GRAND UNIFIED MODELS
AND
NEUTRINO OSCILLATIONS

by

Fahrünisa Neyzi

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1. INTRODUCTION

1.1 The Historical Development of Weak Interaction Physics

Discovery of the neutrino had its origins in nuclear physics. The neutrino was postulated by Pauli[1] in 1933, in order to account for some mysteries in beta decay, which were observed back in 1919. In that year, Chadwick[2] noticed that there was a continuous spectrum of disintegration electrons as well as a well defined cutoff energy when a neutron transformed into a proton during a beta process. The spectrum extended from zero to a definite maximum energy corresponding to the total energy available in the transformatio

In addition to this apparent violation of the principle of conservation of energy, the principle of conservation of both linear and angular momentum seemed to be violated. It looked as though linear momentum was not conserved, because the trajectory of the emerging electron was noncollinear with the trajectory of the nucleus. Furthermore, the neutron was a fermion with spin one-half, whereas the sum of the spins of proton and electron gave one or zero, a composite system which was statistically a boson. Only the electrical balance of the system needed no modification. Finally, to account for these inconsistencies, Pauli postulated the existence of an electrically neutral particle with spin one-half and mass equal to zero.

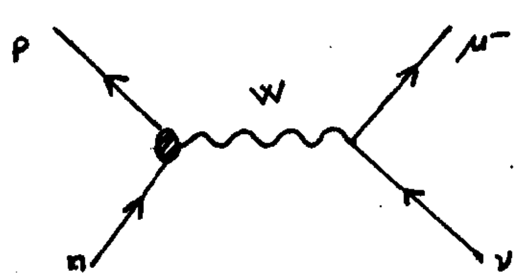
Historically, beta decay was the first manifestation of the weak interaction. In 1934, a year after Pauli postulated the existence of neutrinos Fermi[3] constructed a theory of beta decay, which came very close to being the correct one. Fermi hypothesized a vector interaction in close analogy with quantum electrodynamics without a propagator. The weak interaction was assumed to be a four-fermion vertex interaction with the transition matrix element for beta decay given by :

$$M = \frac{G}{\sqrt{2}} (\bar{\psi}_p \gamma^{\mu} \psi_n)(\bar{\psi}_e \gamma_{\mu} \psi_{\nu}) \qquad (1.1\text{-}1)$$

where γ's are the Dirac matrices and $G/\sqrt{2}$ is the weak interaction coupling constant with the dimensions of inverse mass squared.

Fermi's theory had to be revised, because unlike quantum electrodynamics, it gave only first order diagrams. If this were true, the electron-neutrino elastic crosssection, for example, would rise without limit at high neutrino energies. In addition to a vector interaction, axial vector interactions of the form $\bar{\psi}\gamma^{\mu}\gamma^{5}\psi$ were postulated.

The picture that emerged finally has an intermediate vector boson to mediate the beta process. The matrix element and the diagram for it are as follows:



$\nu + n \rightarrow e^- + p$

Fig. 1

$$M = g^2 [\bar{u}_p (1 - a\gamma_5) \gamma^{\alpha} u_n] \frac{1}{M_W^2 - q^2} [\bar{u}_e (1 - \gamma_5) \gamma_{\alpha} u_{\nu}] \qquad (1.1\text{-}2)$$

where a is the axial vector coefficient approximately equal to 1.25. The problem of a limitless crosssection is remedied in this picture because as momentum q rises, the propagator term suppresses the crosssection.

1.2 Helicity, Charge Conjugation and Parity

The beta decay in which an electron is produced is called a beta minus decay to distinguish it from another beta decay in which positrons are produced. Actually, in beta minus decays anti-neutrinos are produced. The distinction between a neutrino and an anti-neutrino can be made as follows: The spin of a neutrino is anti-parallel to its momentum, whereas the spin of an anti-neutrino is parallel to its momentum. This defines a "handed-ness". We call a neutrino left-handed and an anti-neutrino right-handed. This concept is formulated by defining the helicity, which is the dot product of the spin $\vec{\sigma}$ and momentum $\vec{p}$ divided by the norms of these vectors. For the neutrino helicity is -1 and for the anti-neutrino it is 1. This two component theory of neutrinos, in which half the states of four component fermions is suppressed for a massless particle was developed by Weyl in 1929.* In this scheme, anti-neutrino and neutrino states appear as $1/2\,(1 \pm \gamma_5)\,\psi$.

Assuming the validity of the two component neutrino theory, parity conservation and charge conjugation are violated separately. Parity conservation can be defined as follows: if the mirror reflection of a physical situation is another possible phenomenon, this situation conserves parity. Mathematically, of course the mirror image represents

* See Appendix A

reversing the direction of the spatial vector $\vec{r}$. Parity(P) is conserved in strong and electromagnetic interactions. On the other hand, charge conjugation(C) changes a particle into its anti-particle. It is conserved in strong and electromagnetic interactions also.

The two component theory postulated for neutrinos has two consequences in relation to the C and P operations. The P operation reverses the neutrinos linear momentum, leaving its spin direction unchanged. So we get a neutrino in a positive helicity state, which is not a physical situation according to our hypothesis. Therefore, parity is not conserved. Similarly, charge conjugation is violated, because when we operate on a neutrino state with C, we get an anti-neutrino state with negative helicity. However, CP is conserved together, since it changes a left-handed neutrino into a right-handed anti-neutrino.

1.3 Quarks and Families

Until 1936 electrons, neutrinos and anti-neutrinos were the only known so called leptons. In 1936, Anderson and Neddermayer[4] discovered a cosmic ray particle which acted like an electron but had a larger mass. Later it was called a muon(μ). In 1947, Powell[5] demonstrated the decay of a pion into a muon.

$$\pi^{\pm} \longrightarrow \mu^{\pm} + \nu_1 (\bar{\nu}_1) \qquad (1.3\text{-}1)$$

The pion had another decay mode:

$$\pi^{\pm} \longrightarrow e^{\pm} + \nu_2 (\bar{\nu}_2) \qquad (1.3\text{-}2)$$

It was postulated in the late fifties that the two neutrinos were not the same. ν_1 acted as though it remembered being born with a muon. Therefore, it was called a muon neutrino(ν_μ). ν_2 ,however seemed to couple to an electron. hence was called an electron neutrino(ν_e).

Experiments done in the fifties and sixties verified that the neutrino did exist and the electron neutrino was distinct from the muon neutrino. In 1953, Cowan and Reines[6] performed an experiment in which the direct interaction of a free neutrino was clearly observed, giving proof of the existence of the neutrino. In 1962, Lederman,Schwartz, Steinberger et.al. used muon anti-neutrinos to bombard protons; muons not positrons, were formed. Finally, Perl et.al.[7] found evidence for a heavier lepton tau(τ); a third particle acting like electrons and muons.

By that time, the number of so called "elementary particles" had reached several hundred. They were classified broadly into hadrons, leptons and vector bosons. Only hadrons have strong interactions as well as electromagnetic and weak interactions. In the hadron group, baryons have spin one-half and mesons have integral spins. Leptons include electrons, neutrinos and muons. Electroweak vector bosons consist of the proton and the $W^\pm$ and Z_0 of the weak interactions.

In order to reduce the number of elementary particles, GellMan and Zweig[8] proposed in 1964 the existence of three types of quarks, namely up, down and strange quarks. During the same year Bjorken and Glashow[9] postulated another type or flavor of quarks, called charm. In the quark model, three quarks make a baryon; mesons are composed of a quark and an anti-quark. This model led the way to associating quark multiplets (later called weak isospin multiplets) with leptonic ones.

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{bmatrix} t \\ b \end{bmatrix} : \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{bmatrix} \nu_\tau \\ \tau \end{bmatrix} \qquad (1.3\text{-}3)$$

$$\mathrm{I} \quad \mathrm{II} \quad \mathrm{III} \qquad \mathrm{I} \quad \mathrm{II} \quad \mathrm{III}$$

Each quark multiplet along with its leptonic counterpart is called a family. If we define a family number L_e for electrons and let it equal $+1$ for e^- and ν_e and -1 for e and ν_e, these numbers are conserved in all known reactions. Similarly, the lepton number L_μ seems to be conserved so far.

1.4 Dirac and Majorana Mass Terms

We have seen that the conventional theory of neutrinos rests on two pillars. Firstly that the neutrino is massless and therefore only the left-handed neutrino and the right-handed anti-neutrino exist. Secondly, that family lepton numbers L_e, L_μ etc. and their sums are good quantum numbers conserved in all known reactions.

If neutrinos have nonzero mass and if the mass eigenstates and weak interaction eigenstates do not coincide, then neutrinos will oscillate. If there is such a mismatch, then we can expect a neutrino being produced in a weak process at $t=0$ and thereby being in a pure weak interaction eigenstate, to have a nonzero probability of having turned into:

a) A neutrino from another family

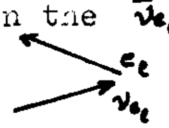
$$\begin{tikzcd} \nu_e \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \nu_\tau \arrow[dl, leftrightarrow] \\ & \nu_\mu & \end{tikzcd}$$

This process conserves total lepton number $L = L_e + L_\mu + L_\tau$, but violates each family lepton number separately.

b) An anti-neutrino from the same or another family

$$\nu_{e_\ell} \longrightarrow (\nu_{e_\ell})^c$$
$$\nu_{e_\ell} \longrightarrow (\nu_{\mu_\ell})^c$$

where c means charge conjugation. This process violates total lepton number by $\pm$ 2 units.

The usual weak interaction Hamiltonian is a sum of the neutral current contribution and a part of the form $G/\sqrt{2}\, j_\alpha J^\alpha$ where j_α is $(\bar{\nu}_{e_\ell} \gamma_\alpha e_\ell) + (\bar{\nu}_{\mu_\ell} \gamma_\alpha \mu_\ell) + (\bar{\nu}_{\tau_\ell} \gamma_\alpha \tau_\ell)$ + a hadronic current term. Clearly, each term conserves both the separate family lepton numbers and the total lepton number. (i.e. $\bar{\nu}_{e_\ell}$ has -1 and e_ℓ has + 1, therefore the sum is zero in the $\bar{\nu}_{e_\ell} \gamma_\alpha e_\ell$ term describing the following vertex : ). Thus if neutrino oscillations are to occur, we need additional terms in the Lagrangian which violate the total L or L_i ; $i = e, \tau, \mu$. If we want to violate just the family lepton number as a good quantum number, we can add just a <u>Dirac</u> term to the above Hamiltonian of the form $\bar{\nu}_R M \nu_L$ where $\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$ and M is a 3X3 mass matrix. This coupling obviously needs right-handed neutrinos which do not exist in the minimal conventional theories.

If however, we allow the change in both the family and the total lepton numbers to be nonzero, then we have a Majorana term in the Hamiltonian of the form $(\bar{\nu}_L)^c M \nu_L$ (or $(\bar{\nu}_R)^c M \nu_R$). This term will violate the total lepton number by ± 2 units).

2. GRAND UNIFIED MODELS

2.1 Introduction

The basic theory that lies behind "grand unification" is that there exists a simple local symmetry group G, which unifies strong, weak and electromagnetic interactions. Because it is a simple group it has only one free gauge coupling constant g, which evolves differently for all three interactions once we come down below the grand unification scale of 10^{15} Gev. At this extremely high momentum scale (which is nevertheless below 10^{19} Gev at which quantum gravitational effects become appreciable), G breaks down to $SU(3)_c \times SU(2) \times U(1)$ which further breaks down to $SU(3)_c \times U(1)_{e-m}$ at lower momenta (~ 100 Gev).

