology of Complex K3 Surfaces

For the singular cohomology groups of $K3$ surface X, since X is connected, $H^0(X,\mathbb{Z}) = \mathbb{Z}$ and since X is oriented, $H^4(X,\mathbb{Z}) = \mathbb{Z}$, by Poincare duality, $H_4(X,\mathbb{Z}) = H_0(X,\mathbb{Z}) = \mathbb{Z}$. Since X is simply connected, $H_1(X,\mathbb{Z}) = \pi_1(X)/[\pi_1(X),\pi_1(X)]$ as the abelianization of the fundamental group $\pi_1(X)$, so $H_1(X,\mathbb{Z})$ vanishes. By Poincare duality, $H^3(X,\mathbb{Z})$ also vanishes. By using universal coefficient theorem and Poincare duality as done previously, we obtain $H^1(X,\mathbb{Z}) = H_3(X,\mathbb{Z}) = 0$. Since $H_1(X,\mathbb{Z})$ has no torsion part, this implies that $H_2(X,\mathbb{Z})$ is also free. $b_2 = 22$ as we computed above, hence $H^2(X,\mathbb{Z}) = H_2(X,\mathbb{Z}) = \mathbb{Z}^{22}$

$K3$ surface X has three important groups, namely Picard group $Pic(X)$, Neron-Severi group $NS(X)$, and Transcendental group $T(X)$.

Let X be a complex manifold of dimension n. The set of holomorphic line bundles on X forms a group, which is isomorphic to $H^1(X, \mathcal{O}^*(X))$. The group structure is induced by the tensor product. The identity element corresponds to the trivial bundle $\mathcal{O}(X)$ and the inverse corresponds to the dual bundle. $H^1(X, \mathcal{O}^*(X))$ is the set of isomorphy classes of line bundles over X.

The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}(X) \to \mathcal{O}^*(X) \to 0$$

gives rise to the long exact sequence.

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}(X)) \to H^1(X, \mathcal{O}^*(X)) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \cdots$$

$$c_1 : H^1(X, \mathcal{O}^*(X)) \to H^2(X, \mathbb{Z})$$

Putting $ker(c_1) = Pic^0(X)$, the identity elements of the Picard group, so $Pic(X)/Pic^0(X)$ is isomorphic to a subgroup of $H^2(X, \mathbb{Z})$, called the Neron-Severi group of X. By the definition of $K3$ surfaces, $H^1(X, \mathcal{O}(X))$ is trivial, hence this mapping is inclusion, $ker(c_1) = Pic^0(X)$ is trivial, so the Neron-Severi group and the Picard group of X coincide for $K3$ surface.

2.3.3 Lattice Structure of K3 Surface

Intersection pairing for $K3$ surface given by the cup product:

$$\smile : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) = \mathbb{Z}$$

which gives rise to a symmetric bilinear form on $H^2(X, \mathbb{Z})$, and since as we computed, $H^2(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{22}$ as a free $\mathbb{Z}$-module, hence $H^2(X, \mathbb{Z})$ inherits a lattice structure.

$H^2(X,\mathbb{Z})$ has three important lattice properties, namely indefinite, even, unimodular.

- By the theorem (2.40), this lattice is unimodular.
- Since the second Betti number $b_2 = b_2^+ + b_2^- = 22$, $b_2^+ = 3$ and $b_2^- = 19$ as we computed in the preceding section, hence the signature of $H^2(X,\mathbb{Z})$ is $sign(H^2(X,\mathbb{Z})) = (3,19)$.
- There is a unique Wu class $u_k \in H^k(X;\mathbb{Z}_2)$ such that for any $x \in H^{n-k}(X;\mathbb{Z}_2)$, $\mathrm{Sq}^k(x) = u_k \smile x$. By using Wu formula [15] ,

$$w_k = \sum_{i+j=k} Sq^i(u_j),$$

$$w_2 = Sq^2u_0 + Sq^1u_1 + Sq^0u_2 = u_1 \smile u_1 + u_2.$$

Using mod(2), and $w_1 = u_1$, we obtain

$$u_2 = w_2 + w_1 \smile w_1.$$

Hence,

$$\mathrm{Sq}^2(x) = u_2 \smile x$$

$$x \smile x = (w_2 + w_1 \smile w_1) \smile x \text{ for any } x \in H^2(X;\mathbb{Z}_2)$$

Since X is orientable, w_1 vanishes, $w_2 = c_1 \mod 2$, but the first chern class vanishes by the definition of $K3$ surface, $c_1(K_X) = c_1(X) = 0$. Therefore, intersection pairing is even.

We can justify also letting each class of a $K3$ lattice by a divisor C, by using Riemann-Roch theorem [4] which states

$$\chi(\mathcal{O}_X(C) = \frac{1}{2}(C^2 + C.K_X) + \chi(\mathcal{O}_X) \tag{2.21}$$

where K_X is a canonical divisor of X. So,

$$(C^2 + C.K_X) \equiv 0 \mod 2,$$

since K_X is a trivial, $K_X.C = 0$, hence $C^2 \equiv 0 \mod 2$. Thus the $K3$ lattice is even.

By the classification theorem (2.21) for the even, indefinite, and unimodular lattices, we conclude that

$$H^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$$

We will call this lattice $K3$-lattice, denoted by Λ.

Picard group $Pic(X)$ and Neron-Severi group $NS(X)$ endowed with lattice structure inherited by c_1.

Theorem 2.43 (Signature Theorem)**.** *Suppose X is a compact surface. If the intersection pairing on $H^2(X, \mathbb{Z})$ is restricted to $H^{1,1}(X)$, then*

- *if b_1 is even, this form is nondegenerate of signature* $(1, h^{1,1} - 1)$
- *if b_1 is odd, this form is nondegenerate of signature* $(0, h^{1,1})$.

Proof. Consider the space $(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$. Since the dimension $dim(H^{2,0}(X)) = p_g$, This space is a $2p_g$ dimensional subspace of $H^2(X, \mathbb{R})$. The intersection pairing on this subspace of $H^2(X, \mathbb{R})$ is positive definite. By using the Hodge decomposition, the orthogonal complement of this subspace in $H^2(X, \mathbb{R})$ is $H^{1,1}(X)$. By using (2.9), and (2.10) we obtain that for b_1 even, $b_2^- = h^{1,1} - 1$, and for b_1 odd, $b_2^+ = h^{1,1}$.

□

Theorem 2.44 (Lefschetz's Theorem on (1, 1)-classes)**.** *Suppose X is a compact surface, then the image of the Picard group by c_1 is equal to $c_1(H^{1,1}(X)) \cap H^2(X, \mathbb{Z})$. Equivalently, $c_1(H^1(X, \mathcal{O}^*(X))$ consists of classes represented by real closed $(1,1)$-forms.*

Proof. We refer to [1, p.142]. $\square$

Hence the Neron-Severi lattice of a $K3$ surface is also as a sublattice $NS(X) = c_1(H^{1,1}(X)) \cap H^2(X, \mathbb{Z})$. Or more precisely,

$$((H^1(X, \mathcal{O}^*(X), \smile) \cong c_1(H^{1,1}(X)) \cap H^2(X, \mathbb{Z})$$

Remark 2.45. The rank of the Neron-Severi lattice of a $K3$ surface is the Picard number and is denoted by $\rho(X)$.

Theorem 2.46. *Suppose X is a K3 surface, then the intersection pairing $\smile$ on $NS(X)$ is non-degenerate and even and its signature is* $(1, \rho(X) - 1)$.

