SIMULTANEOUS AUCTIONS WITH PRIVATE AND COMMON VALUES

DENİZ NEMLİ

BOGAZİÇİ UNIVERSITY

2013

SIMULTANEOUS AUCTIONS WITH PRIVATE AND COMMON VALUES

Thesis submitted to the

Institute for Graduate Studies in the Social Sciences

in partial fulfillment of the requirements for the degree of

Master of Arts

in

Economics

by

Deniz Nemli

Boğaziçi University

2013

Simultaneous Auctions with Private and Common Values

The thesis of Deniz Nemli

has been approved by:

Assist. Prof. Deniz Selman (Thesis Advisor) ______________________

Assoc. Prof. Burak Gürel ______________________

Assist. Prof. Murat Yılmaz ______________________

June 2013

Thesis Abstract

Deniz Nemli, "Simultaneous Auctions with Private and Common Values"

We analyze two simultaneous sealed-bid auctions in which *n* bidders have private values for the good in one auction but a common value for the other, and must choose to participate in at most one auction. The seller in each auction is free to choose whether his good is sold via a first-price or second-price auction. After auction types are announced, bidders simultaneously decide in which auction to participate and their bid in their selected auction. hence bidders bid without knowing the number of other bidders participating in the same auction. Bidders choose auctions according to a cut-off strategy: only those with a sufficiently high private value choose the private auction. For any auction type profile, there is a unique symmetric equilibrium (which is the same for all auction type pro.les) and multiple asymmetric equilibria which vary with auction types. At the unique symmetric equilibrium, revenue equivalence fails to hold in both auctions: the private auction revenue is always strictly higher when it is first-price.

Tez Özeti

Deniz Nemli, "Eş Zamanlı Açık Arttırmalarda Özel ve Genel Değerler"

Değerlendirmesi özel olan bir mal ile genel olan bir diğer malı satmak için yapılan iki eş zamanlı açık arttırmada n sayıda alıcının bir tanesini seçmek zorunda olduğu bir mekanizmayı inceliyoruz. Satıcılar birinci-fiyat ve ikinci-fiyat açık arttırma tiplerinden birini seçtikten sonra, alıcılar hangi açık arttırmaya katılacaklarına ve katıldıkları açık arttırmadaki tekliflerine karar veriyorlar. Açık arttırma seçme dengeleri eşik noktası formunda; özel değerleri belirli bir noktadan yüksek ise özel değerli malı, değilse diğer malı almaya çalışıyorlar. Bütün açık arttırma tip profilleri için aynı olan bir tane simetrik denge var. Bu dengede, gelir denkliği prensibinin tutmadığını ve özel değerli malın birinci-fiyat açık arttırma tipinde olması gerekiyor.

CHAPTER 1

INTRODUCTION

This paper considers a market in which two distinct goods are sold through simultaneous sealed-bid auctions. Bidders have private valuations for one of the goods, for which each bidder's valuation is private information. All bidders have a common (publicly known) valuation for the other good. Bidders are capacity constrained, meaning they can participate in at most one auction. Each seller announces whether his good will be sold through a first-price auction or a second-price auction. Bidders then simultaneously decide in which auction to participate and their bid in their selected auction, meaning that bidders act without knowing the number of other bidders participating in either auction.

Our assumptions based on buyer being capacity constrained and sellers sell their goods simultaneously are motivated by several real world markets. Consider for example the market for houses. A selling homeowner may be better off under certain conditions if all potential buyers were able to pursue her house before some decided to move on to the next one. In reality, there are many other homes being sold at exactly the same time, and coordination frictions inevitably arise. These frictions result in multiple buyers attempting to purchase one house while another house is left uncontested. The academic job market is another example in which many sellers (workers who wish to sell their labor) are forced to sell simultaneously to capacity constrained buyers (universities). Such a mechanism generally leads to inefficient allocation due to the same type of coordination frictions.

We find that bidders choose auctions according to a cut-off strategy: only those with a sufficiently high private valuation choose the private auction. This result has a simple intuition: the payoff to bidders from the private value auction increases with their valuations whereas the payoff from the common value auction is constant. The cut-off strategies also imply a one-dimensional strategy space which considerably simplifies our analysis.

We solve the model by first fixing auction choice cut-off strategies and finding the equilibrium bidding functions given those strategies. Given equilibrium bidding behavior, we then endogenize auction choices and calculate the equilibrium cut-off strategies. An equilibrium cut-off strategy corresponds to the private valuation which makes a bidder's expected payoff from participating in the private auction equal to that of participating in the common auction. Hence, the cut-off values can be calculated by calculating these expected payoffs and solving an indifference condition.

In equilibrium, bidders bid their valuations in a second-price auction regardless of auction choice behavior. In first-price auctions, however, equilibrium bidding strategies depend on auction choice cut-off strategies as well as valuations. In the first-price private value auction, equilibrium bids are equal to the expectation of the second highest valuation conditional on winning; however, the interval from which the private valuations are drawn differs from the standard (single auction) model since bidders' participation in this auction implies that their valuations have to be greater than their cut-off values. In the first-price common value auction, there exists no pure strategy bidding equilibrium since all bidders' (common) valuation for the good is publicly known. We find that the unique equilibrium is in mixed bidding strategies in which bidders choose their bids from continuous distributions with the same connected support. These distributions are constructed to make all (other) bidders indifferent among all bids in this support.

For any auction type profile, there is a unique symmetric auction choice equilibrium, which is the same for all auction type profiles, and multiple asymmetric equilibria which vary with auction types. Revenue equivalence fails to hold in both auctions: the private auction revenue is always strictly higher when it is first-price, while the revenue in the common auction is strictly higher when it is second-price. The intuition for the private value auction is as follows: when second-price, revenue is equal to the expectation of the second highest value *if there are at least two bidders* and zero otherwise; when first price, revenue is equal to the second-price revenue plus a positive term which corresponds to the case in which only one bidder participates. In the common value auction, the second-price revenue is the (common) value times the probability that at least two bidders participate, while first-price revenue is derived from the mixed equilibrium bidding strategies.

The remainder of this paper is organized as follows. We review the related literature in chapter 2. We introduce the model in chapter 3 and show that auction choices are made using cut-off strategies in chapter 4. Finally, Chapter 7 concludes and states possible extensions to our model. Proofs can be found in the Appendix.

CHAPTER 2

LITERATURE REVIEW

We consider two simultaneous auctions (one private-value, one common-value) with incomplete information. Bidders choose auctions and bids simultaneously without observing the number of competitors. Hence, in our model, buyers and not sellers determine the equilibrium prices, is in contrast to a vast majority of the directed search literature. This literature–Burdett, Shi, and Wright (2001) is one of many examples–has focused primarily on cases in which one side of the market posts locations and prices, while the other side observes these prices and decides where to apply. In goods market applications, it is typically the sellers who post prices and consumers who search. In directed labor search, firms post wages and workers search. This paper focuses on cases in which the same side of the market, the buyer side, is in effect responsible for both searching and determining the equilibrium price for a fixed auction profile. To our knowledge, there are four papers closest to considering such a mechanism; Julien, Kennes, and King (2000), Gerding et al 2008, Gavious (2006), Selman (2010). However, in Julien, Kennes, and King (2000)'s (labor market) model, firms decide in a preliminary stage which workers to "bid" on, and then observe how many other firms are bidding on the same worker before choosing their bids. Gerding et al 2008 analyzes a number of simultaneous identical Vickrey auctions each selling complete substitutes to a number of local and global bidders. In our model, we have homogeneous bidders and they all are capacity constrained to bid in only one auction. The most similar paper in the literature to ours is Gavious (2006). In this paper, there are two private-value second price auctions where the one of the goods has higher value than the other. He assumes a private value for the low good is chosen from a distribution and the value for the high good is some constant times the value chosen from the distribution. Here, we do not assume a certain form of relation between the two valuations for the two goods and also each one of the auctions can be either first price or second price. So, our model is a generalization of the model in Gavious' paper. Finally, in Selman (2010), two heterogeneous buyers with commonly known preferences must choose which one of two different goods, a high value good and a low value good, to bid on when the goods are sold through simultaneously held first price auctions. The difference with our model is that we do not assume the auctions are first price, they can also be second price auctions and also in his model both auctions are common value which is not the case with ours.