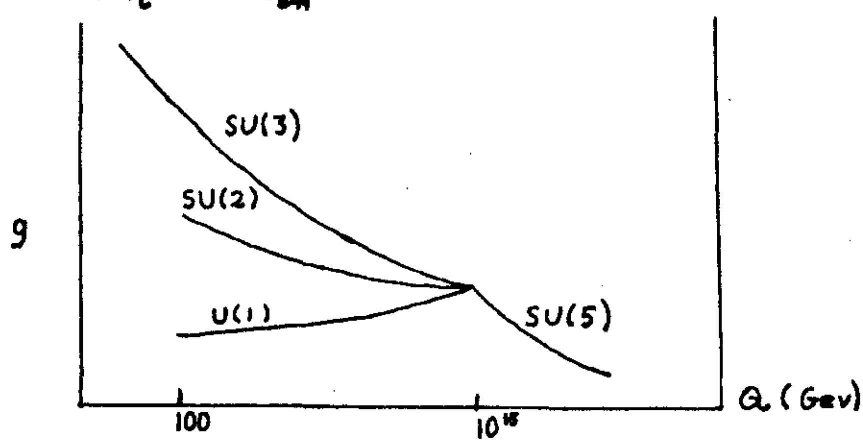


Fig 2. Slow logarithmic variations in the SU(3), SU(2) and U(1) gauge coupling constants

In Grand Unified Theories quarks and leptons share the same representations of G. Furthermore, electric charge operator is a generator of G and when it acts on the multiplet which contains both quarks and leptons, we get a relation between quark and lepton charges. Another consequence of making quarks and leptons share the same representation is that we can expect to find some relationship between their masses and decrease the number of free parameters in the standard SU(3) color plus Weinberg-Salam theories. Since gauge bosons link all particles in a multiplet, quarks and leptons, sitting in the same representation, will interact through these bosons and change into each other. Therefore, baryon and lepton number conservation will be violated.

So far, the only constraints imposed on G were that it has to contain $SU(3)_c \times SU(2) \times U(1)$ as a subgroup and after all the symmetry breaking stages as we move down on the momentum scale, the unified theory must reduce to our low energy standard theory. We impose two other conditions, which limit our choices as to what specific group G will be. First, G must admit a complex representation in order to accomodate the complex representation of fermions in the standard theory. Second, G must be renormalizable. This means that infinities deriving from higher order terms can be compensated by adding a finite number of cancelling terms and redefining only mass terms and coupling constants, so that the final result is a finite physical quantity. These conditions reduce our possibilities a good deal. Since SU(3), SU(2) and U(1) have ranks of 2,1 and 1 respectively, the smallest rank we can allow is 4, and the only group of rank 4 which satisfies these conditions is SU(5). It is the minimal grand unified "scenario".

2.2 The Standard Theory: QCD and Weinberg-Salam Gauge Theories

We have seen that the conventional theory of neutrinos rests on two assumptions. First, that neutrinos are massless therefore only left-handed neutrinos and right-handed anti-neutrinos exist. Second, that family lepton numbers L_μ, L_e, and their sums are good quantum numbers conserved in all known reactions so far. Let us then, examine the standard gauge theory of strong, weak and electromagnetic interactions to see how these phenomenological assumptions fit into the theory and then search for ways of modifying the theory to give us neutrino masses and lepton number violations.

The standard theory asserts that the minimal group needed to describe known phenomena is :

$$G^1 = SU(3)_c \times SU(2)_L \times U(1) \qquad (2.2\text{-}1)$$

where SU(3) (color group) is the gauge group responsible for strong interactions and $SU(2)_L \times U(1)$ is the gauge group of Glashow-Weinberg-Salam[11] responsible for unified weak and electromagnetic interactions.

Fermions, i.e., leptons and quarks, are placed in the simplest possible representations of these groups in the minimal theory. Quarks have three colors red, yellow, blue, and both right-handed and left-handed quarks are triplets under SU(3). Leptons do not participate in strong interactions so they are $SU(3)_c$ singlets. Under SU(2), on the other hand left handed quarks and leptons are doublets. Right-handed quarks and leptons, excluding neutrinos, are singlets. Right-handed neutrinos do not exist. So for each family, we have :

$$\begin{pmatrix} u \\ d \end{pmatrix}_L^{\text{red}} \begin{pmatrix} u \\ d \end{pmatrix}_L^{\text{yellow}} \begin{pmatrix} u \\ d \end{pmatrix}_L^{\text{blue}} \quad ; \; u_R^r, u_R^y, u_R^b \quad d_R^r, d_R^y, d_R^b$$

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad ; \quad e_R \qquad (2.2-2)$$

In addition to fermions which are spin one half particles, the standard theory includes spin one and spin zero particles. These are called gauge bosons and Higgs scalars, respectively.* Gauge bosons are always in the adjoint representation of the group. This principle determines their number. The number of generators in the group equals the number of gauge bosons of the group. SU(3) has $3^2-1=8$ bosons, called gluons; SU(2) X U(1) mix to give $(2^2-1)+1=4$ bosons, and the massless photon of $U(1)_{E-M}$.

Higgs particles are introduced in order to break the gauge symmetry without letting the theory acquire unwanted infinities, i.e. without spoiling "renormalizability".** The G^2 group breaks down into $G' \equiv SU(3)_c \times U(1)_{E-M}$ and the existence of these exact local symmetries results in the conservation of color and electric charge. It is believed that G^2 breaks down to G' through the intervention of the Higgs doublet. Other representations of the Higgs fields are excluded because this theory predicts $\cos\theta = M_W/M_Z$ which has been tested experimentally.[12]

* See Appendix C

** See Appendix B

2.3 Neutrino Masses in the Standard Theory

The conventional fermion masses come from Dirac couplings to the Weinberg-Salam $I = 1/2$ Higgs fields of the form :

$$\left(H_{\substack{I=1/2 \\ \Delta L=0}} \right) \bar{f}_R f_L + H.C. \qquad (2.3\text{-}1)$$

In the absence of a right-handed neutrino field, neutrinos can acquire mass from a majorana coupling of the form:

$$\left(H_{\substack{I=1 \\ \Delta L=2}} \right) f_L f_L + H.C. \qquad (2.3\text{-}2)$$

A majorana mass term is absent in the minimal Weinberg-Salam theory, because it only has a higgs doublet and it conserves lepton number.

However, another theory is proposed by Gelmini and Roncadelli,[13] which has been recently elaborated by Georgi, Glashow and Nussinov.[14] According to this model, a complex Higgs triplet with an electrically neutral component, which can develop a vacuum expectation value is introduced into the standard theory. The triplet has $|\Delta L| = 2$ and gives a majorana mass to neutrinos.

Denoting the complex triplet by a 2 X 2 matrix

$$\begin{bmatrix} \chi^0 & \chi^-/\sqrt{2} \\ \chi^-/\sqrt{2} & \chi^{--} \end{bmatrix} = \chi \qquad (2.3\text{-}3)$$

The usual doublet ϕ by

$$\begin{bmatrix} \phi^0 \\ \phi^- \end{bmatrix} = \phi \qquad (2.3\text{-}4)$$

The covariant derivative by

$$D^{\mu} = \partial^{\mu} + i \frac{e}{\sin\theta} \vec{T}\cdot\vec{W}^{\mu} + i \frac{e}{\cos\theta} S V^{\mu} \tag{2.3-5}$$

where $\vec{T}$ is the generator of the SU(2) and S of U(1);
And defining the action of the generators upon the scalar fields ϕ and χ as:

$$\vec{T}\phi = \vec{\sigma}\,\phi/2$$
$$\vec{T}\chi = \vec{\sigma}\,\chi/2 + \chi\,\vec{\sigma}^{*}/2$$
$$S\phi = -\phi/2\ , \quad S\chi = -\chi \tag{2.3-6}$$

$\vec{\sigma}$'s are the Pauli spin matrices

We can write the most general Lagrangian :

$$\mathcal{L}(\phi,\chi) = (D^{\mu}\phi)^{+} D_{\mu}\phi + tr[(D^{\mu}\chi)^{+} D_{\mu}\chi] - V(\phi,\chi) \tag{2.3-7}$$

where V includes 7 arbitrary parameters λ_i ; $i = 1,...5$ U, V.
V vanishes when $\chi^{0} = V/\sqrt{2}$; $\phi^{0} = U/\sqrt{2}$ (2.3-8)

If we take these to be the vacuum expectation values. then

$$M_W^2 = \frac{e^2}{4\sin^2\theta}(U^2+2V^2) \quad ; \quad M_Z^2 = \frac{e^2}{4\sin^2\theta\cos^2\theta}(U^2+4V^2)$$

$$M_W^2/M_Z^2 = \cos^2\theta\,(U^2+2V^2)/(U^2+4V^2) \tag{2.3-9}$$

But experimentally $M_W/M_Z = \cos\theta$ therefore we deduce that $V \ll U$.

The Higgs doublet and triplet together contain ten real fields. Of these, three are eaten up by the Higgs mechanism, one is the Goldstone boson called Majoron, and the rest are massive. Of these remaining six massive particles, two are neutral, one is singly and the other is doubly charged. Of course, the coupling of χ to leptons is of the form $\overline{\psi}_R^{\ell c}\chi^{+}\psi_L^{\ell}$.

As we have seen, B-L symmetry is spontaneously broken in the model by a small vacuum expectation value of a Higgs triplet. A right-handed neutrino is not introduced, but the left-handed neutrino obtains a majorana mass.

Once we leave the realm of standard theories, other models and rich possibilities open up; leading to theories which go beyond the $G^2 = SU(3)_c \times SU(2) \times U(1)$ group and including it as a subgroup. Advocating that G^2 is a low energy relic of a bigger group which generates all interactions, Grand Unified theorists strive to answer questions unsolved by standard theories; such as why charge is quantized, why there are three families and why there is more than one coupling constant.

2.4 The SU(5) Grand Unified Model[15]

Let us first analyze the SU(5) grand unified model in terms of its gauge boson. Higgs meson and fermion content and see how the representations to which these particles belong decompose under SU(3) X SU(2) subgroups. Gauge bosons necessarily belong to the adjoint representation $24 = 5^2-1$. This representation decomposes under SU(3) X SU(2) as

$$\underline{24} = (3,2) + (\overline{3},2) + (8,1) + (1,3) + (1,1) \qquad (2.4\text{-}1)$$

The part inside curly brackets is familiar. These are the twelve bosons of the standard theory: Eight gluons, singlet under SU(2), and two color singlets combining to form $W^{\pm}$, Z^0, γ. The remaining two (3.2) and ($\overline{3}$.2) represent the isodoublets called X and Y, two superheavy bosons which come in three colors and have charges of 4/3 and 1/3 respectively.

SU(5) breaks down to SU(3) X SU(2) X U(1) through the Higgs 24-plet ϕ ; then SU(3) X SU(2) X U(1) breaks down to SU(3) X U(1) through the Higgs 5-plet H[16]. The vacuum expectation values of these mesons are

$$\underbrace{O(10^{15})\,\mathrm{Gev}}_{v}\left(\begin{array}{ccc|cc} 1 & 0 & 0 & & \\ 0 & 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & 1 & & \\ \hline & & & -3/2 & 0 \\ & 0 & & 0 & -3/2 \end{array}\right) = \langle 0|\phi|0\rangle \qquad (2.4\text{-}2)$$

$$\underbrace{O(100)\,\mathrm{Gev}}_{v_0}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \langle 0|H|0\rangle \qquad (2.4\text{-}3)$$

$\langle 0|\phi|0\rangle$ gives mass to X and Y bosons; $\langle 0|H|0\rangle$ gives mass to the standard $W^{\pm}$ and Z^0 bosons and fermions in the form

$$M_X^2 = M_Y^2 = \frac{25}{8} g^2 v^2 \qquad ; \quad m_{W\pm}^2 = \frac{1}{4} g^2 v_0^2$$

Under SU(3) X SU(2) the 5-plet decomposes as $\underline{5} = (3,1) + (1,2)$. The (1,2) part is the isospin doublet which gives mass to fermions in the standard theory and achieves the breaking down of SU(3) X SU(2) X U(1) to SU(3) X U(1). The (3,1) part has a zero vacuum expectation value, because it is both colored and charged. It is superheavy and it also mediates proton decay.

Fermions are grouped into $\underline{15}$-dimensional reducible representations $\underline{15} = \overline{5} + 10$.

$$\overline{5} = \begin{pmatrix} \bar{d}_r \\ \bar{d}_y \\ \bar{d}_b \\ e^- \\ \nu_e \end{pmatrix}_L = (\overline{3},1) + (1,2) \quad \text{of} \quad SU(3) \times SU(2) \tag{2.4-4}$$

$$10 = 1/\sqrt{2} \begin{pmatrix} 0 & \bar{u}_b & -\bar{u}_y & -u_r & -d_r \\ -\bar{u}_b & 0 & \bar{u}_r & -u_y & -d_y \\ \bar{u}_y & -\bar{u}_r & 0 & -u_b & -d_b \\ u_r & u_y & u_b & 0 & -e^+ \\ d_r & d_y & d_b & e^+ & 0 \end{pmatrix}_L = (3,2) + (\overline{3},1) + (1,1) \tag{2.4-5}$$

Five is the fundamental representation of SU(5) and ten is the antisymmetric part of the product of two fives: $5 \times 5 = 15 + 10$. where 15 is symmetric and 10 is antisymmetric.