Proof. It suffices to consider only the signature, because the intersection pairing $\smile$ inherited from $H^2(X, \mathbb{Z})$, so this sublattice is even and nondegenerate. Since $NS(X) = c_1(H^{1,1}(X)) \cap H^2(X, \mathbb{Z})$, by the signature theorem(2.43), its signature is $(1, \rho(X) - 1)$. $\square$

Definition 2.47. The orthogonal complement of the Neron-Severi lattice $NS(X)$ of a $K3$ surface in the $H^2(X, \mathbb{Z})$-lattice is called the transcendental lattice:

$$T(X) := NS(X)^{\perp} \subset H^2(X, \mathbb{Z})$$

Theorem 2.48. *Suppose X is a K3 surface, then the intersection pairing $\smile$ on $T(X)$ is non-degenerate and even and its signature is* $(2, 20 - \rho(X))$.

Proof. It suffices to consider only the signature, because the intersection pairing $\smile$ inherited from $H^2(X, \mathbb{Z})$, so this sublattice is even and nondegenerate. Since the transcendental lattice is the orthogonal complement of the Neron-Severi lattice $NS(X)$ of a $K3$ surface in the $H^2(X, \mathbb{Z})$-lattice, the signature of the Neron-Severi lattice $NS(X)$ is $(1, \rho(X) - 1)$ and the signature of the $H^2(X, \mathbb{Z})$-lattice $(3, 19)$, by the orthogonality, we obtain its signature $(2, 20 - \rho(X))$.

$\square$

2.4 Enriques Surface

Maib references for this section are [1], [6], [2], [15], [4].

For each Enriques surface Y, there exists a $K3$ surface X and a fixed-point-free involution $\iota : X \hookrightarrow X$ such that the quotient surface X/ι isomorphic to Y i,e,. $Y \cong X/\iota$. Conversely, the universal double covering X of Y is a $K3$-surface.

2.4.1 Basic Definitions and Invariants of Enriques Surface

Definition 2.49 (Enriques Surface)**.** An Enriques surface Y is a smooth projective surface satisfying the following conditions:

- The irregularity $q = h^{0,1} = H^1(X, \mathcal{O}(X)) = 0$
- The canonical line bundle ω_Y is not trivial, i.e., $\omega_Y \ncong \mathcal{O}(Y)$; but $\omega_Y^{\otimes 2} \cong \mathcal{O}_X$. Equivalently, $2K_Y = 0$, but $K_Y \neq 0$ where K_Y is canonical divisor of Y.

We give basic examples of the Enriques surfaces:

Example 2.50.

- Let X be the quartic surface in $\mathbb{P}^3$ defined by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ as in the example (2.42). Let σ be the automorphism of $\mathbb{P}^3$

  $$\sigma : \mathbb{P}^3 \to \mathbb{P}^3$$

  defined by

  $$\sigma(x_0, x_1, x_2, x_3) = (x_0, ix_1, -x_2, -ix_3,).$$

  σ is the automorphism of X, has no fixed points. σ has order two. Since X is a $K3$ surface, then the quotient of X by the involution σ is a Enriques surface.

We need the basic invariants of a Enriques surface Y to obtain the lattice structure of Y. Since $c_1^2(Y) = [K(Y)]^2$, by the definition of a Enriques surface, $c_1^2(Y) = 0$, $p_g = 0$. By using Noether's formula for the holomorphic Euler characteristic $\chi(\mathcal{O}_Y)$ is equal to,

$$\chi(\mathcal{O}(Y)) = \frac{1}{12}(c_1^2(Y)) + c_2(Y)) \tag{2.22}$$

or equivalently,

$$[K(Y)]^2 + \chi(Y) = 12(1 - q + p_g) \tag{2.23}$$

so $\chi(Y) = 12$ and $\chi(\mathcal{O}_Y) = 1$. Since $b_1 = 2q$ and

$$\chi(Y) = \sum_{i=0}^{4} = (-1)^i b_i = 2 - 2b_1 + b_2 \tag{2.24}$$

so $b_2 = 10$. Similarly,

$$c_2[Y] = \chi(Y) = \sum_{i=0}^{4} = (-1)^i b_i = 2 - 2b_1 + b_2 \tag{2.25}$$

so $c_2 = 12$.

Theorem 2.51. *The universal double covering X of Y is a $K3$-surface. For each Enriques surface Y, there exists a $K3$ surface X and a fixed-point-free involution $\iota : X \hookrightarrow X$ such that the quotient surface X/ι isomorphic to Y i,e,. $Y \cong X/\iota$.*

Proof. We refer to [1, p.339]. □

Theorem 2.52. *Suppose Y is a Enriques surface. Then $h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = 0$ and $h^{1,1} = 10$.*

Proof. $b_1 = h^{1,0} + h^{0,1}$, $b_1 = 2q$, so $h^{1,0} = 0$, $b_2 = h^{2,0} + h^{1,1} + h^{0,2} = 0$, by Serre duality, we obtain $b_2 = h^{1,1}$, so $h^{1,1} = 10$. □

Let Y be a Enriques surface with universal double covering $K3$-surface X, then order of $\pi_1(Y)$ is equal to the number of sheets of the covering spaces. Hence

$\pi_1(Y) \cong \mathbb{Z}_2$. Since $H^2(Y,\mathbb{Z})_{tors}$ given by $H_1(Y,\mathbb{Z})_{tors} = \pi_1(Y)$. We have also $b_2 = 10$ as the rank of $H^2(Y,\mathbb{Z})$. Thus

$$H^2(Y,\mathbb{Z}) = \mathbb{Z}^{10} \oplus \mathbb{Z}_2$$

We are well-versed to compute the lattice of Enriques surface Y.

2.4.2 Lattice Structure of Enriques Surface

Intersection pairing for Enriques surface given by the cup product:

$$\smile: H^2(Y,\mathbb{Z}) \times H^2(Y,\mathbb{Z}) \to H^4(Y,\mathbb{Z}) = \mathbb{Z}$$

which gives rise to a symmetric bilinear form on $H^2(Y,\mathbb{Z})$, and since as we computed, $H^2(Y,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{10} \oplus \mathbb{Z}_2$ and even though it is not a free $\mathbb{Z}$-module, but by the linearity of the intersection pairing, it vanishes on the torsion parts. Hence $H^2(Y,\mathbb{Z})$ on the free part inherits a lattice structure.

We will call $H^2(Y,\mathbb{Z})_{free}$ an Enriques lattice E.

Now this lattice has the three important lattice property as in the case of $K3$ surfface, namely unimodular, even, indefinite.

- By the theorem (2.40), this lattice is unimodular.
- By the Hirzebruch signature theorem which states

$$\sigma(Y) = \frac{1}{3}(c_1^2(Y) - 2c_2(Y)) = -8. \tag{2.26}$$

  Since $b_2 = 10$, we can conlude that the signature of Enriques lattice is $(1,9)$.

- Let each class of a Enriques lattice by a divisor C, by using Riemann-Roch theorem [4] which states

$$\chi(\mathcal{O}_Y(C)) = \frac{1}{2}(C^2 + C \cdot K_Y) + \chi(\mathcal{O}_Y) \tag{2.27}$$

  where K_Y is a canonical divisor of Y. So,

$$(C^2 + C \cdot K_Y) \equiv 0 \mod 2,$$

since $C \cdot K_Y$ takes integer values, even though K_Y is not a trivial, $2(C \cdot K_Y) = 2K_Y \cdot C = 0$, hence this implies $K_Y \cdot C = 0$, and $C^2 \equiv 0 \mod 2$. Thus the Enriques lattice is even.

