GEOMETRICAL APPROACHES TO SOLITON EQUATIONS

(Master Thesis)

UĞURHAN MUĞAN

Boğaziçi University
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Uğurhan Muğan

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ABSTRACT

A review of some recent geometrical approaches to the soliton equations is presented. The nonlinear evolution equations which belong to the Ablowitz-Kaup-Newell-Segur scheme are described in terms of a linear connection whose curvature vanishes. The properties of this soliton connection are discussed using exterior differential forms. The existence of infinite number of conservation laws and Bäcklund transformations are considered within this framework. It is shown that soliton equations may also be viewed as embedding problems. A general procedure which associates with the soliton connection two-dimensional surfaces embedded in three-dimensional flat space is outlined. The surfaces associated with the sine-Gordon and the Korteweg-deVries equations are explicitly constructed.

TABLE OF CONTENTS

1. INTRODUCTION

In this thesis we shall be intrested in a class of nonlinear evolution equations which has been extensively studied during the last two decades. The well-known members of this class are the Korteweg-de Vries (KdV) equation, the modified Korteweg-deVries (MKdV) equation, the sine-Gordon equation and the nonlinear Schrödinger equation. This class of equations admit particularly interesting special solutions which are known as solitons and find wide applications in various branches of physics and engineering. Other distinctive features of this class are the following:

1. The initial value problem for these nonlinear partial differential equations can be solved exactly. This is accomplished by using the method of inverse scattering transform which may be regarded as the generalisation of the Fourier analysis to nonlinear problems. In this method one associates with each equation a linear scattering problem in which the unknown variable of the nonlinear equation plays the role of the scattering potential. This unknown variable is determined by linear computations using the inverse scattering theory. The inverse scattering transform was first introduced for the KdV equation by Gardner, Greene, Kruskal and Miura[(1)] and was subsequently generalized by Lax[(2)], Zakharov and Shabat[(3)], Ablowitz, Kaup, Newell and Segur[(4)] and by Calegero and Degasperies[(5)].

2. These evolution equations have Bäcklund transfor-

mations. A Bäcklund transformation relates the solution of a given equation to another solution of the same equation or to a solution of another equation. Historically, this property was first observed for the sine-Gordon equation in the study of pseudospherical surfaces[6].

3. One can associate with each equation an infinite number of conservation laws. Under suitable boundary conditions, these conservation laws give rise to an infinite number of constantsof motion. These provide simple and efficient methods to study the properties of the solutions. The existence of such conserved quantities was first noted by Miura[7] for the KdV equation.

The purpose of the present thesis is to give a survey of some recent goemetric approaches to this class of evolution equations. There are two major reasons for undertaking such a study. First, it is desirable to elucidate the relationship between the nonlinear equations and their associated linear problems. The crucial step in applying the inverse scattering transform to a given equation is the determination of its associated linear problem. Therefore, it is attractive to consider frameworks which encompass both the nonlinear evolution equations and their associated linear problems. Secondly, it is also desirable to gain more insight into relationship between the inverse scattering transform, Bäcklund transformations and the existence of infinite number of conservation laws. With these goals in mind, in Chapter II we introduce the class of equations which belong to the Ablowitz, Kaup, Newell and Segur (AKNS) scheme[4] and outline some of their basic properties. In Chapter III we first follow Crampin, Pirani and Robinson[8] and show that the linear scattering problems for these evolution equations may be described in terms of a linear connection. This soliton connection is represented by a matrix of 1-forms which takes values in the Lie algebra of SL(2,R). In this framework, the requirement that the curvature of this connection vanishes gives rise to the desired nonlinear equations. We then utilise the works of Crampin[9] and Sasaki[10] and interpret the Bäcklund transformations and the existence of the infinite number of conservation laws. In Chapter IV we show that the soliton equations may also be viewed as embedding problems. Utilizing the observations of Lund and Regge[11] and Gürses and Nutku[12] we discuss how one associates with each nonlinear evolution equation and its linear problem, a two-dimensional surface embedded in a three-dimensional flat space. Conversely, from such embedding problems, one can construct nonlinear equations which can be solved using the inverse scattering transform. We conlude the thesis with a discussion of these geometric frameworks.

II. EVOLUTION EQUATIONS SOLVABLE BY INVERSE SCATTERING TRANSFORM

1. Particular Examples

In this section we present the well-known examples of the nonlinear evolution equations and discuss their symmetries.

a. The Kortweg-deVries (KdV) Equation:

$$\phi_t + 6\phi\phi_x + \phi_{xxx} = 0 \quad . \tag{II.1}$$

Here the subscripts denote the partial differentiations with respect to the independent variables t and x. The KdV equation is encountered in the theories of shallow water waves, anharmonic lattice, longitidunal dispersive waves in elastic rods and in plasma physics. This list is not exhaustive; a rather large class of nearly hyperbolic systems has been shown to reduce to KdV equation. There are three different transformations which leave tha KdV equation invariant These are the Galilean transformations:

$$x' = x + \frac{6}{\varepsilon^2} t \ , \quad t' = t \ , \quad \phi'(x',t') = \phi(x,t) + \frac{1}{\varepsilon^2} \ , \tag{II.2}$$

the space-time translations:

$$x' = x + x_0 \ , \quad t' = t + t_0 \ , \quad \phi' = \phi \ , \tag{II.3}$$

and the scale transformations:

$$x' = \eta x \ , \quad t' = \eta^3 t \ , \quad \phi' = \eta^{-2} \phi \ . \tag{II.4}$$

b. The Modified Korteweg-deVries (MKdV) Equation:

$$\phi_t + 6\phi^2\phi_x + \phi_{xxx} = 0 \ . \tag{II.5}$$

This equation has been used to describe acustic waves in certain anharmonic lattices and Alfén waves in collisionless plasma. It can be shown that if ϕ is a solution of the MKdV equation then $(\phi_t + \phi^2)$ satisfies the KdV equation. The MKdV equation shares with the KdV equation the invariance under the transformations (II.2) and (II.3). The scale transformations:

$$x' = \eta x , \quad t' = \eta^3 t , \quad \phi' = \eta^{-1} \phi \quad . \tag{II.6}$$

also leaves the MKdV equation unaltered.

c. The Sine-Gordon Equation:

$$\phi_{xx} - \phi_{tt} = \sin\phi \quad . \tag{II.7}$$

If a simple change of the independent variables is performed in the sine-Gordon equation also takes the form

$$\phi_{xt} = \sin\phi \quad . \tag{II.8}$$

This equation arises in the study of pseudospherical surfaces. It has been used to describe the propogation of a crystal dislocation, Bloch wall motion of magnetic crystals, the propogation of a magnetic flux of Josephson line. It has also been employed in elementary particle theory. The space-time translations (II.3), the two-dimesional Lorentz transformations:

$$x \to \zeta = \frac{x - ut}{\sqrt{1-u^2}} , \quad t' \to \tau = \frac{t - ux}{\sqrt{1-u^2}} , \quad \phi(x,t) \to \phi(\zeta,\tau) = \phi , \tag{II.9}$$

as well as the scale transformations:

$$x' = \eta x , \quad t' = \eta^{-1} t , \quad \phi' = \phi , \tag{II.10}$$

leave the sine-Gordon equation invariant.

d. The Nonlinear Schrödinger Equation:

$$i\phi_t + \phi_{xx} + 2|\phi|^2\phi = 0 \quad . \tag{II.11}$$

Some physical applications are stationary two-dimensional self-focusing of a plane wave, one-dimension self modulation of a monochromatic wave, the self trapping phenomena of non-linear optics, propogation of a heat pulse in a solid and Langmuir waves in plasma. This equation is invariant under (II.3), the scale transformations:

$$x'=\eta x, \qquad t'=\eta^2 t, \qquad \phi'=\eta^{-2}\phi, \tag{II.12}$$

and the transformation:

$$x'=x-2kt, \quad t'=t, \quad \phi=e^{ikx-ik^2t}\phi, \ k=\text{const}. \tag{II.13}$$

