RISK SHARING RULES AND INVESTMENT IN GROUPS

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Risk Sharing Rules and Investment in Groups

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DECLARATION OF ORIGINALITY

I, Gökçen Cangüven, certify that

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- this thesis contains no material that has been submitted or accepted for a degree or diploma in any other educational institutions;
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ABSTRACT

Risk Sharing Rules and Investment in Groups

In this study, we consider a game among n investors, who individually choose to borrow a certain amount to invest in a risky project. We assume that the success probabilities for the risky projects are independent and that the individuals can agree upon a fully enforceable sharing rule. We define four major rules, namely Full Liability (FL), Loss Sharing (LS), Profit Sharing (PS) and Equal Sharing (ES), which differ in terms of the sharing of profits and losses across all investors. We compute the equilibrium investment levels and expected social welfare levels and investigate coalition formation structure under these rules. Our theoretical findings show that the game has a unique dominant strategy Nash equilibrium under each rule. Regarding the total equilibrium investment levels, although the ordering of the rules depends on the parameter values, we show a clear supremacy of ES and LS rules over FL and PS rules. Furthermore, through employing numerical analyses, we demonstrate the dominant structure of ES rule over other sharing rules in terms of individual, utilitarian and egalitarian social welfare levels. Lastly, we find that, with certain constraints on model parameters, two agents with identical risk aversion levels are able to form stable coalitions under each rule.

ÖZET

Risk Paylaşım Kuralları ve Grup Yatırımı

Bu çalışmada, n tane yatırımcının, riskli bir projeye, bireysel olarak borçlanarak yaptıkları yatırımı yansıtan bir oyun inceledik. Riskli projelerin başarılı olma olasılığının her oyuncu için birbirinden farklı olduğunu ve yatırımcıların uygulanabilir paylaşım kuralları üzerinde anlaşabileceklerini varsaydık. Yatırımcılar arasında yatırım sonundaki kar ve kayıpların paylaşımını belirleyen, Tam sorumluluk kuralı, Eşit Paylaşım kuralı, Kayıp Paylaşım kuralı ve Kar Paylaşım kuralı olmak üzere dört kural tanımladık. Bu kurallar altında, toplam borçlanma miktarını ve beklenen refah düzeyini hesapladık. Yatırım oyununun, her bir kural altında tek bir Nash dengesi olduğunu gösterdik. Toplam yatırım miktarının farklı parametre değerleri için değişebileceği sonucuna vardık. Buna rağmen, toplam yatırım miktarının, Eşit Paylaşım ve Kayıp Paylaşım kuralları altında, Tam sorumluluk ve Kar Paylaşım kurallarına göre daha fazla olduğunu gözlemledik. Sayısal analiz yöntemleri kullanarak, beklenen refah düzeyinin Eşit Paylaşım kuralı altında, diğer kurallara göre daha fazla olduğu sonucuna vardık. Son olarak, belirli parametre değerleri için, riskten kaçınma parametreleri aynı olan iki yatırımcının sabit koalisyonlar kurduklarını gösterdik.

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TABLE OF CONTENTS

TABLES

CHAPTER 1

INTRODUCTION

Group liability notion is the main factor in microfinance. It is mainly responsible for access of credit to the poor households without any collateral in many developing countries. The principal group lending feature is to solve adverse selection, information asymmetries and moral hazard problems in credit markets. Microfinance institution (MFI) or non-profit, non-governmental organizations (NGO) offer group loans to shift the responsibility to clients who have an incentive to screen and pressurise each other to make sure proper investments levels and enough exerted effort (Gine and Karlan, 2014). Moreover, it is widely accepted that lenders may recover loans and improve repayment rates through group lending structure (Besley and Coate, 1995), since microcredit contracts require group members to take charge of loan repayment. That is, if some of the group members suffer a failure to repay the individual loan amounts, the remaining group members are expected to reimburse those members suffering negative shocks through informal transfers (Fischer, 2013). Otherwise, in case of default, group members as a whole are not allowed for future borrowing. In this sense, joint liability serves as a substitute for collateral. On the other hand, an extensive debate exists about the disadvantages of group liability. First, group lending structure will instigate some group members to free ride. Second, social sanction and peer pressure may discourage borrowers from borrowing and induce a cost in group formation which will in turn affect profitability of lender institution. For instance, by conducting a field experiment, Gine and Karlan (2007) find that lending at individual level rather than group does not affect the repayment rates, but leads to higher lending amounts by attracting new clients. Therefore, due to the pitfalls of group liability lending, the discussion of the lending structure turns into an argument of individual liability lending.

On the other hand, in the axiomatic literature, there exist a large variety of solution concepts of the bankruptcy problem. These concepts describe a claims vector of the liquidation value of a bankrupt rm among its creditors as solution under bankruptcy rules such as Proportionality, Equal awards, Equal losses and some other rules that belong to the TAL-family (Kıbrıs and Kıbrıs, 2013)[1]. For instance, Kıbrıs and Kıbrıs (2013), analyse bankruptcy rules such as, Proportionality, Equal Awards, Equal Losses, Constrained Equal Awards and Constrained Equal Losses rules, in terms of total investment behaviour and social welfare levels, they induce in equilibrium. They also identify why proportionality has been preferred over other rules.

Additionally, Huddart and Liang (2003) consider how variation across partners' preferences, represented by constant absolute risk aversion utility function, affects the partnership structure. They examine sharing rules that specify how the members divide the output which is produced by the efforts of partners, under three information structure.

Originated from group versus individual lending structure of microfinance institutions and allocation questions in partnerships and bankruptcy literature, we consider a game among n investor. Investors individually choose to borrow an amount s to invest in a risky project under a formal ex ante agreement with full contractual enforcement mechanism. Each contract establishes a unique rule, namely Full

[1]Following O'Neill (1982), many studies analyse bankruptcy and taxation problems. For an extensive survey of the axiomatic literature, we refer the reader to Dagan et al.(1997), Schummer and Thomson (1997), Herrero and Villar (2002), Aumann and Maschler (1985), Moulin (1987), Young (1988), Chun (1988), Chambers and Thomson (2002) and Ju et al.(2007).

Liability rule (FL), Loss Sharing (LS), Profit Sharing (PS) and Equal Sharing (ES) rules. Each rule induces a different game among the investors, with CARA preferences, by regulating the division of profits and losses, after the realization of risky investments. Similar to Kıbrıs and Kıbrıs' (2013) analysis, we compare individual borrowing behaviour (and hence investment behaviour) of the group members and compare total equilibrium investment. Moreover, we explore equilibrium social welfare levels and try to clarify coalition formation of the group members who seek to maximize their own expected utility, induced by the specified rules.

In a recent study, Fischer (2013) compared different contract types, namely individual liability, joint liability and equity-like contracts that borrowers are required to adhere to. While under individual liability contracts the transfers are not mandatory, borrowers have to reimburse their partners through informal transfers under joint liability contracts. In addition, under equity-like contracts, borrowers share the resulting profits and losses equally with full commitment. In particular, by conducting a field experiment, Fischer finds a rising tendency in borrowers risk taking behaviour and therefore higher expected returns under equity-like contracts compared to the other contract types. Therefore, the author emphasize that equity-like contracts should be analysed further as *competent contract structure* in microfinance lending.

This thesis aims to shed a light on the variation of investor's investment decisions and expected welfare levels under certain formal contract structures. Moreover, we question whether mandatory group formation structure can turn into a self enforcing coalition under the formal contracts that regulate the allocation of the investment returns. As representation of the individual lending structure, we define Full Liability rule (FL) under which group members are individually responsible for the results of the risky project. In order to reflect group lending structure we define Loss Sharing (LS), Profit Sharing (PS) and Equal Sharing (ES) rules. Under Loss Sharing rule each group member is responsible for only the failure results of other member as well as his loss. Besides, if his investment is successful, he hold his own profit. Under Profit Sharing rule, they equally share only the total profit with each other and bear the loss individually. Finally, under Equal Sharing rule, all agents share the resulting profits and losses equally.

Much of the microfinance literature focuses on the significance of free-riding, moral hazard and repayment issues in relevance to group size, social ties and so forth. However, we abstract our model from these aspects of microfinance lending for reasons of simplicity. Moreover, we assume that there is no group formation cost and effort is costless. Besides, borrowers make an incentive compatiable, implementable contract under full commitment. So that, we merely try to capture the influence of binding formal contracts on investment behaviour, expected welfare levels and coalition formation structure.

Our theoretical findings show that each specific rule induces a unique dominant strategy Nash equilibrium. Despite the fact that the ordering of the rules varies in model parameter space, we demonstrate the supremacy of ES rule and LS rule over PS rule and FL rule, regarding the total investment level. We perform numerical analysis to investigate the expected individual, egalitarian and utilitarian welfares, generated by each sharing rule in equilibrium. The results of the numerical analysis show that ES rule significantly dominates other rules through attaining higher welfare levels for a larger set of parameter combinations. Furthermore, for two-agent coalitions, we infer that identical agents form stable coalitions for certain set of parameters. So that, both agents achieve higher welfare levels in such coalitions than any other coalition structure, including single groups. Lastly, we discuss agents with different risk aversion levels can form coalitions in a sense both of them are better off compared to the individual case, through regulating investment amounts of agents.

CHAPTER 2

MODEL

The modelling framework is based on Kıbrıs and Kıbrıs (2013). Let N={1,....,n} be the set of agents and index i represent the i'th agent where $i \in N$. Each agent's preferences represented by the following *Constant Absolute Risk Aversion (CARA) utility function* $U_i : \mathbb{R} \rightarrow \mathbb{R}$ on money, where $U_i(x) = -e^{-a_i x}$. We assume each agent $i \in N$ is risk averse; i.e, $a_i > 0$ and $a_1 \leq a_2 \leq .. \leq a_n$ and they are identical in other dimensions.

For each member of our rule family *R* , we consider the investment game as follows: Each agent $i \in N$ initially decides to borrow $s_i \in \mathbb{R}_{>0}$ units of wealth by following a strategy s_i from a strategy set $S_i = \mathbb{R}_{>0}$, to invest on a risky project at an interest rate normalized to 0. After consummation of investments, subject to idiosyncratic shocks, each agent will get a positive return with a *success probability* $p \in (0, 1)$ and this value brings a return $r \in (0, 1]$ with a yield $(1 + r)s_i$. In case of success, he is left with a net return $(1 + r)s_i$-s_i=rs_i after the repayment of initially borrowed amount of s_i. If the investment fails, on the other hand, with *failure probability* (1-p), agents receives a payoff 0 and in this case he left with $-s_i$. We assume there is enough collateral for repayment of the initially borrowed amount. We also assume that failure and success probabilities are independent for each agent in the group, but same in magnitude for each them. After, they sign a formal agreement with a specific rule among our family of rules R that initially determine the amount of share for every contingency, before the resolution of uncertainty.

We define *Loss Sharing Rule (LS)*, *Equal Sharing Rule (ES)*, *Profit Sharing Rule (PS)*, and *Full Liability Rule (FL)* to regulate the division of the profits and losses after the realization of each investment. We additionally compare total equilibrium investment and equilibrium social welfare levels and explore coalition formation which depends on individual welfare levels of the group members, induced by the specified rules.

When agents form groups, their binding contract specifies how each realization of the net return will be shared between them. Let G be the set of group members with positive net return rs_i and $i \in N \backslash G$ be the remaining group members who incur losses. Under *Loss Sharing Rule* each agent is responsible only for the total failure results; that is, merely total negative net return will be distributed equally among the group members. More formally, for each $i \in N$, Loss Sharing Rule is defined as;

$$LS_i(s) = \begin{cases} rs_i - \frac{\sum_{j \in N \backslash G} s_j}{n}, & \text{if } i \in G, \\ \frac{-\sum_{j \in N \backslash G} s_j}{n}, & \text{if } i \in N \backslash G, \end{cases}$$

Under *Equal Sharing Rule*, each agent will share the resulting gains and losses equally. More formally, for each $i \in N$, Equal Sharing Rule is defined as;

$$ES_i(s) = \frac{\sum_{j \in G} rs_j - \sum_{k \in N \backslash G} s_k}{n}$$

Under *Profit Sharing Rule*, the total positive net return will be allocated equally among group members and the loss is incurred individually. More formally, for each $i \in N$, Profit Sharing Rule is defined as;

$$PS_i(s) = \begin{cases} \frac{\sum_{j \in G} rs_j}{n}, & \text{if } i \in G, \\ -s_i + \frac{\sum_{j \in G} rs_j}{n}, & \text{if } i \in N \backslash G, \end{cases}$$

Under the *Full Liability Rule*, every agent makes the investment decision and responsible for the results of the risky project individually. That is; every agent left with his own Profit or loss. More formally, for each $i \in N$, Full Liability Rule is defined as;

$$FL_i(s) = \begin{cases} rs_i, & \text{if } i \in G, \\ -s_i, & \text{if } i \in N \backslash G, \end{cases}$$

Note that, every agent is an expected utility maximizer under the initially agreed fully enforceable contract. The following expected utilities illustrate two-agent cases, where agent *i*'s expected payoff from strategy profile s=(s_i,s_{-i}) is;

1) Under equal sharing rule;

$$U_i^{ES}(s) = -\,pp \cdot e^{-a_i\left(\frac{rs_i + rs_{-i}}{2}\right)} - p(1-p) \cdot e^{-a_i\left(\frac{rs_i - s_{-i}}{2}\right)} - (1-p)p \cdot e^{-a_i\left(\frac{rs_{-i} - s_i}{2}\right)} - (1-p)^2 \cdot e^{-a_i\left(\frac{-s_i - s_{-i}}{2}\right)}$$

2) Under full liability;

$$U_i^{FL}(s) = -p \cdot e^{-a_i r s_i} - (1-p) \cdot e^{a_i s_i}$$

3) Under loss sharing rule;

$$U_i^{LS}(s) = -\,p^2 \cdot e^{-a_i r s_i} - (1-p)p \cdot e^{a_i \frac{s_i}{2}} - p(1-p) \cdot e^{-a_i\left(rs_i - \frac{s_{-i}}{2}\right)} - (1-p)^2 \cdot e^{-a_i\left(\frac{-s_i - s_{-i}}{2}\right)}$$

4) Under Profit Sharing rule;

$$U_i^{PS}(s) = -\,pp \cdot e^{-a_i\left(\frac{rs_i + rs_{-i}}{2}\right)} - p(1-p) \cdot e^{-a_i\left(\frac{rs_i}{2}\right)} - (1-p)p \cdot e^{-a_i\left(\frac{rs_{-i}}{2} - s_i\right)} - (1-p)^2 \cdot e^{a_i s_i}$$

CHAPTER 3

EQUILIBRIUM INVESTMENT LEVEL

In this section, we analyse the Nash Equilibria of each simultaneous game under Loss Sharing rule, Equal Sharing rule, Profit Sharing rule and Full Liability rule. We will employ the culminating Nash Equilibria induced under each regulation, to compare the total investment level, the equilibrium social welfare levels and to analyse the coalition formation structure.

Proposition 3.1. (Equilibrium under Full Liability Rule) The investment game under the Full Liability rule (FL) has the dominant strategy equilibrium $s^* = (s_1{}^* ... s_n{}^*)$, where $\forall i \in N$:

$$s_i{}^* = \begin{cases} \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{a_i(1+r)}, & \text{if } rp > (1-p), \\ 0, & otherwise, \end{cases}$$

Proof: See Appendix.

The preceding proposition shows that there exist a unique dominant Nash Equilibrium under the Full Liability rule. By the resulting unique strictly dominant strategy equilibrium, each agent in the group will choose a positive amount level to invest if and only if $rp > (1-p)$. That is; taking an investment decision is optimal if expected positive return suppresses expected loss. Moreover, $s_i{}^*$ is strictly increasing in the net return rate r and strictly decreasing in agent's own risk aversion level a_i. Note that, also, the positive investment level $s_i{}^*$, each member choose at the unique Nash Equilibrium, is independent from the group size and the other's risk aversion levels, which is inherent and consistent with the full liability notion.

The following proposition shows that under Equal Sharing rule, the game has a unique Nash equilibrium.

Proposition 3.2. (Equilibrium under Equal Sharing rule) The investment game under Equal Sharing Rule (EL) has the dominant strategy equilibrium $s^* = ({s_1}^* ... {s_n}^*)$, where $\forall i \in N$:

$$
{s_i}^* = \begin{cases} \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{n}{a_i(1+r)}, & \text{if } rp > (1-p), \\ 0, & otherwise, \end{cases}
$$

Proof: See Appendix.

The above proposition shows that under Equal Sharing rule, the game has a unique Nash equilibrium.Under inequality $rp > (1-p)$, each group member choose a positive investment level. Similar to the Nash Equilibrium induced by the Full Liability rule, this condition is a comparison of the expected positive return with expected loss. ${s_i}^*$ is strictly increasing in net return rate r, strictly decreasing in agent's own risk aversion level a_i. In addition, under Equal Sharing rule,individual investment level depend on the size of the borrowing group. As groups getting larger, the positive individual investment level growing at a constant rate $\ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{a_i(1+r)}$.

Proposition 3.3. (Equilibrium under Loss Sharing rule) The investment game under the Loss Sharing rule (LS) has the dominant strategy equilibrium $s^* = ({s_1}^* ... {s_n}^*)$, where $\forall i \in N$:

$$
{s_i}^* = \begin{cases} \ln\left(\frac{nrp}{(1-p)}\right) \cdot \frac{1}{a_i(\frac{1}{n}+r)}, & \text{if } nrp > (1-p), \\ 0, & otherwise, \end{cases}
$$

Proof: See Appendix.