Finally, by the classification theorem (2.21) for the even, indefinite, and unimodular lattices, we conclude that the Enriques lattice E,

$$E \cong U \oplus E_8$$

Chapter 3

Which Algebraic K3 Surfaces with Picard Number $\rho(X) = 19$ Cover an Enriques Surface

When X is a complex K3 surface with Picard number $\rho(X)$ over the field $\mathbb{C}$, as we proved theorem (2.48) in the preceding chapter, the transcendental lattice T_X of X has signature $(2; 20 - \rho(X))$. Furthermore, this lattice inherited lattice structure as a sublattice from $H^2(X, \mathbb{Z})$, hence it is even. All lattice can be associated by its Gram matrix as we dealt with in the previous chapter, henceforth we will always associate the transcendental lattice by its Gram matrix, namely

$$\begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix} \tag{3.1}$$

with respect to some basis $\{x, y, z\}$.

As we mentioned earlier Sertöz implemented the following criterion 3.1 in his article [18] to find explicit necessary and sufficient conditions on the parity of entries of the Transcendental lattice T_X so that X covers an Enriques surface when $\rho(X) = 20$, he completely resolved all difficulties arised in case the K3-lattice with $\rho(X) = 20$, and even in the case of $\rho(X) = 20$, explicit necessary and sufficient conditions do not depend on the parity only in the difficult cases.

Theorem 3.1 (Keum's Criterion)**.** *Suppose X is an algebraic K3 surface. Then the followings are equivalent:*

1. *X admits a fixed point free involution.*
2.
   - *There exists a primitive embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$*
   - *$\mathrm{Im}(T_X)^{\perp}$ doesn't contain any vector of self intersection -2 in $\Lambda^- = U \oplus U(2) \oplus E_8(2)$, where U is the even unimodular lattice of signature $(1,1)$ and E_8 is the even unimodular lattice of signature $(0,8)$.*

This theorem also assumes that $\ell(T_X) + 2 \leq \rho(X)$. But this is always satisfied for our cases, that is, $\rho(X) \geq 12$.

Following Sertöz, Lee attacked the problem for finding explicit necessary and sufficient conditions on the entries of T_X so that X covers an Enriques surface when $\rho(X) = 19$, The main difficulties of the problem arise when the entries of T_X given in 3.1 are the following types:

1. Only a is odd.
2. Only b is odd.
3. Only c is odd.
4. Only a and b are odd.
5. Only b and c are odd.
6. Only a and c are odd.
7. Only a; b; and c are odd.

He resolved other cases except these cases as above. But these cases which remain open are the exactly conditions do not depend on the parity only. And yet he showed that all these seven cases are equivalent in his article [14, Lemma 4.1-4.5].

In the following chapters, we seek to find explicit necessary and sufficient conditions on the entries of T_X so that X covers an Enriques surface when $18 \leq \rho(X) \leq 19$.

We are ready to state and prove our first theorem.

Theorem 3.2. *If X is a algebraic K3 surface with Picard number $\rho(X) = 19$ and transcendental lattice given as*

$$\begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix}, \tag{3.2}$$

then X covers an Enriques surface if the following conditions hold:

- *d, e, f are even. a and b are odd, c is negative even.*
- *The form $aX^2 + dXY + bY^2$ is positive definite and does not represent* 1.

Proof. We will consider a particular embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$.

Let $\{x, y, z\}$ be a basis of T_X, $\{u_1, u_2\}$ be a basis of U and $\{v_1, v_2\}$ be a basis of $U(2)$. We can take an element w of $E_8(2)$ which generates a primitive sublattice of $E_8(2)$ isomorphic to $< 2c >$.

Define $\phi : T_X \to \Lambda^-$ by

$$\phi(x) = u_1 + au_2 \tag{3.3}$$

$$\phi(y) = u_1 + (d-a)u_2 + v_1 + \frac{1}{2}(a+b-d)v_2 \tag{3.4}$$

$$\phi(z) = eu_2 + \frac{1}{2}(f-e)v_2 + w \tag{3.5}$$

It can be shown by direct computation that this is an embedding and by using Lemma (2.26), we will prove that this embedding is primitive.

$$\phi(x)\cdot\phi(x) = 2a \tag{3.6}$$

$$\phi(y)\cdot\phi(y) = 2(d-a) + 2(a+b-d) = 2b \tag{3.7}$$

$$\phi(z)\cdot\phi(z) = w^2 = 2c \tag{3.8}$$

$$\phi(x)\cdot\phi(y) = a + (d-a) = d \tag{3.9}$$

$$\phi(x)\cdot\phi(z) = 1.e = e \tag{3.10}$$

$$\phi(y)\cdot\phi(z) = e + (f-e) = f \tag{3.11}$$

To prove that this embedding is primitive,

Note that,

$$A = \begin{bmatrix} 1 & a & 0 & 0 & 0 & \dots & 0 \\ 1 & d-a & 1 & \frac{1}{2}(a+b-d) & 0 & \dots & 0 \\ 0 & e & 0 & \frac{1}{2}(f-e) & w_1 & \dots & w_8 \end{bmatrix}$$

where $w = \sum_{n=1}^{8} w_i e_i \in E_8(2)$, and $e_1, \cdots, e_8$ standard basis for $E_8(2)$.

A is the embedding matrix for the map defined above. Now take first, third, and fifth column, repeatedly first, third and sixth and so on, since w chosen above is primitive element, i.e., $\gcd(w_1, \dots, w_8) = 1$, hence we can conclude that the greatest common divisor of the maximal minors of this embedding matrix is 1, by using Lemma (2.26), this embedding is primitive.

To show that $\phi(T_X)^\perp$ doesn't contain any vector of self intersection -2, Let $f = Xu_1 + x'u_2 + Yv_1 + y'v_2 + e \in \Lambda^-$, where $e \in E_8(2)$ with $e\cdot e = -4k,\ k \geq 0$.

Impose the condition that f lies in the orthogonal complement of $\phi(T_X)$ in Λ^- and that $f\cdot f = -2$.

Solving the equations $f \cdot \phi(x) = 0$, $f \cdot \phi(y) = 0$ for x', y',

$$f \cdot \phi(x) = (Xu_1 + x'u_2 + Yv_1 + y'v_2 + e) \cdot (u_1 + au_2) = 0, \tag{3.12}$$

we obtain that

$$x' = -aX. \tag{3.13}$$

From the equation $f \cdot \phi(y) = 0$,

$$f \cdot \phi(y) = (Xu_1 + x'u_2 + Yv_1 + y'v_2 + e) \cdot (u_1 + (d-a)u_2 + v_1 + \frac{1}{2}(a+b-d)v_2),$$

we obtain also that

$$y' = -\frac{1}{2}(X(d-2a) + Y(a+b-d)) \tag{3.14}$$

and substituting into the equation

$$f \cdot f = 2Xx' + 4Yy' + e \cdot e = -2 \tag{3.15}$$

gives

$$1 - (aX^2 + (d-2a)XY + (a+b-d)Y^2) = 2k \geq 0. \tag{3.16}$$

The binary quadratic form $aX^2 + (d-2a)XY + (a+b-d)Y^2$ has Gram matrix $A = \begin{pmatrix} 2a & d-2a \\ d-2a & 2(a+b-d) \end{pmatrix}$.

Let $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$, by the following transformation ${}^t\theta\, A\, \theta$, we can see that the binary quadratic form $aX^2 + (d-2a)XY + (a+b-d)Y^2$ is equivalent to the form $aX^2 + dXY + bY^2$. Since this is a positive definite form. Equation (3.16) holds if and only if this form represents 1, and then $k = 0$.

If we assume that the form $aX^2 + dXY + bY^2$ does not represent 1, then equation (3.16) cannot be solved, so there is no self intersection -2 vector in the orthogonal complement of $\phi(T_X)$.