There are many papers in which the seller competition is examined such as Burguet and Sakovics (1999), Coles and Eeckhout (2000) and Damianov (2012), however our paper is approaching the problem from buyers' side which is not very common. Burguet and Sakovics (1999) analyzes the competition between two owners of identical goods who wish to sell them to a pool of potential buyers. The sellers compete simultaneously setting reserve prices for their second price sealed bid auctions in their model, whereas in ours sellers compete by choosing auction types. Coles and Eeckhout (2000) has a model in which sellers post prices and then buyers choose between the goods, and after seeing who is their rival, they can walk away. We force bidders decide simultaneously in which auction to participate and how much to bid in that auction.

Our assumption that buyers must choose their bids without learning if they have competition makes this model differ greatly from theirs. Instead, equilibrium bidding behavior in this framework features buyers choosing their bids from continuous distributions with identical and connected support for the first price auction type. Technically, this result is similar to that of Burdett and Judd (1983), in which firms post prices without knowing how many firms they are trying to "outbid" with lower prices, since consumers may or may not search another firm's price. For the second price auction case, as always, the bids are equal to bidder's valuations.

CHAPTER 3

MODEL

There are n buyers and two sellers (P and C) each selling one of two distinct goods. For simplicity we will also call the goods P and C. Buyer $i \in \{1, \ldots, n\}$ has private value $v_i \in [\underline{v}, \overline{v}]$ for good P, where $\overline{v} > \underline{v} > 0$, while all buyers have a common value $x > 0$ for good C. Private values v_i are independently and identically distributed according to a continuously differentiable distribution function F. For now we will assume that the reservation price for both auctions is set to zero.

Sellers P and C each hold one auction to sell their respective goods. Each seller announces whether his good will be sold via a first-price auction (FPA) or a second-price auction (SPA). We assume that the two auctions are held simultaneously, and that each buyer can participate in at most one auction.[1] After auction types are announced, buyers *simultaneously* decide in which auction to participate *and* their bid in their selected auction. Therefore, bidders in both auctions make their bids without knowing the number of other bidders participating in the same auction. A pure strategy for a buyer i with private value v_i (for good P) is given by her auction choice (either P or C) and her bid in that auction–both as functions of v_i and the announced auction types.

Once buyers have chosen their actions, each good is rewarded to the highest bidder; if no bidders participate in one of the auctions, that good is not sold. In the event of a tie in either auction, each buyer is rewarded the good with probability one half. All agents are risk neutral. So, the payoff for a buyer who wins a good is simply equal to her value for the good minus the selling price, while the payoff for a seller is simply equal to the selling price. Any buyer who does not get a good and any seller who receives no bids get zero.

[1] A motivating example for this assumption is a labor market in which capacity constrained firms (buyers) with one job opening cannot feasibly offer a job to two workers (sellers) who could potentially both accept. A possible interpretation is that the disutility for a firm from having to back out of a commitment it cannot meet is sufficiently for it to simply never take the risk of "winning" two workers (goods) by placing multiple bids.

CHAPTER 4

BIDDER EQUILIBRIUM

In this section, we take sellers' auction type announcements as given and solve for the bidder equilibrium. A pure strategy for a bidder i is an auction choice $a_i(v_i) \in \{P, C\}$ and a bid $b_i^{a_i}(v_i)$, both of which are functions of the bidder's private valuation v_i.

Definition 1. Given auction type announcements for both auctions, a bidder equilibrium is a strategy profile $\{a_i, b_i^{a_i}\}_{i=1}^{n}$ such that every bidder i maximizes his expected payoff given the auction types and other bidders' strategies.

In the next session we show that bidders always choose their auction $a_i(v_i)$ using a cut-off strategy.

Auction Choice Cut-off Strategies

The payoff from the private value auction to a bidder increases as his valuation for the good increases, but the payoff from the common value auction does not change with the private valuation of the bidder. Hence, there is a private value(for non-extreme values for $\underline{v}, \overline{v}$ and x) which makes the bidders' payoffs from both auctions the same. This value is the bidders' cut-off value. Above that value, the bidders choose the private value auction because it gives more payoff than the common value auction.

Lemma 1. In all equilibria, each bidder i chooses in which auction to compete according to a reservation strategy $r_i \in [\underline{v}, \overline{v}]$. Bidder i chooses

$$a_i(v_i) = \begin{cases} P & \text{if } v_i > r_i \\ \\ C & \text{if } v_i < r_i. \end{cases}$$

This result implies that, when solving for equilibrium, the auction choice strategy space can be redefined as the one dimensional private valuation interval $[\underline{v}, \overline{v}]$, which significantly eases our analysis.

Next, we will solve for the general equilibria when there are two bidders.

Equilibrium Bidding Strategies

In this section, we fix buyers' auction choices $(r_1, \ldots, r_n)$ and find the equilibrium bidding strategies for both auctions under both auction types. In the first subsection, we analyze second-price auction bidding in both the private value auction and the common value auction. Next, we solve for the equilibrium bidding strategies when the private value auction is first-price in section. Finally, we solve the common value first-price case in the last subsection.

For all auction types, all bidder equilibria are *ex ante* inefficient. This inefficiency arises for one or both of the following reasons: (1) Bidders may fail to coordinate and both bid on the same good, in which case only one good is sold. This occurs with positive probability in most bidding equilibria and, as we will show, all market equilibria of the game. (2) In asymmetric equilibria only, even when bidders bid on different goods, the bidder who has a higher value for the private good may not obtain the private good if he has a higher reservation strategy than the other bidder.

Throughout the section, we use general distribution function F for the private valuations of bidders when describing the model, but to calculate the terms analytically, we use the uniform distribution. We find the equilibria of the bidder game by first calculating the bidding functions for a given cut-off choice profile, then solving for the equilibrium cut-off choice according to the payoffs calculated by using those bidding functions. The symmetric equilibrium cut-off value does not change when the type of the private value auction changes because as we will show, the bidding functions hence the expected payoffs are the same for both types of the private value auction.

We find all equilibria for the case in which both auctions are second-price and show that there are always three equilibria in this case: one symmetric and two asymmetric. In a sense, our model is similar to those of auctions with an uncertain number of bidders. Here, the distribution of number of bidders can be derived directly from the auction choice strategy profile $(r_1, \ldots, r_n)$ and the distribution of private valuations F.

<u>Equilibrium SPA Bidding</u>

It is well known that the only symmetric equilibrium of a second-price auction, including ones in which the number of bidders is uncertain, is the one in which all bidders bid their valuation. Therefore, all bidders who participate in auction P when second-price will bid according to the strategy $b_i^{P;SPA}(v_i) = v_i$, and all bidders who participate in auction C when second-price will bid according to the strategy $b_i^{C;SPA}(v_i) = x$ (for all v_i).

<u>Equilibrium FPA Bidding in the Private Vaule Auction</u>

Fixing auction choice reservation strategies $\mathbf{r} \equiv (r_1, ..., r_n)$ and letting $\{r^1, \ldots, r^D\} = \{r_1, ..., r_n\}$ be the set of D distinct reservation values chosen $(1 \leq D \leq n)$, the equilibrium bidding functions in the private value first-price are derived piecewise, with the bidding function for each $v_i \in [r^d, r^{d+1}]$ being calculated using standard techniques. This is both possible and necessary because the anticipated number of competitors participating in the auction depends on the interval $[r^d, r^{d+1}]$ in which a bidder's private valuation v_i falls.

Proposition 1. Fix auction choice reservation strategies such that $r_1 \leq r_2$. Then, the equilibrium bidding functions are

$$\beta_r(v) = \begin{cases} 0 & \text{if } v < r_2 \\ \\ \frac{1}{F(v)} \int_{r_2}^{v} z f(z)\, dz & \text{if } v > r_2. \end{cases} \tag{1}$$

Proposition states that the optimal bid in our private value FPA is 0 when the bidder has a value less than the maximum cut-off choice, since in this case the bidder wins if he is the only one in that auction. If he has a value greater than the maximum cut-off choice, then he bids expectation of the other bidder's value conditional on winning.

