THE DYNAMIC STABILITY OF AN ELASTIC COLUMN WITH A TIP MASS SUBJECT TO PARAMETRIC EXCITATION

by

ERGUN UYAR

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This Thesis Has Been Approved By

Dr. Ahmet CERANOGLU
(Supervisor)

Dr. Ahmet KUZUCU

Dr. Hüsamettin ALPER

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INTRODUCTION

In recent years, the dynamic analysis of structures has seen more and more prominence in engineering design considerations. Having sufficient knowledge of dynamic stability of stuctures enables us to prevent possible catastrophic failure.

Parametric instability of columns under periodic axial loads has been investigated by many authors [6]. Recently, much attention has been focused on the existence of combination resonances in addition to simple parametric resonances [4]. In most studies, however, only the horizontal accelerations are considered as these would appear to dominate over the effects of the much smaller vertical accelerations [10]. However these vertical accelerations cause instability depending on the dimensions of the system and the amplitude of the excitation frequency.

A column, when excited along its longitudinal axis, may vibrate in a direction transverse to this axis, under small perturbations. For certain relations between the parameters of the system and those of the excitation, the amplitudes of these transverse vibrations will become extremely large, and the column will collapse. It should be emphasized that the failure is caused by the interaction of the various properties of the structure and its excitation although it is a factor. The dependence of this resonance behaviour on the parameters of the system is known as

" parametric resonance ".

A detailed literature survey on the dynamic stability of elastic systems is given in Bolotin`s book [5]. Bolotin made the stability analysis according to the Floquet theory which prevents us to investigate the combination resonances of a structure under dynamic loading. Recently, Iwatsubo, Sugiyama and Ogino [6] analyzed the stability of a uniform elastic column under periodic axial loads for several sets of boundary conditions. Laura et. all. [7] examined the vibrations of a horizontal clamped-free beam with a mass at the free end, such a structure is also known as Beck`s column. The effects of shear and rotatory inertia on Beck`s column were studied by Kounadis and Katsikadelis [8]. In all these studies, analysis has been restricted to those cases where the structure is resting on a stationary foundation and excited by a time dependent tangential or axial load at the free end.

In this work, the dynamic stability of an elastic column with a large mass at the free end will be studied The excitation will be taken to be in the direction of its longitudinal axis, simulating the vertical accelerations of earthquakes, and the subsequent motion of the column transverse to this axis will be investigated. The equation of motion will be derived in Chapter II. In Chapter III we will consider the case of free vibrations and determine the frequency equation of the system. Stability analysis based on a method introduced by Hsu [4] will be given in

Chapter IV. Finally, experimental studies and results will be presented in Chapter V.

CHAPTER II

EQUATION OF MOTION

II.I.Formulation of the Problem:

In this section,the equation of motion governing the transverse vibrations,due to axial excitation,of a structure supporting a large mass will be derived.We will model the structure as a contilevered elastic column with a large mass at its free end,(Fig.2.I)

The parameters of the system will be defined as follows,

M:Mass of the supported body

m:Mass per unit length of the column

l:Length of the column

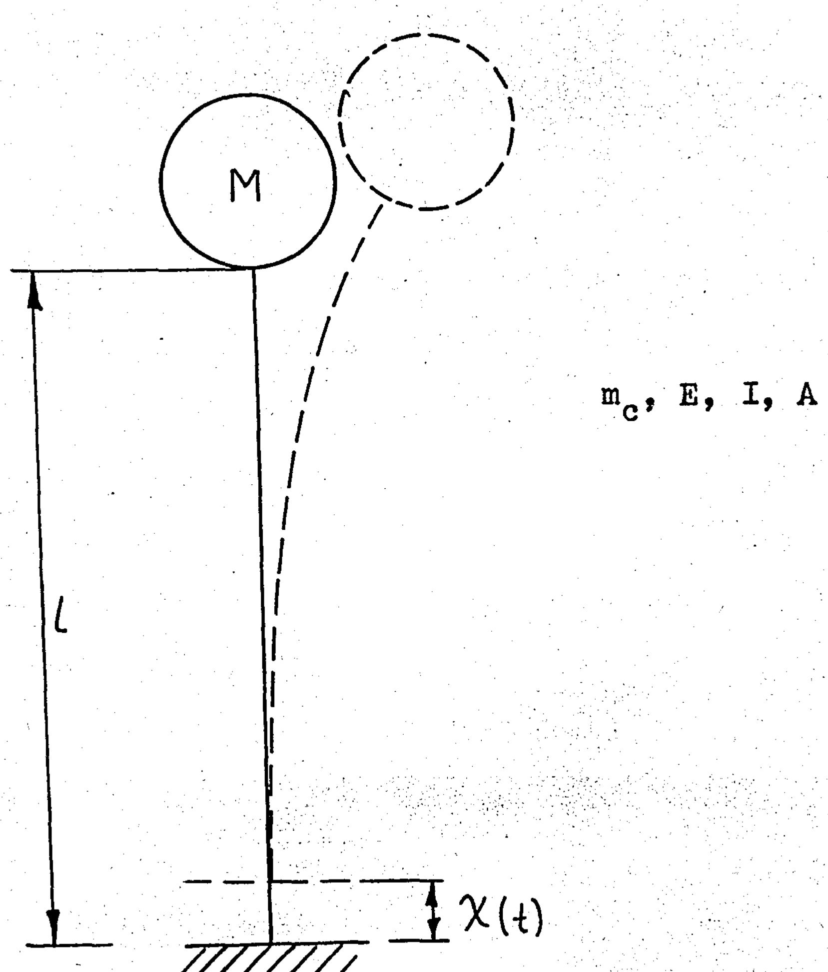
E:Modulus of elasticity of the column

I:Moment of inertia of the column cross-section

A:Cross-sectional area of the column.

In the analysis of the problem, the effects of the rotatory inertia and shear deformation will be neglected and the deflections of the column will be taken to be small in comparison to the smallest dimension of the stucture. The stability of the system will be determined by the boundedness of the solutions to the transverse equation of motion. That is, the column will be said to be unstable when the solutions grow indefinetely in time.

Before we derive the equation of motion, expressions for the longitudinal and transverse components of the acceleration of a material partical will be given in the first part of the following section and the equation of motion will be derived subsequently.



Mathematical Model of the Structure

Fig.(2.I)

II.2. The Equation of Motion :

In order to arrive at the transverse equation of motion of the column, Newton's 2^{nd} Law of Motion will be applied to an element of the column under the assumptions of the classical Bernoulli-Euler theory. To this end, the kinetics of a particle along the column will be discussed first and subsequently the equation of motion will be derived.

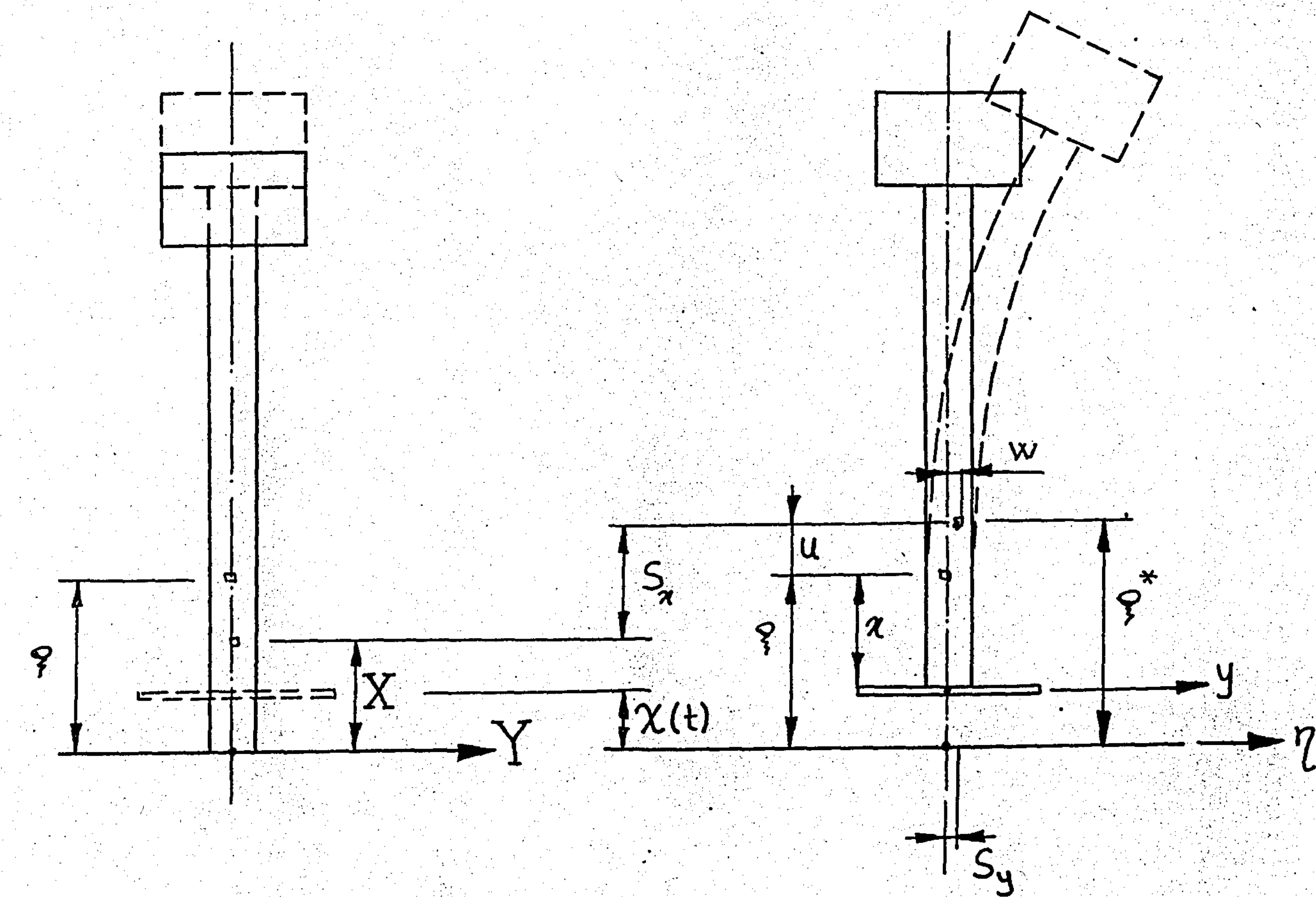
2.I. Kinematics of a Material Particle - Velocity and Acceleration :

Figure (2.2) shows the three successive positions of the system. The first one, which will be referred to as reference configuration (undeformed configuration) is the position at t=0. The system is at rest in the inertial frame fixed to the ground. The second position, a hypothetical one, is the one where the entire structure is rigidly displaced vertically with a time dependency given by $\chi(t)$. Finally, the third position, deformed position, represents the state of the structure at any time t.

The coordinates and the displacements of an arbitrary material particle in each aforementioned position are given below. The first term represents transverse (horizontal) directions.

(X,Y) : Locates a material particle in the reference configuration expressed in a coordinate system measured from a fixed inertial frame.

(ξ,η) : Coordinates of the material particle in the hypothetical configuration (rigid body motion)



reference and intermediate configurations

intermediate and present configurations

Figure (2.2)

measured from the origin of the fixed inertial frame.

(ξ^*, η^*) :Coordinates of the material particle in the deformed position relative to the fixed inertial frame.

(x, y) :Locates the material particle in the undeformed state with respect to the coordinate system moving with the support.

(u, w) :Components of displacement of a material particle relative to its reference position.

The axial and the transverse components of displacement in the hypothetical state can be written as, (Fig.2.I)

$$\left. \begin{aligned} u &= \xi^* - \xi \\ w &= \eta^* - \eta \end{aligned} \right\} \qquad (2.I.I)$$

and the displacements with respect to the reference position are

$$\left. \begin{aligned} S_x &= \xi^* - X = u + \chi \\ S_y &= w = \eta^* - Y \end{aligned} \right\} \qquad (2.I.2)$$

Note that the position of a material particle in the fixed inertial frame is determined by the coordinate X, and its velocity is the rate of change of the displacement function S_x , holding X constant, i.e.,

$$V_x = \left. \frac{\partial S_x}{\partial t} \right|_X \qquad (2.I.3)$$

Since S_x is also a function of ξ and t , applying the chain rule of differentiation to S_x we get,

$$V_x = \frac{\partial S_x}{\partial t} + \frac{\partial S_x}{\partial \rho} \cdot \frac{\partial \rho}{\partial t}$$

where from Eq. (2.I2)

$$\frac{\partial S_x}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial \chi}{\partial t}$$

$$\frac{\partial S_x}{\partial \rho} = \frac{\partial u}{\partial \rho}$$

We also have,

$$\frac{\partial \rho}{\partial t} = \frac{\partial (X + \chi)}{\partial t} \doteq \dot{\chi}$$

Substituting all these relations in the expression for V_x yields

$$V_x = \frac{\partial u}{\partial t} + \dot{\chi}\left(I + \frac{\partial u}{\partial \rho}\right) . \qquad (2.I.4)$$

Neglecting the axial strain $\frac{\partial u}{\partial \rho}$, and the strain velocity $\frac{\partial u}{\partial t}$ compared to $\dot{\chi}$, the expression for the axial component of the velocity of that material particle reduces to

$$V_x \cong \dot{\chi}(t)$$

Showing that the major contribution to the motion is due to the motion of the support, that is

$$\frac{\partial u}{\partial x} \ll \dot{\chi} \qquad \qquad \frac{\partial^2 u}{\partial x^2} \ll \ddot{\chi}$$

Taking the time derivative of the equation (2.I.4) holding X constant will yield the general expression for the longitudinal component of the acceleration,

$$a_x = \left.\frac{\partial V_x}{\partial t}\right|_X = \frac{\partial V_x}{\partial t} + \dot{\chi}\frac{\partial V_x}{\partial \rho}$$

or

$$a_x = \ddot{X} + \left[\frac{\partial^2 u}{\partial t^2} + \ddot{X}\frac{\partial u}{\partial \xi}\right] + \dot{X}\frac{\partial}{\partial \xi}\left[2\frac{\partial u}{\partial t} + \dot{X}\frac{\partial u}{\partial \xi}\right]$$

Neglecting second and the higher order terms the above expression reduces to

$$a_x = \ddot{X}(t) \tag{2.I.5}$$

For the transverse components of velocity and acceleration of the material particle, going through the same procedure yields

$$V_y = \left.\frac{\partial S_y}{\partial t}\right|_X = \frac{\partial S_y}{\partial t} + \frac{\partial S_y}{\partial \xi}\cdot\frac{\partial \xi}{\partial t}$$

$$V_y = \frac{\partial w}{\partial t} + \dot{X}\frac{\partial w}{\partial \xi} \tag{2.I.6}$$

and the general expression for the transverse component of the acceleration is

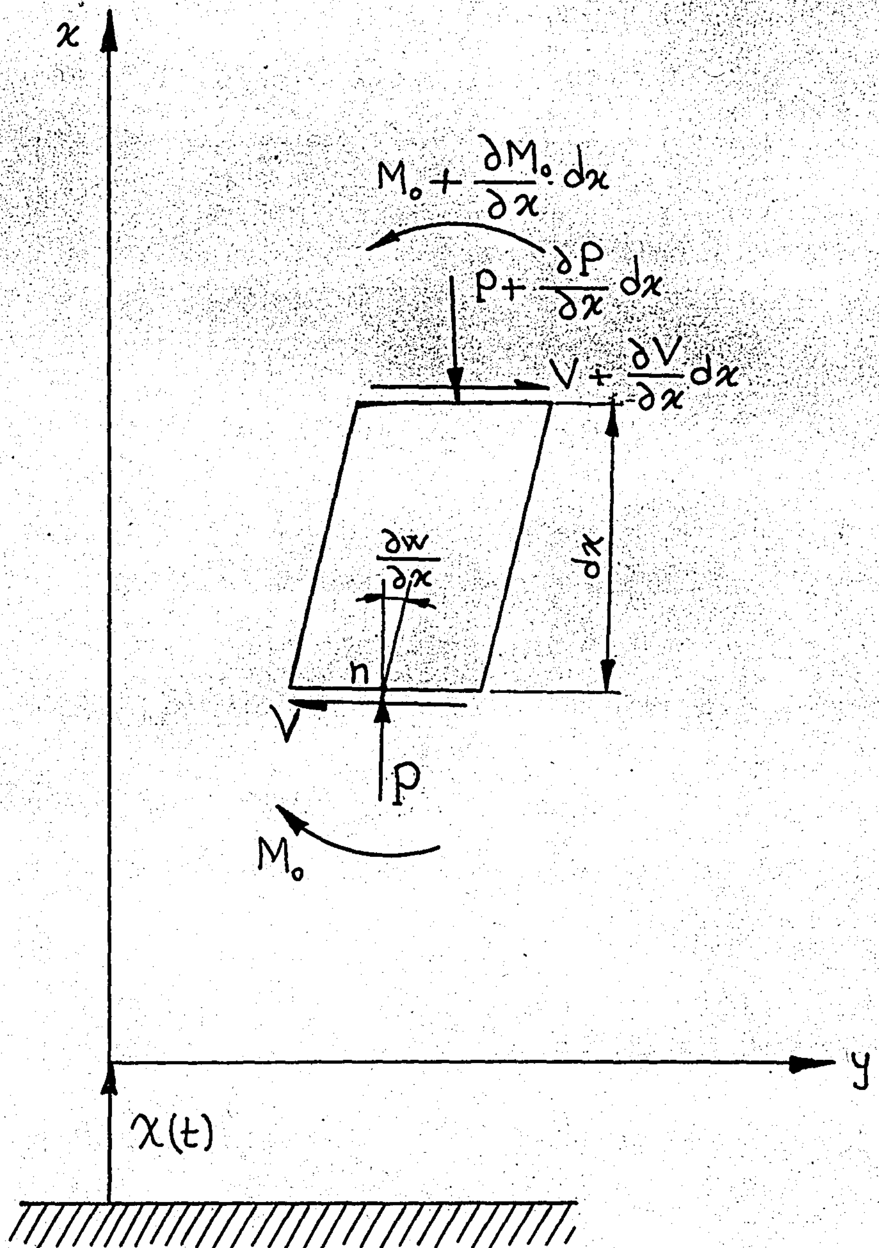
$$a_y = \frac{\partial^2 w}{\partial t^2} + \ddot{X}\frac{\partial w}{\partial \xi} + \dot{X}\frac{\partial}{\partial \xi}\left[2\frac{\partial w}{\partial t} + \dot{X}\frac{\partial w}{\partial \xi}\right]$$

Considering only the first two terms and neglecting the remaining, the expression for the transverse acceleration reduces to

$$a_y = \frac{\partial^2 w}{\partial t^2} + \ddot{X}\frac{\partial w}{\partial \xi} \tag{2.I.7}$$

2.2. The Equation of Motion :

A differential material element of the column which is deformed due to the bending moment is shown in Fig. (2.3) in a coordinate system (x,y) moving with the support. It will be assumed that shear deformation and rotatory inertia effects are negligible. P , V and M_o represent the



Differential Element of the Column

Fig.(2.3)

inertial axial force, shear force and bending moment respectively.

The equations of motion both along the x and y directions obtained by applying Newton's 2^{nd} Law of Motion to the differential element of the column are

$$-\frac{\partial P}{\partial x} = m_c \ddot{\chi} \tag{2.2.1}$$

$$\frac{\partial V}{\partial x} = m_c \left[\frac{\partial^2 w}{\partial t^2} + \ddot{\chi} \frac{\partial w}{\partial x} \right] \tag{2.2.2}$$

where we have used the fact that

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}$$

Summing up the moments about an axis perpendicular to the (x,y) plane and passing through a point n we get

$$\frac{\partial M_o}{\partial x} dx - V dx - \frac{\partial V}{\partial x}(dx)^2 - P \frac{\partial w}{\partial x} dx - \frac{\partial w}{\partial x} \frac{\partial P}{\partial x} (dx)^2 = 0$$

Considering only the first order terms and neglecting the remaining ones we obtain the equation of angular motion of the system as

$$\frac{\partial M_o}{\partial x} - P \frac{\partial w}{\partial x} - V = 0 \tag{2.2.3}$$

Recall the relation between the bending moment M_o and the curvature at the same point for a beam in flextural motion

$$M_o = -EI \frac{\partial^2 w}{\partial x^2} \tag{2.2.4}$$

Eliminating M_o between the equations (2.2.3) and (2.2.4) we get an equation for the shear force

$$-V = EI\frac{\partial^3 w}{\partial x^3} + P\frac{\partial w}{\partial x} \tag{2.2.5}$$

Differentiating the above equation with respect to x and using the equations (2.2.I) and (2.2.2) yields

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + m_c\frac{\partial^2 w}{\partial t^2} = 0 \tag{2.2.6}$$

Note that the axial force P is a function of x and t and is given by

$$P(x,t) = M(g - a_x) + \int_x^l m_c(g - a_x)dx$$

where g is the gravitational acceleration.
Substituting $\ddot{X}$ for a_x the above equation reduces to

$$P(x,t) = \left[M + m_c(l - x)\right]\cdot(g - \ddot{X})$$

Since the beam considered is slender and the mass of the top weight is very large compared to that of column ($M \gg m_c l$) , inertial forces of the column can be neglected. Hence, the expression for the axial force reduces to

$$P(t) = M(g - \ddot{X}) \tag{2.2.7}$$

which is only a function of time. The governing equation of motion for the system described is then obtained by substituting (2.2.7) into the equation (2.2.6)

$$EI\cdot\frac{\partial^4 w}{\partial x^4} + M(g - \ddot{X})\frac{\partial^2 w}{\partial x^2} + m_c\frac{\partial^2 w}{\partial t^2} = 0 \tag{2.2.8}$$

This is a fourth order partial differential equation, hence, there should be four boundary conditions. At the lower end, the beam is built into its foundation meaning that the displacement and the slope must vanish, i.e.,

$$w\Big|_{x=0} = 0 \qquad (2.2.9a)$$

$$\frac{\partial w}{\partial x}\Big|_{x=0} = 0 \qquad (2.2.9b)$$

The other end is free with a large mass. Hence , one of the boundary condition at x=l is zero bending moment

$$EI\frac{\partial^2 w}{\partial x^2}\Big|_{x=l} = 0 \qquad (2.2.9c)$$

To get the second boundary condition, note that the inertial force of the top weight is balanced by shear force at this end of the beam. Thus,

$$V(l,t) = -M a_y(l,t) = -M\left[\frac{\partial^2 w}{\partial t^2} + \ddot{X}\frac{\partial w}{\partial x}\right]_{x=l}$$

is the remaining boundary condition. Substituting (2.2.7) into (2.2.5) and evaluating at x=l will give us the shear force appearing on the left-hand side of the above equation. Thus, the above expression reduces to

$$\left[EI\frac{\partial^3 w}{\partial x^3} + Mg\frac{\partial w}{\partial x} - M\frac{\partial^2 w}{\partial t^2}\right]_{x=l} = 2M\ddot{X}\frac{\partial w}{\partial x}\Big|_{x=l} \qquad (2.2.9d)$$

In this work we will assume that the excitation is a harmonic function of time , that is

$$X(t) = X_0 \cos \Omega t$$

where X_0 and Ω are constants corresponding to the amplitude and the frequency of the forcing function. The expression for the acceleration of the base is then

$$\ddot{X}(t) = -X_0 \Omega^2 \cos \Omega t$$

Substituting $\ddot{X}$ (t) into the Eq. (2.2.8) yields

$$\frac{\partial^4 w}{\partial x^4} + (\beta + \alpha \cos \Omega t) \frac{\partial^2 w}{\partial x^2} + \gamma \frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.2.10)$$

where

$$\left.\begin{aligned} \beta &= \frac{M \cdot g}{E \cdot I} \\ \alpha &= \Omega^2 X_0 \frac{M}{E I} \\ \gamma &= \frac{m_c}{E I} \end{aligned}\right\} \qquad (2.2.11)$$

Equation (2.2.10) together with its boundary conditions (2.2.9) describes the transverse motion of the column in the (x,y) plane subject to a vertical excitation $X(t)$.

Since we will be only interested in the steady-state motion no initial conditions will be prescribed for this problem.

CHAPTER III

FREE VIBRATIONS

In this chapter, the general solution and the frequency equation of the system under consideration will be presented.

III.I. The Equation of Motion and Its Solution :

If the forcing function χ (t) is taken as zero in the equation (2.2.I0), the equation of motion describing the free vibrations case is obtained. Thus Eq.(2.2.I0) reduces to

$$\frac{\partial^4 w}{\partial x^4} + \beta \frac{\partial^2 w}{\partial x^2} + \gamma \frac{\partial^2 w}{\partial t^2} = 0 \qquad (3.I.I)$$

where β and γ are the quantities defined in (2.2.II). The boundary conditions, (Eq.(2.2.9)), can be written as

$$w\Big|_{x=0} = 0 \qquad (3.I.2a)$$

$$\frac{\partial w}{\partial x}\Big|_{x=0} = 0 \qquad (3.I.2b)$$

$$\frac{\partial^2 w}{\partial x^2}\Big|_{x=l} = 0 \qquad (3.I.2c)$$

$$\left(\frac{\partial^3 w}{\partial x^3} + \beta \frac{\partial w}{\partial x} - \frac{\beta}{g} \cdot \frac{\partial^2 w}{\partial t^2}\right)\Big|_{x=l} = 0 \qquad (3.I.2d)$$

The general solution of equation (3.I.I) can be obtained by applying the method of separation of variables. That is , we assume a solution of the form

$$w(x,t) = V(x) \cdot e^{i\omega t} \qquad (3.I.3)$$

Substituting the above equation into Eq.(3.I.I), the equation of motion reduces to an ordinary differential equation of the form

$$\left[\frac{d^4V(x)}{dx^4} + \beta\,\frac{d^2V(x)}{dx^2} - \gamma\,\omega^2 V(x)\right] e^{i\omega t} = 0 \qquad (3.I.4)$$

where ω is the natural frequency of the system. The expressions in brackets must vanish in order to have Eq.(3.I.4) hold true for all time t. Then the problem reduces to an eigenvalue problem of the form

$$\mathcal{L}\,[V] = \omega^2 \gamma\, V \qquad (3.I.5)$$

where $\mathcal{L}$ is a linear differential operator defined by

$$\mathcal{L} \equiv \frac{d^4}{dx^4} + \beta\,\frac{d^2}{dx^2} \qquad (3.I.6)$$

Assuming a solution of the form $V(x) = C e^{\lambda x}$ Eq.(3.I.5) yields the characteristic equation

$$\lambda^4 + \beta\lambda^2 - \omega^2\gamma = 0 . \qquad (3.I.7)$$

The roots of this fourth order equation can be written as

$$\lambda_1 = k_0 \quad , \quad \lambda_2 = -k_0 \quad , \quad \lambda_3 = i\,k_1 \quad , \quad \lambda_4 = -i\,k_1 \qquad (3.I.8)$$

where

$$k_0 = \sqrt{\frac{-\beta + \sqrt{\beta^2 + 4\omega^2\gamma}}{2}} \qquad k_1 = \sqrt{\frac{\beta + \sqrt{\beta^2 + 4\omega^2\gamma}}{2}}$$

Hence the solution for V(x) can be written as

$$V(x) = A \cosh k_0 x + B \sinh k_0 x + C \cos k_1 x + D \sin k_1 x \qquad (3.I.9)$$

where A, B, C, and D are the constants to be determined from the four boundary conditions given by Eq.(3.I.2).