□

The following theorem shows that a lattice M cannot always have an orthogonal splitting into smaller sublattices such that $M = L_1 \oplus L_2$ where L_1, L_2 are sublattices of M, but with respect to some basis, its associated Gram matrix could be turned into the following form:

Theorem 3.3 (Jacobi)**.** *Suppose M is a $\mathbb{Z}$-lattice. Then M has a basis $\{v_1, \cdots, v_n\}$ such that $\beta(v_i, v_j) = 0$ for $|i-j| \geq 2$, and its associated Gram matrix is:*

$$L \cong \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{12} & a_{22} & a_{23} & & \\ & & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n-1,n} & a_{n,n} \end{pmatrix}$$

Proof. We refer to [7, p.126]. □

We will give the proof of this theorem for a lattice M of a rank $r(M) = 3$. To prove generally this theorem is verbatim the same.

Corollary 3.4. *Let T_X be a intersection matrix represented by transcendental lattice as given $T_X = \begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix}$. Then T_X is $\mathbb{Z}$-equivalent to $T'_X = \begin{pmatrix} 2a' & d' & 0 \\ d' & 2b' & f' \\ 0 & f' & 2c' \end{pmatrix}$, noting that $a = a'$ remained fixed.*

Proof. Let $T_X = \begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix}$, if $e = 0$, there is nothing to prove. Let $e \neq 0$, and $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & z & t \end{pmatrix}$, $T'_X = {}^t\theta \, T_X \, \theta = \begin{pmatrix} 2a' & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix}$ where $a' = a$, $b' = bx^2 +$

$fxz+cz^2$, $c' = by^2+fyt+ct^2$, $d' = dx+ez$, $e' = dy+et$, $f' = 2bxy+fxt+fyz+2ctz$. Let $g = \gcd(e,g) \neq 0$, solving equation $dy+et = 0$ with respect to y and t, it can be concluded that $y = e/g$ and $t = -d/g$, since $\gcd(y,t) = 1$, there exist $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & z & t \end{pmatrix} \in GL_3(\mathbb{Z})$ having ${}^t(y,t)$ as its last column. We are done. □

Utilizing Jacobi Theorem 3.3 which states that any lattice over principal ideal domain can be substantially diagonalized, the transcendental lattice T_X of X denoted by its associated Gram matrix as in given (3.1) can be reduced to its triple-diagonal form

$$\begin{pmatrix} 2a & d & 0 \\ d & 2b & f \\ 0 & f & 2c \end{pmatrix} \tag{3.17}$$

as proved in corollary 3.4 above.

We will continue to investigate when the form $aX^2+dXY+bY^2$ represents 1. Before dealing with this form, we will prove the following theorem.

Theorem 3.5. *if a n-ary quadratic form f over $\mathbb{Z}$ such that $\sum_{1\leq i\leq n} c_i x_i^2 + \sum_{1\leq i,j\leq n} 2c_{ij}x_ix_j$ where $c_i, c_{i,j} \in \mathbb{Z}$ and $i \neq j$, represents 1, then f is $\mathbb{Z}$-equivalent to the following form $g(x_1,\cdots,x_n) = x_1^2 + h(x_1,\cdots,x_n)$ where h is a n-ary form not containing the term in x_1^2.*

Proof. Let

$$A = \begin{pmatrix} c_1 & c_{12}+c_{21} & . & . & c_{1n}+c_{n1} \\ c_{12}+c_{21} & c_2 & . & . & . \\ & . & . & . & . \\ & . & . & . & . \\ & . & . & c_{n-1} & c_{n-1,n}+c_{n,n-1} \\ c_{1n}+c_{n1} & . & . & c_{n-1,n}+c_{n,n-1} & c_n \end{pmatrix}$$

be the associated matrix form of the n-ary quadratic form f. Since f represents 1, then there exist $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}$ such that $f(\alpha_1, \cdots, \alpha_n) = 1$. Since the representation of 1 is always primitive, it means that $\gcd(\alpha_1, \cdots, \alpha_n) = 1$. By taking $\alpha_1, \cdots, \alpha_n$ as a first column of a matrix B, we can always construct this matrix θ as an element of $\mathrm{GL}_n(\mathbb{Z})$, then the matrix ${}^t\theta\, A\, \theta$ determine the n-ary quadratic form g which contains x_1^2 and we get $c_1 = 1$.

□

In the next theorem, we will combine the Jacobi theorem (3.3) and the theorem (3.5).

Theorem 3.6. *if a n-ary quadratic form f over $\mathbb{Z}$ such that $\sum_{1\le i\le n} c_i x_i^2 + \sum_{1\le i,j\le n} 2c_{ij}x_ix_j$ where $c_i, c_{i,j} \in \mathbb{Z}$, represents 1, then its associated Gram matrix form is in triple-diagonal Gram matrix L such that the first entry of this matrix $a_{11} = 1$.*

$$L \cong \begin{pmatrix} 1 & a_{12} & & & \\ a_{12} & a_{22} & a_{23} & & \\ & & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n-1,n} & a_{n,n} \end{pmatrix}$$

Proof. It is the direct consequences of the Jacobi theorem (3.3) and the theorem (3.5). □

We can state and prove the corollary of this theorem for our cases.

Corollary 3.7. *Let the associated Gram matrix of the transcendental lattice T_X be as*

$$\begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix},$$

and let the form $aX^2 + dXY + bY^2$ represent 1. *Then T_X is $\mathbb{Z}-$equivalent to*

$$\begin{pmatrix} 2 & d'' & 0 \\ d'' & 2b'' & f'' \\ 0 & f'' & 2c'' \end{pmatrix}$$

.

Proof. Since the form $aX^2+dXY+bY^2$ represents 1, then there exist $\alpha_1, \alpha_2 \in \mathbb{Z}$ such that $a\alpha_1^2+d\alpha_1\alpha_2+b\alpha_2^2 = 1$. Since the representation of 1 is always primitive, it means that $\gcd(\alpha_1, \alpha_2) = 1$. For any given integers $a_1, \cdots, a_n$, there exists an matrix $\theta \in GL_n(\mathbb{Z})$ with $a_1, \cdots, a_n$ as its first column if and only if the greatest common divisor of $\{a_1, \cdots, a_n\}$ is 1 [3, p.163]. Let $\theta = \begin{pmatrix} \alpha_1 & s & k \\ \alpha_2 & t & l \\ 0 & v & m \end{pmatrix} \in GL_3(\mathbb{Z})$. Then every matrix of the form ${}^t\theta T_X \theta$ represents the transcendental lattice of X with respect to some basis. Setting

$$T'_X = {}^t\theta\, T_X\, \theta \quad = \quad \begin{pmatrix} 2a' & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix},$$

the resulting entries in the following forms:

$$a' = a\alpha_1^2 + d\alpha_1\alpha_2 + b\alpha_2^2 = 1,$$

$$b' = as^2 + bt^2 + cv^2 + dst + esv + ftv$$

$$c' = ak^2 + bl^2 + cm^2 + dkl + ekm + flm.$$

By using the corollary 3.4, letting $T'_X = \begin{pmatrix} 2 & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix}$, if $e' = 0$, there is nothing to prove. Let $e' \neq 0$, and $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & y' \\ 0 & z' & t' \end{pmatrix}$, $T''_X = {}^t\theta\, T'_X\, \theta = \begin{pmatrix} 2a'' & d'' & e'' \\ d'' & 2b'' & f'' \\ e'' & f'' & 2c'' \end{pmatrix}$

where $a'' = 1$, $b'' = b'x'^2 + f'x' + c'z'^2$, $c'' = b'y'^2 + f'y't' + c't'^2$, $d'' = d'x' + e'z'$, $e'' = d'y' + e't'$, $f'' = 2b'x'y' + f'x't' + f'y'z' + 2c't'z'$. Let $g' = \gcd(e', g') \neq 0$, solving equation $d'y' + e't' = 0$ with respect to y' and t', it can be concluded that $y = e'/g'$ and $t' = -d'/g'$, since $\gcd(y', t') = 1$, there exist $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & y' \\ 0 & z' & t' \end{pmatrix} \in GL_3(\mathbb{Z})$ having ${}^t(y', t')$ as its last column. We conclude that T_X is $\mathbb{Z}-$equivalent to

$$\begin{pmatrix} 2 & d'' & 0 \\ d'' & 2b'' & f'' \\ 0 & f'' & 2c'' \end{pmatrix}$$

.

