ESSAYS ON BILATERAL TRADE WITH DISCRETE TYPES

A DISSERTATION SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

INDUSTRIAL ENGINEERING

By

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October 2019

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

ESSAYS ON BILATERAL TRADE WITH DISCRETE TYPES

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October 2019

Bilateral trade is probably the most common market interaction problem and can be considered as the simplest form of two sided markets where a seller and a buyer bargain over an indivisible object subject to incomplete information on the reservation values of participants. We treat this problem as a combinatorial optimization problem and re-establish some results of economic theory that are well-known under continuous valuations assumptions for the case of discrete valuations using linear programming techniques.

First, we propose mathematical formulation for the problem under dominant strategy incentive compatibility (DIC) and ex-post individual rationality (EIR) properties. Then we derive necessary and sufficient conditions under which ex-post efficiency can be obtained together with DIC and EIR. We also define a new property called *Allocation Maximality* and prove that the Posted Price mechanism is the only mechanism that satisfies DIC, EIR and allocation maximality. In the final part we consider ambiguity in the problem framework originating from different sets of priors for agents types and derive robust counterparts.

Next, we study the bilateral trade problem with an intermediary who wants to maximize her expected gains. Using network programming we transform the initial linear program into one from which the structure of mechanism is transparent. We then relax the risk-neutrality assumption of the intermediary and consider the problem from the perspective of risk-averse intermediary. The effects of risk-averse approach are presented using computational experiments.

Finally, we broaden the scope of the problem and discuss the case in which the seller is also a producer at the same time and consider benefit and cost functions for the respective parties. Starting by a non-convex optimization problem, we obtain an equivalent convex optimization problem from which the problem is solved easily. We also reconsider the same problem under dominant strategy incentive compatibility and ex-post individual rationality constraints to preserve the practicality of all obtained solutions.

Keywords: Bilateral trade, Mechanism design, Robustness, Ambiguity, ϕ-divergence.

ÖZET

AYRIK TİPLİ İKİ TARAFLI TİCARET ÜZERİNE MAKALELER

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En yaygın pazar etkileşimi olduğunu söyleyebileceğimiz iki taraflı ticaret problemi bir satıcı ve bir alıcının karşılıklı değerlerini bilmedikleri durumda bölünemeyen bir nesne üzerinden pazarlık yaptıkları en basit iki taraflı pazar etkileşimi türüdür. Bu problemi kombinatoryal eniyileme problemi olarak ele alıyoruz ve sürekli değerler varsayımı altında iyi bilinen bazı iktisat teorisi sonuçlarını ayrık tip durumu için doğrusal programlama kullanarak yeniden kuruyoruz.

İlk olarak Baskın Strateji Teşvik Uyumluluğu (BTU) ve Nihai Birey Rasyonelliği (NBR) özellikleri altındaki problem için matematiksel formülasyon öneriyoruz. Sonra nihai verimlilik koşulunun BTU ve NBR ile beraber elde edilebileceği gerek ve yeter koşulları türetiyoruz. Bunun yanında ismi "Allocation Maximality" olan yeni bir özellik tanımlıyoruz ve Posted Price mekanizmasının BTU, NBR ve allocation maximality özelliklerini sağlayan tek mekanizma olduğunu kanıtlıyoruz. Son bölümde, katılımcı tipleri üzerinde tanımlanan olasılık dağılımlarından kaynaklanan belirsizliği problem tanımına alıyoruz ve gürbüz problem çözümlerini buluyoruz.

Buna müteakip kendi kazancını enbüyüklemek isteyen arabulucunun bulunduğu iki taraflı ticaret problemini çalışıyoruz. Ağ programlamasını kullanarak elimizdeki doğrusal formulasyonunu en iyi mekanizmanın anlaşılır olduğu bir duruma getiriyoruz. Daha sonra aracının riske duyarsızlık olduğu varsayımını kaldırıp problemi riskten kaçınan aracı gözünden ele alıyoruz. Riskten kaçınan varsayımının sonuçlar üzerindeki etkilerini hesaplama deneyleriyle sunuyoruz.

Son olarak problem kapsamını genişletip satıcının aynı zamanda üretici olduğu ve ilgili taraflar için fayda ve masraf fonksiyonları düşünülen duruma eğiliyoruz. Dışbükey olmayan bir eniyileme probleminden başlayarak kendisine eşdeğer ve kolayca çözülen bir dışbükey eniyileme problemini elde ediyoruz. Tüm sonuçların uygulanabilirliğini korumak adına aynı problemi Baskın Strateji Teşvik Uyumluluğu ve Nihai Birey Rasyonelliği koşulları altında tekrar ele alıyoruz.

Anahtar sözcükler: İki taraflı ticaret, Mekanizma tasarımı, Gürbüzlük, Belirsizlik, ϕ-divergence.

Acknowledgement

At this moment of accomplishment I would like to express my deepest gratitude to my advisor Prof. Mustafa Çelebi Pınar who always stood behind me and our conversations were the remedy for all discouragements I faced.

I would like to thank Assoc. Prof. Alper Şen, Asst. Prof. Kemal Yıldız, Asst. Prof. Ethem Akyol and Assoc. Prof. Türkmen Göksel for accepting to be a member of my examination committee and providing valuable comments.

My friends Ahmed Burak Paç, Ramez Kian, İrfan Mahmutoğullari, Halenur Şahin, Ece Demirci, Gizem Özbaygın, Esra Koca, Milad Maleki and Parinaz Toufani deserve special thanks for their friendship and support.

I keep my special thanks to Cansu Gülcan, Cemal İlhan, Haşim Özlü, Meltem Peker and Okan Dükkancı for being such nice friends. I feel very lucky to have so many great people around me. This list would not be complete without my dear friend and research colleague Halil İbrahim Bayrak.

It was an honor to be a member of Bilkent University IE Department , and I would like to thank each member of the department.

Finally, I acknowledge the people who mean a lot to me, my mother, father, sister and brother. I salute you all for the selfless love, care, pain and sacrifice you did to shape my life.

I owe thanks to a very special person, my wife Nihal, for her continued and unfailing love and support. You have stood by me through all my travails. Thank you for being my muse, editor, proofreader, and sounding board. But most of all, thank you for being my best friend.

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Chapter 1

Introduction

"Mechanism design is an analytical framework for thinking clearly and carefully about what exactly a given institution can achieve when the information necessary to make decisions is dispersed and privately held." — R. Vohra [1]

In general, mechanism design is about investigating the necessary and sufficient conditions to achieve desired social, environmental or economic outcomes under many assumptions such as individuals' self-interest and incomplete information. It can be said that mechanism design provides an optimization framework in strategic level. In the literature, mechanism design is referred to as a subfield of microeconomics and game theory but there is a distinct difference between game theory and mechanism design. While game theory looks for methods to predict the outcome of a given game, mechanism design takes the reverse path. In mechanism design, we start with a given desirable outcome and try to design a game which produces it.

The main challenge in mechanism design is that the individuals' actual preferences are not publicly observable and the individuals are reluctant to reveal them because they do not find it in their interests to do so. As a result, in one way or another, individuals must be encouraged to reveal this information.

Applications of the mechanism design span a wide and diverse variety of disciplines including health care [2], cloud computing [3], job shop scheduling (truthful job scheduling) [4], electric vehicle charging [5], supply chain management [6], vehicle routing problem [7], etc.

Besides the aforementioned areas mechanism design plays a key role in providing analytical framework for many well known problems from the economics literature such as auctions, provision of public goods, bilateral trade and design of voting procedures and markets. In this thesis we focus on bilateral trading problem and its different variants as combinatorial optimization problems.

APPROACH   Throughout the thesis we undertake an investigation of a celebrated problem of micro-economic theory, bilateral trade problem, using tools of network optimization and linear programming in discrete type spaces (where by *type* we understand the private information parameter value that distinguishes an economic agent). While classical results were obtained using calculus tools (see e.g., [8, 9]) we shall use linear programming duality and finite dimensional convex optimization tools to obtain our results in discrete type spaces. The motivation for this approach is that there is no reason to justify the practice that valuations of economic agents are modeled as a continuum while the discrete type setting is more realistic assumption. We also use game theory to model the strategic interactions and behaviors of rational agents.

1.1 Preliminaries

The following concepts and terms are the backbone of mechanism design, and our models in particular, and we will frequently refer them throughout the thesis.

BILATERAL TRADE PROBLEM Bilateral trading problem is the most common market interaction in which a seller and a buyer bargain over an indivisible object, and the valuation of each agent about the object is private information. For example let us consider a bargaining problem between a risk neutral seller and buyer over an indivisible object. Each individual's valuation about the object is assumed to be an independent random variable and private information. These two individuals will participate in some bargaining mechanism to make a decision about two important issues. Should the object be transferred from the seller to the buyer? If the answer is yes, then what is the transfer price? This well-known problem is referred to as "Bilateral Trading problem" in the mechanism design literature.

INCENTIVE COMPATIBILITY Incentive compatibility is one of the main concepts in the mechanism design literature coined by Hurwitz in 1972 [10] and has several different degrees such as Bayesian incentive compatibility and dominant strategy incentive compatibility (DIC). A mechanism is Bayesian incentive compatible if truth telling is a Bayesian Nash equilibrium and the stronger degree, dominant strategy incentive compatibility, means that the telling the truth is a weakly dominant strategy.

INDIVIDUAL RATIONALITY Like the incentive compatibility individual rationality has also different degrees; interim individual rationality and ex-post individual rationality (EIR) which are defined as follows. Interim individual rationality requires that each individual has non-negative expected gains from the trade and ex-post individual rationality means that regardless of the other agent's type, both traders find it beneficial to participate in the bargain.

EX-POST EFFICIENCY In the context of bilateral trade problem, ex-post efficiency roughly means that the buyer gets the object if and only if the buyer's valuation is higher than the seller's.

COMMON PRIOR ASSUMPTION The assumption that each state of the world is an independent draw from a commonly known distribution is called common prior assumption.

The specific definitions, concepts and notations are discussed in the related chapters.

1.2 Outline of the Thesis

Chapter 2 reviews related work in bilateral trade problem and mechanism design with discrete types. In Chapter 3, we investigate the cases where mechanisms satisfying dominant strategy incentive compatibility and ex-post individual rationality properties can exhibit robust performance in the face of imprecision in prior structure. We start with the general mathematical formulation of the bilateral trade problem with DIC, EIR properties. We derive necessary and sufficient conditions for dominant strategy incentive compatible, ex-post individually rational mechanisms to be ex-post efficient at the same time. Then we define a new property called Allocation Maximality, and prove that the Posted Price mechanisms are the only mechanisms that satisfy DIC, EIR and Allocation Maximal properties. We also show that Posted Price mechanism is not the only mechanism that satisfies DIC and EIR properties. The last part of this chapter introduces different sets of priors for agents' types and consequently allows ambiguity in the problem framework. We derive robust counterparts and solve them numerically for the proposed objective function under box and ϕ-divergence ambiguity specifications. Results suggest that restricting the feasible set to Posted Price mechanisms can decrease the objective value to different extents depending on the uncertainty set.

Chapter 4 is devoted to bilateral trade with risk-averse intermediary. We consider bilateral trade of an object between a seller and a buyer through an intermediary who aims to maximize her expected gains as proposed by Myerson and Satterthwaite [11], in a Bayes-Nash equilibrium framework where the seller and buyer have private, discrete valuations for the object. Using duality of linear network optimization, the intermediary's initial problem is transformed into an equivalent linear programming problem with explicit formulae of expected revenues of the seller and the expected payments of the buyer, from which the optimal mechanism is immediately obtained. Then, an extension of the same problem is considered for a risk-averse intermediary. Through a computational analysis, we observe that the structure of the optimal mechanism is fundamentally changed by switching from risk-neutral to risk-averse environment.

In Chapter 5 we consider an extension for the bilateral trade problem where the seller is also a producer, and the optimal mechanism involves a production quantity on the part of seller. In this chapter departing from a non-convex optimization problem, we obtain an equivalent convex optimization problem from which the problem is solved easily. In the second part of this chapter we change our focus from Bayesian setting to dominant strategy framework. We then give the necessary condition for the positive production level under two assumptions related to probability distributions of agents types and the cost and benefit functions of the buyer and seller, respectively. Finally, Chapter 6 concludes.

1.3 Contributions of the Thesis

Below we summarize the main contributions emerging from our work. The central problem in all following results is bilateral trade.

- We present linear programming formulations for three variants of bilateral trade with discrete types.
- The finite dimensional (a consequence of discrete type spaces) convex optimization formulations given in the thesis have the potential to further the application of modern convex optimization to the problems of economic theory.
- We propose necessary and sufficient conditions so that ex-post efficiency can be obtained together with DIC and EIR properties.
- Defining a new property called Allocation Maximality we prove that the Posted Price mechanisms are the only mechanisms that satisfy DIC, EIR and allocation maximal properties.
- By considering box and ϕ-divergence based sets for priors of agents types, we derive robust counterparts of the problem from the perspective of an ambiguity averse intermediary.
- Using linear (and, in particular network programming) duality, we transform the initial linear program of bilateral trade with intermediary into one from which the structure of the optimal mechanism is transparent.
- By relaxing the risk-neutrality of the intermediary in the bilateral trade with intermediary problem, we propose a stochastic programming formulation for the risk-averse version of the problem and discuss the distinct differences in the structure of optimal mechanisms using numerical results.
- Considering extended version of bilateral trade with intermediary problem where the seller is also a producer, we propose an equivalent convex optimization problem for the initial non-convex one from which the problem is solved easily.

Chapter 2

Literature Review

In this section we provide the related literature review with specific attention to discrete type setting. Mechanism design has been the subject of a substantial number of studies and its literature branches out into diverse directions based on the assumptions, objectives types and possible applications. Before narrowing down our focus to bilateral trading problem it worths to make mention of some important features and assumptions in the literature.

One of the main distinctions in mechanism design literature is the types of participants which categorize the problems into two main, discrete and continuous, types. The relevant private information that each agent has is referred to the type of that agent and assumed to be an independent draw from the type set, say T. Therefore if this type set is continuous we deal with continuous type environment and if it is discrete then we are in discrete types setting. The related studies about mechanism design with discrete types are discussed in sections (2.1) and (2.2).

The other distinguishing feature in the literature is whether there exist a money transfer in the mechanism or not. While in most environments money is used as a medium of compensation there are some cases that monetary compensation is not applicable or even is illegal. The possible institutional and/or ethical considerations can be the reasons for this restriction. Promotion of a faculty member, organ donation, political decisions are among the cases that the decisions must be made without monetary transfer [12, 13, 14].

Another interesting direction which is mostly followed by computer scientists is algorithmic mechanism design where the different preferences of different owners of resources or requests are considered in designing an algorithm in a computer network environment. In fact, algorithmic mechanism design seeks for an algorithm that functions well assuming strategic selfish behavior of each participant. Nisan and Ronen provide a comprehensive presentation of the algorithmic mechanism design in their paper [4].

Besides all these studies, there have been attempts to extend the well known mechanisms in static environment to dynamic ones. For example, Athey and Segal [15] introduce dynamic generalizations for an efficient, budget-balanced, Bayesian incentive compatible mechanism under very general quasi-linear private-value environments. In fact the central problem that the literature of dynamic mechanism tries to address is the design of incentive compatible mechanisms in a dynamic environment in which agents sequentially receive private information over time. Bergemann and Välimäki [16] provide an overview about the basic questions and modeling issues that arise by shifting from static paradigm to a dynamic one.

There are several common measures used in the literature to define the objective function of the problem. Profit and revenue maximizing functions can be referred as the most applied objective types [17, 18, 19, 20]. Other common objective functions are *minmax* or *maxmin* types which are applicable under uncertainty to achieve robust mechanisms or used to reflect the ambiguity averse behaviors of the agents [21, 22, 23]. In addition, some researchers consider a social goal in their studies [24, 25, 26, 27]. Although welfare maximizing objective is the most common goal, some studies consider non-welfare maximizing social goals. For instance, Lavi [28] proposes to consider a social goal different from welfare maximization, namely *makespan* minimization for the task assignment problem in the scheduling domain. The proposed goal aims to construct a balanced allocation, in order to minimize the completion time of the last task.

2.1 Mechanism Design with Discrete Type

Recently Vohra [1, 29] developed a linear programming approach to tackle problems in economics under discrete type spaces. His line of research was then followed by others who investigate some celebrated problems in the literature.

Bayrak and Pınar [30] re-examine the optimal mechanism from [29] and arrive at a conclusion that the second price auction for the sale of a single good through a Bayesian incentive compatible mechanism that maximizes expected revenue of the seller is suboptimal since the principal can do better with a slight modification. They also show that their proposed variant of the second price auction is related to the widely used generalized second price auction mechanism in keyword-auctions for advertising.

Koçyiğit et al. [31] consider maximizing the worst case revenue in an auction with single seller and multiple buyers where all agents are ambiguity-averse, and formulate this problem as a mixed integer programming problem. They also propose a hybrid algorithm to compute the optimal solutions in a significantly shorter times compared to the general purpose MIP solvers.

Bayrak et al. [14] study the allocation problem where a principal has a good to allocate among a set of agents who have a private valuation for receiving the good. In the investigated problem, the principal can check the truthfulness of the agents' value declarations at a cost instead of using monetary transfers. They assume that the agents' valuations are randomly drawn from a discrete set of values, which is not known but can be one of a set of distributions. They also present a robust allocation mechanism by maximizing the worst-case expected value of the principal under two assumptions on the set of distributions.

Augustynczik et al. [27] propose a mechanism design approach for the implementation of biodiversity conservation policies. In their problem the biodiversity is supplied as a single indivisible unit and the government defines a discrete level of biodiversity to be supplied in public forests. They propose a mechanism to levy funds to cover the costs of the biodiversity-oriented forest management. In their setting the agents have quasilinear utilities and are risk-neutral and the proposed mechanism design framework is applied to a temperate forest landscape in southwestern Germany.