By the above proposition, the investment game has a unique Nash equilibrium under Loss Sharing rule. Note that, if $nrp > (1-p)$; i.e, total expected positive return outweigh the expected loss, the dominant strategy for all group members to choose a positive investment level s_i^* at the Nash equilibrium. Also, ${s_i}^*$ is strictly decreasing in agent's individual risk aversion level a_i and is strictly increasing as group size getting larger.

Finally, the following proposition shows that under Profit Sharing rule, the investment game has a unique Nash equilibrium.

Proposition 3.4. (Equilibrium under Profit Sharing rule) The investment game under the Profit Sharing rule (PS) has the dominant strategy equilibrium $s^* = ({s_1}^* ... {s_n}^*)$, where $\forall i \in N$:

$${s_i}^* = \begin{cases} \ln\left(\frac{rp}{n(1-p)}\right) \cdot \frac{1}{a_i(\frac{r}{n}+1)}, & \text{if } rp > n(1-p), \\ 0, & otherwise, \end{cases}$$

Proof: See Appendix.

Note that, if $rp > n(1-p)$ the dominant strategy for all group members is to choose a positive investment level s_i^* at the Nash equilibrium. In other words, if expected positive return offset the total expected loss, group members invest in positive amounts. Also, ${s_i}^*$ is strictly decreasing in agent's individual risk aversion level a_i and the group size. Moreover, the individual level of investment s_i^* is strictly increasing in rate of return r at the equilibrium.

CHAPTER 4

COMPARISON OF EQUILIBRIUM TOTAL INVESTMENT

In this section, we'll compare Loss Sharing rule, Equal Sharing rule, Profit Sharing rule and Full Liability rule in terms of the total investment level they induce in equilibrium. The total investment levels under Full Liability, Equal Sharing, Loss Sharing and Profit Sharing rules are; $\sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{(1+r)}$, $\sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{n}{(1+r)}$, $\sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{nrp}{(1-p)}\right) \cdot \frac{1}{(\frac{1}{n}+r)}$ and $\sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{rp}{n(1-p)}\right) \cdot \frac{1}{(\frac{r}{n}+1)}$, respectively. Obviously, $\sum_{i=1}^{n} \frac{1}{a_i}$ is the common multiplier of each total investment level, induced in equilibrium. Moreover, equilibrium individual investment level does not depend on other group member's investment levels and risk aversion levels. Therefore, we focus on comparison of total investment level for appropriate parameter values of p, r and n. Moreover, we mainly investigate the ordering of the four rules where each group member choose to borrow in positive amounts and we only exemplify other cases.

4.1 Two-Agent case analysis

Firstly, note that, for a given return rate, we observe positive investment levels only for sufficiently high values of success probability for each rule. So that, we solely focus on (p,r) values that satisfy the condition $pr > 2(1-p)$. However, for high values of success probabilities, looking at the individual investment levels, a regular ordering is not observed among the specified rules. Indeed, the ordering differs according to the net return rate r and Proposition 4.1. establishes the relationship between the interest rate r and the sorting of the rules with reference to the total investment levels for two agent case.

Proposition 4.1. (n=2) For a given sufficiently high value of p, $\exists r' \; and \; r''$ such that; $\forall r < r'$, the total investment levels for sharing rule is ordered as follows;

$$TI_{LS} > TI_{ES} > TI_{FL} > TI_{PS}$$

and $\forall r' < r < r''$, the total investment levels ordered as;

$$TI_{ES} > TI_{LS} > TI_{FL} > TI_{PS}$$

and $\forall r'' < r$, the total investment levels ordered as;

$$TI_{ES} > TI_{LS} > TI_{PS} > TI_{FL}$$

Proof: See Appendix.

Observe that, LS and ES rules induce higher total investment level than FL and PS rules in each case. Overall, the superiority of ES rule to other rules is apparent, if, for a given p, net return rate r takes sufficiently high values. Otherwise, LS rule generates higher aggregate investment level. On the other hand, when the likelihood of success of the investment project is substantially improved and net return rate r increase, PS rule takes precedence over FL rule. Otherwise, FL rule guarantees higher total investment. As a numerical example, Table 1 depicts total investment levels induced by each sharing rule for different combination parameter values of r and p for two investor case, where $a_1, a_2 \in \{0.05, 0.1, .., 0.9, 0.95\}$. Note that, multiplying each one of the risk aversion coefficient by the same constant wouldn't change the relative ranking of the rules. Accordingly, Table 1 illustrates the arrangement of the sharing rules with respect to the indicated parameter values.

Table 1. Coalition Structures and Investment Levels

a_1, a_2	p, r	ES	LS	FL	PS
$a_1 = 0.1, a_2 = 0.9$	p=0.9, r=1.5	23.13	18.31	11.56	12.12
$a_1 = 0.3, a_2 = 0.7$	p=0.9, r=1	10.46	9.17	5.23	4.77
$a_1 = 0.4, a_2 = 0.6$	p=0.9, r=0.5	8.35	9.15	4.17	2.70
$a_1 = 0.1, a_2 = 0.9$	p=0.8, r=1.5	15.92	13.80	7.96	6.97
$a_1 = 0.3, a_2 = 0.7$	p=0.8, r=1	6.60	6.60	3.30	2.20
$a_1 = 0.4, a_2 = 0.6$	p=0.8, r=0.5	3.85	5.77	1.92	0
$a_1 = 0.1, a_2 = 0.9$	p=0.6, r=1.5	7.20	8.35	3.60	0.74
$a_1 = 0.3, a_2 = 0.7$	p=0.6, r=1	1.93	3.48	0.96	0
$a_1 = 0.4, a_2 = 0.6$	p=0.6, r=0.5	0	1.68	0	0
$a_1 = 0.1, a_2 = 0.9$	p=0.4, r=1.5	0	3.85	0	0
$a_1 = 0.3, a_2 = 0.7$	p=0.4, r=1	0	0.91	0	0
$a_1 = 0.4, a_2 = 0.6$	p=0.4, r=0.5	0	0	0	0

Notice that, when $p = 0.9, r = 1.5$, PS rule induces higher levels of aggregate investment compared to the FL rule, for indicated risk aversion levels. In other cases, we observe a reverse relation. When $p = 0.9, r = 0.5$ and for low values of success probability, LS rule come to the fore in comparison with ES rule. Moreover, in general for low values of p, agents' investment choices are 0, even for the high net return rates. In brief, upon comparing sharing rules for sufficiently high values of p, one can directly make an inference concerning the overall dominance of ES rule over other set of rules.

4.2 N-Agent case analysis

Note that, we only consider (p,r) values that satisfy the condition $pr > n(1 - p)$, for a given n. In fact, sufficiently high values of success probability is still required for group members to choose positive investment levels as the size of the group getting larger. Under PS rule, the domain of the combination of p and r values, inducing positive investment levels, becomes smaller with an increase in group size. In contrast, under LS rule, this domain is expanding as the size of the group getting larger. Even for the smaller success probabilities and lower net return rate, they tend to invest in positive amounts. For ES and FL rules, this domain of p and r values isn't affected by a change in group size. The following proposition states that, one can not observe a clear ordering of the four rules regarding total investment level they induce at equilibrium. We compare total investment levels in terms of net return rate r and group size n, by choosing sufficiently high values of p.

Proposition 4.2. (n agent case) Let p takes sufficiently high values. Then for a given $r \in (0, 1)$ and for all $n > 1$, ES and LS rules induces higher investment levels than PS and FL rules. Moreover, for a given $r \in (0, 1)$,

i) $\exists n^* \in \mathbb{R}_{>0}$ such that $\forall n \geq n^*$, $TI_{FL} \geq TI_{PS}$, and for all $1 \leq n \leq n^*$ $TI_{PS} \geq TI_{FL}$, where $n \in \mathbb{Z}_{>0}$.

ii) $\exists n^{**} \in \mathbb{R}_{>0}$ such that $\forall n \geq n^{**}$, $TI_{ES} \geq TI_{LS}$, and for all $1 \leq n \leq n^{**}$ $TI_{LS} \geq TI_{ES}$, where $n \in \mathbb{Z}_{>0}$.

Proof: See appendix.

Proposition 6 states that the parameter values the return rate r and group size n place bounds on the arrangement of our family of rules relating to total investment levels, for given sufficiently high values of p. Analogical to the two agent case, the cumulative investment levels under ES rule and LS rule exceed that of under PS and FL rules. Moreover, for the relevant return rates, group size mainly determines the ordering between PS rule and FL rule and that of ES rule and LS rule as stated in case i) and ii), respectively. Table 2 is a numerical illustration of the proposition 6 for different combination parameter values of n, r and p where $a_1, a_2 \in \{0.05, 0.1, .., 0.9, 0.95\}$. Again notice that, rescaling risk aversion coefficients by the same constant wouldn't affect the relative ordering of the rules.

Table 2. Coalition Structures and Investment Levels

a_1, a_2	n, p, r	ES	LS	FL	PS
$a_1 = 0.3, a_2 = 0.8$	n=1, p=0.9, r=0.5	4.59	4.59	4.59	4.59
$a_1 = 0.2, a_2 = 0.4$	n=2, p=0.9, r=1.5	15.61	12.35	7.80	8.18
$a_1 = 0.2, a_2 = 0.4$	n=4, p=0.9, r=1.5	31.23	17.09	7.80	6.63
$a_1 = 0.2, a_2 = 0.4$	n=2, p=0.9, r=0.5	15.04	16.47	7.52	4.86
$a_1 = 0.2, a_2 = 0.4$	n=4, p=0.9, r=0.5	30.08	28.90	7.52	0.78
$a_1 = 0.3, a_2 = 0.8$	n=3, p=0.85, r=1.5	11.77	8.09	3.92	3.18
$a_1 = 0.3, a_2 = 0.8$	n=7, p=0.85, r=1.5	27.46	11.39	3.92	0.73
$a_1 = 0.3, a_2 = 0.8$	n=3, p=0.85, r=0.5	9.54	11.77	3.18	0
$a_1 = 0.3, a_2 = 0.8$	n=7, p=0.85, r=0.5	22.27	21.29	3.18	0
$a_1 = 0.4, a_2 = 0.8$	n=2, p=0.7, r=1.5	3.75	3.64	1.87	1.19
$a_1 = 0.4, a_2 = 0.8$	n=4, p=0.7, r=1.5	7.51	5.65	1.87	0
$a_1 = 0.4, a_2 = 0.8$	n=2, p=0.7, r=0.5	0.77	3.17	0.38	0
$a_1 = 0.4, a_2 = 0.8$	n=4, p=0.7, r=0.5	1.54	7.70	0.38	0

In our example, when $n = 1$, obviously all rules agree with the FL Rule. Consequently, aggregate investment level depends on merely to the parameter values of p and r. Moreover, when $p = 0.9, r = 1.5$, for the two investor case PS rule generates higher level of total investment than FL rule. While, as the group size rise up to four, FL surpasses PS rule for the same parameter values of r and p. On the other hand, for $p = 0.9$, $r = 0.5$, $n = 2$ and $p = 0.85$, $r = 0.5$, $n = 3$, agents choose to invest in greater amounts under LS rule in contrast to ES rule. However, as group size increases to $n = 4$ and $n = 7$, respectively, ES rule outweighs LS rule in accordance with aggregate investment level for the relevant p and r values. Besides, it's apparent that ES rule possesses preeminence among other rules.

CHAPTER 5

EQUILIBRIUM WELFARE

In this section we compare LS, ES, PS and FL rules by focusing on equilibrium individual, egalitarian and utilitarian social welfare levels. Particularly, we elaborate two-agent case for each sharing rule and perform numerical analyses to provide a pairwise comparison of each rule for different parameter values under different welfare levels. Note that, we particularly focus on appropriate values of success probability where in overall, investors choose to borrow in positive amounts for each rule. So that, to make the comparison more interesting we exclude low values of p, where the unique dominant Nash equilibrium strategies are $(s_1, s_2) = (0, 0)$.

Here we use numerical analysis to examine the pairwise relationship of the rules at equilibrium for two agent case with respect to four parameters of interest. We take two rules and compare them according to the individual welfare levels they induce at equilibrium. Then, we report overall percentage of the parameter combinations where the corresponding rules generates strictly higher individual welfare levels compared to the other rule. Particularly, we restrict our attention to a set of variables, where $p \in \{0.4, 0.41, .., 0.98, 0.99\}$, $r \in \{0.01, 0.03, .., 1.47, 1.49\}$ and $a_1, a_2 \in \{0.05, 0.1, .., 0.9, 0.95\}$ and we obtain 1624500 parameter combinations. Note that, the set of risk aversion levels can be scaled up. Indeed, the ordering of the four rules is invariant under such rescaling of risk aversion parameters due to the analytical structure of the objective utility function.

Observation 1. In Table 3, we employ a pairwise comparison of individual welfare levels for the specified rules where group size $n = 2$. In each row, we show the percentage of the size of the parameter combinations where the corresponding rules induces higher welfare levels for both agents.

Table 3. Pairwise Individual Welfare Comparison

Rules of Interest	ES	LS	PS	FL
ES-LS	0.46	0.09		
ES-PS	0.55		0.01	
ES-FL	0.55			0
LS-PS		0.18	0.09	
LS-FL		0.22		0.25
PS-FL			0.09	0.34

Note that, under the observation 1, we solely focus on size of the parameter space where both agent's individual welfare levels are higher for the related sharing rules. So that, we ignore other cases where at most one investor is better off and the other is worse off. Table 3 is a clear demonstration of the dominance of the ES rule over other rules in terms of individual welfare levels. Namely, both agent's expected individual welfares are strictly higher under ES rule compared to LS, PS and FL rules for a significantly larger set of parameters. Clearly, ES generates higher welfare levels than PS and FL rules, for more than half of combination of the values (55%). While this percentage is 46% compared to the LS rule. Indeed, FL rule isn't preferable, since it induces strictly higher welfare level for none of the parameter values. In fact, we state and prove the supremacy of ES rule over FL rule, for a given parameter combination in Lemma 5.0.1. On the other hand, we can explicitly observe the precedence of LS rule over PS rule. Although, for 18 percent of parameter space, LS rule generates higher welfare levels, this portion is considerably small for PS rule which is 0.09 percent. Additionally, the percentage share of the set of the parameter combination slightly higher under FL rule (25%), compared to the LS rule (22%). Evidently, FL rule generates higher individual welfare levels compared to PS rule for larger set of parameters (34%).

Lemma 5.0.1. For a given combination of (p, r) where $pr > (1 - p)$, ES rule generates higher individual welfare levels for both agents in any coalition compared to FL rule.

Proof: See Appendix.

We now discuss the relevance of individual welfare levels with respect to risk aversion levels. Actually, we'll use these inferences of relations as bases, when we investigate the egalitarian and utilitarian welfare levels for identical agents. The following lemma states a general comparison of the individual utility levels for the specified risk aversion levels.

Lemma 5.0.2. (Individual Welfare Comparison where n=2) Assume $a_1 \geq a_2$ and $({s_1}^*, {s_2}^*) > (0, 0)$ be the Nash equilibrium investment levels. Then,

(i) ${U_1}^{FL}({s_1}^*, {s_2}^*) = {U_2}^{FL}({s_1}^*, {s_2}^*)$.

(ii) ${U_1}^{ES}({s_1}^*, {s_2}^*) \leq {U_2}^{ES}({s_1}^*, {s_2}^*)$, where equality satisfied only when $a_1 = a_2$.

(iii) ${U_1}^{LS}({s_1}^*, {s_2}^*) \leq {U_2}^{LS}({s_1}^*, {s_2}^*)$, where equality satisfied only when $a_1 = a_2$.

(iv) ${U_1}^{PS}({s_1}^*, {s_2}^*) \geq {U_2}^{PS}({s_1}^*, {s_2}^*)$, where equality satisfied only when $a_1 = a_2$.

Proof: See Appendix.

5.1 Egalitarian Social Welfare Levels

There are various theoretical measures of a society's collective utility i.e. its social welfare. Various social welfare functions have been suggested, that are functions of a society's individual level utilities, one of which is egalitarian measure of society's welfare. The egalitarian social welfare level, induced by a specific rule R at the Nash equilibrium, equals the utility of the individual who is worst off. More formally;

$$U_{EGL}{}^{R}(p, r, a_1, a_2) = min\{U_1^R(s_1{}^*, s_2{}^*), U_2^R(s_1{}^*, s_2{}^*)\}$$

Similar to the individual welfare case, the ordering of the specified rules regarding the egalitarian social welfare levels contingent upon the parameter values of success probability, net return rate and risk aversion levels. So that, we investigated the relation of the ordering of the rules via carrying out a numerical analyses subject to predefined parameter values of p, r, a_1 and a_2, for the two investor case.

Observation 2. In Table 4, we employ a pairwise comparison of egalitarian welfare levels for the specified rules where group size $n = 2$. In each row, we show the percentage of the size of the parameter combinations where the corresponding rules induces higher welfare levels.

Table 4. Pairwise Egalitarian Welfare Comparison

Rules of Interest	ES	LS	PS	FL
ES-LS	0.83	0.17		
ES-PS	0.77		0.23	
ES-FL	0.55			0.45
LS-PS		0.49	0.51	
LS-FL		0.22		0.78
PS-FL			0.09	0.91

In a similar manner as previous numerical and theoretical comparisons, we see a clear supremacy of ES over other sharing rules concerning egalitarian social welfare levels. Explicitly, the size of the parameter space is nearly four times larger under ES rule (83%) than LS rule (17%). Additionally, while the corresponding percentage of the range of parameters that ES rule generates higher welfare levels is 54% more than that of generated under PS rule; this difference is only 10% for the FL rule. Moreover, the percentage share is nearly the same for PS rule (51%) and LS rule (49%). Obviously, FL rule clearly induces higher egalitarian welfare levels for a larger set of values (78%) compared LS rule (22%). Besides these, the parameter space, where FL rule induces higher egalitarian social welfare level, is substantially larger (91%) than that of induced under PS rule (9%).