$\square$

In the case when the form $aX^2 + dXY + bY^2$ represents 1, we consider the associated Gram matrix of the Transcendental lattice in the following form:

$$\begin{pmatrix} 2 & d & 0 \\ d & 2b & f \\ 0 & f & 2c \end{pmatrix} \tag{3.18}$$

In the Gram matrix of the Transcendental lattice, we will consider the case $d = 0$ in 3.18. So the Gram matrix of the Transcendental lattice is the following form:

So we begin to prove the following theorem.

Theorem 3.8. *If X is a algebraic K3 surface with Picard number $\rho(X) = 19$ and transcendental lattice given as in (3.18), then X covers an Enriques surface if the following conditions hold:*

- *$d = 0$, f is even. c is a negative integer.*

- *The form X^2+bY^2 is positive definite form and represents* 1*, and* $b \neq 1, 2, 4$.

Proof. In this case,

$$T_X = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2b & f \\ 0 & f & 2c \end{pmatrix}$$

Now we are looking to the primitive embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$.

Let $\{x, y, z\}$ be basis for the transcendental lattice T_X. We will construct ϕ by setting $\phi(x) = \alpha$, with

$$\alpha = a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + e \in \Lambda^-$$

with respect to the standard basis of U, $U(2)$ and $E_8(2)$ respectively, and where $e \in E_8(2)$ with $e \cdot e = -4k, \ k \geq 0$.

$< \alpha, \alpha > = 2$ forces a_1 and a_2 to be odd. If $\beta = b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + \omega_2$ is in the orthogonal complement $\alpha^\perp$ of α in Λ^-, then $\alpha \cdot \beta = 0$ implies that b_1 and b_2 are of the same parity. Hence, if we take $\beta, \gamma \in \alpha^\perp$, then $\beta \cdot \gamma \equiv 0 \mod 2$.

We seek for the primitive embedding T_X into $\alpha \oplus \alpha^\perp \subset \Lambda^-$ where the rank of $\alpha^\perp$ is $r(\alpha^\perp) = 11$. By the orthogonality, since the signature of a lattice which generated by α is $(1, 0)$, the signature of $\alpha^\perp$ is $(1, 10)$ in Λ^-.

Suppose $\beta_1, \ldots, \beta_{11}$ is a basis for $\alpha^\perp$, and $B' = (2b_{ij})$, $2b_{ij} = \beta_i \cdot \beta_j$ is the Gram matrix for this basis. Let $B = (b_{ij})$.

Let C be the 12×12-matrix whose rows are the coordinates of $\alpha, \beta_1, \ldots, \beta_{11}$ with respect to the standard basis of Λ^-. Suppose A is the Gram matrix of Λ^-

with respect to its standard basis. We have

$$CA{}^tC = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{pmatrix}. \tag{3.19}$$

Since Λ^- does not have orthogonal element which means that there is no sub-lattices L_1, L_2 in Λ^- such that $\Lambda^- = L_1 \perp L_2$ where rank of L_1, $r(L_1) = 1$, so $\alpha, \beta_1, \dots, \beta_{11}$ is not a basis of Λ^-, $|\det C| > 1$. By lemma (2.34) , $|\det C|$ divides 2, hence is equal to 2.

Therefore, comparing the determinants of both sides,

$$det(C)^2.det(A) = 2.det(B') = 2^{12}.det(B),$$

since $det(C)^2 = 4$ and $det(A) = 2^{10}$, we can conclude that $|\det B| = 1$.

Define a new lattice $L = (\mathbb{Z}^{11}, B)$. L has signature $(\tau^+, \tau^-) = (1, 10)$ and is unimodular. Suppose it is even, then $\tau^+ - \tau^- \not\equiv 0 \mod 8$, this is a contradiction by the classification theorem of indefinite unimodular lattice. Thus L is indefinite, odd, unimodular. By using the classification theorem of indefinite unimodular, odd lattice 2.20, L is isomorphic to $< 1 >^1 \oplus < -1 >^{10}$.

Let

$$\phi : T_X \longrightarrow \alpha \oplus \alpha^\perp \subset \Lambda^-$$

be mapping such that with respect to this new basis of $L(2) \cong \alpha^\perp$,

$$\begin{aligned} \phi(x) &= (1, 0, \dots, 0), \\ \phi(y) &= (0, y_0, \dots, y_{10}), \\ \phi(z) &= (0, z_0, \dots, z_{10}) \end{aligned}$$

such that

$$\phi(y) \cdot \phi(y) = 2y_0^2 - 2y_1^2 - \dots - 2y_{10}^2 = 2b,$$

$$\phi(z) \cdot \phi(z) = 2z_0^2 - 2z_1^2 - \cdots - 2z_{10}^2 = 2c,$$

and,

$$\phi(y) \cdot \phi(z) = 2y_0z_0 - 2y_1z_1 - \cdots - 2y_{10}z_{10} = f,$$

by using lemma(2.26), it is easy to see that in the lattice L, that of investigating the existence of integers $y_0, \ldots, y_{10}$ and $z_0, \ldots, z_{10}$ such that if $y' = (y_0, \cdots, y_{10})$, $z' = (z_0, \cdots, z_{10}) \in L$ then the following conditions are satisfied:

$$\gcd(y_0z_1 - y_1z_0, \ldots, y_iz_j - y_jz_i, \ldots, y_9z_{10} - y_{10}z_9) = 1, \tag{3.20}$$

and from (3.20)

$$\gcd(y_0, \ldots, y_{10}) = 1, \tag{3.21}$$

and similarly,

$$\gcd(z_0, \ldots, z_{10}) = 1, \tag{3.22}$$

$$y'y' = y_0^2 - y_1^2 - \cdots - y_{10}^2 = b, \tag{3.23}$$

$$z'z' = z_0^2 - z_1^2 - \cdots - z_{10}^2 = c, \tag{3.24}$$

$$y'z' = y_0z_0 - y_1z_1 - \cdots - y_{10}z_{10} = \frac{1}{2}f, \tag{3.25}$$

$$\text{if } l \cdot y' = 0 \text{ and } l \cdot z' = 0, \text{ then } l \cdot l \neq -1, \text{ for every } l \in L \tag{3.26}$$

If there exist $y_0, \ldots, y_{10}$ and $z_0, \ldots, z_{10}$, then X covers an Enriques surface. By using the techniques of Vinberg from [19], Sertöz showed in [18] that the problem to investigate the existence of integers $y_0, \ldots, y_{10}$ satisfying 3.21,3.23 and 3.26 reduces to a problem to solve the following conditions:

$$
\begin{aligned}
gcd(y_0,\dots,y_{10}) &= 1,\\
-y_0^2+y_1^2+\cdots+y_{10}^2 &= -b,\\
y_1 \geq \cdots \geq y_{10} &> 0,\\
y_0 \geq y_1+y_2+y_3, &\text{ and}\\
3y_0 > y_1+\cdots+y_{10}&.
\end{aligned}
$$

He gave the following lemma in his article [18],

Let S denote the set of all $y \in L$ satisfying the above conditions.