The general solution of the free vibrations case is then

$$w(x,t)=\left[A\cosh k_0 x + B\sinh k_0 x + C\cos k_1 x + D\sin k_1 x\right]e^{i\omega t} \quad (3.I.10)$$

III.2. Self-adjointness of the System :

Eq.(3.I.5) corresponding to the eigenvalue problem can be put into the non-dimensional form by introducing the variables $z=\frac{x}{l}$, $\tau=\omega^{(n)}t$ corresponding to the non-dimensional lenght and time respectively, $\omega^{(n)}$ is the n^{th} natural frequency of the system. Then the eigenvalue problem takes the form

$$\bar{\mathcal{L}}[\bar{V}]=\left(\frac{\omega}{\omega^{(n)}}\right)^2\cdot\bar{\gamma}\,\bar{V}(z) \quad (3.2.1)$$

with the boundary conditions

$$\bar{V}(0)=0 \quad (3.2.2a)$$

$$\bar{V}'(0)=0 \quad (3.2.2b)$$

$$\bar{V}''(1)=0 \quad (3.2.2c)$$

$$\bar{V}'''(1)+\frac{\bar{\beta}}{g}\cdot\bar{V}'(1)=-\left(\frac{\omega}{\omega^{(n)}}\right)^2\frac{\bar{\beta}}{g}\bar{V}(1) \quad (3.2.2d)$$

where

$$\bar{\mathcal{L}}\equiv\frac{d^4}{dz^4}+\bar{\beta}\frac{d^2}{dz^2}\quad , \qquad (\quad)'\equiv\frac{d(\;)}{dz}$$

and

$$\bar{V}=\frac{V}{l}$$

$$\bar{\beta}=l^2\beta$$

$$\bar{\gamma}=l^4\omega^{(n)^2}\cdot\gamma$$

In general an eigenvalue problem is said to be self-adjoint provided the following relations are satisfied, [9].

$$\int_0^1 r\bar{\mathcal{L}}[s]dz+\sum_{J=1}^{4}\int_0^1 rB_J[s]dz=\int_0^1 s\bar{\mathcal{L}}[r]dz+\sum_{J=1}^{4}\int_0^1 sB_J[r]dz \qquad (3.2.3a)$$

$$\int_0^1 r\bar{\gamma}s\,dz+\sum_{J=1}^{4}\int_0^1 rC_J[s]dz=\int_0^1 s\bar{\gamma}r\,dz+\sum_{J=1}^{4}\int_0^1 sC_J[r]dz \qquad (3.2.3b)$$

where r and s are any two functions satisfying the boundary conditions, while B_j and C_j are operators appearing in the boundary conditions. In our case these operators are

$$B_4\equiv\left.\left(\frac{d}{dz^3}-\frac{\bar{\beta}}{g}\frac{d}{dz}\right)\right|_1 \qquad B_J\equiv 1 \quad J=1,2,3$$

$$C_4\equiv\left.-\frac{\bar{\beta}}{g}\right|_1 \qquad C_J\equiv 0 \quad J=1,2,3$$

It is obvious that the operators $\bar{\gamma}$, B_j (j=I-4), and C_4 are only multiplications by constants, hence Eq.(3.2.3b) is satisfied identically and there remains only to show that the operator $\bar{\mathcal{L}}$ satisfies the relation

$$\int_0^1 r\bar{\mathcal{L}}[s]dz=\int_0^1 s\bar{\mathcal{L}}[r]dz$$

$$\int_0^1 r(s^{(IV)}+\bar{\beta}s^{(II)})dz=\int_0^1 r\frac{d^4s}{dz^4}dz+\int_0^1 r\bar{\beta}\frac{d^2s}{dz^2}dz$$

Integrating the above integrals by parts we get,

$$=r\left.\left(\frac{d^3s}{dz^3}+\bar{\beta}\frac{ds}{dz}\right)\right|_1-s\left.\left(\frac{d^3r}{dz^3}+\bar{\beta}\frac{dr}{dz}\right)\right|_1$$

$$+\int_0^1\frac{d^4r}{dz^4}\cdot s\,dz+\int_0^1\bar{\beta}\frac{d^2r}{dz^2}\cdot s\,dz$$

Using the boundary condition (3.2.2d)

$$= r\cdot\left(-\frac{\omega^2}{\omega_i^{(n)2}}\cdot\frac{\bar{\beta}}{g}\cdot s\right)\Big|_1 - s\left(-\frac{\omega^2}{\omega^{(n)2}}\cdot\frac{\bar{\beta}}{g}\cdot r\right)\Big|_1 + \int_0^1 s\bar{\mathcal{L}}[r]\,dz$$

Thus the self-adjointness of the system has been proven, and it is a known fact that the eigenfunctions of such a system are orthogonal [9]. The orthogonality of eigenfunctions can be stated as

$$\int_0^1 \bar{V}_i(z)\cdot\bar{V}_j(z)\,dz = N_{(i)}\delta_{ij} \qquad (3.2.4)$$

where

$$N_{(i)} = \int_0^1 \bar{V}_i^2(z)\,dz$$

and

$$\delta_{ij}\begin{cases} I & \text{when} \quad i=j \\ 0 & \text{〃} \quad i\neq j \end{cases}$$

Let $q_i(z)$ be the normalized form of the function $\bar{V}_i(z)$ such that

$$q_i(z) = \frac{\bar{V}_i(z)}{\sqrt{N_i}} \qquad (3.2.5)$$

Then $q_i(z)$ form an orthonormal set with the property

$$\int_0^1 q_i(z)\cdot q_j(z)\,dz = \delta_{ij} \qquad (3.2.6)$$

III.3. The Determination of the Natural Frequencies :

Frequency equation of a system is obtained by applying the boundary conditions to the solution of the equation of motion. Thus, applying the boundary condition at $x=0$ yields

$$w(0,t)=0 \longrightarrow A=-C \qquad (3.3.Ia)$$

$$\frac{\partial w(0,t)}{\partial x} = 0 \qquad B = -\frac{k_I}{k_0} \cdot D \tag{3.3.Ib}$$

Hence the solution given by equation (3.I.IO) takes the form

$$w(x,t) = \left[A(\cosh k_0 x - \cos k_1 x) + B\left(\sinh k_0 x - \frac{k_0}{k_1} \sin k_1 x\right)\right] e^{i\omega t} \tag{3.3.2}$$

Now applying the boundary condition at $x=1$ will yield two equations for the unknown constants A and B, that is,

$$0 = A(k_0^2 \cosh k_0 l + k_1^2 \cos k_1 l) + B(k_0^2 \sinh k_0 l + k_0 k_1 \sin k_1 l) \tag{3.3.3a}$$

$$0 = A\left[k_0 k_1^2 \sinh k_0 l - k_1 k_0^2 \sin k_1 l + \frac{\beta}{g}\omega^2(\cosh k_0 l - \cos k_1 l)\right] + B\left[k_0 k_1^2 \cosh k_0 l + k_0^3 \cos k_1 l + \frac{\beta}{g}\left(\sinh k_0 l - \frac{k_0}{k_1} \sin k_1 l\right)\right] \tag{3.3.3b}$$

Equations (3.3.3) can be written in matrix form as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.3.4}$$

where the coefficients

$$\begin{aligned} Q_{11} &= k_0^2 \cosh k_0 l + k_1^2 \cos k_1 l \\ Q_{12} &= k_0 \sinh k_0 l + k_0 k_1 \sin k_1 l \\ Q_{21} &= k_0 k_1^2 \sinh k_0 l - k_0^2 k_1 \sin k_1 l + \frac{\beta}{g}\omega^2(\cosh k_0 l - \cos k_1 l) \\ Q_{22} &= k_0 k_1^2 \cosh k_0 l + k_0^3 \cos k_1 l + \frac{\beta}{g}\omega^2\left(\sinh k_0 l - \frac{k_0}{k_1} \sin k_1 l\right) \end{aligned} \tag{3.3.5}$$

In order to have a non-trivial solution for the equation (3.3.4), the determinant of the coefficient matrix must vanish, i.e.,

$$\det \underset{\approx}{Q} = 0$$

$$Q_{11} \cdot Q_{22} - Q_{12} \cdot Q_{21} = 0$$

Substitution of Q_{ij} (i,j 1,2) into the above equation will yield the desired frequency equation

$$2k_0\omega^2\gamma + \left[k_0(k_0^4 + k_1^4)\operatorname{Cos}k_1 l - \sqrt{\beta^2 + 4\omega^2\gamma}\,\frac{k_0}{k_1}\cdot\frac{\beta}{g}\omega^2 \operatorname{Sin}k_1 l\right]\cdot$$

$$\operatorname{Cosh}k_0 l + \left[\sqrt{\beta^2 + 4\omega^2\gamma}\,\frac{\beta}{g}\omega^2\operatorname{Cos}k_1 l - \beta k_0^2 k_1 \operatorname{Sin}k_1 l\right]\operatorname{Sinh}k_0 l = 0 \qquad (3.3.6)$$

The equation (3.3.6) can be solved numerically for ω when the constants k_0 and k_1 are known for a spesific structure. The roots of the frequency equation $\omega^{(J)}$ (j 1,2,..n) are known as the natural frequencies of the beam-column with a heavy tip mass at the free end.

The constants A and B are related to each other with a constant ratio

$$A = \frac{k_0^2 \operatorname{Sin}k_0 l + k_0 k_1 \operatorname{Sin}k_1 l}{k_0^2\operatorname{Cosh}k_0 l + k_1^2 \operatorname{Cos}k_1 l}\cdot B$$

Therefore the complete solution to the free vibration problem is

$$w(x,t) = \sum_{J=1}^{\infty} A_J\left[\left(\operatorname{Cosh}k_0^{(J)}x - \operatorname{Cos}k_1^{(J)}x\right) - \frac{k_0^{(J)}\operatorname{Cosh}k_0^{(J)}l + k_1^{(J)2}\cdot\operatorname{Cos}k_1^{(J)}l}{k_1^{(J)2}\operatorname{Sinh}k_0^{(J)}l + k_0^{(J)}\operatorname{Sin}k_1^{(J)}l}\right.$$

$$\left.\left(\operatorname{Sinh}k_0^{(J)}x - \frac{k_0^{(J)}}{k_1^{(J)}}\cdot\operatorname{Sin}k_1^{(J)}x\right)\right]e^{i\omega^{(J)}t} \qquad (3.3.7)$$

Each term in this series is known as the normal mode of the free vibration.

CHAPTER IV

STABILITY ANALYSIS

In this chapter, we will find the unstable regions in which instability is taken as the unboundedness of the solutions to the transverse equation of motion. In Section III.3, orthogonality of the normal modes was proven for the free vibration problem, therefore the solution to the forced vibration problem can be taken as a superposition of these normal modes (eigenfunction expansion)[9]. The coefficient of each term in this series expansion will be in general a function of time. Such an approach will yield coupled Mathieu equations to be solved.

The method generalized by C.S. Hsu [4] will be used in this chapter in solving the above-mentioned Mathieu equations. This method enables us to observe both simple and combination resonances while the one introduced by V.V. Bolotin [5] enables us to observe only the former.

In Hsu's method, the boundaries of the stable and unstable regions are found using a combination of perturbation and variation of parameters techniques.

IV.I. Derivation of the Space Independent Equation :

The general equation of motion (2.2.IO) can be written in terms of dimensionless variables z and τ as

$$\frac{\partial^4 \bar{w}}{\partial z^4} + \left[\beta + \bar{\alpha}\, G(\tau)\right] \frac{\partial^2 \bar{w}}{\partial z^2} + \bar{\gamma}\, \frac{\partial^2 \bar{w}}{\partial \tau^2} = 0 \qquad (4.I.I)$$

where $G(\tau) = \cos \nu \tau$, $\tau = \omega^{(n)} . t$

$$\nu = \frac{\Omega}{\omega^{(n)}}$$

$$\bar{\alpha} = l^2 \alpha$$

$$\bar{w} = \frac{w}{l}$$

Note that the i^{th} non-dimensional natural frequency ω_i is

$$\omega_i = \frac{\omega^{(i)}}{\omega^{(n)}}$$

The solution to equation (4.I.I) can be assumed to be of the form

$$w(z, \tau) = \sum_{i=1}^{\infty} \phi_i(\tau) \, . \, q_i(z) \qquad (4.I.2)$$

where $q_i(z)$ is the i^{th} normal mode and $\phi_i(\tau)$ is its corresponding time dependent amplitude. This assumption is known as eigenfunction expansion [9]. By substituting (4.I.2) into the equation (4.I.I) and applying Galerkins method , a system of coupled Mathieu equations whose solutions will determine the stability of the system is obtained.

$$\sum_{i=1}^{\infty} \phi_i(\tau) \bar{\mathcal{L}} \left[q_i(z) \right] + \bar{\gamma} \sum_{i=1}^{\infty} \ddot{\phi}_i(\tau) q_i(z) + \bar{\alpha} G(\tau) \sum_{i=1}^{\infty} \phi_i(\tau) q_i''(z) = 0 \qquad (4.I.3)$$

where $(\dot{\ }) = \frac{d}{d\tau}$, $(\ )' = \frac{d}{dz}$

Multiplying the above equation by $q_j(z)$ and integrating over (0, I) and recalling the orthogonality relation (3.2.6), we get

$$\ddot{\phi}_i(\tau) + \omega_i^2 \phi_i(\tau) + \frac{\overline{\alpha}}{\overline{\delta}} \cdot G(\tau) \sum_{i=1}^{\infty} \phi_i(\tau) \int_0^1 q_i'' \cdot q_J \, dz = 0 \qquad (4.I.4)$$

Defining

$$E_{ij} = \int_0^1 q_j''(z) \cdot q_i(z) \, dz$$

the above equation takes the form

$$\ddot{\phi}_i(\tau) + \omega_i^2 \phi_i(\tau) + \epsilon \overline{\delta} G(\tau) \sum_J E_{iJ} \phi_J(\tau) = 0 \qquad (4.I.5)$$

where

$$\epsilon = \frac{X_o}{l}$$

$$\delta = \frac{M}{m_c \, l}$$

$$\overline{\delta} = \nu^2 \delta$$

and X_o is the amplitude of the excitation of the support.

Defining the following matricies

$$\ddot{\underset{\sim}{\phi}} = \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \cdot \\ \cdot \\ \ddot{\phi}_n \\ \cdot \\ \cdot \end{bmatrix} \qquad \underset{\approx}{\lambda} = \begin{bmatrix} \omega_1^2 & 0 & \cdots & \\ 0 & \omega_2^2 & & \\ \cdot & & \ddots & \\ & & & \omega_n^2 \\ & & & & \ddots \end{bmatrix}$$

$$\underset{\approx}{E} = \begin{bmatrix} E_{11} & E_{12} & \cdot & \cdot & \cdot \\ E_{21} & E_{22} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ & & & E_{nn} & \cdot & \cdot \\ & & & \cdot & \\ & & & \cdot & \end{bmatrix} \qquad \underset{\sim}{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \\ \cdot \end{bmatrix}$$

equation (4.I.5) can be written in matrix form as

$$\ddot{\underset{\sim}{\phi}} + \left[\underset{\approx}{\lambda} + \epsilon \bar{\delta} G(\tau) \underset{\approx}{E} \right] \underset{\sim}{\phi} = \underset{\sim}{0} \qquad (4.I.7)$$

IV.2. The Stability of a System of Coupled Mathieu Equations :

The solution to the equation (4.I.5), according to the method introduced by C.S. Hsu [4], is assumed to be composed of two parts. Since ϵ is a small parameter, the first part of the solution will be the perturbation part. The second part of the solution is found from the undetermined coefficients method and is of the form of the solution to be perturbed about with time dependent coefficients.

The system of equations (4.I.5) can be reduced to first order system of equations by defining a new function $h_i(\tau)$ as

$$\left.\begin{aligned} h_i &= \dot{\phi}_i \\ \dot{h}_i + \omega_i^2 \phi_i &= -\epsilon \bar{\delta} G(\tau) \sum_j E_{ij} \phi_j \end{aligned}\right\} \qquad (4.2.I)$$

Note that if the small parameter ϵ tends to zero the solution of (4.2.I) becomes

$$\phi_i(\tau) = A_i \operatorname{Cos}\omega_i\tau + B_i \operatorname{Sin}\omega_i\tau$$

$$h_i(\tau) = -\omega_i A_i \operatorname{Sin}\omega_i\tau + \omega_i B_i \operatorname{Cos}\omega_i\tau$$

where A_i and B_i are constants.

For $\epsilon = 0$, the above solutions for ϕ_i and h_i can be assumed to be in the form of perturbation about the above solutions with time dependent coefficients. Combining the method of undetermined coefficients with the perturbation method we assume a solution of the form

$$\left.\begin{aligned} \phi_i(\tau) &= A_i(\tau)\,\mathrm{Cos}\,\omega_i\tau + B_i(\tau)\,\mathrm{Sin}\,\omega_i\tau + \sum_{q=1}^{n} \epsilon^{q}\,\phi_i^{(q)}(\tau) \\ h_i(\tau) &= -\omega_i A_i(\tau)\,\mathrm{Sin}\,\omega_i\tau + \omega_i B_i(\tau)\,\mathrm{Cos}\,\omega_i\tau + \sum_{q=1}^{n} \epsilon^{q}\,h_i^{(q)}(\tau) \end{aligned}\right\} \quad (4.2.2)$$

where A_i and B_i as well as $\phi_i^{(q)}$ and $h_i^{(q)}$ are all functions of time.

Substitution of (4.2.2) into (4.2.I) and applying the method of undetermined coefficients yields

$$\dot{A}_i(\tau)\,\mathrm{Cos}\,\omega_i\tau + \dot{B}_i(\tau)\,\mathrm{Sin}\,\omega_i\tau = 0 \quad (4.2.3a)$$

$$-\omega_i\dot{A}_i\,\mathrm{Sin}\,\omega_i\tau + \omega_i\dot{B}_i\,\mathrm{Cos}\,\omega_i\tau + \sum_{q=1}^{n} \epsilon^{q}\left(\ddot{\phi}_i^{(q)} + \omega_i^2\phi_i^{(q)}\right)$$

$$= -\epsilon\bar{\delta}\,G(\tau)\sum_J E_{iJ}\left[A_J\,\mathrm{Cos}\,\omega_J\tau + B_J\,\mathrm{Sin}\,\omega_J\tau + \sum_{q=1}^{n}\epsilon^{q}\phi_J^{(q)}\right] \quad (4.2.3b)$$

Considering only the terms with coefficients upto the first power of ϵ (first approximation), and substituting $G(\tau)$ into equation (4.2.3b) yields a second order differential equation for the time dependent coefficients. Equating the terms having ϵ on both sides of the above equation yields

$$\ddot{\phi}_i^{(1)} + \omega_i^2\phi_i^{(1)} = -\frac{\bar{\delta}}{2}\sum_J E_{iJ}\left\{A_J\left[\mathrm{Cos}(\omega_J-\nu)\tau + \mathrm{Cos}(\omega_J+\nu)\tau\right] + B_J\left[\mathrm{Sin}(\omega_J-\nu)\tau + \mathrm{Sin}(\omega_J+\nu)\tau\right]\right\} \quad (4.2.4)$$

Particular solution of the above equation is

$$\phi_i^{(1)}\Big|_p = -\frac{\bar{\delta}}{2}\sum_J\left\{\frac{E_{iJ}}{\omega_i^2-(\omega_J+\nu)^2}\left[A_J\,\mathrm{Cos}(\omega_J+\nu)\tau + B_J\,\mathrm{Sin}(\omega_J+\nu)\tau\right] + \frac{E_{iJ}}{\omega_i^2-(\omega_J-\nu)^2}\left[A_J\,\mathrm{Cos}(\omega_J-\nu)\tau + B_J\,\mathrm{Sin}(\omega_J-\nu)\tau\right]\right\} \quad (4.2.5)$$

Note that $\phi_i^{(1)}\Big|_p$ being the time dependent coefficient of the solution to the equation (4.I.I) will become infinetely large as $|\omega_i| \to |\omega_j \mp \nu|$ and this corresponds to some sort of a resonance phenomena. The essential feature of the method we have used is to associate resonance causing terms with the variational part of the solution [4]. These troublesome terms can be removed from the perturbation part of the solution. Now let us consider several resonance cases found from the perturbation analysis.

<u>Case I :</u> $\nu = \omega_k + \omega_j + \epsilon\lambda$, $k \neq j$ (λ is a real finite number)

If we set $i=k$ in (4.2.5), it is seen that when the forcing frequency ν is given as above, the j^{th} term ($j=j$) of the second expression on the right-hand side of the k^{th} equation (4.2.5) will go to infinity as $\epsilon \to 0$. Similarly we get the same resonance case by interchanging the indicies as $i=j$ and $j=k$ in the summation. Since resonance case is related with the variational part, we obtain four differential equations for A_k , B_k , A_j and B_j by considering only the variational part of the equation (4.2.3)

$$\dot{A}_k \operatorname{Cos}\omega_k\tau + \dot{B}_k \operatorname{Sin}\omega_k\tau = 0 \qquad (4.2.6a)$$

$$-\omega_k \dot{A}_k \operatorname{Sin}\omega_k\tau + \omega_k \dot{B}_k \operatorname{Cos}\omega_k\tau = -\frac{\epsilon}{2}\,\bar{\delta}\, E_{kj}\Big[A_j \operatorname{Cos}(\omega_k+\epsilon\lambda)\tau - B_j \operatorname{Sin}(\omega_k+\epsilon\lambda)\tau\Big] \qquad (4.2.6b)$$

$$\dot{A}_j \cos\omega_j\tau + \dot{B}_j \sin\omega_j\tau = 0 \tag{4.2.7a}$$

$$-\omega_j \dot{A}_j \sin\omega_j\tau + \omega_j \dot{B}_j \cos\omega_j\tau = -\frac{\epsilon}{2}\bar{\delta} E_{Jk}\Big[A_k \cos(\omega_j+\epsilon\lambda)\tau - B_j \sin(\omega_k+\epsilon\lambda)\tau\Big] \tag{4.2.7b}$$

Solving (4.2.6) for $\dot{A}_k$ and $\dot{B}_k$ we get

$$\dot{B}_k = -\frac{\epsilon\bar{\delta}}{2\omega_k} E_{kJ}\Big[A_J(\cos^2\psi_k \cos\epsilon\lambda\tau - \cos\psi_k \sin\psi_k \cdot \sin\epsilon\lambda\tau) - B_J(\cos\psi_k \sin\psi_k \cdot \cos\epsilon\lambda\tau + \cos^2\psi_k \sin\epsilon\lambda\tau)\Big] \tag{4.2.8a}$$

$$\dot{A}_k = \frac{\epsilon\bar{\delta}}{2\omega_k} \cdot E_{kJ}\Big[A_J(\cos^2\psi_k \sin\psi_k \cos\epsilon\lambda\tau - \sin^2\psi_k \sin\epsilon\lambda\tau) - B_J(\sin^2\psi_k \cos\epsilon\lambda\tau + \cos\psi_k \sin\psi_k \sin\epsilon\lambda\tau)\Big] \tag{4.2.8b}$$

where $\psi_k = \omega_k \cdot \tau$

In order to simplify the equations (4.2.8a - b), we take the average values of the right-hand sides of them with respect to ψ_k over a period of 2π according to the method of Kryloff-Bogoliuboff-Van der Pol [1].* In that calculation A_j and B_j are considered to be constants. In this way we get

$$\left.\begin{aligned} \dot{A}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kJ}\Big[A_J \sin\epsilon\lambda\tau + B_J \cos\epsilon\lambda\tau\Big] \\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kJ}\Big[A_J \cos\epsilon\lambda\tau - B_J \sin\epsilon\lambda\tau\Big] \end{aligned}\right\} \tag{4.2.9a}$$

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* Since $\epsilon\lambda \ll \omega_k$, rapid oscillations do not contribute to the changes occurring in A_j and B_j.

Similarly for $\dot{A}_j$ and $\dot{B}_j$

$$\left.\begin{aligned}\dot{A}_j &= -\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk}\left[A_k \sin\epsilon\lambda\tau + B_k \cos\epsilon\lambda\tau\right]\\ \dot{B}_j &= -\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk}\left[A_k \cos\epsilon\lambda\tau - B_k \sin\epsilon\lambda\tau\right]\end{aligned}\right\} \quad (4.2.9b)$$

We will now solve the coupled equations (4.2.9) for A_k , B_k , A_j and B_j , then the stability problem reduces to determining when these time dependent coefficients appearing in the solution of the equation (4.I.I) remain bounded or increase indefinetely with time.

We will now define the following functions in order to decouple the equations (4.2.9)

$$\left.\begin{aligned} X_1 &= A_k + iB_k \qquad & Y_1 &= A_j + iB_j\\ X_2 &= A_k - iB_k \qquad & Y_2 &= A_j - iB_j\end{aligned}\right\} \quad (4.2.I0)$$

Differentiating (4.2.I0) once with respect to τ and substituting (4.2.9) gives the following system of first order differential equations for X_1 , X_2 , Y_1 and Y_2 .

$$\dot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_2 e^{-i\epsilon\lambda\tau} \quad (4.2.IIa)$$

$$\dot{X}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_1 e^{i\epsilon\lambda\tau} \quad (4.2.IIb)$$

$$\dot{Y}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_2 e^{-i\epsilon\lambda\tau} \quad (4.2.IIc)$$

$$\dot{Y}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_1 e^{i\epsilon\lambda\tau} \quad (4.2.IId)$$

Consider the coupled equations (4.2.IIa) and (4.2.IId) differentiating the first with respect to τ we get

$$\ddot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kJ}\dot{Y}_2 e^{-i\epsilon\lambda\tau} - \frac{\epsilon\bar{\delta}}{4\omega_k} E_{kJ} Y_2 e^{-i\epsilon\lambda\tau}\epsilon\lambda$$

Substituting now $\dot{Y}_2$ (Eq. 4.2.IId) into the above equation yields

$$\ddot{X}_1 + i\epsilon\lambda\dot{X}_1 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J} E_{kJ} E_{Jk} = 0 \qquad (4.2.I2)$$

Similarly we obtain for the others

$$\ddot{Y}_2 - i\epsilon\lambda\dot{Y}_2 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J} E_{kJ} E_{Jk} Y_2 = 0 \qquad (4.2.I3)$$

$$\ddot{X}_2 - i\epsilon\lambda\ddot{X}_2 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J} E_{kJ} E_{Jk} X_2 = 0 \qquad (4.2.I4)$$

$$\ddot{Y}_1 + i\epsilon\lambda\dot{Y}_1 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J} E_{kJ} E_{Jk} Y_1 = 0 \qquad (4.2.I5)$$

The general form of the characteristic equations of the second order differential equations (4.2.I2 - I5) is

$$m^2 \mp (i\epsilon\lambda)m - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J} E_{kJ}\cdot E_{Jk} = 0$$

On the other hand, the general solution of these differential equations is of the form

$$C_1 e^{m_1} + C_2 e^{m_2}$$

where C_1 and C_2 are constants to be determined and m_1 and m_2 are the roots of the characteristic equation which are in general complex numbers with real and imaginary parts. The imaginary parts giving rise to oscillatory type motion do not contribute to instability, however, the real part depending on its sign determines the stability of the structure. Therefore, from equation (4.2.10) we have

$$A_k = \frac{1}{2}(X_1 + X_2) \qquad A_j = \frac{1}{2}(Y_1 + Y_2)$$

$$B_k = \frac{i}{2}(X_2 - X_1) \qquad B_j = \frac{i}{2}(Y_2 - Y_1)$$

Considering the equations (4.1.2), (4.2.5) and the above ones we get

$$w = w(X_1 , X_2 , Y_1 , Y_2)$$

Hence, the instability of the system is determined by the real part of the roots m_1 and m_2 which is equal to

$$\sqrt{-\epsilon^2 \lambda^2 + \frac{\epsilon^2 \bar{\delta}^2}{4\omega_j \omega_k} E_{kj} E_{jk}} \equiv \sqrt{\Delta_I}$$

All the terms except E_{kj} and E_{jk} are positive, therefore the signs of the elements of $\underset{\approx}{E}$ matrix determine whether the system is stable or not. If the sign of the terms under radical sign is negative the solutions are stable, otherwise unstable.

$$\Delta_I \begin{array}{l} < 0 \longrightarrow \text{Stable} \\ > 0 \longrightarrow \text{Unstable} \end{array}$$

If E_{kj} and E_{jk} are of opposite signs, the quantity under the radical sign is always negative and the system is stable.