2. The AKNS System

Given a field $\phi(x,t)$ in one space x and one time t dimensions, consider the following Cauchy problem:

$$\phi_t=K(\phi), \tag{II.14}$$

$$\phi(x,t=0)=\phi_0(x), \tag{II.15}$$

where K is a certain nonlinear operator and the subscript t denotes partial differentiation with respect to time. We wish to solve this nonlinear equation by associating to it a linear scattering problem. For this purpose we introduce a linear operator L which depends on $\phi(x,t)$ and defines an eigenvalue problem by the equation

$$Lv=\eta v, \tag{II.16}$$

where η is the eigenvalue. Suppose the time evolution of the eigenfunctions v is governed by another operator M so that

$$iv_t=Mv. \tag{II.17}$$

By differentiating (II.16) with respect to t and using (II.17) we obtain

$$\eta_t v = -i\{iL_t - [M,L]\}v \ . \tag{II.18}$$

Hence, we see that the eigenvalues will be independent of time if the operators satisfy the Lax condition:

$$iL_t = [M,L] \ . \tag{II.19}$$

Following Ablowitz, Kaup, Newell and Segur (AKNS)[4] we shall be interested only in the operators having the forms

$$L = \begin{pmatrix} \frac{\partial}{\partial x} & -q \\ r & \frac{\partial}{\partial x} \end{pmatrix} , \tag{II.20}$$

$$M = i\begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{II.21}$$

and therefore we shall take

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ . \tag{II.22}$$

For the moment we shall assume that q, r, A, B, C, D are all arbitrary functions of x, t and the eigenvalue η. With these choices (II.16) will take the form

$$v_{1x} - \eta v_1 = q v_2 \ , \tag{II.23a}$$

$$v_{2x} + \eta v_2 = r v_1 \ . \tag{II.23b}$$

These equations will define us a scattering problem and the functions $q(x,t,\eta)$ and $r(x,t,\eta)$ will play the role of a scattering potential. On the other hand, the time dependence of v_1 and v_2 will be governed by the general linear equations

$$v_{1t} = A v_1 + B v_2 \ , \tag{II.24a}$$

$$v_{2t} = C v_1 + D v_2 \ . \tag{II.24b}$$

So far nothing guarantees that η is indepentent of time. After differentiating (II.23) with respect to t, (II.24) with respect to x and setting $\eta_t = 0$, we first note that

$$A_x = -D_x = qC - rB \,, \tag{II.25}$$

which implies

$$A = -D + d(t) \,, \tag{II.26}$$

and without any loss of gereratity we set $d=0$. When this property is taken into account in the equations obtained by cross differentiation, we find that $\eta_t = 0$ is satisfied if the following conditions hold:

$$A_x = qC - rB \,, \tag{II.27a}$$

$$B_x - 2\eta B = q_t - 2Aq \,, \tag{II.27b}$$

$$C_x + 2\eta C = r_t + 2Ar \,. \tag{II.27c}$$

These equations are the compatibility conditions for (II.16) and (II.17) and are equivalent to (II.19). All the nonlinear evolution equations that were mentioned in the previous section can be identified as the special cases of (II.27). For this purpose let us first expand the functions A, B, and C in powers of η:

$$A = a_3\eta^3 + a_2\eta^2 + a_1\eta + a_0 \,, \tag{II.28a}$$

$$B = b_3\eta^3 + b_2\eta^2 + b_1\eta + b_0 \,, \tag{II.28b}$$

$$C = c_3\eta^3 + c_2\eta^2 + c_1\eta + c_0 \,, \tag{II.28c}$$

and substitute (II.28) into (II.27). Comparing the coefficients of powers of η we find

$$A = a_3\eta^3 - a_2\eta - \frac{1}{2}(a_3rq + a_1)\eta - \frac{1}{2}a_2qr$$
$$- (qr_x - q_xr) + a_0 \,, \tag{II.29a}$$

$$B = a_3q\eta^2 - (a_2q - \frac{1}{2}a_3q_x)\eta$$
$$- (a_1q + \frac{1}{2}a_3q^2r + \frac{1}{2}a_2q_x - \frac{1}{4}a_3q_{xx}) \,, \tag{II.29b}$$

$$C = a_3r\eta^2 - (a_2r + \frac{1}{2}a_3r_x)\eta$$
$$-(a_1r + \frac{1}{2}a_3qr^2 + \frac{1}{2}a_2r_x - \frac{1}{4}a_3r_{xx}) \,. \tag{II.29c}$$

We next substitute (II.29) into (II.27b) and (II.27c) which result in the equations

$$q_t - \frac{1}{4} a_3 (q_{xxx} - 6qrq_x) + \frac{1}{2} a_2 (q_{xx} - 2q^2 r)$$
$$- a_1 q_x - 2a_0 q = 0 \ , \qquad \text{(II.30a)}$$
$$r_t - \frac{1}{4} a_3 (r_{xxx} - 6qrr_x) + \frac{1}{2} a_2 (r_{xx} - 2qr^2)$$
$$- a_1 r_x + 2a_0 r = 0 \ . \qquad \text{(II.30b)}$$

Now the choice, $a_0 = a_1 = a_2 = 0$, $a_3 = -4$ and $r = -1$, $q = \phi$ reduce (II.30a) to the KdV equation and the coefficients A, B, C, take the form

$$A = -4\eta^3 - 2\eta\phi - \phi_x \ , \qquad \text{(II.31a)}$$
$$B = -\phi_{xx} - 2\eta\phi_x - 4\eta^2\phi - 2\phi^2 \ , \qquad \text{(II.31b)}$$
$$C = 4\eta^2 + 2\phi \ . \qquad \text{(II.31c)}$$

On the other hand, if we let $a_0 = a_1 = a_2 = 0$, $a_3 = -4$ and $r = -q$, $q = \phi$ we get the MKdV equation together with the relations

$$A = -4\eta^3 - 2\eta\phi \ , \qquad \text{(II.32a)}$$
$$B = -\phi_{xx} - 2\eta\phi_x - 4\eta^2\phi - 2\phi^3 \ , \qquad \text{(II.32b)}$$
$$C = \phi_{xx} - 2\eta\phi_x + 4\eta^2\phi + 2\phi^3 \ . \qquad \text{(II.32c)}$$

The nonlinear Schrödinger equation is obtained by specializing to $a_0 = a_1 = a_3 = 0$, $a_2 = -2i$ and $r = -q^*$, $q = \phi$ and then

$$A = 2i\eta + i\phi^2 \ , \qquad \text{(II.33a)}$$
$$B = i\phi_x + 2i\eta\phi \ , \qquad \text{(II.33b)}$$
$$C = i\phi^*_x - 2i\eta\phi^* \ . \qquad \text{(II.33c)}$$

Similarly, we can also find the evolution equations corresponding to the expansion in inverse powers of η. For example taking

$$A = \frac{a(x,t)}{\eta} \ , \quad B = \frac{b(x,t)}{\eta} \ , \quad C = \frac{c(x,t)}{\eta} \ , \qquad \text{(II.34)}$$

yields

$$a_x = \frac{1}{2}(qr)_t\,, \quad q_{xt} = -4aq\,, \quad r_{xt} = -4ar\,, \qquad (II.35)$$

and the choice:

$$r = q = \frac{\phi_x}{2}\,, \quad a = \frac{1}{4}\cos\phi\,, \quad b = c = \frac{1}{4}\sin\phi\,, \qquad (II.36)$$

gives us the sine-Gordon equation. If we choose

$$r = -q = \frac{\phi_x}{2}\,, \quad a = \frac{1}{4}\cosh\phi\,, \quad b = c = \frac{1}{4}\sinh\phi\,, \qquad (II.37)$$

then we obtain the sinh-Gordon equation.

The nonlinear evolution equations (II.14) which fall into the ANKS scheme are solved by the following procedure: 1) Utilising the initial conditions one first calculates the scattering data (such as the reflection coefficients, discrete eigenvalues etc.) for v at $|x|=\infty$, $t=0$. The potential $q(x,t)$ is a regular function defined for all real values of the variable x and is assumed to vanish asymptotically: $q(x,t) \longrightarrow 0$ as $x \longrightarrow \pm\infty$.
2) Using the asymptotic form of M at $|x|=\infty$ and (II.27) one then determines time evolution of this scattering data. 3)As the nonlinear field $\phi(x,t)$ plays the role of the scattering potential, one constructs $\phi(x,t)$ from the knowledge of the time dependent scattering data using the techniques develope for the inverse scattering problems.

This method of solution is illustrated in Figure I. We shall not go into details of the inverse scattering trans form as there are extensive sources(5),(13) on this subject Here it will be sufficient for our purpose only to note that there is a close similarity between this method of solution of the nonlinear evolution equations and the Fourier analysis of the linear problems. Consider a linear time evolution equation for a field $u(x,t)$ satisfying the initial condition $u(x,0)=u_0(x)$. One may introduce the Fourier transform $\hat{u}(k,t)$ of $u(x,t)$. $\hat{u}(k,0)$ can be calculated from the Fourier transform of $u(x,t)$ for $t=0$ and the initial condition $u_0(x)$. Then if we take the Fourier transform of the given partial diffrential equation we obtain the evolution equation in k-space for the field $\hat{u}(k,t)$ which can be immediatly integrated. The last step which yields the solution $u(x,t)$ of the given linear partial differential equation is the inverse Fourier transform. The adventage of this method is that the time evolution is much simpler in k-space than in x-space. This procedure may be summarized schematically as follows:

$$u(x,0)\longrightarrow\hat{u}(k,0)\longrightarrow\hat{u}(k,t)\longrightarrow u(x,t) \quad .$$

So, from the similarity in the features, we may consider the AKNS scheme as an extension of the Fourier analysis to nonlinear problems.