Before we go on to examine neutrino masses, we note some properties of the minimal grand unified group SU(5). Even though SU(5) is not a safe group, the anomalies of each of the two irreducible representations $\bar{5}$ and 10 cancel. Therefore, this particular reducible representation 15 is renormalizable. In SU(5) the electric charge operator Q is a generator of the group. Therefore, it is traceless and the sum of the electromagnetic charges in any representation must be zero. Taking $\bar{5}$:

$$\left.\begin{array}{r} 3Q_{\bar{d}} + Q_{\bar{e}} + Q_{\nu} = 0 \\ Q_{\nu} = 0 \end{array}\right\} \qquad \begin{array}{l} Q_d = -1/3\, Q_{e^-} \\ Q_d = -1/3 \end{array} \qquad (2.4\text{-}6)$$

Another property of SU(5) is that baryon minus lepton number, B-L, is conserved. B-L global symmetry is a linear combination of two symmetries V_1 and U(1); V_1 global symmetry is defined by the following transformations, which leave the Lagrangian invariant.

$$\left.\begin{array}{l} \psi_{10} \to e^{i\phi}\,\psi_{10} \\ \psi_{\bar{5}} \to e^{-3i\phi}\,\psi_{\bar{5}} \end{array}\right\} \text{Fermions} \qquad \left.\begin{array}{l} H_5 \to e^{-2i\phi}\, H_5 \\ \phi_{24} \to \phi_{24} \end{array}\right\} \text{Higgs} \qquad (2.4\text{-}7)$$

U(1) symmetry in SU(5) is identified with the hypercharge $Y = Q - T_{3L}$ where Q is the charge operator and T_{3L} is the third component of the weak SU(2) isospin. Now even though V_1 global symmetry is spontaneously broken, there are no massless Goldstone bosons around, because a linear combination of and U(1) remains unbroken. To find out what linear combination this is, one solves[17]

$$[aV_1 + bY]\,|H_5\rangle = 0 \qquad (2.4\text{-}8)$$

since the definition of an unbroken symmetry generator L is $L|v\rangle = 0$ where $|v\rangle$ is the vacuum state. Written explicitly,

$$\left\{ a\underbrace{\begin{bmatrix} -2 & & & & \\ & -2 & & 0 & \\ & & -2 & & \\ & 0 & & -2 & \\ & & & & -2 \end{bmatrix}}_{V_1} + b\underbrace{\begin{bmatrix} -1/3 & & & 0 & \\ & -1/3 & & & \\ & & -1/3 & & \\ & 0 & & 1/2 & \\ & & & & 1/2 \end{bmatrix}}_{Y} \right\} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2.4\text{-}9)$$

If $b = 4a$ then

$$-\frac{10}{3}a\left\{ \left(\begin{array}{ccc|cc} 1 & & 0 & & \\ & 1 & & 0 & \\ 0 & & 1 & & \\ \hline & 0 & & 0 & \end{array} \right) \right\} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2.4\text{-}10)$$

We note that there is another global U(1) symmetry left intact after the spontaneous breakdown. This is the $B-L$ global symmetry.

$$B-L \propto aV_1 + bY \propto V_1 + 4Y = 1/5\,(V_1 + 4Y) \qquad (2.4\text{-}11)$$

2.5 Neutrino Masses in the SU(5) Model

In the minimal SU(5) model fermions acquire mass through $\bar{5} \times 10$ and 10×10 couplings.

$$\bar{5} \times 10 = 5 + 45 \qquad (1)$$

$$10 \times 10 = \bar{5} + \overline{45} + 50 \qquad (2)$$

Therefore, the Higgs multiplet giving mass to fermions should belong to $\underline{5}$ or $\underline{45}$ of SU(5). Since down quarks and charged leptons get masses through (1), while up quarks get masses through (2), we can relate the masses of down quarks and charged leptons for each family. Let us use the simplest case, just a Higgs $\underline{5}$. The part of the Lagrangian which denotes the mass of fermions is[18]

$$\mathcal{L}_f = 1/2\,(\chi^\dagger)^{\alpha\beta}\gamma^0 M_1[H_\alpha\psi_\beta - H_\beta\psi_\alpha] - 1/4\,\epsilon^{\alpha\beta\gamma\delta\epsilon}\chi_{\alpha\beta}M_2 H_\gamma \chi_{\delta\epsilon} \tag{2.5-1}$$

where $\psi_\alpha \rightarrow$ fermion $\underline{5}$

$\chi_{\alpha\beta} \rightarrow$ fermion $\underline{10}$

$M_{1,2} \rightarrow$ generator matrices

Concentrating on the first term only and diagonalizing M_1 by rotating ψ and χ, we notice that the vacuum expectation value of the Higgs $\underline{5}$ gives a mass term

$$\mathcal{L}_f \ni \sum_{\beta=1,2,3,4} v_0\,(\chi^+)^{5\beta}\,\gamma^0 \cdot M_1^D\,\psi_\beta \tag{2.5-2}$$

For each family this means:

$$m_d = m_e\ ,\quad m_s = m_\mu\ ,\quad m_b = m_\tau \tag{2.5-3}$$

Thus, we arrive at the crucial issue of neutrino masses in SU(5). In the minimal standard version of the SU(5) model, all neutrino masses are zero. No Dirac mass term $\overline{\nu}_R\nu_L$ is allowed, since there is no right-handed neutrino in the $\overline{5}$+ 10 representations of fermions. On the other hand, the Majorana mass term $\nu_L\nu_L$ is also forbidden, because $B-L$ is a conserved number in SU(5). There are two mechanisms used to introduce neutrino mass into SU(5) theory. One is to put in ν_R by hand as an extra singlet of SU(5). Then, the Lagrangian will include a term proportional to $\overline{\nu}_R H_\alpha \psi^\alpha$ and the vacuum expectation value of the Higgs $\underline{5}$ will give mass to the neutrinos.

Another method is to look at the post SU(5) scale. Since the grand unification mass is only a few orders below the scale where gravity effects become appreciable, we may not be able to ignore the Planck mass scale.[19] We may expect terms with two fermions coupled to two Higgs particles, scaled by an inverse power of the Planck mass. These terms, however, are non-renormalizable.

In addition to equations (1) and (2), fermion masses can arise from a coupling of the form

$$\bar{5} \times \bar{5} = \overline{15} + \overline{10} \qquad (3)$$

Using the first two equations, we have the allowed $f_{\bar{5}} f_{10} H_{\bar{5}}$ and $f_{10} f_{10} H_5$ mass terms, which admit the global U(1) gauge transformation (2.4-7) and conserve B-L number. Equation (3) would have given mass to neutrinos of the form $f_{\bar{5}} f_{\bar{5}} H_{\overline{15}}$ if we had a Higgs <u>15</u> in the minimal SU(5) model.

Using just a Higgs <u>24</u> and a Higgs <u>5</u>, we consider the products of pairs of Higgs representations :

$$5 \times 24 = 5 + 45 + 70$$

$$5 \times 5 = 10 + 15 \qquad (2.5\text{-}4)$$

We may have effective interactions of the form :

$$O(1/m_p)\ f_{\bar{5}} f_{10} H_{\bar{5}} H_{24}$$

$$O(1/m_p)\ f_{10} f_{10} H_5 H_{24}$$

$$O(1/m_p)\ f_{\bar{5}} f_{\bar{5}} H_5 H_5 \qquad (2.5\text{-}5)$$

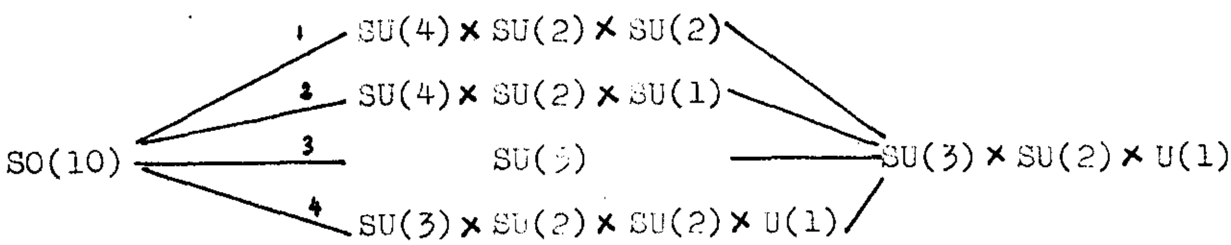
Equation (1) will modify (2.5-3) ; (2) will modify quark masses, but both of these will leave the global symmetry (2.4-7) and thereby B-L conservation intact. Only (3) will generate a Majorana neutrino mass of the order 10^{-5} eV and violate the B-L conservation.

2.6 The SO(10) Grand Unified Model

As we have done for SU(5), we can now examine the next smallest ranking grand unified model, SO(10)[20] in terms of its gauge boson, Higgs meson and fermion content, review their decomposition under SU(5) and finally see what this theory predicts for neutrino masses.

SO(10) is an orthogonal group of rank 5 and the gauge field associated with it transforms in the $45 = (10 \times (10-1))/2$ dimensional adjoint representation, which transforms as a second rank antisymmetric tensor. Under SU(5) $\underline{45}$ decomposes as $24 + 10 + \overline{10} + 1$. The $\underline{24}$ represents the now familiar SU(5) gauge bosons. The remaining 21 bosons are superheavy ones mediating proton decay.

The SO(10) model is very flexible in terms of the Higgs meson content, because there are many ways of breaking SO(10) down to SU(3) X SU(2) X U(1). For example:



(2.6-1)

Concentrating on the most familiar path 3, we note that the minimal set realizing this chain of symmetry breaking is :

$$SO(10) \xrightarrow[\text{Higgs } \underline{16}]{} SU(5) \xrightarrow[\text{Higgs } \underline{45}]{} SU(3)\times SU(2)\times U(1) \xrightarrow[\text{Higgs } \underline{10}]{} SU(3)\times U(1) \qquad (2.6\text{-}2)$$

Since $\underline{45}$ decomposes as $24 + 10 + \overline{10} + 1$ as indicated above. and $\underline{10}$ decomposes as $5 + \overline{5}$ we recover the Higgs mesons of SU(5); namely the 24-plet and 5-plet that accomplish the breakdown of SU(5).

Fermions are all contained in the sixteen dimensional irreducible spinor representation of SO(10). Since $\underline{16}$ decomposes as $\overline{5} + 10 + 1$, we recover the SU(5) fermion content with $\overline{5} + 10$. $\underline{1}$ is the right-handed neutrino. The Higgs particles that can couple to fermions in the SO(10) model, thereby giving mass to fermions, should appear in the decomposition of the product $\underline{16} \times \underline{16}$ under SU(5).

$$16 \times 16 = 10 + 120 + 126$$

The 10 is a vector, the 120 is a third rank antisymmetric tensor and the 126 is a fifth rank antisymmetric tensor. Let us analyze these three representations for their SU(5) content:

$$10 = \overline{5} + 5$$
$$120 = 45 + \overline{45} + 10 + \overline{10} + 5 + \overline{5}$$
$$126 = 50 + 45 + \overline{15} + 10 + \overline{5} + 1 \qquad (2.6\text{-}3)$$

We notice that only the $\underline{126}$ contains an SU(5) singlet component. Therefore, only the $\underline{126}$ can give the right-handed neutrino a mass at the tree level.

$B-L$ operator is a generator of SO(10) denoted by $B-L = 2Q - [T_{3L} + T_{3R}]$. Since it represents a gauge symmetry, it has to be broken at least locally. Furthermore. belongs to the Cartan algebra of the group. Therefore, we cannot break this symmetry by the vacuum expectation value of the adjoint $\underline{45}$. We can break it either through some Higgs mesons(which also couple to fermions) or we can put in some other Higgs mesons explicitly for this purpose.

