FINITE ELEMENT APPLICATION IN THE ANALYSIS OF GROUND -- WATER LOWERING AND WELL YIELD

A MASTER'S THESIS

IN

CIVIL ENGINEERING

Middle East Technical University

By

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February,1988

Approval of the Graduate School of Natural and Applied Sciences.

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TO MY MOTHER

AND

FATHER

The Results of the Method of Solution and the Computer Program in this Thesis belong exclusively to the Middle East Technical University.

ABSTRACT

FINITE ELEMENT APPLICATION
IN THE ANALYSIS
OF GROUND-WATER LOWERING
AND
WELL YIELD

KOŞGAN, Gülgün

M.S. in Civil Eng.
Supervisor: Prof.Dr. A. Altay Birand
February 1988, 96 pages

Partial differential equations may be used to describe a large number of problems in ground-water hydrology. Only a simplified subset of general equations can be solved by analytic means, and these often describe idealized situations that are limited in application.

Numerical solution of these equations using computers offers a logical alternative. Ground-water modelling is needed for numerical solution.

Ground-water modelling begins with a conceptual understanding of physical problem. The next step in modelling is translating the physical system into mathematical terms. These equations, however, are often simplified, using site-specific assumptions, to form a variety of equations subsets. An understanding of these equations and

their associated boundary and initial conditions is necessary before a modelling problem can be formulated.

In this study, the computer program which is written by the thesis author is used to analyse steady-state two dimensional ground-water flow in phreatic aquifers as a result of discharging wells by using finite element method. The analysis is made under three groups, namely,

(a) Horizontal plane flow analysis
(b) Axisymmetric flow analysis
(c) Vertical plane flow analysis

Also, a detailed example is given for the each type of analysis and the comparison of analytical and numerical solution in the case of horizontal plane flow analysis is shown on a given example.

Key words: partial differential equations, ground-water modelling, steady-state two dimensional ground-water flow, phreatic aquifers, finite element method.

ÖZET

SONLU ELEMANLAR METODU KULLANILARAK YERALTI SU SEVİYESİNİN AÇILAN KUYULARLA DÜŞÜRÜLMESİ ÜZERİNE BİR ÇALIŞMA

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Yüksek Lisans Tezi, İnş.Müh.Bölümü
Tez Yöneticisi: Prof.Dr. A. Altay Birand
Şubat 1988, 96 sahife

Yeraltı suyu hidrolojisi ile ilgili problemler genellikle diferansiyel denklemler kullanılarak tanımlanır. Bu denklemlerin, uygulama alanı çok kısıtlı ve indirgenmiş bir bölümü için analitik çözümler uygulanabilir.

Analitik çözümleri olmayan durumların, kompüterler kullanılarak yapılan nümerik çözümleri büyük kolaylıklar sağlar. Nümerik çözümlerde ise sistemin modellenmesi büyük önem taşır.

Modelleme, problemin fiziksel özelliklerinin ortaya konulmasıyla başlar; bundan sonraki basamak ise, fiziksel olarak tanımlanmış modelin, matematiksel ifadeler kullanılarak gösterilmesidir. Oluşturulan denklemlerin , çözümleri var olan denklemlerle karşılaştırılmaları sonucu gerekli basitleştirmeler yapılır, çünkü, denklemlerin çözülebilmesi için tek değişkene bağlı lineer özellik taşıması gerekir. Formülasyona geçilmeden önce ise, son

basamak olarak, elde edilen indirgenmiş denklem kümeleri için sınır şartları belirlenmelidir.

Bu tez çalışmasında, hazırlanan kompüter programıyla, yeraltı suyunun, kuyulardan su çekilmesi sonucu oluşan, iki yönlü durgun akımı, sonlu elemanlar metodu kullanılarak üç alt grup halinde incelenmiştir. Bunlar sırasıyla,

(a) Akımın yatay düzlemde incelenmesi

(b) Akımın aksisimetrik düzlemde incelenmesi

(c) Akımın dikey düzlemde incelenmesi

Aynı zamanda, her grup için detaylı bir örnek verilmiştir. Bunlara ek olarakda, yatay düzlemdeki akımın analitik ve nümerik çözümlerinin karşılaştırılması bir örnek üzerinde gösterilmiştir.