In the case where E_{kj} and E_{jk} are of the same sign, the product of these terms will always be positive. Then we have to examine the inequality

$$-\epsilon^2\lambda^2 + \frac{\epsilon^2\bar{\delta}^2}{4\omega_k\omega_j} E_{kj}E_{jk} > 0$$

for instability. The unstable region is found as

$$-\frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}} < \epsilon\lambda < \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}}$$

Substituting $\epsilon\lambda = \nu - (\omega_j+\omega_k)$ we obtain the boundaries of the unstable regions as

$$(\omega_k+\omega_j) - \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}} < \nu < (\omega_k+\omega_j) + \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}}$$

This resonance case is known as " combination resonance of sum type "[6].

<u>Case II :</u> $\nu = 2\omega_k + \epsilon\lambda \quad , \quad i = j = k$

In this case, when the forcing frequency is equal to or nearly equal to twice any of the natural frequencies of the system, the k^{th} term of the second expression on the right-hand side of the equation (4.2.5) will be infinitely large. Following the same procedure as in Case I , we remove the terms causing resonance phenomena from the equations (4.2.3) and associate them with the variational analysis. Thus we obtain for $\dot{A}_k$ and $\dot{B}_k$

$$\left.\begin{aligned}\dot{A}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k}\cdot E_{kk}\left[A_k \sin\epsilon\lambda\tau + B_k \cos\epsilon\lambda\tau\right]\\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kk}\left[A_k \cos\epsilon\lambda\tau - B_k \sin\epsilon\lambda\tau\right]\end{aligned}\right\}\quad (4.2.16)$$

Differentiating X_1 and X_2 defined in (4.2.10) with respect to τ once we get

$$\dot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kk} X_2 e^{-i\epsilon\lambda\tau} \quad (4.2.17a)$$

$$\dot{X}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kk} X_1 e^{i\epsilon\lambda\tau} \quad (4.2.17b)$$

Elimination of X_1 and X_2 between these two equations will yield

$$\ddot{X}_1 + i\epsilon\lambda\dot{X}_1 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k^2}E_{kk}^2\cdot X_1 = 0 \quad (4.2.18)$$

$$\ddot{X}_2 - i\epsilon\lambda\dot{X}_2 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k^2}E_{kk}^2\cdot X_2 = 0 \quad (4.2.19)$$

The real part of the roots of the characteristic equation of the above differential equations determines the instability and is equal to

$$\sqrt{-\epsilon^2\lambda^2 + \frac{\epsilon^2\bar{\delta}^2}{4\omega_k^2}E_{kk}^2} \equiv \sqrt{\Delta_{II}}$$

Instability will occur when $\Delta_{II} > 0$ or

$$-\epsilon^2\lambda^2 + \frac{\epsilon^2\bar{\delta}^2}{4\omega_k^2}E_{kk}^2 > 0$$

By examining this second order inequality, instability region is obtained as

$$2\omega_k - \frac{\epsilon\bar{\delta}}{2\omega_k}E_{kk} < \nu < 2\omega_k + \frac{\epsilon\bar{\delta}}{2\omega_k}E_{kk}$$

This region is known as " principal instability region " [6].

Case III : $\nu = \omega_j - \omega_k + \epsilon\lambda$, $k \neq j$ $j > k$

When the forcing frequency ν approaches the above values, the j^{th} term of the second expression on the right-hand side of the k^{th} equation of (4.2.5) and k^{th} term of the first expression on the right-hand side of the j^{th} equation of (4.2.5) become unbounded. Removing these resonance causing terms from equations (4.2.4) and associating them with the variational part of (4.2.3), as was done in the first two cases, will yield four differential equations for the functions A_k , B_k , A_j and B_j

$$\dot{A}_k \cdot \cos\omega_k\tau + \dot{B}_k \cdot \sin\omega_k\tau = 0 \qquad (4.2.20a)$$

$$-\dot{A}_k \cdot \sin\omega_k\tau + \dot{B}_k \cdot \cos\omega_k\tau = -\frac{\epsilon\bar{\delta}}{2\omega_k} \cdot E_{kj} \left[A_j \cdot \cos(\omega_k - \epsilon\lambda)\tau \quad B_j \cdot \sin(\omega_k - \epsilon\lambda)\tau \right] \qquad (4.2.20b)$$

$$\dot{A}_j \cos\omega_j\tau + \dot{B}_j \cdot \sin\omega_j\tau = 0 \qquad (4.2.21a)$$

$$-\dot{A}_j \sin\omega_j\tau + \dot{B}_j \cdot \cos\omega_j\tau = -\frac{\epsilon\bar{\delta}}{2\omega_j} E_{jk} \left[A_k \cos(\omega_j + \epsilon\lambda)\tau \quad B_k \sin(\omega_j + \epsilon\lambda)\tau \right] \qquad (4.2.21b)$$

Solving equations (4.2.20) and (4.2.21) for $\dot{A}_k$, $\dot{B}_k$, $\dot{A}_j$ and $\dot{B}_j$ and averaging with respect to ψ_k and ψ_j over a period of 2π yields

$$\dot{A}_k = \frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} \cdot \left[A_j \sin\epsilon\lambda\tau + B_j \cos\epsilon\lambda\tau\right]$$
$$\dot{B}_k = -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj}\left[A_j \cos\epsilon\lambda\tau - B_j \sin\epsilon\lambda\tau\right] \qquad (4.2.22a)$$

$$\dot{A}_j = -\frac{\epsilon\bar{\delta}}{4\omega_j} \cdot E_{jk} \cdot \left[A_k \sin\epsilon\lambda\tau - B_k \cos\epsilon\lambda\tau\right]$$
$$\dot{B}_j = -\frac{\epsilon\bar{\delta}}{4\omega_j} \cdot E_{jk}\left[A_k \cos\epsilon\lambda\tau + B_k \sin\epsilon\lambda\tau\right] \qquad (4.2.22b)$$

If the equations (4.2.22) are substituted into the derivatives of (4.2.IO) we get

$$\dot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_1 e^{i\epsilon\lambda\tau} \qquad (4.2.23a)$$
$$\dot{X}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_2 e^{-i\epsilon\lambda\tau} \qquad (4.2.23b)$$
$$\dot{Y}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_1 e^{-i\epsilon\lambda\tau} \qquad (4.2.23c)$$
$$\dot{Y}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_2 e^{i\epsilon\lambda\tau} \qquad (4.2.23d)$$

Decoupling these equations as before we obtain

$$\ddot{X}_1 - i\epsilon\lambda\dot{X}_1 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj} E_{jk} X_1 = 0 \qquad (4.2.24)$$

$$\ddot{X}_2 + i\epsilon\lambda\dot{X}_2 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj} E_{jk} X_2 = 0 \qquad (4.2.25)$$

$$\ddot{Y}_1 + i\epsilon\lambda\dot{Y}_1 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj} E_{jk} Y_1 = 0 \qquad (4.2.26)$$

$$\ddot{Y}_2 - i\epsilon\lambda\dot{Y}_2 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj} E_{jk} Y_2 = 0 \qquad (4.2.27)$$

Once again , the real parts of the roots of the characteristic equations of the above equations are the same, and equal to

$$\sqrt{-\epsilon^2\lambda^2 - \frac{\epsilon^2 \lambda^2}{4\omega_j\omega_k} E_{jk} E_{kj}} = \sqrt{\Delta_{III}}$$

Instability will occur when $\Delta_{III} > 0$ or

$$-\epsilon^2\lambda^2 - \frac{\epsilon^2 \bar{\delta}^2}{4\omega_j\omega_k} E_{jk} E_{kj} > 0$$

It is obvious that when E_{kj} and E_{jk} are of the same sign there would be no instability but if they are of opposite signs instability will occur in the following range of ν

$$(\omega_j-\omega_k) - \frac{\epsilon\bar{\delta}}{2}\sqrt{-\frac{E_{kj} E_{jk}}{\omega_k \omega_j}} < \nu < (\omega_j-\omega_k) - \frac{\epsilon\bar{\delta}}{2}\sqrt{-\frac{E_{kj} E_{jk}}{\omega_k \omega_j}}$$

This resonance case is known as " combination resonance of difference type " [6].

Case IV : $\nu = \epsilon\lambda$, $i=j=k$

This case corresponds to very small forcing frequency. The k^{th} term of the first and second expressions on the right-hand side of the k^{th} equation of (5.2.5) will become very large. Following a similar procedure as in the previous cases, differential equations for $\dot{A}_k$ and $\dot{B}_k$ are obtained as

$$\left.\begin{aligned} \dot{A}_k &= \frac{\epsilon\bar{\delta}}{2\omega_k} E_{kk} B_k \cos\epsilon\lambda\tau \\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{2\omega_k} E_{kk} A_k \cos\epsilon\lambda\tau \end{aligned}\right\} \qquad (4.2.28)$$

Similarly we get

$$\dot{X}_1 + \frac{i\epsilon\bar{\delta}}{2\omega_k} E_{kk} \cdot \cos\epsilon\lambda\tau \cdot X_1 = 0 \qquad (4.2.29)$$

$$\dot{X}_2 - \frac{i\epsilon\bar{\delta}}{2\omega_k} E_{kk} \cdot \cos\epsilon\lambda\tau \cdot X_2 = 0 \qquad (4.2.30)$$

By applying separation of variables method the above equations are solved and the results become

$$X_1 = C_1 \cdot \exp\left(-\frac{i\bar{\delta}}{2\omega_k\lambda} E_{kk} \sin\epsilon\lambda\tau\right) \qquad (4.2.31a)$$

$$X_2 = C_2 \cdot \exp\left(\frac{i\bar{\delta}}{2\omega_k\lambda} E_{kk} \sin\epsilon\lambda\tau\right) \qquad (4.2.31b)$$

Upon examining (4.2.31), it is seen that for this case instability uccurs only if E_{kk} or ω_k have imaginary parts, thus the case of very small forcing frequecy does not affect the stability of the structure.

CHAPTER V

THEORETICAL and EXPERIMENTAL STUDIES

The stability criteria derived in Chapter IV will now be applied to specific structures and the experimental results will be compared with theory. The roots of the frequency equations are found numerically and the elements of $\underset{\approx}{E}$ matricies are evaluated by numerical integrations. The boundaries of the unstable regions are found for each case according to the inequalities given in Chapter IV. These regions will then be graphically presented.

V.I. Theoretical Calculations :

Circular steel and brass bars have been selected for the theoretical and experimental studies in order to make easy interpretations of the results. All the bars are in equal lenght and diameter supporting equal tip masses. The columns having diameters of 2 mm are tested with the weights of 90,I00,II0 and I20 gr and the ones with diameters of 3 mm are tested with the weights of 250,300 and 350 gr. Modulus of elasticity of the bars are

$$E_{steel}=2.I\ IO^{6}\ kg/cm^{2} \qquad E_{brass}=I.I2\ IO^{6}\ kg/cm^{2}$$

and the corresponding densities are

$$d_{steel}=7.8\ gr/cm^{3} \qquad d_{brass}=8.7\ gr/cm^{3}$$

Lengths of the columns were L=270 mm in all cases.

The natural frequencies of a given structure are found from the equation (3.3.6) by giving values to ω from zero onward. A root is found each time the left-hand side of the frequency equation chances sign . Some of these roots are tabulated in Table (5.I).

The perturbation method is valid if $\epsilon\bar{\delta}E_{ij} \ll 1$ or

$$\frac{X_o}{l} . E_{ij} \nu^2 . \frac{M}{m_c l} \ll 1.$$

The other restriction on the criteria developed in Chapter IV is about the "averaging technique" which states that $\omega_i \gg \epsilon\lambda$ where λ is finite. Therefore the parameters of the system must be properly chosen or the amplitude of the excitation should be chosen as small as possible in order to preserve the validity of the above inequalities.

The elements of $\underset{\approx}{E}$ matrix are computed from

$$E_{ij} = \int_0^1 q_i(z) \cdot q_j''(z)\, dz = \frac{\int_0^1 \bar{V}_i(z) . \bar{V}_j''(z)\, dz}{\left[\int_0^1 \bar{V}_i^2(z)\, dz\right]^{1/2} \cdot \left[\int_0^1 \bar{V}_j^2(z)\, dz\right]}$$

A computer program is given in the Appendix for the evaluation of the elements of $\underset{\approx}{E}$ by numerical integration using the trapezoidal method.*

The difference type of combination resonance will not occur for this kind of systems due to the fact that the matrix E has no negative elements. The instability range is then

$$(\omega_k+\omega_j) - \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}\, E_{jk}}{\omega_j\, \omega_k}} < \nu < (\omega_k+\omega_j) + \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}\, E_{jk}}{\omega_j\, \omega_k}}$$

If $k \neq j$ it is called combination resonance of sum type

* See Appendix for the computer programs used.

and if k=j it is known as simple resonance. The expected unstable frequencies of the selected models are given in Table (5.2).

The first four mode shapes for the specimens are schematically shown below.

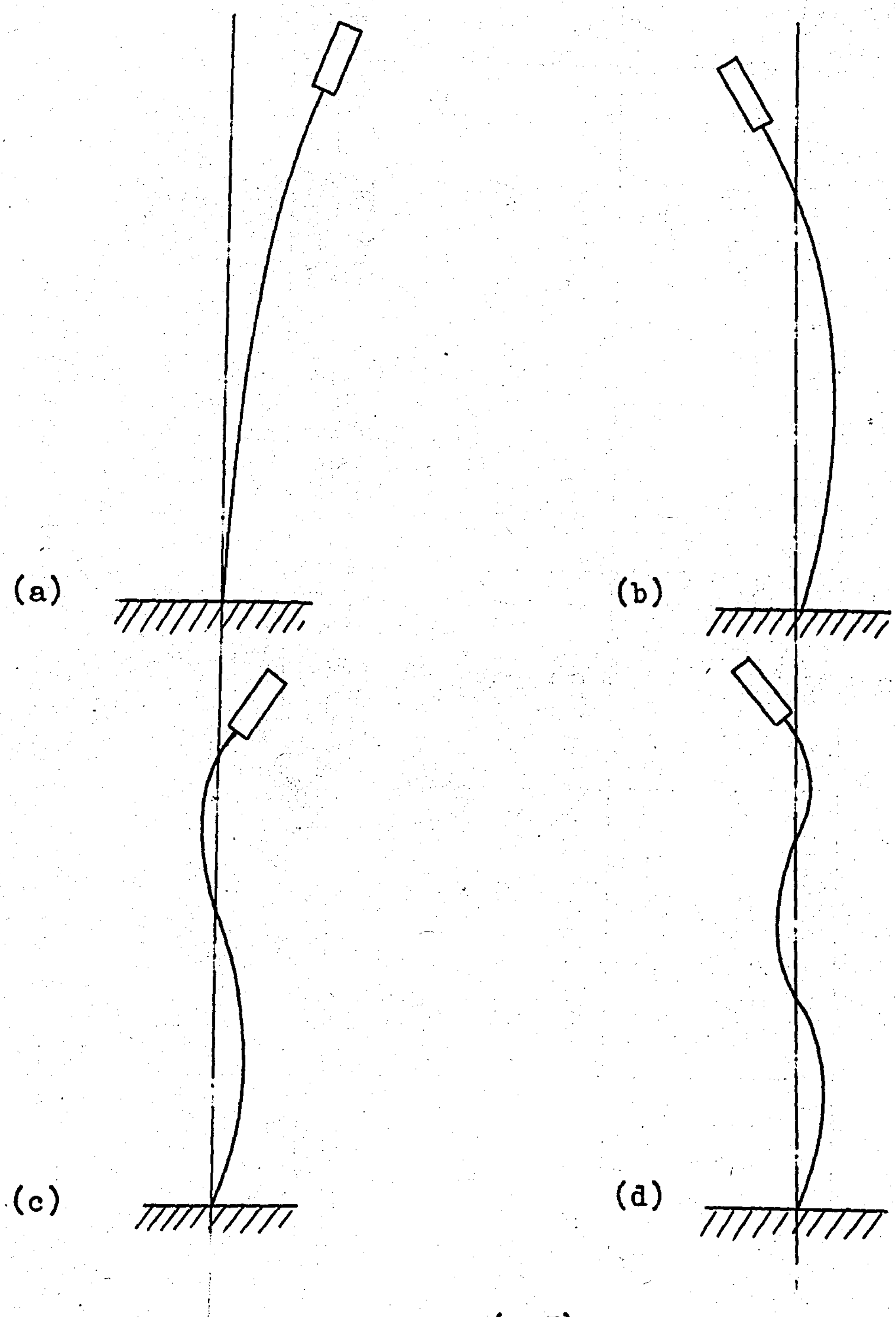


Fig.(5.I)

Diameter (mm)	Tip Mass (gr)	BRASS									
		$2\omega_1$		$\omega_1+\omega_2$		$2\omega_2$		$\omega_1+\omega_3$		$\omega_2+\omega_3$	
		The.	Ex.	The.	Ex.	The.	Exp.	The.	Exp.	The.	Exp.
2	90	3.20	-	60.53	41	117.86	115	194.20	215	251.53	260
	100	2.98	-	60.26	40	117.54	110	193.92	210	251.20	255
	110	2.80	-	60.03	37	117.26	105	193.67	205	250.90	250
	120	2.60	-	59.78	34	116.96	100	193.41	200	250.59	245
3	250	4.76	-	91.72	55	178.68	150	292.99	-	379.95	360
	300	4.26	-	91.24	50	178.22	144	292.47	-	379.45	355
	350	3.88	-	90.82	45	177.76	140	292.03	-	378.97	350

Theoretical and experimental unstable frequencies (Hz)

Table (5.2)

Theoretical unstable regions found from the computer program for the case

Tip mass : M = I20 gr.

Length of C. : L = 270 mm.

Diameter of C: D = 2mm.

Material of C: Brass

are illustrated in Fig.(5.2). The elements of E_{ij} matrix where i and j taken upto 3 are given below.

$$E_{jk} = \begin{bmatrix} 0.0046995 & 0.0185771 & 0.0611296 \\ 0.0042487 & 0.0186938 & 0.0648045 \\ 0.0033405 & 0.0178469 & 0.0675349 \end{bmatrix}$$

By chosing the first natural frequency as the normalizing frequency the first three non-dimensional natural frequencies are

$\omega_1 =$ I.

$\omega_2 = 44.817$

$\omega_3 = 147.207$

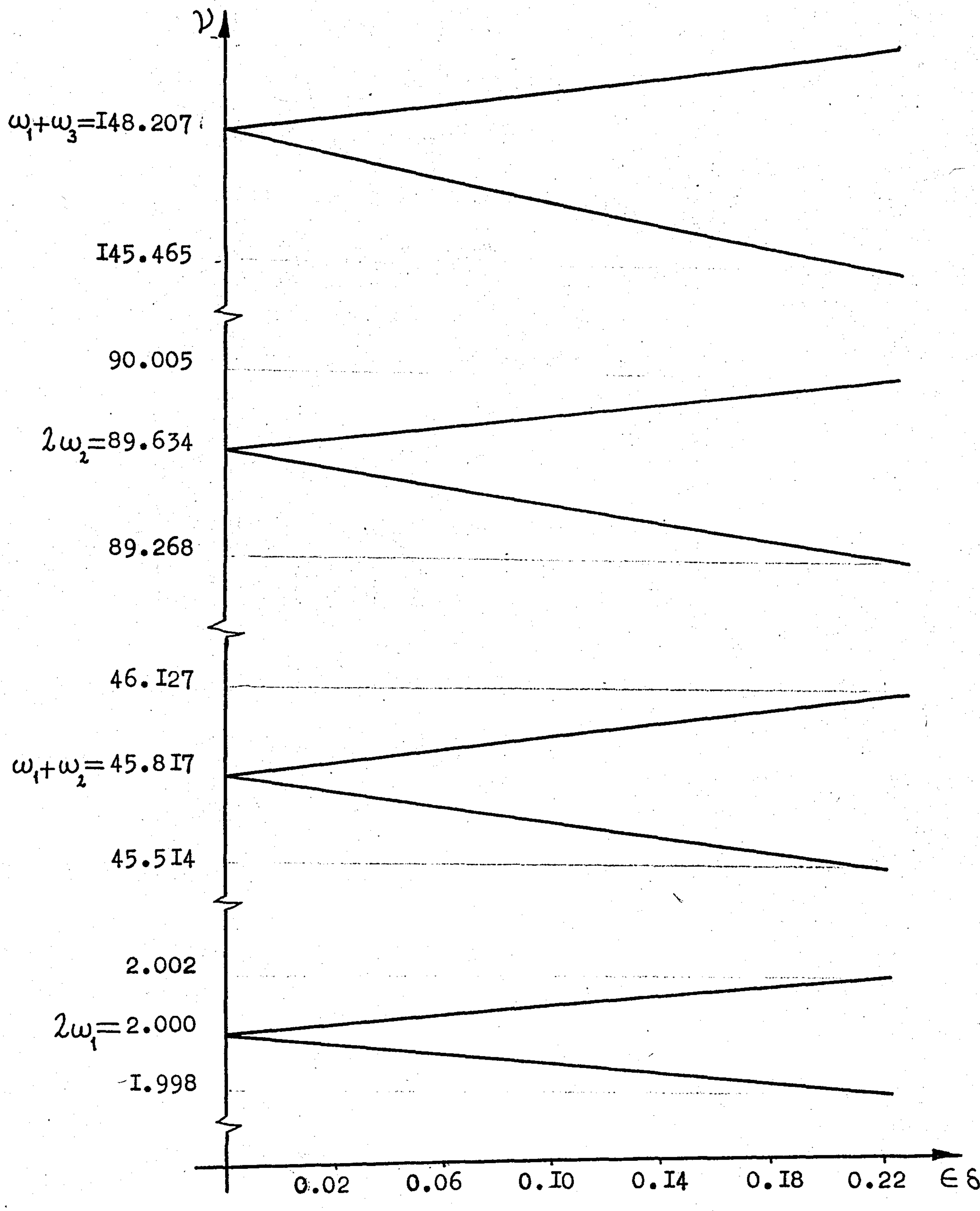

ν
ω₁+ω₃=I48.207
I45.465
90.005
2ω₂=89.634
89.268
46.I27
ω₁+ω₂=45.8I7
45.5I4
2.002
2ω₁=2.000
I.998
0.02
0.06
0.I0
0.I4
0.I8
0.22
εδ

Fig.(5.2)

Theoretical unstable regions for the case

Tip mass : $M = I20$ gr.

Length of C. : $L = 270$ mm.

Diameter of C.: $D = 2$ mm.

Material of C.: Steel

are shown in the Fig.(5.3)

The elements of the E_{ij} matrix are,

$$E_{ij} = \begin{bmatrix} 0.0054095 & 0.0I9I745 & 0.06I9598 \\ 0.0049742 & 0.0I92865 & 0.0654339 \\ 0.004024I & 0.0I8439I & 0.068II37 \end{bmatrix} .$$

and the non-dimensional natural frequencies are

$$\omega_1 = I.$$

$$\omega_2 = 42.046$$

$$\omega_3 = I37.093$$

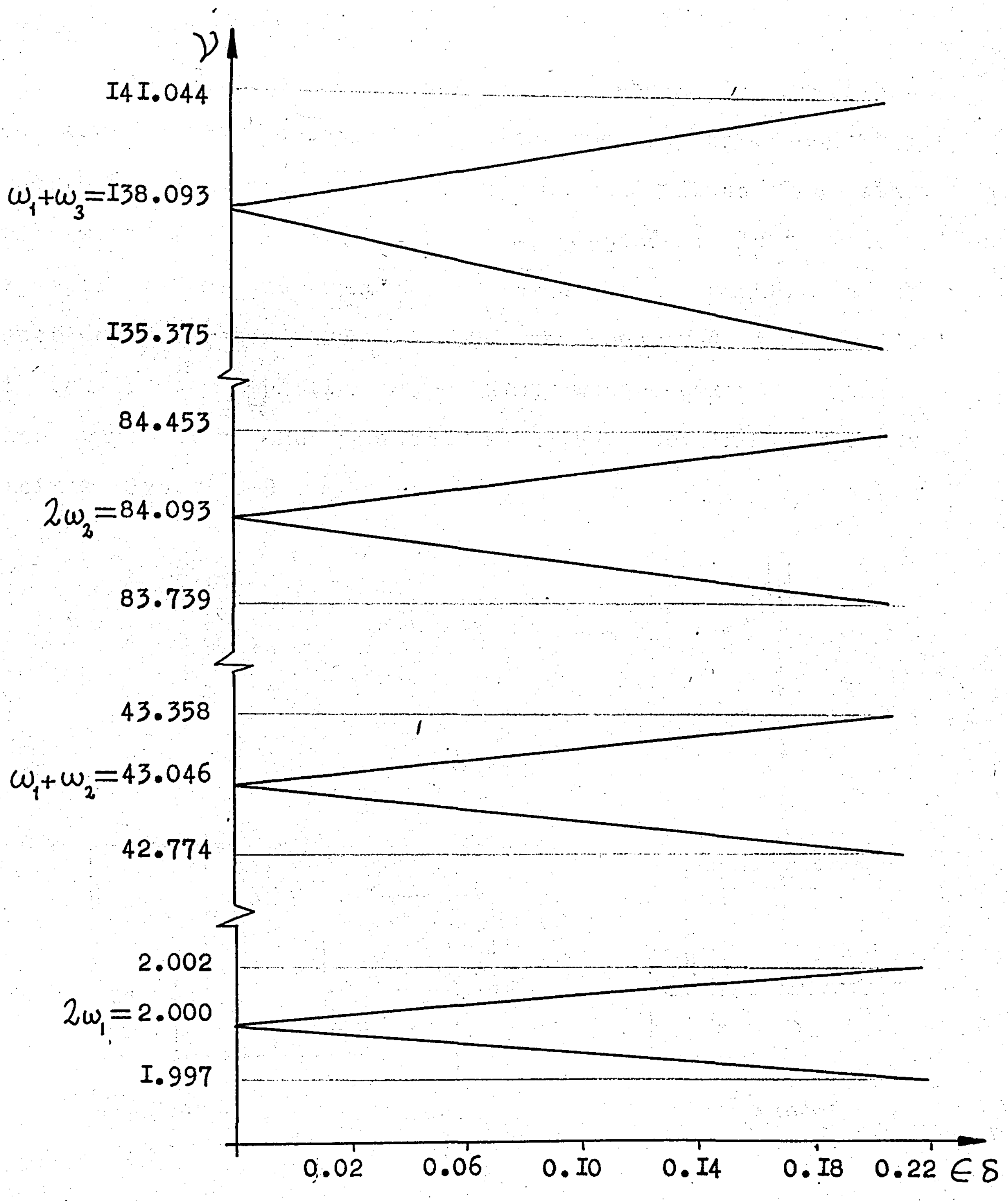

ν
I4I.044
ω1+ω3=I38.093
I35.375
84.453
2ω2=84.093
83.739
43.358
ω1+ω2=43.046
42.774
2.002
2ω1=2.000
I.997
0.02
0.06
0.I0
0.I4
0.I8
0.22
εδ

Fig.(5.3)

V.2.I. Experimental Set-up :

The experimental apparatus is shown in Fig.(5.4). The arrangement depicted satisfies the boundary conditions described in analytical section, one end fixed the other is free with a large mass. The time dependent load (excitation) is applied to the column by means of a shaker as shown. Function Generator Type TM 50I is connected to the input of the Power Amplifier Type 2706 whose output supplies sine wave with the required amplitude to the Vibration Exciter Type 4809.