If we choose the first way we can use the part of Higgs $\underline{126}$ that transforms as $\underline{1}$ or $\underline{15}$ under SU(5). These are the ones that couple to left and right-handed neutrinos, respectively.[21] For the second method, we can use a Higgs 16-plet, which has both neutral entries getting a nonzero vacuum expectation value.[22] If only one does, then $B-L$ will still be conserved.

2.7 Neutrino Masses in the SO(10) Model

As we have seen, the right-handed neutrino exists in SO(10), therefore the $\bar{\nu}_R \nu_L$ Dirac mass term is allowed. Furthermore, because $B-L$ is a generator of the group and there are no massless Goldstone bosons around, it must be violated. If $B-L$ is broken, $\nu_L \nu_L$ and $\nu_R \nu_R$ type Majorana terms are allowed. So, SO(10) does predict neutrino masses naturally, unless there are some secret symmetries preventing it. We shall go through two models in detail : The Georgi-Nanopoulos model[23] and the Witten[24] model.

Georgi and Nanopoulos introduce an extra neutral lepton singlet E_L which couples to the fermion 16-plet through the Higgs 16-plet. As we have seen before this Higgs field has two neutral components which give a vacuum expectation value and break $B-L$ symmetry globally. If only one neutral component has a nonzero vacuum expectation value, the will be broken locally only and E_L and ν_R will gain a large Dirac mass and our familiar left-handed neutrino will remain massless. However, if the other neutral component of the Higgs 16-plet also gets a vacuum expectation value, then ν_L will get a small Majorana mass in the form $E_L \nu_L$.

This addition of an extra singlet as in the SU(5) model is a little arbitrary and this defect is remedied in Witten's model. Witten's argument goes like this: in SO(10), the right-handed and left-handed neutrinos will couple to get Dirac masses comparable to the usual quark and lepton masses. Since we know that the left-handed neutrinos are relatively light, if not massless, then this large Dirac mass term must be avoided. This can be done by giving the right-handed neutrino a large Majorana mass. In matrix form, without the Majorana mass, the mass matrix would be :

$$\begin{array}{c|c} & \begin{array}{cc} \nu_R & \nu_L \end{array} \\ \begin{array}{c} \nu_R \\ \nu_L \end{array} & \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \end{array} \tag{2.7-1}$$

where m has the magnitude of a quark or lepton mass. However, when we add a large Majorana mass M , then the mass matrix becomes :

$$\begin{array}{c|c} & \begin{array}{cc} \nu_R & \nu_L \end{array} \\ \begin{array}{c} \nu_R \\ \nu_L \end{array} & \begin{pmatrix} M & m \\ m & 0 \end{pmatrix} \end{array} \qquad M \gg m \tag{2.7-2}$$

The eigenvalues are approximately M and m^2/M , these being the masses of the right and left-handed neutrinos, respectively.

As we have seen the only Higgs multiplets that can couple to fermions are those that appear in the product <u>16</u> x <u>16</u>, namely <u>10</u>, <u>120</u> and <u>126</u>. Since only <u>126</u> contains an SU(5) singlet and since the right-handed neutrino is a singlet, only 126 can give it mass at the tree level. However, this value of mass M will then be a free parameter. To avoid this Witten tries other methods. His proposal is that the right-handed neutrino receives a mass at the two loop level.

In the minimal form of the SO(10) model only Higgs 10-plet and gauge field 45 couple directly to fermions. Also, Higgs 16-plet is necessary both to break SO(10) down to SU(5) and to break B-L symmetry. Furthermore, in order to compensate for the Higgs 126-plet, which is a fifth rank tensor, we need a vector and two second rank tensors. So we need a 10 and two 45's coupling to fermions. In addition, we are allowed to use Higgs 16-plet, but not in direct coupling to fermions. Given these rules, the two loop diagram is as follows :

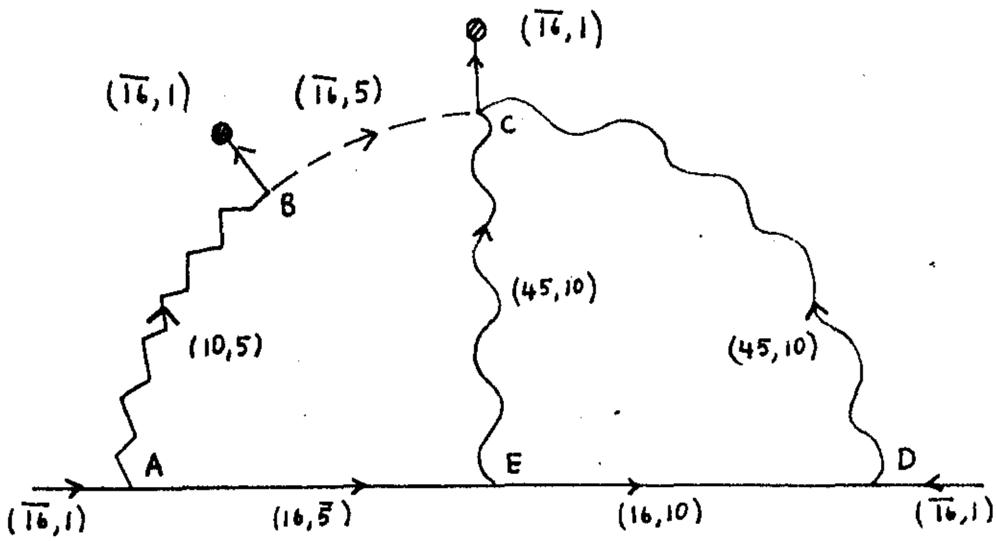


Fig. 3. The two loop diagram that gives mass to the right-handed neutrino. Shown in parentheses are the SO(10) and SU(5) transformation properties of each field.

Gauge field A_μ

Fermion

Higgs 10-plet

Higgs 16-plet

Higgs vacuum expectation value

A :

$$16 \times 10 \supset \overline{16} \qquad (SO(10))$$

$$5 \times \bar{5} \supset 1 \qquad (SU(5))$$

$$\mathcal{L} \supset g_{yukawa}\, \psi \bar{\psi} H_{10}$$

B :

$$\overline{16} \times \overline{16} \supset 10$$

$$5 \times 1 \supset 5$$

$$\mathcal{L} \supset \mu H_{\overline{16}} H_{\overline{16}} H_{10}$$

C :

$$\overline{16} \times 45 \times 45 \supset \overline{16}$$

$$5 \times 10 \times 10 \supset 1$$

$$\mathcal{L} \supset g^2 [(\vec{\tau} \cdot A_\mu) H_{16}]^* [(\vec{\tau} \cdot A_\mu) H_{16}]$$

D :

$$16 \times \overline{16} \supset 45$$

$$1 \times 10 \supset 10$$

$$\mathcal{L} \supset g \bar{\psi} \gamma_\mu (\vec{A}_\mu \cdot \vec{\tau}) \psi$$

E :

$$16 \times 45 \supset 16$$

$$10 \times 10 \supset \bar{5}$$

$$\mathcal{L} \supset g \bar{\psi} \gamma_\mu (\vec{A}_\mu \cdot \vec{\tau}) \psi \qquad (2.7-3)$$

Let us estimate the mass that the right-handed neutrino receives from this diagram. We let $[K]$ denote the contribution from the loop integral. We consider all the coupling constants at the vertices and the vacuum expectation values of the Higgs fields. So we write a general expression :

$$m_{\nu_R} \sim g^4 g_{yukawa}\, \mu \langle \phi_{16} \rangle \langle \phi_{16} \rangle [K] \qquad (2.7-4)$$

Since g_{yukawa} is a free parameter, in order to estimate its value we note that the Higgs 10 couples to both quarks and vector bosons

$$g_{yukawa} \langle \phi_{10} \rangle \psi \bar{\psi} \sim m_q$$
$$g^2 \langle \phi_{10} \rangle \langle \phi_{10} \rangle A_\mu A_\mu \sim m_w^2 \qquad (2.7\text{-}5)$$

where m_q is the mass of quarks and m_w is the W boson mass. From (2.7-5) we get

$$g_{yukawa} = g \frac{m_q}{m_w} \qquad (2.7\text{-}6)$$

We find the contribution from the integrals using the following formula

$$[K] = \Lambda^k$$

$$k = p - (3/2)m - n \qquad (2.7\text{-}7)$$

p = external lines $(=4)$

m = fermion lines $(=2)$

n = boson lines $(=2)$

Λ = cut off

We get $k = -1$. All masses appearing in this diagram (vacuum expectation values and cut offs) are superheavy masses. Let us denote this scale as M. Using (2.7-6) and (2.7-7), (2.7-4) becomes

$$m_{\nu_R} \sim g^4 \left(g\, m_q / m_w \right) \mu\, M^2 / M \qquad (2.7\text{-}8)$$

If we use the following numbers

$$
\begin{aligned}
&M \sim 10^{15}\ \text{Gev}\\
&g^2 \sim \alpha/\pi \sim 2\times 10^{-3}\\
&g \sim .05\\
&\mu \sim 1\\
&M_W \sim 20\ \text{Gev}
\end{aligned}
\qquad (2.7\text{-}9)
$$

then we get

$$m_{\nu_R} \sim 4\times 10^{-6} \times .05 \times .05 \times 10^{15}\, m_q \sim 10^{7}\, m_q \qquad (2.7\text{-}10)$$

We can estimate the mass that the left-handed neutrino gives from this process, since we know that $m_{\nu_L} = m_q^2/m_{\nu_R}$ So,

$$m_{\nu_L} \sim m_q^2 \times (M_W/m_q)^2 \times 1/g\mu \times (\alpha/\pi)^{-3} \times 1/M \qquad (2.7\text{-}11)$$

With our previous numbers this means that for each generation $m_{\nu_L} = 10^{-7}\, m_q$. Or explicitly :

$m_{\nu_L} \sim 1\,eV$, $m_{\nu_\mu} \sim 100\,eV$, $m_{\nu_\tau} \sim 1\text{-}10\ keV$.

These estimates, although consistent with laboratory bounds on these masses, violate cosmological constraints, which say that the sum of all neutrino masses must be less than 40 eV. Hence, the estimated neutrino masses must be suppressed by a factor of 10^2 , which means that the right-handed neutrino mass must be larger by a factor of 10^2. One obvious way of achieving this is by estimating to be 10^{19} GeV, i.e., the Planck mass.

In order to estimate corrections from higher order diagrams, we next consider one possible three loop process. Since 10 X 10 contains a <u>45</u> instead of getting an effective <u>126</u> from 10 X 45 X 45, we get it from 10 X 10 X 10 X 45. So we consider the following process :

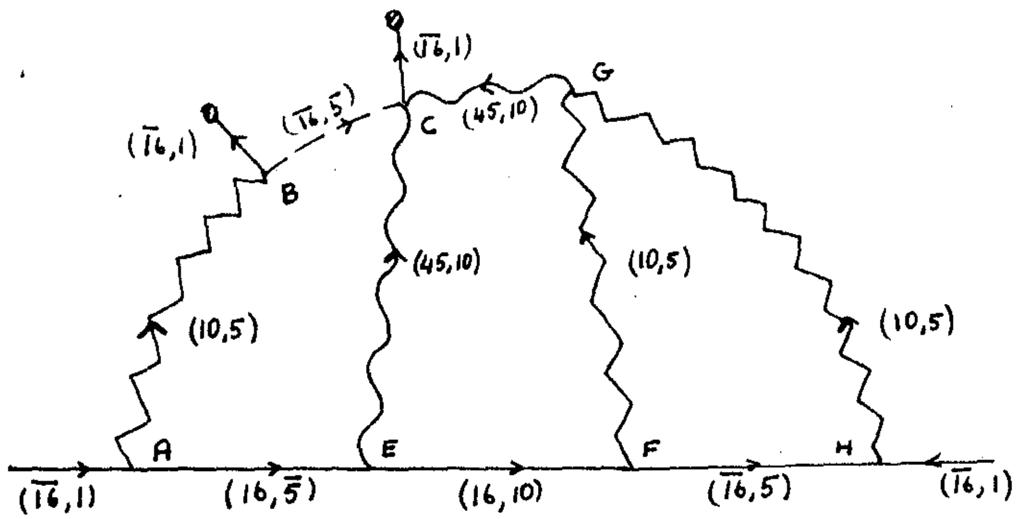


Fig 4. The three loop process that gives mass to the right-handed neutrino

We need to check only three vertices this time since the others are just like the two loop process.