Duives et al. [32] apply mechanism design approach for the sequencing problem. They consider a single-server setting where jobs require compensation for waiting and waiting cost is private information to the jobs. The proposed model aims to find a Bayes–Nash incentive compatible mechanism that minimizes the total expected payments to the jobs. They also show that the problem is solvable in polynomial time, by a version of Smith's rule. Later, Hoeksma and Uetz [33] studied generalized version of the sequencing problem where the types of job-agents including processing times and waiting costs are private to the jobs. They also showed that the problem can be solved in polynomial time by linear programming techniques.

Li et al. [34] investigate mechanism design applicability in assembly production systems. The authors propose a contracting mechanism for the assembler's contract design problem. The objective is to maximize the assembler's expected profit and the dominant strategy incentive compatible consideration guarantees that all suppliers truthfully reveal their own production costs. In order to simplify the proposed mechanism they introduce a hybrid mechanism. In the proposed hybrid mechanism the complexity of the contract offered to a given supplier is related the importance of that supplier to the assembler's overall profit.

2.2 Bilateral Trade Problem

One of the pioneering studies in bilateral trading problem was done by Myerson and Satterthwaite [11]. A well-cited result of Myerson and Satterthwaite shows that it is impossible to design an ex-post efficient Bayesian transfer mechanism for an object between a seller and a buyer with private valuations, with the following properties: both parties reveal their true valuations in equilibrium and both parties find it beneficial to participate. The result is known as the Myerson and Satterthwaite impossibility theorem. However, the same reference establishes that an optimal mechanism – optimal from the view point of the intermediary – can be defined where both parties achieve non-negative utilities in expectation, and declare their true valuations in equilibrium.

Later, Hagerty and Rogerson [35] criticized this study in particular and mechanisms with common prior assumption in general for the following reasons: most of the time it is hard to derive exactly the traders' priors or it is possible that we encounter with a variety of priors over time. So the authors proposed an alternative mechanism which shows robust performance with respect to variations in prior structure.

In their mechanism, the Bayesian incentive compatible and interim individual rationality properties are replaced with dominant strategy incentive compatibility and ex-post individual rationality, respectively.

The aforementioned pioneering studies have been inspiring for many researchers to study mechanisms for bilateral trading problem. However, there is only a handful of research papers concerning the bilateral trading problem under discrete type setting. When we look at the literature on bilateral trade problem with discrete types we notice that most of the works focus on Bayesian incentive compatible and interim individually rational mechanisms.

Matsuo [36] considers a bargaining problem between one seller and one buyer for a single object when both agents have two-type private values. The author then finds necessary and sufficient conditions on the agent beliefs so that budget balanced, ex-post efficient mechanism is achievable with Bayesian incentive compatibility and individual rationality properties.

Othman and Sandholm [37] use automated mechanism design to investigate how often the impossibility occurs over discrete valuation domains. They draw samples with respect to different distributions to check the feasibility of ex-post efficient bilateral trade. The main finding of the paper is that in the settings with large numbers of possible valuations (approaching the continuous case) the impossibility appears generally but as the cardinality of the type set decreases the impossibility is observed less frequent.

Kos and Manea [38] prove that there exists an ex-post efficient, budget balanced mechanism if and only if a VCG-like mechanism does not run an expected deficit. The authors also consider the multiple buyers case and the effect of an additional buyer to the existence of ex-post Efficient mechanism. Lastly, the authors deal with the mechanism maximizing total gains from trade.

Flesch et al. [39] focus on ex-post individually rational mechanisms and show that Ex-post efficiency is possible if the cardinality of the type set is less than or equal to five. Later, Flesch et al. [40] proved that for any ex-post efficient mechanism, there exists prior distributions such that it is also Bayesian incentive compatible and interim individually rational.

To the best of our knowledge there are only two studies in the literature that consider dominant strategy incentive compatible and ex-post individually rational mechanisms with discrete types; Carroll [41] and Pınar [42]. Carroll [41] considers a nontrivial case when each agent has two types and shows that first-best welfare (ex-post efficiency) is infeasible while Pınar [42] considers the robust trade mechanisms in the presence of an intermediary and gives the characterization of the optimal robust trade as the solution of a simple linear program when budget balance requirement is relaxed to feasibility.

Against this background we investigate different versions of bilateral trade problem. We use linear programming duality and finite dimensional convex optimization tools to obtain our results in discrete type spaces. The problem is also explored based on different objective functions and under risk-averse and ambiguity-averse agents.

Chapter 3

Robust Bilateral Trade with Discrete Types

The purpose of present chapter is to reconsider properties and results of robust mechanism design for bilateral trading problem under discrete framework, and various specifications for the set of priors. The main contributions and novelty of the current chapter can be summarized as follows. Note that all findings and results are for discrete type setting:

- We propose necessary and sufficient conditions so that ex-post efficiency can be obtained together with DIC and EIR.
- We show by an example that Posted Price mechanisms are not the only DIC, EIR mechanisms, which is the case in continuous type space as proved by [35].
- We define a new property called Allocation Maximality and prove that the Posted Price mechanisms are the only mechanisms that satisfy DIC, EIR and Allocation Maximality properties.

- We consider ambiguity in the problem framework originating from different sets of priors for agents types. Then robust counterparts from the perspective of an ambiguity averse intermediary are derived, and related computational results are discussed.

The rest of this chapter proceeds as follows. In the next section we define the proposed problem and give the related assumptions and concepts. We then formulate the bilateral trade problem under DIC, EIR properties with discrete types. In Section 3.1, we also provide intuition about the necessary and sufficient conditions for a DIC, EIR mechanism to also be ex-post efficient. In Section 3.2, the relations between the newly defined Allocation Maximal property and Posted Price mechanisms are scrutinized, and we prove that the Posted Price mechanisms are the only Allocation Maximal DIC, EIR mechanisms. In Section 3.3, we derive the robust counterparts for the bilateral trade problem while the intermediary wants to maximize seller's expected revenue. The proposed models consider ambiguity under box and ϕ-divergence based sets, respectively. In Section 3.4, computational results are provided, and the performance of the proposed models are compared in terms of their objective function value. Finally, Section 3.5 concludes.

The results of this chapter are published in *Euro Journal on Computational Optimization.*

3.1 Problem Statement

Suppose there is a risk neutral seller who owns an object and a risk neutral buyer who wishes to buy that object. Let i and j denote the value of the object to the seller and the buyer, respectively. These valuations are privately kept by traders. The value that each trader assigns to the object is called type of that trader. The type of each trader is an independent draw from the set $T = \{1, 2, ..., m\}$[1]. Variables p and x are defined to be trade probability and expected payment value, respectively, while g_{ij}^r is the probability mass function for the payment r conditional on the agents types i, j. A mechanism that is dominant strategy incentive compatible and ex-post individually rational should satisfy the following system of non-linear inequalities:

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \qquad \forall i, j, k \in T \tag{3.1}$$

$$jp_{ij} - x_{ij} \geq jp_{ik} - x_{ik} \qquad \forall i, j, k \in T \tag{3.2}$$

$$x_{ij} = p_{ij} \sum_r r g_{ij}^r \qquad \forall i, j \in T \tag{3.3}$$

$$\sum_{r=i}^{j} g_{ij}^r = 1 \qquad \forall i, j \in T \tag{3.4}$$

$$g_{ij}^r \geq 0 \qquad \forall r, i, j \in T \tag{3.5}$$

$$p_{ij} \leq 1 \qquad \forall i, j \in T \tag{3.6}$$

$$p_{ij} \geq 0 \qquad \forall i, j \in T. \tag{3.7}$$

Note that a continuous analog of these constraints is also the starting point of [35]. Obviously, constraints (3.6) and (3.7) ensure that trade probability is between zero and one. Constraint (3.3) calculates the expected payment from trade probability and payment distribution. Constraints (3.4) and (3.5) force g_{ij}^r variables to define a valid probability mass function. It is enough to consider g_{ij}^r variables for $i \leq r \leq j$ because we are interested in EIR mechanisms. Finally, constraints (3.1) and (3.2) represent the dominant strategy incentive compatibility for the seller and the buyer, respectively. These constraints ensure that reporting a different type other than the actual one will result in utility which is less than or equal to the case when the type is truthfully reported for all possible types. It is clear that we are only interested in the mechanisms in which the optimal strategy is to report truthfully. In order to have a linear system of inequalities we want to take out the g_{ij}^r variable

[1]We work with more general discrete type sets in Proposition 3.1 below. However, we prefer the simple type set T not to encumber the notation.

and solve the problem over x_{ij} and p_{ij}. Note that x_{ij} variable should be zero if $p_{ij} = 0$, and otherwise x_{ij} is bounded below and above by ip_{ij} and jp_{ij}, respectively. Therefore, using the following system does not eliminate any EIR mechanisms and also gets rid of the nonlinear equality:

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \qquad \forall i, j, k \in T \tag{3.1}$$

$$jp_{ij} - x_{ij} \geq jp_{ik} - x_{ik} \qquad \forall i, j, k \in T \tag{3.2}$$

$$x_{ij} - ip_{ij} \geq 0 \qquad \forall i, j \in T \tag{3.8}$$

$$jp_{ij} - x_{ij} \geq 0 \qquad \forall i, j \in T \tag{3.9}$$

$$p_{ij} \leq 1 \qquad \forall i, j \in T \tag{3.6}$$

$$p_{ij} \geq 0 \qquad \forall i, j \in T. \tag{3.7}$$

Constraints (3.8) and (3.9) bound the expected payment variable so that it satisfies the EIR conditions. Given a mechanism satisfying the above system, one can easily find the set of all EIR payment distributions g_{ij}^r for all $p_{ij} > 0$ using the following system:

$$\sum_{r=i}^{j} r g_{ij}^r = x_{ij}/p_{ij} \quad \forall i, j \in T$$

$$\sum_{r=i}^{j} g_{ij}^r = 1 \quad \forall i, j \in T$$

$$g_{ij}^r \geq 0 \quad \forall r, i, j \in T.$$

Therefore, we continue our search for DIC, EIR mechanisms by considering the latter system. Next, we will look into the system of inequalities (3.2) and (3.9) which corresponds to the dual constraints of a shortest path problem:

$$jp_{ij} - jp_{ik} \geq x_{ij} - x_{ik} \qquad \forall i, j, k \in T \tag{3.2}$$

$$jp_{ij} \geq x_{ij} \qquad \forall i, j \in T. \tag{3.9}$$

This system is separable for each $i \in T$ so that we can consider each of them separately. Introducing a vertex for each type j and an arc between every successive type $(j+1, j)$ of length $jp_{ij} - jp_{ij+1}$, we will obtain the network in Figure 3.1 for all $i \in T$ (also introduce a dummy node zero).

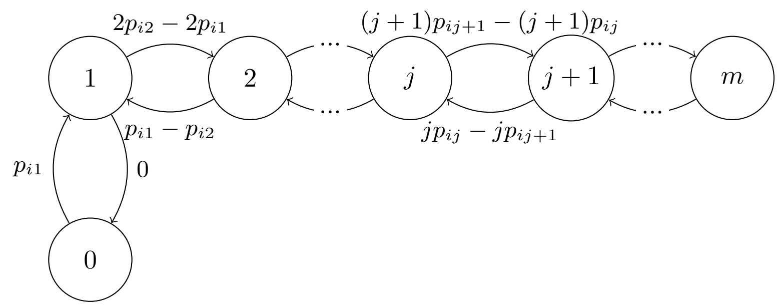

Figure 3.1: Network of types where only the arcs between successive nodes are drawn

Note that this network contains only a subset of the arcs defined by constraints (3.2) and (3.9). Thus, if the corresponding primal shortest path problem is unbounded, constraints (3.2) and (3.9) are infeasible. Then we should not have any negative cost cycles in the network. Let us consider the length of the cycle $j \to j+1 \to j$:

$$(j+1)p_{ij+1} - (j+1)p_{ij} + jp_{ij} - jp_{ij+1} = p_{ij+1} - p_{ij} \geq 0.$$

A network with non-negative cycle costs means that p_{ij} variable should be non-decreasing in $j \in T$. Besides, it can be shown that all shortest paths of the network are represented in the given figure. To see this, consider the length of $j \to j+1 \cdots \to k$ in the given network:

$$\begin{aligned}(j+1)p_{ij+1} - (j+1)p_{ij} + \cdots + kp_{ik} - kp_{ik-1} &= kp_{ik} - (j+1)p_{ij} - \sum_{l=j+1}^{k-1} p_{il} \\ &= kp_{ik} - kp_{ij} - \sum_{l=j+1}^{k-1} (p_{il} - p_{ij}),\end{aligned}$$

which is less than or equal to $kp_{ik} - kp_{ij}$, length of the arc (j,k), since p_{ij} variables are monotone increasing in j. Now we consider the path $j \to j -$

$1 \cdots \to k$:

$$
\begin{aligned}
(j-1)p_{ij-1} - (j-1)p_{ij} + \cdots + kp_{ik} - kp_{ik+1} \quad &= kp_{ik} - (j-1)p_{ij} + \sum_{l=k+1}^{j-1} p_{il} \\
&= kp_{ik} - kp_{ij} + \sum_{l=k+1}^{j-1} (p_{il} - p_{ij}),
\end{aligned}
$$

which is again less than or equal to $kp_{ik} - kp_{ij}$. Since this is true for all arcs, all shortest paths are represented in Figure 3.1. We use this fact in the following manner: take $p_{i0} = 0$, $x_{i0} = 0$ and sum up the constraints corresponding to the shortest path from node 0 to j which is actually the tightest upper bound on x_{ij} variable:

$$
\sum_{k=1}^{j}(kp_{ik} - kp_{ik-1}) = jp_{ij} - \sum_{k=1}^{j-1} p_{ik} \geq x_{ij}.
$$

Similarly by summing up the constraints corresponding to the shortest path from node j to 0, we will obtain:

$$
\sum_{k=1}^{j}(k-1)(p_{ik-1} - p_{ik}) = -(j-1)p_{ij} + \sum_{k=1}^{j-1} p_{ik} \geq -x_{ij},
$$

which turns out to be the tightest lower bound on x_{ij} implied by constraints (3.2) and (3.9). Our analysis on the dual shortest path problem for the buyer's DIC and EIR constraints led us to a relaxation as follows:

$$
\begin{aligned}
& p_{im} \geq p_{im-1} \geq \cdots \geq p_{i2} \geq p_{i1} && \forall i \in T \\
& jp_{ij} - \sum_{k=1}^{j-1} p_{ik} \geq x_{ij} \geq (j-1)p_{ij} - \sum_{k=1}^{j-1} p_{ik} && \forall i, j \in T.
\end{aligned}
$$

Vohra [1] made extensive use of this duality relation to transform the buyer's Bayesian incentive compatibility and interim individual rationality constraints.

Now, we also apply a similar approach to the seller's DIC, EIR constraints which can be written as:

$$
\begin{aligned}
& ip_{kj} - ip_{ij} \geq x_{kj} - x_{ij} && \forall i, j, k \in T && (3.1) \\
& -ip_{ij} \geq -x_{ij} && \forall i, j \in T. && (3.8)
\end{aligned}
$$

Again consider these constraints as the dual of a shortest path problem. For all $j \in T$, this time we will obtain the network in Figure 3.2. Dummy node $m+1$ is connected to node m and p_{im+1}, x_{im+1} are equal to zero.

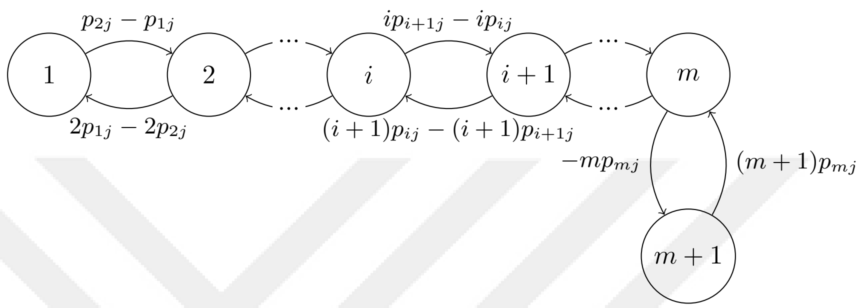

Figure 3.2: Network of types for constraints (3.1) and (3.8)

Let us calculate the cost of path $i \to i+1 \cdots \to m \to m+1$:

$$\sum_{k=i}^{m}(kp_{k+1j} - kp_{kj}) = -ip_{ij} - \sum_{k=i+1}^{m} p_{kj},$$

which is obviously less than the cost of arc $(i, m+1)$ for any $i \in T$. After constructing the network, we utilize the same set of arguments in order to find the following set of inequalities:

$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{m-1j} \geq p_{mj} \qquad \forall j \in T$$
$$\sum_{k=i+1}^{m} p_{kj} + ip_{ij} \leq x_{ij} \leq \sum_{k=i+1}^{m} p_{kj} + (i+1)p_{ij} \quad \forall i, j \in T.$$

No negative cost cycle argument requires p_{ij} to be monotone decreasing on i, and it can be shown that all shortest paths are contained in the given network. The only difference from the previous analysis is that we find the lower bound on x_{ij} by considering the path from node i to $m+1$ following the arcs in Figure 3.2 and upper bound is given by the path from $m+1$ to i.

At this point, we introduce the relaxed formulation which should be satisfied by any DIC, EIR mechanism:

$$p_{im} \geq p_{im-1} \geq \cdots \geq p_{i2} \geq p_{i1} \qquad \forall i \in T \tag{3.10}$$

$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{m-1j} \geq p_{mj} \qquad \forall j \in T \tag{3.11}$$

$$jp_{ij} - \sum_{k=1}^{j-1} p_{ik} \geq x_{ij} \geq (j-1)p_{ij} - \sum_{k=1}^{j-1} p_{ik} \qquad \forall i,j \in T \tag{3.12}$$

$$\sum_{k=i+1}^{m} p_{kj} + ip_{ij} \leq x_{ij} \leq \sum_{k=i+1}^{m} p_{kj} + (i+1)p_{ij} \qquad \forall i,j \in T \tag{3.13}$$

$$p_{ij} \leq 1 \qquad \forall i,j \in T \tag{3.6}$$

$$p_{ij} \geq 0 \qquad \forall i,j \in T. \tag{3.7}$$

A trivial solution of the above system is to set all trading probabilities to zero. Although we do not allow any trade in this mechanism, it satisfies the DIC and EIR conditions. Nobody is ex-post worse off by participating in the trade, and each trader's dominant strategy set contains reporting one's true type. We present three examples in Figure 3.3 in order to investigate the relation between DIC, EIR mechanisms and the relaxed formulation, where $m = 5$. These examples only specify allocation rules but we also need transfer rules to check if the mechanism satisfies DIC, EIR constraints or not. As we shall see below, the relaxed formulation helps us track down the DIC, EIR transfer rules.