We next provide an analysis for the two investor case where agents are identical in terms of risk aversion. Obviously, at the symmetric case, both agents achieve the same individual welfare levels under each rule. Since agent's individual welfare under FL rule is independent of a_i; egalitarian welfare level inherently does not depend on risk aversion levels, which is only depend on success probability p and net return rate r. For ES, LS and PS rules, when agents are identical in terms of risk aversion level, they attain the same and maximum egalitarian welfare levels (see Figure 1). This fact is generalized in the following proposition.

Proposition 5.1.1. (Egalitarian Welfare Comparison where n=2) Assume $({s_1}^*, {s_2}^*) > (0, 0)$ be the Nash equilibrium investment levels. Then, egalitarian welfare level, induced by specific rule R, where agents choose to follow positive dominant Nash equilibrium strategy, is maximized when $a_1 = a_2$.

Proof: By definition of ${U_{EGL}}^R(p, r, a_1, a_2) = min\{U_1^R({s_1}^*, {s_2}^*), U_2^R({s_1}^*, {s_2}^*)\}$, it's clear that the proof is a direct implication of Lemma 5.0.2.

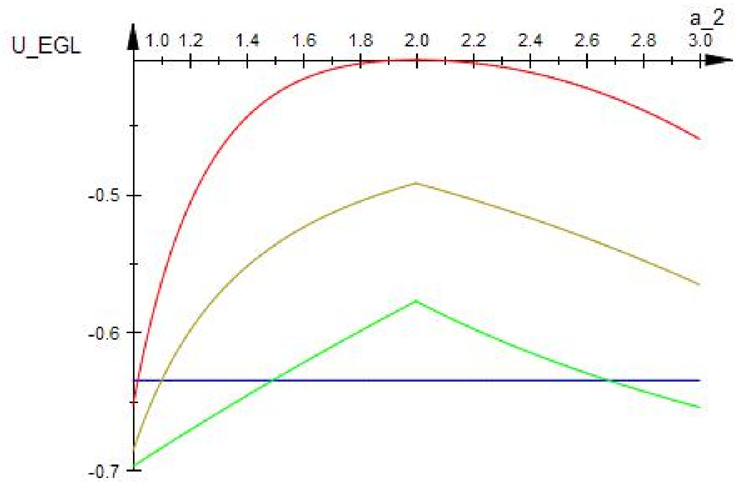

Fig. 1: Egalitarian welfare levels under FL (blue), PS (green), LS (brown) and ES (red) for the parameter values are p=0.9, r=0.9 and $a_1 = 2$.

5.2 Utilitarian Social Welfare Levels

In this part, we consider utilitarian welfare levels which sums the utility of each individual in order to obtain group's overall welfare. More formally, the total utility of a group, formed by two agents, choose to follow the equilibrium investment level strategy is defined as follows;

$$U_{UTL}{}^{R}(p, r, a_1, a_2) = U_1^R({s_1}^*, {s_2}^*) + U_2^R({s_1}^*, {s_2}^*)$$

Similar to the previous cases, we identify the ordering of the specified rules by using a numerical method, for the determinated 1624500 combination of parameter space of success probability p, net return rate r and risk aversion levels of the agents. We simply focus on two agent case to examine the comparison of our family of rules.

Observation 3. In Table 5, we employ a pairwise comparison of utilitarian welfare levels for the specified rules where group size $n = 2$. In each row, we show the percentage of the size of the parameter values from the set of all possible combinations of values where the corresponding rules induces higher welfare levels.

Table 5. Pairwise Utilitarian Welfare Comparison

Rules of Interest	ES	LS	PS	FL
ES-LS	0.80	0.20		
ES-PS	0.77		0.23	
ES-FL	0.61			0.39
LS-PS		0.49	0.51	
LS-FL		0.34		0.66
PS-FL			0.35	0.65

Table 5 clearly depicts that ES still retains the priority among other set of rules, associated with the utilitarian welfare level it induces at equilibrium. ES rule explicitly dominates LS rule through achieving a larger set of parameter space where utilitarian welfare level is higher (80%) than LS rule(20%). Moreover, the parameter set in which ES rule induces higher welfare levels, account for 77% of our parameter space; while this ration is only 23% for PS rule. Although, we can see a contraction of the range of the parameter values as to LS rule and PS rule case. Namely, ES rule still generates higher welfare levels (61%) than FL rule (39%). Besides these, PS rule and FL rule surpasses LS rule, through constituting a larger share of our set of variables which are 51% and 66%, respectively. Also, FL rule outweighs PS rule, by inducing higher welfare levels for a wider set of parameter values.

We end up this section with presenting a discussion similar to the previous cases where agents have identical risk aversion levels. It's apparent that utilitarian welfare level does not depend on risk aversion levels, which is constant under Full Liability Rule. As can be seen from Figure 2, the relation between ES, LS, FL shows

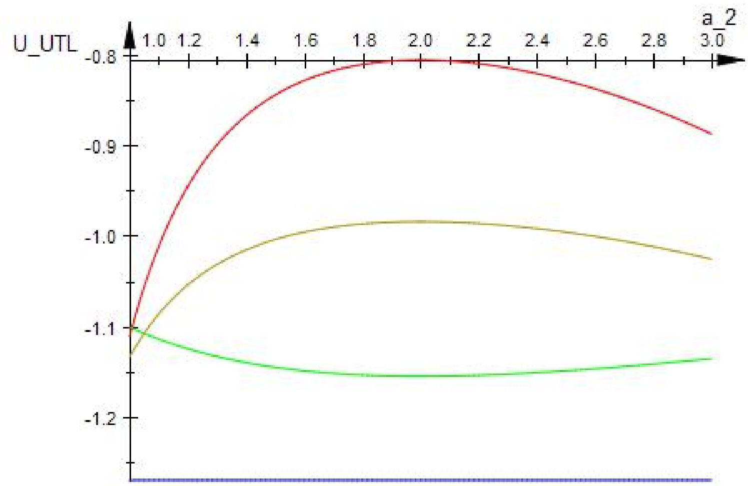

Fig. 2: Utilitarian welfare levels under ES (blue), LS (green), FL (brown) and PS (red) for the parameter values are p=0.9, r=0.9 and $a_1 = 2$

similar patterns like egalitarian case with the exception of PS rule. Indeed, utilitarian welfare levels reach the maximum amount at the symmetric case, i.e, when $a_1 = a_2 = 2$. As opposed to this fact, the welfare level under Profit Sharing rule is minimized when agents are identical in terms of risk aversion. This fact is stated in Proposition 5.2.1..

Proposition 5.2.1. (Utilitarian Welfare Comparison where n=2) Assume $({s_1}^*, {s_2}^*) > (0, 0)$ be the Nash equilibrium investment levels. Then, utilitarian welfare level, induced by Equal Sharing, Loss Sharing and Full Liability rules is maximized and induced by Profit Sharing rule is minimized when $a_1 = a_2$.

Proof: By definition of ${U_{UTL}}^R(p, r, a_1, a_2) = U_1^R({s_1}^*, {s_2}^*) + U_2^R({s_1}^*, {s_2}^*)$, it's clear that the proof is direct implication of Lemma 5.0.2.

CHAPTER 6

COALITION FORMATION

In this section, we investigate a group formation model, in which agents belonging in same population with independent outcomes, drawn from some joint distribution. Agents form groups to maximize their ex ante utility under a prespecified rule. Within this population, agents are ranked by their risk aversion level in the Arrow-Pratt sense. That is, for any two agent, one is more risk averse than the other according to their risk aversion. We also allow group members be homogeneous with the same risk preferences and be heterogeneous with distinct risk preferences. Moreover, we assume there is equal mass of borrowers of every type. We identify equilibrium allocations of agents to specify the composition and size of these groups by comparing each agent's utility level. More specifically, we shall focus attention on core partition P (i.e., group assignments of individuals with possibly some single groups). These partitions are stable in a sense that there is no any other group formation possibility for matched or unmatched rational individuals such that any individual can improve his payoff by getting involve in another risk-sharing group or by becoming unmatched. More formally;

Definition: A coalition structure $\Pi = \{P_1, P_2, .., P_S\}$ $(S \leq |N| is$ a positive integer) is a partition of N. That is, $P_s \neq \emptyset$ for any s $\in \{1, .., S\}$, $\bigcup_{s=1}^{S} P_s = N$and $P_i \cap P_j = \emptyset$ for any $i, j \in \{1, .., S\}$ with $i \neq j$. For some coalition structure Π and any player s let $\Pi(s) = \{P \in \Pi : s \in P\}$ be the set of her partners. Denote the collection of all coalition structures in N by $\psi(N)$. Similarly define $\psi(T)$ for any $T \subset N$ with $T \neq \emptyset$.

Note that, each player's expected payoff depends only on the members investment decision of his coalition. That is there is no spillover affect or payoff externalities that influence the coalition members' payoffs. Moreover, we also assume there is no cost of forming groups and no incentive for players to free-ride which is mediated through fully enforceable contracts.

6.1 Characterization

For analytical tractability, we focus on the two agent case. Note that individuals who remain single receive their random income $-s_i$ or rs_i. Since everyone is an expected utility maximizer, income x yields $U_i(x)$.

Moreover, recall that if agents $\{i, j\}$ choose to match and share risk, they can sign a binding agreement, ex-ante, prior to the realization of their outcomes under Equal Sharing, Loss Sharing and Profit Sharing Rules that specify how the resulting returns will be shared. By focusing on individual rationality behaviour, we compare each group members ex-ante payoffs under the agreed contract upon a specified sharing rule to their respective ex-ante payoffs when single.

Example 1. (Equal Sharing Rule) Let $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.3$ and $a_4 = 0.5$ be the risk aversion levels of agents. Table 6 illustrates some coalition structures and related expected payoffs, by denoting agents with their risk aversion levels where $p = 0.8$ and $r = 0.3$.

As can be seen from Table 6, in general when two agent form coalition, the utility of the risk loving agent is higher; although, the more risk averse agent strictly worse off compared to the single case. But in some cases, like the group formation structure of $\{a_2 = 0.2, a_3 = 0.3\}$, both agents are strictly better off, for $p = 0.8$ and $r = 0.3$ values. Although the group structure is changing for the different parameter values and the risk aversion levels, under ES rule expected utility levels increases, as the difference between risk aversion levels of the agents in the same group getting smaller. Moreover, it's a general fact that the utility levels of the agents with identical risk aversion are maximized under Equal Sharing rule. Indeed, agents prefer not to form groups with those who is different in terms of risk aversion and prefer not to stay single due to an decrease in expected welfare levels. We state this fact in Proposition 6.1..

Table 6. Coalition Structures and Related Expected Payoffs under ES

Π_{ES} \ a_i	$a_1 = 0.1$	$a_2 = 0.2$	$a_3 = 0.3$	$a_4 = 0.5$
$\{\{a_1\},\{a_2\}\}$	-0.9971	-0.9971		
$\{\{a_1,a_2\}\}$	-0.9950	-0.9973		
$\{\{a_1,a_1\}\}$	-0.9943			
$\{\{a_2\},\{a_4\}\}$		-0.9971		-0.9971
$\{\{a_2,a_4\}\}$		-0.9953		-1.001
$\{\{a_2,a_2\}\}$		-0.9943		
$\{\{a_1\},\{a_3\}\}$	-0.9971		-0.9971	
$\{\{a_1,a_3\}\}$	-0.9955		-1.0068	
$\{\{a_3,a_3\}\}$			-0.9943	
$\{\{a_2\},\{a_3\}\}$		-0.9971	-0.9971	
$\{\{a_2,a_3\}\}$		-0.9946	-0.9950	
$\{\{a_4,a_4\}\}$				-0.9943

Proposition 6.1. (n=2) For p and r values which satisfy the condition $pr > 1 - p$, $\forall(p, r)$, agents form stable coalitions under Equal Sharing rule whenever $a_1 = a_2$.
Proof: See Appendix.

We also explore the consequences of external regulation of limiting or augmenting investment amounts for the two agent group formation case. Accordingly, we put upper bound for the risk loving agent's allowed investment level on $(s_1{}^*, s_2{}^*)$. In this case, his utility is strictly decreasing if this bound approaches the risk averse agents individual investment level $s_2{}^*$; in contrast, the risk averse agent gains from this limitation. On the other hand, if we force the risk averse agent to borrow in large amounts on $({s_1}^*, {s_2}^*)$ through putting lower bounds, his utility decreases as this bound getting larger. Although, this regulation causes risk loving agent's utility to increase on the defined interval. This fact is stated in Lemma 6.1.1..

Lemma 6.1.1. For $a_1 < a_2$;

i) If ${s_2}^* < \overline{s_1} < {s_1}^*$, then ${U_1}^{ES}(\overline{s_1}, {s_2}^*)$ decreases strictly and ${U_2}^{ES}(\overline{s_1}, {s_2}^*)$ increases strictly as $\overline{s_1}$ decreases on $({s_2}^*, {s_1}^*)$.

ii) If ${s_2}^* < \underline{s_2} < {s_1}^*$, then ${U_1}^{ES}({s_1}^*, \underline{s_2})$ increases strictly and ${U_2}^{ES}({s_1}^*, \underline{s_2})$ decreases strictly as $\underline{s_2}$ increases on $({s_2}^*, {s_1}^*)$.

Proof: See Appendix.

In fact, we show that, whenever $a_1 < a_2$ and $({s_1}^*, {s_2}^*) > (0, 0)$, ${U_1}^{ES}({s_1}^*, {s_2}^*) > {U_2}^{ES}({s_1}^*, {s_2}^*)$ (see Lemma 5.0.1.). Then, by putting an upper bound for relatively risk loving agent ($\overline{s_1} < {s_1}^*$), while the other agent chooses to borrow the Nash equilibrium investment level, external enforcement mechanisms can turn into self enforcing informal arrangements under Equal Sharing rule. Since both agents are better off in such coalition structure compared to the individual case. For instance, for $p = 0.8$ and $r = 0.3$ values when $a_1 = 0.2$, $a_2 = 0.5$, individual investment levels are $({s_1}^*, {s_2}^*) = (1.40, 0.56)$ and their respective welfare levels are $({U_1}^{ES}({s_1}^*, {s_2}^*), {U_2}^{ES}({s_1}^*, {s_2}^*) = (-0.9953, -1.001)$. When we limit the relatively risk loving agents borrowing amount and make them invest $(\overline{s_1}, {s_2}^*) = (1, 0.56)$, in this case their welfare levels are $({U_1}^{ES}(\overline{s_1}, {s_2}^*), {U_2}^{ES}(\overline{s_1}, {s_2}^*) = (-0.9955, -0.9961)$. Notice that, although first agent is a bit worse off compared to the initial case, both agent's utility levels are strictly higher than individual utility level which is ${U_i}^{FL}({s_i}^*) = -0.9971$, for $i = 1, 2$.

Example 2. (Loss Sharing Rule) Let $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.3$ and $a_4 = 0.5$ be the risk aversion levels of agents. Table 7 illustrates some coalition structures and related expected payoffs, by denoting agents with their risk aversion levels where $p = 0.9$ and $r = 0.9$.

Table 7. Coalition Structures and Related Expected Payoffs under LS

Π_{LS} \ a_i	$a_1 = 0.1$	$a_2 = 0.2$	$a_3 = 0.3$	$a_4 = 0.5$
$\{\{a_1\},\{a_2\}\}$	-0.6348	-0.6348		
$\{\{a_1,a_2\}\}$	-0.4476	-0.6859		
$\{\{a_1,a_1\}\}$	-0.4922			
$\{\{a_2\},\{a_4\}\}$		-0.6348		-0.6348
$\{\{a_2,a_4\}\}$		-0.4411		-0.8840
$\{\{a_2,a_2\}\}$		-0.4922		
$\{\{a_1\},\{a_3\}\}$	-0.6348		-0.6348	
$\{\{a_1,a_3\}\}$	-0.4371		-1.2098	
$\{\{a_3,a_3\}\}$			-0.4922	
$\{\{a_2\},\{a_3\}\}$		-0.6348	-0.6348	
$\{\{a_2,a_3\}\}$		-0.4601	-0.5655	
$\{\{a_4,a_4\}\}$				-0.4922

Table 7 depicts, when two agent with different risk aversion levels form coalitions, generally, the expected utility of the relatively risk loving agent is higher. While, the more risk averse agent is strictly worse off compared to the single case. Yet, like the group formation structure under Equal Sharing Rule, under the two agent coalition $\{a_2 = 0.2, a_3 = 0.3\}$, payoffs of both agent are higher for $p = 0.9$ and $r = 0.9$ values, compared to individual payoffs. Similar to the Equal Sharing case, the expected utility levels of agent depends on the group structure and parameter values. For the two agent case, under Loss Sharing rule, an increase in the other agent's risk aversion level has opposite effects on the utility levels of two agents. For instance, as the second agent's risk aversion level increase, relatively risk loving first agent's willingness to match with a safer investor increase. On the other hand, the same variation in the agent 2's risk aversion level refrain him to match with a potential risk seeking partner because of an decrease in his welfare level. Proposition 6.2. states a general fact for the described values of p and r.

Proposition 6.2. (n=2) For p and r values which satisfy the condition $pr > 1 - p$, $\forall(p, r)$, agents form stable coalitions under Loss Sharing rule whenever $a_1 = a_2$.

Proof: See Appendix.

Moreover, we investigate the implications of external regulation of investment levels for Loss Sharing rule. Similar to the equal sharing case, we put upper bound on the investment level for relatively risk loving agent on $({s_1}^*, {s_2}^*)$. A decrease of this upper bound results in a decrease in the individual welfare of the less risk averse agent. While, this regulation lead to an increase in the more risk averse agent individual utility level. Furthermore, putting an upper bound on borrowing amounts for the more risk averse agent causes a loss in both agent's individual utility levels. We state and prove this fact in Lemma 6.1.2..