F : $10 \times \overline{16} \supset 16$ (SO(10))

$5 \times 5 \supset 10$ (SU(5))

$\mathcal{L} \supset g_{yukawa}\, \psi \overline{\psi} H_{\overline{10}}$

G : $10 \times 10 \supset 45$

$5 \times 5 \supset 10$

$$\mathcal{L} \supset g\, [\partial_\mu \phi^* (A_\mu \times \phi)] \qquad (2.7\text{-}12)$$

H: $\overline{16} \times \overline{16} \supset 10$

$5 \times 1 \supset 5$

$\mathcal{L} \supset g_{yukawa}\, \psi \overline{\psi} H_{\overline{10}}$

We can estimate the right-handed neutrino mass

$$m_{\nu_R} \sim g^3_{yukawa}\; g^4 \mu \;\langle \phi_{16} \rangle \langle \phi_{16} \rangle [K] \qquad (2.7\text{-}13)$$

Using (2.7-6) and (2.7-7) we get

$$\begin{aligned} m_{\nu_R} &\sim \left(\frac{m_q}{m_W} \right)^3 g^3 g^4 \mu \frac{M^2}{M} \\ &\sim \left(\frac{m_q}{m_W} \right)^3 \left(\frac{\alpha}{\pi} \right)^3 g \mu M \end{aligned} \qquad (2.7\text{-}14)$$

Using our previous numbers and letting m_q be around 1 Gev.
(2.7-14) gives

$$\begin{aligned} m_{\nu_R} &\sim (.05)^3 \times (8 \times 10^{-9}) \times .05 \times 10^{15} \quad \text{Gev} \\ &\sim 50 \text{ Gev} \\ &\sim 50\, m_q \end{aligned} \qquad (2.7\text{-}15)$$

Comparing $50\, m_q$ with $10^7\, m_q$ we conclude that the higher order corrections do not contribute to any significant degree.

3. NEUTRINO OSCILLATIONS

So far we have seen how some theories predict nonzero neutrino masses. Now let us direct our attention to one possible effect of such a prediction. If neutrinos have mass they may display the phenomenon of neutrino oscillations.[47] This phenomenon occurs because mass eigenstates and weak interaction eigenstates do not coincide.

Let $|\nu_\sigma\rangle$ be a mass eigenstate of the Hamiltonian, i.e., $H|\nu_\sigma\rangle = E_\sigma|\nu_\sigma\rangle$. Then we can express $|\nu_\ell\rangle$, the weak interaction eigenstate as a superposition of these:

$$|\nu_\ell\rangle = \sum_\sigma U_{\ell\sigma}|\nu_\sigma\rangle \qquad \ell = e, \mu .. \quad \sigma = 1,2 .. \tag{3-1}$$

and,

$$|\nu_\sigma\rangle = \sum_\ell U_{\ell\sigma}|\nu_\ell\rangle \tag{3-2}$$

where U is an orthogonal n x n matrix for a theory in which the Lagrangian is C-P invariant and n is the number of families existing in nature.

Let a neutrino produced in a weak interaction and thereby in a pure state $|\nu_\ell\rangle$ at time 0 be given by $|\nu_\ell(t)\rangle$ at a later time t. Then

$$|\nu_\ell(t)\rangle = e^{-iHt}|\nu_\ell\rangle = \sum_\sigma U_{\ell\sigma} e^{-iE_\sigma t}|\nu_\sigma\rangle \tag{3-3}$$

After we substitute for $|\nu_\ell\rangle$ in terms of $|\nu_\sigma\rangle$. Now let us go back and substitute for $|\nu_\sigma\rangle$ in terms of a primed set $|\nu_{\ell'}\rangle$

$$|\nu_\ell(t)\rangle = \sum_\sigma U_{\ell\sigma} e^{-iE_\sigma t}\left(\sum_{\ell'} U_{\ell'\sigma}|\nu_{\ell'}\rangle\right) . \tag{3-4}$$

The probability amplitude of finding ν_ℓ changed into $\nu_{\ell'}$ after a time t is given by : $\sum_\sigma U_{\ell\sigma}U_{\ell'\sigma}e^{-iE_\sigma t}$.

And the transition amplitude is given by the real part of the square of this term :

$$P_{e\leftrightarrow e'} = \sum_{\sigma\sigma'} U_{e\sigma} U_{e'\sigma} U_{e\sigma'} U_{e'\sigma'} \cos(E_\sigma - E_{\sigma'})t \qquad (3\text{-}5)$$

Now to see what these formulas mean. let us look at a simple theory with two families only. A general form for a 2 x 2 orthogonal matrix is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \begin{aligned} U_{e1} &= \cos\theta \\ U_{e2} &= \sin\theta \\ U_{\mu 1} &= -\sin\theta \\ U_{\mu 2} &= \cos\theta \end{aligned} \qquad (3\text{-}6)$$

The transition probability for an electron neutrino to change into a muon is given by :

$$\sum_{\substack{\sigma,\sigma' \\ \sigma,\sigma' = 1,2}} U_{e\sigma} U_{\mu\sigma'} U_{\mu\sigma} U_{e\sigma'} \cos(E_\sigma - E_{\sigma'})t \qquad (3\text{-}7)$$

Writing this out we get :

$$\begin{aligned} & U_{e1} U_{\mu 1} U_{e1} U_{\mu 1} \cos 0 + U_{e1} U_{\mu 1} U_{e2} U_{\mu 2} \cos(E_1 - E_2)t + \\ & U_{e2} U_{\mu 2} U_{e1} U_{\mu 1} \cos(E_2 - E_1)t + U_{e2} U_{\mu 2} U_{e2} U_{\mu 2} \cos 0 \\ = \; & 2\cos^2\theta \sin^2\theta - 2\cos^2\theta \sin^2\theta \; (E_1 - E_2)t \\ = \; & 1/2 \sin^2 2\theta \left(1 - \cos(E_1 - E_2)t\right) \end{aligned} \qquad (3\text{-}8)$$

Similarly. the transition probability of finding an electron neutrino at a time t after "creation" in a weak process is $1 - 1/2 \sin^2 2\theta \left(1 - \cos(E_1 - E_2)t\right)$. This is equal to 1 only if the mixing angle θ is 0 or $E_1 = E_2$, i.e.. mass eigenstates are degenerate.

For this simple case we can show that:

$$\tan 2\theta = 2m_{\bar{\mu}e} / m_{\bar{\mu}\mu} - m_{\bar{e}e}$$

$$m_{1,2} = 1/2 \left(m_{\bar{e}e} + m_{\bar{\mu}\mu} \pm \sqrt{(m_{\bar{e}e} - m_{\bar{\mu}\mu})^2 + 4(m_{\bar{\mu}e})^2}\right)^{*} \qquad (3\text{-}9)$$

* See Appendix D

For the oscillations to take place we need $\theta \neq 0$ and $m_1 \neq m_2$. This will happen if $m_{\bar{\mu}e}$ and at least one of the parameters $m_{\bar{e}e}$ and $m_{\bar{\mu}\mu}$ is nonzero.

4. CONCLUSION

We can now briefly mention some phenomenological manifestations of nonzero neutrino masses. The first is the famous "solar neutrino puzzle". Davis and collaborators have been conducting an experiment in which solar neutrinos, hitting a detector target of chlorine at a depth of 4400 metres convert some chlorine to argon.

$$\nu_\ell + {}^{37}Cl \longrightarrow e^- + {}^{37}Ar \tag{4-1}$$

Extracting argon and counting the number of "events" the experimentors found that the solar neutrino flux is much smaller than it should be. One possible solution would be that solar neutrinos are changing into other neutrinos on their way from the sun to the earth. Therefore, we on earth cannot detect them all in an experiment designed to trace only electron neutrinos.

Of course, the greatest loophole in this experiment is that it is very hard to predict the intensity expected in the absence of oscillations. It turns out that in this particular Chlorine-Argon experiment, the uncertainty affecting the flux is quite large. To remedy this, a Gallium experiment is designed. The expected flux can be calculated in a reliable way if low energy ($E < .4$ Mev) neutrinos emitted in the $p+p \rightarrow d+e^+ + \nu_e$ reaction can be detected. The Gallium experiment is sensitive to low energy neutrinos. The reaction goes like this :

$$\nu_e + {}^{71}Ga \longrightarrow {}^{71}Ge + e^- \tag{4-2}$$

Gallium replaces chlorine and germanium replaces argon. One big obstacle against the realization of this experiment is financial; the amount of gallium required is more than the annual global production.

In addition to the apparent deficit of solar neutrinos, the existence of substantial amounts of non-luminous mass in the universe may be an indication for a nonzero neutrino mass. Of course, this mass does not have to reside in the form of massive neutrinos. However, if the neutrino masses explain the "missing mass" in the universe, this would imply that the neutrino mass is significant enough to provide sufficient mass to stop the explosion of the universe and make it collapse back onto itself.

All this is rather far-fetched. However, it is obvious that the neutrino mass problem is closely related to some cosmological problems. The most stringent bound on neutrino mass is of cosmological origin. If the total neutrino mass remaining from the big bang is not to exceed the total cosmic mass density, the sum of all neutrino masses must be less than 40 eV. This close connection to cosmological mysteries makes the neutrino mass question and all possible manifestations of it, like neutrino oscillations, very intriguing.

I would like to thank my advisor Metin Arık
for his guidance and patience.

APPENDIX A : THE TWO COMPONENT THEORY OF THE NEUTRINO

A massive spin 1/2 particle obeys the Dirac equation

$$\left(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m\right)\psi = 0 \qquad \text{(A-1)}$$

where ψ is a four component spinor, describing the two spin states of the particle wave function and the two spin states of the anti-particle wave function.

Using the following Weyl representation for the gamma matrices :

$$\gamma^{0} = \begin{pmatrix} 0 & -\underset{\sim}{I} \\ -\underset{\sim}{I} & 0 \end{pmatrix} \qquad \gamma^{k} = \begin{pmatrix} 0 & \underset{\sim}{\sigma}^{k} \\ -\underset{\sim}{\sigma}^{k} & 0 \end{pmatrix} \qquad k = 1,2,3 \qquad \text{(A-2)}$$

with the wave function ψ written as :

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{(A-3)}$$

where u and v are two component spinors, we can express the Dirac equation as two coupled equations

$$\begin{aligned} i\,\partial u/\partial t + i\sigma\cdot\nabla u &= -mv \\ i\,\partial v/\partial t - i\sigma\cdot\nabla v &= -mu \end{aligned} \qquad \text{(A-4)}$$

If the fermion is massless, these two equations are coupled. Using $\hbar = c = 1$ notation :

$$\langle i\,\partial/\partial t\rangle = E \qquad \langle i\,.\,\nabla\rangle = \langle \vec{p}\rangle$$

For a massless particle $\langle E\rangle = \langle\vec{p}\rangle$ (A-5)

The equations (A-4) mean that

$$\langle\vec{\sigma}\cdot\vec{p}\rangle = +\langle\vec{p}\rangle \qquad \text{(A-6)}$$

for the u spinor and

$$\langle \vec{\sigma} \cdot \vec{p} \rangle = -\langle \vec{p} \rangle \qquad \text{(A-7)}$$

for the v spinor. That is, v represents a left-handed, positive helicity neutrino with anti-parallel spin and momentum. On the other hand, u is a right-handed neutrino with parallel spin and momentum.