Anahtar kelimeler: diferansiyel denklemler, yeraltı suyu modellemesi, iki yönlü durgun yeraltı suyu akımı, sonlu elemanlar metodu.

ACKNOWLEDGEMENT

The author wishes to express sincere gratitude to Prof.Dr. A. Altay Birand for his guidance and valuable suggestions throughout the research.

Very special thanks are extended to Prof.Dr. Doğan Altınbilek and Asst.Prof.Dr. Nurkan Karahanoğlu for their suggestions and comments.

Thanks are also extended to Research Assistant Oğuz Çalışan for his help in using graphics software, and Mrs. Naile Canbaz for typing the manuscript .

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NOMENCLATURE

x, y, z	:	labels of global coordinate system
$\bar{x}, \bar{y}$	:	labels of element coordinate system
K	:	permeability of an aquifer
Q	:	rate of flow(volume of water flowing per unit time)
A	:	cross-sectional area of filter
L	:	length of filter
I	:	hydraulic gradient
h	:	piezometric head
q	:	specific discharge (volume of water flowing per unit time through a unit cross-sectional area)
p	:	pressure
γ	:	specific weight of water
z	:	elevation head
ϕ	:	piezometric head or potential
∇	:	(grad) operator or (div) operator
θ	:	angle between the horizontal and phreatic surface
s	:	label of direction tangent to the phreatic surface
h_o	:	initial elevation of ground-water surface from impervious bottom.
h_s	:	height of seepage line at the well face
h_w	:	height of water surface in the well
N	:	rate of precipitation
ρ	:	density of water
S_o	:	specific volume storativity

$\bar{Q}$: volume of water which is added or substracted per unit time and unit volume

$\delta(.)$: Dirac delta function

T : transmissivity of aquifer

$\bar{T}$: average constant transmissivity

$\overset{o}{T}$: deviation from average transmissivity

u : name of a dummy variable

f_o : rate of pumping

$\hat{K}$: name of a constant obtained by K/f_o

ϕ_o : potential value at the boundaries of an aquifer

D : label of domain

x_o, y_o: x and y coordinates of pumping well in analytic solution

L : an operator ($K\nabla^2$)

λ : eigen value

Γ : label of domain in eigen function description

f : name of a dummy function of x

g : name of a dummy function of y

A,B,C: names of constants

η : name of a dummy constant

a_{nm} : name of a dummy constant

a : length of side of a rectangular domain in x-direction

b : length of side of a rectangular domain in y-direction

G : Green function

A_{nm} : name of a dummy constant

I : functional name

U_1, U_2: dummy variable names

F : dummy function name

$N(x,y)$: linear interpolation function

δ : value of a linear interpolation function

a,b,c : names of constants in the polynomial describing the potential surface

Δ : area of a finite element

ψ : streamline function

U_i : generalized displacement at a point i

r : radial distance from a point to a well

1. INTRODUCTION

Numerical models provide the most general tool for the analysis of ground-water applications. They are not subject to many of the restrictive assumptions required for familiar analytical solutions. In spite of the flexibility of numerical models, their mathematical basis is actually less sophisticated than that of analytical methods.

In this thesis, one of the most commonly used numerical tecniques,which is finite element method is used for a special type of problem of ground-water lowering with discharging wells. On the other hand, understanding of this special case leads us to solve variety of problems on the same field.

To develop a numerical model of physical system (in our case, an aquifer), it is first necessary to understand how that system behaves. This understanding takes the form of laws and concepts. These concepts and laws are then translated into mathematical expressions, usually partial differential equations, with boundary and initial conditions. In our case boundary conditions became more essential due to analysis of flow, independent on time (steady state).

Numerical solution of such a problem involves approximating continuous (defined at every point) partial differential equations with a set of discrete equations in space. Thus, the region of interest are divided in some fashion, resulting in an equation or set of equations for each subregion.

In the following parts of the study, analytical and numerical solution of the problem, will be discussed, and compared with a number of examples.

2.1. Ground-water Motion

The following parts are partly summarized from Bear,J.,1972, otherwise it is stated.