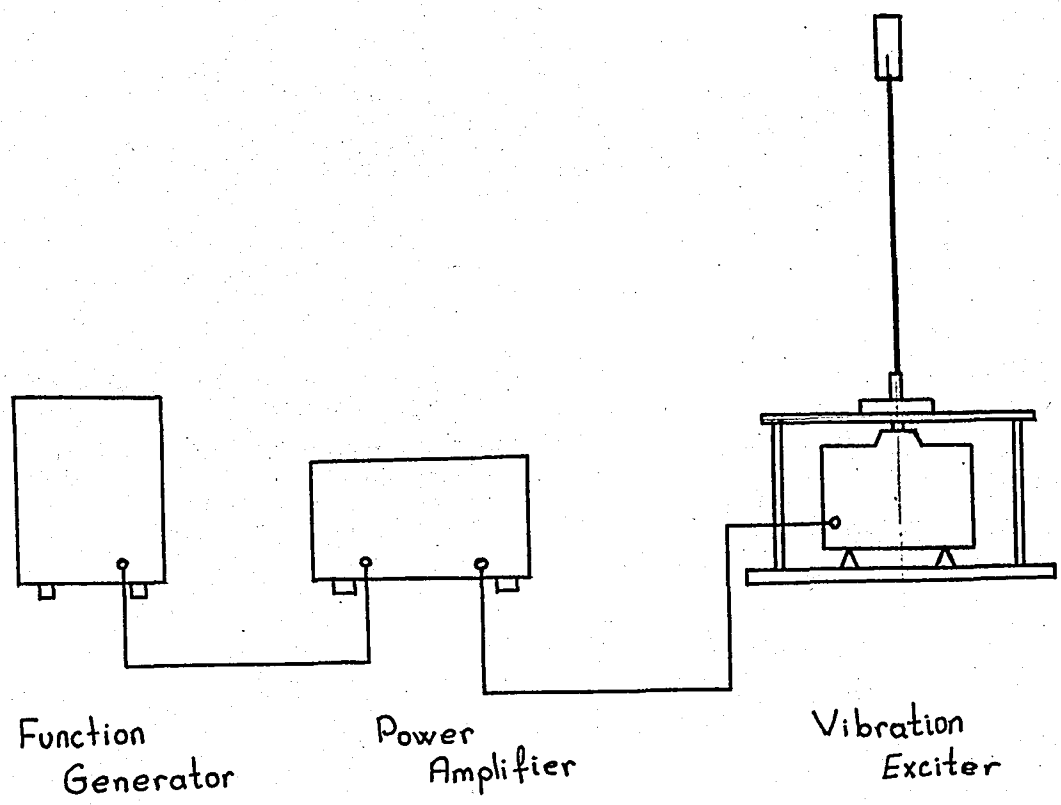


Fig. (5.4)

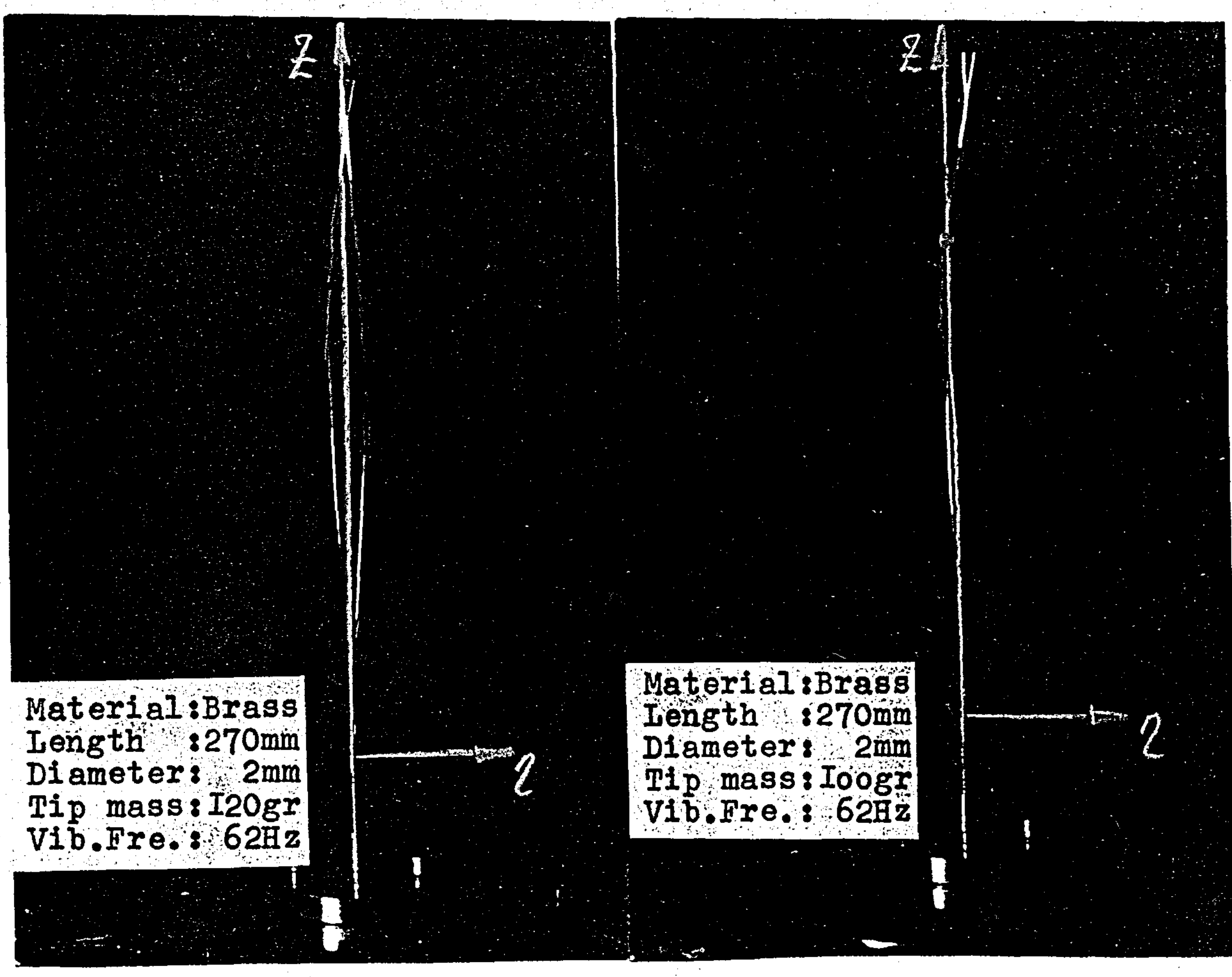
Diameter (mm)	Tip Mass (gr)	STEEL									
		$2\omega_1$		$\omega_1 + \omega_2$		$2\omega_2$		$\omega_1 + \omega_3$		$\omega_2 + \omega_3$	
		The.	Ex.	The.	Ex.	The.	Exp.	The.	Exp.	The.	Exp.
2	90	4.78	-	88.40	108	172.02	155	282.07	315	365.69	350
	100	4.54	-	88.18	100	171.82	150	281.82	305	365.46	347
	110	4.24	-	87.90	95	171.56	145	281.53	295	365.19	345
	120	4.06	-	87.68	90	171.30	140	281.31	285	364.93	342
3	250	6.80	-	133.35	-	259.90	205	424.87	-	551.42	-
	300	6.16	-	132.80	-	259.44	200	424.31	-	550.95	-
	350	5.66	-	132.66	-	249.06	195	423.84	-	550.54	-

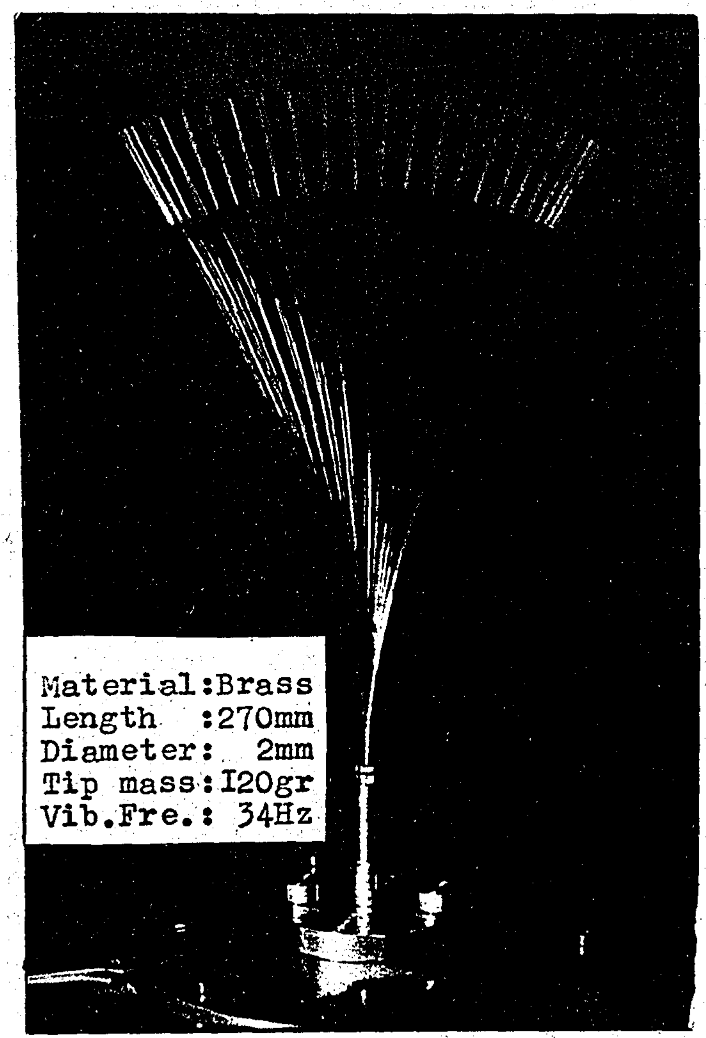
Theoretical and experimental unstable frequencies (Hz)

Table (5.3)

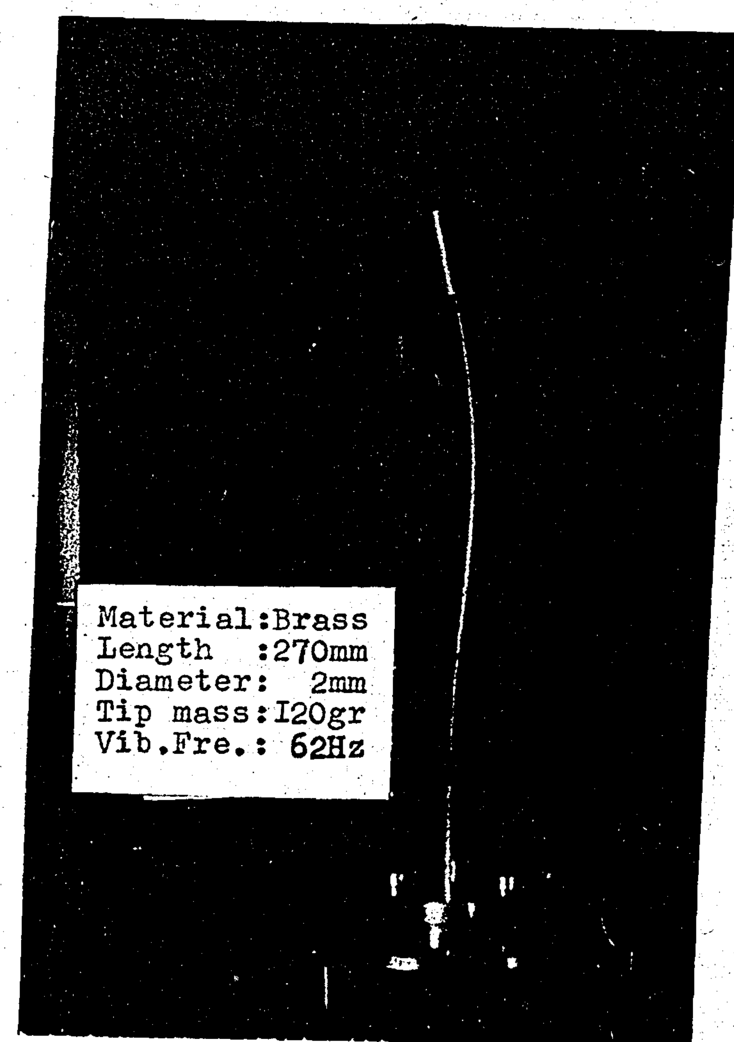
V.2.2. Experiments :

Coordinates of the nodes change with the constant $\delta = M/m_c.l$. In the reference [7] , some of the non-dimensional coordinates were given for the various values of δ . We have found experimentally the change in coordinates of the nodes with the increasing tip mass M. It is experimentally verified that the coordinate z increases as the tip mass M increases.

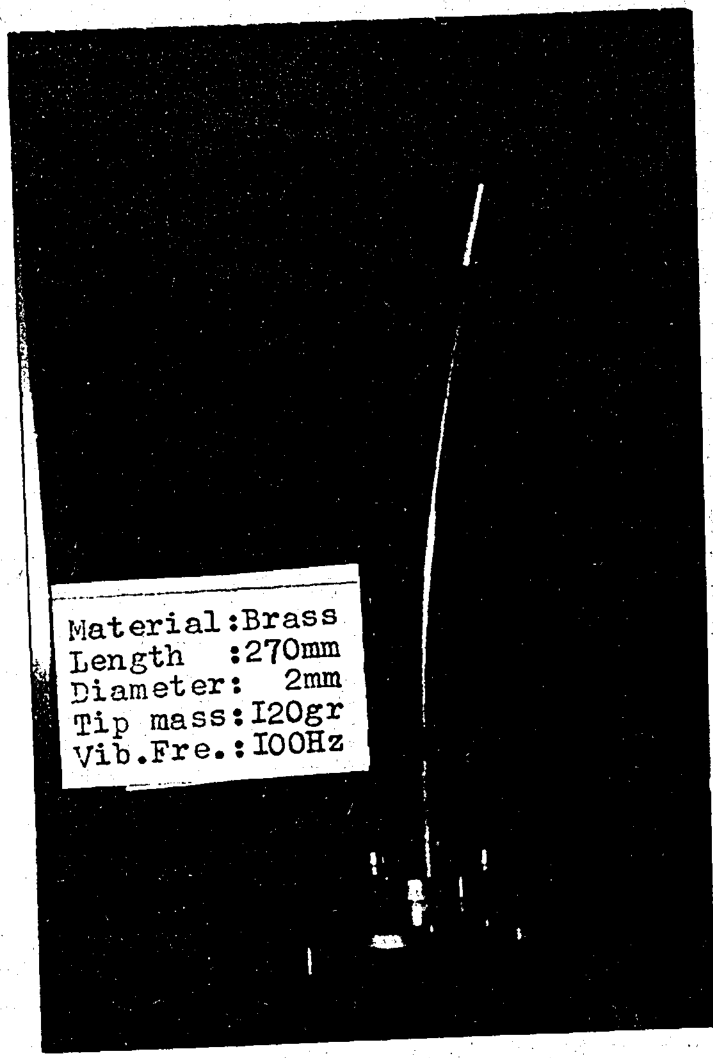




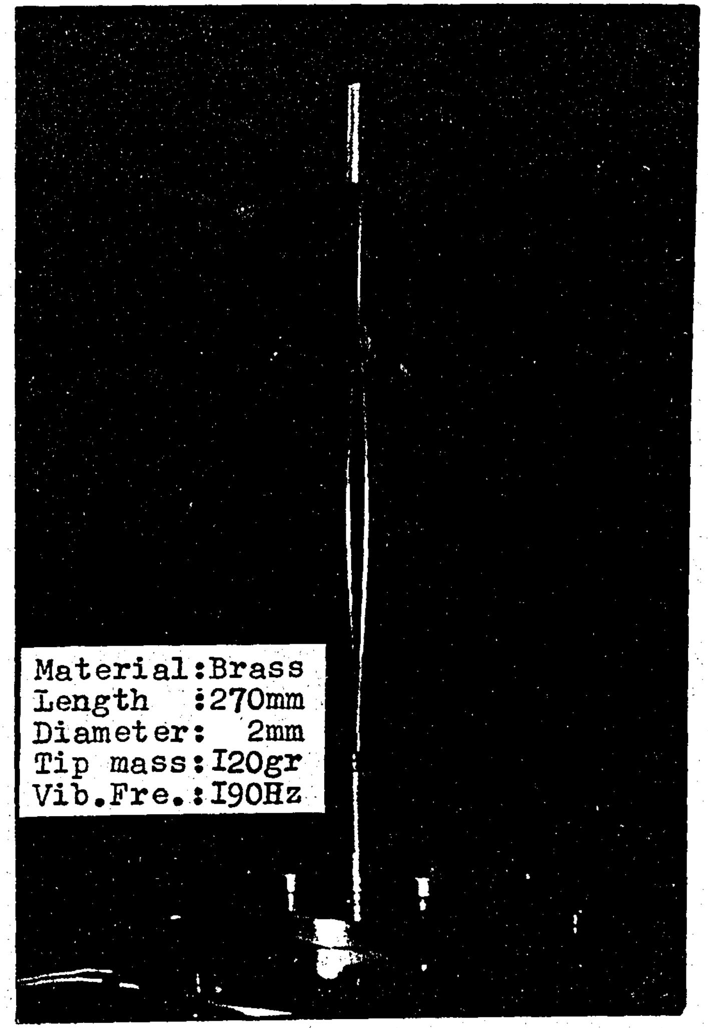
Instability



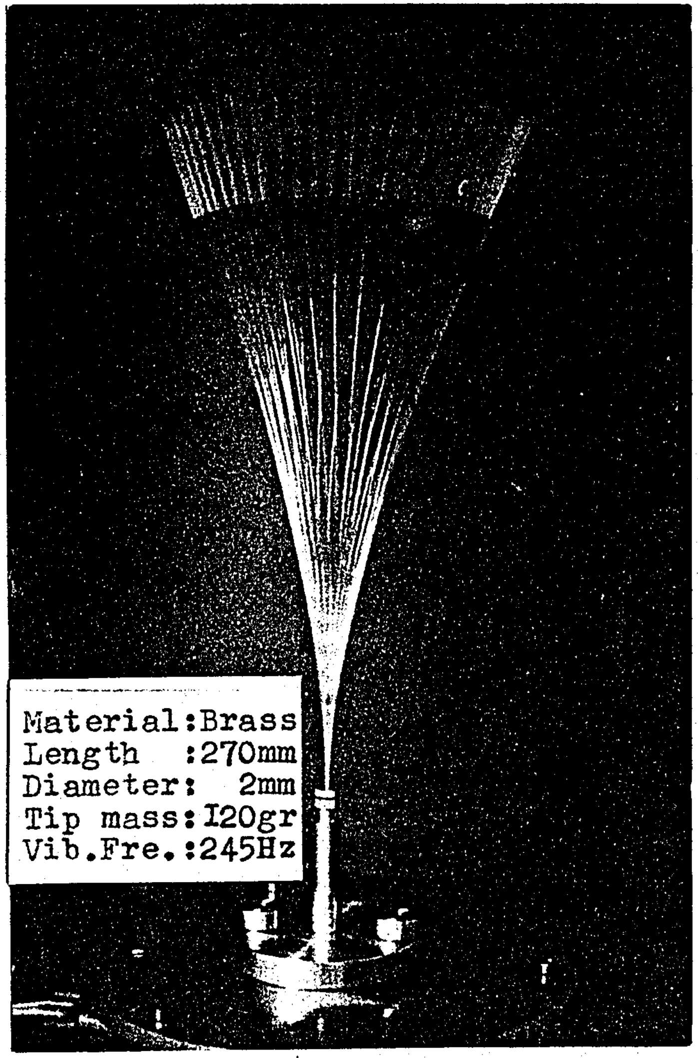
Mode shape



Instability



Mode shape



Instability

Mode shape

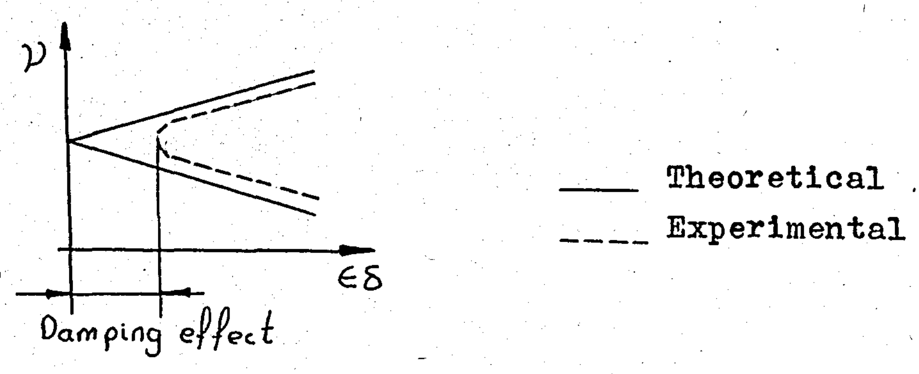
CHAPTER VI

CONCLUDING REMARKS

In this work, the importance of combination resonances on the stability of a structure composed of a cantilevered elastic column supporting a large mass at the free end and subjected to vertical harmonic excitation at the support has been examined. Theoretical results along with the experimental results were presented and compared.

It was found that for a system described in the analytical section combination resonance of difference type does not occur due to the characteristics of the elements of the coupling matrix $\underset{\sim}{E}$.

Experiments showed that, internal material damping allows resonance phenomena to begin at the higher values of $\epsilon\delta$ such that (or excitation amplitude)



The effects of the tip mass, Young Modulus of the column and length of the column on the natural frequencies and the stability of the column have been examined. Experimentally found natural frequencies are more appropriate to the theoretical ones than the frequencies cousing instability. This is due to the fact that, we have made an additional approximation in the stability analysis.Rotatory inertia and shear deformation effects cause the differences between the experimental and theoretical natural frequencies.

APPENDIX

COMPUTER PROGRAMS

```
      DOUBLE PRECISION UM,Z,FZ,QUO,SO,SI,TI,T2,T3,T4,T5,T6,T7,A,
     $B,SQRT,DCOS,DSIN,DABS,ABSTOT,TOT,SAL
      DIMENSION FZ(4IOO),ABSFZ(4IOO)
C     ROOTS OF FREQUENCY EQUATION : FZ=0
C     Z=NATURAL FREQUENCY
      READ(5,*) G,AL,W,E,RHO,AI,AREA
      EI=E*AI
      UM=RHO*AREA/G
      A=UM/EI
      B=W/EI
      J=0
      Z=0.
      DO IOO I I,4000
      QUO=DSQRT((B**2)+4.*A*(Z**2))
      SO=DSQRT((-B-QUO)/2.)
      SI=DSQRT((B-QUO)/2.)
      TI=2.*SO*A*(Z**2)
      SAL=SI*AL
      T2=SO*((SO**4)-(SI**4))*DCOS(SAL)
      T3=QUO*(SO/SI)-(B/G)*(Z**2)*DSIN(SAL)
      XI=SO*AL
      T4=(T2-T3)*COSH(XI)
      T5=QUO*(B/G)*(Z**2)*DCOS(SAL)
      T6=B*(SO**2)*SI*DSIN(SAL)
      T7=(T5-T6)*SINH(XI)
      FZ(I)=T4-T7-TI
      Z=Z-I.
      ABSFZ(I)=DABS(FZ(I))
      IF(I.EQ.1) GO TO 100
      J=I-1
      ABSTOT=ABSFZ(I)-ABSFZ(J)
      TOT=DABS(FZ(I)-FZ(J))
      IF(ABSTOT.GT.TOT) GO TO 80
      GO TO 100
```

```
 80 WRITE(6,60) Z,FZ(J),FZ(I)
 60 FORMAT(5X,`Z=`,D28.16,10X,`FZ=`,D28.16,5X,`FZ2=`,D28.16,/)
100 CONTINUE
    STOP
    END
```

```
C       CALCULATION OF R-MATRIX BY NUMERICAL INTEGRATION
C       TRAPEZOIDAL METHOD
        DIMENSION W(5),SO(5),S (5),BK(5),CK(5),AK(5),T(5),TT(5,5)
       $QUO(5),U(5),E(5,5),UW(5),V(5),F(5,5),R(5,5),FF(5),RR(5),PN(5)
        REAL MG
        READ(5,*) (W(I),I=1,3)
        READ(5,*) G,AL,MG,S,RHO,AI,AREA
        UM=RHO*AREA/G
        EI=S*AI
        A=UM/EI
        B=MG/EI
C       N=NUMBER OF MODES CONSIDERED
C       M=NUMBER OF INTERVALS FOR INTEGRATION
C       H=DELTA Z
        N=3
        M=5000
        L=M-1
        H=AL/5000.
        DO 10 I=1,N
        UO(I)=SQRT((B**2)-4.*A*(W(I)**2))
        SO(I)=SQRT((-B-QUO(I))/2.)
        S1(I)=SQRT((B-QUO(I))/2.)
        BK(I)=(SO(I)**2)*COSH(SO(I)*AL)+(S1(I)**2)*COS(S1(I))*AL)
        CK(I)=(SO(I)**2)*SINH(SO(I)*AL)+SO(I)*S1(I)*SIN(S1(I)*AL)
        AK(I)=BK(I)/CK(I)
     10 CONTINUE
        DO 100 I=1,N
        DO 100 J=1,N
        Z=0.
        DO 200 K=1,L
        V(I)=COSH(SO(I)*Z)-COS(S1(I)*Z)+AK(I)*(SINH(SO(I)*Z)-(SO(I)/
       $S1(I))*SIN(S1(I)*Z))
        U(J)=(SO(J)**2)*COSH(SO(J)*Z)+S1(J)**2)*COS(S1(J)*Z)+AK(J)*(
       $(SO(J)**2)*SINH(SO(J)*Z)+SO(J)*S1(J)*SIN(S1(J)*Z))
        F(I,J)=V(I)*U(J)
        FF(I)=V(I)*V(I)
```

```
      IF(Z.EQ.O.) GO TO 500
      IF(Z.EQ.AL) GO TO 550
      GO TO 600
  500 T(I,J)=0.5*E(I,J)+T(I,J)
      TT(I)=0.5*FF(I)+TT(I)
      GO TO 650
  600 T(I,J)=F(I,J)+T(I,J)
      TT(I)=FF(I)+TT(I)
  650 CONTINUE
      Z=Z-H
  200 CONTINUE
      R(I,J)=H*T(I,J)
      RR(I)=H*TT(I)
      PN(I)=SQRT(RR(I))
  110 CONTINUE
  100 CONTINUE
      DO 75 I=1,3
   75 WRITE(6,64) (R(I,J),J=1,3)
      WRITE(6,66) (PN(I),I=1,3)
   66 FORMAT (5X,`PN(1)=`,F15.9,/;5X,PN(2)=`,F15.9,/,`PN(3)=`,F15.9)
      DO 400 I=1,3
      DO 400 J=1,3
      E(I,J)=R(I,J)/PN(I)/PN(J)
  400 CONTINUE
      DO 40  I=1,3
   40 WRITE(6,64) (E(I,J),J=1,3)
   64 FORMAT(//,3(F 5.9,5X))
C     UNSTABLE  REGION  BOUNDARIES
      DO 450 I=1,3
  450 UW(I)=W(I)/5./W(1)
      WRITE(6,68) (UW(I),I=1,3)
   68 FORMAT(/,5X,`UW(1)=`,F8.5,/,5X,`UW(2)=`,F8.5,/,5X,`UW(3)=`,
     $F8.5,/)
      DO 800 I=1,3
      DO 900 J=1,3
      IF(I.EQ.2.AND.J.EQ.1) GO TO 900
```

```
      IF(I.EQ.1.AND.J.EQ.3) GO TO 900
      IF(I.EQ.2.AND.J.EQ.3) GO TO 900
      UU=UW(I)-UW(J)
      WRITE(6,67) I,J,UU
   67 FORMAT(10X,`UU=UW( `,I1,`)+UW(`,I1,`)=`,F10.5)
      EPSD=0.
  850 EPSD=EPSD-0.02
      DEL1=SQRT(E(I,J)*E(J,I)/UW(I)/UW(J))
      DEL2=SQRT(1.-2.*(UW(I)-UW(J))*EPSD*DEL1)
      DEL3=SQRT(1.-2.*(UW(I)-UW(J))*EPSD*DEL1)
      ALTS=(DEL2-1.)/EPSD/DEL
      USTS=(1.-DEL3)/EPSD/DEL
      WRITE(6,65) ALTS,USTS,EPSD
   65 FORMAT(/,5X,`ALT SINIR=`,F12.8,10X,`UST SINIR=`,F12.8,5X,
     $`EPSD=`,F5.2)
      IF(EPSD.GT.0.2) GO TO 900
      GO TO 850
  900 CONTINUE
  800 CONTINUE
      STOP
      END
```

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This Thesis Has Been Approved By

Dr. Ahmet CERANOGLU
(Supervisor)

Dr. Ahmet KUZUCU

Dr. Hüsamettin ALPER

THE DYNAMIC STABILITY OF AN ELASTIC COLUMN WITH A TIP MASS SUBJECT TO PARAMETRIC EXCITATION

by

ERGUN UYAR

Submitted to the Faculty of the School of Engineering in Partial Fulfillment of the Requirements for the Degree of

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1982

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Ergun Uyar

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INTRODUCTION

In recent years, the dynamic analysis of structures has seen more and more prominence in engineering design considerations. Having sufficient knowledge of dynamic stability of stuctures enables us to prevent possible catastrophic failure.