In the Weyl representation,

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \underset{\sim}{I} & 0 \\ 0 & -\underset{\sim}{I} \end{pmatrix} \qquad \text{(A-8)}$$

So that we can write :

$$\tfrac{1}{2}(1+\gamma_5)\,\psi = \binom{u}{0} \equiv \psi^R$$

$$\tfrac{1}{2}(1-\gamma_5)\,\psi = \binom{0}{v} \equiv \psi^L \qquad \text{(A-9)}$$

Experimental evidence suggests that the neutrino appears only as ψ^L in weak interactions. In order to find the anti-neutrino wave function, we note that a fermion in an electromagnetic field obeys the following equation :

$$\left[\left(i\frac{\partial}{\partial x_\mu} - eA_\mu \right) \gamma^\mu - m \right] \psi = 0 \qquad \text{(A-10)}$$

Whereas an anti-fermion obeys the equation

$$\left[\left(i\frac{\partial}{\partial x_\mu} + eA_\mu \right) \gamma^\mu - m \right] \psi_c = 0 \qquad \text{(A-11)}$$

Taking the complex conjugate of (A-10) we get

$$\left[-\left(i\frac{\partial}{\partial x_\mu} + eA_\mu \right) \gamma^{*\mu} - m \right] \psi^* = 0 \qquad \text{(A-12)}$$

Operating with G, we get

$$\left[-\left(i\frac{\partial}{\partial x_\mu} + eA_\mu\right) G\gamma^{*\mu} G^{-1} - m\right] G\psi^* = 0$$

(A-13)

This equation will be similar to (A-11) if we let

$$-G\gamma^{*\mu} G^{-1} = \gamma^\mu$$
$$G\psi^* = \psi_c$$

(A-14)

Solving for G using the Weyl representation for the gamma matrices, we get

$$G = i\gamma^2 = \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{bmatrix}$$

$$\psi^L = G\psi^{L*} = i\gamma^2 \begin{bmatrix} 0 \\ 0 \\ V_1^* \\ V_2^* \end{bmatrix} = \begin{bmatrix} V_2^* \\ -V_1^* \\ 0 \\ 0 \end{bmatrix}$$

(A-15)

which represents a right-handed particle.

APPENDIX B : SPONTANEOUS SYMMETRY BREAKDOWN

The symmetry of the hamiltonian of a quantum mechanical system is not necessarily obvious from the ground state of the system. For example, nuclear forces are rotationally invariant; however, the ground state of a nucleus is not necessarily so, i.e., it is not of spin zero. An example of a system which, unlike nuclei, is of infinite spatial extent is the Heisenberg ferromagnet, an infinite array of spin two magnetic dipoles with spin-spin interactions between neighbouring dipoles. The total hamiltonian is rotationally invariant, but the ground state is a state in which all dipoles are lined up in one arbitrary direction. Someone living inside such a magnet would never discover that the hamiltonian was rotationally invariant.

Generalizing to relativistic quantum mechanics, we substitute "vacuum" for the ground state and some "internal" symmetry for the rotational invariance. So, if we conclude that the laws of nature may possess symmetries hidden from us because the vacuum is not invariant under them. This is called "spontaneous symmetry breakdown".

Let us investigate spontaneous spymmetry breakdown in some systems. For a single real scalar field ϕ , the Lagrangian density is

$$\mathcal{L} = 1/2\,(\partial_\mu \phi\, \partial^\mu \phi) - \mu^2/2\, \phi^2 - \lambda/4\, \phi^4 \qquad \text{(B-1)}$$

This Lagrangian has a discrete symmetry $\phi \rightarrow - \phi$

Let us consider the potential $V(\phi)$ and find its minimum :

$$V(\phi) = \mu^2 \phi^2/2 + \lambda/4\ \phi^4$$

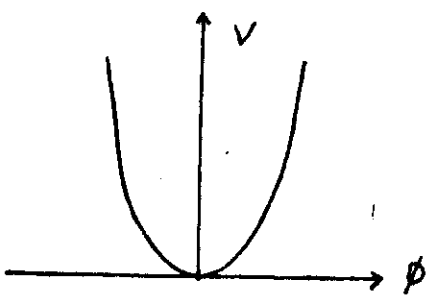
$$\partial V/\partial \phi = 0 = [\mu^2 + \lambda \phi^2]\,\phi \qquad \text{(B-2)}$$

$$\partial^2 V/\partial \phi^2 = \mu^2 + 3\lambda\phi^2 > 0 \quad \text{to be a minimum}$$

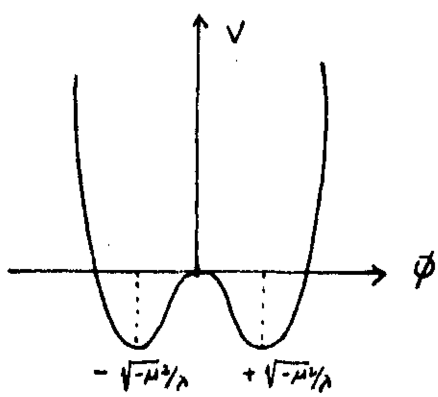
There are two cases :

$$\mu^2 > 0 \qquad \langle \phi \rangle_{min} = 0 \qquad \text{(B-3)}$$

$$\mu^2 < 0 \qquad \langle \phi \rangle_{min} = \pm \sqrt{-\mu^2/\lambda} \qquad \text{(B-4)}$$



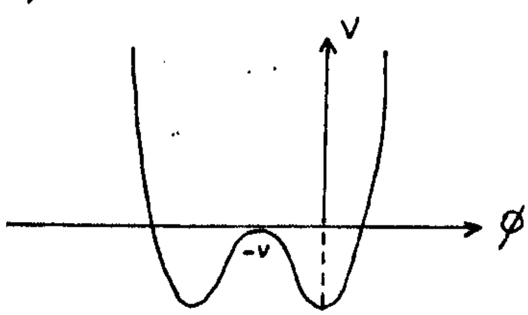
$\mu^2 > 0$



$\mu^2 < 0$

We let μ^2 be less than zero. Choosing the positive root for conventional reasons, $\langle \phi \rangle_{min} = \sqrt{-\mu^2/\lambda}$

Now the vacuum is not at zero anymore. Therefore, we define a new field ϕ' for which the vacuum is at zero.



$$\phi' = \phi - \sqrt{-\mu^2/\lambda} \equiv \phi - \nu \tag{B-5}$$

Substituting for ϕ in terms of ϕ' and ν

$$\mathcal{L} = 1/2\ (\partial^\mu \phi' \partial_\mu \phi') - 1/2\, \mu^2 (\phi' + \nu)^2 - \lambda/4\ (\phi' + \nu)^4 \tag{B-6}$$

Using $\nu^2 = -\dfrac{\mu^2}{\lambda}$ we get

$$\mathcal{L} = 1/2\ (\partial^\mu \phi' \partial_\mu \phi') + \mu^2 \phi'^2 - \lambda \nu \phi'^3 - 1/4\, \lambda \phi'^4 + \text{constants} \tag{B-7}$$

Now ϕ' has a positive mass $-2\mu^2$ and because we have a ϕ'^3 term the Lagrangian does not exhibit the reflection symmetry of the original Lagrangian.

Let us next consider a complex scalar field ϕ

$$\mathcal{L} = 1/2\, (\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2/2\ \phi^* \phi - \lambda/4\ (\phi^* \phi)^2$$

$$\mu^2 < 0 \tag{B-8}$$

This Lagrangian is invariant under the transformation $\phi \rightarrow e^{i\theta} \phi$, where θ is not a function of the space-time.

$$V(\phi, \phi^*) = \mu^2/2\ \phi^* \phi + \lambda/4\ (\phi^* \phi)^2$$

$$\partial V / \partial \phi^* = \left(\mu^2/2 + \lambda/2\ \phi^* \phi \right) \phi$$

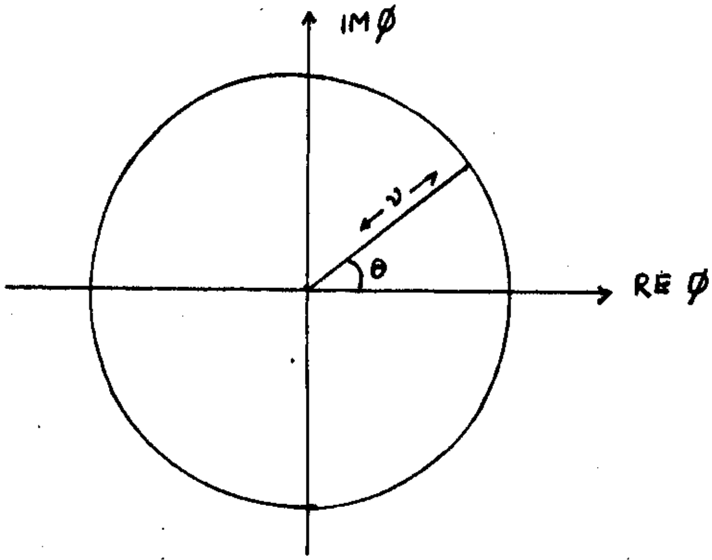
$$\partial V / \partial \phi = \left(\mu^2/2 + \lambda/2\ \phi^* \phi \right) \phi^* \tag{B-9}$$

Since μ^2 is less than zero,

$$\mu^2/2 + \lambda/2\, \phi^* \phi = 0$$

$$|\phi|^2 = -\mu^2/\lambda \equiv \nu^2 \qquad \text{(B-10)}$$

So the minimum is a circle with radius ν in the ϕ plane.



Since the phase angle θ is arbitrary we can take it to be zero. We now make the substitution

$$\phi = e^{i\xi/\nu}\,(\nu + \eta) \qquad \text{(B-11)}$$

where ξ represents perturbations in the angular direction and η represents perturbations in the radial direction. Doing perturbation about $\mathrm{Re}\,\phi = \nu$ and $\mathrm{Im}\,\phi = 0$ for small values of ξ and η is equivalent to the equation

$$\phi = \nu + \eta + i\xi \qquad \text{(B-12)}$$

Substituting this value for in the original Lagrangian we get

$$\mathcal{L} = 1/2 \left[(\partial_\mu \xi \partial^\mu \xi) + (\partial_\mu \eta \partial^\mu \eta) \right] + \mu^2 \eta^2 + \cdots \qquad (B-13)$$

So we get a mass term for η but not for ξ . The mass η gains is due to trying to make displacements in the radial directions against restoring forces. ξ has no mass; it corresponds to displacements around the circle, the minimum surface where there are no restoring forces.

We now let ϕ be an n-component real field whose equation of motion can be derived from the Lagrangian

$$\mathcal{L} = 1/2 \, (\partial_\mu \phi^i \partial^\mu \phi^i) - 1/2 \, \mu^2 (\phi^i \phi^i) - \lambda/4 \, (\phi^i \phi^i)^2 \qquad (B-14)$$

This Lagrangian is invariant under the group $O(n)$

$$V(\phi^i) = 1/2 \, \mu^2 (\phi^i \phi^i) + \lambda/4 \, (\phi^i \phi^i)^2$$

$$\partial V / \partial \phi^i = \left[\mu^2 + \lambda |\phi|^2 \right] \phi^i = 0 \qquad (B-15)$$

$$|\phi|^2 = -\mu^2/\lambda \equiv \nu^2$$

We can choose to satisfy (B-15) by letting ϕ_i for $i = 1, 2 \ldots n-1$ be zero and ϕ_n be ν . Then the vacuum has a lower symmetry than the Lagrangian; the vacuum is invariant under the group $O(n-1)$

$O(n)$ group has $1/2\, n(n-1)$, $O(n-1)$ group has $1/2 (n-1)(n-2)$ generators. The difference in the number of generators, therefore, is $n-1$. These $n-1$ generators correspond to the broken symmetry. We let L_{ij} $i,j \neq n$ be the $O(n-1)$ generators and $k_i \equiv L_{in}$ be the $n-1$ unbroken generators, where

$$[L_{ij}]_{kl} = -i[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}] \tag{B-16}$$

We define the fields η and ξ_i ; $i = 1,2 \ldots n-1$

$$\phi = e^{i\xi_i k_i/v} \begin{bmatrix} 0 \\ \vdots \\ v+\eta \end{bmatrix} \tag{B-17}$$

Up to terms quadratic in the fields we have

$$\phi \approx \left[I + i\frac{\xi_1 k_1}{v} + i\frac{\xi_2 k_2}{v} \ldots \right] \begin{bmatrix} 0 \\ \vdots \\ v+\eta \end{bmatrix} \tag{B-18}$$

Since k_j has a minus i in the j th row, n th column and a plus i in the n th row, j th column, k_j operating on $\begin{bmatrix} 0 \\ \vdots \\ v+\eta \end{bmatrix}$ gives a vector with the only nonzero component in its j th row. So

$$\phi \approx \begin{bmatrix} i\frac{\xi_1}{v}(-i(v+\eta)) \\ \vdots \\ v+\eta \end{bmatrix} \sim \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-1} \\ v+\eta \end{bmatrix} \tag{B-19}$$

Substituting ξ_i for ϕ_i for $i \neq n$ and $v+\eta$ for ϕ_n, in (B-14), we get a Lagrangian with a mass term only for η. Since $n-1$ ξ_i have no mass we get $n-1$ massless "Goldstone bosons" corresponding to $n-1$ broken generators.