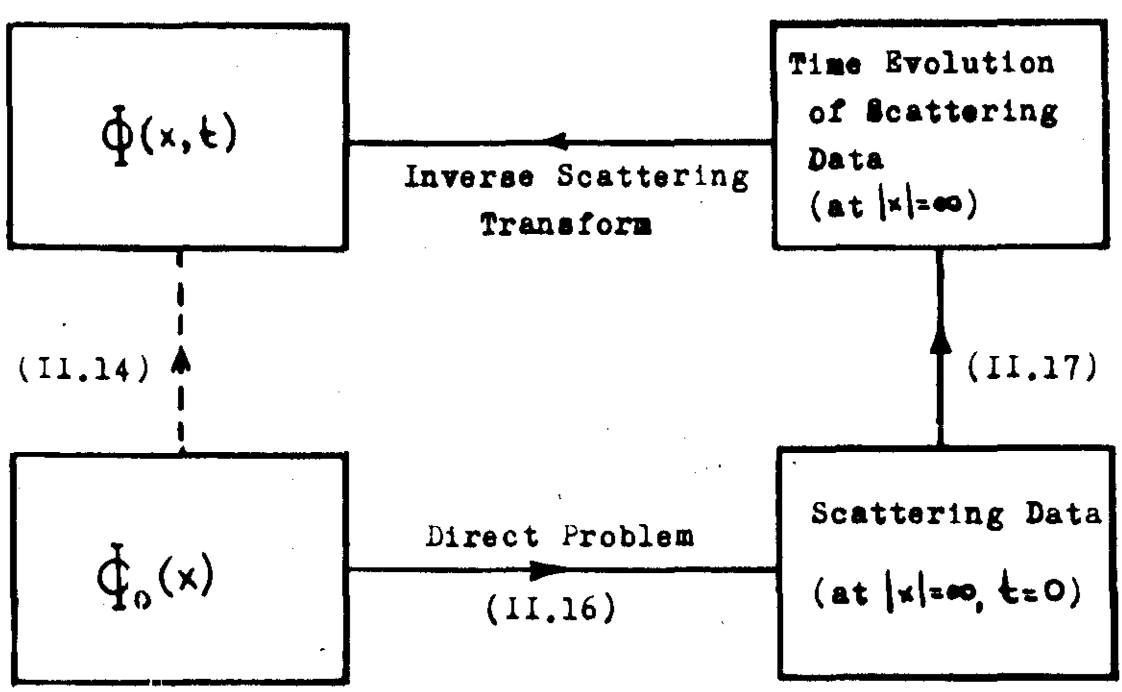

Figure 1

3. Bäcklund Transformations

One of the basic properties of the nonlinear partial differential equations which can be solved by inverse scattering transform is that each one admits Bäcklund transformations. Roughly speaking, Bäcklund transformation is a pair of first order differential equation which relate a solution of a higher order differential equation to another solution of the same equation, or to a solution of another differential equation. Utilising these first order equations one may construct new solutions from a given solution of the partial differential equation. In many problems the trivial solution $\Phi=0$ exists and this solution may be employed as the initial step in constructing new solutions. Moreover, in certain cases new solutions can even be constructed algebraically from a set of known solutions. An important example of such a procedure is the theorem of permutability for the sine-Gordon equation[14].

Let us introduce a new variable γ by

$$\gamma = \frac{v_1}{v_2} \quad , \tag{II.38}$$

Then the associated linear problems (II.23) and (II.24) respectively take the form

$$\gamma_x = 2\eta\gamma + q - r\gamma^2 \quad , \tag{II.39a}$$

$$\gamma_t = B + 2A\gamma - C\gamma^2 \; . \tag{II.39b}$$

These equations are called the Riccati form of (II.23) and (II.24). To obtain the Bäcklund transformations, we follow Konno and Wadati[15] and introduce γ' which is assumed to satisfy (II.39a) with a new potential $q'(x)$. The new potential is defined as

$$q'(x)=q(x)-f(\gamma,\eta)\ . \tag{II.40}$$

Next step is to eliminate γ' in (II.39) and (II.40). Here we shall consider only the following three classes:

Class 1: $r=-1\ ,\quad q=\phi\ .$

When we apply this procedure, the Riccati equation (II.39a) becomes

$$\gamma_x=2\eta\gamma+\phi+\gamma^2\ . \tag{II.41}$$

If we take the function γ' and the new field ϕ' as

$$\gamma'=-\gamma-2\eta\ , \tag{II.42a}$$

$$\phi'(x)=\phi(x)+2\frac{\partial}{\partial x}\left(-\gamma-2\eta\right), \tag{II.42b}$$

γ', ϕ' satisfy the Riccati form (II.41). By introducing $\phi=-\omega_x$ and $\phi'=-\omega'_x$, (II.42b) can be written as

$$\frac{\partial}{\partial x}\left(\omega-\omega'\right)=-2\frac{\partial}{\partial x}\left(\gamma+2\eta\right)\ , \tag{II.43}$$

and then we have

$$\omega-\omega'=-2\left(\gamma+2\eta\right)+2\,k(t)\ , \tag{II.44}$$

where $k(t)$ comes from the integration. The integrability condition for $(\omega-\omega')$ implies that $k(t)$ is a constant. Next, combining (II.42b), which can be expressed as

$$\gamma_x=\frac{1}{2}\left(\omega'_x-\omega_x\right)\ , \tag{II.45}$$

with (II.41) we obtain

$$\omega'_x+\omega_x=4\eta\gamma+2\gamma^2\ . \tag{II.46}$$

After taking the square of (II.44) and choosing the integration constant k as $-\eta$ we now have

$$\frac{1}{2}\left(\omega-\omega'\right)^2=2\left[\gamma^2+2\gamma\eta+\eta^2\right]\ . \tag{II.47}$$

This choice, results in one of the equations of the

Bäcklund transformation:

$$\omega_x+\omega'_x=-2\eta^2+\frac{(\omega-\omega')^2}{2} \quad . \tag{II.48a}$$

Substitution of γ from (II.47) to the other Riccati form (II.39b) gives us the other equation of the pair:

$$\omega'_t-\omega_t=2B+4A\left[\frac{\omega'-\omega}{2}-\eta\right]-2C\left[\frac{\omega'-\omega}{2}-\eta\right]. \tag{II.48b}$$

When we substitute A, B and C given in (II.31) these are the Bäcklund transformations for the KdV equation.

Hence, we get two equations which relate the partial derivatives of ω,ω' to ω and ω', so they can be used to calculate $\omega'(x,t)$ from the knowledge of $\omega(x,t)$ without using the inverse scattering transform. Another solution $\omega''(x,t)$ can be obtained from the knowledge of $\omega'(x,t)$ and so on.

Class ii: $r=-q\,,\quad q=\phi\,.$

The Riccati form (II.41a) takes the form

$$\gamma_x=2\eta\gamma+\phi+\phi\gamma^2 \quad . \tag{II.49}$$

If we choose γ' and ϕ' as

$$\gamma'=\frac{1}{\gamma} \quad , \tag{II.50a}$$

$$\phi'(x)=\phi(x)-2\frac{\partial}{\partial x}\tan^{-1}\gamma \quad , \tag{II.50b}$$

then γ' satisfies the (II.49) with ϕ'. Using the procedure and the notations that are outlined for class i, we get the Bäcklund transformations for class ii:

$$\omega_x+\omega'_x=-2\eta\sin(\omega-\omega') \quad , \tag{II.51a}$$

$$\omega_t-\omega'_t=(C-B)-(B+C)\cos(\omega-\omega')+2A\sin(\omega-\omega'). \tag{II.51b}$$

(II.51) are the Bäcklund transformations for the modified KdV equation when A, B and C given in (II.32) are substituted into the proper places. The Bäcklund transformations for the sine-Gordon equation can be obtained by substituting $\omega = \frac{\phi}{2}$ and A, B and C given in (II.22). These can be written as

$$\frac{1}{2}(\phi_x + \phi'_x) = -2\eta \sin\left[\frac{1}{2}(\phi - \phi')\right] , \tag{II.52a}$$

$$\frac{1}{2}(\phi_t - \phi'_t) = -\frac{1}{2\eta} \sin\left[\frac{1}{2}(\phi + \phi')\right] . \tag{II.52b}$$

Class iii. $r = -q^*$, $q = \phi$.

The Riccati form becomes

$$\gamma_x = 2\eta\gamma + \phi + \phi^* \gamma^2 . \tag{II.53}$$

If we choose γ' and ϕ' as

$$\gamma' = \frac{1}{\gamma^*} , \tag{II.54a}$$

$$\phi' = \phi + 2\frac{\gamma^2(\gamma^*_x) - \gamma_x}{1 - |\gamma|^4} , \tag{II.54b}$$

then γ' with ϕ' satisfies (II.53) for real η. We find that

$$\phi' + \phi = -4\eta \frac{\gamma}{1 + |\gamma|^2} , \tag{II.55}$$

which may be inverted to give

$$\gamma = -\frac{2\eta + \sqrt{4\eta^2 - |\phi' + \phi|^2}}{\phi'^* + \phi^*} . \tag{II.56}$$

When we next substitute (II.56) in the Riccati forms of this class and use the A, B and C given in (II.33), the Bäcklund transformations for nonlinear Schrödinger equation are obtained.

$$\phi_x + \phi'_x = (\phi - \phi')\sqrt{4\eta^2 - |\phi + \phi'|^2} , \tag{II.57a}$$

$$\begin{aligned} \phi_t + \phi'_t &= i(\phi_x - \phi'_x)\sqrt{4\eta^2 - |\phi + \phi'|^2} \\ &\quad + i(\phi + \phi')\left(|\phi + \phi'|^2 + |\phi - \phi'|^2\right) . \end{aligned} \tag{II.57b}$$