Lemma 6.1.2. For $a_1 < a_2$;

i) If ${s_2}^* < \overline{s_1} < {s_1}^*$, then ${U_1}^{LS}(\overline{s_1}, {s_2}^*)$ decreases strictly and ${U_2}^{LS}(\overline{s_1}, {s_2}^*)$ increases strictly as $\overline{s_1}$ decreases on $({s_2}^*, {s_1}^*)$.

ii) If ${s_2}^* < \underline{s_2} < {s_1}^*$, then ${U_1}^{LS}({s_1}^*, \underline{s_2})$ and ${U_2}^{LS}({s_1}^*, \underline{s_2})$ both decreases strictly as $\underline{s_2}$ increases on $({s_2}^*, {s_1}^*)$.

Proof: See Appendix.

By Lemma 5.0.1, when agents choose to follow interior Nash Equilibrium strategies, if $a_1 < a_2$, then ${U_1}^{LS}({s_1}^*, {s_2}^*) > {U_2}^{LS}({s_1}^*, {s_2}^*)$. Then, through putting upper bound on the relatively risk seeking agent's borrowing amounts, two agent form coalitions such that both of them better off with respect to the individual case.

As an illustration of this fact, for $p = r = 0.9$ and $a_1 = 0.1$, $a_2 = 0.2$ parameter values, individual investment levels are $({s_1}^*, {s_2}^*) = (19.89, 9.94)$ and the corresponding utility levels are $({U_1}^{LS}({s_1}^*, {s_2}^*), {U_2}^{LS}({s_1}^*, {s_2}^*))$ $= (-0.4476, -0.6859)$. If, the less risk averse agent is forced to borrow $\overline{s_1} = 15$, although other agent still choose to follow Nash equilibrium strategy; then individual utilities are $({U_1}^{LS}(\overline{s_1}, {s_2}^*), {U_2}^{LS}(\overline{s_1}, {s_2}^*) = (-0.4736, -0.5670)$. Clearly, in such coalition both agent's are better off compared to the individual utility level which is ${U_i}^{FL}({s_i}^*) = -0.6348$, for $i = 1, 2$.

Example 3. (Profit Sharing Rule) Let $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.3$ and $a_4 = 0.5$ be the risk aversion levels of agents. Table 8 illustrates some coalition structures and related expected payoffs, by denoting agents with their risk aversion levels where $p = 0.9$ and $r = 0.9$.

Unlike previous cases, when two agent with different risk aversion levels form coalitions which is illustrated in Example 3, the expected utility of the relatively risk loving agent is lower. Despite, the more risk averse agent strictly better off compared to the individual case. When they come together to form groups with agents who have identical risk aversion levels, we observe a rising tendency in individual expected utility levels. Moreover, under Profit Sharing rule, as relatively more risk averse agent's risk aversion level increases, more risk averse agent's expected utility level increase. However, such an increase cause a decline in his partner's expected utility level. Proposition 6.3. generalizes this fact for two agent group formations under PS rule.

Table 8. Coalition Structures and Related Expected Payoffs under PS

Π_{PS} \ a_i	$a_1 = 0.1$	$a_2 = 0.2$	$a_3 = 0.3$	$a_4 = 0.5$
$\{\{a_1\},\{a_2\}\}$	-0.6348	-0.6348		
$\{\{a_1,a_2\}\}$	-0.6969	-0.4039		
$\{\{a_1,a_1\}\}$	-0.5775			
$\{\{a_2\},\{a_4\}\}$		-0.6348		-0.6348
$\{\{a_2,a_4\}\}$		-0.7241		-0.3416
$\{\{a_2,a_2\}\}$		-0.5775		
$\{\{a_1\},\{a_3\}\}$	-0.6348		-0.6348	
$\{\{a_1,a_3\}\}$	-0.7429		-0.2914	
$\{\{a_3,a_3\}\}$			-0.5775	
$\{\{a_2\},\{a_3\}\}$		-0.6348	-0.6348	
$\{\{a_2,a_3\}\}$		-0.6542	-0.4813	
$\{\{a_4,a_4\}\}$				-0.5775

Proposition 6.3. (n=2) For p and r values which satisfy the condition $pr > 2(1-p)$, $\forall(p,r)$, agents form stable coalitions under Profit Sharing rule whenever $a_1 = a_2$.

Proof: See Appendix.

We explore the consequences of putting an upper bound and lower bound on relatively less risk averse and more risk averse agent's borrowing amounts. For the Profit Sharing rule, limiting the borrowing amount of less risk averse agent result in an reduction in both agent's utility level. On the contrary, impelling more risk averse agent to borrow in higher amounts than the Nash equilibrium amount lead to a loss in his utility levels. Nevertheless, this enforcement mechanism result in an increase in utility of the relatively risk loving agent. Lemma 6.1.3. states this fact;

Lemma 6.1.3. For $a_1 < a_2$;

i) If ${s_2}^* < \overline{s_1} < {s_1}^*$, then ${U_1}^{PS}(\overline{s_1}, {s_2}^*)$ and ${U_2}^{PS}(\overline{s_1}, {s_2}^*)$ decreases strictly, as $\overline{s_1}$ decreases on $({s_2}^*, {s_1}^*)$.

ii) If ${s_2}^* < \underline{s_2} < {s_1}^*$, then ${U_1}^{PS}({s_1}^*, \underline{s_2})$ increases strictly and ${U_2}^{PS}({s_1}^*, \underline{s_2})$ decreases strictly, as $\underline{s_2}$ increases on $({s_2}^*, {s_1}^*)$.

Proof: See Appendix.

We end up this section by an implication of the preceding proposition. In Lemma 5, we show that at the Nash Equilibrium, if $a_1 < a_2$, then ${U_1}^{PS}({s_1}^*, {s_2}^*) < {U_2}^{PS}({s_1}^*, {s_2}^*)$. Then as an application of Proposition 6.6, if more risk averse agent borrow in higher amounts than the interior Nash equilibrium amount, agents in the same coalition with distinct risk aversion levels will be better off in relevance to individual case. For example, at equilibrium, individual investment levels are $({s_1}^*, {s_2}^*) = (9.46, 3.21)$ and the corresponding utility levels are $({U_1}^{PS}({s_1}^*, {s_2}^*), {U_2}^{PS}({s_1}^*, {s_2}^*) = (-0.7429, -0.2914)$, where $p = r = 0.9$ and $a_1 = 0.1$, $a_2 = 0.3$. When, more risk averse agent borrow an amount $\underline{s_2} = 7.5$, while the other agent invest the optimum amount; individual utilities become $({U_1}^{PS}({s_1}^*, \underline{s_2}), {U_2}^{PS}({s_1}^*, \underline{s_2}) = (-0.6274, -0.4397)$. Then, since ${U_i}^{FL}({s_i}^*) = -0.6348$, for $i = 1, 2$, it' obvious that both agents are better off in such coalition structure compared to the individual case.

CHAPTER 7

CONCLUSION

In this study, we consider an investment game among n investors with CARA preferences. Accordingly, we define four major sharing rules, namely Full Liability (FL), Loss Sharing (LS), Profit Sharing (PS) and Equal Sharing (ES), which regulates the allocation of the resulting profits and losses. We mainly attempt to explore how these rules affect the individual and aggregate investment behaviour and expected social welfare levels, as well as coalition formation structure.

Our theoretical findings show that, each rule induces a unique dominant Nash equilibrium in terms of individual investment levels. Specifically, individual investment amounts mainly depend on success probability, net return rate, group size and investors own risk aversion level.

Secondly, when each investor chooses to borrow in positive amounts, regarding aggregate investment amount, the ranking of the rules related to parameter values. Nonetheless, in any group formation under ES rule and LS rule, investors risk taking behaviour increases compared to the PS and FL rules. On the other hand, for sufficiently high values of r,in any group formation ES rule explicitly dominates other rules; while for low values of r, the sorting depends on the the group size. It's worth emphasizing that our theoretical findings are consistent with Fischer's (2013) empirical results.

Furthermore, we employ numerical analysis to study the pairwise comparison of sharing rules for the two investor case. The superiority of the ES rule over other rules is the most remarkable point of our numerical comparison. In terms of individual welfare levels, while the set of parameter combinations account for almost half of our parameter space, the percentage of parameter combinations is considerably small for PS rule. Besides, ES rule dominates other rules regarding egalitarian and utilitarian social welfare levels. Additionally, we demonstrate for the identical two-agent coalitions, egalitarian welfare levels are maximized under each rule. On the other hand, for the same coalition structures, utilitarian welfare levels, induced by ES and LS rules are maximized and minimized under PS rule.

Lastly, we show that two-agent coalition formation of investors with identical risk taking behaviour is stable for certain success probability and net return rates. That is, there is no other coalition structure possibility such that both investor are better off in the new formation. Besides, we address the question of whether investors with distinct risk aversion levels unilaterally and voluntarily agree on an arrangement, so that both of them are better off. Accordingly, we illustrate via numerical examples that such formations are possible under each rule.

APPENDIX

Proof (Proposition 3.1): Agent i's utility under Full Liability Rule is;

$$U_i^{FL}(s_i) = -pe^{-a_i r s_i} - (1-p)e^{a_i s_i}$$

The unconstrained maximizer of this expression is;

$$s_i^* = \ln\left(\frac{rp}{1-p}\right) \cdot \frac{1}{a_i(1+r)}$$

Then, the best response function(the strictly dominant strategy) defined as follows;

$$b_i(s_-i) = \begin{cases} s_i^*, & \text{if } \ln\left(\frac{rp}{1-p}\right) > 0 \\ 0, & otherwise \end{cases}$$

Notice that, $b_i(s_-i)$ is independent of other agents investment levels and risk aversion levels.

Assumption: Fix p for agent i. Let $A_j = \{p_j, (1-p_j)\}, \forall j \in N$. and the Cartesian product $\mathcal{A} = A_1 \times ...A_{i-1} \times A_{i+1} \times ...A_n$ be the set of ordered (n-1)-tuples $(\rho_1, ..\rho_{i-1}, \rho_{i+1}, .., \rho_n)$, where $\rho_k \in A_k$ for each $k \in N/\{i\}$.

Proof: (Proposition 3.2) By Assumption 1, for all $\rho_k \in A_k$ for each $k \in N/\{i\}$ define;

$$\Omega_{ES} : \{\overline{\rho}_t = \prod_{N/\{i\}} \rho_k |, \forall\ \rho_k \in A_k\ and\ k \in N/\{i\}\} \rightarrow ES_{-i}(\overline{s}^t_{-i})$$

where $\overline{s}^t_{-i}$ is the net return that corresponds to $\overline{\rho}_t$ and $ES_{-i}(\overline{s}^t_{-i}) = \frac{\sum_{j\in G/\{i\}} rs_j - \sum_{k\in N\setminus G/\{i\}} s_k}{n}$. Observe that, Ω_{ES} is a well defined bijection. Then, agent i's utility under Equal Sharing Rule is;

$$\begin{aligned} U_i^{ES}(s^*) =& p \cdot -e^{-a_i r s_i/n} \left[\sum_{t=1}^{2^{n-1}} \left(\overline{\rho}_t u_i(ES_{-i}(\overline{s}^t_{-i})) \right) \right] \\ &+ (1-p) \cdot -e^{a_i s_i/n} \left[\sum_{t=1}^{2^{n-1}} \left(\overline{\rho}_t u_i(ES_{-i}(\overline{s}^t_{-i})) \right) \right] \end{aligned}$$

where $u_i(x) = -e^{-a_i x}$ for any $i \in N$. The unconstrained maximizer of this expression is;

$$s_i{}^* = \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{n}{a_i(1+r)}$$

Then, the best response function(the strictly dominant strategy) defined as follows;

$$b_i(s_-i) = \begin{cases} s_i{}^*, & \text{if } \ln\left(\frac{rp}{1-p}\right) > 0 \\ 0, & otherwise \end{cases}$$

Again notice that, $b_i(s_-i)$ is independent of other agents investment levels and risk aversion levels.

Proof: (Proposition 3.3) By Assumption 1, for all $\rho_k \in A_k$ for each $k \in N/\{i\}$ define;

$$\Omega_{LS} : \{\overline{\rho}_t = \prod_{N/\{i\}} \rho_k |, \forall\ \rho_k \in A_k\ and\ k \in N/\{i\}\} \to LS_{-i}(\overline{s}^t_{-i})$$

where $\overline{s}^t_{-i}$ is the net return that corresponds to $\overline{\rho}_t$ and $LS_{-i}(\overline{s}^t_{-i}) = \frac{\sum_{j\in N\setminus G/\{i\}} -s_j}{n}$. Observe that, Ω_{LS} is a well-defined bijection. Then, agent i's utility under Loss Sharing Rule is;

$$U_i^{LS}(s^*) = p \cdot -e^{-a_i r s_i}\left[\sum_{t=1}^{2^{n-1}} \left(\overline{\rho}_t u_i(LS_{-i}(\overline{s}^t_{-i}))\right)\right] + (1-p) - e\frac{a_i s_i}{n}\left[\sum_{t=1}^{2^{n-1}} \left(\overline{\rho}_t u_i(LS_{-i}(\overline{s}^t_{-i}))\right)\right]$$

where $u_i(x) = -e^{-a_i x}$ for any $i \in N$. The unconstrained maximizer of this expression is;

$$s_i{}^* = \ln\left(\frac{nrp}{(1-p)}\right) \cdot \frac{1}{a_i(\frac{1}{n}+r)}$$

Then, the best response function defined as follows;

$$b_i(s\_i) = \begin{cases} s_i{}^*, & \text{if } \ln\left(\frac{nrp}{(1-p)}\right) > 0 \\ 0, & otherwise \end{cases}$$

Proof: (Proposition 3.4) By Assumption 1, for all $\rho_k \in A_k$ for each $k \in N/\{i\}$ define;

$$\Omega_{PS} : \{\overline{\rho}_t = \prod_{N/\{i\}} \rho_k |, \forall\, \rho_k \in A_k \; and \; k \in N/\{i\}\} \rightarrow PS_{-i}(\overline{s}^t_{-i})$$

where $\overline{s}^t_{-i}$ is the net return that corresponds to $\overline{\rho}_t$ and $PS_{-i}(\overline{s}^t_{-i}) = \frac{\sum_{j\in G\setminus\{i\}} rs_j}{n}$. Observe that, Ω_{PS} is a well-defined bijection. Then, agent i's utility under Profit Sharing Rule is;

$$\begin{aligned} U_i^{PS}(s^*) =& p \cdot -e^{-a_i r s_i/n}\left[\sum_{t=1}^{2^{n-1}}\left(\overline{\rho}_t u_i(PS_{-i}(\overline{s}^t_{-i}))\right)\right] \\ &+(1-p)\cdot -e^{a_i s_i}\left[\sum_{t=1}^{2^{n-1}}\left(\overline{\rho}_t u_i(PS_{-i}(\overline{s}^t_{-i}))\right)\right] \end{aligned}$$

where $u_i(x) = -e^{-a_i x}$ for any $i \in N$. The unconstrained maximizer of this expression is;

$$s_i^* = \ln\left(\frac{rp}{n(1-p)}\right)\cdot\frac{1}{a_i(\frac{r}{n}+1)}$$

Then, the best response function defined as follows;

$$b_i(s\_i) = \begin{cases} s_i^*, & \text{if } \ln\left(\frac{rp}{n(1-p)}\right) > 0 \\ 0, & otherwise \end{cases}$$

Proof: (Proposition 4.1) Proof is by pairwise comparison of each rule, where n=2. Note that, we focus on sufficiently high values of p which is necessary for positive investment levels. W.L.O.G let $\sum_{i=1}^{n}\frac{1}{a_i} = 1$ and fix p^*. Then,

(1)$\forall r > 0, TI_{ES} > TI_{FL}$. Since;

$$\sum_{i=1}^{n}\frac{1}{a_i}\ln\left(\frac{rp^*}{1-p^*}\right)\cdot\frac{2}{(1+r)} > \sum_{i=1}^{n}\frac{1}{a_i}\ln\left(\frac{rp^*}{1-p^*}\right)\cdot\frac{1}{(1+r)} \Leftrightarrow 2 > 1$$

(2) $\forall r > 0$, in order to show $TI_{ES} > TI_{PS}$, consider the graph of Υ_{ES} and Υ_{PS} defined by;

$$(p, r) \xrightarrow{\Upsilon_{ES}} \left(p, r, \ln\left(\frac{rp}{1-p}\right) \cdot \frac{2}{(1+r)}\right)$$

and

$$(p, r) \xrightarrow{\Upsilon_{PS}} \left(p, r, \ln\left(\frac{rp}{2(1-p)}\right) \cdot \frac{1}{(\frac{r}{2}+1)}\right)$$

in the domain $(p, r) \in (0, 1) \times (0, 1)$. Observe that $\Upsilon_{(.)}$ functions are continuous on $\mathbb{R}^3$. Then, it's suffice to show that $\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{ES}(p^*, r) - \Upsilon_{PS}(p^*, r)) > 0$ where $d : (0, 1) \times (0, 1) \mapsto \mathbb{R}$ is a metric in the three-dimensional Euclidean space. Then,

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{ES}(p^*, r) - \Upsilon_{PS}(p^*, r)) > 0 \\
\Leftrightarrow &\ln\left(\frac{rp^*}{1-p^*}\right) \cdot \frac{2}{(1+r)} - \ln\left(\frac{rp^*}{2(1-p^*)}\right) \cdot \frac{1}{(\frac{r}{2}+1)} > 0 \\
\Leftrightarrow &\underbrace{\ln\left(\frac{rp^*}{1-p^*}\right)}_{>0} \underbrace{\left[\frac{2}{(1+r)} - \frac{2}{(2+r)}\right]}_{\sigma(\mathrm{r})} + \underbrace{\ln 2 \frac{1}{(\frac{r}{2}+1)}}_{>0} > 0
\end{aligned}$$

Since $\sigma(r) > 0$ on $r \in (0, 1)$, above inequality holds. Therefore,

$\forall r > 0,\ TI_{ES} > TI_{PS}$.