Parametric instability of columns under periodic axial loads has been investigated by many authors [6]. Recently, much attention has been focused on the existence of combination resonances in addition to simple parametric resonances [4]. In most studies, however, only the horizontal accelerations are considered as these would appear to dominate over the effects of the much smaller vertical accelerations [10]. However these vertical accelerations cause instability depending on the dimensions of the system and the amplitude of the excitation frequency.

A column, when excited along its longitudinal axis, may vibrate in a direction transverse to this axis, under small perturbations. For certain relations between the parameters of the system and those of the excitation, the amplitudes of these transverse vibrations will become extremely large, and the column will collapse. It should be emphasized that the failure is caused by the interaction of the various properties of the structure and its excitation although it is a factor. The dependence of this resonance behaviour on the parameters of the system is known as

" parametric resonance ".

A detailed literature survey on the dynamic stabilit of elastic systems is given in Bolotin`s book [5]. Bolotin made the stability analysis according to the Floquet theory which prevents us to investigate the combination resonances of a structure under dynamic loading. Recently, Iwatsubo, Sugiyama and Ogino [6] analyzed the stability of a uniform elastic column under periodic axial loads for several sets of boundary conditions. Laura et. all. [7] examined the vibrations of a horizontal clamped-free beam with a mass at the free end, such a structure is also known as Beck`s column. The effects of shear and rotatory inertia on Beck`s column were studied by Kounadis and Katsikadelis [8]. In all these studies, analysis has been restricted to those cases where the structure is resting on a stationary foundation and excited by a time dependent tangential or axial load at the free end.

In this work, the dynamic stability of an elastic column with a large mass at the free end will be studied The excitation will be taken to be in the direction of its longitudinal axis, simulating the vertical accelerations of earthquakes, and the subsequent motion of the column transverse to this axis will be investigated. The equation of motion will be derived in Chapter II. In Chapter III we will consider the case of free vibrations and determine the frequency equation of the system. Stability analysis based on a method introduced by Hsu [4] will be given in

Chapter IV. Finally, experimental studies and results will be presented in Chapter V.

CHAPTER II

EQUATION OF MOTION

II.I.Formulation of the Problem:

In this section,the equation of motion governing the transverse vibrations,due to axial excitation,of a structure supporting a large mass will be derived.We will model the structure as a contilevered elastic column with a large mass at its free end,(Fig.2.I)

The parameters of the system will be defined as follows,

M:Mass of the supported body

m:Mass per unit length of the column

l:Length of the column

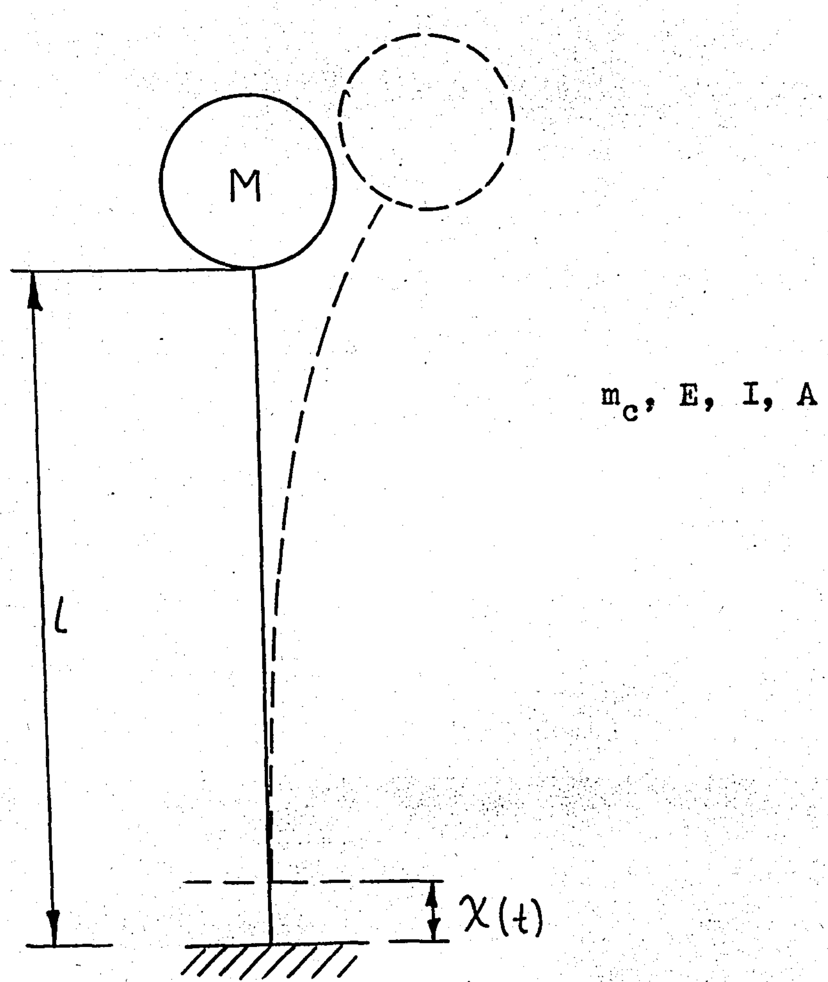
E:Modulus of elasticity of the column

I:Moment of inertia of the column cross-section

A:Cross-sectional area of the column.

In the analysis of the problem, the effects of the rotatory inertia and shear deformation will be neglected and the deflections of the column will be taken to be small in comparison to the smallest dimension of the stucture. The stability of the system will be determined by the boundedness of the solutions to the transverse equation of motion. That is, the column will be said to be unstable when the solutions grow indefinetely in time.

Before we derive the equation of motion, expressions for the longitudinal and transverse components of the acceleration of a material partical will be given in the first part of the following section and the equation of motion will be derived subsequently.



Mathematical Model of the Structure

Fig.(2.I)

II.2. The Equation of Motion :

In order to arrive at the transverse equation of motion of the column, Newton's 2^{nd} Law of Motion will be applied to an element of the column under the assumptions of the classical Bernoulli-Euler theory. To this end, the kinetics of a particle along the column will be discussed first and subsequently the equation of motion will be derived.

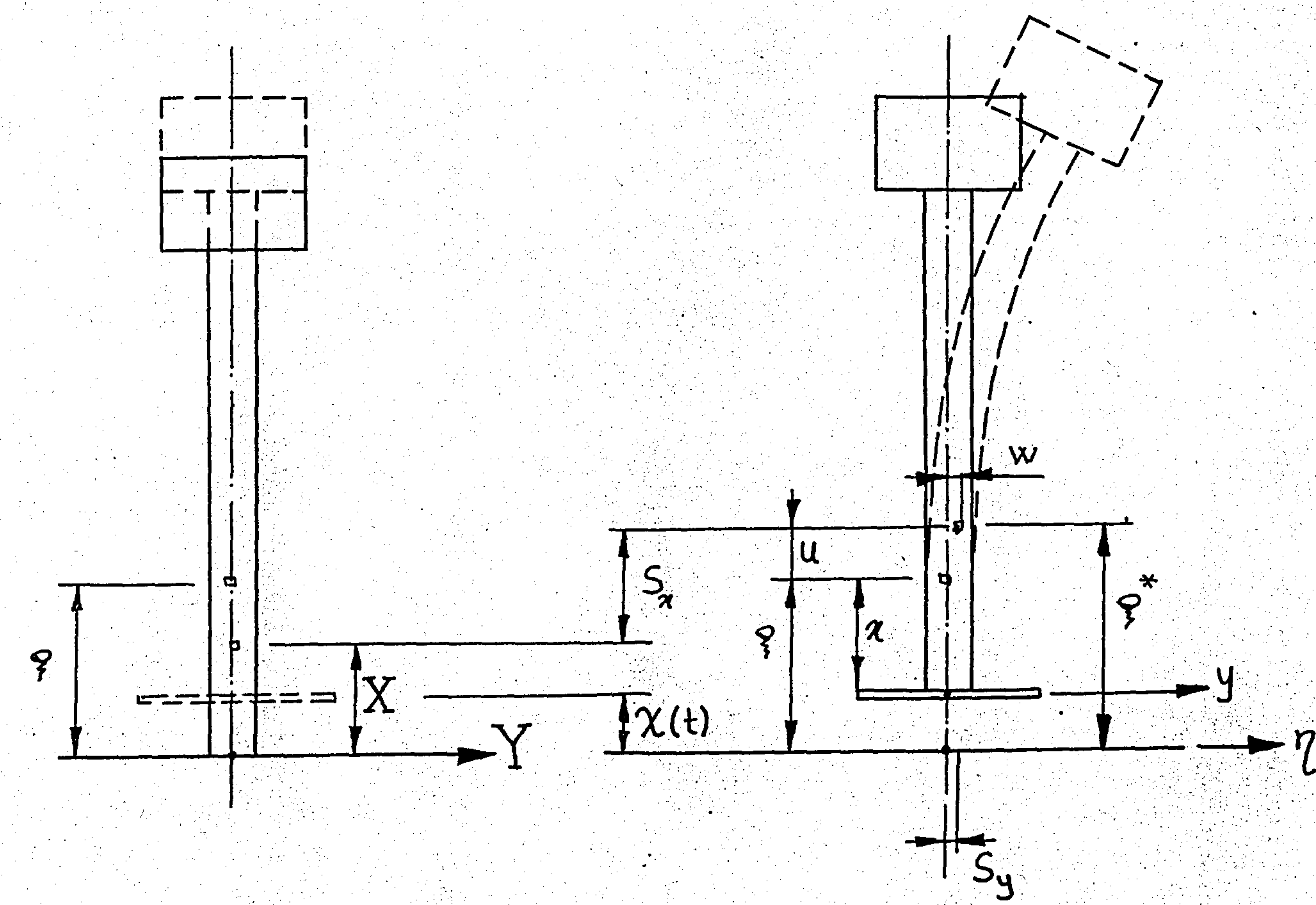
2.I. Kinematics of a Material Particle - Velocity and Acceleration :

Figure (2.2) shows the three successive positions of the system. The first one, which will be referred to as reference configuration (undeformed configuration) is the position at t=0. The system is at rest in the inertial frame fixed to the ground. The second position, a hypothetical one, is the one where the entire structure is rigidly displaced vertically with a time dependency given by $\chi(t)$. Finally, the third position, deformed position, represents the state of the structure at any time t.

The coordinates and the displacements of an arbitrary material particle in each aforementioned position are given below. The first term represents transverse (horizontal) directions.

(X,Y) : Locates a material particle in the reference configuration expressed in a coordinate system measured from a fixed inertial frame.

(ξ,η) : Coordinates of the material particle in the hypothetical configuration (rigid body motion)



reference and intermediate configurations

intermediate and present configurations

Figure (2.2)

measured from the origin of the fixed inertial frame.

(ξ^*, η^*) :Coordinates of the material particle in the deformed position relative to the fixed inertial frame.

(x,y) :Locates the material particle in the undeformed state with respect to the coordinate system moving with the support.

(u,w) :Components of displacement of a material particle relative to its reference position.

The axial and the transverse components of displacement in the hypothetical state can be written as, (Fig.2.I)

$$\left.\begin{aligned} u &= \xi^* - \xi \\ w &= \eta^* - \eta \end{aligned}\right\} \qquad (2.I.I)$$

and the displacements with respect to the reference position are

$$\left.\begin{aligned} S_x &= \xi^* - X = u + \chi \\ S_y &= w = \eta^* - Y \end{aligned}\right\} \qquad (2.I.2)$$

Note that the position of a material particle in the fixed inertial frame is determined by the coordinate X, and its velocity is the rate of change of the displacement function S_x , holding X constant, i.e.,

$$V_x = \left.\frac{\partial S_x}{\partial t}\right|_X \qquad (2.I.3)$$

Since S_x is also a function of ξ and t , applying the chain rule of differentiation to S_x we get,

$$V_x = \frac{\partial S_x}{\partial t} + \frac{\partial S_x}{\partial \rho} \cdot \frac{\partial \rho}{\partial t}$$

where from Eq. (2.I2)

$$\frac{\partial S_x}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial \chi}{\partial t}$$

$$\frac{\partial S_x}{\partial \rho} = \frac{\partial u}{\partial \rho}$$

We also have,

$$\frac{\partial \rho}{\partial t} = \frac{\partial (X+\chi)}{\partial t} \doteq \dot{\chi}$$

Substituting all these relations in the expression for V_x yields

$$V_x = \frac{\partial u}{\partial t} + \dot{\chi}\left(I + \frac{\partial u}{\partial \rho}\right) . \qquad (2.I.4)$$

Neglecting the axial strain $\frac{\partial u}{\partial \rho}$, and the strain velocity $\frac{\partial u}{\partial t}$ compared to $\dot{\chi}$, the expression for the axial component of the velocity of that material particle reduces to

$$V_x \cong \dot{\chi}(t)$$

Showing that the major contribution to the motion is due to the motion of the support, that is

$$\frac{\partial u}{\partial x} \ll \dot{\chi} \qquad\qquad \frac{\partial^2 u}{\partial x^2} \ll \ddot{\chi}$$

Taking the time derivative of the equation (2.I.4) holding X constant will yield the general expression for the longitudinal component of the acceleration,

$$a_x = \left.\frac{\partial V_x}{\partial t}\right|_X = \frac{\partial V_x}{\partial t} + \dot{\chi}\frac{\partial V_x}{\partial \rho}$$

or

$$a_x = \ddot{X} + \left[\frac{\partial^2 u}{\partial t^2} + \ddot{X}\frac{\partial u}{\partial \xi}\right] + \dot{X}\frac{\partial}{\partial \xi}\left[2\frac{\partial u}{\partial t} + \dot{X}\frac{\partial u}{\partial \xi}\right]$$

Neglecting second and the higher order terms the above expression reduces to

$$a_x = \ddot{X}(t) \tag{2.I.5}$$

For the transverse components of velocity and acceleration of the material particle, going through the same procedure **yields**

$$V_y = \left.\frac{\partial S_y}{\partial t}\right|_X = \frac{\partial S_y}{\partial t} + \frac{\partial S_y}{\partial \xi}\cdot\frac{\partial \xi}{\partial t}$$

$$V_y = \frac{\partial w}{\partial t} + \dot{X}\frac{\partial w}{\partial \xi} \tag{2.I.6}$$

and the general expression for the transverse component of the acceleration is

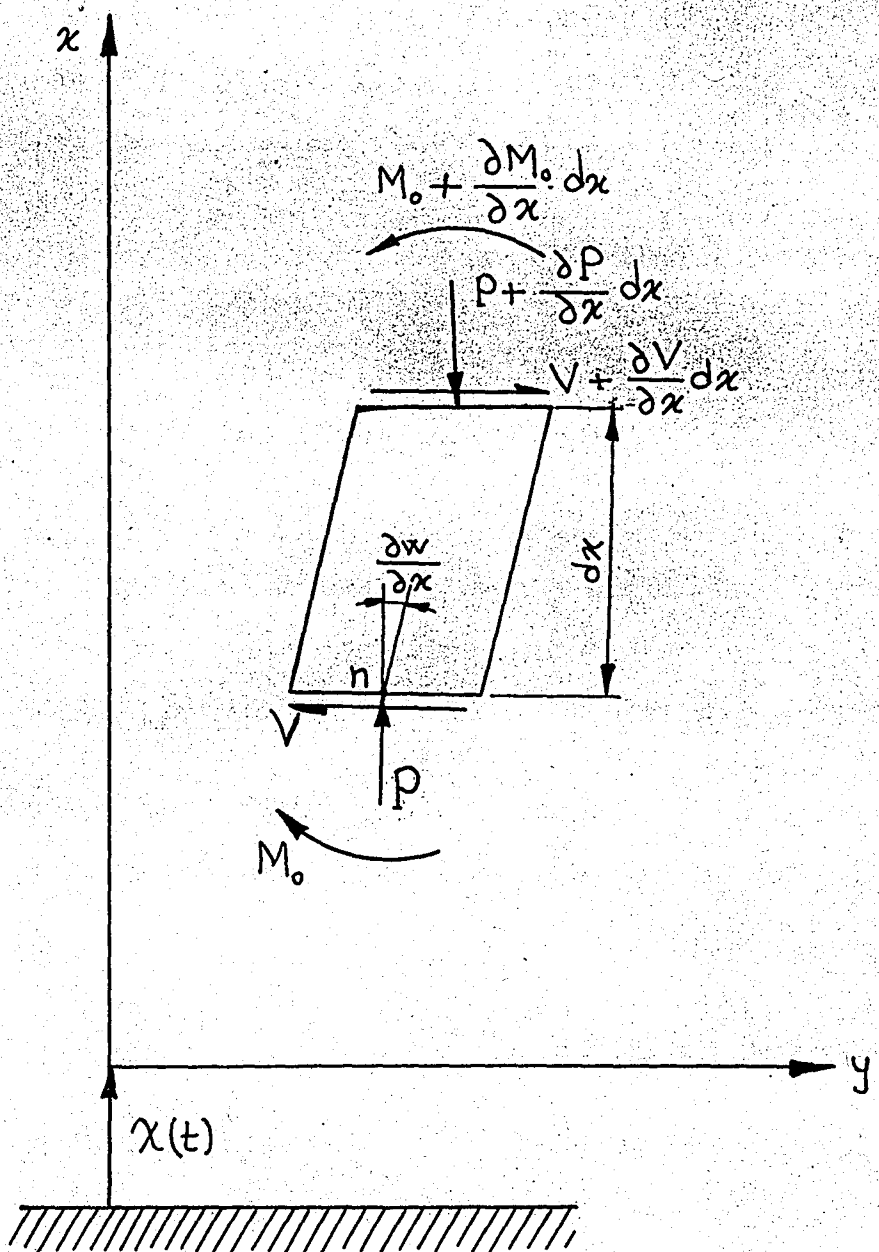
$$a_y = \frac{\partial^2 w}{\partial t^2} + \ddot{X}\frac{\partial w}{\partial \xi} + \dot{X}\frac{\partial}{\partial \xi}\left[2\frac{\partial w}{\partial t} + \dot{X}\frac{\partial w}{\partial \xi}\right]$$

Considering only the first two terms and neglecting the remaining, the expression for the transverse acceleration reduces to

$$a_y = \frac{\partial^2 w}{\partial t^2} + \ddot{X}\frac{\partial w}{\partial \xi} \tag{2.I.7}$$

2.2. The Equation of Motion :

A differential material element of the column which is deformed due to the bending moment is shown in Fig. (2.3) in a coordinate system (x,y) moving with the support. It will be assumed that shear deformation and rotatory inertia effects are negligible. P , V and M_o represent the



Differential Element of the Column

Fig.(2.3)

inertial axial force, shear force and bending moment respectively.

The equations of motion both along the x and y directions obtained by applying Newton's 2nd Law of Motion to the differential element of the column are

$$-\frac{\partial P}{\partial x} = m_c \ddot{X} \tag{2.2.I}$$

$$\frac{\partial V}{\partial x} = m_c \left[\frac{\partial^2 w}{\partial t^2} + \ddot{X} \frac{\partial w}{\partial x} \right] \tag{2.2.2}$$

where we have used the fact that

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}$$

Summing up the moments about an axis perpendicular to the (x,y) plane and passing through a point n we get

$$\frac{\partial M_o}{\partial x} dx - V dx - \frac{\partial V}{\partial x}(dx)^2 - P \frac{\partial w}{\partial x} dx - \frac{\partial w}{\partial x}\frac{\partial P}{\partial x}(dx)^2 = 0$$

Considering only the first order terms and neglecting the remaining ones we obtain the equation of angular motion of the system as

$$\frac{\partial M_o}{\partial x} - P \frac{\partial w}{\partial x} - V = 0 \tag{2.2.3}$$

Recall the relation between the bending moment M_o and the curvature at the same point for a beam in flextural motion

$$M_o = -EI \frac{\partial^2 w}{\partial x^2} \tag{2.2.4}$$

Eliminating M_o between the equations (2.2.3) and (2.2.4) we get an equation for the shear force

$$-V = EI\frac{\partial^3 w}{\partial x^3} + P\frac{\partial w}{\partial x} \qquad (2.2.5)$$

Differentiating the above equation with respect to x and using the equations (2.2.I) and (2.2.2) yields

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + m_c\frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.2.6)$$

Note that the axial force P is a function of x and t and is given by

$$P(x,t) = M(g - a_x) + \int_x^l m_c(g - a_x)\,dx$$

where g is the gravitational acceleration.
Substituting $\ddot{X}$ for a_x the above equation reduces to

$$P(x,t) = \left[M + m_c(l - x)\right]\cdot(g - \ddot{X})$$

Since the beam considered is slender and the mass of the top weight is very large compared to that of column ($M \gg m_c\, l$) , inertial forces of the column can be neglected. Hence, the expression for the axial force reduces to

$$P(t) = M(g - \ddot{X}) \qquad (2.2.7)$$

which is only a function of time. The governing equation of motion for the system described is then obtained by substituting (2.2.7) into the equation (2.2.6)

$$EI\cdot\frac{\partial^4 w}{\partial x^4} + M(g - \ddot{X})\frac{\partial^2 w}{\partial x^2} + m_c\frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.2.8)$$

This is a fourth order partial differential equation, hence, there should be four boundary conditions. At the lower end, the beam is built into its foundation meaning that the displacement and the slope must vanish, i.e.,

$$w\Big|_{x=0} = 0 \tag{2.2.9a}$$

$$\frac{\partial w}{\partial x}\Big|_{x=0} = 0 \tag{2.2.9b}$$

The other end is free with a large mass. Hence , one of the boundary condition at x=l is zero bending moment

$$EI\frac{\partial^2 w}{\partial x^2}\Big|_{x=l} = 0 \tag{2.2.9c}$$

To get the second boundary condition, note that the inertial force of the top weight is balanced by shear force at this end of the beam. Thus,

$$V(l,t) = -M a_y(l,t) = -M\left[\frac{\partial^2 w}{\partial t^2} + \ddot{X}\frac{\partial w}{\partial x}\right]_{x=l}$$

is the remaining boundary condition. Substituting (2.2.7) into (2.2.5) and evaluating at x=l will give us the shear force appearing on the left-hand side of the above equation. Thus, the above expression reduces to

$$\left[EI\frac{\partial^3 w}{\partial x^3} + Mg\frac{\partial w}{\partial x} - M\frac{\partial^2 w}{\partial t^2}\right]_{x=l} = 2M\ddot{X}\frac{\partial w}{\partial x}\Big|_{x=l} \tag{2.2.9d}$$

In this work we will assume that the excitation is a harmonic function of time , that is

$$X(t) = X_o \cos \Omega t$$

where X_o and Ω are constants corresponding to the amplitude and the frequency of the forcing function. The expression for the acceleration of the base is then

$$\ddot{X}(t) = -X_o \Omega^2 \cos \Omega t$$

Substituting $\ddot{X}$ (t) into the Eq. (2.2.8) yields

$$\frac{\partial^4 w}{\partial x^4} + (\beta + \alpha \cos \Omega t)\frac{\partial^2 w}{\partial x^2} + \gamma \frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.2.10)$$

where

$$\left.\begin{aligned} \beta &= \frac{M \cdot g}{E \cdot I} \\ \alpha &= \Omega^2 X_o \frac{M}{EI} \\ \gamma &= \frac{m_c}{EI} \end{aligned}\right\} \qquad (2.2.11)$$

Equation (2.2.10) together with its boundary conditions (2.2.9) describes the transverse motion of the column in the (x,y) plane subject to a vertical excitation X(t).

Since we will be only interested in the steady-state motion no initial conditions will be prescribed for this problem.

CHAPTER III

FREE VIBRATIONS

In this chapter, the general solution and the frequency equation of the system under consideration will be presented.

III.I. The Equation of Motion and Its Solution :

If the forcing function χ (t) is taken as zero in the equation (2.2.I0), the equation of motion describing the free vibrations case is obtained. Thus Eq.(2.2.I0) reduces to

$$\frac{\partial^4 w}{\partial x^4} + \beta \frac{\partial^2 w}{\partial x^2} + \gamma \frac{\partial^2 w}{\partial t^2} = 0 \qquad (3.I.I)$$

where β and γ are the quantities defined in (2.2.II). The boundary conditions, (Eq.(2.2.9)), can be written as

$$w\Big|_{x=0} = 0 \qquad (3.I.2a)$$

$$\frac{\partial w}{\partial x}\Big|_{x=0} = 0 \qquad (3.I.2b)$$

$$\frac{\partial^2 w}{\partial x^2}\Big|_{x=l} = 0 \qquad (3.I.2c)$$

$$\left(\frac{\partial^3 w}{\partial x^3} + \beta \frac{\partial w}{\partial x} - \frac{\beta}{g} \cdot \frac{\partial^2 w}{\partial t^2}\right)\Big|_{x=l} = 0 \qquad (3.I.2d)$$

The general solution of equation (3.I.I) can be obtained by applying the method of separation of variables. That is , we assume a solution of the form

$$w(x,t) = V(x) \cdot e^{i\omega t} \qquad (3.I.3)$$

Substituting the above equation into Eq.(3.I.I), the equation of motion reduces to an ordinary differential equation of the form

$$\left[\frac{d^4V(x)}{dx^4} + \beta\,\frac{d^2V(x)}{dx^2} - \gamma\,\omega^2 V(x)\right] e^{i\omega t} = 0 \qquad (3.I.4)$$

where ω is the natural frequency of the system. The expressions in brackets must vanish in order to have Eq.(3.I.4) hold true for all time t. Then the problem reduces to an eigenvalue problem of the form

$$\mathcal{L}\,[V] = \omega^2 \gamma\, V \qquad (3.I.5)$$

where $\mathcal{L}$ is a linear differential operator defined by

$$\mathcal{L} \equiv \frac{d^4}{dx^4} + \beta\,\frac{d^2}{dx^2} \qquad (3.I.6)$$

Assuming a solution of the form $V(x) = Ce^{\lambda x}$ Eq.(3.I.5) yields the characteristic equation

$$\lambda^4 + \beta\lambda^2 - \omega^2\gamma = 0 . \qquad (3.I.7)$$

The roots of this fourth order equation can be written as

$$\lambda_1 = k_0 \quad , \quad \lambda_2 = -k_0 \quad , \quad \lambda_3 = i k_1 \quad , \quad \lambda_4 = -i k_1 \qquad (3.I.8)$$

where

$$k_0 = \sqrt{\frac{-\beta + \sqrt{\beta^2 + 4\omega^2\gamma}}{2}} \qquad k_1 = \sqrt{\frac{\beta + \sqrt{\beta^2 + 4\omega^2\gamma}}{2}}$$

Hence the solution for V(x) can be written as

$$V(x) = A\,\mathrm{Cosh}\,k_0 x + B\,\mathrm{Sinh}\,k_0 x + C\,\mathrm{Cos}\,k_1 x + D\,\mathrm{Sin}\,k_1 x \qquad (3.I.9)$$

where A, B, C, and D are the constants to be determined from the four boundary conditions given by Eq.(3.I.2).