4. Infinite Number of Conservation Laws

By a conservation law we understand an equation having the form

$$T_t + N_x = 0 \,, \tag{II.58}$$

where T and N are function of the solution $\phi(x,t)$ of the nonlinear evolution equation. The function T is called the conserved density and N the conserved flow or flux. Then the functional

$$I = \int_{-\infty}^{+\infty} T[\phi(x,t)]\, dx \quad , \tag{II.59}$$

will be constant of motion provided the integral exists and the integrand satisfies certain boundary conditions at infinity. The conservation laws associated with a given nonlinear equation can be generated by utilising the associated linear problem. For this purpose let us introduce two variables

$$\gamma_1 = \frac{v_2}{v_1} \quad , \tag{II.60a}$$

$$\gamma_2 = \frac{v_1}{v_2} \quad . \tag{II.60b}$$

The variable γ_2 has already been considered in Section 3. The linear equations (II.23) and (II.24) can be used to obtain the equations

$$\gamma_{1_x} = -2\eta\gamma_1 + r - q\gamma_1^2 \quad , \tag{II.61a}$$

$$\gamma_{1_t} = C - 2A\gamma_1 - B\gamma_1^2 \quad , \tag{II.61b}$$

$$\gamma_{2_x} = 2\eta\gamma_2 + q - r\gamma_2^2 \quad , \tag{II.62a}$$

$$\gamma_{2_t} = B + 2A\gamma_2 - C\gamma_2^2 \quad . \tag{II.62b}$$

From the compatibility conditions $\gamma_{i_{xt}} = \gamma_{i_{tx}}$ $(i=1,2)$ and (II.27) one finds the following conservation laws:

$$(q\gamma_1)_t = (A + B\gamma_1)_x \quad , \tag{II.63a}$$

$$(r\gamma_2)_t = (-A + C\gamma_2)_x \quad . \tag{II.63b}$$

The coefficients of dx in $d\gamma_1$ and $d\gamma_2$, after respectively multiplying with q and r, give the equations for the conserved densities:

$$2\eta(q\gamma_1) = rq - (q\gamma_1)^2 - q\left(\frac{q\gamma_1}{q}\right)_x , \qquad \text{(II.64a)}$$

$$2\eta(r\gamma_2) = -rq + (r\gamma_2)^2 + r\left(\frac{r\gamma_2}{r}\right)_x . \qquad \text{(II.64b)}$$

Let us seek the power series solution of (II.64) in inverse powers of η

$$q\gamma_1 = \sum_{n=1}^{\infty} f_n \eta^{-n} . \qquad \text{(II.65)}$$

This expansion gives rise to the following recursion relation

$$f_{m+1} = \frac{1}{2}\left[(rq)\,\delta_{m,0} - \sum_{k=1}^{m-1} f_k f_{m-k} - q\left(\frac{f_m}{q}\right)_x\right] , \qquad \text{(II.66)}$$

which gives rise to

$$m=0 \;:\; f_1 = \frac{rq}{2} ,$$

$$m=1 \;:\; f_2 = \left(\frac{1}{2}\right)^2 r_x q ,$$

$$m=2 \;:\; f_3 = -\left(\frac{1}{2}\right)^3 \left[r^2q^2 + q_x r_x - (q r_x)_x\right] ,$$

$$m=3 \;:\; f_4 = \left(\frac{1}{2}\right)^4 \left\{ r r_x q^2 + \frac{1}{q}\left[q(q_x r_x)_x - (q r_x)_{xx} q + q_x (q r_x)_x + (q^3 r^2)_x + q_x^2 r_x\right]\right\} ,$$

etc. In this way the determination of density $T = q\gamma_1$ is completed. Then the flux $N = A + B\gamma_1$ can easily be determined because, when T is known one may use the relation

$$N = A + \frac{1}{q} BT . \qquad \text{(II.67)}$$

Finally, substituting these expressions back into (II.63a) the equating the coefficients of the same powers of η one obtains explicitly the aforementioned infinite set of conservation laws. Once T is determined the constant of motion can be obtained from (II.59).

III. THE SOLITON CONNECTION

1. Identification of the Connection

Basic equations of the associated linear problem can be rewritten in the form

$$\frac{\partial}{\partial x}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}+\begin{bmatrix} -\eta & -q \\ -r & \eta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad , \tag{III.1}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}+\begin{bmatrix} -A & -B \\ -C & A \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0 \quad . \tag{III.2}$$

Following Crampin, Pirani, and Robinson(8), we note that (III.1) and (III.2) can be identified as the two components of the equation

$$\frac{\partial v^R}{\partial x^m}+\Gamma^R_{Sm}v^S=0 \quad , \tag{III.3}$$

where $R, S, v = 1,2$; $m = n = 1,2$ and the summation convention is employed. Here we take $x^1 = x$, $x^2 = t$ and set

$$-\Gamma^R_{S1}=\begin{bmatrix} \eta & q \\ r & -\eta \end{bmatrix} \quad , \quad -\Gamma^R_{S2}=\begin{bmatrix} A & B \\ C & -A \end{bmatrix} \quad . \tag{III.4}$$

Therefore (III.3) can be interpreted as the property that the vector field v^R is covariantly constant:

$$\nabla_m v^R=0 \quad , \tag{III.5}$$

where ∇_m is the covariant derivative. This interpretation requires that the Γ^R_{Sm} be idetified as the components of a linear connection on the space on which v is defined. Let us denote by Γ the connection one-forms

$$\Gamma=\Gamma^R_{Sm}\,dx^m \quad . \tag{III.6}$$

This matrix of one-forms has three independent one-form entries:

$$\Gamma = \begin{bmatrix} \theta^0 & \theta^1 \\ \theta^2 & -\theta^0 \end{bmatrix} ,$$

where

$$\theta^0 = -(\eta dx + A\, dt) \quad , \qquad \text{(III.7a)}$$

$$\theta^1 = -(q dx + B dt) \quad , \qquad \text{(III.7b)}$$

$$\theta^2 = -(r dx + C\, dt) \quad . \qquad \text{(III.7c)}$$

We shall call this connection the soliton connection.

Note that the soliton connection Γ may be expressed

$$\Gamma = \theta^\alpha X_\alpha \quad , \qquad \alpha = 0, 1, 2 \, , \qquad \text{(III.8)}$$

where X_α are the infinitesimal generators of sl(2,R); the Lie algebra of the group SL(2,R). Consider the 2x2 unmodular real matrices,

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad , \qquad \alpha\delta - \beta\gamma = 1 \; . \qquad \text{(III.9)}$$

S is a general element of the group SL(2,R). When we introduce the one parameter subgroup of SL(2,R) as

$$g_0 = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{bmatrix} , \quad g_1 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} , \quad g_2 = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} , \qquad \text{(III.10)}$$

we can write any element of the group SL(2,R) as the product

$$S = g_0\, g_1\, g_2 \quad , \qquad \text{(III.11)}$$

assuming $\delta \neq 0$. The infinitesimal generators of the Lie algebra of the group SL(2,R) are obtained as

$$X_0 = \frac{d}{d\lambda} g_0 \Big|_{\lambda=0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad X_1 = \frac{d}{da} g_1 \Big|_{a=0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad X_2 = \frac{d}{db} g_2 \Big|_{b=0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \qquad \text{(III.12)}$$

and they have the commutation relations:

$$[X_0, X_1] = 2X_1 \, , \quad [X_0, X_2] = -2X_2 \, , \quad [X_1, X_2] = X_0 \, . \qquad \text{(III.13)}$$

Hence the soliton connection takes values in the Lie algebra of the group SL(2,R).

Having the identified the soliton connection we can now define the curvature two-form by

$$R = d\Gamma + \Gamma\wedge\Gamma \quad , \tag{III.14}$$

which also takes values in the Lie algebra of SL(2,R). In (III.14) d denotes the exterior derivative and in the second term matrix and exterior multiplications are to be understood. When the curvature two-form is expanded in terms of the generators of the Lie algebra of SL(2,R) as $R = R^\alpha X_\alpha$ we have

$$\begin{aligned} R^0 &= d\theta^0 + \theta^1\wedge\theta^2 \\ &= (-A_x + qC - rB)\, dx\wedge dt \quad , \end{aligned} \tag{III.15a}$$

$$\begin{aligned} R^1 &= d\theta^1 + 2\theta^0\wedge\theta^1 \\ &= (q_t - B_x + 2\eta B - 2qA)\, dx\wedge dt \quad , \end{aligned} \tag{III.15b}$$

$$\begin{aligned} R^2 &= d\theta^2 - 2\theta^0\wedge\theta^2 \\ &= (r_t - C_x - 2\eta C + 2rA)\, dx\wedge dt \quad , \end{aligned} \tag{III.15c}$$

that is,

$$\begin{aligned} R = dx\wedge dt \{ &(-A_x + qC + rB) X_1 + (q_t - B_x + 2\eta B \\ &- 2qA) X_2 + (r_t - C_x - 2\eta C + 2rA) X_2 \} \end{aligned} \tag{III.16}$$

We note that vanishing of the curvature two-form gives the compatibility conditions (II.27) which were obtained in the previous chapter. For example, when one substitutes the values of A, B, C, q and r which are given in (II.31), (II.32) and (II.34), (II.36), one gets, for the KdV equation:

$$R = dx\wedge dt \{ (\phi_t + 6\phi\phi_x + \phi_{xxx}) X_1 \} \quad . \tag{III.17}$$

For the modified KdV equation the curvature two-form is

$$R = dx\wedge dt \{ (\phi_t + 6\phi^2\phi_x + \phi_{xxx}) (X_1 - X_2) \} \quad , \tag{III.18}$$

and for the sine-Gordon equation one finds

$$R = dx\wedge dt \{ (\phi_{xt} - \sin\phi) (X_2 - X_1) \} \quad . \tag{III.19}$$

These results can be summarized as follows: The linear equations associated with the nonlinear evolution equations can be written as the vanishing of the covariant derivative of a real two-component vector field. In other words, the linear equations may be described in terms of a linear connection which takes its values in sl(2,R). The vanishing of the curvature two-form of this connection gives the nonlinear evolution equations.