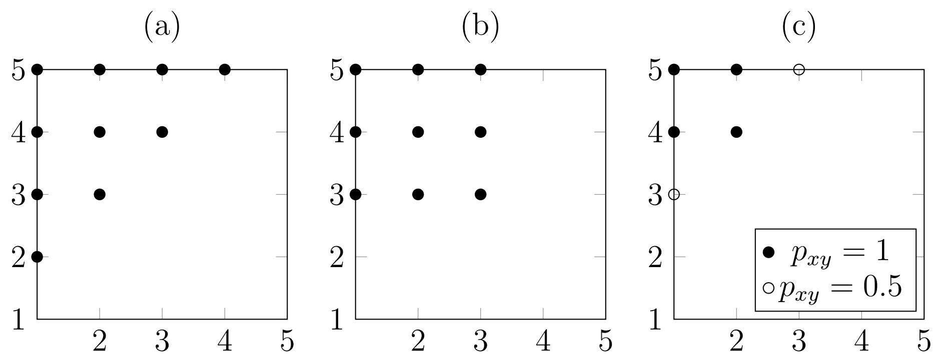

Figure 3.3: Trade probabilities with different properties; (a) ex-post efficient mechanism, (b) Posted Price mechanism, (c) Neither ex-post efficient nor Posted Price mechanism.

Ex-post efficiency dictates that the trade should take place if and only if the buyer has a higher valuation than the seller. Example (a) in Figure 3.3 illustrates an ex-post efficient allocation where the tie break rule leaves the good to the seller. It is easy to check that ex-post efficient mechanism (with any tie break rule) is not feasible in the relaxed formulation because of the constraints (3.12) and (3.13). Therefore, we can conclude that there does not exist any DIC, EIR and Ex-post efficient mechanism when both agents have type set $T = \{1, 2, 3, 4, 5\}$. However this is not true in general, and the following proposition gives conditions using general discrete type sets T_b and T_s (not necessarily the first m integers), for buyer and seller respectively, so that ex-post efficiency can be obtained together with DIC and EIR.

Proposition 3.1. *For finite type sets T_b and T_s with strictly positive elements, there exists a DIC, EIR, Ex-post efficient mechanism if and only if the convex hull of agents' efficient type sets which are defined as $T_b^* = \{b_k \in T_b | b_k > s_l$ for some $s_l \in T_s\}$ and $T_s^* = \{s_k \in T_s | s_k < b_l$ for some $b_l \in T_b\}$ have finite intersection.*

Proof. Assume that there exists a DIC, EIR and Ex-post Efficient mechanism

(p^*, x) but convex hull of sets T_b^* and T_s^* have infinite intersection. Then there exist $b_j \in T_b^*$ and $s_i \in T_s^*$ such that b_j is strictly less than s_i. By definition of efficient type sets, there exist types $s_l \in T_s$ and $b_k \in T_b$ satisfying $s_l < b_j$ and $b_k > s_i$. Then we can write $s_l < b_j < s_i < b_k$ so that $p_{lj} = p_{lk} = p_{ik} = 1$ holds. We know from Lemma 3.1 that $x_{lj} = x_{lk} = x_{ik}$ should also hold in order to satisfy DIC constraints. Given all these information, let us check EIR constraints. We see that $b_j \geq x_{lj} \geq s_l$ and $b_k \geq x_{ik} \geq s_i$ cannot be satisfied together with $x_{lj} = x_{ik}$ since we have $b_j < s_i$. Hence there is no transfer rule we can use together with p^* to have a DIC, EIR mechanism. This is a contradiction.
Now we start from efficient type sets T_b^* and T_s^* whose convex hulls have finite intersection. If both efficient type sets are empty, we have a trivial case $b_m \leq s_1$ where seller always values the good more. Then any Posted Price mechanism imposes Ex-post Efficiency. In the nontrivial case, both sets are nonempty and minimum type, $\underline{b}$, in T_b^* should be bigger than or equal to maximum type, $\bar{s}$, in T_s^*. Here any Posted Price mechanism with unique price $x \in [\bar{s}, \underline{b}]$ will be Ex-post efficient. Since all Posted Price mechanisms are DIC, EIR the proof is complete. □

As an immediate result of this proposition, if the buyer and seller have a common type set $T = \{1, 2, \ldots, m\}$, which is the case in the current study, Ex-post efficiency can be obtained when $m \leq 3$. In three types case, the posted price will be equal to 2 and efficient types will be $\{1, 2\}$ for the seller and $\{2, 3\}$ for the buyer. Adding an extra type 4 will result in efficient type sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ whose convex hulls have infinite intersection.

The other two examples in Figure 3.3, (b) and (c), only specify allocation variables but one can use the relaxed formulation to elicit transfer variables. When p_{ij} values of example (b) are written in the relaxed formulation, it is easy to see that the only feasible solution is setting x_{ij} equal to three whenever p_{ij} is equal to one. This is actually the Posted Price mechanism with price set to three and it is a DIC, EIR mechanism. Similarly, when we use p_{ij} values in example (c), we see that the relaxed formulation gives $x_{13} = 1$, $x_{24} = 3$, $x_{35} = 2$. For other transfer variables, we find following intervals, $x_{15} \in [2.5, 3.5]$, $x_{14} \in [2.5, 3]$, $x_{25} \in [3, 3.5]$. We use another characteristic from DIC mechanisms to find the unique solution in this case.

Lemma 3.1. *When all elements in finite type set T are strictly positive, any DIC mechanism has $x_{ij} = x_{kj}$ if and only if $p_{ij} = p_{kj}$ holds for all $i, j, k \in T$. Similarly, $x_{ij} = x_{ik}$ holds if and only if $p_{ij} = p_{ik}$ is satisfied for all $i, j, k \in T$.*

Proof. Truthful reporting is a weakly dominant strategy if the following set of constraints are satisfied:

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \qquad \forall i, j, k \in T \tag{3.1}$$

$$jp_{ij} - x_{ij} \geq jp_{ik} - x_{ik} \qquad \forall i, j, k \in T. \tag{3.2}$$

For any pair of types $i, k \in T$, we have the following two constraints from inequality (3.1):

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \qquad \forall j \in T$$

$$x_{kj} - kp_{kj} \geq x_{ij} - kp_{ij} \qquad \forall j \in T.$$

If $x_{ij} = x_{kj}$ holds, we end up with $i(p_{kj} - p_{ij}) \geq 0$ and $k(p_{ij} - p_{kj}) \geq 0$. Then for any $j \in T$, we should also have $p_{ij} = p_{kj}$ since all elements in T are strictly positive. Other parts can be proven similarly. □

The intuition behind Lemma 3.1 is that whenever one of these equalities holds, there is a profitable deviation for some type if the other equality does not hold. Therefore, transfer rule in example (c) should be $x_{15} = x_{14} = x_{25} = x_{24} = 3$. Along with this transfer rule, example (c) satisfies DIC, EIR constraints. Note that finding DIC, EIR transfer rules from the relaxed formulation is not generally easy.

Therefore we found a DIC, EIR mechanism, example (c), which is not a Posted Price mechanism. Recall that according to [35] every DIC, EIR mechanism is a Posted Price mechanism when agents have continuous type space. Our example (c) showed that DIC, EIR constraints for the discrete type space are also satisfied by other solutions, a testimony to the discrepancy between continuous and discrete type space. In the following section we will use the proposed relaxed formulation to show that Posted Price mechanisms can be formulated exactly.

3.2 Posted Price and Allocation Maximal Mechanisms

In this section, we show that using the constraints of the relaxed formulation, we can formulate Posted Price mechanisms. We start our discussion by referring to the following set of inequalities as the final relaxed formulation (FRF). We get rid of transfer variables and use their upper and lower bounds given in (3.12) and (3.13) to come up with constraint (3.14). Obviously any DIC and EIR mechanism should satisfy FRF:

$$p_{im} \geq p_{im-1} \geq \cdots \geq p_{i2} \geq p_{i1} \qquad \forall i \in T \tag{3.10}$$

$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{m-1j} \geq p_{mj} \qquad \forall j \in T \tag{3.11}$$

$$(j-i)p_{ij} \geq \sum_{k=1}^{j-1} p_{ik} + \sum_{k=i+1}^{m} p_{kj} \qquad \forall i, j \in T \tag{3.14}$$

$$p_{ij} \leq 1 \qquad \forall i, j \in T \tag{3.6}$$

$$p_{ij} \geq 0 \qquad \forall i, j \in T. \tag{3.7}$$

First, we investigate another set of examples when $T = \{1, 2, 3\}$ in order to clarify the relation between DIC, EIR mechanisms and FRF. Monotonicity and bounding constraints for p variables are obviously satisfied for all three examples in Figure 3.4. We will check the constraint (3.14) for ex-post efficient example (d):

$$2p_{13} \geq p_{11} + p_{12} + p_{23} + p_{33} \quad \text{gives} \quad 2 = 2$$
$$p_{12} \geq p_{11} + p_{22} \quad \text{gives} \quad 1 \geq 0$$
$$p_{23} \geq p_{22} + p_{33} \quad \text{gives} \quad 1 \geq 0.$$

Example (d) is a Posted Price mechanism with unique price two but its tie break rule awards the good to the seller unlike example (e). Posted Price mechanism in example (e) has another characteristic apart from being DIC, EIR, ex-post efficient. It satisfies the constraint (3.14) with equality for all $i, j \in T$. It is easy to see that example (f) also satisfies the constraint (3.14) with equality and we cannot increase any p_{ij} variable without decreasing another one first. A mechanism with no trade also satisfies constraint (3.14) with equality but we can increase p_{1m} as long as $m > 1$. When the cardinality of the type set gets bigger than three we no longer have ex-post Efficiency. However in this case how much efficiency one can capture becomes a relevant question. To answer this question we define the concept of Allocation Maximality and prove that a feasible mechanism in the FRF is Allocation Maximal only if it is a Posted Price mechanism.

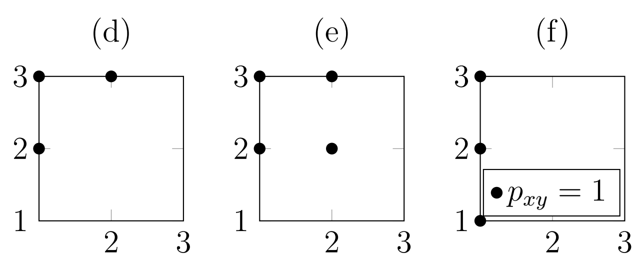

Figure 3.4: Trade probabilities with different properties; (d) Ex-post efficient mechanism, (e) Posted Price mechanism with unique price 2, (f) Posted Price mechanism with unique price 1.

Definition 3.1. *An allocation rule, p^*, that is feasible in FRF is Allocation Maximal if and only if there does not exist any other mechanism, p, feasible in FRF such that $p_{ii} \geq p^*_{ii}$ for all $i \in T$ and $p_{kk} > p^*_{kk}$ for some $k \in T$.*

In order to show the structure of Allocation Maximal mechanisms of FRF we will need the following result.

Lemma 3.2. *The following two equations are equivalent for mechanisms feasible in FRF.*

$$(j-i)p_{ij} = \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} \qquad \forall i,j \in T \tag{3.15}$$

$$p_{ij} = \sum_{k=i}^{j} p_{kk} \qquad \forall i,j \in T. \tag{3.16}$$

Proof. Firstly notice that we can change the constraint (3.14) with the following:

$$(j-i)p_{ij} \geq \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} \quad \forall i,j \in T.$$

We only need to consider p variables that satisfy $i \leq j$ in the right hand side. This is because constraint (3.14) forces p_{ij} to be zero if $i > j$ is satisfied. Now we can continue with the proof.

Equivalence is obvious for the cases when i is greater than or equal to j since neither constraint is restrictive in this case. Therefore, we will consider remaining cases. Assume that (3.15) holds for all $i, j \in T$. We will use induction to show that if (3.15) holds, then (3.16) also holds. For the base case, $j = i + 1$, equivalence is simple:

$$p_{ij} = \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} = \sum_{k=i}^{j} p_{kk}.$$

Assume that (3.16) holds for all $i, j \in T$ such that $j \leq i + q$. Then consider $j = i + q + 1$:

$$\begin{aligned}
(q+1)p_{ij} &= \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} = \sum_{k=i}^{j-1}\sum_{l=i}^{k} p_{ll} + \sum_{k=i+1}^{j}\sum_{l=k}^{j} p_{ll} \\
&= \sum_{k=i}^{j-1}(j-k)p_{kk} + \sum_{k=i+1}^{j}(k-i)p_{kk} = (j-i)\sum_{k=i}^{j} p_{kk} \\
&= (q+1)\sum_{k=i}^{j} p_{kk}.
\end{aligned}$$

Now assume that (3.16) holds for all $i, j \in T$. Then we can rewrite the right hand side of (3.15) as:

$$\begin{aligned}
&\sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} = \sum_{k=i}^{j-1}\sum_{l=i}^{k} p_{ll} + \sum_{k=i+1}^{j}\sum_{l=k}^{j} p_{ll} \\
&= \sum_{k=i}^{j-1}(j-k)p_{kk} + \sum_{k=i+1}^{j}(k-i)p_{kk} = (j-i)\sum_{k=i}^{j} p_{kk} = (j-i)p_{ij}.
\end{aligned}$$

□

Proposition 3.2. *An allocation rule that is feasible in FRF is Allocation Maximal if and only if p_{1m} is equal to one and $p_{ij} = \sum_{k=i}^{j} p_{kk}$ holds for all $i, j \in T$.*

Proof. Assume that p is Allocation Maximal but equality (3.16) is not satisfied. Then using Lemma 3.2, we also know equality (3.15) is not satisfied for some $i, j \in T$. We will show that we can increase some p_{ii} and still get feasibility in FRF which contradicts the Allocation Maximality of p.

First, notice that such a profile would have strictly positive difference, $j - i$. If difference is less than or equal to zero then equality (3.15) should be satisfied because of the monotonicity and non-negativity constraints. Then we only need to consider profiles with $j - i > 0$. Consider the profile (x, y) which does not satisfy equality (3.15) and have the minimum difference, $y - x$, among all such profiles:

$$(y - x)p_{xy} > \sum_{n=x}^{y-1} p_{xn} + \sum_{n=x+1}^{y} p_{ny}.$$

Then we know that (3.15) holds for all profiles (k, l) such that $(l - k) < (y - x)$. Using the induction argument from the proof of Lemma 3.2, we can show that equivalence holds for such profiles:

$$(l - k)p_{kl} = \sum_{n=k}^{l-1} p_{kn} + \sum_{n=k+1}^{l} p_{nl} \quad \forall k, l \in T \text{ such that } (l - k) < (y - x)$$

$$p_{kl} = \sum_{n=k}^{l} p_{nn} \quad \forall k, l \in T \text{ such that } (l - k) < (y - x).$$

For profile (x, y), we can write the following:

$$(y - x)p_{xy} > \sum_{n=x}^{y-1} p_{xn} + \sum_{n=x+1}^{y} p_{ny} = (y - x)\sum_{n=x}^{y} p_{nn}.$$

Then using this result and constraint (3.14), we can conclude that:

$$p_{ij} > \sum_{n=i}^{j} p_{nn} \quad \forall i, j \in T \text{ such that } j \geq y \text{ and } i \leq x,$$

which means that $p_{1m} > \sum_{n=1}^{m} p_{nn}$. Now define $\epsilon = 1 - \sum_{n=1}^{m} p_{nn}$ so that we can exhibit a contradiction using p^* defined as follows:

$$p^*_{nn} = p_{nn} + \epsilon/m \qquad \forall n \in T,$$

$$p^*_{ij} = \sum_{n=i}^{j} p^*_{nn} \qquad \forall i, j \in T.$$

Because of the construction of p^*_{ij} variables, we know that monotonicity constraints hold and constraint (3.15) and (3.16) are satisfied with equality. Since $p^*_{ii} > p_{ii}$ for all $i \in T$, the existence of p^* contradicts the Allocation Maximality of p.

Now assume that p is Allocation Maximal, $p_{ij} = \sum_{n=i}^{j} p_{nn}$ holds for all $i, j \in T$ but p_{1m} is less than one. Then we can construct a new allocation rule p^* that is feasible in FRF by increasing p_{nn} for all $n \in T$ by $\epsilon = (1-p_{1m})/m$ as above. By Definition 3.1, p is not Allocation Maximal. This is a contradiction.