Equilibrium FPA Bidding in the Common Vaule Auction

When the common value auction is FPA, the bidders will not bid their valuation x to avoid getting 0 payoff. Also they will not bid 0 with probability 1 because in that case the other bidder can easily outbid him by bidding a small amount $\varepsilon > 0$ and win the object. Hence, in equilibrium, they will play mixed strategy drawn from a distribution function which is determined from the payoffs and has a continuous support depending on the minimum of the two cut-off choices.

Proposition 2. The equilibrium bidding function in the common value auction when it is FPA will be mixed strategies drawn from distribution functions Λ_1, Λ_2 satisfying

$$\Lambda_1(b;r) = \begin{cases} \left(\frac{b}{x-b}\right)\frac{1-F(r_1)}{F(r_1)} & \text{if } b \in [0, xF(r_1)] \\ 1 & \text{if } b > xF(r_1) \end{cases} \tag{2}$$

and

$$\Lambda_2(b;r) = \begin{cases} \frac{F(r_2)-F(r_1)}{F(r_2)} & \text{if } b = 0 \\ \left(\frac{b}{x-b}\right)\frac{1-F(r_2)}{F(r_2)} + \left(\frac{x}{x-b}\right)\frac{F(r_2)-F(r_1)}{F(r_2)} & \text{if } b \in [0, xF(r_1)] \\ 1 & \text{if } b > xF(r_1), \end{cases} \tag{3}$$

Equilibrium Auction Choice

Bidders decide their equilibrium auction choices according to the indifference points between payoffs of two auctions. Let r_1, r_2 be the cut-off strategies of bidders, and assume wlog $r_1 \leq r_2$. To calculate the payoffs, we use the bidding functions found in the previous sections: the pure bidding strategy (1) for the private value FPA, and the mixed bidding strategies (2) and (3) for the common value FPA.

The payoff functions associated with these and the trivial SPA bidding strategies are

$$U_r^{P,FPA}(v) = \begin{cases} (v - \beta_r(v)) F(r) & \text{if } v < r_2 \\ \\ (v - \beta_r(v)) F(v) & \text{if } v > r_2 \end{cases}$$
$$= \begin{cases} vF(r_2) & \text{if } v < r_2 \\ \\ vF(v) - \int_{r_2}^{v} zf(z)\,dz & \text{if } v > r_2 \end{cases}$$

$$U_r^{P,SPA}(v) = \begin{cases} vF(r_2) & \text{if } v < r_2 \\ \\ \left(v - \frac{1}{F(v)} \int_{r_2}^{v} zf(z)\,dz\right) F(v) & \text{if } v > r_2. \end{cases}$$

We see that the expected payoff from the private value auction to bidders is the same regardless of the auction types. So, let

$$U_r^P(v) \equiv U_r^{P,FPA}(v) = U_r^{P,SPA}(v).$$

The expected utility from the common value auction when FPA is

$$U_r^{C,FPA} = x(1 - F(r_1)),$$

and when SPA is

$$U_{i,r}^{C,SPA} = x\left(1 - F\left(r_j\right)\right),$$

where $i \neq j$. We see that bidder 2(say bidder 2 choose r_2) gets the same payoff in both the FPA and the SPA, but bidder 1 (i.e. the one with the lower value of the two cut-off choices), meaning he is less likely to participate in the common calue auction) gets a strictly higher expected payoff if the auction is a FPA. Because a positive payoff is achieved in the SPA iff the other bidder does not participate, bidder 1 is at a relative disadvantage since bidder 2's likelihood of participating is higher. On the other hand, in the FPA they receive identical expected payoffs since the asymmetry in participation likelihood is precisely compensated for by the asymmetric mixed bidding strategies Λ_1 and Λ_2.

Symmetric Equilibrium

At the symmetric equilibrium, we see that the expected payoff from each auction does not change when the auction type changes. Hence, for each auction type profile, the symmetric equilibrium cut-off choice should be the same. The following proposition states that.

Proposition 3. The value of r^*, the symmetric equilibrium cut-off choice, is the same for all auction profiles $(SPA, SPA), (FPA, SPA), (SPA, FPA), (FPA, FPA)$. And the indifference condition r satisfy is

$$U_{r^*}^{P}\left(r^*\right) = U_{r^*}^{C}$$
$$\Leftrightarrow r^* F\left(r^*\right) = x\left(1 - F\left(r^*\right)\right)$$
$$\Leftrightarrow F\left(r^*\right) = \frac{x}{x + r^*}.$$

For F uniform on $[\underline{v}, \overline{v}]$, this gives us the symmetric equilibrium cut-off choice:

$$r^* = \frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right).$$

Asymmetric Equilibrium

When both auctions are second-price, we calculated all equilibria when the distribution function is uniform. The unique symmetric equilibrium is the same as stated in the previous proposition. In the following proposition, we show that there are always two asymmetric equilibria which changes according to values of $\underline{v}, \overline{v}$ and x. We see that $(\overline{v}, \underline{v})$ can also be an equilibrium if x is high enough, so that one of the bidders agree on getting the common value good all the time. It is the worst case for the sellers because in each auction there will be only one bidder and the sellers get 0 from selling their good. So, sellers will want coordination friction between bidders when both auctions are second-price.

Proposition 4. There always exist exactly three Nash Equilibria, given by the set

$$NE = \{(r^*, r^*), (r', r''), (r'', r')\},$$

where $r' > r''$ and

$$r^* = \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right),$$

$$(r', r'') = \begin{cases} (\overline{v}, \underline{v}) & \text{if } x \geq \frac{\overline{v} - \underline{v}}{2} \\ \left(r^{BR}(\underline{v}), \underline{v}\right) & \text{if } x < \frac{\overline{v} - \underline{v}}{2} \text{ and } r^{BR}(\underline{v}) \geq r^{\max} \\ (r^2, r_1^1) & \text{if } x < \frac{\overline{v} - \underline{v}}{2} \text{ and } r^{BR}(\underline{v}) < r^{\max}, \end{cases}$$

where

$$r^{BR}\left(\underline{v}\right) = \underline{v} + \sqrt{2x\left(\overline{v} - \underline{v}\right)},$$

$$r^{\max} = \frac{\overline{v}x + \underline{v}^2}{\underline{v} + x},$$

$$r_2 = \frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right) + \left(\underline{v} + x\right),$$

$$r_1 = \frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right) - \left(\underline{v} + x\right).$$

CHAPTER 5

MARKET EQUILIBRIUM

As we know the symmetric equilibrium and the bidding functions, we can calculate the expected revenues from both auctions.

Revenue Comparison

The revenue equivalence principle states that under certain assumptions, FPA and SPA yields the same revenue when valuations are private, which is equal to the expectation of the second highest value. This principle does not apply to our model because in our model there is an uncertainty in the number of bidders. With that uncertainty, the probability of the seller who has the private value good getting a positive revenue changes according to the auction types. When the private value auction is SPA, the seller gets the second highest value with the probability that there are at least two bidders, whereas when it is FPA, the seller gets the same thing plus the expected bid when there is one bidder. Hence, the revenue principle fails due to the fact that the revenue from the private value auction is higher when it is FPA rather than SPA. In this section, we will show that analytically.

Revenue from the Private Value Auction

When the private value auction is second-price, the expected revenue is

$$\begin{aligned} E\left[R_P^{SPA}\right] &= \Pr\{\text{there are two bidders}\}\, E\left[\text{second highest valuation}\right] \\ &= \left(1-F\left(r_1\right)\right)\left(1-F\left(r_2\right)\right) E\left[\min\left\{v_1, v_2 \mid v_1 > r_1, v_2 > r_2\right\}\right]. \end{aligned}$$

When F is uniform, at the symmetric equilibrium r^*, this becomes

$$\begin{aligned} E\left[R_P^{SPA}\right] &= \left(\frac{\overline{v}-r^*}{\overline{v}-\underline{v}}\right)^2 \frac{2\left(\overline{v}-\underline{v}\right)^2}{\left(\overline{v}-r^*\right)^2} \int_{r^*}^{\overline{v}} \frac{z\left(\overline{v}-\underline{v}\right)-z\left(z-\underline{v}\right)}{\left(\overline{v}-\underline{v}\right)^2} dz \\ &= \left(\frac{\overline{v}-r^*}{\overline{v}-\underline{v}}\right)^2 \left(\frac{\overline{v}+2r^*}{3}\right). \end{aligned}$$

Note that this is equal to the probability that both bidders' values are higher than r^*, meaning both bidders choose the private value auction, times expected value of the second highest valuation. It is basically the second highest valuation times the probability of getting it.