Note that the connection one-form Γ is not unique for a given nonlinear evolution equation. We can obtain new forms of the connection by gauge transformations

$$\Gamma \longrightarrow \Gamma' = S^{-1}\Gamma S + S^{-1}dS \quad , \qquad \text{(III.20)}$$

where S is an element of the group SL(2,R). When the connection transforms as in (III.20), curvature two-form R behaves as

$$R \longrightarrow R' = S^{-1}RS = 0 \quad . \qquad \text{(III.21)}$$

The corresponding change in the vector field is given by

$$v \longrightarrow v' = S^{-1}v \quad . \qquad \text{(III.22)}$$

The action of the one-parameter subgroups of SL(2,R) on the connection one-forms are as follows:

$S = g_0$:

$$\theta^0 \longrightarrow \theta^{0\prime} = \theta^0 + d\lambda \quad , \qquad \text{(III.23a)}$$

$$\theta^1 \longrightarrow \theta^{1\prime} = e^{2\lambda}\theta^1 \quad , \qquad \text{(III.23b)}$$

$$\theta^2 \longrightarrow \theta^{2\prime} = e^{-2\lambda}\theta^2 \quad , \qquad \text{(III.23c)}$$

$S = g_1$:

$$\theta^0 \longrightarrow \theta^{0\prime} = \theta^0 - a\theta^0 \quad , \qquad \text{(III.24a)}$$

$$\theta^1 \longrightarrow \theta^{1\prime} = 2a\theta^0 + \theta^1 - a^2\theta^2 - da \quad , \qquad \text{(III.24b)}$$

$$\theta^2 \longrightarrow \theta^{2\prime} = \theta^2 \quad , \qquad \text{(III.24c)}$$

$S=g_2$:

$$\theta^0 \to \theta^{0\prime} = \theta^0 + b\theta^1 \quad , \tag{III.25a}$$
$$\theta^1 \to \theta^{1\prime} = \theta^1 \quad , \tag{III.25b}$$
$$\theta^2 \to \theta^{2\prime} = -2b\theta^0 - b^2\theta^1 + \theta^2 - db \; . \tag{III.25c}$$

2. Interpretation of Bäcklund Transformations

In the previous section we have seen that the vanishing of the curvature two-form constructed from the soliton connection gives us the nonlinear partial differential equations. The condition $R=0$ can be written as the following three equations:

$$d\theta^0 + \theta^1 \wedge \theta^2 = 0 \quad , \tag{III.26a}$$
$$d\theta^1 + 2\,\theta^0 \wedge \theta^1 = 0 \quad , \tag{III.26b}$$
$$d\theta^2 - 2\theta^0 \wedge \theta^2 = 0 \quad . \tag{III.26c}$$

Now consider the left invariant one-forms of SL(2,R):

$$L = X^{-1}dX = \begin{bmatrix} \omega^0 & \omega^1 \\ \omega^2 & -\omega^0 \end{bmatrix} \quad , \tag{III.27}$$

where X is a general element of the group SL(2,R). The Maurer-Cartan equations for SL(2,R) are:

$$dL + L\wedge L = \begin{bmatrix} d\omega^0 + \omega^1\wedge\omega^2 & d\omega^1 + 2\omega^0\wedge\omega^1 \\ d\omega^2 - 2\omega^0\wedge\omega^2 & -(d\omega^0 + \omega^1\wedge\omega^2) \end{bmatrix} = 0 \, , \tag{III.28}$$

which can be written as

$$d\omega^\alpha + \frac{1}{2} C_\beta{}^\alpha{}_\gamma \, \omega^\beta \wedge \omega^\gamma = 0 \quad , \tag{III.29}$$

where $C_\beta{}^\alpha{}_\gamma$ are the structure constant of SL(2,R). Therefore, (III.26) are formally same as the Maurer-Cartan equations for the left invariant one-forms of SL(2,R). However it should be noted that θ^α are one-forms defined on the two-dimensional space R^2. Using the Frobenius Theorem, one can find a SL(2,R) valued matrix G such that

$$\Gamma = G^{-1} dG \quad . \tag{III.30}$$

The meaning of this statement is that G is a mapping from an open set of $\mathbb{R}^2$ into SL(2,R):

$$\chi : \mathbb{R}^2 \longrightarrow SL(2,\mathbb{R}) \quad ,$$

and this map induces a mapping χ^* from space of one-forms on SL(2,R) into the space of one-forms on $\mathbb{R}^2$. Therefore

$$\chi^*(\omega^\alpha) = \theta^\alpha \quad .$$

We may write (III.30) as

$$dG = G\Gamma \quad . \tag{III.31}$$

Let the bottom row of G be denoted by $(v_2, -v_1)$. Then

$$dv_2 = v_2\theta^0 - v_1\theta^2 \quad , \qquad -dv_1 = v_2\theta^1 + v_1\theta^0 \quad . \tag{III.32}$$

These are equivalent to the associated linear equations (II.23) and (II.24).

Using the Iwasawa decomposition[16] any element of the group SL(2,R) can be written uniquely as a product of an upper triangular matrix and a rotation matrix. Therefore we may set

$$G = TR^{-1} \quad , \tag{III.33}$$

where T is an upper triangular matrix valued function and R is a rotation matrix valued function on $\mathbb{R}^2$. We shall choose them as

$$T = \begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} \quad , \qquad R = \begin{bmatrix} \cos\frac{1}{2}\psi & \sin\frac{1}{2}\psi \\ -\sin\frac{1}{2}\psi & \cos\frac{1}{2}\psi \end{bmatrix} . \tag{III.34}$$

When we construct $T^{-1}dT$ from (III.33) we obtain

$$T^{-1}dT = R^{-1}dR + R^{-1}\Gamma R \quad . \tag{III.35}$$

The left hand side of (III.35) is also upper triangular. So it implies that the lower left corner element of right hand side vanishes:

$$d\psi + (\theta^1 - \theta^2) = 2\theta^0 \sin\psi + (\theta^1 + \theta^2)\cos\psi \quad . \qquad \text{(III.36)}$$

This equation can be written as two first order partial differential equations for the rotation angle ψ. From θ^α dependence it contains a solution of the given nonlinear partial differential equation and its partial derivatives. If θ^α contains a known solution then (III.36) gives a relation between the known solution and the rotation angle ψ. This observation suggests that (III.36) must be linked to the Bäcklund transformations associated with the nonlinear evolution equation. We shall now show that the Bäcklund transformations can be interpreted as the gauge transformation which makes connection one-form Γ an upper triangular matrix[9].

If we rewrite the matrix G in (III.33) expilicitly,

$$G = \begin{bmatrix} a\cos\frac{\psi}{2} - b\sin\frac{\psi}{2} & a\sin\frac{\psi}{2} + b\cos\frac{\psi}{2} \\ \frac{1}{a}\cos\frac{\psi}{2} & \frac{1}{a}\sin\frac{\psi}{2} \end{bmatrix} , \qquad \text{(III.37)}$$

the ratio of the elements of bottom row $(v_2, -v_1)$ of G gives

$$\tau = tg\frac{\psi}{2} = -\frac{v_2}{v_1} \quad . \qquad \text{(III.38)}$$

When one takes the exterior derivative of τ and uses (III.26) one obtain

$$d\tau = 2\tau\theta^0 - \tau^2\theta^1 + \theta^2 \quad , \qquad \text{(III.39)}$$

which is equivalent to Riccati form of (II.23) and (II.24). Crampin[9] has observed that τ is a pseudopotential for the soliton equation, in the sense of Wahlquist and Estabrook[17].