APPENDIX C : THE HIGGS MECHANISM

If we have a Lagrangian which possesses global symmetry, and if the minimum of our potential is not at zero, then we get massless Goldstone bosons. However, if the symmetry is local, then we need to introduce a vector field into our Lagrangian to make it gauge invariant. Then, spontaneously broken symmetry plus local gauge invariance will lead to an exception to the Goldstone theorem. Previously we had as many massless bosons as the broken generators. Now, these massless bosons will be eaten up by the vector fields to give us as many massive vector mesons as the broken generators. The vector bosons which remain massless will correspond to the unbroken symmetry of the Lagrangian.

Let us briefly summarize some requirements for a local gauge invariant theory.

$$\phi(x) \rightarrow U(\theta(x))\,\phi(x) \rightarrow \exp\{-iL\cdot\theta(x)\}\,\phi \qquad \text{(C-1)}$$

L is the appropiate matrix representation of the symmetry group. We see that the number of components of the column vector ϕ should equal the dimension of the matrix L.

$$\begin{aligned} L^{i}{}_{jk} &= -ic^{ijk} \\ D_\mu &= \partial_\mu - igL\cdot A_\mu(x) \end{aligned} \qquad \text{(C-2)}$$

D_μ is the covariant derivative and A_μ is the vector field. We see that the number of generators must equal the number of components of the vector field.

$$L.A_\mu \rightarrow U(\theta)\, L.A_\mu\, U^{-1}(\theta) - i/g\, (\partial_\mu U(\theta))\, U^{-1}(\theta)$$
$$F_{\mu\nu}{}^{i} \rightarrow \partial_\mu A_\nu{}^{i} - \partial_\nu A_\mu{}^{i} + g c_{ijk} A_{\mu j} A_{\nu k} \qquad \text{(C-3)}$$

Let us consider a Lagrangian invariant under a U(1) transformation of the form $\phi \rightarrow e^{-i\theta}\phi$

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \qquad \text{(C-4)}$$

Following the above prescriptions for a local gauge invariant theory and letting $L = 1,\ g = e$ we construct

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\,\phi^*(\partial_\mu - ieA_\mu)\,\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - 1/4\, F_{\mu\nu} F^{\mu\nu}$$
$$\mu^2 < 0 \qquad \text{(C-5)}$$

Under the local transformations of the form

$$\phi \rightarrow e^{-i\theta(x)}\phi$$
$$A_\mu \rightarrow A_\mu - 1/e\, \partial_\mu \theta(x) \qquad \text{(C-6)}$$

$\mathcal{L}$ is invariant.

We give a nonzero vacuum expectation value to ϕ not ϕ^*

$$\langle \phi \rangle_0 = \nu/\sqrt{2} \qquad \nu^2 = -\mu^2/\lambda$$
$$\phi = \exp(i\xi/\nu)(\nu+\eta)/\sqrt{2}$$
$$\approx 1/\sqrt{2}\,(\nu+\eta+i\xi) \qquad \text{(C-7)}$$

Substituting into (C-5) we get

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\, \partial^{\mu}\eta\, \partial_{\mu}\eta + \tfrac{1}{2}\, \partial_{\mu}\xi\, \partial^{\mu}\xi + \tfrac{1}{2}\, e^{2}\nu^{2} A_{\mu} A_{\mu}$$
$$- e\nu A_{\mu}\partial^{\mu}\xi + \mu^{2}\eta^{2} + \ldots \qquad \text{(C-8)}$$

Because of the $e\nu A_{\mu}\partial^{\mu}\xi$ term, this result is hard to interpret. So we let our gauge function $\Theta(x)$ be $\xi(x)/\nu$

$$\phi \rightarrow e^{-i\xi/\nu}\phi \rightarrow e^{-i\xi/\nu}\left[e^{i\xi/\nu}\,(\nu+\eta)/\sqrt{2}\right]$$

$$\phi \rightarrow (\nu+\eta)/\sqrt{2}$$

$$A_{\mu} \rightarrow A_{\mu} - 1/e\nu\ \partial_{\mu}\xi \qquad \text{(C-9)}$$

Substituting into (C-8) we get a mass term $\tfrac{1}{2} e^{2}\nu^{2} A_{\mu}' A_{\mu}'$ for the redefined vector field. There is not a mass term for ξ . ξ , corresponding to the broken symmetry, has disappeared and A_{μ} has grown massive. Originally we had ϕ, ϕ^{*} and two polarizations for A_{μ}, adding up to four degrees of freedom. Now, we have a massive vector field with three degrees of freedom plus η leaving us with four degrees of freedom again.

Now let us consider a non-abelian example. We have a Lagrangian invariant under the SU(2) group. Following our gauge prescriptions and remembering that for SU(2) $c^{ijk} = e^{ijk}$ we have

$$D_{\mu}\vec{\phi} = (\partial_{\mu} - ig\vec{\underset{\sim}{L}}\cdot\vec{A}_{\mu})\vec{\phi}$$
$$= (\partial_{\mu} - ig(\underset{\sim}{L}_{1} A_{\mu 1} + \underset{\sim}{L}_{2} A_{\mu 2} + \underset{\sim}{L}_{3} A_{\mu 3})\vec{\phi} \qquad \text{(C-10)}$$

Here each $\underset{\sim}{L}$ is a matrix acting on a vector $\vec{\phi}$

$$(\underset{\sim}{L}\,\vec{\phi})_i = L_{ik}\,\phi_k \tag{C-11}$$

So using $L^m_{ik} = -i\,\epsilon_{mik}$ we get

$$\begin{aligned} D_\mu \phi_i &= \partial_\mu \phi_i - ig\, L^m_{ik} A^m_\mu \phi_k \\ &= \partial_\mu \phi_i - g\, \epsilon_{mik} A^m_\mu \phi_k \\ &= \partial_\mu \phi_i + g\, \epsilon_{imk} A^m_\mu \phi_k \end{aligned} \tag{C-12}$$

$$\mathcal{L} = 1/2\,(D_\mu \phi_i)(D^\mu \phi_i) - V(\phi^2) \tag{C-13}$$

The potential V has a minimum at $\phi_3 = v$

$$\langle \phi \rangle_{min} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \equiv \vec{v} \tag{C-14}$$

We also write our three generators explicitly

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{C-15}$$

We see that $\underset{\sim}{L}_1\vec{v}$ and $\underset{\sim}{L}_2\vec{v}$ are nonzero and $\underset{\sim}{L}_3\vec{v}$ is zero; so only $\underset{\sim}{L}_3$ remains as an unbroken symmetry generator. $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ are our broken symmetry generators and we define ξ_1 and ξ_2, associated with the unbroken symmetry. We parameterize ϕ as

$$\phi = \exp\left\{ i/\nu\, (\xi_1 L^1 + \xi_2 L^2)\right\} \begin{pmatrix} 0 \\ 0 \\ \nu+\eta \end{pmatrix} \approx \begin{pmatrix} -\xi_2 \\ \xi_1 \\ \nu+\eta \end{pmatrix} \qquad \text{(C-16)}$$

We make our gauge transformation

$$U = \exp\left\{ -i/\nu\, (\xi_1 L_1 + \xi_2 L_2)\right\}$$

$$\phi' = U\phi = \begin{pmatrix} 0 \\ 0 \\ \nu+\eta \end{pmatrix}$$

$$[\, L.A_\mu'\,]\,\phi' = [\, UL.A_\mu U^{-1} - i/g\,(\partial_\mu U)U^{-1}]\, U\phi \qquad \text{(C-17)}$$

Ignoring terms higher than quadratic in the fields, we first calculate $(UL.A_\mu U^{-1})U\phi$ term

$$UL.A_\mu \begin{pmatrix} -\xi_2 \\ \xi_1 \\ \nu+\eta \end{pmatrix} = U[L_1 A_{\mu 1} + L_2 A_{\mu 2} + L_3 A_{\mu 3}] \begin{pmatrix} -\xi_2 \\ \xi_1 \\ \nu+\eta \end{pmatrix} \qquad \text{(C-18)}$$

$$= U\left[A_{\mu 1}\begin{pmatrix} 0 \\ -i(\nu+\eta) \\ i\xi_1 \end{pmatrix} + A_{\mu 2}\begin{pmatrix} i(\nu+\eta) \\ 0 \\ i\xi_2 \end{pmatrix} + A_{\mu 3}\begin{pmatrix} -i\xi_1 \\ -i\xi_2 \\ 0 \end{pmatrix} \right] \qquad \text{(C-19)}$$

$$= \left(I - \frac{i\xi_1 L_1}{\nu} - \frac{i\xi_2 L_2}{\nu} \right) \begin{bmatrix} iA_{\mu 2}(\nu+\eta) - iA_{\mu 3}\xi_1 \\ -i\, A_{\mu 1}(\nu+\eta) - i\, A_{\mu 3}\xi_2 \\ i\, A_{\mu 1}\xi_1 + iA_{\mu 2}\xi_2 \end{bmatrix} \qquad \text{(C-20)}$$

$$= \begin{bmatrix} iA_{\mu 2}(\nu+\eta) - iA_{\mu 3}\xi_1 \\ -iA_{\mu 1}(\nu+\eta) - iA_{\mu 3}\xi_2 \\ iA_{\mu 1}\xi_1 + iA_{\mu 2}\xi_2 \end{bmatrix} - i\frac{\xi_1}{\nu}\begin{bmatrix} 0 \\ A_{\mu 1}\xi_1 + A_{\mu 2}\xi_2 \\ A_{\mu 1}(\nu+\eta) + A_{\mu 3}\xi_2 \end{bmatrix} - i\frac{\xi_2}{\nu}\begin{bmatrix} -A_{\mu 1}\xi_1 - A_{\mu 2}\xi_2 \\ 0 \\ A_{\mu 2}(\nu+\eta) - A_{\mu 3}\xi_1 \end{bmatrix} \quad \text{(C-21)}$$

$$= ig\,(UL.A_\mu U^{-1})U\phi = g\begin{bmatrix} A_{\mu 2}(\nu+\eta) - A_{\mu 3}\xi_1 \\ -A_{\mu 1}(\nu+\eta) - A_{\mu 3}\xi_2 \\ 0 \end{bmatrix} \quad \text{(C-22)}$$

Now we calculate $i/g\,(\partial_\mu U)(U^{-1})U\phi$.

$$i/g\,(\partial_\mu U)(U^{-1})U\phi = +i/g\,(\partial_\mu U)\begin{pmatrix} -\xi_2 \\ \xi_1 \\ \nu+\eta \end{pmatrix} \quad \text{(C-23)}$$

$$= i/g\left[-i/\nu\,(\partial_\mu \xi_1 L_1 + \partial_\mu \xi_2 L_2)\right]\left[I - i/\nu\,\xi_1 L_1 - i\xi_2 L_2/\nu\right]\begin{bmatrix} -\xi_2 \\ \xi_1 \\ \nu+\eta \end{bmatrix} \quad \text{(C-24)}$$

$$= 1/g\nu\,(\partial_\mu \xi_1 L_1 + \partial_\mu \xi_2 L_2)\left[\begin{pmatrix} -\xi_2 \\ \xi_1 \\ \nu+\eta \end{pmatrix} - \frac{i}{\nu}\xi_1\begin{pmatrix} 0 \\ -i(\nu+\eta) \\ i\xi_1 \end{pmatrix} - \frac{i\xi_2}{\nu}\begin{pmatrix} i(\nu+\eta) \\ 0 \\ i\xi_2 \end{pmatrix}\right] \quad \text{(C-25)}$$

$$= 1/g\nu\,(\partial_\mu \xi_1 L_1 + \partial_\mu \xi_2 L_2)\begin{bmatrix} 0 \\ 0 \\ \nu+\eta \end{bmatrix} \quad \text{(C-26)}$$

$$= 1/g\nu \left[\partial_\mu \xi_1 \begin{pmatrix} 0 \\ -i(\nu+\eta) \\ 0 \end{pmatrix} + \partial_\mu \xi_2 \begin{pmatrix} i(\nu+\eta) \\ 0 \\ 0 \end{pmatrix} \right] \tag{C-27}$$

$$= i/g\nu \begin{pmatrix} \partial_\mu \xi_2 (\nu+\eta) \\ -\partial_\mu \xi_1 (\nu+\eta) \\ 0 \end{pmatrix} \tag{C-28}$$

$$ig\left(i/g\,(\partial_\mu U)(U^{-1})\,\phi' \right) = -1/\nu \begin{pmatrix} \partial_\mu \xi_2 (\nu+\eta) \\ -\partial_\mu \xi_1 (\nu+\eta) \\ 0 \end{pmatrix} \tag{C-29}$$