When the private value auction is first-price, the expected revenue is given by the following

$$\begin{aligned} E\left[R_P^{FPA}\right] &= \Pr\{\text{there is one bidder with value greater than } r_2\} E[\text{bid}] \\ &\quad + \Pr\{\text{there are two bidders with values bigger than } r_2\} E[\text{max of bids}] \\ &= \Pr\{v_1 > r_2, v_2 < r_2\} E[\beta(v_1) \mid v_1 > r_2] \\ &\quad + \Pr\{v_1 < r_2, v_2 > r_2\} E[\beta(v_2) \mid v_2 > r_2] \\ &\quad + \Pr\{v_1, v_2 > r_2\} E[\beta(v_{\max}) \mid v_{\max} \equiv \max\{v_1, v_2\} \text{ where } v_1, v_2 > r_2]. \end{aligned}$$

When F is uniform, at the symmetric equilibrium r^*, this becomes

$$\begin{aligned} E\left[R_P^{FPA}\right] &= 2\left(\frac{r^* - \underline{v}}{\overline{v} - \underline{v}}\right)\left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)\left(\frac{\overline{v} - \underline{v}}{\overline{v} - r^*}\right)\int_{r^*}^{\overline{v}} \frac{z^2 - (r^*)^2}{2(z - \underline{v})}\left(\frac{1}{\overline{v} - \underline{v}}\right) dz \\ &\quad + \int_{r^*}^{\overline{v}} \frac{z^2 - (r^*)^2}{2(z - \underline{v})} 2\left(\frac{z - \underline{v}}{\overline{v} - \underline{v}}\right)\left(\frac{1}{\overline{v} - \underline{v}}\right) dz \\ &= \underbrace{\left(\frac{r^* - \underline{v}}{\overline{v} - \underline{v}}\right)\left(\frac{1}{\overline{v} - \underline{v}}\right)\begin{pmatrix} \left(\underline{v}^2 - (r^*)^2\right)\ln(\overline{v} - \underline{v}) + \overline{v}\underline{v} \\ +\frac{1}{2}\overline{v}^2 - \left(\underline{v}^2 - (r^*)^2\right)\ln(r^* - \underline{v}) - \underline{v}r^* - \frac{1}{2}(r^*)^2 \end{pmatrix}}_{>0} \\ &\quad + E\left[R_P^{SPA}\right] \\ &> E\left[R_P^{SPA}\right]. \end{aligned}$$

This proves the following proposition.

Proposition 5. Revenue from the private value auction is greater when it is FPA rather than SPA.

Revenue from the Common Value Auction

When the common value auction is second-price, the expected revenue is

$$E\left[R_C^{SPA}\right] = F(r_1)\, F(r_2)\, x.$$

When F is uniform, at the symmetric equilibrium, this becomes

$$E\left[R_C^{SPA}\right] = \left(\frac{r^* - \underline{v}}{\overline{v} - \underline{v}}\right)^2 x.$$

Here again the expected revenue is equal to the probability of both bidders come to the common value auction times the second highest bid which is equal to the value of the good in that auction, x.

When it is first-price, the expected revenue is

$$\begin{aligned} E\left[R_C^{FPA}\right] &= \Pr\{\text{there is exactly one bidder}\}\, E[\text{bid of that bidder}] \\ &\quad + \Pr\{\text{there are two bidders}\}\, E[\text{maximum of the bids}] \\ &= \left(F(r_1)(1 - F(r_2)) + (1 - F(r_1))\, F(r_2)\right) \int_0^{F(r_1)x} b\Lambda'(b)\, db \\ &\quad + F(r_1)\, F(r_2) \int_0^{F(r_1)x} 2b\Lambda'(b)\, \Lambda(b)\, db. \end{aligned}$$

When F is uniform, at the symmetric equilibrium, this becomes

$$\begin{aligned} E\left[R_C^{FPA}\right] &= 2\left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)^2 x \left(\frac{r^* - \underline{v}}{\overline{v} - r^*} + \ln\left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)\right) \\ &\quad + \left(\frac{r^* - \underline{v}}{\overline{v} - \underline{v}}\right) x \left(3\left(\frac{(r^* - \underline{v})(r^* - 2\overline{v} + \underline{v})}{(\overline{v} - \underline{v})^2}\right)^2 - 2\left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)^2 \ln\left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)\right). \end{aligned}$$

Conjecture 1. The revenue from the common value auction is higher if it is SPA rather than FPA.

Seller Equilibrium

We see from the previous section that the seller who has the private value auction will choose the first-price auction, and we believe that the other seller should choose the second-price auction. Those two imply that we cannot have a market equilibrium in which the private value auction is second-price and the common value auction is first-price. Hence, any market equilibrium should include sellers choosing first-price for the private value auction and second-price for the common value auction and bidders choose auctions and bidding functions accordingly.

CHAPTER 6

EFFICIENCY

We can define a general welfare measure as follows:

$$\begin{aligned} E\left[W\right] &= \sum_{i\neq j} \Pr\left\{P \text{ sold to } i,\, C \text{ sold to } j\right\} \left(E\left[v_i \mid i \text{ wins } P\right] + x\right) \\ &\quad + \sum_{i} \Pr\left\{P \text{ sold to } i,\, C \text{ not sold}\right\} \left(E\left[v_i \mid i \text{ wins } P\right]\right) \\ &\quad + \sum_{i} \Pr\left\{P \text{ not sold},\, C \text{ sold to } i\right\} x. \end{aligned}$$

In equillibrium with $n = 2$ and $r_1 \leq r_2$ this becomes:

$$\begin{aligned} E\left[W\right] &= F\left(r_1\right)\left(1 - F\left(r_2\right)\right)\left(\frac{1}{1 - F\left(r_2\right)} \int_{r_2}^{\overline{v}} z f\left(z\right) dz + x\right) \\ &\quad + \left(1 - F\left(r_1\right)\right) F\left(r_2\right)\left(\frac{1}{1 - F\left(r_1\right)} \int_{r_1}^{\overline{v}} z f\left(z\right) dz + x\right) \\ &\quad + \left(F\left(r_2\right) - F\left(r_1\right)\right)\left(1 - F\left(r_2\right)\right) \frac{1}{1 - F\left(r_2\right)} \int_{r_2}^{\overline{v}} z f\left(z\right) dz \\ &\quad + \left(1 - F\left(r_2\right)\right)\left(1 - F\left(r_2\right)\right)\left(\frac{1}{1 - F\left(r_2\right)}\right)^2 \int_{r_2}^{\overline{v}} z g_{(2)}\left(z\right) dz \\ &\quad + F\left(r_1\right) F\left(r_2\right) x \end{aligned}$$

which can be rewritten as

$$\begin{aligned} E\left[W\right] &= \underbrace{\left[F\left(r_1\right) + F\left(r_2\right) - F\left(r_1\right) F\left(r_2\right)\right] x}_{\text{welfare from } C} \\ &\quad \underbrace{+ F\left(r_2\right) \int_{r_2}^{\overline{v}} z f\left(z\right) dz + F\left(r_2\right) \int_{r_1}^{\overline{v}} z f\left(z\right) dz + \int_{r_2}^{\overline{v}} z g_{(2)}\left(z\right) dz}_{\text{welfare from } P}. \end{aligned}$$

For $r_1 = r_2 = r^*$ this becomes

$$\begin{aligned} E\left[W\right] &= \left[2F\left(r^*\right) - F^2\left(r^*\right)\right] x + 2F\left(r^*\right) \int_{r^*}^{\overline{v}} z f\left(z\right) dz + \int_{r^*}^{\overline{v}} z g_{(2)}\left(z\right) dz \\ &= \left[1 - \left(1 - F\left(r^*\right)\right)^2\right] x + 2F\left(r^*\right) \int_{r^*}^{\overline{v}} z f\left(z\right) dz + \int_{r^*}^{\overline{v}} z g_{(2)}\left(z\right) dz \end{aligned}$$

When F is uniform, this becomes

$$E\left[W\right] = \left[1 - \left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)^2\right] x + 2\left(\frac{r^* - \underline{v}}{\overline{v} - \underline{v}}\right)\left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)\left(\frac{\overline{v} + r^*}{2}\right) + \left(\frac{\overline{v} - r^*}{\overline{v} - \underline{v}}\right)^2\left(\frac{2\overline{v} + r^*}{3}\right).$$

CHAPTER 7

CONCLUSION

In our model, we show that coordination frictions between bidders cause inefficiency, but sellers benefit from them since having both bidders at their auction increases their payoff. Equilibrium bidder behavior differs from the standard case in the first-price auction whereas it stays the same in a second-price auction.