Let us expilicitly demonstrate this interpretation of the Bäcklund transformations in the two standart examples. First, consider the sine-Gordon equation. Using (III.7) and (II.35) the connection one-form for this equation can easily be constructed. It is given by

$$\Gamma = \begin{bmatrix} \eta dx + \frac{1}{4\eta}\cos\phi\, dt & -\frac{1}{2}\phi_x dx + \frac{1}{4\eta}\sin\phi\, dt \\ \frac{1}{2}\phi_x dx + \frac{1}{4\eta}\sin\phi\, dt & -(\eta dx + \frac{1}{4\eta}\cos\phi\, dt) \end{bmatrix}, \qquad \text{(III.40)}$$

Let us now utilise the gauge freedom of the soliton connection and perform a transformation on Γ with the matrix

$$S = \begin{bmatrix} \cos\frac{1}{4}\phi & -\sin\frac{1}{4}\phi \\ \sin\frac{1}{4}\phi & \cos\frac{1}{4}\phi \end{bmatrix} \quad . \qquad \text{(III.41)}$$

This transformation brings Γ into the form

$$\Gamma = \begin{bmatrix} -\cos\frac{\phi}{2}(\eta dx + \frac{1}{4\eta}dt) & (\frac{1}{4}\phi_x + \eta\sin\frac{\phi}{2})dx - (\frac{1}{4}\phi_t + \frac{1}{4\eta}\sin\frac{\phi}{2})dt \\ (-\frac{1}{4}\phi_x + \eta\sin\frac{\phi}{2})dx + (\frac{1}{4}\phi_t - \frac{1}{4\eta}\sin\frac{\phi}{2})dt & \cos\frac{\phi}{2}(\eta dx + \frac{1}{4\eta}dt) \end{bmatrix} \qquad \text{(III.42)}$$

Now, the application of (III.36) gives us

$$d\psi + \frac{1}{2}(\phi_x dx + \phi_t dt) = -(2\eta\cos\frac{\phi}{2}dx - \frac{1}{2\eta}\cos\frac{\phi}{2}dt)\sin\psi + (2\eta\sin\frac{\phi}{2}dx - \frac{1}{2\eta}\sin\frac{\phi}{2}dt)\cos\psi, \qquad \text{(III.43)}$$

which results in the following pair of first order partial differential equations:

$$\psi_x + \frac{1}{2}\phi_x = 2\eta\left(\sin\frac{\phi}{2}\cos\psi - \cos\frac{\phi}{2}\sin\psi\right) , \qquad \text{(III.44a)}$$

$$\psi_t - \frac{1}{2}\phi_t = -\frac{1}{2\eta}\left(\sin\frac{\phi}{2}\cos\psi + \cos\frac{\phi}{2}\sin\psi\right) . \qquad \text{(III.44b)}$$

With the definition $\psi = \phi/2$ these equations are, of course, same as the ones given in (II.52).

As the second example we take the KdV equation $\phi_t + 12\phi\phi_x + \phi_{xxx} = 0$. Note that the coefficient of $\phi\phi_x$ is different from (II.1). The possible difference comes from the choice of $r=-2$ instead of $r=-1$ in (II.29) and (II.30). With this different choice, the connection one-form for this equation can easily be constructed by using (III.7) and the new forms of A, B, C in (II.29). It is given by

$$\Gamma = \begin{bmatrix} -\eta dx + (4\eta^3 + 4\phi\eta + 2\phi_x)dt & -\phi dx + (4\phi\eta^2 + 2\phi_x\eta + 4\phi^2 + \phi_{xxx})dt \\ 2dx - (8\eta^2 + 8\phi)dt & \eta dx - (4\eta^3 + 4\phi\eta + 2\phi_x)dt \end{bmatrix} \qquad \text{(III.45)}$$

Again, by using the gauge freedom of the soliton connection one may perform sequential transformations on Γ with the matrices.

$$S_1=\begin{bmatrix}\sqrt{2} & 0\\ 0 & 1/\sqrt{2}\end{bmatrix},\quad S_2=\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix},\quad S_3=\begin{bmatrix}1 & 0\\ -\eta & 1\end{bmatrix}. \tag{III.46}$$

Then the new form of Γ, after setting $\lambda=\eta^2$ is

$$\Gamma=\begin{bmatrix}2\phi_x dt & dx-4(\phi+\lambda)dt\\ (\lambda-2\phi)dx+[4(\phi+\lambda)(2\phi-\lambda)+2\phi_{xx}]dt & -2\phi_x dt\end{bmatrix}. \tag{III.47}$$

Therefore, (III.39) can be written as

$$\tau_x=\lambda-2\phi-\tau^2\ , \tag{III.48a}$$

$$\tau_t=-4\tau\phi_x+4(\phi+\lambda)\tau^2+4(\phi+\lambda)(2\phi-\lambda) +2\phi_{xx}\ . \tag{III.48b}$$

It can be verified easily that if ϕ satisfies the KdV equation given in the above form then

$$\phi'=\lambda-\phi-\tau^2\ , \tag{III.49}$$

is also solution of the same equation. By introducing $\phi=-\omega_x$ and $\phi'=-\omega'_x$ and omitting the constant of integration (III.49) takes the familiar form

$$-\omega'_x-\omega_x=\lambda-(\omega-\omega')^2\ . \tag{III.50a}$$

Utilizing (III.48b) the second equation of the pair is obtained as

$$\omega'_t+\omega_t=4(\phi'^2+\phi\phi'+\phi^2) +2(\omega'-\omega)(\omega'_{xx}-\omega_{xx})\ . \tag{III.50b}$$

3. The Conservation Laws

In this section we shall again consider the variables γ_1 and γ_2 that were defined by (II.60a) and (II.60b). We first note that the Riccati equations (II.61) for γ_1 can be combined into the Pfaffian equation

$$\lambda_1 = d\gamma_1 - \theta^1 \gamma_1^2 - 2\theta^0 \gamma_1 + \theta^2 = 0 \ . \tag{III.51}$$

Similarly, from (II.62) and the definition of the soliton connection, we can easily write

$$\lambda_2 = d\gamma_2 - \theta^2 \gamma_2^2 + 2\theta^0 \gamma_2 + \theta^1 = 0 \ . \tag{III.52}$$

We next investigate the integrability conditions for (III.51) and (III.52). By taking the exterior derivatives of λ_1 and λ_2 and using vanishing of the curvature of the soliton connection we find that, in order to be completely integrable, λ_1 and λ_2 must satisfy the necessary and sufficient conditions

$$d\lambda_1 = -2\lambda_1 \wedge (\theta^0 + \gamma_1 \theta^1) \ , \tag{III.53}$$

$$d\lambda_2 = 2\lambda_2 \wedge (\theta^0 - \gamma_2 \theta^1) \ . \tag{III.54}$$

Hence we see that for the solutions of (III.51) and (III.52) the one-forms

$$j_1 = \theta^0 + \gamma_1 \theta^1 \ , \tag{III.55a}$$

$$j_2 = -\theta^0 + \gamma_2 \theta^2 \ , \tag{III.55b}$$

must be closed

$$dj_1 = 0 \ , \tag{III.56a}$$

$$dj_2 = 0 \ . \tag{III.56b}$$

If these equations are expanded in terms of the basis two-form $dx \wedge dt$ one of course, regains the conservation laws (II.63a) and (II.63b). Therefore, we observe that the soliton connection provides an elegant way of expressing the conservation laws.

IV. SOLITON EQUATIONS AS EMBEDDING PROBLEMS

1. Fundamental Equations of Surface Theory

Let M be a three-dimensional flat space. At each point $\vec{P} \in M$ introduce the orthonormal basis vectors $\vec{e}_i$ ($i=1,2,3$) satisfying the conditions

$$\vec{e}_i \cdot \vec{e}_j = \eta_{ij} \quad , \tag{IV.1}$$

where the metric is given as

$$\eta_{ij} = \mathrm{diag}\,(1, \epsilon, \epsilon') \quad , \tag{IV.2}$$

and $\epsilon^2 = \epsilon'^2 = 1$. The indicators ϵ, ϵ' are introduced in order to handle the different choices of the signature of the metric. Obviously, the usual Euclidean space correspond to the case $\epsilon = \epsilon' = 1$.

Let $d\vec{P}$ be a small displacement. This displacement is a vector valued one-form and we can expand it in terms of $\vec{e}_i$ as

$$d\vec{P} = \omega^i \vec{e}_i \quad , \tag{IV.3}$$

where ω^i are the dual basis one-forms. We can do a similar expansion for the displacements $d\vec{e}_i$ in the basis vectors themselves:

$$d\vec{e}_i = \omega^k{}_i \, \vec{e}_k \quad . \tag{IV.4}$$

As we are considering only the flat spaces, the operator d can be interpreted as the exterior derivative. Taking the exterior derivative of (IV.1) and noting that $d\eta_{kj} = 0$ we obtain

$$\omega_{ij} + \omega_{ji} = 0 \quad . \tag{IV.5}$$

On the other hand, taking the exterior derivative of (IV.3) and $d(d\vec{P})=0$ gives

$$d\omega^k + \omega^k{}_i \wedge \omega^i = 0 \quad . \tag{IV.6}$$

Similarly, $d(d\vec{e}_i)=0$ results in the equation

$$\theta^k{}_j = d\omega^k{}_j + \omega^k{}_i \wedge \omega^i{}_j = 0 \quad , \tag{IV.7}$$

which expresses that for the flat spaces the curvature two-form is zero. (IV.3), (IV.4), (IV.5) are known as the Cartan's structure equations and (IV.6), (IV.7) are the integrability conditions.