Now assume that we have an allocation rule that is feasible in FRF and it satisfies $p_{1m} = 1$ and $p_{ij} = \sum_{n=i}^{j} p_{nn}$ holds for all $i, j \in T$. Using Lemma 3.2, we also have equality (3.15) satisfied for all $i, j \in T$. Assume to the contrary that there exists a p^* feasible in FRF such that $p^*_{ii} \geq p_{ii}$ for all $i \in T$ and $p^*_{kk} > p_{kk}$ for some $k \in T$. Then we have the following inequality:

$$\sum_{n=1}^{m} p^*_{nn} > \sum_{n=1}^{m} p_{nn} = p_{1m} = 1.$$

Using induction argument as in the proof of Lemma 3.2, one can also show that $p^*_{ij} \geq \sum_{n=i}^{j} p^*_{nn}$ should hold for any $i, j \in T$. Therefore, $p^*_{1m} \geq \sum_{n=1}^{m} p^*_{nn} > 1$, which means p^* is not feasible in FRF and this is a contradiction. $\square$

We now show that all Allocation Maximal allocation rules in FRF are Posted Price mechanisms. We first need to define the Posted Price mechanism in general form. The seller (or the intermediary, if there is one) announces that he will post a price according to some distribution F and its probability mass function f. After observing the posted price, the buyer and the seller decide if they want to trade or not. Assuming that agents always favor trade more than status quo, we can write the Posted Price mechanism as:

$$p_{ij} = F(j) - F(i-1), \quad x_{ij} = \sum_{n=i}^{j} n f_n, \qquad \forall i, j \in T.$$

In other words, trade probability, p_{ij} is equal to the probability that posted price is in the set $\{i, i+1, \ldots, j-1, j\}$. Transfer value, x_{ij}, is equal to expected payment with respect to posted price probability mass function. The above definition of Posted Price mechanism allows the seller (intermediary) to pick a price distribution which will enable him to randomize the posted price he will announce.

Proposition 3.3. *A DIC, EIR mechanism is Allocation Maximal if and only if it is a Posted Price mechanism with the price mass function* $\sum_{n=1}^{m} f(n) = 1$ *where trade is preferred to status quo.*

Proof. Assume that a DIC, EIR mechanism (p, x) is Allocation Maximal. Then allocation rule p should be feasible in the FRF. By Proposition 3.2, we have $p_{1m} = 1$ and $p_{ij} = \sum_{n=i}^{j} p_{nn}$ holds for all $i, j \in T$. From constraints (3.12) and (3.13), we can write the following bounds for the transfer rule:

$$jp_{ij} - \sum_{k=i}^{j-1} p_{ik} \geq x_{ij} \geq \sum_{k=i+1}^{j} p_{kj} + ip_{ij}$$

$$j\sum_{n=i}^{j} p_{nn} - \sum_{k=i}^{j-1}\sum_{n=i}^{k} p_{nn} \geq x_{ij} \geq \sum_{k=i+1}^{j}\sum_{n=k}^{j} p_{nn} + i\sum_{n=i}^{j} p_{nn}$$

$$j\sum_{n=i}^{j} p_{nn} - \sum_{k=i}^{j-1}(j-k)p_{nn} \geq x_{ij} \geq \sum_{k=i+1}^{j} (k-i)p_{nn} + i\sum_{n=i}^{j} p_{nn}$$

$$\sum_{n=i}^{j} np_{nn} \geq x_{ij} \geq \sum_{n=i}^{j} np_{nn}.$$

We see that there is only one transfer rule feasible in the relaxation. This mechanism is equivalent to the following Posted Price mechanism with probability mass function f:

$$f_i = p_{ii} \quad \forall i \in T \qquad \Rightarrow \qquad p_{ij} = F(j) - F(i-1), \quad x_{ij} = \sum_{n=i}^{j} nf_n, \qquad \forall i, j \in T.$$

Since p_{1m} is equal to one, we have $\sum_{n=1}^{m} f(n) = 1$. This mechanism awards the good to the buyer when both agents have the same type equal to the posted price. In other words, trade is preferred to status quo where seller keeps the good. Since we utilized Proposition 3.2 giving necessary and sufficient conditions, the proof is complete. □

Corollary 3.1. *The following system of equations is DIC-EIR implementable and every feasible solution is a Posted Price mechanism where trade is preferred to status quo.*

$$x_{ij} = jp_{ij} - \sum_{k=i}^{j-1} p_{ik} \qquad \forall\, i, j \in T \tag{3.17}$$

$$x_{ij} = ip_{ij} + \sum_{l=i+1}^{j} p_{lj} \qquad \forall\, i, j \in T \tag{3.18}$$

$$p_{ij} \le 1 \qquad \forall i, j \in T \tag{3.6}$$

$$p_{ij} \ge 0 \qquad \forall i, j \in T. \tag{3.7}$$

$$(3.10)\ ,\ (3.11).$$

The proof directly follows from Lemma 3.2 and the definition of Posted Price mechanism. Restricting the allocation variables to be binary gives all Posted Price mechanisms with unique price where trade is preferred to status quo. Giving positive probability to more than one price might not be preferable due to practical concerns. Therefore, we will also investigate Posted Price mechanisms with a unique posted price and analyze its performance compared to Posted Price mechanism with not necessarily unique price in Section 5.

3.3 Bilateral Trading under Ambiguity

Until this point, we were interested in the general characteristics of DIC, EIR mechanisms. However, such analysis does not give specific information that a seller would need in practice. In order to specify the optimal trade probabilities and expected transfers, we need an objective function and an assumption about the priors. By relaxing the unique common prior assumption, which is commonly used in the literature, we introduce ambiguity into the problem framework. To deal with non-unique prior we consider bilateral trading problem from the perspective of an ambiguity-averse seller.

As in the paper by Gilboa and Schmeidler [43], we maximize the worst case expected utility of the seller subject to DIC, EIR constraints. The bilateral trade problem with ambiguity-averse agents was also considered by De Castro and Yannelis in [44]. The authors show that when all agents are ambiguity-averse, for some class of max-min preferences DIC, EIR mechanisms are Ex-post efficient. For other examples of mechanism design problems with ambiguity, we refer to [22] and [45]. In the following two sections we consider two types of ambiguity specifications. The first set based on interval uncertainty is one of the most widely used polyhedral uncertainty sets in robust combinatorial optimization literature. Interval uncertainty sets have been applied for a variety of problems in the fields of economics, production, transportation, etc. The reader may refer to study by Kouvelis and Yu [46] for use of robustness approach in different environments. The second set is constructed based on ϕ-divergence ambiguity sets which reflects distributional robustness. As the uncertainty set constructed around the nominal distribution covers all possible probability distributions in that range, the ϕ-divergence based ambiguity region is in accordance with the DIC concept of robust mechanism design.

3.3.1 Bilateral Trading Mechanism under Box Ambiguity Set

In this section we derive the robust counterpart for bilateral trading problem under box ambiguity set. First let us write our objective function as follows:

$$\max_{x,p\in X}\left\{\min_{h\in U}\sum_{i,j}h_{ij}\left(x_{ij}-ip_{ij}\right)\right\}, \tag{3.19}$$

where h_{ij} is density of joint distribution of agents type, X contains the constraints acting on p and x depending on the model used, and U is a set of ambiguity for the prior h and defined as follows:

$$U=\left\{l_{ij}\le h_{ij}\le u_{ij}\ ,\ \sum_{i,j}h_{ij}=1\right\}.$$

In this step we propose a linear programming model for the robust counterpart of this problem using Lagrangian duality. Let us consider the inner part of equation (3.19) separately as follows:

$$\begin{aligned}&\min_{l_{ij}\le h_{ij}\le u_{ij}}\sum_{i,j}h_{ij}(x_{ij}-ip_{ij})\\&s.t:\sum_{i,j}h_{ij}=1.\end{aligned}$$

then the Lagrangian can be written as:

$$\mathcal{L}(h,\mu)=\sum_{i,j}h_{ij}(x_{ij}-ip_{ij})+\mu(\sum_{i,j}h_{ij}-1),$$

and the dual function is:

$$g(\mu)=\min_{h}\mathcal{L}(h,\mu)=-\mu+\min_{h}\sum_{i,j}h_{ij}(x_{ij}-ip_{ij}+\mu),$$

so the Lagrange dual problem is:

$$\max_{\mu}-\mu+\sum_{i,j}\left(l_{ij}(x_{ij}-ip_{ij}+\mu)_{+}+u_{ij}(x_{ij}-ip_{ij}+\mu)_{-}\right),$$

as a result we obtain the following optimization problem as the robust counterpart problem:

$$\max_{x,p\in X,\mu,a,b}\sum_{i,j} -\mu + l_{ij}a_{ij} - u_{ij}b_{ij}$$

$$\begin{aligned} s.t: \quad & x_{ij} - ip_{ij} + \mu = a_{ij} - b_{ij} && \forall i,j \in T, \\ & a_{ij}, b_{ij} \geq 0 && \forall i,j \in T. \end{aligned}$$

3.3.2 Bilateral Trading Mechanism under ϕ-divergence Ambiguity Set

In this section we derive robust counterpart for our objective function under ϕ-divergence-based ambiguity region. Using ϕ-divergence measures, we probabilistically ensure that the ambiguity set contains the true distribution with a desired level of confidence. This is the main advantage of ambiguity sets based on ϕ-divergence measures over those based on box ambiguity. The reader can refer to [47] and [48] for other advantages and applications related to ϕ-divergence measures in robust optimization problems, specially in data-driven setting. The construction of the uncertainty region from the given data is out of scope of this study. However, we refer the interested reader to [49] which explains how to obtain an approximate uncertainty set for probability vectors h around nominal distribution, $\hat{h}$, as confidence set of confidence level at least $(1-\alpha)$, for example. ϕ-divergence measures are commonly used to reflect the distance between two probability distributions and defined as follows:

The ϕ-divergence measure between two probability distributions $h = (h_1, ..., h_n)^T \geq 0$ and $g = (g_1, ..., g_n)^T \geq 0$ in $\mathbb{R}^n$ is

$$I_\phi(h,g) = \sum_{i=1}^{n} h_i \ \phi(\frac{h_i}{g_i}), \ \phi \in \Phi,$$

where Φ is the class of all convex functions $\phi(t)$, $t \geq 0$ such that $\phi(1) = 0$, $0\phi(0/0) = 0$ and $0\phi(p/0) = \lim_{u\to\infty} \phi(u)/u$.

We suppose that h comes from an uncertainty set constructed around a prior which can be derived from historical data, forecasting, simulation, etc., and four well-known ϕ-divergence functionals are applied as a measure of distance. Table 3.1 shows their characteristics (See [49] for other specifications and choices for ϕ). The reader may also refer to [50] for detailed and comprehensive review on this subject.

Consider the following robust linear constraint:

$$(a + Bh)^T x \leq d \qquad \forall h \in \mathcal{M}, \tag{3.20}$$

where $a \in \mathrm{I\!R}^{\mathrm{n}}$, $B \in \mathrm{I\!R}^{\mathrm{n}\times\mathrm{m}}$, $d \in \mathrm{I\!R}$ are given parameters; $h \in \mathrm{I\!R}^{\mathrm{m}}$ is the uncertain parameter; $x \in \mathrm{I\!R}^{\mathrm{n}}$ is the optimization vector and the uncertainty region $\mathcal{M}$ is given by

$$\mathcal{M} = \big\{h \in \mathrm{I\!R}^{\mathrm{m}} |\ h \geq 0,\ e^T h = 1,\ I_\phi(h, g) \leq \rho\big\}, \tag{3.21}$$

where ρ controls the ambiguity level. The large value of ρ means that our confidence in data is low, and small value for ρ indicates that we trust in data.

Ben-Tal et al. [49] prove that:

Theorem 3.1. *A vector $x \in \mathrm{I\!R}$ satisfies (3.20) with uncertainty region $\mathcal{M}$ such that $h \in \mathcal{M}$ if and only if there exist $\eta \in \mathrm{I\!R}$ and $\lambda \in \mathrm{I\!R}$ such that (x, λ, η) satisfies*

$$\begin{cases} a^T x + \eta + \rho\lambda + \lambda \sum\limits_{i=1}^{m} h_i \phi^*(\frac{b_i^T x - \eta}{\lambda}) \leq d, \\ \lambda \geq 0. \end{cases}$$

In Theorem 3.1, b_i are the ith column of B and $\phi^* : \mathrm{I\!R} \to \mathrm{I\!R} \cup \{\infty\}$ is the conjugate function of ϕ which is defined as follows:

$$\phi^*(s) = \sup_{t\geq 0}\{st - \phi(t)\}.$$

Table 3.1: ϕ-Divergence Measures

Divergence measure	$\phi(t)$	$\phi^*(s)$	$I_\phi(h,g)$
Burg entropy	$-log(t)+t-1$	$-log(1-s),\ s<1$	$\sum_i g_i log(\frac{g_i}{h_i})$
Kullback-Leibler	$t\,log(t)-t+1$	e^s-1	$\sum_i h_i log(\frac{h_i}{g_i})$
χ^2-distance	$\frac{1}{t}(t-1)^2$	$2-2\sqrt{1-s},\ s<1$	$\sum_i \frac{(h_i-g_i)^2}{h_i}$
Hellinger-distance	$(\sqrt{t}-1)^2$	$\frac{s}{1-s}, s<1$	$\sum_i(\sqrt{h_i}-\sqrt{g_i})^2$

Now let us reconsider the objective function of proposed problem with the uncertainty region defined by $\mathcal{M}$ as follows:

$$\max_{x,p\in X}\left\{\min_{h\in\mathcal{M}}\sum_{i,j}h_{ij}\left(x_{ij}-ip_{ij}\right)\right\},$$

which is equal to:

$$\max_{x,p\in X,h\in\mathcal{M},\beta}\left\{\beta \mid \sum_{i,j}h_{ij}\left(x_{ij}-ip_{ij}\right)\geq\beta\right\}. \tag{3.22}$$

Using Theorem 3.1 and Table 3.1 we can derive the robust counterpart for (3.22) with different divergence measures as follows:

<u>Burg entropy:</u>

$$\max_{x,p\in X,\lambda\geq 0,\eta}\left\{-\eta-\rho\lambda-\lambda\sum_{i,j}\left(h_{ij}\left(-log(1-(\frac{-\left(x_{ij}-ip_{ij}\right)-\eta}{\lambda}))\right)\right)\right\},$$

<u>Kullback-Leibler:</u>

$$\max_{x,p\in X,\lambda\geq 0,\eta}\left\{-\eta-\rho\lambda-\lambda\sum_{i,j}\left(h_{ij}\left(e^{(\frac{-\left(x_{ij}-ip_{ij}\right)-\eta}{\lambda})}-1\right)\right)\right\},$$

χ^2-distance:

$$\max_{x,p\in X,\lambda\geq 0,\eta}\left\{-\eta-\rho\lambda-\lambda\sum_{i,j}\left(h_{ij}\left(2-2\sqrt{1-(\frac{-(x_{ij}-ip_{ij})-\eta}{\lambda})}\right)\right)\right\},$$

Hellinger-distance:

$$\max_{x,p\in X,\lambda\geq 0,\eta}\left\{-\eta-\rho\lambda-\lambda\sum_{i,j}\left(h_{ij}\left[\frac{(\frac{-(x_{ij}-ip_{ij})-\eta}{\lambda})}{1-(\frac{-(x_{ij}-ip_{ij})-\eta}{\lambda})}\right]\right)\right\}.$$

We solve these models numerically and the results are reported and discussed in the next section.

3.4 Computational Results

In this section we present the computational results related to the problems with the objective functions discussed in previous section. For each problem we construct three models with different constraint sets. Model 1 is the general model for robust bilateral trading model and considers the constraints (3.1), (3.2) and (3.6)-(3.9). We construct Model 2 by considering the constraints given in Corollary 1. This set of constraints lead to Posted Price mechanisms. In Model 3, we consider the same constraints as in Model 2 but p_{ij}'s are defined as binary variables and as a result Model 3 is even tighter than Model 2. This modification results in Posted Price mechanism with unique price which is more applicable. We consider these three models in our computational study to investigate how objective function value is changed if we want to apply the Posted Price mechanism.

In each table, first column is labeled with "m" which denotes the cardinality of set T. The second column entitled "h-distribution" specifies the distribution that h comes from. We consider two types of distributions for this purpose, "Uniform" stands for the uniform distribution such that $h_{ij} = 1/m^2$ and "Normal" refers to the normal distribution with $N \sim (\frac{m}{2}, (\frac{m}{8})^2)$. The last three columns provide objective function values for Models 3, Model 2 and Model 1, respectively. The value between parenthesis in the "$OF_3(x^*)$" column is the unique price that have to be posted in Model 3 at optimality. The problem instances were formulated in GAMS 23.3.3 and solved using BARON ([51]) and COINIPOPT ([52]) solvers.

Table 3.2: Results for models without ambiguity

m	h-distribution	$OF_3(x^*)$	OF_2	OF_1
5	Uniform	0.480 (4)	0.480	0.500
	Normal	0.448 (5)	0.448	0.456
10	Uniform	0.840 (7)	0.840	0.861
	Normal	0.942(8)	0.942	0.953
15	Uniform	1.222 (11)	1.222	1.237
	Normal	1.263 (11)	1.263	1.280
20	Uniform	1.592 (14)	1.592	1.609
	Normal	1.557 (14)	1.557	1.573

In the Table 3.2, we give results for the problem without ambiguity. This helps us to have a clear insight about the behavior of the problem with ambiguity.

Table 3.3: Results for models under box ambiguity

m	h-distribution	r	$OF_3(x^*)$	OF_2	OF_1
5	Uniform	0.5	0.240 (4)	0.240	0.250
		0.25	0.360 (4)	0.360	0.375
		0.1	0.432 (4)	0.432	0.450
	Normal	0.5	0.224 (5)	0.224	0.228
		0.25	0.336 (5)	0.336	0.342
		0.1	0.403 (5)	0.403	0.410
10	Uniform	0.5	0.420 (8)	0.420	0.431
		0.25	0.630 (8)	0.630	0.646
		0.1	0.756 (8)	0.756	0.775
	Normal	0.5	0.471 (8)	0.471	0.477
		0.25	0.707 (8)	0.707	0.715
		0.1	0.848 (8)	0.848	0.858
15	Uniform	0.5	0.611 (11)	0.611	0.619
		0.25	0.917 (11)	0.917	0.928
		0.1	1.100 (11)	1.100	1.114
	Normal	0.5	0.631 (11)	0.631	0.640
		0.25	0.947 (11)	0.947	0.960
		0.1	1.137 (11)	1.137	1.152
20	Uniform	0.5	0.796 (14)	0.796	0.804
		0.25	1.194 (14)	1.194	1.207
		0.1	1.433 (14)	1.433	1.448
	Normal	0.5	0.778 (14)	0.778	0.787
		0.25	1.167 (14)	1.167	1.180
		0.1	1.401 (14)	1.401	1.416

In the Table 3.3, the results for the problem under box ambiguity set are illustrated. The "r" column defines the range of the interval by specifying the upper and lower bounds using the following formulae: $u_{ij} = h_{ij}(1 + r)$ and $l_{ij} = h_{ij}(1 - r)$. We set three values of 0.1, 0.25 and 0.5 for "r" which reflect low, medium and high ambiguity, respectively. Results suggest that it is optimal for Posted Price mechanisms to have unique price.