The general solution of the free vibrations case is then

$$w(x,t)=\left[A\,\mathrm{Cosh}\,k_o x+B\,\mathrm{Sinh}\,k_o x+C\,\mathrm{Cos}\,k_1 x+D\,\mathrm{Sin}\,k_1 x\right]e^{i\omega t} \quad (3.I.I0)$$

III.2. Self-adjointness of the System :

Eq.(3.I.5) corresponding to the eigenvalue problem can be put into the non-dimensional form by introducing the variables $z=\frac{x}{l}$, $\tau=\omega^{(n)}t$ corresponding to the non-dimensional lenght and time respectively, $\omega^{(n)}$ is the n^{th} natural frequency of the system. Then the eigenvalue problem takes the form

$$\bar{\mathcal{L}}[\bar{V}]=\left(\frac{\omega}{\omega^{(n)}}\right)^2\cdot\bar{\gamma}\,\bar{V}(z) \quad (3.2.I)$$

with the boundary conditions

$$\bar{V}(0)=0 \quad (3.2.2a)$$

$$\bar{V}'(0)=0 \quad (3.2.2b)$$

$$\bar{V}''(1)=0 \quad (3.2.2c)$$

$$\bar{V}'''(1)+\frac{\bar{\beta}}{g}\cdot\bar{V}'(1)=-\left(\frac{\omega}{\omega^{(n)}}\right)^2\frac{\bar{\beta}}{g}\bar{V}(1) \quad (3.2.2d)$$

where $\bar{\mathcal{L}}\equiv\frac{d^4}{dz^4}+\bar{\beta}\frac{d^2}{dz^2}$, $(\quad)'\equiv\frac{d(\ )}{dz}$

and

$$\bar{V}=\frac{V}{l}$$

$$\bar{\beta}=l^2\beta$$

$$\bar{\gamma}=l^4\,\omega^{(n)^2}\cdot\gamma$$

In general an eigenvalue problem is said to be self-adjoint provided the following relations are satisfied, [9].

$$\int_0^1 r\bar{\mathcal{L}}[s]dz+\sum_{j=1}^{4}\int_0^1 rB_j[s]dz=\int_0^1 s\bar{\mathcal{L}}[r]dz+\sum_{j=1}^{4}\int_0^1 sB_j[r]dz \qquad (3.2.3a)$$

$$\int_0^1 r\bar{\gamma}s\,dz+\sum_{j=1}^{4}\int_0^1 rC_j[s]dz=\int_0^1 s\bar{\gamma}r\,dz+\sum_{j=1}^{4}\int_0^1 sC_j[r]dz \qquad (3.2.3b)$$

where r and s are any two functions satisfying the boundary conditions, while B_j and C_j are operators appearing in the boundary conditions. In our case these operators are

$$B_4\equiv\left(\frac{d}{dz^3}-\frac{\bar{\beta}}{g}\frac{d}{dz}\right)\Big|_1 \qquad B_j\equiv 1 \quad j=1,2,3$$

$$C_4\equiv-\frac{\bar{\beta}}{g}\Big|_1 \qquad C_j\equiv 0 \quad j=1,2,3$$

It is obvious that the operators $\bar{\gamma}$, B_j (j=1-4), and C_4 are only multiplications by constants, hence Eq.(3.2.3b) is satisfied identically and there remains only to show that the operator $\bar{\mathcal{L}}$ satisfies the relation

$$\int_0^1 r\bar{\mathcal{L}}[s]dz=\int_0^1 s\bar{\mathcal{L}}[r]dz$$

$$\int_0^1 r\left(s^{(IV)}+\bar{\beta}s^{(II)}\right)dz=\int_0^1 r\frac{d^4s}{dz^4}dz+\int_0^1 r\bar{\beta}\frac{d^2s}{dz^2}dz$$

Integrating the above integrals by parts we get,

$$=r\left(\frac{d^3s}{dz^3}+\bar{\beta}\frac{ds}{dz}\right)\Big|_1-s\left(\frac{d^3r}{dz^3}+\bar{\beta}\frac{dr}{dz}\right)\Big|_1+\int_0^1\frac{d^4r}{dz^4}\cdot s\,dz+\int_0^1\bar{\beta}\frac{d^2r}{dz^2}\cdot s\,dz$$

Using the boundary condition (3.2.2d)

$$= r\left(-\frac{\omega^2}{\omega_i^{(n)2}}\frac{\bar{\beta}}{g}\cdot s\right)\Big|_1 - s\left(-\frac{\omega^2}{\omega^{(n)2}}\cdot\frac{\bar{\beta}}{g}\cdot r\right)\Big|_1 + \int_0^1 s\bar{\mathcal{L}}[r]\,dz$$

Thus the self-adjointness of the system has been proven, and it is a known fact that the eigenfunctions of such a system are orthogonal [9]. The orthogonality of eigenfunctions can be stated as

$$\int_0^1 \bar{V}_i(z)\cdot\bar{V}_j(z)\,dz = N_{(i)}\delta_{ij} \tag{3.2.4}$$

where

$$N_{(i)} = \int_0^1 \bar{V}_i^2(z)\,dz$$

and

$$\delta_{ij}\begin{cases} I & \text{when} \quad i=j \\ 0 & \text{''} \quad i\neq j \end{cases}$$

Let $q_i(z)$ be the normalized form of the function $\bar{V}_i(z)$ such that

$$q_i(z) = \frac{\bar{V}_i(z)}{\sqrt{N_i}} \tag{3.2.5}$$

Then $q_i(z)$ form an orthonormal set with the property

$$\int_0^1 q_i(z)\cdot q_j(z)\,dz = \delta_{ij} \tag{3.2.6}$$

III.3. The Determination of the Natural Frequencies :

Frequency equation of a system is obtained by applying the boundary conditions to the solution of the equation of motion. Thus, applying the boundary condition at $x=0$ yields

$$w(0,t)=0 \longrightarrow A=-C \tag{3.3.Ia}$$

$$\frac{\partial w(0,t)}{\partial x} = 0 \qquad B = -\frac{k_1}{k_0} \cdot D \tag{3.3.1b}$$

Hence the solution given by equation (3.1.10) takes the form

$$w(x,t) = \left[A(\operatorname{Cosh} k_0 x - \operatorname{Cos} k_1 x) + B\left(\operatorname{Sinh} k_0 x - \frac{k_0}{k_1}\operatorname{Sin} k_1 x\right)\right] e^{i\omega t} \tag{3.3.2}$$

Now applying the boundary condition at $x=1$ will yield two equations for the unknown constants A and B, that is,

$$0 = A(k_0^2 \operatorname{Cosh} k_0 l + k_1^2 \operatorname{Cos} k_1 l) + B(k_0^2 \operatorname{Sinh} k_0 l + k_0 k_1 \operatorname{Sin} k_1 l) \tag{3.3.3a}$$

$$0 = A\left[k_0 k_1^2 \operatorname{Sinh} k_0 l - k_1 k_0^2 \operatorname{Sin} k_1 l + \frac{\beta}{g}\omega^2(\operatorname{Cosh} k_0 l - \operatorname{Cos} k_1 l)\right]$$
$$+ B\left[k_0 k_1^2 \operatorname{Cosh} k_0 l + k_0^3 \operatorname{Cos} k_1 l + \frac{\beta}{g}\left(\operatorname{Sinh} k_0 l - \frac{k_0}{k_1}\operatorname{Sin} k_1 l\right)\right] \tag{3.3.3b}$$

Equations (3.3.3) can be written in matrix form as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.3.4}$$

where the coefficients

$$Q_{11} = k_0^2 \operatorname{Cosh} k_0 l + k_1^2 \operatorname{Cos} k_1 l$$

$$Q_{12} = k_0 \operatorname{Sinh} k_0 l + k_0 k_1 \operatorname{Sin} k_1 l$$

$$Q_{21} = k_0 k_1^2 \operatorname{Sinh} k_0 l - k_0^2 k_1 \operatorname{Sin} k_1 l + \frac{\beta}{g}\omega^2(\operatorname{Cosh} k_0 l - \operatorname{Cos} k_1 l) \tag{3.3.5}$$

$$Q_{22} = k_0 k_1^2 \operatorname{Cosh} k_0 l + k_0^3 \operatorname{Cos} k_1 l + \frac{\beta}{g}\omega^2\left(\operatorname{Sinh} k_0 l - \frac{k_0}{k_1}\operatorname{Sin} k_1 l\right)$$

In order to have a non-trivial solution for the equation (3.3.4), the determinant of the coefficient matrix must vanish, i.e.,

$$\det \underset{\approx}{Q} = 0$$

$$Q_{11} \cdot Q_{22} - Q_{12} \cdot Q_{21} = 0$$

Substitution of Q_{ij} (i,j 1,2) into the above equation will yield the desired frequency equation

$$2k_o\omega^2\gamma + \left[k_o(k_o^4 + k_1^4)\operatorname{Cos}k_1 l - \sqrt{\beta^2 + 4\omega^2\gamma}\ \frac{k_o}{k_1}\cdot\frac{\beta}{g}\omega^2 \operatorname{Sin}k_1 l\right]\cdot$$

$$\operatorname{Cosh}k_o l + \left[\sqrt{\beta^2 + 4\omega^2\gamma}\ \frac{\beta}{g}\omega^2\operatorname{Cos}k_1 l - \beta k_o^2 k_1 \operatorname{Sin}k_1 l\right]\operatorname{Sinh}k_o l = 0 \qquad (3.3.6)$$

The equation (3.3.6) can be solved numerically for ω when the constants k_o and k_1 are known for a spesific structure. The roots of the frequency equation $\omega^{(J)}$ (j 1,2,..n) are known as the natural frequencies of the beam-column with a heavy tip mass at the free end.

The constants A and B are related to each other with a constant ratio

$$A = \frac{k_o^2 \operatorname{Sin}k_o l + k_o k_1 \operatorname{Sin}k_1 l}{k_o^2 \operatorname{Cosh}k_o l + k_1^2 \operatorname{Cos}k_1 l}\cdot B$$

Therefore the complete solution to the free vibration problem is

$$w(x,t) = \sum_{J=1}^{\infty} A_J\left[\left(\operatorname{Cosh}k_o^{(J)}x - \operatorname{Cos}k_1^{(J)}x\right) - \frac{k_o^{(J)}\operatorname{Cosh}k_o^{(J)}l + k_1^{(J)2}\cdot\operatorname{Cos}k_1^{(J)}l}{k_1^{(J)2}\operatorname{Sinh}k_o^{(J)}l + k_o^{(J)}\operatorname{Sin}k_1^{(J)}l}\right.$$

$$\left.\left(\operatorname{Sinh}k_o^{(J)}x - \frac{k_o^{(J)}}{k_1^{(J)}}\cdot\operatorname{Sin}k_1^{(J)}x\right)\right]e^{i\omega^{(J)}t} \qquad (3.3.7)$$

Each term in this series is known as the normal mode of the free vibration.

CHAPTER IV

STABILITY ANALYSIS

In this chapter, we will find the unstable regions in which instability is taken as the unboundedness of the solutions to the transverse equation of motion. In Section III.3, orthogonality of the normal modes was proven for the free vibration problem, therefore the solution to the forced vibration problem can be taken as a superposition of these normal modes (eigenfunction expansion)[9]. The coefficient of each term in this series expansion will be in general a function of time. Such an approach will yield coupled Mathieu equations to be solved.

The method generalized by C.S. Hsu[4] will be used in this chapter in solving the above-mentioned Mathieu equations. This method enables us to observe both simple and combination resonances while the one introduced by V.V. Bolotin [5] enables us to observe only the former.

In Hsu's method, the boundaries of the stable and unstable regions are found using a combination of perturbation and variation of parameters techniques.

IV.I. Derivation of the Space Independent Equation :

The general equation of motion (2.2.IO) can be written in terms of dimensionless variables z and τ as

$$\frac{\partial^4 \bar{w}}{\partial z^4} + \left[\beta + \bar{\alpha}\, G(\tau)\right] \frac{\partial^2 \bar{w}}{\partial z^2} + \bar{\gamma}\, \frac{\partial^2 \bar{w}}{\partial \tau^2} = 0 \qquad (4.I.I)$$

where $G(\tau) = \cos \nu \tau \quad , \quad \tau = \omega^{(n)} . t$

$$\nu = \frac{\Omega}{\omega^{(n)}}$$

$$\bar{\alpha} = l^2 \alpha$$

$$\bar{w} = \frac{w}{l}$$

Note that the i^{th} non-dimensional natural frequency ω_i is

$$\omega_i = \frac{\omega^{(i)}}{\omega^{(n)}}$$

The solution to equation (4.I.I) can be assumed to be of the form

$$w(z,\tau) = \sum_{i=1}^{\infty} \phi_i(\tau) \, . \, q_i(z) \qquad (4.I.2)$$

where $q_i(z)$ is the i^{th} normal mode and $\phi_i(\tau)$ is its corresponding time dependent amplitude. This assumption is known as eigenfunction expansion [9]. By substituting (4.I.2) into the equation (4.I.I) and applying Galerkins method , a system of coupled Mathieu equations whose solutions will determine the stability of the system is obtained.

$$\sum_{i=1}^{\infty} \phi_i(\tau) \bar{\mathcal{L}} \left[q_i(z) \right] + \bar{\gamma} \sum_{i=1}^{\infty} \ddot{\phi}_i(\tau) q_i(z) + \bar{\alpha} G(\tau) \sum_{i=1}^{\infty} \phi_i(\tau) q_i''(z) = 0 \qquad (4.I.3)$$

where $(\dot{\ }) = \frac{d}{d\tau} \quad , \quad (\ )' = \frac{d}{dz}$

Multiplying the above equation by $q_j(z)$ and integrating over (0, I) and recalling the orthogonality relation (3.2.6), we get

$$\ddot{\phi}_i(\tau) + \omega_i^2\,\phi_i(\tau) + \frac{\bar{\alpha}}{\bar{\gamma}}\cdot G(\tau)\sum_{i=1}^{\infty}\phi_i(\tau)\int_0^1 q_i''\cdot q_J\,dz = 0 \qquad (4.I.4)$$

Defining
$$E_{ij} = \int_0^1 q_j''(z)\cdot q_i(z)\,dz$$
the above equation takes the form

$$\ddot{\phi}_i(\tau) + \omega_i^2\,\phi_i(\tau) + \in\bar{\delta}\,G(\tau)\sum_J E_{iJ}\,\phi_J(\tau) = 0 \qquad (4.I.5)$$

where
$$\in = \frac{X_o}{l}$$
$$\delta = \frac{M}{m_c\ l}$$
$$\bar{\delta} = \nu^2\delta$$

and X_o is the amplitude of the excitation of the support.

Defining the following matricies

$$\ddot{\underset{\sim}{\phi}} = \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \cdot \\ \cdot \\ \ddot{\phi}_n \\ \cdot \\ \cdot \end{bmatrix} \qquad \underset{\approx}{\lambda} = \begin{bmatrix} \omega_1^2 & 0 & \cdots & \\ 0 & \omega_2^2 & & \\ \cdot & & \ddots & \\ & & & \omega_n^2 \\ & & & & \cdot \end{bmatrix}$$

$$\underset{\approx}{E} = \begin{bmatrix} E_{11} & E_{12} & \cdot & \cdot & \cdot \\ E_{21} & E_{22} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ & & & E_{nn} & \cdot & \cdot \\ & & & \cdot & \\ & & & \cdot & \end{bmatrix} \qquad \underset{\sim}{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \\ \cdot \end{bmatrix}$$

equation (4.I.5) can be written in matrix form as

$$\ddot{\underset{\sim}{\phi}} + \left[\underset{\approx}{\lambda} + \epsilon \bar{\delta} G(\tau) \underset{\approx}{E} \right] \underset{\sim}{\phi} = \underset{\sim}{0} \qquad (4.I.7)$$

IV.2. The Stability of a System of Coupled Mathieu Equations :

The solution to the equation (4.I.5) , according to the method introduced by C.S. Hsu [4], is assumed to be composed of two parts. Since ϵ is a small parameter, the first part of the solution will be the perturbation part. The second part of the solution is found from the undetermined coefficients method and is of the form of the solution to be perturbed about with time dependent coefficients.

The system of equations (4.I.5) can be reduced to first order system of equations by defining a new function $h_i(\tau)$ as

$$\left.\begin{aligned} h_i &= \dot{\phi}_i \\ \dot{h}_i + \omega_i^2 \phi_i &= -\epsilon \bar{\delta} G(\tau) \sum_j E_{ij} \phi_j \end{aligned}\right\} \qquad (4.2.I)$$

Note that if the small parameter ϵ tends to zero the solution of (4.2.I) becomes

$$\phi_i(\tau) = A_i \cos\omega_i\tau + B_i \sin\omega_i\tau$$

$$h_i(\tau) = -\omega_i A_i \sin\omega_i\tau + \omega_i B_i \cos\omega_i\tau$$

where A_i and B_i are constants.

For $\epsilon = 0$, the above solutions for ϕ_i and h_i can be assumed to be in the form of perturbation about the above solutions with time dependent coefficients. Combining the method of undetermined coefficients with the perturbation method we assume a solution of the form

$$\left.\begin{aligned} \phi_i(\tau) &= A_i(\tau)\cos\omega_i\tau + B_i(\tau)\sin\omega_i\tau + \sum_{q=1}^{n}\epsilon^q \phi_i^{(q)}(\tau) \\ h_i(\tau) &= -\omega_i A_i(\tau)\sin\omega_i\tau + \omega_i B_i(\tau)\cos\omega_i\tau + \sum_{q=1}^{n}\epsilon^q h_i^{(q)}(\tau) \end{aligned}\right\} \quad (4.2.2)$$

where A_i and B_i as well as $\phi_i^{(q)}$ and $h_i^{(q)}$ are all functions of time.

Substitution of (4.2.2) into (4.2.I) and applying the method of undetermined coefficients yields

$$\dot{A}_i(\tau)\cos\omega_i\tau + \dot{B}_i(\tau)\sin\omega_i\tau = 0 \quad (4.2.3a)$$

$$-\omega_i\dot{A}_i\sin\omega_i\tau + \omega_i\dot{B}_i\cos\omega_i\tau + \sum_{q=1}^{n}\epsilon^q\left(\ddot{\phi}_i^{(q)} + \omega_i^2\phi_i^{(q)}\right)$$
$$= -\epsilon\bar{\delta}G(\tau)\sum_J E_{iJ}\left[A_J\cos\omega_J\tau + B_J\sin\omega_J\tau + \sum_{q=1}^{n}\epsilon^q\phi_J^{(q)}\right] \quad (4.2.3b)$$

Considering only the terms with coefficients upto the first power of ϵ (first approximation), and substituting $G(\tau)$ into equation (4.2.3b) yields a second order differential equation for the time dependent coefficients. Equating the terms having ϵ on both sides of the above equation yields

$$\ddot{\phi}_i^{(1)} + \omega_i^2\phi_i^{(1)} = -\frac{\bar{\delta}}{2}\sum_J E_{iJ}\left\{A_J\left[\cos(\omega_J-\nu)\tau + \cos(\omega_J+\nu)\tau\right]\right.$$
$$\left. + B_J\left[\sin(\omega_J-\nu)\tau + \sin(\omega_J+\nu)\tau\right]\right\} \quad (4.2.4)$$

Particular solution of the above equation is

$$\phi_i^{(1)}\Big|_p = -\frac{\bar{\delta}}{2}\sum_J\left\{\frac{E_{iJ}}{\omega_i^2-(\omega_J+\nu)^2}\left[A_J\cos(\omega_J+\nu)\tau + B_J\sin(\omega_J+\nu)\tau\right]\right.$$
$$\left. + \frac{E_{iJ}}{\omega_i^2-(\omega_J-\nu)^2}\left[A_J\cos(\omega_J-\nu)\tau + B_J\sin(\omega_J-\nu)\tau\right]\right\} \quad (4.2.5)$$

Note that $\phi_i^{(1)}\Big|_p$ being the time dependent coefficient of the solution to the equation (4.I.I) will become infinetely large as $|\omega_i| \rightarrow |\omega_j \mp \nu|$ and this corresponds to some sort of a resonance phenomena. The essential feature of the method we have used is to associate resonance causing terms with the variational part of the solution [4]. These troublesome terms can be removed from the perturbation part of the solution. Now let us consider several resonance cases found from the perturbation analysis.

Case I : $\nu = \omega_k + \omega_j + \epsilon\lambda$, $k \neq j$ (λ is a real finite number)

If we set $i=k$ in (4.2.5), it is seen that when the forcing frequency ν is given as above, the j^{th} term ($j=j$) of the second expression on the right-hand side of the k^{th} equation (4.2.5) will go to infinity as $\epsilon \rightarrow 0$. Similarly we get the same resonance case by interchanging the indicies as $i=j$ and $j=k$ in the summation. Since resonance case is related with the variational part, we obtain four differential equations for A_k, B_k, A_j and B_j by considering only the variational part of the equation (4.2.3)

$$\dot{A}_k \, \text{Cos}\,\omega_k\tau + \dot{B}_k \, \text{Sin}\,\omega_k\tau = 0 \qquad (4.2.6a)$$

$$-\omega_k \dot{A}_k \text{Sin}\,\omega_k\tau + \omega_k \dot{B}_k \text{Cos}\,\omega_k\tau = -\frac{\epsilon}{2}\,\bar{\delta}\, E_{kj}\Big[A_j \text{Cos}(\omega_k+\epsilon\lambda)\tau - B_j \text{Sin}(\omega_k+\epsilon\lambda)\tau\Big] \qquad (4.2.6b)$$

$$\dot{A}_j \cos\omega_j\tau + \dot{B}_j \sin\omega_j\tau = 0 \qquad (4.2.7a)$$

$$-\omega_j \dot{A}_j \sin\omega_j\tau + \omega_j \dot{B}_j \cos\omega_j\tau = -\frac{\epsilon}{2}\bar{\delta} E_{jk}\Big[A_k \cos(\omega_j + \epsilon\lambda)\tau - B_j \sin(\omega_k + \epsilon\lambda)\tau\Big] \qquad (4.2.7b)$$

Solving (4.2.6) for $\dot{A}_k$ and $\dot{B}_k$ we get

$$\dot{B}_k = -\frac{\epsilon\bar{\delta}}{2\omega_k} E_{kJ}\Big[A_J(\cos^2\psi_k \cos\epsilon\lambda\tau - \cos\psi_k \sin\psi_k \cdot \sin\epsilon\lambda\tau) - B_J(\cos\psi_k \sin\psi_k \cdot \cos\epsilon\lambda\tau + \cos^2\psi_k \sin\epsilon\lambda\tau)\Big] \qquad (4.2.8a)$$

$$\dot{A}_k = \frac{\epsilon\bar{\delta}}{2\omega_k} \cdot E_{kJ}\Big[A_J(\cos^2\psi_k \sin\psi_k \cos\epsilon\lambda\tau - \sin^2\psi_k \sin\epsilon\lambda\tau) - B_J(\sin^2\psi_k \cos\epsilon\lambda\tau + \cos\psi_k \sin\psi_k \sin\epsilon\lambda\tau)\Big] \qquad (4.2.8b)$$

where $\psi_k = \omega_k \cdot \tau$

In order to simplify the equations (4.2.8a - b), we take the average values of the right-hand sides of them with respect to ψ_k over a period of 2π according to the method of Kryloff-Bogoliuboff-Van der Pol [1].* In that calculation A_j and B_j are considered to be constants. In this way we get

$$\left.\begin{aligned} \dot{A}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kJ}\Big[A_J \sin\epsilon\lambda\tau + B_J \cos\epsilon\lambda\tau\Big] \\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kJ}\Big[A_J \cos\epsilon\lambda\tau - B_J \sin\epsilon\lambda\tau\Big] \end{aligned}\right\} \qquad (4.2.9a)$$

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* Since $\epsilon\lambda \ll \omega_k$, rapid oscillations do not contribute to the changes occurring in A_j and B_j.

Similarly for $\dot{A}_j$ and $\dot{B}_j$

$$\left.\begin{aligned}\dot{A}_j &= -\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk}\left[A_k \sin\epsilon\lambda\tau + B_k \cos\epsilon\lambda\tau\right]\\ \dot{B}_j &= -\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk}\left[A_k \cos\epsilon\lambda\tau - B_k \sin\epsilon\lambda\tau\right]\end{aligned}\right\} \quad (4.2.9b)$$

We will now solve the coupled equations (4.2.9) for A_k , B_k , A_j and B_j , then the stability problem reduces to determining when these time dependent coefficients appearing in the solution of the equation (4.I.I) remain bounded or increase indefinetely with time.

We will now define the following functions in order to decouple the equations (4.2.9)

$$\left.\begin{aligned} X_1 &= A_k + iB_k \qquad & Y_1 &= A_j + iB_j\\ X_2 &= A_k - iB_k \qquad & Y_2 &= A_j - iB_j\end{aligned}\right\} \quad (4.2.IO)$$

Differentiating (4.2.IO) once with respect to τ and substituting (4.2.9) gives the following system of first order differential equations for X_1 , X_2 , Y_1 and Y_2 .

$$\dot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_2 e^{-i\epsilon\lambda\tau} \quad (4.2.IIa)$$

$$\dot{X}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_1 e^{i\epsilon\lambda\tau} \quad (4.2.IIb)$$

$$\dot{Y}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_2 e^{-i\epsilon\lambda\tau} \quad (4.2.IIc)$$

$$\dot{Y}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_1 e^{i\epsilon\lambda\tau} \quad (4.2.IId)$$

Consider the coupled equations (4.2.IIa) and (4.2.IId) differentiating the first with respect to τ we get

$$\ddot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k}E_{kJ}\dot{Y}_2 e^{-i\epsilon\lambda\tau} - \frac{\epsilon\bar{\delta}}{4\omega_k}E_{kJ}Y_2 e^{-i\epsilon\lambda\tau}\epsilon\lambda$$

Substituting now $\dot{Y}_2$ (Eq. 4.2.IId) into the above equation yields

$$\ddot{X}_1 + i\epsilon\lambda\dot{X}_1 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J}E_{kJ}E_{Jk} = 0 \qquad (4.2.I2)$$

Similarly we obtain for the others

$$\ddot{Y}_2 - i\epsilon\lambda\dot{Y}_2 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J}E_{kJ}E_{Jk}Y_2 = 0 \qquad (4.2.I3)$$

$$\ddot{X}_2 - i\epsilon\lambda\ddot{X}_2 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J}E_{kJ}E_{Jk}X_2 = 0 \qquad (4.2.I4)$$

$$\ddot{Y}_1 + i\epsilon\lambda\dot{Y}_1 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J}E_{kJ}E_{Jk}Y_1 = 0 \qquad (4.2.I5)$$

The general form of the characteristic equations of the second order differential equations (4.2.I2 - I5) is

$$m^2 \mp (i\epsilon\lambda)m - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_J}E_{kJ}\cdot E_{Jk} = 0$$

On the other hand, the general solution of these differential equations is of the form

$$C_1 e^{m_1} + C_2 e^{m_2}$$

where C_1 and C_2 are constants to be determined and m_1 and m_2 are the roots of the characteristic equation which are in general complex numbers with real and imaginary parts. The imaginary parts giving rise to oscillatory type motion do not contribute to instability , however , the real part depending on its sign determines the stability of the structure. Therefore , from equation (4.2.I0) we have

$$A_k = \frac{1}{2}(X_1 + X_2) \qquad A_j = \frac{1}{2}(Y_1 + Y_2)$$

$$B_k = \frac{i}{2}(X_2 - X_1) \qquad B_j = \frac{i}{2}(Y_2 - Y_1)$$

Considering the equations (4.I.2) , (4.2.5) and the above ones we get

$$w = w(X_1 , X_2 , Y_1 , Y_2)$$

Hence , the instability of the system is determined by the real part of the roots m_1 and m_2 which is equal to

$$\sqrt{-\epsilon^2\lambda^2 + \frac{\epsilon^2 \bar{\delta}^2}{4\omega_j\omega_k} E_{kj} E_{jk}} \equiv \sqrt{\Delta_I}$$

All the terms except E_{kj} and E_{jk} are positive , therefore the signs of the elements of $\underset{\approx}{E}$ matrix determine whether the system is stable or not. If the sign of the terms under radical sign is negative the solutions are stable, otherwise unstable.

$$\Delta_I \begin{matrix} < 0 \longrightarrow \text{Stable} \\ > 0 \longrightarrow \text{Unstable} \end{matrix}$$

If E_{kj} and E_{jk} are of opposite signs, the quantity under the radical sign is always negative and the system is stable.