Our model can be extended in several ways. The common value auction can be generalized to a private value auction, so that the common value case becomes a special case. The number of sellers can be increased to decrease inefficiency, or sellers may be allowed to change the reservation price to increase their expected revenue in the second price case. We have a conjecture from our calculations on revenues that the revenue from the common value auction is higher when it is SPA rather than FPA, but we didn't prove it. By comparing those revenues, since we have comparison in the private value auction, we can find the optimal mechanism for the sellers.

APPENDICES

APPENDIX A: PROOFS OF THE PROPOSITIONS AND THEOREMS

Proof of Lemma 1. Assume for a contradiction that there is no such $r_i \in [\underline{v}, \overline{v}]$. Then, for all r in $[\underline{v}, \overline{v}]$; there exist $v' < r$ and $v'' > r$ such that, for a fixed r_j;

$$\overline{u}_i^H (v', r_j) > \overline{u}_i^L (v', r_j),$$
$$\overline{u}_i^H (v'', r_j) > \overline{u}_i^L (v'', r_j).$$

Since $\overline{u}_i^H (v_i, r_j)$ is strictly increasing in v_i, we have

$$\overline{u}_i^H (v'', r_j) > \overline{u}_i^H (r, r_j) > \overline{u}_i^H (v', r_j),$$

which gives us a contradiction since

$$\overline{u}_i^L (r, r_j) = \overline{u}_i^L (v', r_j) < \overline{u}_i^H (v', r_j) < \overline{u}_i^H (r, r_j) = \overline{u}_i^L (r, r_j).$$

Proof of Proposition 1. Bidders will choose the bid which maximizes their expected payoff which is given by

$$U_r^{P,FPA} (v) = \begin{cases} (v - \beta (v)) F (r) & \text{if } v < r_2 \\ \\ (v - \beta (v)) F (v) & \text{if } v > r_2. \end{cases}$$

We easily see that when a bidder's value is less than r_2, her bid does not affect her probability of win, $F(r)$. So, he will bid 0 to maximize his payoff. However, when his value is greater than r_2, he will maximize his payoff function $(v - \beta (v)) F (v)$. We will take the derivative of the payoff function with respect to $b = \beta (v)$ and then find b which makes the derivative equal to zero since this is a concave function and finally introduce the boundary condition $\beta(r_2) = 0$:

$$\begin{aligned} -F\left(\beta^{-1}(b)\right) + \frac{f\left(\beta^{-1}(b)\right)(v-b)}{\beta'\left(\beta^{-1}(b)\right)} &= 0 \\ &\Leftrightarrow \beta'(v) F(v) = f(v)(v-b) \\ &\Leftrightarrow \beta'(v) F(v) + f(v) b = f(v) v \\ &\Leftrightarrow (\beta(v) F(v))' = f(v) v \\ &\Leftrightarrow \beta(v) = \frac{1}{F(v)} \int_{r_2}^{v} f(z) z dz. \end{aligned}$$

Proof of Proposition 2. Suppose $r_1 \neq r_2$. The probability that bidder i will participate in this auction is $F(r_i)$. Let $\Lambda_i(b)$ be the mixed strategy bidding distribution chosen by bidder i *conditional on participating.* (Assume for now these are continuous.) Then, for bidder 1, the payoff from bidding b_1 is

$$\begin{aligned} \Pi_1^B(b_1) &= (x - b_1) \Pr\{\text{win} \mid b_1\} \\ &= (x - b_1)(1 - F(r_2) + F(r_2)\Lambda_2(b_1)). \end{aligned}$$

Similarly,

$$\Pi_2^B(b_2) = (x - b_2)(1 - F(r_1) + F(r_1)\Lambda_1(b_2)).$$

By the same argument as in Burdett-Judd, we know that a necessary condition is that the max support of both bidding distributions must be the same: $\bar{b}_1 = \bar{b}_2 \equiv \bar{b}$. Then, by definition, $\Lambda_1(\bar{b}) = \Lambda_2(\bar{b}) = 1$, so

$$\begin{aligned} \Pi_i^B(\bar{b}) &= (x - \bar{b})\left(1 - F(r_j) + F(r_j)\Lambda_j(\bar{b})\right) \\ &= x - \bar{b}. \end{aligned}$$

Again, by the same argument as in Burdett-Judd, we know that a necessary condition is that the min support of both bidding distributions must be the same and equal to zero: $\underline{b}_1 = \underline{b}_2 = 0$. But then we have

$$\begin{aligned} \Pi_i^B(0) &= (x-0)\left(1-F(r_j)+F(r_j)\Lambda_j(0)\right) \\ &= x\left(1-F(r_j)+F(r_j)\Lambda_j(0)\right). \end{aligned}$$

Because it must hold that $\Pi_i^B\left(\overline{b}\right) = \Pi_i^B(0)$, we now have

$$x-\overline{b} = x\left(1-F(r_i)+F(r_i)\Lambda_i(0)\right)$$

for both $i = 1, 2$. Because either $\Lambda_1(0) = 0$ or $\Lambda_2(0) = 0$ must hold (otherwise there will be a positive probability of a tie and hence a profitable deviation), let us set, without loss of generality, $\Lambda_1(0) = 0$ and solve our two equations for our two unknowns $\overline{b}$ and $\Lambda_2(0)$:

$$\begin{aligned} x-\overline{b} &= x\left(1-F(r_1)\right), \\ x-\overline{b} &= x\left(1-F(r_2)+F(r_2)\Lambda_2(0)\right). \end{aligned}$$

Solving yields

$$\overline{b} = xF(r_1)$$

and

$$x-xF(r_1) = x\left(1-F(r_2)+F(r_2)\Lambda_2(0)\right)$$

$$\Leftrightarrow \Lambda_2(0) = \frac{F(r_2)-F(r_1)}{F(r_2)}.$$

So in the (unique) equilibrium, bidder 2 (i.e. the bidder who bids zero with positive probability) must also be the bidder with the higher value (i.e. $r_2 > r_1$). We can now solve for $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$. For all $b \in \left[0, \overline{b}\right]$, it must hold that

$$\Pi_i^B(b) = x\left(1-F(r_1)\right),$$

so we can write

$$\left(x-b\right)\left(1-F\left(r_j\right)+F\left(r_j\right)\Lambda_i\left(b\right)\right)=x\left(1-F\left(r_1\right)\right)$$

$$\Leftrightarrow \Lambda_i\left(b\right)=\frac{\frac{x(1-F(r_1))}{x-b}-\left(1-F\left(r_j\right)\right)}{F\left(r_j\right)}$$

This yields

$$\begin{aligned}\Lambda_1\left(b\right) &= \frac{\frac{x(1-F(r_1))}{x-b}-\left(1-F\left(r_1\right)\right)}{F\left(r_1\right)} \\ &= \left(\frac{b}{x-b}\right)\frac{1-F\left(r_1\right)}{F\left(r_1\right)}.\end{aligned}$$

and

$$\begin{aligned}\Lambda_2\left(b\right) &= \frac{\frac{x(1-F(r_1))}{x-b}-\left(1-F\left(r_2\right)\right)}{F\left(r_2\right)} \\ &= \left(\frac{b}{x-b}\right)\frac{1-F\left(r_2\right)}{F\left(r_2\right)}+\left(\frac{x}{x-b}\right)\frac{F\left(r_2\right)-F\left(r_1\right)}{F\left(r_2\right)}.\end{aligned}$$