Results for the problem under different ϕ-divergence measures are summarized in the Tables 3.4-3.7. The column ρ is the same parameter introduced in the set definition (3.21) which determines the uncertainty region around h. The three values that ρ can take are 0.1, 0.01 and 0.001, which correspond to high, medium and low ambiguity, respectively.

As to be expected, the first observation is that as the ambiguity increases, we see that the objective function value decreases for all models and instances. Similarly, when ambiguity decreases the difference between objective function values in all models also decreases and in low level of ambiguity the objective function values for Model 2 and Model 3 are equal in most cases. This valuable result means that when we encounter low level of ambiguity the proposed "Posted Price mechanism with unique price" which is quite common practice can provide a solution without significant loss of profit. We also observe that in the absence of ambiguity Model 2 and Model 3 provide the same solution which means that the Posted Price mechanisms with unique price are the optimal mechanisms. However this is not the case for the models with ambiguity.

In Table 3.8, we summarize the amount of profit loss in percentage caused by the application of the Posted Price mechanism. The "Uncertainty set" column specifies the considered uncertainty set. The "Min", "Max" and "Avg." labels stand for the minimum, maximum and average profit loss in percentage, respectively, considering the instances presented in Tables 3.3 - 3.7. The "Unique

Table 3.4: Results for models under Burg Entropy divergence measure

m	h-distribution	ρ	$OF_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.168 (4)	0.173	0.196
		0.01	0.358 (4)	0.358	0.378
		0.001	0.439 (4)	0.439	0.459
	Normal	0.1	0.146 (4)	0.169	0.195
		0.01	0.318 (5)	0.328	0.346
		0.001	0.404 (5)	0.404	0.419
10	Uniform	0.1	0.284 (7)	0.302	0.323
		0.01	0.620 (7)	0.622	0.642
		0.001	0.766 (7)	0.767	0.788
	Normal	0.1	0.326 (7)	0.343	0.361
		0.01	0.692 (8)	0.695	0.710
		0.001	0.858 (8)	0.858	0.869
15	Uniform	0.1	0.400 (11)	0.427	0.448
		0.01	0.883 (11)	0.892	0.911
		0.001	1.107 (11)	1.107	1.123
	Normal	0.1	0.412 (10)	0.439	0.461
		0.01	0.918 (11)	0.921	0.940
		0.001	1.146 (11)	1.146	1.164
20	Uniform	0.1	0.461 (12)	0.552	0.572
		0.01	1.154 (14)	1.159	1.177
		0.001	1.444 (14)	1.444	1.460
	Normal	0.1	0.423 (12)	0.539	0.559
		0.01	1.124 (14)	1.127	1.146
		0.001	1.409 (14)	1.409	1.427

Table 3.5: Results for models under Kullback-Leibler divergence measure

m	h-distribution	ρ	$OF_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.125 (4)	0.138	0.165
		0.01	0.352 (4)	0.352	0.373
		0.001	0.438 (4)	0.438	0.458
	Normal	0.1	0.107 (4)	0.142	0.170
		0.01	0.312(4)	0.322	0.341
		0.001	0.403 (5)	0.403	0.418
10	Uniform	0.1	0.204 (7)	0.234	0.259
		0.01	0.609 (7)	0.612	0.633
		0.001	0.765 (7)	0.765	0.786
	Normal	0.1	0.249 (7)	0.270	0.293
		0.01	0.681 (8)	0.684	0.700
		0.001	0.857 (8)	0.857	0.868
15	Uniform	0.1	0.283 (10)	0.328	0.351
		0.01	0.866 (11)	0.876	0.894
		0.001	1.105(11)	1.105	1.121
	Normal	0.1	0.295 (10)	0.337	0.362
		0.01	0.900 (11)	0.904	0.924
		0.001	1.144 (11)	1.144	1.162
20	Uniform	0.1	0.363 (13)	0.421	0.444
		0.01	1.132 (14)	1.138	1.157
		0.001	1.441 (14)	1.441	1.458
	Normal	0.1	0.353 (12)	0.413	0.435
		0.01	1.101 (14)	1.106	1.125
		0.001	1.407 (14)	1.407	1.424

Table 3.6: Results for models under χ^2-distance divergence measure

m	h-distribution	ρ	$OF_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.200 (4)	0.200	0.277
		0.01	0.394 (4)	0.394	0.414
		0.001	0.451 (4)	0.451	0.471
	Normal	0.1	0.226 (4)	0.241	0.254
		0.01	0.356 (5)	0.360	0.378
		0.001	0.417 (5)	0.417	0.430
10	Uniform	0.1	0.442 (7)	0.449	0.469
		0.01	0.685 (7)	0.687	0.707
		0.001	0.787 (7)	0.788	0.809
	Normal	0.1	0.492 (8)	0.506	0.521
		0.01	0.766 (8)	0.766	0.779
		0.001	0.883 (8)	0.883	0.894
15	Uniform	0.1	0.626 (10)	0.640	0.660
		0.01	0.983 (11)	0.986	1.004
		0.001	1.141 (11)	1.141	1.156
	Normal	0.1	0.646 (11)	0.660	0.681
		0.01	1.019 (11)	1.020	1.038
		0.001	1.180 (11)	1.180	1.198
20	Uniform	0.1	0.802 (14)	0.831	0.850
		0.01	1.283 (14)	1.284	1.302
		0.001	1.487 (14)	1.487	1.504
	Normal	0.1	0.786 (14)	0.809	0.829
		0.01	1.251 (14)	1.251	1.269
		0.001	1.453 (14)	1.453	1.470

Table 3.7: Results for models under Hellinger-distance divergence measure

m	h-distribution	ρ	$OF_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.066 (4)	0.087	0.106
		0.01	0.309 (4)	0.309	0.329
		0.001	0.422 (4)	0.422	0.442
	Normal	0.1	0.056 (4)	0.094	0.113
		0.01	0.272 (4)	0.284	0.306
		0.001	0.385 (5)	0.386	0.403
10	Uniform	0.1	0.107 (7)	0.147	0.166
		0.01	0.531 (7)	0.535	0.556
		0.001	0.735 (7)	0.736	0.757
	Normal	0.1	0.138 (7)	0.172	0.191
		0.01	0.592 (8)	0.602	0.618
		0.001	0.823 (8)	0.823	0.834
15	Uniform	0.1	0.150 (9)	0.205	0.223
		0.01	0.754 (10)	0.764	0.784
		0.001	1.060 (11)	1.060	1.077
	Normal	0.1	0.155 (10)	0.210	0.229
		0.01	0.780 (11)	0.789	0.809
		0.001	1.098 (11)	1.098	1.116
20	Uniform	0.1	0.192 (12)	0.263	0.280
		0.01	0.939 (14)	0.992	1.011
		0.001	1.382 (14)	1.382	1.399
	Normal	0.1	0.188 (12)	0.256	0.274
		0.01	0.951 (14)	0.964	0.985
		0.001	1.349 (14)	1.349	1.360

Posted Price" column represents the difference between objective function values of Model 3 and Model 1, and the "Posted Price" column provides the difference between objective function values of Model 2 and Model 1. For example in the uncertainty set defined by Burg Entropy divergence measure, on average we lose 7.2% of the objective function value for optimal DIC, EIR mechanism if we insist on a Posted Price mechanism with unique price.

Table 3.8: Profit loss in percentage for different models

Uncertainty set	Unique Posted Price (OF_3)			Posted Price (OF_2)		
	Min	Max	Avg.	Min	Max	Avg.
Box	1.0	4.0	1.8	1.0	4.0	1.8
Burg Entropy	1.1	25.1	7.2	1.1	13.3	3.9
Kullback-Leibler	1.2	37.1	9.2	1.2	16.5	4.8
χ^2	1.1	27.8	4.6	1.1	27.8	3.6
Hellinger	1.0	50.4	14.2	1.0	18.0	5.5

3.5 Conclusion

In this chapter, we focused on the robust bilateral trade problem with discrete types. First, we formulated a general model for DIC, EIR mechanisms and considered its relaxation which proved to be useful in two different ways. Given any allocation rule, the relaxation can be used to find transfer rules that give DIC, EIR mechanisms. Besides, constraints of the relaxation can be used to formulate Posted Price mechanisms which are DIC and EIR. On the other hand, we showed that ex-post efficiency can be obtained together with DIC and EIR if and only if convex hull of agents' efficient type sets have finite intersection. When agents share the same type set with cardinality greater than or equal to four, ex-post efficiency is infeasible but one can consider allocation maximal mechanisms. We showed that the Posted Price mechanisms are not the only DIC, EIR mechanisms but they are the only ones satisfying allocation maximality together with DIC, EIR. Lastly, we introduced different sets of priors and considered the problem in the shoes of ambiguity averse intermediary. To manage the ambiguity in the probability distribution of agents types we derived robust counterparts for the proposed objective function under box and ϕ-divergence ambiguity specifications. We also examined the performance of the proposed robust models based on an extensive numerical study.

Chapter 4

Bilateral Trade with Risk-Averse Intermediary using Linear Network Optimization

In this chapter we start by proposing a linear programming formulation for the bilateral trading problem where risk-neutral intermediary wants to benefit from the difference between the payment of the buyer and the transfer to the seller and maximize her expected gains.

The contribution of the present chapter is twofold: the first one is to re-establish some results of economic theory, that are well-known, under continuous valuations assumptions for the case of discrete valuations of both parties in a bilateral trade framework through an intermediary. While these classical results were obtained using calculus tools (see e.g., [8, 9]) we move on to use linear (and, in particular network programming) duality to transform the initial linear program into one from which the structure of the optimal mechanism is transparent. To be precise, we show that in a mechanism optimal for an intermediary maximizing her expected net gains the trade takes place whenever the value declarations of the buyer and the seller are at least a minimum value apart. This result is consistent with that of the aforementioned study by Myerson and Satterthwaite [8].

As the second contribution, we relax the risk-neutrality assumption of the intermediary and consider bilateral trading problem from the perspective of a risk-averse intermediary. In order to quantify the associated risks we use Conditional Value-at-Risk coherent risk measure and propose a stochastic programming formulation. Then, to investigate the effects of risk-aversion approach we conduct a computational experiment. The results show that the objective function value and allocation rule are affected by the passage from risk-neutral intermediary to risk-averse one.

The rest of this chapter is organized as follows. In Section 4.1, we define the bilateral trading with intermediary problem, provide the related assumptions and concepts and formulate the problem with discrete types. We then propose an equivalent linear programming formulation and investigate the structure of optimal mechanisms. In Section 4.2, a risk-averse optimization model based on Conditional Value-at-Risk coherent risk measure is presented. Section 4.3 is devoted to computational results and compares risk-neutral and risk-averse models in terms of objective function value and allocation rule. Finally, Section 4.4 presents our summary and conclusions.

The results of this chapter are published in *NETWORKS* journal.

4.1 Problem Statement

We work in the Myerson and Satterthwaite [8] bilateral trade setting where incentive constraints are represented by the requirement that truthful reporting of valuations by agents be a Bayesian equilibrium. While the reference [8] is usually cited for the impossibility theorem (with the exception of [9]), we shall concentrate on bilateral trade through an intermediary (*broker* in the jargon of [8]) in this section.

Let us start with the notation. For the sake of simplicity in the formulae we assume that the set of valuations for the seller are $\{1, \ldots, m\}$ while that of the buyer is $\{1, \ldots, n\}$[1]. The seller's probability mass function is denoted as $(f_1, \ldots, f_m)$, and that of the buyer is $(g_1, \ldots, g_n)$. The value that each trader assigns to the object is referred to as the type of that trader. As part of the intermediary's mechanism design problem the decision variables p_{ij} are used to represent the probability that the object is transferred from the seller to the buyer when the seller declares type i and the buyer declares type j. Both f and g vectors are assumed known. Now, we define the decision variables that represents the monetary transfers. Let x_{ij} denote the revenue of the seller when she declares type i and the buyer declares type j, and y_{ij} the payment of the buyer in that case. The remaining variables $\bar{x}_i$, $\bar{y}_j$, $\bar{p}_i$ and $\bar{q}_j$ are defined as follows. The expected revenue of the seller when his/her valuation is equal to i is denoted $\bar{x}_i$, and is given as:

$$\bar{x}_i = \sum_{j=1}^{n} x_{ij} g_j, \quad \forall\, i = 1, \ldots, m. \tag{4.1}$$

Similarly, the expected payment of the buyer when his/her valuation is equal to j, denoted $\bar{y}_j$, is given by

$$\bar{y}_j = \sum_{i=1}^{m} y_{ij} f_i, \quad \forall\, j = 1, \ldots, n. \tag{4.2}$$

The probability of the seller selling the object when his/her valuation is i is

$$\bar{p}_i = \sum_{j=1}^{n} p_{ij} g_j, \quad \forall\, i = 1, \ldots, m. \tag{4.3}$$

[1] This is not a serious restriction. It is only for the sake of simplicity. One can also consider other more general, discrete valuations, which will only change some of the formulae in a straightforward manner.

The probability of the buyer getting the object when his/her valuation is j is

$$\bar{q}_j = \sum_{i=1}^{m} p_{ij} f_i, \quad \forall\, j = 1, \ldots, n. \tag{4.4}$$

The intermediary (broker) aims to maximize her expected gain —the intermediary's gain is the positive difference (retained by her) between the payment of the buyer and the transfer to the seller— subject to Bayesian incentive compatibility and individual rationality constraints on the part of both the seller and the buyer. Hence, we have the following linear programming problem:

$$\max_{x,y,\bar{x},\bar{y},p,\bar{p},\bar{q}} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij} - x_{ij}) f_i g_j$$

subject to (Incentive compatibility for the seller:)

$$\bar{x}_i - i\bar{p}_i \geq \bar{x}_k - i\bar{p}_k \quad \forall k, i = 1, \ldots, m, \tag{4.5}$$

(Incentive compatibility for the buyer:)

$$j\bar{q}_j - \bar{y}_j \geq j\bar{q}_\ell - \bar{y}_\ell \quad \forall \ell, j = 1, \ldots, n, \tag{4.6}$$

(Individual rationality for both:)

$$\bar{x}_i - i\bar{p}_i \geq 0, \quad \forall\, i = 1, \ldots, m, \tag{4.7}$$

$$j\bar{q}_j - \bar{y}_j \geq 0, \quad \forall\, j = 1, \ldots, n, \tag{4.8}$$

in addition to non-negativity restrictions on all p_{ij}s and all p_{ij}s bounded above by one, and defining equations (4.1)–(4.4). Constraints (4.5) and (4.6) ensure that each agent will receive highest expected utility by truthfully reporting his/her type. The left-hand side of both constraints represents the utility gained by truthful reporting of the type while the right-hand side gives the utility expression for reporting any arbitrary type. Therefore truthful reporting can never be worse in terms of utility than concealing one's type. Constraints (4.7) and (4.8) guarantee that any type receives non-negative expected utility by participating in the mechanism. I.e., nobody is worse off by taking part in the mechanism. We refer to the above linear program as (BT1). Now, we will explain the connection to shortest path problem which Vohra [53] introduced for the Bayesian incentive compatibility and interim individual rationality constraints of the buyer. Then we will go into the details as we show that the same idea is also applicable for the seller's constraints. First rewrite the constraints for buyer:

$$j\bar{q}_j - j\bar{q}_\ell \geq \bar{y}_j - \bar{y}_\ell \quad \forall \ell, j = 1, \ldots, n,$$
$$j\bar{q}_j \geq \bar{y}_j \quad \forall j = 1, \ldots, n.$$

Consider a network with set of nodes $\{0, 1, \ldots, n\}$ that contains buyer types and a dummy node 0. Every pair of nodes, (j, ℓ) will be connected by one arc having cost $j\bar{q}_j - j\bar{q}_\ell$. Then above system of constraints with variables $\bar{y}_j$ are nothing but the dual of shortest path problem in this network. Strong duality of linear programming enables us to ensure the feasibility of dual system by simply requiring shortest path problem to be bounded below (see, e.g., [54]). Using this insight, we know that costs of the arcs should not give rise to any negative cost cycles. This is satisfied if and only if variables $\bar{q}_j$ are monotone increasing. Another result uses the same idea utilized by label correcting algorithm which gives the optimal shortest path when there are no negative cost cycles. We know that labels given to each node are updated according to dual constraints. At optimality, label of a node cannot be greater than the shortest path length to that node. Structure of the arc costs at hand gives a unique tight upper bound for each variable $\bar{y}_j$. Hence a direct mechanism which is interim individually rational for the buyer should satisfy the following set of constraints:

$$\bar{q}_n \geq \bar{q}_{n-1} \geq \ldots \geq \bar{q}_1 \geq 0,$$
$$\bar{y}_j \leq j\bar{q}_j - \sum_{\ell=1}^{j-1} \bar{q}_\ell, \quad \forall\, j = 1, \ldots, n.$$

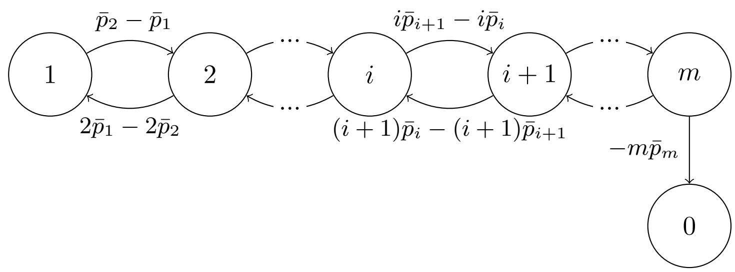

Figure 4.1: Network for the seller's constraints (3.1) and (3.8)

Constraints for the seller can be written as shown below, and we will use the network in Figure 4.1 to analyze the corresponding shortest path problem:

$$i\bar{p}_k - i\bar{p}_i \geq \bar{x}_k - \bar{x}_i \quad \forall k, i = 1, \ldots, m,$$
$$-i\bar{p}_i \geq -\bar{x}_i, \quad \forall\, i = 1, \ldots, m.$$

Although every pair of nodes in the network are connected by an arc, only the arcs represented in Figure 4.1 will construct the shortest paths. This is a direct result of the feasibility condition for the dual problem which is the absence of negative cost cycles. Consider the cost of cycle $i \to i+1 \to i$:

$$i\bar{p}_{i+1} - i\bar{p}_i + (i+1)\bar{p}_i - (i+1)\bar{p}_{i+1} = \bar{p}_i - \bar{p}_{i+1},$$

which holds for any $i \in \{1, \ldots, m-1\}$. Hence we should have variables $\bar{p}_i$ to be monotone decreasing in order to ensure the feasibility of dual system.