(3) $\forall r > 0$, we can show $TI_{LS} > TI_{FL}$ by a similar discussion to the above case, where;

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{LS}(p^*, r) - \Upsilon_{FL}(p^*, r)) > 0 \\
\Leftrightarrow &\ln\left(\frac{2rp^*}{1-p^*}\right) \cdot \frac{1}{((1/2)+r)} - \ln\left(\frac{rp^*}{1-p^*}\right) \cdot \frac{1}{(1+r)} > 0 \\
\Leftrightarrow &\underbrace{\ln\left(\frac{rp^*}{1-p^*}\right)}_{>0} \underbrace{\left(\frac{1}{((1/2)+r)} - \frac{1}{(r+1)}\right)}_{\sigma(\mathrm{r})} + \underbrace{\ln 2 \frac{1}{(\frac{1}{2}+r)}}_{>0} > 0
\end{aligned}$$

Again, since $\sigma(r) > 0$ on $r \in (0,1)$, above inequality holds. $\forall r > 0,\ TI_{LS} > TI_{FL}$.

(4) $\forall r > 0, TI_{LS} > TI_{PS}$. Observe that;

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{LS}(p^*, r) - \Upsilon_{PS}(p^*, r)) > 0 \\
&\Leftrightarrow \ln\left(\frac{2rp^*}{1-p^*}\right) \cdot \frac{1}{((1/2)+r)} - \ln\left(\frac{rp^*}{2(1-p^*)}\right) \cdot \frac{1}{((r/2)+1)} > 0 \\
&\Leftrightarrow \underbrace{\ln\left(\frac{rp^*}{1-p^*}\right)}_{>0} \underbrace{\left(\frac{1}{((1/2)+r)} - \frac{1}{((r/2)+1)}\right)}_{\sigma(\text{r})} \\
&+ \underbrace{\ln 2\left(\frac{1}{((1/2)+r)} + \frac{1}{((r/2)+1)}\right)}_{>0} > 0
\end{aligned}$$

The above inequality holds, since $\sigma(r) > 0$ on $r \in (0,1)$.

(5) To prove the existence $r' \in (0,1)$ where $TI_{ES} > TI_{LS}$ for $1 > r > r'$ and $TI_{LS} > TI_{ES}$ for $r' > r$, we need to show that there exist a partition of set (0,1) which is a division of two connected, non-overlapping and non-empty cells such that $\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{ES}(p^*, r) - \Upsilon_{LS}(p^*, r))$ take either positive values for any value in one cell or take negative values for any value in other cell, for some sufficiently high value of p^* where,

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{ES}(p^*, r) - \Upsilon_{LS}(p^*, r)) \\
&= \ln\left(\frac{rp^*}{1-p^*}\right) \cdot \frac{2}{(1+r)} - \ln\left(\frac{2rp^*}{(1-p^*)}\right) \cdot \frac{1}{((1/2)+r)} \\
&= \ln\left(\frac{rp^*}{1-p^*}\right)\left(\frac{2}{(1+r)} - \frac{1}{((1/2)+r)}\right) - \ln 2 \frac{1}{((1/2)+r)}
\end{aligned}$$

Observe that, $d(\Upsilon_{ES}(p^*, r) - \Upsilon_{LS}(p^*, r))$ is a continuous mapping on domain $(0, 1) \times (0, 1)$. Moreover, $\frac{\partial d(\Upsilon_{ES}(p^*,r)-\Upsilon_{LS}(p^*,r))}{\partial r} =$ $\frac{(1+r)((1/2)+r)}{(1+r)^2((1/2)+r)^2} - \ln\left(\frac{rp^*}{1-p^*}\right) \cdot \frac{(r^2-(1/2))}{(1+r)^2((1/2)+r)^2} + ln2\frac{(r+1)^2}{(1+r)^2((1/2)+r)^2} > 0$ and the values $d(\Upsilon_{ES}(p^*, r) - \Upsilon_{LS}(p^*, r))$ lies within an interval $(-2 * ln(2), ln(-p^*/(p^* - 1))/3 - (2 * ln(2))/3)$ where limit points take negative and positive values for $r \in (0, 1)$ and for a given sufficiently high $p^* \in (0, 1)$. Then, since $d(\Upsilon_{ES}(p^*, r) - \Upsilon_{LS}(p^*, r))$ is continuous for relevant parameter values; by intermediate value theorem $d(\Upsilon_{ES}(p^*, r') - \Upsilon_{LS}(p^*, r')) = 0$, for some $r' \in (0, 1)$ and for some sufficiently high $p^* \in (0, 1)$, which proves the existence of partition set of (0,1).

(6) Finally, we claim that, there exists $r'' \in (0, 1)$ s.t. $TI_{PS} > TI_{FL}$ for $1 > r > r''$ and $TI_{FL} > TI_{PS}$ for $r'' > r$ (6). Similar to the previous discussion, we have to show the existence of a partition set of (0,1) s.t. $\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{FL}(p^*, r) - \Upsilon_{PS}(p^*, r))$ take either positive values in one cell or take negative values in other cell, for some p^* where,

$$\sum_{i=1}^{n} \frac{1}{a_i} d(\Upsilon_{FL}(p^*, r) - \Upsilon_{PS}(p^*, r))$$

$$= \ln\left(\frac{rp^*}{1-p^*}\right) \cdot \frac{1}{1+r} - \ln\left(\frac{rp^*}{2(1-p^*)}\right) \cdot \frac{1}{(r/2)+1}$$

$$= \ln\left(\frac{rp^*}{1-p^*}\right)\left(\frac{1}{1+r} - \frac{1}{(r/2)+1}\right) + \ln 2\frac{1}{(1+(r/2))}$$

Again, the distance function $d(\Upsilon_{FL}(p^*, r) - \Upsilon_{PS}(p^*, r))$ is a continuous on domain $(0, 1) \times (0, 1)$ and $\frac{\partial d(\Upsilon_{FL}(p^*,r)-\Upsilon_{PS}(p^*,r))}{\partial r} = -\frac{1}{2(1+r)((r/2)+1)} - \frac{ln2}{2((r/2)+1)^2}$ $-\ln\left(\frac{rp^*}{1-p^*}\right)\left(\frac{1}{(1+r)^2} - \frac{1}{2((r/2)+1)^2}\right) < 0$. The values $d(\Upsilon_{FL}(p^*, r) - \Upsilon_{PS}(p^*, r))$ lies within an interval $((2ln(2))/3 - ln(p/(1 - p))/6, ln(2))$ where limit points take negative and positive values for $r \in (0, 1)$ values and for a given sufficiently high

$p^* \in (0,1)$. Then, since $d(\Upsilon_{FL}(p^*,r) - \Upsilon_{PS}(p^*,r))$ is continuous relevant parameter values, by intermediate value theorem $d(\Upsilon_{FL}(p^*,r'') - \Upsilon_{PS}(p^*,r'')) = 0$ for some $r'' \in (0,1)$ and for some sufficiently high $p^* \in (0,1)$, which proves the existence of partition set of (0,1).

$\therefore$ (1),(2),(3),(4),(5) and (6) together proves proposition 4.1.

Proof: (Proposition 4.2) W.L.O.G assume $\sum_{i=1}^{n} \frac{1}{a_i} = 1$. Let p takes sufficiently high values. Fix (p^*, r^*), where p^* is sufficiently high. If $n = 1$, then it's clear that $TI_{ES} = TI_{LS} = TI_{FL} = TI_{PS}$. For $n > 1$, the proof is by pairwise comparison of each rule.

(1)$\forall n > 1, TI_{ES} > TI_{FL}$. Since;

$$TI_{ES} = \sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{n}{(1+r^*)} > \sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{1}{(1+r^*)} = TI_{FL}$$
$$\Leftrightarrow n > 1$$

which holds by assumption.

(2)$\forall n > 1, TI_{ES} > TI_{PS}$. Since;

$$TI_{ES} = \sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{n}{(1+r^*)} > \sum_{i=1}^{n} \frac{1}{a_i} \ln\left(\frac{r^*p^*}{n(1-p^*)}\right)\frac{n}{(n+r^*)} = TI_{PS}$$
$$\Leftrightarrow \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{1}{(1+r^*)} > \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{1}{(n+r^*)} - ln2\frac{1}{(n+r^*)}$$

which clearly holds for $n > 1$, for a given (p^*, r^*).

(3) For $TI_{ES} - TI_{LS}$ case, for a given $r \in (0,1)$ we claim that $\exists n^{**} \in \mathbb{R}_{>0}$ such that $\forall n \geq n^{**}, TI_{ES} \geq TI_{LS}$ and for $1 \leq n \leq n^{**}, TI_{LS} \geq TI_{ES}$ where $n \in \mathbb{Z}$. Firstly, for arbitrary (p^*, r^*), $\frac{\partial TI_{ES}(p^*,r^*,n)}{\partial n} = ln\left(\frac{pr}{(1-p)}\right)\frac{1}{1+r}$ is constant. Observe that,

$\frac{\partial TI_{LS}(p^*,r^*,n)}{\partial n} = \frac{1}{n(r+\frac{1}{n})} + \frac{ln\left(\frac{npr}{(1-p)}\right)}{n^2(r+\frac{1}{n})^2} > 0$ and
$\frac{\partial^2 TI_{LS}(p^*,r^*,n)}{\partial n^2} = \frac{2}{\sigma_2} - \frac{1}{n^2(r+\frac{1}{n})} - \frac{\sigma_1}{\sigma_2} + \frac{\sigma_1}{n^4(r+\frac{1}{n})^3} = \frac{2-(nr+1)}{(nr+1)^3} + \frac{(-nr)2\ln\left(\frac{nrp}{(1-p)}\right)}{(nr+1)^4} < 0$ where $\sigma_1 = 2ln\left(\frac{npr}{(1-p)}\right)$ and $\sigma_2 = n^3(r+\frac{1}{n})^2$ which implies TI_{LS} is strictly increasing concave function on $n \in (0,\infty)$. Now let $\zeta = \{n | TI_{ES}(p^*,r^*,n) = TI_{LS}(p^*,r^*,n)\}$ for a given (p^*,r^*). Then there are three cases: i) If $\zeta = \{1\}$, then take $n^{**} = 1$. Notice that, $\forall n \geq n^{**}$, $TI_{ES} \geq TI_{LS}$. ii) If $\zeta = \{1, \underline{n}\}$ where $\underline{n} < 1$, then take $n^{**} = 1$. Notice that, $\forall n \geq n^{**}$, $TI_{ES} \geq TI_{LS}$. iii) If $\zeta = \{1, \overline{n}\}$ where $\overline{n} > 1$, then take $n^{**} = \overline{n}$. Then, $\forall n \geq n^{**}$, $TI_{ES} \geq TI_{LS}$ and for $\forall n \in \mathbb{Z}_{>0}$ on $1 \leq n \leq n^{**}$, $TI_{LS} \geq TI_{ES}$.

(4) $\forall n > 1, TI_{LS} > TI_{FL}$. Since;

$$\begin{aligned} TI_{LS} = \ln\left(\frac{nr^*p^*}{1-p^*}\right)\frac{n}{(1+nr^*)} &> \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{1}{(1+r^*)} = TI_{FL} \\ \Leftrightarrow \ln(n)\frac{n}{1+nr^*} &> \ln\left(\frac{r^*p^*}{1-p^*}\right)\left[\frac{1}{(1+r^*)} - \frac{n}{(1+nr^*)}\right] \\ \Leftrightarrow \ln(n)n &> \ln\left(\frac{r^*p^*}{1-p^*}\right)\frac{(1-n)}{(1+r^*)} \end{aligned}$$

Notice that above inequality clearly holds for $n > 1$.

(5) $\forall n > 1, TI_{LS} > TI_{PS}$. Since;

$$\begin{aligned} TI_{LS} = \ln\left(\frac{nr^*p^*}{1-p^*}\right)\frac{n}{(1+nr^*)} &> \ln\left(\frac{r^*p^*}{n(1-p^*)}\right)\frac{n}{(n+r^*)} = TI_{PS} \\ \Leftrightarrow \ln(n)\left[\frac{1}{(1+nr^*)} + \frac{1}{(r^*+n)}\right] &> \ln\left(\frac{r^*p^*}{1-p^*}\right)\left[\frac{1}{(r^*+n)} - \frac{1}{(1+nr^*)}\right] \\ \Leftrightarrow \ln(n)(1+r^*)(1+n) &> \ln\left(\frac{r^*p^*}{1-p^*}\right)(1-n)(1-r^*) \end{aligned}$$

which obviously holds for $n > 1$.

(6) For $TI_{PS} - TI_{FL}$ case, for a given $r \in (0,1)$, we claim that $\exists n^* \in \mathbb{R}_{>0}$ such that $\forall n \geq n^*, TI_{FL} \geq TI_{PS}$ and for $1 \leq n \leq n^*, TI_{PS} \geq TI_{FL}$ where $n \in \mathbb{Z}_{>0}$. Firstly, note that for a given (p^*, r^*), TI_{FL} is constant. Observe that $\frac{\partial TI_{PS}(p^*,r^*,n)}{\partial n} = \frac{\left[r^* \ln\left(\frac{p^* r^*}{n(1-p^*)}\right) - n((r^*/n)+1)\right]}{n^2((r^*/n)+1)^2} = \frac{r^*\left[\ln\left(\frac{p^*}{(1-p^*)}\right) + \ln\left(\frac{n}{r^*}\right) - 1 - \frac{n}{r^*}\right]}{n^2((r^*/n)+1)^2}$ takes positive and negative values for some finite $n' \in (0, \infty)$. Since,

$\lim_{n \to 0^+} \frac{\partial TI_{PS}(p^*,r^*,n)}{\partial n} = -\frac{\left(-\infty - \ln\left(\frac{p^* r^*}{(1-p^*)}\right)\right)}{r^*} = \infty$,

$\lim_{n \to n'} \frac{\partial TI_{PS}(p^*,r^*,n)}{\partial n} = -\frac{\left(-r^* - r^* \ln\left(\frac{p^* r^*}{n'(1-p^*)}\right) + n'\right)}{(r^* + n')^2} < 0$ for some finite $n' \in (0, \infty)$ and $\lim_{n \to +\infty} \frac{\partial TI_{PS}(p^*,r^*,n)}{\partial n} = 0$. Moreover, $\frac{\partial^2 TI_{PS}(p^*,r^*,n)}{\partial n^2} = \frac{n^2 - r^{*2} - 2r^* n \ln\left(\frac{p^* r^*}{n'(1-p^*)}\right)}{(r^*+n)^4}$ also takes negative and positive values for some finite $n' \in (0, \infty)$. Because,

$\lim_{n \to 0^+} \frac{\partial^2 TI_{PS}(p^*,r^*,n)}{\partial n^2} = -\frac{1}{r^{*2}}$ and

$\lim_{n \to n'} \frac{\partial^2 TI_{PS}(p^*,r^*,n)}{\partial n^2} = -\frac{2n' r^* \ln\left(\frac{p^* r^*}{n'(1-p^*)}\right) + r^{*2} - (n')^2}{(r^* + n')^4} > 0$ for appropriate values of n', for a given sufficiently high (p^*, r^*). Hence, there exist a maximum n_{max} where $\frac{\partial^2 TI_{PS}(p^*,r^*,n_{max})}{\partial n^2} < 0$. Then, since for $n = 1$ $TI_{PS} = TI_{FL}$, there are three cases: i)If $n_{max} = 1$, then take $n^* = n_{max} = 1$. Notice that for $n > 1, TI_{FL} \geq TI_{PS}$. ii) If $n_{max} < 1$, then take $n^* = 1$. Again for $n > 1, TI_{FL} \geq TI_{PS}$. iii) If $n_{max} > 1$, then there exist $n^* > 1$ such that for $1 \leq n < n^*, TI_{PS} \geq TI_{FL}$ and for $n > n^*$, $TI_{FL} \geq TI_{PS}$.

$\therefore$ (1), (2), (3), (4), (5) and (6) together proves proposition 4.2.