Lemma 3.9. *For every positive integer N, other than $1, 2$ and 4, there is an $y \in S$ such that $y \cdot y = -N$.*

Proof. We refer to [18].

□

Thus, this implies that $\phi(T_X)^{\perp}$ doesn't contain a self intersection -2 vector whenever $b \neq 1, 2, 4$.

It only suffices to check that isometry and primitivity properties to complete the proof.

Before proving the existence of such integers satisfying the set of equations above, we need the famous Lagrange's four-square theorem:

Theorem 3.10. *Every natural number can be represented as the sum of four integer squares, that is, $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$.*

Proof. We refer to [7, p.185]. □

Let the vectors

$$Y = (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}) \tag{3.27}$$

$$Z = (z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}) \tag{3.28}$$

such that

$$y_5 = \frac{1}{2}f, y_6 = 1, y_7 = y_8 = y_9 = y_{10} = 0 \tag{3.29}$$

where y_0, y_1, y_2, y_3, y_4 are free variables. And

$$z_0 = z_1 = z_2 = z_3 = z_4 = z_6 = 0, z_5 = -1 \tag{3.30}$$

where z_7, z_8, z_9, z_{10} are free variables.

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 - y_5^2 - y_6^2 - y_7^2 - y_8^2 - y_9^2 - y_{10}^2 = b \tag{3.31}$$

Since y_0, y_1, y_2, y_3, y_4 are free variables, there exists y_0 such that $y_0^2 > b + f^2$, then by the Lagrange theorem 3.10, the equation 3.31 has infinite solutions.

Similarly,

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 - z_5^2 - z_6^2 - z_7^2 - z_8^2 - z_9^2 - z_{10}^2 = c, \tag{3.32}$$

Since z_7, z_8, z_9, z_{10} are free variables, and $-(1+c) \geq 0$, then by the Lagrange theorem 3.10, the equation 3.32 has infinite solutions. Lastly,

$$Y.Z = \frac{1}{2}f \tag{3.33}$$

This proves that the mapping ϕ is an embedding.

Finally to prove the primitivity of the embedding, it suffices to check the greatest common divisor of maximal minors of the embedding matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_0 & y_1 & y_2 & y_3 & y_4 & \frac{1}{2}f & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & z_7 & z_8 & z_9 & z_{10} \end{pmatrix} \tag{3.34}$$

Taking the first, seventh, and eighth columns, it is obvious that its determinant is 1, hence by using the lemma 2.26, this embedding is primitive.

To conclude that we found the primitive embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$ and $\mathrm{Im}(T_X)^\perp$ doesn't contain any vector of self intersection -2 in $\Lambda^- = U \oplus U(2) \oplus E_8(2)$, thus by the Keum's theorem 3.1, X admits a fixed point free involution σ . As we know that X/σ is a Enriques surface. This is the end of the proof of the theorem 3.8.

$\square$

In the theorems 3.2, 3.8 we showed above, the integral positive quadratic form floats around. Hence, we seek to find criteria under which condition 3×3 the Gram matrix of transcendental lattice T_X can be transformed into 3×3 matrix T'_X such that 2×2 matrix in 3×3 matrix T'_X is represented by positive definite binary quadratic form. For this purpose, we will state the following corollary.

We recall that for $\theta = \begin{pmatrix} x & y & z \\ s & t & v \\ k & l & m \end{pmatrix} \in GL_3(\mathbb{Z})$, every matrix of the form $T'_X = {}^t\theta\, T_X\, \theta = \begin{pmatrix} 2a' & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix}$ represents the transcendental lattice of X with respect to some basis and T'_X is $\mathbb{Z}$-equivalent to T_X.

Before we proceed with the lemma, one of the entries on the main diagonal of the Gram matrix of T_X, namely a or b or c, can be chosen as positive or negative by the signature of T_X which is $(2, 1)$.

Lemma 3.11. *Let T_X be a intersection matrix represented by transcendental lattice given in*

$$\begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix} \tag{3.35}$$

with $a > 0$. Then T_X is $\mathbb{Z}$-equivalent to T'_X with $a > 0$, $b > 0$ and $c = c$

Proof. if we assume that $b > 0$, there is nothing to prove. Let $b \leq 0$,

$\theta = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z})$, and $T'_X = {}^t\theta\, T_X\, \theta$ is $\mathbb{Z}$-equivalent to T_X. $T'_X = {}^t\theta\, T_X\, \theta = \begin{pmatrix} 2a' & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix}$ where $a' = a$, $b' = b + \alpha^2 a + \alpha d$, $c' = c$, $d' = d + \alpha a$, $e' = e$, $f' = f + \alpha e$.

Case 1: $d \geq 0$, $b \neq 0$, let $\alpha = nb$, where $n \in \mathbb{Z}_{<0}$. Then $b' = b + (nb)^2 a + nbd > 0$.

Case 2: $d < 0$, $b \neq 0$, let $\alpha = nb$, where $n \in \mathbb{Z}_{>0}$. Then $b' = b + (nb)^2 a + nbd > 0$

Case 3: $d > 0$, or $d < 0$, $b = 0$, let $\alpha = nd$, where $n \in \mathbb{Z}_{<0}$. Then $b' = b + (nb)^2 a + nbd > 0$.

Case 4: $d = 0$, $b = 0$, let $\alpha = n$, where $n \in \mathbb{Z}_{>0}$ Then $b' = (n)^2 a > 0$. □

Lemma 3.12. *Let T'_X be a intersection matrix represented by transcendental lattice as given in 3.35 with $a > 0$, $b > 0$. Then T'_X is $\mathbb{Z}$-equivalent to T''_X with $a > 0$, $b > 0$ and $c > 0$, so is T_X.*

Proof. As in lemma 3.11, applying $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z})$ to T'_X in lemma 3.11, where $\alpha \in \mathbb{Z}$, and $T''_X = {}^t\theta\, T'_X\, \theta$ is $\mathbb{Z}$-equivalent to T'_X, hence T''_X is $\mathbb{Z}$-equivalent to T_X. □

Corollary 3.13. *Let $\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & z & t \end{pmatrix} \in GL_3(\mathbb{Z})$ and T_X be a intersection matrix represented by transcendental lattice as given in 3.35, then 2×2 matrix in*

3×3 the Gram matrix is represented by positive definite binary quadratic form if $be^2 - def + cd^2 \geq 0$, and particularly, $def \leq 0$.

Proof. Let $T_X = \begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix}$, using lemma 3.11 and lemma 3.12, T_X can always be changed such that $a > 0$, $b > 0$ and $c > 0$. By corollary 3.4, if $be^2 - def + cd^2 \geq 0$, then we can conclude that 2×2 matrix in 3×3 the Gram matrix is represented by positive definite binary quadratic form. In the case of $def \leq 0$, $be^2 - def + cd^2$ is always greater or equal to zero.