Now we can show that all shortest paths are represented in the Figure 4.1 by comparing the cost of path $i \to i+1 \to \cdots \to s-1 \to s$:

$$i\bar{p}_{i+1} - i\bar{p}_i + (i+1)\bar{p}_{i+2} - (i+1)\bar{p}_{i+1} + \cdots + (s-1)\bar{p}_s - (s-1)\bar{p}_{s-1} = (s-1)\bar{p}_s - i\bar{p}_i - \sum_{k=i+1}^{s-1} \bar{p}_k,$$

to the cost of arc (i, s) that is $i\bar{p}_s - i\bar{p}_i$. Since $\bar{p}_i$ variables should be monotone decreasing and nonnegative, the result follows.

In order to find the tightest bound on $\bar{x}_i$ variables, let node 0 to be the destination node and use label correcting algorithm to find shortest paths from all nodes to node 0. Set the label for node 0, d_0 to zero and use the arcs in Figure 4.1 to find the bounds. Since we already know the shortest paths, we know that labels will be assigned according to following equation where c_{i0} is the cost of path $i \to i+1 \to \cdots \to m-1 \to m \to 0$:

$$\begin{aligned}
c_{i0} + d_i &= d_0, \\
i\bar{p}_{i+1} - i\bar{p}_i + (i+1)\bar{p}_{i+2} &- (i+1)\bar{p}_{i+1} + \ldots \\
+(m-1)\bar{p}_m - (m-1)\bar{p}_{m-1} &- m\bar{p}_m + d_i = 0, \\
d_i &= i\bar{p}_i + \sum_{k=i+1}^{m} \bar{p}_k.
\end{aligned}$$

We found that d_i is the tightest lower bound on $\bar{x}_i$ since $c_{ij} + d_i \geq d_j$ should be satisfied for any arc (i, j).

Next we see that $\bar{x}_i$ and $\bar{y}_j$ will be optimized at their respective tightest bound values. Hence we can write the objective as:

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - x_{ij})f_i g_j &= \sum_{j=1}^{n} g_j \bar{y}_j - \sum_{i=1}^{m} f_i \bar{x}_i \\
&= \sum_{j=1}^{n} g_j (j\bar{q}_j - \sum_{\ell=1}^{j-1} \bar{q}_\ell) - \sum_{i=1}^{m} f_i (i\bar{p}_i + \sum_{k=i+1}^{m} \bar{p}_k) \\
&= \sum_{j=1}^{n} g_j j \bar{q}_j - \sum_{j=1}^{n} g_j \sum_{\ell=1}^{j-1} \bar{q}_\ell - \sum_{i=1}^{m} f_i i \bar{p}_i - \sum_{i=1}^{m} f_i \sum_{k=i+1}^{m} \bar{p}_k \\
&= \sum_{j=1}^{n} g_j j \bar{q}_j - \sum_{\ell=1}^{n-1} \sum_{j=\ell+1}^{n} g_j \bar{q}_\ell - \sum_{i=1}^{m} f_i i \bar{p}_i - \sum_{k=2}^{m}\sum_{i=1}^{k-1} f_i \bar{p}_k \\
&= \sum_{j=1}^{n} g_j j \bar{q}_j - \sum_{\ell=1}^{n-1}(1 - G_\ell)\bar{q}_\ell - \sum_{i=1}^{m} f_i i \bar{p}_i - \sum_{k=2}^{m} F_{k-1}\bar{p}_k \\
&= \sum_{j=1}^{n} g_j (j - \frac{1 - G_j}{g_j})\bar{q}_j - \sum_{i=1}^{m} f_i (i + \frac{F_{i-1}}{f_i})\bar{p}_i,
\end{aligned}$$

where we define the cumulative probabilities $G_j = \sum_{\ell=1}^{j} g_\ell$, $\forall j = 1, \ldots, n,$, and $F_i = \sum_{k=1}^{i} f_k, \forall\, i = 1, \ldots, m$. Let

$$\Phi_j = j - \frac{1 - G_j}{g_j}, \quad \forall j = 1, \ldots, n,$$

and

$$\Gamma_i = i + \frac{F_{i-1}}{f_i}, \quad \forall\, i = 1, \ldots, m.$$

Theorem 4.1. *1. The problem (BT1) is equivalently solved by solving the following linear program, referred to as (BT2), in variables* $p_{ij}, \bar{p}_i, \bar{q}_i$, *(where* $0 \leq p_{ij} \leq 1$ *for all* i, j*):*

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} (\Phi_j - \Gamma_i) p_{ij} f_i g_j$$

subject to

$$\bar{p}_i = \sum_{j=1}^{n} p_{ij} g_j, \qquad \forall\, i = 1, \ldots, m,$$
$$\bar{q}_j = \sum_{i=1}^{m} p_{ij} f_i, \qquad \forall\, j = 1, \ldots, n,$$
$$\bar{p}_1 \geq \bar{p}_2 \geq \ldots \geq \bar{p}_m \geq 0,$$
$$\bar{q}_n \geq \bar{q}_{n-1} \geq \ldots \geq \bar{q}_1 \geq 0.$$

2. If Φ_j *and* Γ_i *are both monotone increasing, then the optimal mechanism is as follows;*

$$p_{ij} = \begin{cases} 1, & \text{if } \Phi_j \geq \Gamma_i, \\ 0, & \text{otherwise}, \end{cases}$$
$$y_{ij} = j p_{ij} - \sum_{\ell=1}^{j-1} p_{i\ell},$$
$$x_{ij} = i p_{ij} + \sum_{k=i+1}^{m} p_{kj}.$$

Proof.

The first part of the theorem should be immediate due to the network arguments. In order to prove the second part, ignore the expected allocation variables and monotonicity constraints. Then we need to maximize the objective function subject to $p_{ij} \in [0, 1]$ constraints. Optimal solution will be setting p_{ij} equal to one whenever $\Phi_j \geq \Gamma_i$ holds. It should be zero otherwise. Under the monotone Φ_j and Γ_i assumption, we see that optimal p_{ij} values are monotone increasing in j and monotone decreasing in i. This means that expected allocation variables satisfy the monotonicity constraints. Solution given in part 2 should also be optimal for the $BT2$ problem. □

Compare now the above theorem to the derivation in Chapter 7 of [9] where Spulber defines the analogous Γ_i value as $i + \frac{F_i}{f_i}$ for all $i \in [c_0, c_1]$[2]. While it is not immediately apparent at a first glance, one cannot simply apply the result for continuous type space to the discrete type space since the definition of Γ_i changes in the passage to discrete type space.

The proof also established that the marginal probability of the buyer getting the object is increasing (weakly) with his/her valuation, whereas the marginal probability of the seller losing the object is diminishing with his/her valuation of the object.

As an immediate consequence of Theorem 4.1, we have that if trade takes place then the slack between buyer and seller valuations is at least equal to a certain quantity. However, this does not mean that every time the minimum slack is observed there is trade.

[2]Continuous set of possible buyer types which Spulber refers as opportunity cost.

4.1.1 Illustrative Example

For purposes of illustration consider an example with 9 types for both the buyer and the seller, i.e., the seller values the object as one of $i \in \{1, \ldots, 9\}$ with probabilities $(0.3, 0.3, 0.3, 0.03, 0.02, 0.02, 0.01, 0.01, 0.01)$. The buyer values the object as one of $j \in \{1, \ldots, 9\}$ with probabilities $(0.3, 0.21, 0.2, 0.2, 0.03, 0.02, 0.02, 0.01, 0.01)$. Trade takes place for the (i, j) pairs

$$(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),$$
$$(2,4),(2,5),(2,6),(2,7),(2,8),(2,9) \text{ and } (3,7),(3,8),(3,9)$$

which are the only pairs where the difference $\Phi_j - \Gamma_i$ is non-negative, and hence the corresponding p_{ij} is set to one at optimality. However, considering an example with 20 equally likely types given as $\{1, \ldots, 20\}$ for both seller and buyer, trade takes place whenever the difference between buyer and seller valuations is at least equal to 10, which is in agreement with the example given in [8, 9].

4.2 Bilateral Trade with Risk-Averse Intermediary

In the previous section we proposed a mathematical model which maximizes the expected gain of intermediary. In fact we assumed that the intermediary is completely neutral toward risk and evaluates the uncertain gain function by its expectation. However, we know that the gain of the intermediary is a random outcome and depends on the seller's and the buyer's valuations. Consequently it is quite possible that some cases with low probability have significant impacts on the decision of the risk-neutral intermediary. In addition, in many real-life application especially in economic setting people -even those who do not know much about probabilistic events- are risk-averse to some degree. So it would be perfectly rational if we substitute the risk-neutral intermediary with a risk-averse one in our problem.

Before developing a risk-averse optimization model for our problem we need to answer the question of how we will measure the risk. In fact we need a risk measure which will map from the set of random variables in our problem into real numbers. There are remarkable studies in the literature proposing various risk measures. While each has its own attractive features they may also lack some desired properties. The reader may refer to [55] for comprehensive review on the modeling and optimization of risk-averse preferences. However, the concept of a coherent risk measure which was introduced by Artzner et al. [56] has attracted a special attention because of its desirable properties. A popular example of coherent risk measures is Conditional Value-at-Risk (CVaR) which is widely used in finance applications and is also a good candidate for bilateral trade with a risk-averse intermediary. The CVaR was developed as a remedy to alleviate some problems associated with the Value-at-Risk (VaR) measure commonly used in finance. VaR is a quantile risk measure that gives the loss amount of a financial portfolio for a specified probability. VaR is not in general a convex function in the portfolio positions. Another criticism leveled against VaR was that it ignored the potential magnitude of portfolio losses once the losses hit the critical VaR value. CVaR measures the expected value of portfolio losses given that the critical VaR value has been reached. CVaR is a convex function of portfolio positions and it can also be computed using linear programming in discrete probability spaces. To define the CVaR risk measure let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the sample space, $\mathcal{F}$ is the set of all events defined on Ω and $\mathbb{P}$ is the corresponding probability distribution. We also define $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{F}, \mathbb{P})$ as the set of all probability measures on $(\Omega, \mathcal{F})$. Now the conditional value-at-Risk at confidence level $\alpha \in (0, 1]$ for a random variable Z, $(CVaR_\alpha(Z) : \mathcal{M} \to \mathbb{R})$, is defined as

[57, 58]:

$$\text{CVaR}_\alpha(Z) = \max_{\eta \in \mathbb{R}} \left\{ \eta - \frac{1}{1-\alpha} \mathbb{E}[(\eta - Z)_+] \right\}. \tag{4.9}$$

The optimization problem (4.9) can be linearized using the following form:

$$\begin{aligned} \text{CVaR}(Z) = \max_{\vartheta, \eta} \quad & \eta - \frac{1}{1-\alpha} \sum_{\omega \in \Omega} p_\omega \vartheta_\omega, \\ \text{subject to} \quad & \vartheta_\omega \geq \eta - z_\omega, \quad \forall \omega \in \Omega. \\ & \vartheta_\omega \geq 0, \quad \forall \omega \in \Omega. \end{aligned} \tag{4.10}$$

where z_ω's are the realization of the Z with corresponding probability of p_ω.

Now we can formulate the risk-averse optimization model for our problem as follows:

$$\max_{\vartheta, \eta} \quad \eta - \frac{1}{1-\alpha} \sum_{i=1}^{m} \sum_{j=1}^{n} f_i g_j \vartheta_{ij}, \tag{4.11}$$

$$\text{subject to} \quad z_{ij} = y_{ij} - x_{ij}, \qquad \forall\, i = 1, \ldots, m, \quad \forall j = 1, \ldots, n, \tag{4.12}$$

$$\vartheta_{ij} \geq \eta - z_{ij}, \qquad \forall\, i = 1, \ldots, m, \quad \forall j = 1, \ldots, n, \tag{4.13}$$

$$\vartheta_{ij} \geq 0, \qquad \forall\, i = 1, \ldots, m, \quad \forall j = 1, \ldots, n, \tag{4.14}$$

$$(4.1) - (4.8),$$

in addition that all p_{ij}s are bounded below and above by zero and one, respectively.

As the seller and the buyer remain risk-neutral, the incentive compatibility and individual rationality constraints, (4.1)-(4.8) are also valid for this model.

4.3 Computational Results

In this section we provide the numerical results related to the bilateral trade problem with risk-averse and risk-neutral intermediary. Table 4.1 summarizes these results. The first column of the table entitled "T" specifies the cardinality of the sets where both seller and buyer valuations come from. The second column with "$\sim dis$" label identifies the probability distribution of the seller and buyer valuations. We consider three specifications of distributions for this purpose. "U" stands for the uniform case where $f_i = \frac{1}{T}$ for all i and $g_j = \frac{1}{T}$ for all j. "N" mimics the normal distribution and is defined as follows; at first step we assign the value of $|T|$ to the seller's probability mass function with median index, say $f_{\hat{i}}$. Then, for the remaining elements assign the value using $f_{\hat{i}\pm k} = |T| - 2.|k|$ formula and in the last step normalize them such that all elements sum up to one. The same approach is applied for the buyer's mass functions. Finally, "DnI" refers to the case with increasing (decreasing) values of the seller's (buyer's) mass function probability. The applied rules are $f_i = |i|$ and $g_j = |T| - |j| + 1$, then we also need to normalize the obtained values. The third column with three sub-columns provides the objective function value given by (4.11) and in fact is the gain of the risk-averse intermediary with three different confidence level. The last column illustrates the expected gain of the risk-neutral intermediary. The value between parenthesis, Δ, states the minimum difference between seller's and buyer's valuations where transaction may happen. For example, if the value is three, it means that in the optimal mechanism the transaction will not happen if the difference between the buyer's and seller's valuation is less than three.

We can observe from the table that when the level of risk-aversion increases the related objective function value decreases and the risk-neutral model gives higher objective value compared to the risk-averse one, which were the expected results. We also observe that when level of risk-aversion increases the

Table 4.1: Intermediary's gains under risk-averse and risk-neutral approaches

T	$\sim dis$	$Risk\ Averse\ (\Delta)$ $\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$Risk\ Neutral\ (\Delta)$
5	U	0.266 (2)	0.228 (1)	0.213 (1)	0.320 (2)
	N	0.229 (1)	0.208 (1)	0.191 (1)	0.260 (2)
	D\I	0.083 (1)	0.074 (1)	0.063 (1)	0.106 (2)
10	U	0.426 (3)	0.373 (3)	0.324 (2)	0.550 (5)
	N	0.349 (2)	0.318 (2)	0.295 (1)	0.411 (3)
	D\I	0.089 (1)	0.077 (1)	0.065 (1)	0.124 (3)
15	U	0.574 (5)	0.499 (4)	0.428 (3)	0.746 (7)
	N	0.443 (3)	0.407 (3)	0.374 (2)	0.522 (5)
	D\I	0.108 (2)	0.091 (1)	0.077 (1)	0.158 (5)
20	U	0.732 (6)	0.633 (5)	0.541 (3)	0.962 (10)
	N	0.566 (5)	0.517 (4)	0.470 (3)	0.675 (7)
	D\I	0.127 (2)	0.106 (2)	0.088 (1)	0.189 (7)

Δ value decreases. This means that when the intermediary becomes more risk-averse he prefers to allow the transaction to occur with smaller difference between the valuations of the seller and the buyer.

Another concluding remark which is not present in the table is that, in all instances presented in Table 4.1 the optimal value of the $p_{ij}s$ are zero or one in the risk-neutral model but they take fractional value between zero and one in some cases of the risk-averse model. As a result the optimal mechanism changes for the risk-averse intermediary who decides the allocation through a lottery in some realizations of the types.

4.4 Conclusion

In this chapter, we dealt with bilateral trade with intermediary problem of microeconomic theory using tools of linear (network) programming duality. The problem is one of designing an optimal mechanism from the viewpoint of an intermediary benefiting from the difference between the buyer and seller valuations of the good to be exchanged. More precisely, a seller and a buyer, each withholding private information about an object, exchange it through an intermediary maximizing her expected gains from trade. In a Bayes-Nash equilibrium framework, we derived the optimal exchange mechanism using linear network optimization duality under discrete valuations of the two parties. Then we considered an extension where the intermediary also seeks to avoid risk. We also presented numerical results and discussed the main difference between the optimal mechanism structures of the problems.

Chapter 5

Intermediated Trade with Production

In this section, we shall extend the results of the previous section to the case of a seller who is a producer, following [9]. This extension is important as it finds application in marketing channel. To be more precise the problem environment can be considered as one-level distribution channel where ownership of goods is transfered from the point of production to the point of consumption by intermediary of a broker. The buyer has a willingness-to-buy parameter, and the producer a cost parameter. Both parameters are private information to the respective parties. The mechanism that is designed by the broker determines the level of production, the payment from the buyer to the intermediary, and the payment from the intermediary to the producer while revealing the buyer's willingness-to-buy and the producer's cost and also maximizing the expected revenue of the intermediary.