In the case where E_{kj} and E_{jk} are of the same sign, the product of these terms will always be positive. Then we have to examine the inequality

$$-\epsilon^2\lambda^2 + \frac{\epsilon^2 \bar{\delta}^2}{4\omega_k\omega_j} E_{kj} E_{jk} > 0$$

for instability. The unstable region is found as

$$-\frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}} < \epsilon\lambda < \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}}$$

Substituting $\epsilon\lambda = \nu - (\omega_j+\omega_k)$ we obtain the boundaries of the unstable regions as

$$(\omega_k+\omega_j) - \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}} < \nu < (\omega_k+\omega_j) + \frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}E_{jk}}{\omega_k\omega_j}}$$

This resonance case is known as " combination resonance of sum type "[6].

<u>Case II :</u> $\nu = 2\omega_k + \epsilon\lambda$, $i = j = k$

In this case, when the forcing frequency is equal to or nearly equal to twice any of the natural frequencies of the system, the k^{th} term of the second expression on the right-hand side of the equation (4.2.5) will be infinitely large. Following the same procedure as in Case I , we remove the terms causing resonance phenomena from the equations (4.2.3) and associate them with the variational analysis. Thus we obtain for $\dot{A}_k$ and $\dot{B}_k$

$$\left.\begin{aligned}\dot{A}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k}\cdot E_{kk}\left[A_k \operatorname{Sin}\epsilon\lambda\tau + B_k \operatorname{Cos}\epsilon\lambda\tau\right] \\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kk}\left[A_k \operatorname{Cos}\epsilon\lambda\tau - B_k \operatorname{Sin}\epsilon\lambda\tau\right]\end{aligned}\right\} \quad (4.2.16)$$

Differentiating X_1 and X_2 defined in (4.2.10) with respect to τ once we get

$$\dot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kk} X_2 e^{-i\epsilon\lambda\tau} \quad (4.2.17a)$$

$$\dot{X}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kk} X_1 e^{i\epsilon\lambda\tau} \quad (4.2.17b)$$

Elimination of X_1 and X_2 between these two equations will yield

$$\ddot{X}_1 + i\epsilon\lambda\dot{X}_1 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k^2}E_{kk}^2 \cdot X_1 = 0 \quad (4.2.18)$$

$$\ddot{X}_2 - i\epsilon\lambda\dot{X}_2 - \frac{\epsilon^2\bar{\delta}^2}{16\omega_k^2}E_{kk}^2 \cdot X_2 = 0 \quad (4.2.19)$$

The real part of the roots of the characteristic equation of the above differential equations determines the instability and is equal to

$$\sqrt{-\epsilon^2\lambda^2 + \frac{\epsilon^2\bar{\delta}^2}{4\omega_k^2}E_{kk}^2} \equiv \sqrt{\Delta_{II}}$$

Instability will occur when $\Delta_{II} > 0$ or

$$-\epsilon^2\lambda^2 + \frac{\epsilon^2\bar{\delta}^2}{4\omega_k^2}E_{kk}^2 > 0$$

By examining this second order inequality, instability region is obtained as

$$2\omega_k - \frac{\epsilon\bar{\delta}}{2\omega_k}E_{kk} < \nu < 2\omega_k + \frac{\epsilon\bar{\delta}}{2\omega_k}E_{kk}$$

This region is known as " principal instability region " [6].

Case III : $\nu = \omega_j - \omega_k + \epsilon\lambda$, $k \neq j$ $j > k$

When the forcing frequency ν approaches the above values, the j^{th} term of the second expression on the right-hand side of the k^{th} equation of (4.2.5) and k^{th} term of the first expression on the right-hand side of the j^{th} equation of (4.2.5) become unbounded. Removing these resonance causing terms from equations (4.2.4) and associating them with the variational part of (4.2.3), as was done in the first two cases, will yield four differential equations for the functions A_k , B_k , A_j and B_j

$$\dot{A}_k \cdot \cos\omega_k\tau + \dot{B}_k \cdot \sin\omega_k\tau = 0 \qquad (4.2.20a)$$

$$-\dot{A}_k \cdot \sin\omega_k\tau + \dot{B}_k \cdot \cos\omega_k\tau = -\frac{\epsilon\bar{\delta}}{2\omega_k} \cdot E_{kj} \left[A_j \cdot \cos(\omega_k - \epsilon\lambda)\tau \right. \\ \left. B_j \cdot \sin(\omega_k - \epsilon\lambda)\tau \right] \qquad (4.2.20b)$$

$$\dot{A}_j \cos\omega_j\tau + \dot{B}_j \cdot \sin\omega_j\tau = 0 \qquad (4.2.21a)$$

$$-\dot{A}_j \sin\omega_j\tau + \dot{B}_j \cdot \cos\omega_j\tau = -\frac{\epsilon\bar{\delta}}{2\omega_j} E_{jk} \left[A_k \cos(\omega_j + \epsilon\lambda)\tau \right. \\ \left. B_k \sin(\omega_j + \epsilon\lambda)\tau \right] \qquad (4.2.21b)$$

Solving equations (4.2.20) and (4.2.21) for $\dot{A}_k$, $\dot{B}_k$, $\dot{A}_j$ and $\dot{B}_j$ and averaging with respect to ψ_k and ψ_j over a period of 2π yields

$$\left.\begin{aligned}\dot{A}_k &= \frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj}\cdot\left[A_j \sin\epsilon\lambda\tau + B_j \cos\epsilon\lambda\tau\right]\\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj}\left[A_j \cos\epsilon\lambda\tau - B_j \sin\epsilon\lambda\tau\right]\end{aligned}\right\} \quad (4.2.22a)$$

$$\left.\begin{aligned}\dot{A}_j &= -\frac{\epsilon\bar{\delta}}{4\omega_j}\cdot E_{jk}\cdot\left[A_k \sin\epsilon\lambda\tau - B_k \cos\epsilon\lambda\tau\right]\\ \dot{B}_j &= -\frac{\epsilon\bar{\delta}}{4\omega_j}\cdot E_{jk}\left[A_k \cos\epsilon\lambda\tau + B_k \sin\epsilon\lambda\tau\right]\end{aligned}\right\} \quad (4.2.22b)$$

If the equations (4.2.22) are substituted into the derivatives of (4.2.IO) we get

$$\dot{X}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_1 e^{i\epsilon\lambda\tau} \quad (4.2.23a)$$

$$\dot{X}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_k} E_{kj} Y_2 e^{-i\epsilon\lambda\tau} \quad (4.2.23b)$$

$$\dot{Y}_1 = -i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_1 e^{-i\epsilon\lambda\tau} \quad (4.2.23c)$$

$$\dot{Y}_2 = i\frac{\epsilon\bar{\delta}}{4\omega_j} E_{jk} X_2 e^{i\epsilon\lambda\tau} \quad (4.2.23d)$$

Decoupling these equations as before we obtain

$$\ddot{X}_1 - i\epsilon\lambda\dot{X}_1 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj}\, E_{jk}\, X_1 = 0 \quad (4.2.24)$$

$$\ddot{X}_2 + i\epsilon\lambda\dot{X}_2 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj}\, E_{jk}\, X_2 = 0 \quad (4.2.25)$$

$$\ddot{Y}_1 + i\epsilon\lambda\dot{Y}_1 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj}\, E_{jk}\, Y_1 = 0 \quad (4.2.26)$$

$$\ddot{Y}_2 - i\epsilon\lambda\dot{Y}_2 + \frac{\epsilon^2\bar{\delta}^2}{16\omega_k\omega_j} E_{kj}\, E_{jk}\, Y_2 = 0 \quad (4.2.27)$$

Once again , the real parts of the roots of the characteristic equations of the above equations are the same, and equal to

$$\sqrt{-\epsilon^2\lambda^2 - \frac{\epsilon^2\lambda^2}{4\omega_j\omega_k} E_{jk} E_{kj}} = \sqrt{\Delta_{III}}$$

Instability will occur when $\Delta_{III} > 0$ or

$$-\epsilon^2\lambda^2 - \frac{\epsilon^2\bar{\delta}^2}{4\omega_j\omega_k} E_{jk} E_{kj} > 0$$

It is obvious that when E_{kj} and E_{jk} are of the same sign there would be no instability but if they are of opposite signs instability will occur in the following range of ν

$$(\omega_j-\omega_k) - \frac{\epsilon\bar{\delta}}{2}\sqrt{-\frac{E_{kj}E_{jk}}{\omega_k\omega_j}} < \nu < (\omega_j-\omega_k) - \frac{\epsilon\bar{\delta}}{2}\sqrt{-\frac{E_{kj}E_{jk}}{\omega_k\omega_j}}$$

This resonance case is known as " combination resonance of difference type " [6].

Case IV : $\nu = \epsilon\lambda$, $i=j=k$

This case corresponds to very small forcing frequency. The k^{th} term of the first and second expressions on the right-hand side of the k^{th} equation of (5.2.5) will become very large. Following a similar procedure as in the previous cases , differential equations for $\dot{A}_k$ and $\dot{B}_k$ are obtained as

$$\left.\begin{aligned} \dot{A}_k &= \frac{\epsilon\bar{\delta}}{2\omega_k} E_{kk} B_k \cos\epsilon\lambda\tau \\ \dot{B}_k &= -\frac{\epsilon\bar{\delta}}{2\omega_k} E_{kk} A_k \cos\epsilon\lambda\tau \end{aligned}\right\} \qquad (4.2.28)$$

Similarly we get

$$\dot{X}_1 + \frac{i\epsilon\bar{\delta}}{2\omega_k} E_{kk} \cdot \cos\epsilon\lambda\tau \cdot X_1 = 0 \qquad (4.2.29)$$

$$\dot{X}_2 - \frac{i\epsilon\bar{\delta}}{2\omega_k} E_{kk} \cdot \cos\epsilon\lambda\tau \cdot X_2 = 0 \tag{4.2.30}$$

By applying separation of variables method the above equations are solved and the results become

$$X_1 = C_1 \cdot \exp\left(-\frac{i\bar{\delta}}{2\omega_k\lambda} E_{kk} \sin\epsilon\lambda\tau\right) \tag{4.2.3Ia}$$

$$X_2 = C_2 \cdot \exp\left(\frac{i\bar{\delta}}{2\omega_k\lambda} E_{kk} \sin\epsilon\lambda\tau\right) \tag{4.2.3Ib}$$

Upon examining (4.2.3I) , it is seen that for this case instability uccurs only if E_{kk} or ω_k have imaginary parts , thus the case of very small forcing frequecy does not affect the stability of the structure.

CHAPTER V

THEORETICAL and EXPERIMENTAL STUDIES

The stability criteria derived in Chapter IV will now be applied to specific structures and the experimental results will be compared with theory. The roots of the frequency equations are found numerically and the elements of $\underset{\approx}{E}$ matricies are evaluated by numerical integrations. The boundaries of the unstable regions are found for each case according to the inequalities given in Chapter IV. These regions will then be graphically presented.

V.I. Theoretical Calculations :

Circular steel and brass bars have been selected for the theoretical and experimental studies in order to make easy interpretations of the results. All the bars are in equal lenght and diameter supporting equal tip masses. The columns having diameters of 2 mm are tested with the weights of 90, I00, II0 and I20 gr and the ones with diameters of 3 mm are tested with the weights of 250, 300 and 350 gr. Modulus of elasticity of the bars are

$$E_{steel} = 2.I\ IO^6\ kg/cm^2 \qquad E_{brass} = I.I2\ IO^6\ kg/cm^2$$

and the corresponding densities are

$$d_{steel} = 7.8\ gr/cm^3 \qquad d_{brass} = 8.7\ gr/cm^3$$

Lengths of the columns were L=270 mm in all cases.

The natural frequencies of a given structure are found from the equation (3.3.6) by giving values to ω from zero onward. A root is found each time the left-hand side of the frequency equation chances sign . Some of these roots are tabulated in Table (5.I).

The perturbation method is valid if $\epsilon\bar{\delta}E_{ij}\ll 1$ or

$$\frac{X_o}{l}.E_{ij}\,\nu^2.\frac{M}{m_c l}\ll 1.$$

The other restriction on the criteria developed in Chapter IV is about the "averaging technique" which states that $\omega_i \gg \epsilon\lambda$ where λ is finite.Therefore the parameters of the system must be properly chosen or the amplitude of the excitation should be chosen as small as possible in order to preserve the validity of the above inequalities.

The elements of $\underset{\approx}{E}$ matrix are computed from

$$E_{ij}=\int_0^1 q_j(z)\cdot q_j''(z)\,dz=\frac{\int_0^1 \bar{V}_i(z)\cdot\bar{V}_j''(z)\,dz}{\left[\int_0^1 \bar{V}_i^2(z)\,dz\right]^{1/2}\cdot\left[\int_0^1 \bar{V}_j^2(z)\,dz\right]}$$

A computer program is given in the Appendix for the evaluation of the elements of $\underset{\approx}{E}$ by numerical integration using the trapezoidal method.*

The difference type of combination resonance will not occur for this kind of systems due to the fact that the matrix E has no negative elements.The instability range is then

$$(\omega_k+\omega_j)-\frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}\,E_{jk}}{\omega_j\,\omega_k}}<\nu<(\omega_k+\omega_j)+\frac{\epsilon\bar{\delta}}{2}\sqrt{\frac{E_{kj}\,E_{jk}}{\omega_j\,\omega_k}}$$

If $k\neq j$ it is called combination resonance of sum type

* See Appendix for the computer programs used.

and if k=j it is known as simple resonance. The expected unstable frequencies of the selected models are given in Table (5.2).

The first four mode shapes for the specimens are schematically shown below.

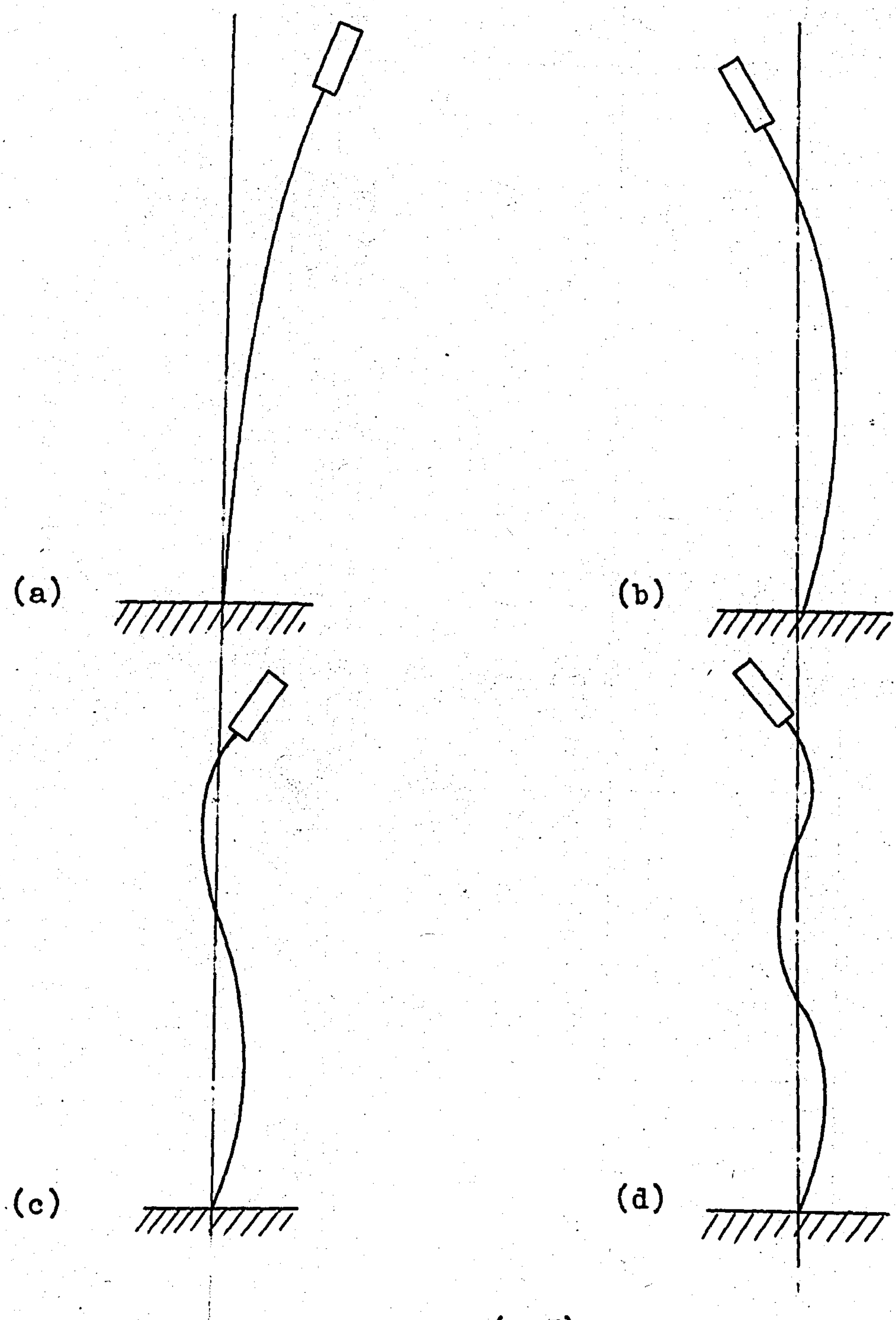


Fig.(5.I)

Diameter (mm)	Tip Mass (gr)	BRASS									
		$2\omega_1$		$\omega_1 + \omega_2$		$2\omega_2$		$\omega_1 + \omega_3$		$\omega_2 + \omega_3$	
		The.	Ex.	The.	Ex.	The.	Exp.	The.	Exp.	The.	Exp.
2	90	3.20	-	60.53	41	117.86	115	194.20	215	251.53	260
	100	2.98	-	60.26	40	117.54	110	193.92	210	251.20	255
	110	2.80	-	60.03	37	117.26	105	193.67	205	250.90	250
	120	2.60	-	59.78	34	116.96	100	193.41	200	250.59	245
3	250	4.76	-	91.72	55	178.68	150	292.99	-	379.95	360
	300	4.26	-	91.24	50	178.22	144	292.47	-	379.45	355
	350	3.88	-	90.82	45	177.76	140	292.03	-	378.97	350

Theoretical and experimental unstable frequencies (Hz)

Table (5.2)

Theoretical unstable regions found from the computer program for the case

Tip mass : $M = 120$ gr.

Length of C. : $L = 270$ mm.

Diameter of C: $D = 2$mm.

Material of C: Brass

are illustrated in Fig.(5.2). The elements of E_{ij} matrix where i and j taken upto 3 are given below.

$$E_{jk} = \begin{bmatrix} 0.0046995 & 0.0185771 & 0.0611296 \\ 0.0042487 & 0.0186938 & 0.0648045 \\ 0.0033405 & 0.0178469 & 0.0675349 \end{bmatrix}$$

By chosing the first natural frequency as the normalizing frequency the first three non-dimensional natural frequencies are

$$\omega_1 = 1.$$

$$\omega_2 = 44.817$$

$$\omega_3 = 147.207$$

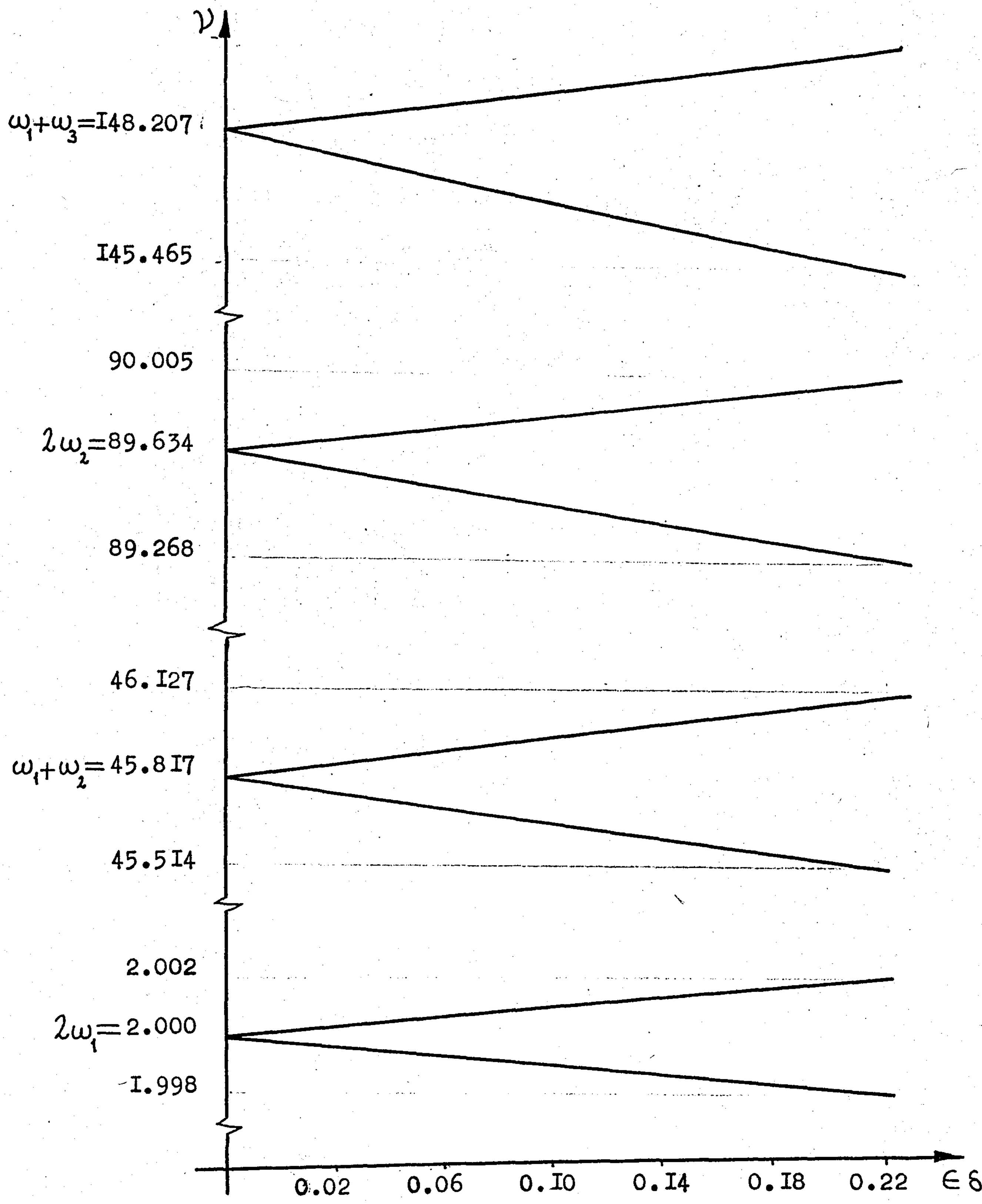

ν
ω1+ω3=148.207
145.465
90.005
2ω2=89.634
89.268
46.127
ω1+ω2=45.817
45.514
2.002
2ω1=2.000
1.998
0.02
0.06
0.10
0.14
0.18
0.22
εδ

Fig.(5.2)

Theoretical unstable regions for the case

Tip mass : $M = 120$ gr.

Length of C. : $L = 270$ mm.

Diameter of C.: $D = 2$ mm.

Material of C.: Steel

are shown in the Fig.(5.3)

The elements of the E_{ij} matrix are,

$$E_{ij} = \begin{bmatrix} 0.0054095 & 0.0191745 & 0.0619598 \\ 0.0049742 & 0.0192865 & 0.0654339 \\ 0.0040241 & 0.0184391 & 0.0681137 \end{bmatrix}.$$

and the non-dimensional natural frequencies are

$\omega_1 = 1.$

$\omega_2 = 42.046$

$\omega_3 = 137.093$

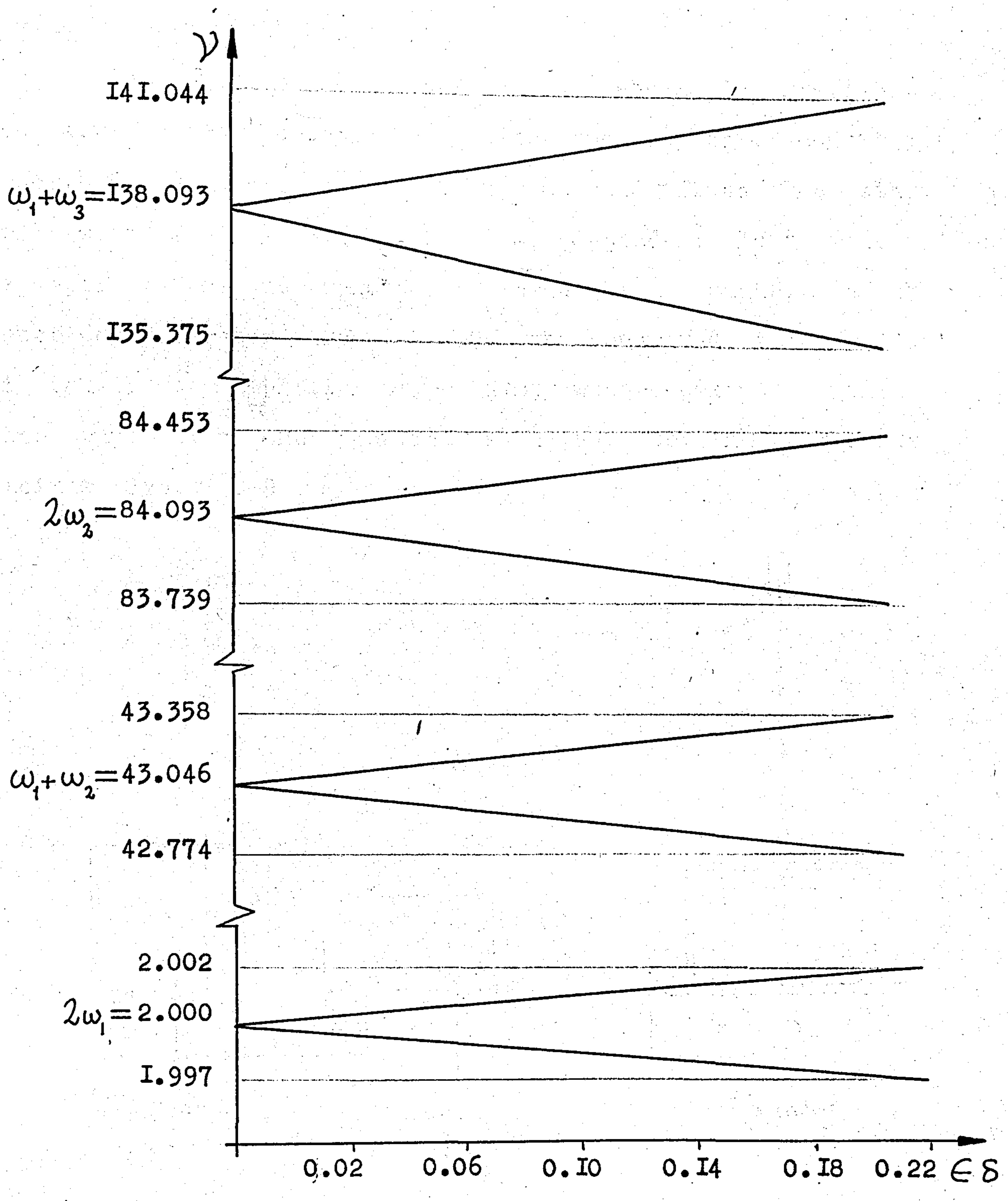

ν
I41.044
ω1+ω3=I38.093
I35.375
84.453
2ω2=84.093
83.739
43.358
ω1+ω2=43.046
42.774
2.002
2ω1=2.000
I.997
0.02
0.06
0.I0
0.I4
0.I8
0.22
∈δ

Fig.(5.3)

V.2.I. Experimental Set-up :

The experimental apparatus is shown in Fig.(5.4). The arrangement depicted satisfies the boundary conditions described in analytical section, one end fixed the other is free with a large mass. The time dependent load (excitation) is applied to the column by means of a shaker as shown. Function Generator Type TM 50I is connected to the input of the Power Amplifier Type 2706 whose output supplies sine wave with the required amplitude to the Vibration Exciter Type 4809.