Proof: (Lemma 5.0.1.) For a given (p, r) such that $pr \neq (1-p)$, to show agents are better off in any two agent coalition compared to the full liability case, we must show that;

$$|U_i{}^{FL}(s_i{}^*)| - |U_i^{2ES}(s_i{}^*, s_{-i}{}^*)| = -p^2 e^{\frac{-a_i r}{2}\left(\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)+\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_{-i}(1+r)}\right)\right)}$$
$$-(1-p)pe^{\frac{-a_i}{2}\left(r\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)-\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_{-i}(1+r)}\right)\right)}$$
$$-(1-p)pe^{\frac{-a_i}{2}\left(-\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)+r\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_{-i}(1+r)}\right)\right)}$$
$$-(1-p)^2 e^{\frac{a_i}{2}\left(\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)+\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_{-i}(1+r)}\right)\right)}$$
$$+(1-p)e^{a_i\frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}} + pe^{-a_i r\frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}} > 0$$

where $U_i{}^{FL}(s_i{}^*)$ is i-th agent's individual utility level and $U_i^{2ES}(s_i{}^*, s_{-i}{}^*)$ is i-th agent's utility level in 2-agent coalition under ES rule. Then, since $1+x-y \leq e^{x-y}$, $\forall$ x,y $\in \mathbb{R}$, it's enough to show that

$$(1-p)\left(1+\frac{\ln(rp)}{(r+1)}-\frac{\ln(1-p)}{(r+1)}\right)+p\left(1+\frac{r\ln(1-p)}{(r+1)}-\frac{r\ln(pr)}{(r+1)}\right)$$
$$-p(1-p)\left(1-\left(r-\frac{a_i}{a_{-i}}\right)\frac{\ln(rp)}{(1+r)}+\left(r-\frac{a_i}{a_{-i}}\right)\frac{\ln(1-p)}{(1+r)}\right)$$
$$-p(1-p)\left(1+\left(1-\frac{a_i r}{a_{-i}}\right)\frac{\ln(rp)}{(1+r)}-\left(1-\frac{a_i r}{a_{-i}}\right)\frac{\ln(1-p)}{(1+r)}\right)$$
$$-p^2\left(1-\left(r+\frac{a_i r}{a_{-i}}\right)\frac{\ln(rp)}{(1+r)}+\left(r+\frac{a_i r}{a_{-i}}\right)\frac{\ln(1-p)}{(1+r)}\right)$$
$$-(1-p)^2\left(1+\left(1+\frac{a_i}{a_{-i}}\right)\frac{\ln(rp)}{(1+r)}-\left(1+\frac{a_i}{a_{-i}}\right)\frac{\ln(1-p)}{(1+r)}\right)>0$$

Then this inequality reduces to the following inequality, which is strictly bigger than 0, for any a_i, a_{-i};

$$\frac{a_i}{a_{-i}}\underbrace{(pr+p-1)}_{\Omega(p,r)}\underbrace{\ln\left(\frac{rp}{(1-p)}\right)}_{\varphi(p,r)}\frac{1}{(1+r)}>0$$

Observe that, the signs of the terms $\Omega(p,r)$ and $\varphi(p,r)$ agree.

Proof: (Lemma 5.0.2.) Assume $a_2 \leq a_1$.

(i)

$$\begin{aligned}{U_1}^{FL}({s_1}^*) = &- pe^{-a_i r\left(\ln\left(\frac{rp}{(1-p)}\right)\frac{1}{a_i(1+r)}\right)} - (1-p)e^{a_i\left(\ln\left(\frac{rp}{(1-p)}\right)\frac{1}{a_i(1+r)}\right)}\\ &- pe^{-r\left(\ln\left(\frac{rp}{(1-p)}\right)\frac{1}{(1+r)}\right)} - (1-p)e^{\left(\ln\left(\frac{rp}{(1-p)}\right)\frac{1}{(1+r)}\right)}\end{aligned}$$

Note that, above expression obviously doesn't depend on risk aversion levels. Then ${U_1}^{FL}({s_1}^*) = {U_2}^{FL}({s_1}^*)$.

(ii) We want to show ${U_1}^{ES}({s_1}^*, {s_2}^*) \leq {U_2}^{ES}({s_1}^*, {s_2}^*)$, where ${s_i}^* = \ln\left(\frac{rp}{(1-p)}\right)\frac{2}{a_i(1+r)}$ for $i = 1, 2$. Then, this is equivalent to show that;

$$\begin{aligned}&{U_2}^{ES}({s_1}^*, {s_2}^*) - {U_1}^{ES}({s_1}^*, {s_2}^*) = p^2 e^{-a_1\left(\frac{r{s_1}^*+r{s_2}^*}{2}\right)} + p(1-p)e^{-a_1\left(\frac{r{s_1}^*-{s_2}^*}{2}\right)}\\ &+ p(1-p)e^{-a_1\left(\frac{r{s_2}^*-{s_1}^*}{2}\right)} + (1-p)^2 e^{-a_1\left(\frac{-{s_1}^*-{s_2}^*}{2}\right)}\\ &- p^2 e^{-a_2\left(\frac{r{s_1}^*+r{s_2}^*}{2}\right)} - p(1-p)e^{-a_2\left(\frac{r{s_2}^*-{s_1}^*}{2}\right)}\\ &- p(1-p)e^{-a_2\left(\frac{r{s_1}^*-{s_2}^*}{2}\right)} - (1-p)^2 e^{-a_2\left(\frac{-{s_1}^*-{s_2}^*}{2}\right)} \geq 0\end{aligned}$$

Then expanding and rearranging the terms of preceding expression results in:

$$
\begin{aligned}
&U_2{}^{ES}(s_1{}^*, s_2{}^*) - U_1{}^{ES}(s_1{}^*, s_2{}^*) = \\
&\left(-pe^{\frac{-a_2 r s_1{}^*}{2}} + (p-1)e^{\frac{a_2 s_1{}^*}{2}}\right) \cdot \left(pe^{\frac{-a_2 r s_2{}^*}{2}} + (1-p)e^{\frac{a_2 s_2{}^*}{2}}\right) \\
&+ \left(pe^{\frac{-a_1 r s_2{}^*}{2}} + (1-p)e^{\frac{a_1 s_2{}^*}{2}}\right) \cdot \left(pe^{\frac{-a_1 r s_1{}^*}{2}} + (1-p)e^{\frac{a_1 s_1{}^*}{2}}\right) \geq 0 \\
&\Leftrightarrow \sigma \underbrace{\left(-pe^{\frac{-a_2 r s_1{}^*}{2}} + (p-1)e^{\frac{a_2 s_1{}^*}{2}} + pe^{\frac{-a_1 r s_2{}^*}{2}} + (1-p)e^{\frac{a_1 s_2{}^*}{2}}\right)}_{\varphi} \geq 0
\end{aligned}
$$

where $\sigma = \left(pe^{\frac{-a_2 r s_2{}^*}{2}} + (1-p)e^{\frac{a_2 s_2{}^*}{2}}\right) = \left(pe^{\frac{-a_1 r s_1{}^*}{2}} + (1-p)e^{\frac{a_1 s_1{}^*}{2}}\right)$. Then observe that, $\varphi \geq 0$ since by assumption $a_2 \leq a_1$. Hence $U_2{}^{ES}(s_1{}^*, s_2{}^*) - U_1{}^{ES}(s_1{}^*, s_2{}^*) \geq 0$. Moreover, clearly above equality holds when $a_1 = a_2$.

(iii) We want to show $U_1{}^{LS}(s_1{}^*, s_2{}^*) \leq U_2{}^{LS}(s_1{}^*, s_2{}^*)$, where $s_i{}^* = \ln\left(\frac{2rp}{(1-p)}\right)\frac{1}{a_i(\frac{1}{2}+r)}$ for $i = 1, 2$. Then, this is equivalent to show that;

$$
\begin{aligned}
&U_2{}^{LS}(s_1{}^*, s_2{}^*) - U_1{}^{LS}(s_1{}^*, s_2{}^*) = p^2 e^{-a_1 r s_1{}^*} + p(1-p)e^{-a_1\left(r s_1{}^* - \frac{s_2{}^*}{2}\right)} \\
&+ p(1-p)e^{-a_1\left(\frac{-s_1{}^*}{2}\right)} + (1-p)^2 e^{-a_1\left(\frac{-s_1{}^* - s_2{}^*}{2}\right)} \\
&- p^2 e^{-a_2 r s_2{}^*} - p(1-p)e^{-a_2\left(r s_2{}^* - \frac{s_1{}^*}{2}\right)} \\
&- p(1-p)e^{-a_2\left(\frac{-s_2{}^*}{2}\right)} - (1-p)^2 e^{-a_2\left(\frac{-s_1{}^* - s_2{}^*}{2}\right)} \geq 0
\end{aligned}
$$

Then expanding and rearranging the terms of preceding expression results in:

$$pp\underbrace{\left(e^{-a_1 rs_1{}^*}-e^{-a_2 rs_2{}^*}\right)}_{=0}+p(1-p)\underbrace{\left(e^{-a_1\left(rs_1{}^*-\frac{s_2{}^*}{2}\right)}-e^{-a_2\left(rs_2{}^*-\frac{s_1{}^*}{2}\right)}\right)}_{\geq 0}$$
$$+p(1-p)\underbrace{\left(e^{a_1\frac{s_1{}^*}{2}}-e^{a_2\frac{s_2{}^*}{2}}\right)}_{=0}+(1-p)^2\underbrace{\left(e^{a_1\left(\frac{s_1{}^*+s_2{}^*}{2}\right)}-e^{a_2\left(\frac{s_1{}^*+s_2{}^*}{2}\right)}\right)}_{\geq 0}\geq 0$$

Clearly above inequality holds, since $a_2 \leq a_1$. Additionally, it's obvious that, equality holds when $a_1 = a_2$.

(iv) We want to show ${U_1}^{PS}(s_1{}^*, s_2{}^*) \geq {U_2}^{PS}(s_1{}^*, s_2{}^*)$, where $s_i{}^* = \ln\left(\frac{rp}{2(1-p)}\right)\frac{1}{a_i(\frac{r}{2}+1)}$ for $i = 1, 2$. Then, this is equivalent to show that;

$$\begin{aligned}
&{U_1}^{PS}(s_1{}^*, s_2{}^*) - {U_1}^{PS}(s_1{}^*, s_2{}^*) = p^2 e^{-a_2\left(\frac{rs_1{}^*+rs_2{}^*}{2}\right)} + p(1-p)e^{-a_2\frac{rs_2{}^*}{2}}\\
&+ p(1-p)e^{-a_2\left(\frac{rs_1{}^*}{2}-s_2{}^*\right)} + (1-p)^2 e^{-a_2(-s_2{}^*)}\\
&- p^2 e^{-a_1\left(\frac{rs_1{}^*+rs_2{}^*}{2}\right)} - p(1-p)e^{-a_1\frac{rs_1{}^*}{2}}\\
&- p(1-p)e^{-a_1\left(\frac{rs_2{}^*}{2}-s_1{}^*\right)} - (1-p)^2 e^{-a_1(-s_1{}^*)} \geq 0
\end{aligned}$$

Then expanding and rearranging the terms of preceding expression results in:

$$pp\underbrace{\left(e^{-a_2\left(\frac{rs_1{}^*+rs_2{}^*}{2}\right)}-e^{-a_1\left(\frac{rs_1{}^*+rs_2{}^*}{2}\right)}\right)}_{\geq 0}+(1-p)p\underbrace{\left(e^{-a_2\frac{rs_2{}^*}{2}}-e^{-a_1\frac{rs_1{}^*}{2}}\right)}_{=0}$$
$$+p(1-p)\underbrace{\left(e^{-a_2\left(\frac{rs_1{}^*}{2}-s_2{}^*\right)}-e^{-a_1\left(\frac{rs_2{}^*}{2}-s_1{}^*\right)}\right)}_{\geq 0}+(1-p)^2\underbrace{\left(e^{-a_2(-s_2{}^*)}-e^{-a_1(-s_1{}^*)}\right)}_{=0}$$

Observe that above inequality holds by assumption $a_2 \leq a_1$ and equality holds when $a_1 = a_2$.

Proof: (Proposition 6.1) Assume N={1,2} and a_1, a_2 be the risk aversion levels. Let (p,r) values satisfy the condition $pr > (1-p)$. When both agent invest the positive Nash Equilibrium investment amount $s^* = ({s_1}^*, {s_2}^*)$ where p take sufficiently high values, the utility of agent-1 equal to:

$$\begin{aligned}
{U_1}^{ES}({s_1}^*, {s_2}^*) = &- pp \cdot e^{\left(-a_1 \cdot r \cdot \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{(1+r)} \cdot \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\right)} \\
&- p(1-p) \cdot e^{\left(-a_1 \cdot \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{(1+r)} \cdot \left(\frac{r}{a_1} - \frac{1}{a_2}\right)\right)} \\
&- (1-p)p \cdot e^{\left(-a_1 \cdot \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{(1+r)} \cdot \left(-\frac{1}{a_1} + \frac{r}{a_2}\right)\right)} \\
&- (1-p)(1-p)e^{\left(a_1 \cdot \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{1}{(1+r)} \cdot \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\right)}
\end{aligned}$$

where ${s_i}^* = \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{2}{a_i(1+r)}$ for i=1,2.

Taking the first derivative of ${U_1}^{ES}({s_1}^*, {s_2}^*)$ w.r.t. a_1 yields;

$$\begin{aligned}
\frac{\partial {U_1}^{ES}({s_1}^*, {s_2}^*)}{\partial a_1} = &\; ppe^{-a_1 r(\sigma_3+\sigma_2)} \left(r(\sigma_3+\sigma_2) - \sigma_1\right) - \frac{(1-p)^2 \sigma_4 e^{a_1(\sigma_3+\sigma_2)}}{a_2(r+1)} \\
&- \frac{(1-p)p\sigma_4 e^{a_1(\sigma_2-\sigma_1)}}{a_2(r+1)} + \frac{(1-p)p\sigma_4 r e^{a_1\left(\sigma_3 - \left(\frac{r\sigma_4}{a_2(r+1)}\right)\right)}}{a_2(r+1)}
\end{aligned}$$

where $\sigma_1 = \frac{r\sigma_4}{a_1(r+1)}$, $\sigma_2 = \frac{\sigma_4}{a_2(r+1)}$, $\sigma_3 = \frac{\sigma_4}{a_1(r+1)}$ and $\sigma_4 = \ln\left(\frac{rp}{1-p}\right)$. Then, $\frac{\partial {U_1}^{ES}({s_1}^*, {s_2}^*)}{\partial a_1} = 0$ (and by symmetry $\frac{\partial {U_2}^{ES}({s_1}^*, {s_2}^*)}{\partial a_2} = 0$) where $a_1 = a_2$.

Moreover, the second derivative of $U_1{}^{ES}(s_1{}^*, s_2{}^*)$ w.r.t. a_1 yields;

$$\frac{\partial^2 U_1{}^{ES}(s_1{}^*, s_2{}^*)}{\partial a_1{}^2} = \frac{-pe^{a_1(\sigma_3-\sigma_2)}\sigma_5{}^2(1-p)}{\sigma_1} - \frac{e^{a_1(\sigma_4+\sigma_3)}\sigma_5{}^2(1-p)^2}{\sigma_1} - p^2e^{-a_1r(\sigma_4+\sigma_3)}(r(\sigma_4+\sigma_3)-\sigma_2)^2 - \frac{pr^2e^{a_1(\sigma_4-r\sigma_3)}\sigma_5{}^2(1-p)}{\sigma_1}$$

where $\sigma_1 = a_2{}^2(1+r)^2$, $\sigma_2 = \frac{r\sigma_5}{a_1(r+1)}$, $\sigma_3 = \frac{\sigma_5}{a_2(r+1)}$, $\sigma_4 = \frac{\sigma_5}{a_1(r+1)}$, $\sigma_5 = \ln\left(\frac{rp}{1-p}\right)$. Observe that, $\frac{\partial^2 U_1{}^{ES}(s_1{}^*,s_2{}^*)}{\partial a_1{}^2} < 0$ (and by symmetry $\frac{\partial^2 U_2{}^{ES}(s_1{}^*,s_2{}^*)}{\partial a_2{}^2} < 0$). Then $U_1{}^{ES}(s_1{}^*, s_2{}^*)$ has a unique maximum at $a_1 = a_2$.

Finally, to prove, agents with identical risk aversion levels $a_1 = a_2 = a_i$ are better off in the 2-agent coalition compared to the individual case, we must show that;

$$\begin{aligned}|U_i{}^{FL}(s_i{}^*)| - |U_i^{2ES}(s_1{}^*, s_2{}^*)| &= (1-p)e^{a_i\frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}} + pe^{-a_ir\frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}} \\ &- p^2e^{\frac{-a_ir}{2}\left(\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)+\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)\right)} \\ &- (1-p)pe^{\frac{-a_i}{2}\left(r\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)-\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)\right)} \\ &- (1-p)pe^{\frac{-a_i}{2}\left(-\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)+r\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)\right)} \\ &- (1-p)^2e^{\frac{a_i}{2}\left(\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)+\ln\left(\frac{rp}{(1-p)}\right)\left(\frac{2}{a_i(1+r)}\right)\right)} > 0\end{aligned}$$

where $U_i{}^{FL}(s_i{}^*)$ is i-th agent's individual utility level and $U_i^{2ES}(s_1{}^*, s_2{}^*)$ is i-th agent's utility level in 2-agent coalition under Equal Sharing rule. Then, since

$1 + x - y \leq e^{x-y}$, $\forall$ x,y $\in \mathbb{R}$, it's enough to show that,

$$\begin{aligned}
&(1-p)\left(1+\frac{\ln(rp)}{(r+1)}-\frac{\ln(1-p)}{(r+1)}\right)+p\left(1+\frac{r\ln(1-p)}{(r+1)}-\frac{r\ln(pr)}{(r+1)}\right)\\
&-p(1-p)\left(1+\frac{(1-r)\ln(rp)}{(1+r)}-\frac{(1-r)\ln(1-p)}{(1+r)}\right)\\
&-p(1-p)\left(1+\frac{(1-r)\ln(rp)}{(1+r)}-\frac{(1-r)\ln(1-p)}{(1+r)}\right)\\
&-p^2\left(1-\frac{2r\ln(rp)}{(1+r)}+\frac{2r\ln(1-p)}{(1+r)}\right)\\
&-(1-p)^2\left(1+\frac{2\ln(rp)}{(1+r)}-\frac{2\ln(1-p)}{(1+r)}\right)>0
\end{aligned}$$

Then this inequality reduces to the following inequality which is strictly bigger than 0:

$$(pr+p-1)\ln\left(\frac{rp}{(1-p)}\right)\frac{1}{(1+r)}>0$$

Proof: (Lemma 6.1.1.)Assume $a_1 < a_2$. then,

Claim 1: $\frac{\partial U_1^{ES}(\overline{s_1},{s_2}^*)}{\partial \overline{s_1}} = \frac{a_1 p^2 r e^{-a_1 r((\overline{s_1}/2)+\sigma_1)}}{2} - \frac{a_1(1-p)^2 e^{a_1((\overline{s_1}/2)+\sigma_1)}}{2} + \frac{a_1 p(p-1) e^{a_1((\overline{s_1}/2)-\frac{r\sigma_2}{a_2(r+1)})}}{2} - \frac{a_1 pr(p-1) e^{-a_1((r\overline{s_1}/2)-\frac{\sigma_2}{a_2(r+1)})}}{2}$ is strictly bigger than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for ${s_2}^* < \overline{s_1} < {s_1}^*$ values where $\sigma_1 = \frac{\sigma_2}{a_2(r+1)}$ and $\sigma_2 = ln\left(\frac{pr}{1-p}\right)$.