Proof of Proposition 3. Since the payoffs are the same when the equilibrium auction choice is symmetric, it is obvious that the symmetric cut-off choice will be the same for each of the auction type profiles. When F is uniform, the cut-off choice should satisfy

$$\begin{aligned}\frac{r^*-\underline{v}}{\overline{v}-\underline{v}} &= \frac{x}{x+r^*} \\ &\Leftrightarrow \left(r^*-\underline{v}\right)\left(x+r^*\right)=\overline{v}x-\underline{v}x \\ &\Leftrightarrow \left(r^*\right)^2+r^*\left(x-\underline{v}\right)-\underline{v}x=\overline{v}x-\underline{v}x \\ &\Leftrightarrow \left(r^*+\frac{x-\underline{v}}{2}\right)^2-\left(\frac{x-\underline{v}}{2}\right)^2=\overline{v}x \\ &\Leftrightarrow r^*=\frac{\underline{v}-x+\sqrt{4\overline{v}x+\left(\underline{v}-x\right)^2}}{2}.\end{aligned}$$

*Proof of Proposition 4.*_ given r_2, player 1's payoffs become

$$\begin{aligned}
\overline{u}_1^H(v_1,r_2) &= \Pr\{win\}(v_1 - E[price \mid win]) \\
&= \Pr\{a_2(v_2)=L\}(v_1 - E[price \mid a_2(v_2)=L]) \\
&\quad + \Pr\{a_2(v_2)=H \text{ and } 1\ wins\}(v_1 - E[price \mid a_2(v_2)=H \text{ and } 1\ wins]) \\
&= \underbrace{\Pr\{v_2<r_2\}}_{=\frac{r_2-\underline{v}}{\overline{v}-\underline{v}}}\left(v_1 - \underbrace{E[price \mid a_2(v_2)=L]}_{=0}\right) \\
&\quad + \underbrace{\Pr\{r_2<v_2<v_1\}}_{=\max\left\{\frac{v_1-r_2}{\overline{v}-\underline{v}},0\right\}}\left(v_1 - \underbrace{E[price \mid r_2<v_2<v_1]}_{=\frac{v_1+r_2}{2}}\right) \\
&= \begin{cases} \left(\frac{r_2-\underline{v}}{\overline{v}-\underline{v}}\right)v_1 & \text{if } v_1<r_2 \\ \frac{r_2-\underline{v}}{\overline{v}-\underline{v}}(v_1) + \frac{v_1-r_2}{\overline{v}-\underline{v}}\left(v_1-\frac{v_1+r_2}{2}\right) & \text{if } v_1>r_2 \end{cases} \\
&= \begin{cases} \left(\frac{r_2-\underline{v}}{\overline{v}-\underline{v}}\right)v_1 & \text{if } v_1<r_2 \\ \frac{r_2-\underline{v}}{\overline{v}-\underline{v}}(v_1) + \frac{(v_1-r_2)^2}{2(\overline{v}-\underline{v})} & \text{if } v_1>r_2 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\overline{u}_1^L(v_1,r_2) &= \Pr\{a_2=L\}\cdot 0 + \Pr\{a_2=H\}x \\
&= \left(\frac{\overline{v}-r_2}{\overline{v}-\underline{v}}\right)x.
\end{aligned}$$

Player 1 will therefore choose auction H iff

$$\overline{u}_1^H(v_1,r_2) \geq \overline{u}_1^L(v_1,r_2). \tag{4}$$

To find the equilibrium, first we will find best response functions.

Let $r^{\min}$ denote the value of r_2 for which player 1 is indifferent between the two auctions precisely when $v_1=\overline{v}$. We can find it by solving the following equation:

$$\overline{u}_1^H\left(\overline{v}, r^{\min}\right) = \overline{u}_1^L\left(\overline{v}, r^{\min}\right)$$
$$\Leftrightarrow \frac{r^{\min} - \underline{v}}{\overline{v} - \underline{v}}\left(\overline{v}\right) + \frac{\left(\overline{v} - r^{\min}\right)^2}{2\left(\overline{v} - \underline{v}\right)} = \left(\frac{\overline{v} - r^{\min}}{\overline{v} - \underline{v}}\right)x$$
$$\Leftrightarrow \left(r^{\min} - \underline{v}\right)\overline{v} + \frac{\left(\overline{v} - r^{\min}\right)^2}{2} = \left(\overline{v} - r^{\min}\right)x$$

The roots of the equation are

$$r^{\min} = -x \pm \sqrt{-\overline{v}^2 + 2\overline{v}x + 2\overline{v}\underline{v} + x^2}.$$

So, the only possible solution is

$$r^{\min} = \sqrt{x^2 - \overline{v}^2 + 2\overline{v}x + 2\overline{v}\underline{v}} - x.$$

This is in $[\underline{v}, \overline{v}]$ if and only if $\frac{\overline{v}-\underline{v}}{2} \leq x$.

Let $r^{\max}$ denote the value of r_2 for which player 1 is indifferent between the two auctions precisely when $v_1 = \underline{v}$. That is, define $r^{\max}$ such that

$$\overline{u}_1^H\left(\underline{v}, r^{\max}\right) = \overline{u}_1^L\left(\underline{v}, r^{\max}\right).$$

We can find it by solving the following equation for r_2 :

$$\left(\frac{r_2 - \underline{v}}{\overline{v} - \underline{v}}\right)\underline{v} = \left(\frac{\overline{v} - r_2}{\overline{v} - \underline{v}}\right)x$$

So, we get

$$r^{\max} = \frac{\overline{v}x + \underline{v}^2}{\underline{v} + x},$$

which is always in the interval $[\underline{v}, \overline{v}]$.

When $v_1 < r_2$, this inequality becomes

$$\begin{aligned}\left(\frac{r_2-\underline{v}}{\overline{v}-\underline{v}}\right)v_1 &\geq \left(\frac{\overline{v}-r_2}{\overline{v}-\underline{v}}\right)x \\ &\Leftrightarrow\ v_1 \geq \left(\frac{\overline{v}-r_2}{r_2-\underline{v}}\right)x,\end{aligned}$$

so

$$r_1^{BR}(r_2) = \left(\frac{\overline{v}-r_2}{r_2-\underline{v}}\right)x$$

for the interval $r_2 \in [v_1, r^{\max})$, so precisely when $r_2 < r^{\max}$ and

$$\begin{aligned} r_1^{BR}(r_2) &\leq r_2 \\ \Leftrightarrow \left(\frac{\overline{v}-r_2}{r_2-\underline{v}}\right)x &\leq r_2 \\ \Leftrightarrow r_2 &\geq \frac{1}{2}\left(\underline{v}-x+\sqrt{(\underline{v}-x)^2+4\overline{v}x}\right). \end{aligned} \tag{5}$$

Hence, we can see already that

$$(r^*, r^*) = \left(\frac{1}{2}\left(\underline{v}-x+\sqrt{(\underline{v}-x)^2+4\overline{v}x}\right), \frac{1}{2}\left(\underline{v}-x+\sqrt{(\underline{v}-x)^2+4\overline{v}x}\right)\right)$$

is the symmetric Nash equilbrium.

When $v_1 > r_2$ then the inequality (4) becomes

$$\begin{aligned} \frac{r_2-\underline{v}}{\overline{v}-\underline{v}}(v_1)+\frac{(v_1-r_2)^2}{2(\overline{v}-\underline{v})} &\geq \left(\frac{\overline{v}-r_2}{\overline{v}-\underline{v}}\right)x \\ \Leftrightarrow (r_2-\underline{v})v_1+\frac{(v_1-r_2)^2}{2} &\geq (\overline{v}-r_2)x \\ \Leftrightarrow v_1 &\geq \underline{v}+\sqrt{2\overline{v}x+\underline{v}^2-r_2^2-2xr_2}, \end{aligned}$$

so

$$r_1^{BR}(r_2) = \underline{v}+\sqrt{2\overline{v}x+\underline{v}^2-r_2^2-2xr_2}$$

for the interval $r_2 \in (r^{\min}, v_1]$, so precisely when $r_2 > r^{\min}$ and

$$
\begin{aligned}
& r_1^{BR}(r_2) \geq r_2 \\
\Leftrightarrow\ & \underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - r_2^2 - 2xr_2} \geq r_2 \\
\Leftrightarrow\ & r_2 \leq \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right),
\end{aligned}
$$

which is the negation of the condition in (5), as expected.