Putting (C-29) and (C-22) together, we find

$$D_\mu \vec{\phi}' = \left[\begin{pmatrix} 0 \\ 0 \\ \partial_\mu \eta \end{pmatrix} + g \begin{pmatrix} A_{\mu 2}(\nu+\eta) - A_{\mu 3}\xi_1 \\ -A_{\mu 1}(\nu+\eta) - \xi_2 A_{\mu 3} \\ 0 \end{pmatrix} - \frac{1}{\nu} \begin{pmatrix} \partial_\mu \xi_2 (\nu+\eta) \\ -\partial_\mu \xi_1 (\nu+\eta) \\ 0 \end{pmatrix} \right] \tag{C-30}$$

Using (C-30) we can now calculate the Lagrangian in terms of the transformed variables

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \eta\, \partial^\mu \eta + \tfrac{1}{2} \partial_\mu \xi'\, \partial^\mu \xi' + \tfrac{1}{2} \partial_\mu \xi^2\, \partial^\mu \xi^2 + \tfrac{1}{2} g^2 \nu^2 (A_{\mu 1}^2 + A_{\mu 2}^2) - g\nu\, (A_{\mu 1} \partial_\mu \xi_1 + A_{\mu 2} \partial_\mu \xi_2) \tag{C-31}$$

From the $\frac{1}{2} g^2 v^2 (A_{\mu 1}^2 + A_{\mu 2}^2)$ term we see that the vector mesons corresponding to the broken symmetry generators have acquired mass.

We are now ready to discuss the general case. We have a Lagrangian invariant under a group G. There are N generators, therefore there are also N gauge mesons A_μ^α $\alpha = 1, \ldots N$. We choose an n-dimensional representation for these generators. So we have n scalar fields ϕ_i $i = 1 \ldots n$.

Let us suppose that there is a subgroup of G called S with M generators that leave the vacuum invariant. So we have $N-M$ generators for which $\underset{\sim}{L}^\alpha \vec{v}$ is nonzero. $\vec{v}$ is the vacuum. It is an n component vector which makes $V(\phi^2)$ a minimum.

We parameterize ϕ

$$\phi = \exp\left(\sum_\alpha i \xi_\alpha L^\alpha / v\right)(\vec{v} + \vec{\eta}) \qquad \alpha = 1 \ldots N-M \tag{C-32}$$

$\vec{\eta}$ represents the $n-(N-M)$ fields. Next we make the following gauge transformation

$$U = \exp\left(\sum_\beta i\left(-\xi_\beta L^\beta / v\right)\right) \qquad \beta = 1 \ldots N-M \tag{C-33}$$

As a result, we get $N-M$ gauge mesons gaining mass by eating up $N-M$ Goldstone bosons. M remaining vector bosons stay massless.

Let us now check the overall degrees of freedom. We started out with n scalar particles and N massless vector mesons. Therefore we had $n+2N$ degrees of freedom. After the spontaneous symmetry breaking we are left with $N-M$ massive, M massless vector bosons and $n-[N-M]$ η fields.

Writing explicitly

$$2N + n = (N-M) + M + n - [N-M] \qquad (C-54)$$

We see that the overall degrees of freedom remain unchanged, as expected.

APPENDIX D : TWO FAMILY OSCILLATIONS

In order to make allowance for the lepton number violation, the weak interaction Lagrangian must have an additional part of the form

$$\mathcal{L}_1 = m_{\bar{e}e}\,\bar{\nu}^c_{eR}\,\nu_{eL} + m_{\bar{\mu}\mu}\,\bar{\nu}^c_{\mu R}\,\nu_{\mu L} + m_{\bar{\mu}e}\,(\bar{\nu}^c_{\mu R}\,\nu_{eL} + \bar{\nu}^c_{eR}\,\nu_{\mu L}) + \text{H.C.} \quad (D-1)$$

Here we shall consider only Majorana fields. We define

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \end{pmatrix} \quad \nu^c_R = \begin{pmatrix} \nu_{eR}{}^c \\ \nu_{\mu R}{}^c \end{pmatrix} \quad M = \begin{pmatrix} m_{\bar{e}e} & m_{\bar{\mu}e} \\ m_{\bar{\mu}e} & m_{\bar{\mu}\mu} \end{pmatrix} \quad (D-2)$$

We can write the full Lagrangian including the standard interaction terms as follows:

$$\mathcal{L}_T = \bar{\nu}^c_R\, M\,(\nu_L + \nu^c_R) + \bar{\nu}_L\, M\,(\nu^c_R + \nu_L) \equiv \bar{\chi} M \chi$$

$$\chi = \nu_L + \nu^c_R = \begin{pmatrix} \nu_{eL} + \nu_{eR}^c \\ \nu_{\mu L} + \nu_{\mu R}^c \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad (D-3)$$

In order to diagonalize the mass matrix, we let

$$\chi = U\phi$$

$$\mathcal{L}_T = \bar{\chi} M \chi = \bar{\phi}\, U^T M U \phi \quad (D-4)$$
$$= \bar{\phi}\, M_D\, \phi$$

M_D is the diagonal mass matrix :

$$M_D = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = U^T M U \qquad \text{(D-5)}$$

If we assume that the Lagrangian is CP invariant, then U is an orthogonal matrix which has the following general form

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \text{(D-6)}$$

Using $\chi = U\phi$ and (D-3) we get

$$\chi = \begin{pmatrix} \nu_{eL} + \nu_{eR}^c \\ \nu_{\mu L} + \nu_{\mu R}^c \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_{1L} + \phi_{1R} \\ \phi_{2L} + \phi_{2R} \end{pmatrix} \qquad \text{(D-7)}$$

Taking only the left components we get

$$\begin{aligned} \nu_{eL} &= \cos\theta\, \phi_{1L} + \sin\theta\, \phi_{2L} \\ \nu_{\mu L} &= -\sin\theta\, \phi_{1L} + \cos\theta\, \phi_{2L} \end{aligned} \qquad \text{(D-8)}$$

θ denotes the degree of mixing of the Majorana fields and $\phi_{1,2}$'s are the mass eigenstates with masses m_1 and m_2 respectively. Using (D-4) we get

$$M = \begin{pmatrix} m_{\bar{e}e} & m_{\bar{\mu}e} \\ m_{\bar{\mu}e} & m_{\bar{\mu}\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{(D-9)}$$

$$\begin{pmatrix} m_{ee} & m_{\bar{\mu}e} \\ m_{\bar{\mu}e} & m_{\bar{\mu}\mu} \end{pmatrix} = \begin{pmatrix} \cos^2\theta m_1 + \sin^2\theta m_2 & \sin\theta\cos\theta(m_2-m_1) \\ \cos\theta\sin\theta(m_2-m_1) & \sin^2\theta m_1 + \cos^2\theta m_2 \end{pmatrix} \quad \text{(D-9)}$$

We note from (D-9) that

$$\begin{aligned} m_{\bar{\mu}\mu} - m_{ee} &= \cos^2\theta\,(m_2-m_1) - \sin^2\theta(m_2-m_1) \\ &= (m_2-m_1)(\cos^2\theta - \sin^2\theta) \\ &= (m_2-m_1)\cos 2\theta \end{aligned} \quad \text{(D-10)}$$

Using the value we found for in (D-9), we get

$$\frac{2m_{\bar{\mu}e}}{m_{\bar{\mu}\mu}-m_{ee}} = \frac{2(m_2-m_1)\sin\theta\cos\theta}{(m_2-m_1)\cos 2\theta}$$

$$= \tan 2\theta \quad \text{(D-11)}$$

We now try to solve for m_1 and m_2 using (D-9).

$$m_{ee} + m_{\bar{\mu}\mu} = m_1 + m_2 \quad \text{(D-12)}$$

$$\begin{aligned} (m_1-m_2)^2 &= (m_1-m_2)^2\left[(\cos 2\theta)^2 + (\sin 2\theta)^2\right] \\ &= (m_{\bar{\mu}\mu} - m_{ee})^2 + 4(m_{\bar{\mu}e})^2 \\ (m_1-m_2) &= \left[(m_{\bar{\mu}\mu} - m_{ee})^2 + 4(m_{\bar{\mu}e})^2\right]^{1/2} \end{aligned} \quad \text{(D-13)}$$

Using (D-12) and (D-13)

$$m_{\substack{1\\2}} = \tfrac{1}{2}\left(m_{ee} + m_{\bar{\mu}\mu} \pm \sqrt{(m_{ee} - m_{\bar{\mu}\mu})^2 + 4(m_{\bar{\mu}e})^2}\right)$$

REFERENCES

1) Emilio Segre, Nuclei and Particles . Second Edition, 391 (1977).

2) D. Cheng, G.K.O'Neill. Elementary Particle Physics , 7 (1979).

3) E. Fermi "Versuch Einer Theorie der B-Strahlen.I", Z. Phys 88 , 161 (1934).

4) Anderson and Neddermeyer "Cloud Chamber Observation of Cosmic Rays at 4300 Meters Elevation and Near Sea Level". Phys. Rev 50, 263 (1936).

5) C.M.G. Lattes H. Muirhead G.P.S. Occhialini and C.F. Powell. "Processes Involving Charged Mesons" Nature 159 694 (1947).

6) C.L. Cowan and F. Reines "Detection of the Free Neutrino", Phys. Rev 92 830 (1953).

7) M.L. Perl et al "Evidence for Anomalous Lepton Production in Annihilation". Phys. Rev. Lett. 35, 1489 (1975).

8) M. Gell-Mann, "A Schematic Model of Baryons and Mesons", Phys. Lett 8, 214 (1964).

9) B.J. Bjorken and S.L. Glashow, "Elementary Particles and SU(4)", Phys. Let 11, 255 (1964).

10) H. Georgi, H.R. Quinn and S. Weinberg, Physics Rev. Lett. 33, 451 (1974).

11) S. Weinberg. Phys. Rev. Lett. 19, 1264 (1967).

12) See references in D.V.Nanopoulos CERN preprint TH.2896 (1980).

13) G.B. Gelmini and M. Roncadelli Phys. Lett. [illegible] 411 (1981).

14) H.M.Georgi.S.L.Glashow and S.Nussinov Harvard University preprint HUTP-81/A026.

15) H. Georgi and S.L.Glashow Phys. Rev. Lett. 32, 438 (1974).

16) A.J.Buras, J.Ellis,M.K.Gaillard and D.V.Nanopoulos, Nucl. Phys. B135 66 (1978).

17) D.V. Nanopoulos CERN preprint TH.2896 (1980).

18) J. Ellis CERN preprint TH.2942 (1980).

19) J. Ellis and M.K.Gaillard, Phys. Lett. 88B 315 (1979).
R.Barbieri,J.Ellis and M.K.Gaillard, Phys. Lett. 90B, 249 (1980).

20) M.S. Chanowitz, J. Ellis and M.K.Gaillard, Nucl. Phys. B128 506 (1977).
H. Georgi and D.V. Nanopoulos, Nucl. Phys. B155, 52 (1979) and Phys. Lett. 82B 392 (1979).

21) See references in (17).

22) H. Georgi and D.V. Nanopoulos. Ref 17.

23) See references in (22).

24) E. Witten Phys. Lett. 91B. 81 (1980).

25) S.M. Bilenky and B. Pontecorvo. Phys. Rep. 41 No 4 (1978) .