Now, consider a two-dimensional surface embedded in M. Choose a moving frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ at each point $\vec{P}$ of the surface in such a way that $\vec{e}_1$ and $\vec{e}_2$ are tangent and $\vec{e}_3$ is normal to the surface. Since $\vec{P}$ is constrained to move in the surface, $d\vec{P}$ must lie in the tangent plane, so the sur-
is defined by

$$\omega^3 = 0 \quad , \tag{IV.8}$$

and therefore as it is stated

$$d\vec{P} = \omega^1 \vec{e}_1 + \omega^2 \vec{e}_2 \quad . \tag{IV.9}$$

When we distinguish the tangent vectors and the normal (IV.4) can be written as

$$d\vec{e}_\alpha = \omega^\beta{}_\alpha \vec{e}_\beta + \omega^3{}_\alpha \vec{e}_3 \quad , \qquad (\alpha, \beta = 1,2) \, . \tag{IV.10}$$

This equation is known as the Gauss equation. From $\vec{e}_3 \cdot \vec{e}_3 = 0$ we have $\vec{e}_3 \cdot d\vec{e}_3 = 0$ so $d\vec{e}_3$ must also lie in the tangent plane. We thus obtain the Weingarten equation:

$$d\vec{e}_3 = \omega^\alpha{}_3 \vec{e}_\alpha \quad . \tag{IV.11}$$

Again from seperation of the equations for the surface and its normal, the integrability conditions can be written in the form

$$d\omega^\alpha + \omega^\alpha{}_\beta \wedge \omega^\beta = 0 \quad , \tag{IV.12}$$

$$d\omega^3 = \omega^1{}_3 \wedge \omega^1 + \omega^2{}_3 \wedge \omega^2 = 0 \quad , \tag{IV.13}$$

and vanishing of curvature two-form implies that

$$d\omega^1{}_2 + \omega^1{}_3 \wedge \omega^3{}_2 = 0 \quad , \tag{IV.14a}$$

$$d\omega^1{}_3 + \omega^1{}_2 \wedge \omega^2{}_3 = 0 \quad , \tag{IV.14b}$$

$$d\omega^2{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 = 0 \quad , \tag{IV.14c}$$

which are called the Gauss-Mainardi-Codassi equations. Let us introduce the notation

$$\Pi^1 = \omega^1{}_3 \quad , \qquad \Pi^2 = \omega^2{}_3 \ . \tag{IV.15}$$

Then (IV.13) and (IV.14) can be written, as follows:

$$\Pi^1 \wedge \omega^1 + \Pi^2 \wedge \omega^2 = 0 \quad , \tag{IV.16}$$

$$d\omega^1{}_2 - \epsilon\epsilon' \Pi^1 \wedge \Pi^2 = 0 \quad , \tag{IV.17a}$$

$$d\Pi^1 + \omega^1{}_2 \wedge \Pi^2 = 0 \quad , \tag{IV.17b}$$

$$d\Pi^2 - \epsilon\, \omega^1{}_2 \wedge \Pi^1 = 0 \quad . \tag{IV.17c}$$

The one-forms Π^1 and Π^2 can be expressed as linear combinations of ω^1 and ω^2. Because of the relation (IV.16), we have a symmetry in the coefficients:

$$\Pi^1 = k\omega^1 + l\omega^2 \quad , \tag{IV.18a}$$

$$\Pi^2 = \epsilon l \omega^1 + n\omega^2 \ . \tag{IV.18b}$$

Since there is only one linearly independent two-form on the surface, we must have

$$\Pi^1 \wedge \Pi^2 = \epsilon\epsilon' K \omega^1 \wedge \omega^2 \quad , \tag{IV.19}$$

where

$$K = \epsilon\epsilon' (kn - \epsilon l) \quad , \tag{IV.20}$$

is a scalar and called the Gaussian curvature. Similarly, $\omega^2 \wedge \Pi^1 - \omega^1 \wedge \Pi^2$ is a two-form on the surface and

$$\omega^2 \wedge \Pi^1 - \omega^1 \wedge \Pi^2 = 2H\, \omega^1 \wedge \omega^2 \quad , \tag{IV.21}$$

defines a scalar H called the mean curvature of the surface. Using (IV.18) the mean curvature can be written as

$$2H = \epsilon k + \epsilon' n \quad . \qquad \text{(IV.22)}$$

The charecteristic roots of the matrix

$$\begin{pmatrix} k & l \\ \epsilon l & n \end{pmatrix} \quad ,$$

are called the principal curvatures κ_1 and κ_2 of the surface. We consequently have

$$2H = \kappa_1 + \kappa_2 \quad , \qquad \text{(IV.23)}$$

$$K = \epsilon\epsilon' \kappa_1 \kappa_2 \quad . \qquad \text{(IV.24)}$$

Returning back to (IV.17a) we see that

$$d\omega^1{}_2 - K\omega^1 \wedge \omega^2 = 0 \quad , \qquad \text{(IV.25)}$$

and the Riemannian curvature two form $\theta^1{}_2$ of the surface is

$$\theta^1{}_2 = d\omega^1{}_2 = K\omega^1 \wedge \omega^2 \quad . \qquad \text{(IV.26)}$$

The first fundamental form of the surface is defined by

$$ds_1^2 = d\vec{P} . d\vec{P} = \omega^1 \otimes \omega^1 + \epsilon\, \omega^2 \otimes \omega^2 \quad , \qquad \text{(IV.27)}$$

where $\otimes$ denotes the tensor product. The first fundamental form refers to the intrinsic properties of the surface. The information about how the surface is embedded in the three-dimensional flat manifold is contained in the second fundamental form (the extrinsic curvature). This quantity is defined as

$$ds_2^2 = - d\vec{e}_3 . d\vec{P} = - \left(\pi^1 \otimes \omega^1 + \pi^2 \otimes \omega^2 \right) . \qquad \text{(IV.28)}$$

Finally, let us note that the classical form of the equations of the surface theory can be obtained by choosing

$$\omega^1 = \sqrt{E}\, du + \left(F/\sqrt{E}\right) dv \quad , \qquad \text{(IV.29a)}$$

$$\omega^2 = \left(H/\sqrt{E}\right) dv \quad , \qquad \text{(IV.29b)}$$

where u, v are the local coordinates, E, F, G are the metric functions and $H^2 = EG - F^2$. Also choosing

$$\Pi^1 = P\,du + Q\,dv \quad , \tag{IV.30a}$$

$$\Pi^2 = R\,du + S\,dv \quad , \tag{IV.30b}$$

the second fundamental form can be expressed as

$$ds_2^2 = L\,du^2 + 2M\,du\,dv + N\,dv^2 \quad . \tag{IV.31}$$

2. The Sine-Gordon Equation

Consider a two-dimensional surface having the line element

$$ds^2 = \sin^2\theta\, dv^2 + \cos^2\theta\, du^2 \quad , \tag{IV.32}$$

where $\theta = \theta(u,v)$ and u, v are the local coordinates on the surface. Choosing the orthonormal basis one-forms as

$$\omega^1 = \sin\theta\, dv \quad , \tag{IV.33a}$$

$$\omega^2 = \cos\theta\, du \quad , \tag{IV.33b}$$

the connection one-form of the surface and its curvature two-form can easily be constructed. In the coordinate basis they are given by

$$\omega^1{}_2 = \theta_u\, dv + \theta_v\, du \quad , \tag{IV.34}$$

$$\theta^1{}_2 = (\theta_{vv} - \theta_{uu})\, dv \wedge du \quad . \tag{IV.35}$$

Let us demand that the surface is a pseudospherical one having the Gaussian curvature $K = -1$. From (IV.14) and (IV.19) we then have

$$\theta^1{}_2 = -\omega^1 \wedge \omega^2 \quad , \tag{IV.36}$$

and therefore obtain

$$\theta_{uu} - \theta_{vv} = \sin\theta\cos\theta \quad , \tag{IV.37}$$

which reduce to the canonical form (II.7) of the sine-Gordon equation by setting $\phi = 2\theta$. Hence we see that the pseudospherical surfaces are intimately related with the sine-Gordon equation. It is now natural to ask if the associated linear scattering problem can be included into this geometrical framework. Lund and Regge[11] has given a detailed discussion of this problem. To show how the associated linear problem fits into the picture, we shall assume that the pseudospherical surface is embedded in a three-dimensional flat space having the signature. $\epsilon = -\epsilon' = 1$. Let us again consider the connection one-form (III.42) for the sine-Gordon equation. After performing a scale transformation we have

$$\theta^0 = -\cos\theta\,(dx' + \tfrac{1}{4}dt') \quad , \tag{IV.38a}$$

$$\theta^1 = \tfrac{1}{2}(\theta_{x'}dx' - \theta_{t'}dt') + \sin\theta\,(dx' - \tfrac{1}{4}dt') \quad , \tag{IV.38b}$$

$$\theta^2 = -\tfrac{1}{2}(\theta_{x'}dt' - \theta_{t'}dt') + \sin\theta\,(dx' - \tfrac{1}{4}dt') \quad , \tag{IV.38c}$$

where we have used $\theta = \frac{\phi}{2}$ and put primes to indicate the new coordinates. We next define the coordinates

$$u = 2\,(x' + \tfrac{1}{4}t') \quad , \tag{IV.39a}$$

$$v = 2\,(x' - \tfrac{1}{4}t') \quad , \tag{IV.39b}$$

and perform another gauge transformation with the matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad . \tag{IV.40}$$

The new form of the connection one-forms are given by

$$\theta^{0'} = \frac{i}{2}(\theta_u dv - \theta_v du) \quad , \tag{IV.41a}$$

$$\theta^{1'} = \frac{1}{2}\sin\theta\,dv - \frac{i}{2}\cos\theta\,du \quad , \tag{IV.41b}$$

$$\theta^{2'} = \frac{1}{2}\sin\theta\,dv + \frac{i}{2}\cos\theta\,du \quad . \tag{IV.41c}$$