□

Chapter 4

Which Algebraic K3 Surfaces with Picard Number $\rho(X) = 18$ Do Cover an Enriques Surface

When X is an algebraic K3 surface with Picard number $\rho(X) = 18$ over the field $\mathbb{C}$, the transcendental lattice T_X of X is denoted by its intersection matrix

$$\begin{pmatrix} 2a & e & f & g \\ e & 2b & h & i \\ f & h & 2c & j \\ g & i & j & 2d \end{pmatrix} \tag{4.1}$$

with respect to a basis $\{x, y, z, t\}$. This transcendental lattice T_X of X has signature $(2; 20 - \rho(X)) = (2; 2)$. By the theorem 3.3, the transcendental lattice T_X of X denoted by its intersection matrix can be reduced to its triple-diagonal form:

$$\begin{pmatrix} 2a & e & 0 & 0 \\ e & 2b & f & 0 \\ 0 & f & 2c & g \\ 0 & 0 & g & 2d \end{pmatrix} \tag{4.2}$$

Theorem 4.1. *If X is an algebraic K3 surface with Picard number $\rho(X) = 18$*

and transcendental lattice given as

$$\begin{pmatrix} 2a & e & 0 & 0 \\ e & 2b & f & 0 \\ 0 & f & 2c & g \\ 0 & 0 & g & 2d \end{pmatrix},$$

then X covers an Enriques surface if the following conditions hold:

- *a is a positive integer, c is a positive even integer. b, d are negative even integers, e, f, g are even integers.*
- *The form $aX^2 + cY^2$ is positive definite and does not represent* 1.

Proof. We will consider a particular embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$.

Let $\{x, y, z, t\}$ be a basis of T_X , $\{u_1, u_2\}$ be a basis of U and $\{v_1, v_2\}$ be a basis of $U(2)$. We can take two elements w_1 and w_2 of $E_8(2)$ which generate a primitive sublattice of $E_8(2)$ isomorphic to $< 2b > \oplus < 2d >$ by the corollary 2.36.

Define $\phi : T_X \to \Lambda^-$ by

$$\phi(x) = u_1 + au_2, \tag{4.3}$$

$$\phi(y) = eu_2 + \frac{1}{2}fv_1 + w_1, \tag{4.4}$$

$$\phi(z) = \frac{1}{2}cv_1 + v_2 \tag{4.5}$$

$$\phi(t) = \frac{1}{2}gv_1 + w_2 \tag{4.6}$$

where $w_1, w_2 \in E_8(2)$ such that $w_1^2 = 2b$, $w_2^2 = 2d$

It can be shown by direct computation that this is an embedding. To prove that this embedding is primitive,

Note that,

$$A = \begin{bmatrix} 1 & a & 0 & 0 & 0 & \dots & 0 \\ 0 & e & \frac{1}{2}f & 0 & x_1 & \dots & x_8 \\ 0 & 0 & \frac{1}{2}c & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2}g & 0 & y_1 & \dots & y_8 \end{bmatrix}$$

where $w_1 = \sum_{n=1}^{8} x_i e_i \in E_8(2)$, $w_2 = \sum_{n=1}^{8} y_i e_i \in E_8(2)$ and $e_1, \cdots, e_8$ standard basis for $E_8(2)$.

A is the embedding matrix for the map defined above. Now take first and fourth column, and take any other two columns from 5th to 12th. By the theorem 2.36, the embedding of $< 2b > \oplus < 2d >$ into $E_8(2)$ is primitive, and by using 2.26 about characterization of primitive embedding of lattices, the greatest common divisor of the maximal minors of this embedding matrix is 1, hence we can conclude that the greatest common divisor of the maximal minors of this embedding matrix A is also 1. Again by using 2.26, this embedding is primitive.

Finally, to prove orthogonal complement of the image of ϕ in Λ^- contains no self intersection -2 vector, let $f = Xu_1 + x'u_2 + Yv_1 + y'v_2 + e \in \Lambda^-$, where $e' \in E_8(2)$ with $e' \cdot e' = -4k,\ k \geq 0$.

Impose the condition that f lies in the orthogonal complement of $\phi\,(T_X)$ in Λ^- and that $f \cdot f = -2$.

Solving the equations $f \cdot \phi(x) = 0$, $f \cdot \phi(z) = 0$ for x', y', from the equation

$$f \cdot \phi(x) = (Xu_1 + x'u_2 + Yv_1 + y'v_2 + e) \cdot (u_1 + au_2) = 0, \tag{4.7}$$

we obtain that

$$x' = -aX. \tag{4.8}$$

From the equation $f \cdot \phi(z) = 0$,

$$f \cdot \phi(y) = (Xu_1 + x'u_2 + y'v_1 + Yv_2 + e) \cdot (\frac{1}{2}cv_1 + v_2),$$

we obtain also that

$$y' = -\frac{1}{2}cY \tag{4.9}$$

and substituting into the equation

$$f \cdot f = 2Xx' + 4Yy' + e \cdot e = -2 \tag{4.10}$$

gives

$$1 - (aX^2 + cY^2) = 2k \geq 0. \tag{4.11}$$

Since this is positive definite, equation (4.11) holds if and only if this form represents 1, and then $k = 0$.

If we assume that the form $aX^2 + cY^2$ does not represent 1, then equation (4.11) cannot be solved, so there is no self intersection -2 vector in the orthogonal complement of $\phi(T_X)$. We are done.

□

Theorem 4.2. *If X is a $K3$ surface with a transcendental lattice given as*

$$\begin{pmatrix} 2a & e & 0 & 0 \\ e & 2b & f & 0 \\ 0 & f & 2c & g \\ 0 & 0 & g & 2d \end{pmatrix},$$

then X covers an Enriques surface if the following conditions hold:

- *a, b, d are even; and g or f is odd; e is even; $b, d < 0$.*

Proof. We will consider a particular embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$.

Let $\{x, y, z, t\}$ be a basis of T_X , $\{u_1, u_2\}$ be a basis of U and $\{v_1, v_2\}$ be a basis of $U(2)$. We can choose two elements w_1 and w_2 of $E_8(2)$ which generate a primitive sublattice of $E_8(2)$ isomorphic to $< 2b > \oplus < 2d >$ by the corollary 2.36.

Define $\phi : T_X \to \Lambda^-$ by

$$\phi(x) = v_1 + \frac{1}{2}av_2, \tag{4.12}$$

$$\phi(y) = fu_1 + \frac{1}{2}ev_2 + w_1, \tag{4.13}$$

$$\phi(z) = cu_1 + u_2, \tag{4.14}$$

$$\phi(t) = gu_1 + w_2. \tag{4.15}$$

It can be shown by direct computation that this is an embedding.

To prove that this embedding is primitive,

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & a/2 & 0 & \dots & 0 \\ f & 0 & 0 & e/2 & x_1 & \dots & x_8 \\ c & 1 & 0 & 0 & 0 & \dots & 0 \\ g & 0 & 0 & 0 & y_1 & \dots & y_8 \end{bmatrix}$$

as an embedding matrix, where $w_1 = \sum_{n=1}^{8} x_i e_i \in E_8(2)$, $w_2 = \sum_{n=1}^{8} y_i e_i \in E_8(2)$ and $e_1, \cdots, e_8$ standard basis for $E_8(2)$. Now take two column, and take other three columns from fifth to twelfth. By the theorem 2.36, the embedding of $< 2b > \oplus < 2d >$ into $E_8(2)$ is primitive, and by using Sertoz Theorem 2.26, the greatest common divisor of the maximal minors of this embedding matrix is 1, hence we can conclude that the greatest common divisor of the maximal minors of the embedding matrix A is also 1. Hence by using Sertoz Theorem 2.26 again, this embedding is primitive.

Finally, to prove orthogonal complement of the image of $\phi(T_X)$ in Λ^- contains no self intersection -2 vector. Suppose g is odd, then let $s = X_1.u_1 + X_2.u_2 + X_3.v_1 + X_4.v_2 + w$ be an element in the orthogonal complement of $\phi(T_X)$, then

$$s.\phi(t) = X_2.g + w.w_2 = 0. \tag{4.16}$$

Since $w.w_2$ is even and g is odd, by solving the equation 4.16 with respect to mod 2, X_2 must be the form $2k$ for some $k \in \mathbb{Z}$.