Pf: By assumption, ${s_1}^* = \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{2}{a_1(1+r)} > \overline{s_1}$. Then,

$$
\begin{aligned}
&\ln\left(\frac{rp}{(1-p)}\right) > \frac{a_1\overline{s_1}}{2} + \frac{a_1 r\overline{s_1}}{2} \\
&\Leftrightarrow \ln(pr) + \ln\left(e^{\frac{-a_1 r\overline{s_1}}{2}}\right) + \ln\left(pe^{-a_1 r\sigma_1} + (1-p)e^{a_1\sigma_1}\right) \\
&> \ln(1-p) + \ln\left(e^{\frac{a_1\overline{s_1}}{2}}\right) + \ln\left(pe^{-a_1 r\sigma_1} + (1-p)e^{a_1\sigma_1}\right) \\
&\Leftrightarrow \ln\left(p^2 re^{-a_1 r\left(\frac{\overline{s_1}}{2}+\sigma_1\right)} + (1-p)pre^{-a_1\left(\frac{r\overline{s_1}}{2}-\sigma_1\right)}\right) \\
&> \ln\left((1-p)pe^{a_1 r\left(\frac{\overline{s_1}}{2}-r\sigma_1\right)} + (1-p)^2 e^{a_1\left(\frac{\overline{s_1}}{2}+\sigma_1\right)}\right) \\
&\Leftrightarrow \frac{\partial U_1^{ES}(\overline{s_1}, {s_2}^*)}{\partial\overline{s_1}} > 0
\end{aligned}
$$

Claim 2: $\frac{\partial U_2^{ES}(\overline{s_1},{s_2}^*)}{\partial\overline{s_1}} = \frac{a_2p^2re^{-a_2r((\overline{s_1}/2)+\sigma_1)}}{2} - \frac{a_2(1-p)^2e^{a_2((\overline{s_1}/2)+\sigma_1)}}{2} + \frac{a_2p(p-1)e^{a_2((\overline{s_1}/2)-\frac{r\sigma_2}{a_2(r+1)})}}{2} - \frac{a_2pr(p-1)e^{-a_2((r\overline{s_1}/2)-\frac{\sigma_2}{a_2(r+1)})}}{2}$ is strictly smaller than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for ${s_2}^* < \overline{s_1} < {s_1}^*$ values where $\sigma_1 = \frac{\sigma_2}{a_2(r+1)}$ and $\sigma_2 = ln\frac{pr}{1-p}$.

Pf: By assumption, ${s_2}^* = \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{2}{a_2(1+r)} < \overline{s_1}$. Then,

$$
\begin{aligned}
&\ln\left(\frac{rp}{(1-p)}\right) < \frac{a_2\overline{s_1}}{2} + \frac{a_2 r\overline{s_1}}{2} \\
&\Leftrightarrow \ln(pr) + \ln\left(e^{\frac{-a_2 r\overline{s_1}}{2}}\right) + \ln\left(pe^{-a_2 r\sigma_1} + (1-p)e^{a_2\sigma_1}\right) \\
&< \ln(1-p) + \ln\left(e^{\frac{a_2\overline{s_1}}{2}}\right) + \ln\left(pe^{-a_2 r\sigma_1} + (1-p)e^{a_2\sigma_1}\right) \\
&\Leftrightarrow \ln\left(p^2 re^{-a_2 r\left(\frac{\overline{s_1}}{2}+\sigma_1\right)} + (1-p)pre^{-a_2\left(\frac{r\overline{s_1}}{2}-\sigma_1\right)}\right) \\
&< \ln\left((1-p)pe^{a_2 r\left(\frac{\overline{s_1}}{2}-r\sigma_1\right)} + (1-p)^2 e^{a_2\left(\frac{\overline{s_1}}{2}+\sigma_1\right)}\right) \\
&\Leftrightarrow \frac{\partial U_2^{ES}(\overline{s_1}, {s_2}^*)}{\partial\overline{s_1}} < 0
\end{aligned}
$$

Claim 3: $\frac{\partial U_1^{ES}({s_1}^*,\underline{s_2})}{\partial\underline{s_2}} = \frac{a_1p^2re^{-a_1r((\underline{s_2}/2)+\sigma_1)}}{2} - \frac{a_1(1-p)^2e^{a_1((\underline{s_2}/2)+\sigma_1)}}{2} + \frac{a_1p(p-1)e^{a_1((\underline{s_2}/2)-\frac{r\sigma_2}{a_1(r+1)})}}{2} - \frac{a_1pr(p-1)e^{-a_1((r\underline{s_2}/2)-\frac{\sigma_2}{a_1(r+1)})}}{2}$ is strictly higher than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for $s_2{}^* < \underline{s_1} < s_1{}^*$ values where $\sigma_1 = \frac{\sigma_2}{a_1(r+1)}$ and $\sigma_2 = ln\frac{pr}{1-p}$.

Pf: By assumption, $s_1{}^* = \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{2}{a_1(1+r)} > \underline{s_2}$.Then,

$$\ln\left(\frac{rp}{(1-p)}\right) > \frac{a_1 r \underline{\underline{s_2}}}{2} + \frac{a_1 \underline{\underline{s_2}}}{2}$$

$$\Leftrightarrow \ln(pr) + \ln\left(e^{\frac{-a_1 r \underline{\underline{s_2}}}{2}}\right) + \ln\left(pe^{-a_1 r\sigma_1} + (1-p)e^{a_1\sigma_1}\right)$$

$$> \ln(1-p) + \ln\left(e^{\frac{a_1 \underline{\underline{s_2}}}{2}}\right) + \ln\left(pe^{-a_1 r\sigma_1} + (1-p)e^{a_1\sigma_1}\right)$$

$$\Leftrightarrow \ln\left(p^2 r e^{-a_1 r\left(\frac{\underline{\underline{s_2}}}{2}+\sigma_1\right)} + (1-p)pre^{-a_1\left(\frac{r\underline{\underline{s_2}}}{2}-\sigma_1\right)}\right)$$

$$> \ln\left((1-p)pe^{a_1\left(\frac{\underline{\underline{s_2}}}{2}-r\sigma_1\right)} + (1-p)^2 e^{a_1\left(\frac{\underline{\underline{s_2}}}{2}+\sigma_1\right)}\right)$$

$$\Leftrightarrow \frac{\partial U_1^{ES}(s_1{}^*), \underline{\underline{s_2}}}{\partial \underline{s_2}} > 0$$

Claim 4: $\frac{\partial U_2(s_1{}^*, \underline{s_2})}{\partial \underline{s_2}} = \frac{a_2 p^2 r e^{-a_2 r((\underline{s_2}/2)+\sigma_1)}}{2} - \frac{a_2(1-p)^2 e^{a_2((\underline{s_2}/2)+\sigma_1)}}{2} + \frac{a_2 p(p-1)e^{a_2((\underline{s_2}/2)-\frac{r\sigma_2}{a_1(r+1)})}}{2} - \frac{a_2 pr(p-1)e^{-a_2((r\underline{s_2}/2)-\frac{\sigma_2}{a_2(r+1)})}}{2}$ is strictly smaller than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for $s_2{}^* < \underline{s_2} < s_1{}^*$ values where $\sigma_1 = \frac{\sigma_2}{a_1(r+1)}$ and $\sigma_2 = ln\frac{pr}{1-p}$.

Pf: By assumption, $s_2{}^* = \ln\left(\frac{rp}{(1-p)}\right) \cdot \frac{2}{a_2(1+r)} < \underline{s_2}$. Then,

$$\ln\left(\frac{rp}{(1-p)}\right) < \frac{a_2 r \underline{\underline{s_2}}}{2} + \frac{a_2 \underline{\underline{s_2}}}{2}$$

$$\Leftrightarrow \ln(pr) + \ln\left(e^{\frac{-a_2 r \underline{\underline{s_2}}}{2}}\right) + \ln\left(pe^{-a_2 r\sigma_1} + (1-p)e^{a_2\sigma_1}\right)$$

$$< \ln(1-p) + \ln\left(e^{\frac{a_2 \underline{\underline{s_2}}}{2}}\right) + \ln\left(pe^{-a_2 r\sigma_1} + (1-p)e^{a_2\sigma_1}\right)$$

$$\Leftrightarrow \ln\left(p^2 r e^{-a_2 r\left(\frac{\underline{\underline{s_2}}}{2}+\sigma_1\right)} + (1-p)pre^{-a_2\left(\frac{r\underline{\underline{s_2}}}{2}-\sigma_1\right)}\right)$$

$$< \ln\left((1-p)pe^{a_2\left(\frac{\underline{\underline{s_2}}}{2}-r\sigma_1\right)} + (1-p)^2 e^{a_2\left(\frac{\underline{\underline{s_2}}}{2}+\sigma_1\right)}\right)$$

$$\Leftrightarrow \frac{\partial U_2^{ES}(s_1{}^*), \underline{\underline{s_2}}}{\partial \underline{s_2}} < 0$$

Proof: (Proposition 6.2) Let p and r parameter values satisfy the condition $pr + p > 1$, N={1,2} and $a_1 = a_2 = a_i$ be the risk aversion levels. To prove, agents with identical risk aversion levels are better off in the 2-agent coalition, for $i \in N$ we must show that;

$$
\begin{aligned}
|U_i^{FL}(s_i^*)| - |U_i^{LS}(s_1^*, s_2^*)| &= (1-p)e^{a_i \frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}} + pe^{-ra_i \frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}} \\
&- p(1-p)e^{a_i \frac{\ln\left(\frac{2rp}{(1-p)}\right)}{2a_i\left(\frac{1}{2}+r\right)}} - p(1-p)e^{a_i\left(\frac{\ln\left(\frac{2rp}{(1-p)}\right)}{2a_i\left(\frac{1}{2}+r\right)} - \frac{r\ln\left(\frac{2rp}{(1-p)}\right)}{a_i\left(\frac{1}{2}+r\right)}\right)} \\
&- p^2 e^{-ra_i \frac{\ln\left(\frac{2rp}{(1-p)}\right)}{a_i\left(\frac{1}{2}+r\right)}} - (1-p)^2 e^{a_i \frac{\ln\left(\frac{2rp}{(1-p)}\right)}{a_i\left(\frac{1}{2}+r\right)}} > 0
\end{aligned}
$$

where $U_i^{FL}(s_i^*)$ is i-th agent's individual utility level and $U_i^{LS}(s_1^*, s_2^*)$ is i-th agent's utility level in 2-agent coalition under Loss Sharing rule. Then, since $1 + x - y \le e^{x-y}$, $\forall$ x,y $\in \mathbb{R}$, whenever $a_1 = a_2$, it's enough to show that,

$$
\begin{aligned}
&(1-p)\left(1 + \frac{\ln(rp)}{r+1} - \frac{\ln(1-p)}{r+1}\right) + p\left(1 + \frac{r\ln(1-p)}{r+1} - \frac{r\ln(pr)}{r+1}\right) \\
&- p(1-p)\left(1 + \frac{\ln(2rp)}{2\left(\frac{1}{2}+r\right)} - \frac{\ln(1-p)}{2\left(\frac{1}{2}+r\right)}\right) \\
&- p(1-p)\left(1 + \frac{(1-2r)\ln(2rp)}{2\left(\frac{1}{2}+r\right)} - \frac{(1-2r)\ln(1-p)}{2\left(\frac{1}{2}+r\right)}\right) \\
&- p^2\left(1 + \frac{r\ln(1-p)}{\left(\frac{1}{2}+r\right)} - \frac{r\ln(2pr)}{\left(\frac{1}{2}+r\right)}\right) - (1-p)^2\left(1 + \frac{\ln(2pr)}{\left(\frac{1}{2}+r\right)} - \frac{\ln(1-p)}{\left(\frac{1}{2}+r\right)}\right) \\
&> 0
\end{aligned}
$$

This inequality reduces to the following inequality which is strictly bigger than 0:

$$\underbrace{(1-pr-p)\frac{\ln(rp)}{r+1}}_{>0} + \underbrace{(pr+p-1)}_{>0}\underbrace{\left(\frac{\ln(1-p)}{r+1} - \frac{\ln(1-p)}{\left(\frac{1}{2}+r\right)}\right)}_{>0}$$
$$+ (pr+p-1)\underbrace{\frac{\ln(2rp)}{\left(\frac{1}{2}+r\right)}}_{>0} > 0$$

Then, $|U_i{}^{FL}(s_1{}^*)| - |U_i{}^{LS}(s_1{}^*, s_2{}^*)| > 0$, when $a_1 = a_2$. Moreover, we must show that there is no other stable group formation structure, in which agents with different risk aversion levels are better off. Assume there exist a group formation $\{\{a_1, a_2\}\}$ where $a_1 < a_2$ such that 1_{st} and 2_{nd} agent are better of compared to the group formations $\{\{a_1, a_1\}\}$ and $\{\{a_2, a_2\}\}$, respectively. First, since, $\frac{\partial U_1{}^{LS}(s_1,s_2)}{\partial a_2} = \frac{a_1 e^{a_1\left(\frac{\sigma_3}{2a_1(r+(1/2))}+\sigma_2\right)}\sigma_3(1-p)^2}{\sigma_1} + \frac{a_1 p(1-p)e^{a_1\left(-\frac{r\sigma_3}{a_1(r+(1/2))}+\sigma_2\right)}\sigma_3}{\sigma_1} > 0$ where $\sigma_1 = 2a_2{}^2(r+(1/2))$, $\sigma_2 = \frac{\sigma_3}{2a_2(r+(1/2))}$ and $\sigma_3 = \ln\left(\frac{2pr}{1-p}\right)$, relatively risk loving 1_{st} agent will be strictly better of whenever his partner's risk aversion level increase. However, relatively risk averse agent will be worse off in such group formation; since $\frac{\partial U_2{}^{LS}(s_1,s_2)}{\partial a_2} = -\frac{p(1-p)e^{a_2\left(\frac{\sigma_1}{\sigma_2}-\frac{r\sigma_1}{a_2(r+(1/2))}\right)}\sigma_1}{\sigma_2} - \frac{(1-p)^2 e^{a_2\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_1}{2a_2(r+(1/2))}\right)}\sigma_1}{\sigma_2} < 0$, $\sigma_2 = 2a_1(r+(1/2))$, where $\sigma_1 = \ln\left(\frac{2pr}{1-p}\right)$. Then this lead to unstable group formation structure. Secondly, since, $\frac{\partial U_2{}^{LS}(s_1,s_2)}{\partial a_1} = \frac{a_2 e^{a_2\left(\frac{\sigma_3}{2a_2(r+(1/2))}+\sigma_2\right)}\sigma_3(1-p)^2}{\sigma_1} + \frac{a_2 p(1-p)e^{a_2\left(-\frac{r\sigma_3}{a_2(r+(1/2))}+\sigma_2\right)}\sigma_3}{\sigma_1} > 0$, where $\sigma_1 = 2a_1{}^2(r+(1/2))$, $\sigma_2 = \frac{\sigma_3}{2a_1(r+(1/2))}$ and $\sigma_3 = \ln\left(\frac{2pr}{1-p}\right)$, 2_{nd} agent's utility will increase if his partner's risk aversion level increase. But in this case, 1_{st} agent's utility will decrease since;

$\frac{\partial U_1{}^{LS}(s_1,s_2)}{\partial a_1} = -\frac{p(1-p)e^{a_1\left(\frac{\sigma_1}{\sigma_2}-\frac{r\sigma_1}{a_1(r+(1/2))}\right)}\sigma_1}{\sigma_2} - \frac{(1-p)^2 e^{a_1\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_1}{2a_1(r+(1/2))}\right)}\sigma_1}{\sigma_2} < 0$, where $\sigma_2 = 2a_2(r+(1/2))$, $\sigma_1 = \ln\left(\frac{2pr}{1-p}\right)$. Then, we have contradiction.

Proof: (Proposition 6.1.2.)Assume $a_1 < a_2$. then, *Claim 1*: $\frac{\partial U_1^{LS}(\overline{s_1},s_2{}^*)}{\partial \overline{s_1}} = a_1p^2re^{-a_1r\overline{s_1}} - \frac{-a_1(1-p)^2e^{a_1((\overline{s_1}/2)+\sigma_1)}}{2} - \frac{a_1p(1-p)e^{a_1((\overline{s_1}/2)}}{2} + a_1pr(1-p)e^{-a_1(r\overline{s_1}-\sigma_1)}$ is strictly bigger than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for $s_2{}^* < \overline{s_1} < s_1{}^*$ values where and $\sigma_1 = ln\left(\frac{2pr}{1-p}\right)\frac{1}{2a_2(r+(1/2))}$.