We now make the following identifications.

$$\omega^1{}_2 = -2i\theta^{0'} = \theta_u dv + \theta_v du \quad , \tag{IV.42a}$$

$$\pi^1 = -(\theta^{1'} + \theta^{2'}) = -\sin\theta\,dv \quad , \tag{IV.42b}$$

$$\pi^2 = i(\theta^{2'} - \theta^{1'}) = -\cos\theta\,du \quad , \tag{IV.42c}$$

and verify that, with ω^1 and ω^2 given in (IV.33),

$$\pi^1\wedge\omega^1+\pi^2\wedge\omega^2=0 \quad , \tag{IV.43}$$

$$\pi^1\wedge\pi^2=\omega^1\wedge\omega^2=0 \quad . \tag{IV.44}$$

We are therefore justifed in taking as the second fundamental form of the surface the quadratic form

$$ds_2^2=\sin^2\theta\, dv^2+\cos^2\theta\, du^2 \quad . \tag{IV.45}$$

This result may be summarized as follows: Using the soliton connection we have mapped the sine-Gordon equation and its associated linear problem to the problem of embedding pseudospherical surfaces in a certain three-dimensional flat space. It should be noted that the nonlinear sine-Gordon equation only refers to an intrinsic property of the two-dimensional surfaces; to the Riemannian curvature.

3. Gürses-Nutku Construction

It has been recently shown(12) that the above geometrical construction can be extended to all nonlinear evolution equations belonging to the AKNS scheme. General precription is to take

$$\omega^1{}_2=-\frac{2i}{\sqrt{\epsilon}}\,\theta^0 \quad , \tag{IV.46a}$$

$$\pi^1=i\sqrt{\epsilon'}\,(\theta^1+\theta^2) \quad , \tag{IV.46b}$$

$$\pi^2=\sqrt{\epsilon\epsilon'}\,(-\theta^1+\theta^2) \quad . \tag{IV.46c}$$

It can be easily shown that with these identifications (IV.17) is nothing but the Maurer-Cartan equations

$$d\theta^\alpha+\frac{1}{2}C_\beta{}^\alpha{}_\gamma\,\theta^\beta+\theta^\gamma=0 \quad , \tag{IV.47}$$

which was previously considered in Section 2 of Chapter III. Hence given a nonlinear equation and its associated linear problem one may first conruct the soliton connection and where we have omitted the primes on the transformed variables. Then the gauge transformation by the matrix g_0 given in (III.10) for

$$e^{\lambda} = \eta^{-1/2} \ , \tag{IV.51}$$

gives the η independent one-forms. For later convenience, one can simplify the η independent one-forms by another gauge transformation with the matrix g_1 which is also given in (III.10), for $a=-1$. These transformations bring the elements of the soliton connection into the form

$$\theta^0 = \phi_x \, dt \ , \tag{IV.52a}$$

$$\theta^1 = (1-\phi)dx + (\phi_{xx} + 2\phi + 2\phi^2 - 4)dt \ , \tag{IV.52b}$$

$$\theta^2 = dx - (4+2\phi)\,dt \ . \tag{IV.52c}$$

Utilizing the invariance property of the KdV equation under the Galilean transformations given in (II.2) and performing yet another gauge transformation with the matrix

$$S = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \ , \tag{IV.53}$$

one obtains the elements of the soliton connection as follows:

$$\theta^0 = -\phi_x \, dt \ , \tag{IV.54a}$$

$$\theta^1 = dx - 2\phi \, dt \ , \tag{IV.54b}$$

$$\theta^2 = -\phi \, dx + (2\phi^2 + \phi_{xx})\,dt \ . \tag{IV.54c}$$

Let us choose the basis one-forms ω^α as

$$\omega^1 = \left(\frac{\phi}{2} + 1\right)dt + \frac{1}{4}\,dx \ , \tag{IV.55a}$$

$$\omega^2 = \left(\frac{\phi}{2} - 1\right)dt + \frac{1}{4}\,dx \ . \tag{IV.55b}$$

Then the connection one-form $\omega^1{}_2$ can be found by using (IV.12). It is given by

$$\omega^1{}_2 = 2\phi_x \, dt \ . \tag{IV.56}$$

If one compares this connection one-form with the identification (IV.46a) obtains that

$$\epsilon = -1 \quad . \tag{IV.57}$$

The identifications (IV.46b) and (IV.46c) and the Gauss-Mainardi-Codazzi equations for the basis given in (IV.55) lead to

$$\epsilon' = -1 \quad . \tag{IV.58}$$

Hence the metric of the space is determined. From (II.46b) and (II.46c) one can find π^1 and π^2 as

$$\pi^1 = (1-\phi)dx + (2\phi^2 - 2\phi + \phi_{xx})\,dt \quad , \tag{IV.59}$$

$$\pi^2 = -(1+\phi)dx + (2\phi^2 + 2\phi + \phi_{xx})\,dt \quad . \tag{IV.60}$$

Therefore, the first and second fundemental forms of the surface which is related to the KdV equation are given by

$$ds_1^2 = 2\phi\,dt^2 + dt\,dx \quad , \tag{IV.61}$$

$$ds_2^2 = -\frac{1}{2}dx^2 - 2(\phi^2 + \phi_{xx})\,dt^2 + 2\phi\,dx\,dt \, . \tag{IV.62}$$

Finally, the Gaussian curvature of the surface is

$$K = -4\phi_{xx} \quad . \tag{IV.63}$$

The embedding problem of a surface with the first and second fundemental forms given by (IV.61) and (IV.62) is equivalent to the KdV equation and its associated linear problem.

V. DISCUSSION

In this thesis we have given a survey of some recent geometrical approaches to a class of nonlinear evolution equations. These equations were called soliton equations because of their physically interesting soliton solutions. The geometrical approaches were found attractive because they encompose both the nonlinear equations and their associated linear problems. In our discussion of the soliton equations two gedmetrical notions played important roles. First of these was the introduction of a connection and the second was the association of two-dimensional surfaces embedded in three-dimensional flat spaces. Before the discussion of these approaches well-known examples of the evolution equations were exhibited in Chapter II. Some of their physical applications and their invariance proper ties were also mentioned. In the same Chapter a summary of the AKNS scheme was given and Bäcklund transformations, conservation laws were discussed. In Chapter III the evolution equations were interpreted at the level of a connection. At this level the linear eigenvalue problem associeted with a given nonlinear equation was shown to be equivalent to the vanishing of the covariant derivative of a two-component vectorfield. This enabled us to identify the soliton connection which was represented by a SL(2,R) Lie algebra valued one-form. In this framework the nonlinear evolution equations were obtained by demanding that the curvature constructed from the soliton connection vanishes. It should be noted that the same costruction could be carried out by specifying the gauge group is SL(2,C) rather then SL(2,R). In fact, in (IV.40) we had assumed this generalization. The gauge freedom associated with the soliton connection was employed in the discussion of the Bäcklund transformations. Bäcklund transformations were interpreted as the gauge transformations which bring the soliton connection into the form of an upper triangular matrix. The existence of the conservation laws also found a simple explanation; they were the consequence of the existence of two closed one-forms.

In Chapter IV it was shown that the soliton equations may alternatively viewed as embedding problems. For this purpose the fundamental equations of the surface theory[18] [19] were given in terms of exterior differential forms. Then a general procedure which associates with the soliton connection two-dimensional surfaces embedded in three-dimensional flat spaces were outlined. This procedure was explicitly applied to the sine-Gordon and the KdV equation and the corresponding surfaces were constructed. Here it is intresting to note that there is another approach, due to Sasaki[10] which interpretes all the soliton equations as describing only the pseudospherical surfaces in three-dimensional flat space. In our notation, Sasaki's prescription is to take

$$\theta^0 = -\frac{1}{2}\,\omega^2 \quad , \tag{V.1}$$

$$\theta^1 = \frac{1}{2}\,(\omega^1{}_2 + \omega^1) \quad , \tag{V.2}$$

$$\theta^2 = -\frac{1}{2}\,(\omega^1{}_2 - \omega^1) \quad , \tag{V.3}$$

and assume $K=-1$. As can be verified easily, this approach has the undesirable feature that the evolution equations have to be assumed in the construction of the connection one-forms of the surfaces. This is in contrast to the Gürses Nutku prescription where the evolution equations refer only to the curvatures of the two-dimensional surfaces.

An attractive feature of the embedding approach is that it enables us to obtain simple generalizations of the well-known evolution equations. For example, applicable surfaces to the KdV surfaces were recently constructed[12]. These are surfaces hacing the same intrinsic geometry us the KdV surfaces but a different embedding in the three dimensional flat space. This gives rise to a pair of coupled nonlinear equations which generalizes the KdV equation and falls into the AKNS scheme. On the other hand, a generalization of the sine-Gordon equation has been obtained by considering the surfaces of Guichard[12]. The surfaces of Guichard are a generalization of the pseudo-spherical surfaces and provide a completely integrable system which again falls into the AKNS scheme. It is an interesting problem to seek similar generalizations of the other well-known evolution equations. It is also an interesting problem to consider in detail the Bäcklund transformations and conservation laws within the embedding approach.

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