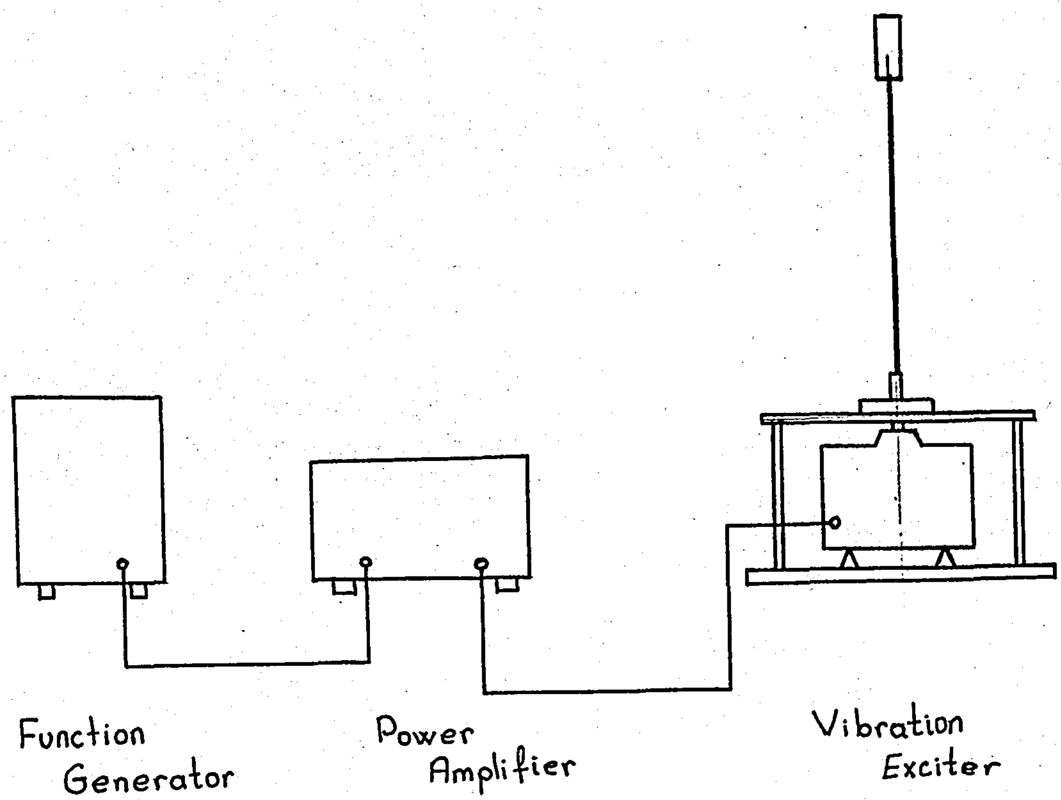


Fig. (5.4)

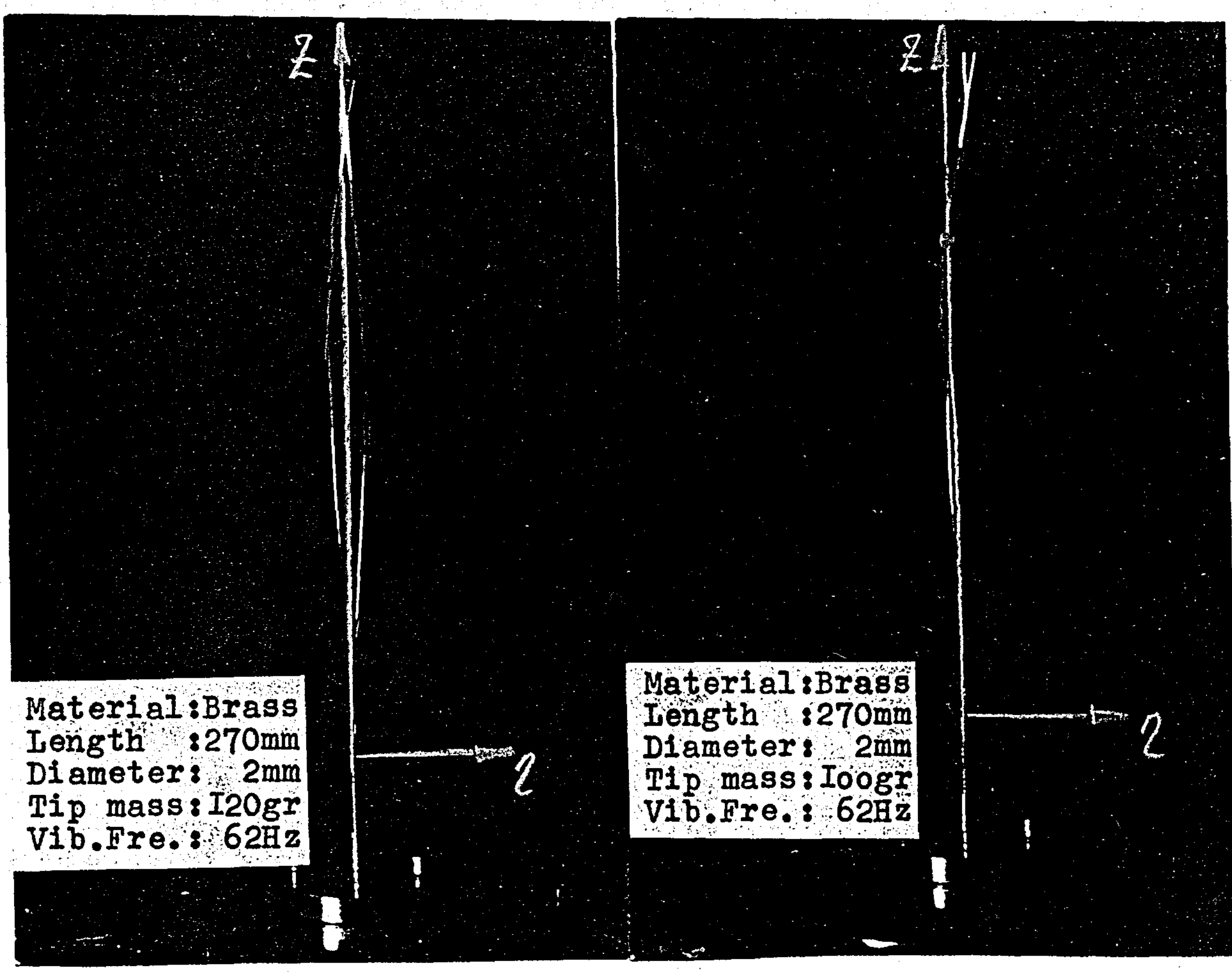
Diameter (mm)	Tip Mass (gr)	STEEL									
		$2\omega_1$		$\omega_1 + \omega_2$		$2\omega_2$		$\omega_1 + \omega_3$		$\omega_2 + \omega_3$	
		The.	Ex.	The.	Ex.	The.	Exp.	The.	Exp.	The.	Exp.
2	90	4.78	-	88.40	108	172.02	155	282.07	315	365.69	350
	100	4.54	-	88.18	100	171.82	150	281.82	305	365.46	347
	110	4.24	-	87.90	95	171.56	145	281.53	295	365.19	345
	120	4.06	-	87.68	90	171.30	140	281.31	285	364.93	342
3	250	6.80	-	133.35	-	259.90	205	424.87	-	551.42	-
	300	6.16	-	132.80	-	259.44	200	424.31	-	550.95	-
	350	5.66	-	132.66	-	249.06	195	423.84	-	550.54	-

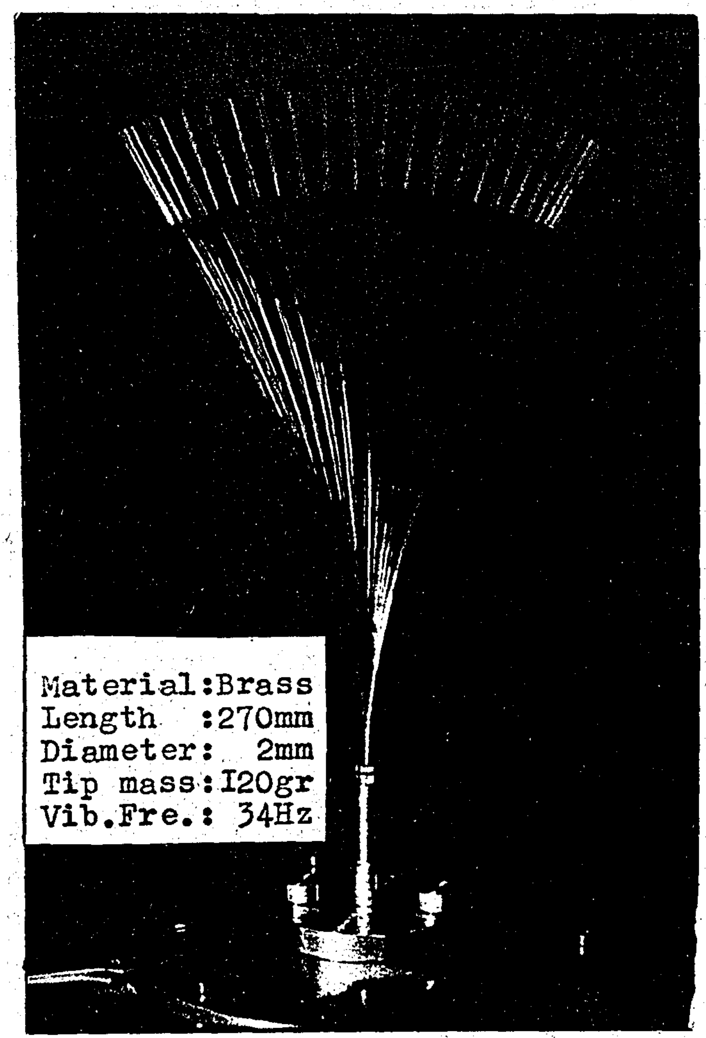
Theoretical and experimental unstable frequencies (Hz)

Table (5.3)

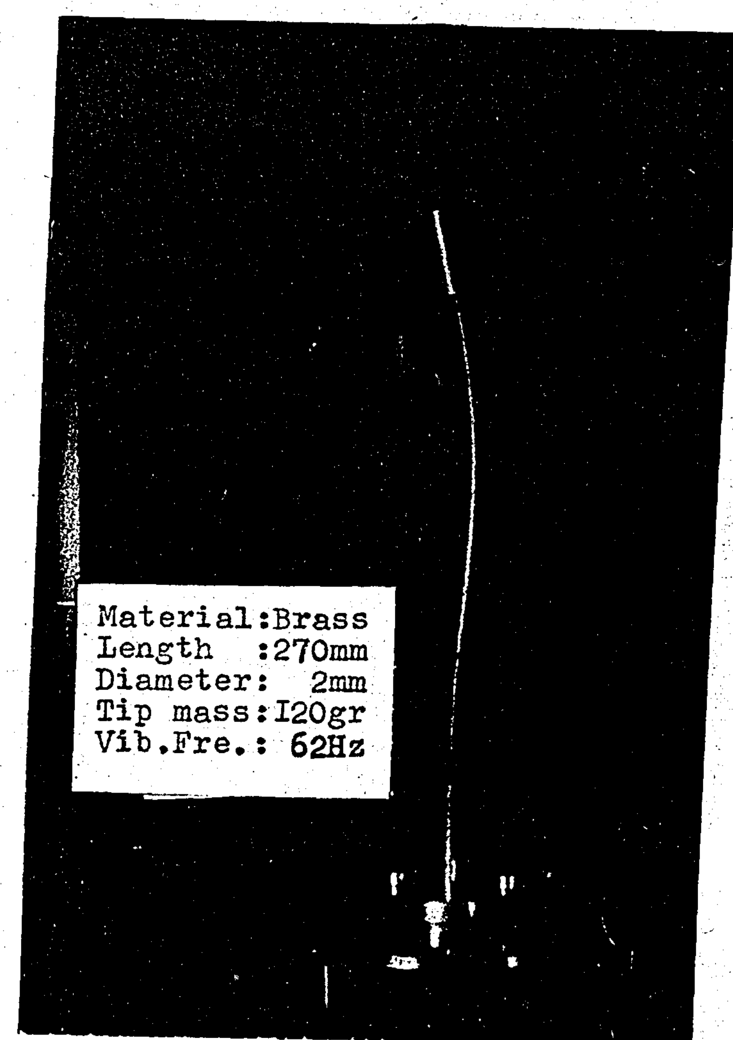
V.2.2. Experiments :

Coordinates of the nodes change with the constant $\delta = M/m_c.l$. In the reference [7] , some of the non-dimensional coordinates were given for the various values of δ . We have found experimentally the change in coordinates of the nodes with the increasing tip mass M. It is experimentally verified that the coordinate z increases as the tip mass M increases.



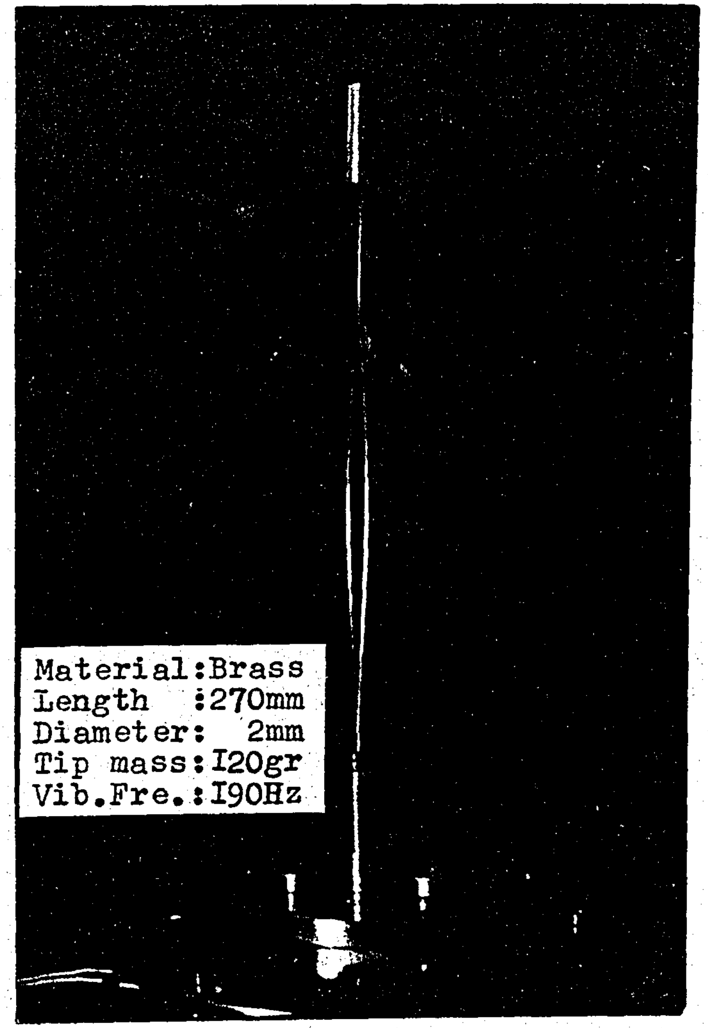


Instability

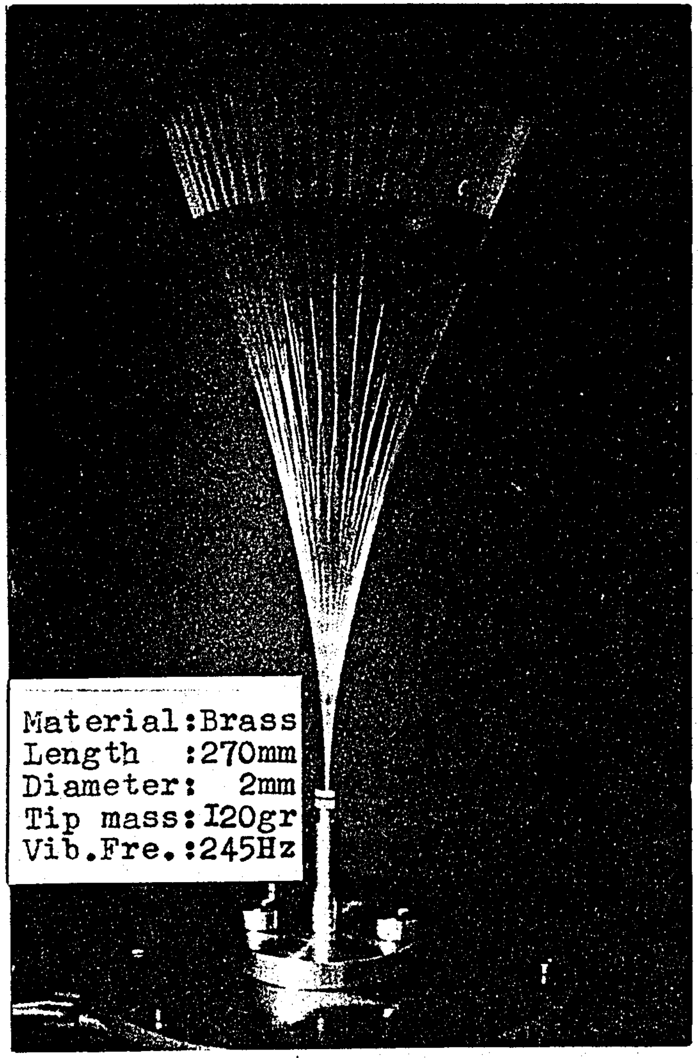


Mode shape

Instability



Mode shape



Instability

Mode shape

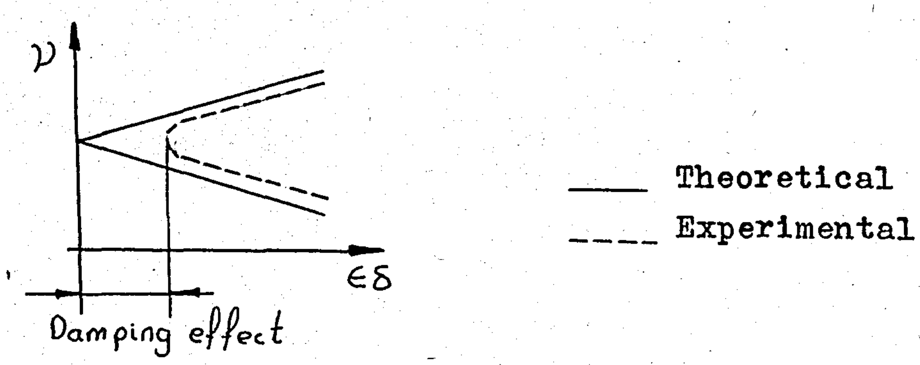
CHAPTER VI

CONCLUDING REMARKS

In this work, the importance of combination resonances on the stability of a structure composed of a cantilevered elastic column supporting a large mass at the free end and subjected to vertical harmonic excitation at the support has been examined.Theoretical results along with the experimental results were presented and compared.

It was found that for a system described in the analytical section combination resonance of difference type does not occur due to the characteristics of the elements of the coupling matrix $\underset{\sim}{E}$.

Experiments showed that,internal material damping allows resonance phenomena to begin at the higher values of $\epsilon\delta$ such that (or excitation amplitude)



The effects of the tip mass,Young Modulus of the column and length of the column on the natural frequencies and the stability of the column have been examined.Experimentally found natural frequencies are more appropriate to the theoretical ones than the frequencies cousing instability. This is due to the fact that,we have made an additional approximation in the stability analysis.Rotatory inertia and shear deformation effects cause the differences between the experimental and theoretical natural frequencies.

APPENDIX

COMPUTER PROGRAMS

```
      DOUBLE PRECISION UM,Z,FZ,QUO,SO,SI,TI,T2,T3,T4,T5,T6,T7,A,
     $B,SQRT,DCOS,DSIN,DABS,ABSTOT,TOT,SAL
      DIMENSION FZ(4IOO),ABSFZ(4IOO)
C     ROOTS OF FREQUENCY EQUATION : FZ=0
C     Z=NATURAL FREQUENCY
      READ(5,*) G,AL,W,E,RHO,AI,AREA
      EI=E*AI
      UM=RHO*AREA/G
      A=UM/EI
      B=W/EI
      J=0
      Z=0.
      DO IOO I I,4000
      QUO=DSQRT((B**2)+4.*A*(Z**2))
      SO=DSQRT((-B-QUO)/2.)
      SI=DSQRT((B-QUO)/2.)
      TI=2.*SO*A*(Z**2)
      SAL=SI*AL
      T2=SO*((SO**4)-(SI**4))*DCOS(SAL)
      T3=QUO*(SO/SI)-(B/G)*(Z**2)*DSIN(SAL)
      XI=SO*AL
      T4=(T2-T3)*COSH(XI)
      T5=QUO*(B/G)*(Z**2)*DCOS(SAL)
      T6=B*(SO**2)*SI*DSIN(SAL)
      T7=(T5-T6)*SINH(XI)
      FZ(I)=T4-T7-TI
      Z=Z-I.
      ABSFZ(I)=DABS(FZ(I))
      IF(I.EQ.1) GO TO 100
      J=I-1
      ABSTOT=ABSFZ(I)-ABSFZ(J)
      TOT=DABS(FZ(I)-FZ(J))
      IF(ABSTOT.GT.TOT) GO TO 80
      GO TO 100
```

```
 80 WRITE(6,60) Z,FZ(J),FZ(I)
 60 FORMAT(5X,`Z=`,D28.16,10X,`FZ=`,D28.16,5X,`FZ2=`,D28.16,/)
100 CONTINUE
    STOP
    END
```

```
C       CALCULATION OF R-MATRIX BY NUMERICAL INTEGRATION
C       TRAPEZOIDAL METHOD
        DIMENSION W(5),SO(5),S (5),BK(5),CK(5),AK(5),T(5),TT(5,5)
       $QUO(5),U(5),E(5,5),UW(5),V(5),F(5,5),R(5,5),FF(5),RR(5),PN(5)
        REAL MG
        READ(5,*) (W(I),I=1,3)
        READ(5,*) G,AL,MG,S,RHO,AI,AREA
        UM=RHO*AREA/G
        EI=S*AI
        A=UM/EI
        B=MG/EI
C       N=NUMBER OF MODES CONSIDERED
C       M=NUMBER OF INTERVALS FOR INTEGRATION
C       H=DELTA Z
        N=3
        M=5000
        L=M-1
        H=AL/5000.
        DO 10 I=1,N
        UO(I)=SQRT((B**2)-4.*A*(W(I)**2))
        SO(I)=SQRT((-B-QUO(I))/2.)
        S1(I)=SQRT((B-QUO(I))/2.)
        BK(I)=(SO(I)**2)*COSH(SO(I)*AL)+(S1(I)**2)*COS(S1(I))*AL)
        CK(I)=(SO(I)**2)*SINH(SO(I)*AL)+SO(I)*S1(I)*SIN(S1(I)*AL)
        AK(I)=BK(I)/CK(I)
     10 CONTINUE
        DO 100 I=1,N
        DO 100 J=1,N
        Z=0.
        DO 200 K=1,L
        V(I)=COSH(SO(I)*Z)-COS(S1(I)*Z)+AK(I)*(SINH(SO(I)*Z)-(SO(I)/
       $S1(I))*SIN(S1(I)*Z))
        U(J)=(SO(J)**2)*COSH(SO(J)*Z)+S1(J)**2)*COS(S1(J)*Z)+AK(J)*(
       $(SO(J)**2)*SINH(SO(J)*Z)+SO(J)*S1(J)*SIN(S1(J)*Z))
        F(I,J)=V(I)*U(J)
        FF(I)=V(I)*V(I)
```

```
      IF(Z.EQ.O.) GO TO 500
      IF(Z.EQ.AL) GO TO 550
      GO TO 600
 500  T(I,J)=0.5*E(I,J)+T(I,J)
      TT(I)=0.5*FF(I)+TT(I)
      GO TO 650
 600  T(I,J)=F(I,J)+T(I,J)
      TT(I)=FF(I)+TT(I)
 650  CONTINUE
      Z=Z-H
 200  CONTINUE
      R(I,J)=H*T(I,J)
      RR(I)=H*TT(I)
      PN(I)=SQRT(RR(I))
 110  CONTINUE
 100  CONTINUE
      DO 75 I=1,3
  75  WRITE(6,64) (R(I,J),J=1,3)
      WRITE(6,66) (PN(I),I=1,3)
  66  FORMAT (5X,`PN(1)=`,F15.9,/;5X,PN(2)=`,F15.9,/,`PN(3)=`,F15.9)
      DO 400 I=1,3
      DO 400 J=1,3
      E(I,J)=R(I,J)/PN(I)/PN(J)
 400  CONTINUE
      DO 40  I=1,3
  40  WRITE(6,64) (E(I,J),J=1,3)
  64  FORMAT(//,3(F 5.9,5X))
C     UNSTABLE  REGION  BOUNDARIES
      DO 450 I=1,3
 450  UW(I)=W(I)/5./W(1)
      WRITE(6,68) (UW(I),I=1,3)
  68  FORMAT(/,5X,`UW(1)=`,F8.5,/,5X,`UW(2)=`,F8.5,/,5X,`UW(3)=`,
     $F8.5,/)
      DO 800 I=1,3
      DO 900 J=1,3
      IF(I.EQ.2.AND.J.EQ.1) GO TO 900
```

```
      IF(I.EQ.1.AND.J.EQ.3) GO TO 900
      IF(I.EQ.2.AND.J.EQ.3) GO TO 900
      UU=UW(I)-UW(J)
      WRITE(6,67) I,J,UU
   67 FORMAT(10X,`UU=UW( `,I1,`)+UW(`,I1,`)=`,F10.5)
      EPSD=0.
  850 EPSD=EPSD-0.02
      DEL1=SQRT(E(I,J)*E(J,I)/UW(I)/UW(J))
      DEL2=SQRT(1.-2.*(UW(I)-UW(J))*EPSD*DEL1)
      DEL3=SQRT(1.-2.*(UW(I)-UW(J))*EPSD*DEL1)
      ALTS=(DEL2-1.)/EPSD/DEL
      USTS=(1.-DEL3)/EPSD/DEL
      WRITE(6,65) ALTS,USTS,EPSD
   65 FORMAT(/,5X,`ALT SINIR=`,F12.8,10X,`UST SINIR=`,F12.8,5X,
     $`EPSD=`,F5.2)
      IF(EPSD.GT.0.2) GO TO 900
      GO TO 850
  900 CONTINUE
  800 CONTINUE
      STOP
      END
```

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