So player 1's best response correspondence when $x \leq \frac{\overline{v}-\underline{v}}{2}$ is

$$
r_1^{BR}(r_2) = \begin{cases}
\underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - r_2^2 - 2xr_2} & \text{if } r_2 \leq r^* \\
\left(\frac{\overline{v}-r_2}{r_2-\underline{v}}\right)x & \text{if } r^* \leq r_2 \leq r^{\max} \\
\underline{v} & \text{if } r_2 > r^{\max}
\end{cases}
$$

while when $x \geq \frac{\overline{v}-\underline{v}}{2}$ it is

$$
r_1^{BR}(r_2) = \begin{cases}
\overline{v} & \text{if } r_2 \leq r^{\min} \\
\underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - r_2^2 - 2xr_2} & \text{if } r^{\min} \leq r_2 \leq r^* \\
\left(\frac{\overline{v}-r_2}{r_2-\underline{v}}\right)x & \text{if } r^* \leq r_2 \leq r^{\max} \\
\underline{v} & \text{if } r_2 > r^{\max}.
\end{cases}
$$

To understand the shape of the best response functions, we take the first and second derivative of both parts, and get

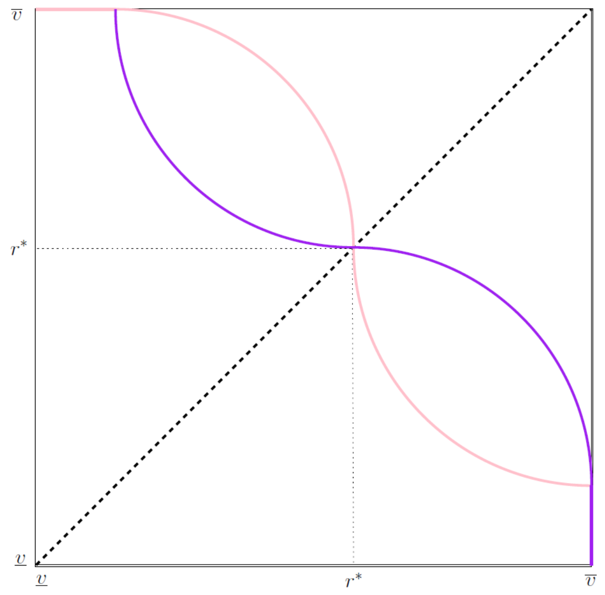
We have

$$\frac{\partial r_1^{BR}(r_2)}{\partial r_2} = \begin{cases} 0 & \text{if } r_2 \leq r^{\min} \\ -\left(\frac{r_2+x}{\sqrt{2\overline{v}x+\underline{v}^2-r_2^2-2xr_2}}\right) & \text{if } r^{\min} \leq r_2 \leq r^* \\ -\left(\frac{(\overline{v}-\underline{v})x}{(r_2-\underline{v})^2}\right) & \text{if } r^* \leq r_2 \leq r^{\max} \\ 0 & \text{if } r_2 > r^{\max}, \end{cases}$$

and

$$\frac{\partial^2 r_1^{BR}(r_2)}{\partial r_2^2} = \begin{cases} 0 & \text{if } r_2 \leq r^{\min} \\ -\left(\frac{2\overline{v}x+\underline{v}^2+r_2^2+2rx+2x^2}{\left(2\overline{v}x+\underline{v}^2-r_2^2-2xr_2\right)\sqrt{2\overline{v}x+\underline{v}^2-r_2^2-2xr_2}}\right) & \text{if } r^{\min} \leq r_2 \leq r^* \\ \frac{2}{r_2-\underline{v}} & \text{if } r^* \leq r_2 \leq r^{\max} \\ 0 & \text{if } r_2 > r^{\max}. \end{cases}$$

Hence, we deduce that the best response function is decreasing in the other bidder's cut-off choice, concave when the other bidder's cut-off choice is less than r^*, convex in the remaining interval.



The slope of the best response function at $r^*$is

$$
\begin{aligned}
\frac{\partial r_2^{BR}(r^*)}{\partial r_1} &= -\left(\frac{(\overline{v}-\underline{v})\,x}{(r^*-\underline{v})^2}\right) \\
&= -\left(\frac{(\overline{v}-\underline{v})\,x}{\frac{1}{4}\left(-\underline{v}-x+\sqrt{4\overline{v}x+(\underline{v}-x)^2}\right)^2}\right) \\
&= -\left(\frac{4\,(\overline{v}-\underline{v})\,x}{\left(-\underline{v}-x+\sqrt{4\overline{v}x+(\underline{v}-x)^2}\right)^2}\right) \\
&= -\left(\frac{2\,(\overline{v}-\underline{v})\,x}{\underline{v}^2+x^2+2\overline{v}x-(\underline{v}+x)\sqrt{4\overline{v}x+(\underline{v}-x)^2}}\right).
\end{aligned}
$$

An illustrative graph is below to show the best response functions:

Also, we see that the slope of the best response functions at r^* is always greater than 1.

$$
\begin{aligned}
\left|\frac{\partial r_2^{BR}(r^*)}{\partial r_1}\right| &> 1 \\
&\Leftrightarrow \left|\frac{2(\overline{v}-\underline{v})x}{\underline{v}^2+x^2+2\overline{v}x-(\underline{v}+x)\sqrt{4\overline{v}x+(\underline{v}-x)^2}}\right|>1 \\
&\Leftrightarrow 2(\overline{v}-\underline{v})x>\left|\underline{v}^2+x^2+2\overline{v}x-(\underline{v}+x)\sqrt{4\overline{v}x+(\underline{v}-x)^2}\right| \\
&\Leftrightarrow 2\overline{v}x-2\underline{v}x>(\underline{v}+x)\sqrt{4\overline{v}x+(\underline{v}-x)^2}-\underline{v}^2-x^2-2\overline{v}x \\
&\Leftrightarrow (\underline{v}-x)^2+4\overline{v}x>(\underline{v}+x)\sqrt{4\overline{v}x+(\underline{v}-x)^2} \\
&\Leftrightarrow \sqrt{4\overline{v}x+(\underline{v}-x)^2}>\underline{v}+x \\
&\Leftrightarrow 4\overline{v}x+(\underline{v}-x)^2>(\underline{v}+x)^2 \\
&\Leftrightarrow 4\overline{v}x>4\underline{v}x \\
&\Leftrightarrow \overline{v}>\underline{v}.
\end{aligned}
$$

Now, we will find the equilibria by intersecting best response functions in each of the following cases:

1. $x \geq \frac{\overline{v}-\underline{v}}{2}$.

   (a) When $r_2 < r^{\min}$, player 1's best response is $\overline{v}$. To $r_1 = \overline{v}$, player 2's best response is $\underline{v} < r^{\min}$. Hence we get a Nash equilibrium: $(\overline{v}, \underline{v})$.

   (b) When $r^{\min} < r_2 < r^*$, player 1's best response is $r_1^{BR}(r_2) = \underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - r_2^2 - 2xr_2}$. To that, player 2's best response is $r_2^{BR}(r_1) = \left(\frac{\overline{v}-r_1}{r_1-\underline{v}}\right)x$. We can look for equilibria by setting:

$$
\begin{aligned}
& r_2^{BR}\left(r_1^{BR}(r_2)\right) = r_2 \\
\Leftrightarrow\ & \left(\frac{\overline{v}-r_1^{BR}(r_2)}{r_1^{BR}(r_2)-\underline{v}}\right)x = r_2 \\
\Leftrightarrow\ & \left(\frac{\overline{v}-\underline{v}-\sqrt{2\overline{v}x+\underline{v}^2-r_2^2-2xr_2}}{\sqrt{2\overline{v}x+\underline{v}^2-r_2^2-2xr_2}}\right)x = r_2 \\
\Leftrightarrow\ & r_2 \in \{r^*, r^*-(\underline{v}+x)\}.
\end{aligned}
$$

The first solution yields the symmetric equilibirum which we had found earlier. For the second solution–call it $r_2^{**} = r^* - (\underline{v} + x)$–we first check that $r_2^{**} \in \left(\max\left\{\underline{v}, r^{\min}\right\}, r^*\right)$. It is immediate from observation that that $r_2^{**} < r^*$. Now we need to check that $r_2^{**} > \max\left\{\underline{v}, r^{\min}\right\}$.