5.1 Problem Statement

We denote by i the willingness-to-buy (or, type) parameter of the buyer and let the first m integers denote its values. The probability of buyer type i is denoted $f_i > 0$. We use j to represent to cost parameter and the first n integers the values of the cost parameter of the producer. The probability of producer type j is denoted $g_j > 0$. The buyer has a benefit function $B(q, i)$ from output q_{ij}. The producer has a cost function $C(q, j)$. We shall assume that B has the form

$$B(q_{ij}, i) = iq_{ij} + \phi(q_{ij})$$

where ϕ is a concave, continuously differentiable univariate function. Likewise, we shall confine ourselves with cost functions C of the form:

$$C(q_{ij}, j) = jq_{ij} + \gamma(q_{ij})$$

where γ is a convex, continuously differentiable univariate function. This form of functions is referred to as *quasi-linear* in the economics literature.

The broker seeks a mechanism maximizing his/her expected net gain while eliciting incentive compatibility and individual rationality on the part of the buyer and the producer, respectively.

We define the following decision variables. We have a_{ij} to denote the payment by the buyer to the intermediary when the willingness-to-buy is i and the cost is j (or, type realization (i, j)), and likewise, b_{ij} the transfer from the intermediary to the producer under the type realization (i, j). Let

$$a_i^1 = \sum_{j=1}^{n} a_{ij} g_j, \forall i = 1, \ldots, m, \tag{5.1}$$

$$b_j^2 = \sum_{i=1}^{m} b_{ij} f_i, \forall j = 1, \ldots, n \tag{5.2}$$

denote the variables representing the respective expected transfer payments. Then we have the following optimization model referred to as (BtQ):

$$\max_{q_{ij},a_{ij},b_{ij},a_i^1,b_j^2} \sum_{i=1}^{m}\sum_{j=1}^{n} f_i g_j (a_{ij}-b_{ij}) \tag{5.3}$$

subject to

$$\sum_{j=1}^{n} g_j B(q_{ij},i) - a_i^1 \geq \sum_{j=1}^{n} g_j B(q_{kj},i) - a_k^1, \forall k \neq i, \forall k, i = 1,\ldots,m \tag{5.4}$$

$$\sum_{j=1}^{n} g_j B(q_{ij},i) - a_i^1 \geq 0, \forall i = 1,\ldots,m \tag{5.5}$$

$$b_j^2 - \sum_{i=1}^{m} f_i C(q_{ij},j) \geq b_\ell^2 - \sum_{i=1}^{m} f_i C(q_{i\ell},j), \forall \ell \neq j, \forall \ell, j = 1,\ldots,n \tag{5.6}$$

$$b_j^2 - \sum_{i=1}^{m} f_i C(q_{ij},j) \geq 0, \forall j = 1,\ldots,n \tag{5.7}$$

with equations (5.1)-(5.2) and non-negativity of all variables. We note that problem (BtQ) is a non-convex optimization problem. However, we shall show below that it can be solved by solving an equivalent and much simpler (almost) unconstrained convex optimization problem. Thus, problem (BtQ) has hidden convexity. Denote by $\mathbf{Q}$ the matrix with entries q_{ij}, and by F_i and G_j, the cumulative probabilities, respectively. We begin with a simple observation that is useful here and later.

Lemma 5.1. *If all $a_k \geq 0$ (with $b_k \geq a_k$), $f_k \geq 0$ for all $k \in \mathcal{K}$, then the following optimization problem*

$$\min_{y_k} \sum_{k=1}^{m} f_k y_k$$

subject to

$$a_k \leq y_{k+1} - y_k \leq b_k, \forall k = 1,\ldots,m-1,$$

$$y_k \geq 0, \forall k = 1,\ldots,m,$$

admits an optimal solution of the form $y_1^* = 0$ and $y_k^* = \sum_{i=1}^{k-1} a_i$ for $k = 2,\ldots,m$.

Proof. By a change of variables $\Delta_i = y_{i+1} - y_i$ for all $i = 1, \ldots, m-1$, and observing that we can always set $y_1 = 0$ at optimality, we obtain the equivalent problem

$$\min_{\Delta_i} \sum_{i=2}^{m} f_i \left(\sum_{k=1}^{i-1} \Delta_k \right)$$

subject to

$$a_i \leq \Delta_i \leq b_i, \forall i = 1, \ldots, m-1,$$

which is solved at $\Delta_i^* = a_i, i = 1, \ldots, m-1$. □

Theorem 5.1. *Output level* $\mathbf{Q}^*$ *in an optimal mechanism is computed by maximizing the concave function*

$$\Psi(\mathbf{Q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} f_i g_j \left[B(q_{ij}, i) - C(q_{ij}, j) \right] - \sum_{i=1}^{m-1} (1 - F_i) \sum_{j=1}^{n} g_j q_{ij} - \sum_{j=1}^{n-1} G_j \sum_{i=1}^{m} f_i q_{ij+1}$$

over non-negativity restrictions on q_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Proof. First, we eliminate the variables a_{ij} and b_{ij} from the objective function using the defining equations (5.1) and (5.2), and get the equivalent objective function:

$$\max \sum_{i=1}^{m} f_i a_i^1 - \sum_{j=1}^{n} g_j b_j^2$$

over the constraints of non-negativity and (5.4)-(5.5)-(5.6)-(5.7). Now we make the following changes of variables inspired from [59]:

$$y_i \equiv \sum_{j=1}^{n} g_j B(q_{ij}, i) - a_i^1, \forall i = 1, \ldots, m,$$

$$x_j \equiv b_j^2 - \sum_{i=1}^{m} f_i C(q_{ij}, j), \forall j = 1, \ldots, n.$$

Kerkkamp et al. [59] refer to the newly defined variables as *information rent* variables. Substituting for a_i^1 and b_j^2 in the objective function and constraints we obtain the equivalent model after straightforward algebra:

$$\max \sum_{i=1}^{m} f_i(\sum_{j=1}^{n} g_j B(q_{ij}, i) - y_i) - \sum_{j=1}^{n} g_j(\sum_{i=1}^{m} f_i C(q_{ij}, j) + x_j)$$

subject to

$$y_i - y_k \geq (i-k) \sum_{j=1}^{n} g_j q_{kj}, \forall k \neq i, k, i = 1, \ldots, m,$$

$$x_j - x_\ell \geq (\ell - j) \sum_{i=1}^{m} f_i q_{i\ell}, \forall \ell \neq j, \ell, j = 1, \ldots, n,$$

and non-negativity of y_i and x_j, for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ by the individual rationality constraints. Notice that the above model is now in a desired form since we have a concave function to maximize with linear constraints. This simplification is due to the assumed forms of the functions B and C. However, the problem has still too many constraints to yield a useful structure. On the other hand, we can reduce the number of constraints by observing that we have to consider only the consecutive incentive compatibility (IC) constraints. To see this, let us examine the consecutive constraints of the buyer that are re-written as range constraints for fixed q_{ij}s:

$$\sum_{j=1}^{n} g_j q_{ij} \leq y_{i+1} - y_i \leq \sum_{j=1}^{n} g_j q_{i+1j}, \forall i = 1, \ldots, m-1.$$

For ease of notation let us define $c_i = \sum_{j=1}^{n} g_j q_{ij}$. Hence the above range constraints are

$$c_i \leq y_{i+1} - y_i \leq c_{i+1}, \forall i = 1, \ldots, m-1.$$

Feasibility dictates that one should have c_is weakly monotone increasing, i.e., $0 \leq c_1 \leq c_2 \leq \ldots \leq c_m$. Then it is immediate to observe that any y_i values satisfying the consecutive incentive compatibility constraints also satisfy all incentive compatibility constraints for the buyer. To show this, let us consider types i and k, not consecutive with $i < k$. For any feasible y values satisfying the consecutive IC constraints we have the chain of range inequalities:

$$c_i \leq y_{i+1} - y_i \leq c_{i+1},$$

$$c_{i+1} \leq y_{i+2} - y_{i+1} \leq c_{i+2},$$

$$\vdots$$

$$c_{k-1} \leq y_k - y_{k-1} \leq c_k.$$

Adding the inequalities, and simplifying we obtain the range inequality

$$\sum_{l=i}^{k-1} c_l \leq y_k - y_i \leq \sum_{l=i+1}^{k} c_l.$$

However, by monotonicity of c_i we have that the above inequality implies

$$(k-i)c_i \leq y_k - y_i \leq (k-i)c_k,$$

which is exactly the IC constraint for the pair (i,k). By a similar reasoning we can also confine ourselves to consecutive incentive compatibility constraints for the producer:

$$\sum_{i=1}^{m} f_i q_{ij+1} \leq x_j - x_{j+1} \leq \sum_{i=1}^{m} f_i q_{ij}, \forall j = 1, \ldots, n-1.$$

We denote by d_j the quantity $\sum_{i=1}^{m} f_i q_{ij}$. Hence, the inequalities above are re-written for convenience as

$$d_{j+1} \leq x_j - x_{j+1} \leq d_j, \forall j = 1, \ldots, n-1.$$

The next step in the proof is to consider, for fixed $\mathbf{Q}$ the problem

$$\min_{y_i} \sum_{i=1}^{m} f_i y_i$$

subject to

$$c_i \leq y_{i+1} - y_i \leq c_{i+1}, \forall i = 1, \ldots, m-1,$$

and non-negativity of y_is. By the previous lemma, this results in optimal y_i^*s as $y_1^* = 0$ and

$$y_i^* = \sum_{k=1}^{i-1} c_k, i = 2, \ldots, m.$$

By the same token, for fixed $\mathbf{Q}$ we consider the problem

$$\min_{x_j} \sum_{j=1}^{n} g_j x_j$$

subject to

$$d_{j+1} \leq x_j - x_{j+1} \leq d_j, \forall j = 1, \ldots, n-1,$$

where d_j are non-negative and weakly monotone decreasing. Using a proof technique similar to that of Lemma 1 above, an optimal solution has $x_n^* = 0$ and

$$x_j^* = \sum_{l=j+1}^{n} d_l, \forall j = 1, \ldots, n-1.$$

Now, when we substitute the y_i^* and x_j^* values into the objective function, and recalling the definitions of c_i and d_j parameters we obtain the function Ψ after simple algebra. □

We note that the expression for Ψ given in the theorem above is precisely (mutatis mutandis) the discrete type space version of the objective function expression in equation (28) on pp. 120 of [9] for continuous type space. The proof above shows, among other things, that the information rent increases for buyers with increasing willingness-to-buy while it decreases for producers with increasing cost parameter.

Consider an example with ten types for both the buyer and the producer with the following quadratic functions

$$B(q, i) = iq - q^2/2,$$

and

$$C(q, j) = jq + q^2/2.$$

The probabilities f_i are as follows:

$$(0.4, 0.3, 0.17, 0.05, 0.02, 0.02, 0.01, 0.01, 0.01, 0.01),$$

while the g_j are specified as

$$(0.55, 0.18, 0.12, 0.02, 0.06, 0.02, 0.02, 0.01, 0.01, 0.01).$$

Solving the first-order conditions (which result in a simple linear equation for each variable) for each q_{ij} and zeroing out the result if it is less than zero, we obtain the following result: trade takes place at the respective valuations

$$(3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1), (8, 2), (9, 1), (9, 2), (10, 1)(10, 2), (10, 3)$$

with output levels

$$(0.618, 0.643, 0.643, 1.5, 1.5, 2.5, 0.472, 3.5, 1.472, 4.5, 2.472, 0.458).$$

respectively, in the same order of the couples above.

5.2 DIC, EIR Mechanism for Intermediated Trade with Production

In the previous section departing from a non-convex optimization problem, we obtained an equivalent convex one that can be solved easily. However, considering Bayesian incentive compatibility and interim individual rationality properties in some rare cases may lead to impractical solutions. To have a clear idea let us consider the following example with four types for both buyer and producer with the following quadratic functions as the benefit and cost functions, respectively.

$$B(q, i) = iq - q^2/2,$$

and

$$C(q, j) = jq + q^2/2.$$

The probabilities f_i are as follows:

$$(0.235, 0.269, 0.254, 0.242),$$

while the g_j are specified as

$$(0.243, 0.240, 0.246, 0.276).$$

By solving the problem under Bayesian incentive compatibility and interim individual rationality considerations we obtain the following result. The unrepresented values corresponding to the remaining decision variables are zero.

q_{ij}	1	2
3	0.262	0
4	0.75	0.246

i	$\hat{a}i$
1	0
2	0
3	0.175
4	0.750

j	$\hat{b}j$
1	0.461
2	0.134
3	0
4	0

Figure 5.1: Results for an example of intermediated trade with production under BIC and IIR constraints

Although the result seems operational in terms of expected transfer payments but by investigating the obtained solution for each realization of agents types we noticed impracticality in the payments and amount of production. We observe that there exist cases in which there is money transfer without any production, or agents do not pay or receive money while they buy or sell. These cases are clear from the following results related to the optimal values of a_{ij} and b_{ij}.

(i,j)	a_{ij}	(i,j)	b_{ij}
(3,1)	0.718	(1,1)	1.96
(4,1)	3.087	(1,2)	0.569

Figure 5.2: Results for an example of intermediated trade with production

For example, if the buyer has type one, $i = 1$, and seller has type two, $j = 2$, there is a money transfer from the buyer to the broker, $b_{12} = 0.569$, while the seller receives no money. More interestingly the corresponding production level, q_{12}, is also zero which means there is no production but the buyer pays money and receives nothing instead.

Therefore, to impede these outcomes and have a mechanism that provides practical solutions relating to the money transfers and production levels we replace the Bayesian incentive compatibility and interim individual rationality with dominant strategy incentive compatibility and ex-post individual rationality, respectively.

Then we have the following optimization model that maximizes the expected utility of the intermediary subject to DIC and EIR constraints:

$$(M1) \qquad \max_{q_{ij},a_{ij},b_{ij}} \sum_{i=1}^{m} \sum_{j=1}^{n} f_i g_j (a_{ij} - b_{ij}) \tag{5.8}$$

subject to

$$B(q_{ij}, i) - a_{ij} \geq B(q_{kj}, i) - a_{kj}, \forall k \neq i \forall i, j, k \tag{5.9}$$

$$B(q_{ij}, i) - a_{ij} \geq 0, \forall i, j \tag{5.10}$$

$$b_{ij} - C(q_{ij}, j) \geq b_{i\ell} - C(q_{i\ell}, j), \forall \ell \neq j, \forall i, j, \ell \tag{5.11}$$

$$b_{ij} - C(q_{ij}, j) \geq 0, \forall i, j \tag{5.12}$$

and non-negativity of all variables.

Now if we resolve the the last example under the DIC and EIR constraints we obtain the following solution which is practical in terms of money transfer and production level. In all realizations if there is money transaction it happens for the seller and buyer simultaneously and positive production level is only possible with positive money transfer.

q_{ij}	1	2
3	0.262	0
4	0.75	0.246

(i,j)	a_{ij}
(3,1)	0.718
(4,1)	2.175
(4,2)	0.925

(i,j)	b_{ij}
(3,1)	0.331
(4,1)	1.559
(4,2)	0.553

Figure 5.3: Results for an example of intermediated trade with production under DIC, EIR constraints

However, using duality of network optimization and linear programming techniques, this initial problem is transformed into an equivalent problem from which the necessary condition for the mechanisms is achieved under some assumptions.

Theorem 5.2.

The problem (M1) is equivalently solved by solving the following problem, referred to as (M2), in variable $q_{ij} \geq 0$:

$$(M2) \quad \max_{q_{ij}} \sum_{i=1}^{m} \sum_{j=1}^{n} f_i g_j \{B(q_{ij}, i) - \sum_{l=1}^{i-1} q_{lj} - C(q_{ij}, j) - \sum_{l=j+1}^{m} q_{il}\} \tag{5.13}$$

subject to

$$q_{i+1j} \geq q_{ij} \quad \forall i, j \tag{5.14}$$

$$q_{ij-1} \geq q_{ij} \quad \forall i, j \tag{5.15}$$

Proof. We start the proof by considering constraints (5.9) and (5.10) which are corresponding to the dual constraints of a shortest path problem defined based on the following network in which each buyer type, i, is represented by one node and each consecutive pairs of nodes are connected to each other with an arc of length $B(q_{ij}, i) - B(q_{i+1j,i+1})$. Note that this system is separable for each j so that we can consider each of them separately and the network contains only a subset of the arcs defined by constraints (5.9) and (5.10).