Similarly, suppose f is odd, then let $s = X_1.u_1 + X_2.u_2 + X_3.v_1 + X_4.v_2 + w$ be an element in the orthogonal complement of $\phi(T_X)$, then

$$s.\phi(y) = X_2.f + w.w_2 = 0. \tag{4.17}$$

Since $w.w_2$ is even and f is odd, by solving the equation 4.17 with respect to mod 2, X_2 must be the form $2k$ for some $k \in \mathbb{Z}$.

Thus $s.s = 2X_1.X_2 + 4X_3.X_4 + w^2 \equiv 0 \mod 4$ and hence cannot be -2. We are done.

□

Chapter 5

Which Algebraic K3 Surfaces with Picard Number $\rho(X) = 18$ Do Not Cover an Enriques Surface

In this section, our purpose is to show following theorems:

Theorem 5.1. *If X is an algebraic K3 surface with an Gram matrix of T_X given in*

$$\begin{pmatrix} 2a & e & f & g \\ e & 2b & h & i \\ f & h & 2c & j \\ g & i & j & 2d \end{pmatrix}, \tag{5.1}$$

then X does not cover an Enriques surface if any of the following conditions hold:

Type I

1. *a, b, c are even and efh is odd.*

2. *a, b, d are even and egi is odd.*

3. *a, c, d are even and fgj is odd.*

4. b, c, d *are even and* hij *is odd.*

Type II

1. a, b *are odd and* e *is odd.*
2. a, c *are odd and* f *is odd.*
3. a, d *are odd and* g *is odd.*
4. b, c *are odd and* h *is odd.*
5. b, d *are odd and* i *is odd.*
6. c, d *are odd and* j *is odd.*

Type III

1. a, h *are odd and* b, c *and* ef *are even.*
2. b, f *are odd and* a, c *and* eh *are even.*
3. c, e *are odd and* a, b *and* fh *are even.*
4. a, i *are odd and* b, d *and* eg *are even.*
5. b, g *are odd and* a, d *and* ei *are even.*
6. d, e *are odd and* a, b *and* gi *are even.*
7. a, j *are odd and* c, d *and* fg *are even.*
8. c, g *are odd and* a, d *and* fj *are even.*
9. d, f *are odd and* a, c *and* gj *are even.*
10. b, j *are odd and* c, d *and* hi *are even.*
11. c, i *are odd and* b, d *and* hj *are even.*

12. d, h *are odd and* b, c *and* ij *are even.*

Type IV

1. a, h *are even and* b, c *and* $e + f$ *are odd.*
2. b, f *are even and* a, c *and* $e + h$ *are odd.*
3. c, e *are even and* a, b *and* $f + h$ *are odd.*
4. a, i *are even and* b, d *and* $e + g$ *are odd.*
5. b, g *are even and* a, d *and* $e + i$ *are odd.*
6. d, e *are even and* a, b *and* $g + i$ *are odd.*
7. a, j *are even and* c, d *and* $f + g$ *are odd.*
8. c, g *are even and* a, d *and* $f + j$ *are odd.*
9. d, f *are even and* a, c *and* $g + j$ *are odd.*
10. b, j *are even and* c, d *and* $h + i$ *are odd.*
11. c, i *are even and* b, d *and* $h + j$ *are odd.*
12. d, h *are even and* b, c *and* $i + j$ *are odd.*

Proof. Let $\{x, y, z, t\}$ be a basis of the transcendental lattice T_X and let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the standard bases of U and $U(2)$, respectively. Then in case of all types above, the embedding of T_X into Λ^- leads us to a contradiction,i.e. there is no such embedding into Λ^-.
Consider the mapping $\phi : T_X \hookrightarrow \Lambda^-$ defined generically;

$$\phi(x) = a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + w_1 \tag{5.2}$$

$$\phi(y) = b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + w_2 \tag{5.3}$$

$$\phi(z) = c_1u_1 + c_2u_2 + c_3v_1 + c_4v_2 + w_3 \tag{5.4}$$

$$\phi(t) = d_1u_1 + d_2u_2 + d_3v_1 + d_4v_2 + w_4 \tag{5.5}$$

where a_i, b_i, c_i, d_i are integers and $w_i \in E_8(2)$. Since it is an embedding we have that,

$$\phi(x) \cdot \phi(x) = 2a_1a_2 + 4a_3a_4 + w_1^2 = 2a \tag{5.6}$$

$$\phi(y) \cdot \phi(y) = 2b_1b_2 + 4b_3b_4 + w_2^2 = 2b \tag{5.7}$$

$$\phi(z) \cdot \phi(z) = 2c_1c_2 + 4c_3c_4 + w_3^2 = 2c \tag{5.8}$$

$$\phi(t) \cdot \phi(t) = 2d_1d_2 + 4d_3d_4 + w_4^2 = 2d \tag{5.9}$$

$$\phi(x) \cdot \phi(y) = a_1b_2 + a_2b_1 + 2a_3b_4 + 2a_4b_3 + w_1w_2 = e \tag{5.10}$$

$$\phi(x) \cdot \phi(z) = a_1c_2 + a_2c_1 + 2a_3c_4 + 2a_4c_3 + w_1w_3 = f \tag{5.11}$$

$$\phi(x) \cdot \phi(t) = a_1d_2 + a_2d_1 + 2a_3d_4 + 2a_4d_3 + w_1w_4 = g \tag{5.12}$$

$$\phi(y) \cdot \phi(z) = b_1c_2 + b_2c_1 + 2b_3c_4 + 2b_4c_3 + w_2w_3 = h \tag{5.13}$$

$$\phi(y) \cdot \phi(t) = b_1d_2 + b_2d_1 + 2b_3d_4 + 2b_4d_3 + w_2w_4 = i \tag{5.14}$$

$$\phi(z) \cdot \phi(t) = c_1d_2 + c_2d_1 + 2c_3d_4 + 2c_4d_3 + w_3w_4 = j \tag{5.15}$$

A_k is the 3×3 matrix obtained from 4×4 Gram matrix A of T_X by removing the k-th row and k-th column of A. Notice that in each type the conditions represent parities of A_1, A_2, A_3, A_4. So it is enough to consider only one case in each type.

Type I

Since a is even by using (5.6) we can say that either a_1 or a_2 is even. We also have the same for b_1, b_2 and c_1, c_2. Without loss of generality, we may assume that a_1 is even. Then since e and f is odd, a_2, b_1 are odd by (5.10), and a_2, c_1 are odd by (5.11). Hence, b_2, c_2 are even. Then h should be even, which leads us to a contradiction.

For other cases in Type I, what we have done for this case will work verbatim. Hence T_X has no embedding into Λ^-.

Type II

Since a and b are odd, by (5.6) a_1, a_2 and by (5.7) b_1, b_2 are odd. This forces e being even in the equation (5.10). This leads us to a contradiction.

For other cases in Type II, what we have done for this case will work verbatim. Hence T_X has no embedding into Λ^-.

Type III

Since a and h are odd, a_1, a_2 are odd by (5.6) and $b_1c_2 + b_2c_1$ are odd by (5.13). Since b and c are even, then there are two cases:

1. b_1 and c_2 are even, b_2 and c_1 are odd,
2. b_2 and c_1 are even, b_1 and c_2 are odd.

In both cases, it forces e and f to be odd. This leads us to a contradiction.

For other cases in Type III, what we have done for this case will work verbatim. Hence T_X has no embedding into Λ^-.

Type IV

Without loss of generality, we may assume that e is odd. a_1 or a_2 are even by (5.6), b_1, b_2 are odd by (5.7) and c_1, c_2 are odd by (5.8), then it forces f being odd in (5.11). Thus, this leads us to a contradiction.

For other cases in Type IV, what we have done for this case will work verbatim. Hence T_X has no embedding into Λ^-.

$\square$

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