Here, $\max(r^{\min}, \underline{v}) = r^{\min}$, and so we need

$$\begin{aligned} \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right) - (\underline{v} + x) &> \sqrt{x^2 - \overline{v}^2 + 2\overline{v}x + 2\overline{v}\underline{v}} - x \\ &\iff x > \frac{1}{2}\overline{v} - \underline{v}, \end{aligned}$$

this cannot happen since

$$\begin{aligned} x &> \frac{1}{2}\overline{v} - \underline{v} \\ \text{since we have } \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right) - (\underline{v} + x) &> \underline{v} \\ &\implies x > \frac{9}{2}\underline{v} - \underline{v} > \underline{v}. \end{aligned}$$

Hence, if $r^{\min} > \underline{v}$, we do not have a well-defined best response, and cannot have an equilibrium.

(c) When $r_2 = r^*$, player 1's best response is $r_1^{BR}(r_2) = \left(\frac{\overline{v} - r_2}{r_2 - \underline{v}}\right) x$. To that, player 2's best response is $r_2^{BR}(r_1) = \left(\frac{\overline{v} - r_1}{r_1 - \underline{v}}\right) x$. From this case we get the symmetric Nash equilibrium:

$$(r^*, r^*) = \left(\frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right), \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right)\right).$$

(d) When $r^* < r_2 < r^{\max}$, the same as case (b). No equilibrium.

(e) When $r^{\max} < r_2$, we get a Nash equilibrium $(\underline{v}, \overline{v})$.

2. $x \leq \frac{\overline{v} - \underline{v}}{2}$...

(a) *(This case does not exist by definition.)*

(b) When $r_2 < r^*$, player 1's best response is $r_1^{BR}(r_2) = \underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - r_2^2 - 2xr_2}$. To that, player 2's best response is $r_2^{BR}(r_1) = \left(\frac{\overline{v}-r_1}{r_1-\underline{v}}\right)x$. As in case 1(b) above, we set $r_2^{BR}\left(r_1^{BR}(r_2)\right) = r_2$ to yield

$$r_2 \in \{r^*, r^* - (\underline{v} + x)\},$$

The first solution again yields the symmetric equilibirum which we had found earlier. Again, call the second solution $r_2^{**} = r^* - (\underline{v} + x)$–we again must check that $r_2^{**} \in \left(\max\left\{\underline{v}, r^{\min}\right\}, r^*\right)$. In this case ($x \leq \frac{\overline{v}-\underline{v}}{2}$), $\max\left\{\underline{v}, r^{\min}\right\} = \underline{v}$. Again, it is immediate from observation that that $r_2^{**} < r^*$, so we are left to check that;

$$\begin{aligned}
&\underline{v} < r_2^{**} \\
\Leftrightarrow\ &\underline{v} < \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right) - (\underline{v} + x) \\
\Leftrightarrow\ &\frac{3}{2}\underline{v} + \frac{3}{2}x < \frac{1}{2}\sqrt{(\underline{v} - x)^2 + 4\overline{v}x} \\
\Leftrightarrow\ &9\underline{v}^2 + 18\underline{v}x + 9x^2 < \underline{v}^2 - 2\underline{v}x + x^2 + 4\overline{v}x \\
\Leftrightarrow\ &2\underline{v}^2 + 5\underline{v}x + 2x^2 < \overline{v}x \\
\Leftrightarrow\ &\overline{v} > \frac{2\underline{v}^2}{x} + 2x + 5\underline{v}.
\end{aligned}$$

When this condition holds, the implied equilibrium is

$$(r_1^*, r_2^{**}),$$

where

$$r_2^{**} = \frac{1}{2}\left(\underline{v} - x + \sqrt{(\underline{v} - x)^2 + 4\overline{v}x}\right) - (\underline{v} + x)$$

and

$$\begin{aligned} r_1^* &= r_1^{BR}\left(r_2^{**}\right) \\ &= \underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - \left(r_2^{**}\right)^2 - 2xr_2^{**}} \\ &= \frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right) + \left(\underline{v} + x\right). \end{aligned}$$

Now, we should confirm that $r_1^* \in (r^*, r^{\max})$. It is obvious that $r_1^* > r^*$.

$$\begin{aligned} r_1^* &< r^{\max} \\ \Leftrightarrow\ & \frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right) + \left(\underline{v} + x\right) < \frac{\overline{v}x + \underline{v}^2}{\underline{v} + x} \\ \Longleftrightarrow\ & \overline{v} > \frac{2\underline{v}^2}{x} + 2x + 5\underline{v}, \end{aligned}$$

which is the same condition with the one we found in checking the interval of r_2.

Hence, we found an equilibrium

$$\left(r_1^* = r^* + \left(\underline{v} + x\right), r_2^{**} = r^* - \left(\underline{v} + x\right)\right)$$

when $\overline{v} > \frac{2\underline{v}^2}{x} + 2x + 5\underline{v}$.

(c) When $r_2 = r^*$, we again get the symmetric equilibrium

$$(r^*, r^*) = \left(\frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right), \frac{1}{2}\left(\underline{v} - x + \sqrt{\left(\underline{v} - x\right)^2 + 4\overline{v}x}\right)\right).$$

(d) When $r^* < r_2 < r^{\max}$, same as case (b).

(e) When $r^{\max} < r_2$, player 1's best response is $\underline{v}$. To that, player 2's best response is

$$\begin{aligned} r_2^{BR}\left(\underline{v}\right) &= \underline{v} + \sqrt{2\overline{v}x + \underline{v}^2 - r_1^2 - 2xr_1} \\ &= \underline{v} + \sqrt{2\overline{v}x - 2x\underline{v}}. \end{aligned}$$

Hence,

$$(\underline{v}, \underline{v} + \sqrt{2\overline{v}x - 2x\underline{v}})$$

is a Nash equilibrium iff

$$\begin{aligned} \underline{v} + \sqrt{2\overline{v}x - 2x\underline{v}} &> \frac{\overline{v}x + \underline{v}^2}{\underline{v} + x}. \\ &\Leftrightarrow \quad x < \frac{1}{4}\overline{v} - \frac{5}{4}\underline{v} - \frac{1}{4}\sqrt{(\overline{v} - \underline{v})(\overline{v} - 9\underline{v})}. \end{aligned}$$

BIBLIOGRAPHY

Ashlagi, I., D. Monderer & M. Tennenholtz (2011). "Simultaneous Ad Auctions." *Mathematics of Operations Research*, Vol. 36, pp. 1--13.

Burdett, K. & S. Shi & R. Wright (2001). "Pricing and Matching with Frictions." *The Journal of Political Economy*, Vol. 109.

Burdett, K. & K. Judd (1983). "Equilibrium Price Dispersion." *Econometrica*, Vol. 51, 955-970.

Burguet, R. & J. Sakovics (1999). "Imperfect Competition in Auction Designs." *International Economic Review*, Vol. 40.

Coles, M. G. & J. Eeckhout (2000). "Heterogeneity as a Coordination Device", UPF *Economics & Business,* Working Paper.

Celik, G. & O. Yılankaya (2009). "Optimal Auctions with Simultaneous and Costly Participation", *The B.E. Journal of Theoretical Economics*, Vol. 9.

Damianov, S. D. (2010). "Seller competition by mechanism design." *Econ Theory* 51: 105-137.

Gavious, A. (2006). "Separating Equilibria in Public Auctions", Ben-Gurion University of the Negev.

Gerding, E. H., R. K. Dash, A. Byde & N. R. Jennings (2008). "Optimal Strategies for Bidding Agents Participating in Simultaneous Vickrey Auctions with Perfect Substitutes", *Journal of Artificial Intelligence Research*, 32 (2008) 939-982.

Kennes, J. & I. King & B. Julien (2005). "Directed Search without Price Directions", University of Otago, Department of Economics, 2005.

Peters, M. & S. Severinov (1997). "Competition among Sellers Who Offer Auctions Instead of Prices", *Journal of Economic Theory*, 75, 141 179.

Selman, D. (2010). "Coordination Frictions and Heterogeneity in Markets with Bidding." working paper, Boğaziçi University.

Tan, G. & O.Yılankaya (2006). "Equilibria in Second Price Auctions with Participation Costs", *Journal of Economic Theory*, 130 205--219.