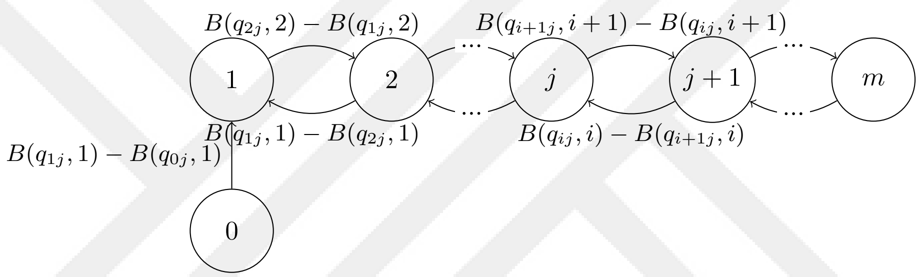

Figure 5.4: Network of buyer types where only the arcs between successive nodes are drawn

Clearly we should not have any negative cycle in the network given by Figure 5.4, otherwise the system of constraints (5.9) and (5.10) would be infeasible. Let us consider the length of the cycle $i \to i+1 \to i$:

$$
\begin{aligned}
&B(q_{i+1j}, i+1) - B(q_{ij}, i+1) + B(q_{ij}, i) - B(q_{i+1j}, i) \\
&= (i+1)q_{i+1j} + \phi(q_{i+1j}) - (i+1)q_{ij} - \phi(q_{ij}) + iq_{ij} + \phi(q_{ij}) - iq_{i+1j} - \phi(q_{i+1j}) \\
&= q_{i+1j} - q_{ij} \geq 0.
\end{aligned}
$$

A network with non-negative cycle costs means that q_{ij} variable should be non-decreasing in i . Besides, it can be shown that all shortest paths of the network are represented in the given

network. To verify this, consider the length of $i \to i+1 \cdots \to k$ in the given network:

$$
\begin{aligned}
&B(q_{i+1j}, i+1) - B(q_{ij}, i+1) + \cdots + B(q_{kj}, k) - B(q_{k-1j}, k) \\
&= (i+1)q_{i+1j} + \phi(q_{i+1j}) - (i+1)q_{ij} - \phi(q_{ij}) + \\
&\cdots + kq_{kj} + \phi(q_{kj}) - kq_{k-1j} - \phi(q_{k-1j}) \\
&= B(q_{kj}, k) - B(q_{ij}, k) - \sum_{l=i+1}^{k-1} (q_{lj} - q_{ij}),
\end{aligned}
$$

which is less than or equal to $B(q_{kj}, k) - B(q_{ij}, k)$, length of the arc (i, k), since q_{ij} variables are non-decreasing in i.
Now we consider the path $i \to i-1 \cdots \to k$:

$$
\begin{aligned}
&B(q_{i-1j}, i-1) - B(q_{ij}, i-1) + \cdots + B(q_{kj}, k) - B(q_{k+1j}, k) \\
&= (i-1)q_{i-1j} + \phi(q_{i-1j}) - (i-1)q_{ij} - \phi(q_{ij}) + \\
&\cdots + kq_{kj} + \phi(q_{kj}) - kq_{k+1j} - \phi(q_{k+1j}) \\
&= B(q_{kj}, k) - B(q_{ij}, k) + \sum_{l=k+1}^{i-1} (q_{lj} - q_{ij}),
\end{aligned}
$$

which is again less than or equal to $B(q_{kj}, k) - B(q_{ij}, k)$. Since this is true for all arcs, all shortest paths are represented in Figure 5.4. We use this fact in the following manner: take $q_{0j} = 0$, $a_{0j} = 0$ and sum up the constraints corresponding to the shortest path from node 0 to i which is actually the tightest upper bound on a_{ij} variable:

$$
\sum_{k=1}^{i} (B(q_{kj}, k) - B(q_{k-1j}, k)) = B(q_{ij}, i) - \sum_{l=1}^{i-1} q_{lj} \geq a_{ij}.
$$

Similarly by summing up the constraints corresponding to the shortest path from node i to 0, we will obtain:

$$
\sum_{k=1}^{i} B(q_{k-1j}, k-1) - B(q_{kj}, k-1) = B(q_{ij}, i-1) - \sum_{l=1}^{i-1} q_{lj} \leq a_{ij},
$$

which turns out to be the tightest lower bound on a_{ij} implied by constraints (5.9) and (5.10). Our analysis on the dual shortest path problem for the buyer's DIC and EIR constraints led us to a relaxation as follows:

$$q_{1j} \leq q_{2j} \leq \cdots \leq q_{i-1j} \leq q_{ij} \qquad \forall i = 1, \ldots, m.$$
$$B(q_{ij}, i-1) - \sum_{l=1}^{i-1} q_{lj} \leq a_{ij} \leq B(q_{ij}, i) - \sum_{l=1}^{i-1} q_{lj} \quad \forall i = 1, \ldots, m \quad \text{and} \quad \forall j = 1, \ldots, n.$$

Now if we apply the similar approach for the seller's DIC and EIR restrictions given by constraints (5.11) and (5.12) we obtain the following set of constraints.

$$q_{im} \geq q_{im-1} \geq \cdots \geq q_{ij+1} \geq q_{ij} \qquad \forall j = 1, \ldots, n.$$
$$C(q_{ij}, j+1) + \sum_{l=j+1}^{m} q_{il} \geq b_{ij} \geq C(q_{ij}, j) + \sum_{l=j+1}^{m} q_{il} \quad \forall i = 1, \ldots, m \quad \text{and} \quad \forall j = 1, \ldots, n.$$

The optimization problem , $M2$, is easily derived using the obtained bounds and monotonicity of variable q with trivial calculations. □

As an immediate result of this theorem we can claim the following corollary:

Corollary 5.1. *If the following two conditions are satisfied, then* $i - j \geq \frac{m-1}{2}$ *is the necessary condition for* $q_{ij} \geq 0$.

1. $\phi(q_{ij}) - \gamma(q_{ij}) \leq 0, \quad \forall qij; \quad \forall i = 1, \ldots, m$ *and* $\forall j = 1, \ldots, n$

2. f_i *and* g_j *are uniformly distributed* $\forall i = 1, \ldots, m$ *and* $\forall j = 1, \ldots, n$

Proof. Follows directly from linear algebra. □

5.3 Conclusion

In this chapter we considered an extension for the intermediated bilateral trade problem where the seller is also a producer, and the optimal mechanism involves a production quantity on the part of seller. Starting by a non-convex optimization problem, we obtained an equivalent convex one that can be solved easily. To preserve the practicality of all obtained solutions we reconsidered the same problem under dominant strategy incentive compatibility and ex-post individual rationality constraints and proposed mathematical formulation. Using duality of network programming we also provided the necessary condition for the optimal mechanisms. This necessary condition is valid for the cases that agents types are uniformly distributed and the concave part of the benefit function is greater than or equal to the convex part of the cost function for all realization of production level.

Chapter 6

Conclusion

In this dissertation we studied three bilateral trade problems under different transaction frameworks and objective functions including ambiguity averse and risk averse ones. The substantive difference that distinguishes our problems from those in the literature is considering the agents type sets as discrete spaces. Our decision to switch from continuous type setting to discrete one has two main reasons; 1) to make the type-sets assumption more realistic and 2) treat the problem as a combinatorial optimization and provide the conducive ground for applying linear programming techniques.

Focusing on the bilateral trade as a central problem, first in Chapter 3 we reconsidered properties and results of robust mechanism design for bilateral trading problem under discrete framework, and various specifications for the set of priors. To that purpose we proposed a mathematical formulation for the problem under dominant strategy incentive compatibility and ex-post individual rationality properties. Then we derived necessary and sufficient conditions under which ex-post efficiency can be obtained together with DIC and EIR. We also defined a new property called Allocation Maximality and proved that the Posted Price mechanism is the only mechanism that satisfies DIC, EIR and allocation maximality. In the final part we imposed ambiguity into the problem framework originating from different sets of priors for agents types based on box and ϕ-divergence ambiguity specifications and derived corresponding robust counterparts.

Next, in Chapter 4 we investigated the bilateral trade problem with an intermediary who wants to maximize her expected gains. We proposed a linear mathematical formulation for the problem, and then using network programming duality transformed that initial formulation into one that enables a transparent mechanism structure. We then relaxed the risk-neutrality of the intermediary and reconsidered the problem with existence of a risk-averse intermediary. We also presented numerical results and discussed the main difference in the structures of optimal mechanisms in both problems.

As a final problem, we extended intermediated bilateral trade problem to the case where the seller is also a producer. Therefore, the mechanism determines the level of production, the payment from the buyer to the intermediary, and the payment from the intermediary to the producer. Starting from a non-convex optimization problem, we obtained an equivalent convex one. To preserve the practicality of all obtained solutions we reconsidered the problem under dominant strategy incentive compatibility and ex-post individual rationality constraints. We also provided the necessary condition for the optimal mechanisms for the special case where agent types are uniformly distributed and the concave part of the benefit function is greater than or equal to the convex part of the cost function for all realizations of production level.

Bibliography

[1] R. V. Vohra, *Mechanism Design: a linear programming approach*, vol. 47. Cambridge Univ Pr, 2011.

[2] P. Toulis and D. C. Parkes, "Design and analysis of multi-hospital kidney exchange mechanisms using random graphs," *Games and Economic Behavior*, vol. 91, pp. 360–382, 2015.

[3] A. S. Prasad and S. Rao, "A mechanism design approach to resource procurement in cloud computing," *IEEE Transactions on Computers*, vol. 63, pp. 17–30, Jan 2014.

[4] N. Nisan and A. Ronen, "Algorithmic mechanism design," *Games and Economic Behavior*, vol. 35, no. 1, pp. 166 – 196, 2001.

[5] E. H. Gerding, V. Robu, S. Stein, D. C. Parkes, A. Rogers, and N. R. Jennings, "Online mechanism design for electric vehicle charging," in *Proceedings of the 10th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS'11)*, pp. 811–818, 2011.

[6] S. Oh and O. Ozer, "Mechanism design for capacity planning under dynamic evolutions of asymmetric demand forecasts," *Management Science*, vol. 59, no. 4, pp. 987–1007, 2013.

[7] M. Saleh, A. Soeanu, S. Ray, M. Debbabi, J. Berger, and A. Boukhtouta, "Mechanism design for decentralized vehicle routing problem," in *Proceedings of the 27th Annual ACM Symposium on Applied Computing*, pp. 749–754, ACM, 2012.

[8] R. B. Myerson and M. A. Satterthwaite, "Efficient mechanisms for bilateral trading," *Journal of economic theory*, vol. 29, no. 2, pp. 265–281, 1983.

[9] D. F. Spulber, *Market microstructure: intermediaries and the theory of the firm*. Cambridge University Press, 1999.

[10] L. Hurwicz, *The design of resource allocation mechanisms*, pp. 3–38. Cambridge University Press, 1977.

[11] R. B. Myerson and M. A. Satterthwaite, "Efficient mechanisms for bilateral trading," *Journal of Economic Theory*, vol. 29, no. 2, pp. 265 – 281, 1983.

[12] E. Ben-Porath, E. Dekel, and B. L. Lipman, "Optimal allocation with costly verification," *American Economic Review*, vol. 104, pp. 3779–3813, December 2014.

[13] M. Olson and D. Porter, "An experimental examination into the design of decentralized methods to solve the assignment problem with and without money," *Economic Theory*, vol. 4, no. 1, pp. 11–40, 1994.

[14] H. I. Bayrak, K. Güler, and M. Ç. Pınar, "Optimal allocation with costly inspection and discrete types under ambiguity," *Optimization Methods and Software*, pp. 1–20, 2017.

[15] S. Athey and I. Segal, "An efficient dynamic mechanism," *Econometrica*, vol. 81, no. 6, pp. 2463–2485, 2013.

[16] D. Bergemann and J. Välimäki, "Dynamic mechanism design: An introduction," *Journal of Economic Literature*, vol. 57, no. 2, pp. 235–74, 2019.

[17] R. B. Myerson, "Optimal auction design," *Mathematics of Operations Research*, vol. 6, pp. 58–73, 1981.

[18] J. Lu, "Optimal auctions with asymmetric financial externalities," *Games and Economic Behavior*, vol. 74, no. 2, pp. 561 – 575, 2012.

[19] D. L. Post, S. S. Coppinger, and G. B. Sheble, "Application of auctions as a pricing mechanism for the interchange of electric power," *IEEE Transactions on Power Systems*, vol. 10, pp. 1580–1584, Aug 1995.

[20] J. Hartline and A. Karlin, "Profit maximization in mechanism design," *Algorithmic Game Theory*, vol. 331, 2007.

[21] A. L. Bodoh-Creed, "Ambiguous beliefs and mechanism design," *Games and Economic Behavior*, vol. 75, no. 2, pp. 518 – 537, 2012.

[22] S. Bose, E. Ozdenoren, and A. Pape, "Optimal auctions with ambiguity," *Theoretical Economics*, vol. 1, no. 4, pp. 411–438, 2006.

[23] G. Carroll, "Robustness in mechanism design and contracting," *Annual Review of Economics*, vol. 11, no. 1, pp. 139–166, 2019.

[24] F. J. Bierbrauer and M. F. Hellwig, "Mechanism design and voting for public-good provision," *MPI Collective Goods Preprint*, no. 2011/31, 2011.

[25] D. Chakrabarty and C. Swamy, "Welfare maximization and truthfulness in mechanism design with ordinal preferences," in *Proceedings of the 5th conference on Innovations in theoretical computer science*, pp. 105–120, ACM, 2014.

[26] P. Samadi, H. Mohsenian-Rad, R. Schober, and V. W. S. Wong, "Advanced demand side management for the future smart grid using mechanism design," *IEEE Transactions on Smart Grid*, vol. 3, pp. 1170–1180, Sep. 2012.

[27] A. L. D. Augustynczik, R. Yousefpour, and M. Hanewinkel, "Climate change and the provision of biodiversity in public temperate forests : A mechanism design approach for the implementation of biodiversity conservation policies," *Journal of Environmental Management*, vol. 246, pp. 706–716, 2019.

[28] R. Lavi, *Mechanism Design*, pp. 5510–5523. New York, NY: Springer New York, 2009.

[29] R. V. Vohra, "Optimization and mechanism design," *Mathematical Programming*, vol. 134, no. 1, pp. 283–303, 2012.

[30] H. I. Bayrak and M. Ç. Pınar, "Generalized second price auction is optimal for discrete types," *Economics Letters*, vol. 141, pp. 35–38, 2016.

[31] Ç. Koçyiğit, H. I. Bayrak, and M. Ç. Pınar, "Robust auction design under multiple priors by linear and integer programming," *Annals of Operations Research*, pp. 1–21, 2017.

[32] J. Duives, B. Heydenreich, D. Mishra, R. Müller, and M. Uetz, "On optimal mechanism design for a sequencing problem," *Journal of Scheduling*, vol. 18, pp. 45–59, Feb 2015.

[33] R. Hoeksma and M. Uetz, "Optimal mechanism design for a sequencing problem with two-dimensional types," *Operations Research*, vol. 64, no. 6, pp. 1438–1450, 2016.

[34] Z. Li, J. K. Ryan, L. Shao, and D. Sun, "Incentive-compatible in dominant strategies mechanism design for an assembler under asymmetric

information," *Production and Operations Management*, vol. 28, no. 2, pp. 479–496, 2019.

[35] K. M. Hagerty and W. P. Rogerson, "Robust trading mechanisms," *Journal of Economic Theory*, vol. 42, no. 1, pp. 94 – 107, 1987.

[36] T. Matsuo, "On incentive compatible, individually rational, and ex post efficient mechanisms for bilateral trading," *Journal of Economic theory*, vol. 49, no. 1, pp. 189–194, 1989.

[37] A. Othman and T. Sandholm, "How pervasive is the myerson-satterthwaite impossibility?," in *Proceedings of the 21st International Jont Conference on Artifical Intelligence*, IJCAI'09, (San Francisco, CA, USA), pp. 233–238, Morgan Kaufmann Publishers Inc., 2009.

[38] N. Kos and M. Manea, "Efficient trade mechanisms with discrete values," tech. rep., Working paper, 2009.

[39] J. Flesch, M. Schröder, and A. J. Vermeulen, "The bilateral trade model in a discrete setting," 2013.

[40] J. Flesch, M. Schröder, and D. Vermeulen, "Implementable and ex-post ir rules in bilateral trading with discrete values," *Mathematical Social Sciences*, vol. 84, pp. 68–75, 2016.

[41] G. Carroll, "Information acquisition and robust trading mechanisms," 2017.

[42] M. Ç. Pınar, "Robust trade mechanisms over 0-1 polytopes," *Journal of Combinatorial Optimization*, 2017.

[43] I. Gilboa and D. Schmeidler, "Maxmin expected utility with non-unique prior," *Journal of mathematical economics*, vol. 18, no. 2, pp. 141–153, 1989.

[44] L. I. De Castro and N. C. Yannelis, "Ambiguity aversion solves the conflict between efficiency and incentive compatibility," tech. rep., Discussion Paper, Center for Mathematical Studies in Economics and Management Science, 2010.

[45] M. Ç. Pınar and C. Kızılkale, "Robust screening under ambiguity," *Mathematical Programming*, vol. 163, pp. 273–299, May 2017.

[46] P. Kouvelis and G. Yu, *Robust discrete optimization and its applications*, vol. 14. Springer Science & Business Media, 2013.

[47] G. Bayraksan and D. K. Love, "Data-driven stochastic programming using phi-divergences," in *The Operations Research Revolution*, pp. 1–19, INFORMS, 2015.

[48] A. Ben-Tal, D. Bertsimas, and D. B. Brown, "A soft robust model for optimization under ambiguity," *Operations Research*, vol. 58, no. 4-part-2, pp. 1220–1234, 2010.

[49] A. Ben-Tal, D. Den Hertog, A. De Waegenaere, B. Melenberg, and G. Rennen, "Robust solutions of optimization problems affected by uncertain probabilities," *Management Science*, vol. 59, no. 2, pp. 341–357, 2013.

[50] L. Pardo, *Statistical inference based on divergence measures*, vol. 185. Chapman & Hall/CRC, 2005.

[51] M. Tawarmalani and N. V. Sahinidis, "A polyhedral branch-and-cut approach to global optimization," *Mathematical Programming*, vol. 103, pp. 225–249, 2005.

[52] A. Wächter and L. T. Biegler, "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical Programming*, vol. 106, pp. 25–57, Mar 2006.

[53] R. V. Vohra, "Optimization and mechanism design," *Mathematical Programming*, vol. 134, no. 1, pp. 283 – 303, 2012.

[54] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, "Network flows: theory, algorithms, and applications," 1993.

[55] P. Krokhmal, M. Zabarankin, and S. Uryasev, "Modeling and optimization of risk," *Surveys in Operations Research and Management Science*, vol. 16, no. 2, pp. 49 – 66, 2011.

[56] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent measures of risk," *Mathematical Finance*, vol. 9, no. 3, pp. 203–228, 1999.

[57] R. T. Rockafellar and S. Uryasev, "Optimization of conditional value-at-risk," *Journal of risk*, vol. 2, pp. 21–42, 2000.

[58] R. T. Rockafellar and S. Uryasev, "Conditional value-at-risk for general loss distributions," *Journal of banking & finance*, vol. 26, no. 7, pp. 1443–1471, 2002.

[59] R. Kerkkamp, W. van den Heuvel, and A. Wagelmans, "Two-echelon supply chain coordination under information asymmetry with multiple types," *Omega*, vol. 76, pp. 137 – 159, 2018.