Pf: By assumption, $s_1{}^* = \ln\left(\frac{2rp}{(1-p)}\right)\frac{1}{a_1(r+(1/2))} > \overline{s_1}$. Then,

$$\ln\left(\frac{2a_1rp}{a_1(1-p)}\right) > \frac{a_1\overline{s_1}}{2} + a_1r\overline{s_1}$$

$$\Leftrightarrow \ln(a_1pr) + \ln\left(e^{-a_1r\overline{s_1}}\right) + \ln\left(p+(1-p)e^{a_1\sigma_1}\right)$$

$$> \ln\left(\frac{a_1(1-p)}{2}\right) + \ln\left(e^{\frac{a_1\overline{s_1}}{2}}\right) + \ln\left(p+(1-p)e^{a_1\sigma_1}\right)$$

$$\Leftrightarrow \ln\left(a_1p^2re^{-a_1r\overline{s_1}} + a_1(1-p)pre^{-a_1(r\overline{s_1}-\sigma_1)}\right)$$

$$> \ln\left(\frac{a_1(1-p)pe^{\frac{a_1\overline{s_1}}{2}}}{2} + \frac{a_1(1-p)^2e^{a_1\left(\frac{\overline{s_1}}{2}+\sigma_1\right)}}{2}\right)$$

$$\Leftrightarrow \frac{\partial U_1^{LS}(\overline{s_1},s_2{}^*)}{\partial \overline{s_1}} > 0$$

Claim 2: $\frac{\partial U_2^{LS}(\overline{s_1},s_2{}^*)}{\partial \overline{s_1}} < 0$. Since, it's obvious that $\frac{\partial U_2^{LS}(\overline{s_1},s_2{}^*)}{\partial \overline{s_1}} = \frac{-a_2(1-p)pe^{a_2\left(\frac{\overline{s_1}}{2}-r\sigma_1\right)}}{2} - \frac{a_2(1-p)^2e^{a_2\left(\frac{\overline{s_1}}{2}+\frac{\sigma_1}{2}\right)}}{2} < 0$, where $\sigma_1 = ln\left(\frac{2pr}{1-p}\right)\frac{1}{a_2(r+(1/2))}$.

Claim 3: $\frac{\partial U_2^{LS}(s_1{}^*,\underline{s_2})}{\partial \underline{s_2}} = a_2p^2re^{-a_2r\underline{s_2}} - \frac{a_2(1-p)^2e^{a_2\left(\underline{s_2}/2)+\sigma_1\right)}}{2} - \frac{a_2p(1-p)e^{\frac{a_2\underline{s_2}}{2}}}{2} + a_2pr(1-p)e^{-a_2\left(r\underline{s_2}-\sigma_1\right)}$ is strictly smaller than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for $s_2{}^* < \underline{s_2} < s_1{}^*$ values where and $\sigma_1 = ln\left(\frac{2pr}{1-p}\right)\frac{1}{2a_1(r+(1/2))}$.

Pf: By assumption, $s_2{}^* = \ln\left(\frac{2rp}{(1-p)}\right)\frac{1}{a_2(r+(1/2))} < \underline{s_2}$. Then,

$$\ln\left(\frac{2a_2rp}{a_2(1-p)}\right) < \frac{a_2\underline{s_2}}{2} + a_2r\underline{s_2}$$

$$\Leftrightarrow \ln(a_2pr) + \ln\left(e^{-a_2r\underline{s_2}}\right) + \ln\left(p+(1-p)e^{a_2\sigma_1}\right)$$

$$< \ln\left(\frac{a_2(1-p)}{2}\right) + \ln\left(e^{\frac{a_2\underline{s_2}}{2}}\right) + \ln\left(p+(1-p)e^{a_2\sigma_1}\right)$$

$$\Leftrightarrow \ln\left(a_2p^2re^{-a_2r\underline{s_2}} + a_2(1-p)pre^{-a_2\left(r\underline{s_2}-\sigma_1\right)}\right)$$

$$< \ln\left(\frac{a_2(1-p)pe^{\frac{a_2\underline{s_2}}{2}}}{2} + \frac{a_2(1-p)^2e^{a_2\left(\frac{\underline{s_2}}{2}+\sigma_1\right)}}{2}\right)$$

$$\Leftrightarrow \frac{\partial U_2^{LS}({s_1}^*,\underline{s_2})}{\partial\underline{s_2}} < 0$$

Claim 4: $\frac{\partial U_1^{LS}({s_1}^*,\underline{s_2})}{\partial\underline{s_2}} < 0$. Since, it's obvious that
$\frac{\partial U_1^{LS}({s_1}^*,\underline{s_2})}{\partial\underline{s_2}} = \frac{-a_1(1-p)pe^{a_1\left(\frac{\underline{s_2}}{2}-r\sigma_1\right)}}{2} - \frac{a_1(1-p)^2e^{a_1\left(\frac{\underline{s_2}}{2}+\frac{\sigma_1}{2}\right)}}{2} < 0$, where
$\sigma_1 = ln\left(\frac{2pr}{1-p}\right)\frac{1}{a_1(r+(1/2))}$.

Proof: (Proposition 6.3) Let p and r parameter values satisfy the condition $pr > 2(1-p)$, N=$\{$1,2$\}$ and $a_1 = a_2 = a_i$ be the risk aversion levels. To prove, agents with identical risk aversion levels are strictly better off in the 2-agent coalition, for $i \in N$ we must show that;

$$|{U_i}^{FL}({s_i}^*)| - |{U_i}^{PS}({s_1}^*,{s_2}^*)| = (1-p)e^{a_i\left(\frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}\right)} + pe^{-ra_i\left(\frac{\ln\left(\frac{rp}{(1-p)}\right)}{a_i(1+r)}\right)}$$

$$- p(1-p)e^{a_i\left(\frac{-r\ln\left(\frac{rp}{2(1-p)}\right)}{2a_i\left(\frac{r}{2}+1\right)}\right)} - p(1-p)e^{a_i\left(\frac{\ln\left(\frac{rp}{2(1-p)}\right)}{a_i\left(\frac{r}{2}+1\right)} - \frac{r\ln\left(\frac{rp}{2(1-p)}\right)}{2a_i\left(\frac{r}{2}+1\right)}\right)}$$

$$- p^2e^{a_i\left(-r\frac{\ln\left(\frac{rp}{2(1-p)}\right)}{a_i\left(\frac{r}{2}+1\right)}\right)} - (1-p)^2e^{a_i\left(\frac{\ln\left(\frac{rp}{2(1-p)}\right)}{a_i\left(\frac{r}{2}+1\right)}\right)} > 0$$

where $U_i^{FL}(s_i{}^*)$ is i-th agent's individual utility level and $U_i^{PS}(s_1{}^*, s_2{}^*)$ is i-th agent's utility level in 2-agent coalition under Profit Sharing rule. Since $1 + x - y \leq e^{x-y}$, $\forall$ x,y $\in \mathbb{R}$, it's enough to show that;

$$
\begin{aligned}
&(1-p)\left(1+\frac{\ln(rp)}{r+1}-\frac{\ln(1-p)}{r+1}\right)+p\left(1+\frac{r\ln(1-p)}{r+1}-\frac{r\ln(pr)}{r+1}\right)\\
&-p(1-p)\left(1+\frac{r\ln(2(1-p))}{2\left(\frac{r}{2}+1\right)}-r\frac{\ln(pr)}{2\left(\frac{r}{2}+1\right)}\right)\\
&-p(1-p)\left(1+\left(1-\frac{r}{2}\right)\frac{\ln pr}{\left(\frac{r}{2}+1\right)}-\left(1-\frac{r}{2}\right)\frac{\ln(2(1-p))}{\left(\frac{r}{2}+1\right)}\right)\\
&-p^2\left(1+\frac{r\ln(2(1-p))}{\left(\frac{r}{2}+1\right)}-\frac{r\ln(pr)}{\left(\frac{r}{2}+1\right)}\right)\\
&-(1-p)^2\left(1+\frac{\ln(pr)}{\left(\frac{r}{2}+1\right)}-\frac{\ln(2(1-p))}{\left(\frac{r}{2}+1\right)}\right)>0
\end{aligned}
$$

This inequality reduces to the following expression which is strictly bigger than 0:

$$
2\underbrace{(pr+p-1)}_{>0}\underbrace{\left(\frac{\ln(pr)-\ln 2-ln(1-p)}{(r+2)}\right)}_{>0}+\underbrace{(pr+p-1)}_{>0}\underbrace{\left(\frac{\ln(pr)-ln(1-p)}{(r+1)}\right)}_{>0}
$$

Then, $|U_i^{FL}(s_1{}^*, s_2{}^*)| - |U_i^{PS}(s_1{}^*, s_2{}^*)| > 0$, when $a_1 = a_2$. Moreover, we must show that there is no other stable group formation structure, in which agents with different risk aversion levels are better off. Assume there exist a group formation $\{\{a_1,a_2\}\}$, where $a_1 < a_2$ such that 1_{st} and 2_{nd} agent are better of compared to the group formations $\{\{a_1,a_1\}\}$ and $\{\{a_2,a_2\}\}$, respectively. Firstly, more risk averse agent's utility strictly increases as he form a group with relatively risk seeking partner.

Since, $\frac{\partial U_2{}^{PS}(s_1,s_2)}{\partial a_1} = -\frac{a_2 p(1-p) r\sigma_1 e^{a_2\left(\frac{\sigma_1}{a_2((r/2)+1)} - \frac{r\sigma_1}{2a_1((r/2)+1)}\right)}}{\sigma_2}$
$-\frac{a_2 p^2 r\sigma_1 e^{-ra_2\left(\frac{\sigma_1}{2a_1((r/2)+1)} + \frac{\sigma_1}{2a_2((r/2)+1)}\right)}}{\sigma_2} < 0$, where $\sigma_1 = \ln\left(\frac{pr}{2(1-p)}\right)$ and $\sigma_2 = 2a_1{}^2((r/2)+1)$. However, as difference between risk aversion level increases for the sake of more risk averse agent, this increase will harm relatively risk loving agent; since,

$\frac{\partial U_1{}^{PS}(s_1,s_2)}{\partial a_1} = p^2 e^{-a_1 r\sigma_1}\left(r\sigma_1 - \frac{r\sigma_2}{2a_1((r/2)+1)}\right) + \frac{p(1-p)r\sigma_2 e^{a_1\left(\frac{\sigma_2}{a_1((r/2)+1)} - \frac{r\sigma_2}{2a_2((r/2)+1)}\right)}}{2a_2((r/2)+1)} > 0$,

where $\sigma_1 = \frac{\sigma_2}{a_1((r/2)+1)} + \frac{\sigma_2}{2a_2((r/2)+1)}$ and $\sigma_2 = \ln\left(\frac{pr}{2(1-p)}\right)$. Secondly, relatively risk loving agent's utility increases as he form a group with less risk averse agent. Since:

$\frac{\partial U_1{}^{PS}(s_1,s_2)}{\partial a_2} =$

$-\frac{a_1 p(1-p) r\sigma_1 e^{a_1\left(\frac{\sigma_1}{a_1((r/2)+1)} - \frac{r\sigma_1}{2a_2((r/2)+1)}\right)}}{\sigma_2} - \frac{a_1 p^2 r\sigma_1 e^{-ra_1\left(\frac{\sigma_1}{2a_1((r/2)+1)} + \frac{\sigma_1}{2a_2((r/2)+1)}\right)}}{\sigma_2} < 0$, where $\sigma_1 = \ln\left(\frac{pr}{2(1-p)}\right)$ and $\sigma_2 = 2a_2{}^2((r/2)+1)$. But, such group formation will strictly harm relatively risk averse agent. Since;

$\frac{\partial U_2{}^{PS}(s_1,s_2)}{\partial a_2} = p^2 e^{-a_2 r\sigma_1}\left(r\sigma_1 - \frac{r\sigma_2}{2a_2((r/2)+1)}\right) + \frac{p(1-p)r\sigma_2 e^{a_2\left(\frac{\sigma_2}{a_2((r/2)+1)} - \frac{r\sigma_2}{2a_1((r/2)+1)}\right)}}{2a_1((r/2)+1)} > 0$,

where $\sigma_1 = \frac{\sigma_2}{a_1((r/2)+1)} + \frac{\sigma_2}{2a_2((r/2)+1)}$ and $\sigma_2 = \ln\left(\frac{pr}{2(1-p)}\right)$. Then, we have contradiction.

Proof: (Lemma 6.1.3)Assume $a_1 < a_2$. then, *Claim 1.* $\frac{\partial U_1^{PS}(\overline{s_1},s_2{}^*)}{\partial \overline{s_1}} =$
$-a_1 p(1-p)e^{a_1(\overline{s_1}-r\sigma)} - a_1(1-p)^2 e^{a_1\overline{s_1}} + \frac{a_1 p^2 r e^{-a_1 r\left(\frac{\overline{s_1}}{2}+\sigma\right)}}{2} + \frac{-a_1 p(1-p) r e^{\left(-\frac{a_1 r\overline{s_1}}{2}\right)}}{2}$ is strictly bigger than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for $s_2{}^* < \overline{s_1} < s_1{}^*$ values where and $\sigma = ln\left(\frac{pr}{2(1-p)}\right)\frac{1}{2a_2(1+(r/2))}$.

Pf: By assumption, $s_1{}^* = \ln\left(\frac{rp}{2(1-p)}\right)\frac{1}{a_1(1+(r/2))} > \overline{s_1}$. Then,

$$\ln\left(\frac{a_1 rp}{2a_1(1-p)}\right) > \frac{a_1 r\overline{s_1}}{2} + a_1\overline{s_1}$$

$$\Leftrightarrow \ln\left(\frac{a_1 pr}{2}\right) + \ln\left(e^{\frac{-a_1 r\overline{s_1}}{2}}\right) + \ln\left((1-p) + pe^{-a_1 r\sigma}\right)$$

$$> \ln(a_1(1-p)) + \ln\left(e^{a_1\overline{s_1}}\right) + \ln\left((1-p) + pe^{-a_1 r\sigma}\right)$$

$$\Leftrightarrow \ln\left(\frac{a_1 p^2 re^{-a_1 r\left(\frac{\overline{s_1}}{2}+\sigma\right)}}{2} + \frac{a_1 p(1-p)re^{\left(\frac{-a_1 r\overline{s_1}}{2}\right)}}{2}\right)$$

$$> \ln\left(a_1 p(1-p)e^{a_1(\overline{s_1}-r\sigma)} + (1-p)^2 a_1 e^{a_1\overline{s_1}}\right)$$

$$\Leftrightarrow \frac{\partial U_1^{PS}(\overline{s_1}, {s_2}^*)}{\partial\overline{s_1}} > 0$$

Claim 2. $\frac{\partial U_2^{PS}(\overline{s_1},{s_2}^*)}{\partial\overline{s_1}} > 0$. Since, it's obvious that $\frac{\partial U_2^{PS}(\overline{s_1},{s_2}^*)}{\partial\overline{s_1}} = \frac{a_2 p^2 re^{-a_2 r\left(\frac{\overline{s_1}}{2}+\frac{\sigma}{2}\right)}}{2} + \frac{a_2 pr(1-p)e^{a_2\left(\frac{r\overline{s_1}}{2}-\sigma\right)}}{2} > 0$, where $\sigma = ln\left(\frac{pr}{2(1-p)}\right)\frac{1}{a_2(1+(r/2))}$.

Claim 3. $\frac{\partial U_2^{PS}({s_1}^*,\underline{s_2})}{\partial\underline{s_2}} = -a_2 p(1-p)e^{a_2(\underline{s_2}-r\sigma)} - a_2(1-p)^2 e^{a_2\underline{s_2}} + \frac{a_2 p^2 re^{-a_2 r\left(\frac{\underline{s_2}}{2}+\sigma\right)}}{2} + \frac{a_2 p(1-p)re^{\left(\frac{-a_2 r\underline{s_2}}{2}\right)}}{2}$ is strictly smaller than 0 for sufficiently high values of p, for any $r \in (0,1)$ and for ${s_2}^* < \underline{s_2} < {s_1}^*$ values where and $\sigma = ln\left(\frac{pr}{2(1-p)}\right)\frac{1}{2a_1(1+(r/2))}$.

Pf: By assumption, ${s_2}^* = \ln\left(\frac{rp}{2(1-p)}\right)\frac{1}{a_2(1+(r/2))} < \underline{s_2}$. Then,

$$\ln\left(\frac{a_2rp}{2a_2(1-p)}\right) < \frac{a_2r\underline{s_2}}{2} + a_2\underline{s_2}$$

$$\Leftrightarrow \ln\left(\frac{a_2pr}{2}\right) + \ln\left(e^{\frac{-a_2r\underline{s_2}}{2}}\right) + \ln\left((1-p) + pe^{-a_2r\sigma}\right)$$

$$< \ln(a_2(1-p)) + \ln\left(e^{a_2\underline{s_2}}\right) + \ln\left((1-p) + pe^{-a_2r\sigma}\right)$$

$$\Leftrightarrow \ln\left(\frac{a_2p^2re^{-a_2r\left(\frac{\underline{s_2}}{2}+\sigma\right)}}{2} + \frac{a_2p(1-p)re^{\left(\frac{-a_2r\underline{s_2}}{2}\right)}}{2}\right)$$

$$< \ln\left(a_2p(1-p)e^{a_2(\underline{s_2}-r\sigma)} + (1-p)^2a_2e^{a_2\underline{s_1}}\right)$$

$$\Leftrightarrow \frac{\partial U_2^{PS}({s_1}^*, \underline{s_2})}{\partial \underline{s_2}} < 0$$

Claim 4. $\frac{\partial U_1^{PS}({s_1}^*,\underline{s_2})}{\partial \underline{s_2}} > 0$. Since, it's obvious that $\frac{\partial U_1^{PS}({s_1}^*,\underline{s_2})}{\partial \underline{s_2}} = \frac{a_1p^2re^{-a_1r\left(\frac{\underline{s_2}}{2}+\frac{\sigma}{2}\right)}}{2} + \frac{a_1p(1-p)re^{-a_1\left(\frac{r\underline{s_2}}{2}-\sigma\right)}}{2} > 0$, where $\sigma = ln\left(\frac{pr}{2(1-p)}\right)\frac{1}{a_1(1+(r/2))}$.

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