As part of hydrologic cycle, ground-water is always in motion from regions of natural and artificial replenishment, to those of natural and artificial discharge.

One of the main characteristics of ground-water motion is that it occurs at very, sometimes extremely, low velocities. However, because of the large cross-sectional areas through which this motion takes place, large quantities of water are transported. The word, flow, throughout the text will mean saturated flow. In saturated flow, water completely fills the void space of the considered porous medium domain.

In an aquifer, flow takes place through a complex network. However, when dealing with flow in an aquifer, the microscopic flow patterns inside individual pores are overlooked and some fictitious average flow which takes place in the porous medium comprising the aquifer is considered.

By doing so, the concept of a continuum is employed. The reason for employing the continuum approach in flow through a porous medium is that it is practically impossible to describe in any exact mathematical manner the complicated geometry of the solid surfaces that bound the flow domain. Therefore, the values assigned to a point in the continuum, or macroscopic level of descriptions, are averaged ones, taken over the representative elementary volume centered at that point.

Homogenity and isotropy with respect to seepage of a porous medium refer to its property named permeability (K).

A porous medium domain is said to be homogeneous if its permeability is the same at all its points. Otherwise, the domain is heterogeneous. If, the permeability at a considered point is independent of direction, the medium is said to be isotropic at that point. Otherwise, it is anisotropic.

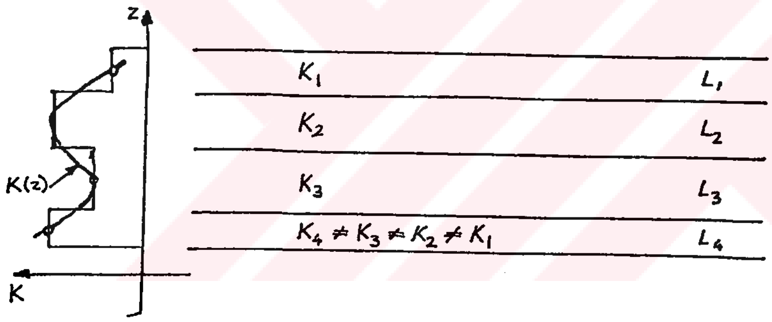


Figure 2.1. A layered aquifer

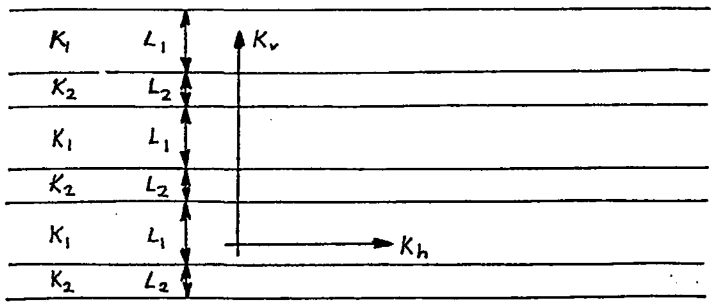


Figure 2.2. Aquifer composed of alternating layers exhibits anisotropy with $K_v \neq K_h$

Usually there are two types of inhomogeneous aquifer domains. Type 1, with a gradual change in transmissivity, and Type 2, with abrupt changes across well-defined surfaces of discontinuity.

Figure 2.1 shows how a layered aquifer (Type 1 inhomogeneity) may be considered as an inhomogeneous aquifer with a gradual variation of permeability.

In many cases aquifers are anisotropic. This may happen, for example, when the sediments comprising the aquifer are such (e.g., flat shaped mica particles) that when deposited, the resulting porous medium has a higher permeability in one direction (usually the horizontal one, unless later tilting of the formation occurs) than in other directions.

An inhomogeneous material composed of alternating layers of different textures (Fig. 2.2) is equivalent in its behaviour to an homogeneous anisotropic medium.

However, in order for a stratified formation of this kind to be considered as an equivalent homogeneous anisotropic aquifer, the thickness of the individual layers must be much smaller than lengths of interest.

2.1.1. Darcy's Law

In 1856, Henry Darcy investigated the flow of water in vertical homogeneous sand filters.

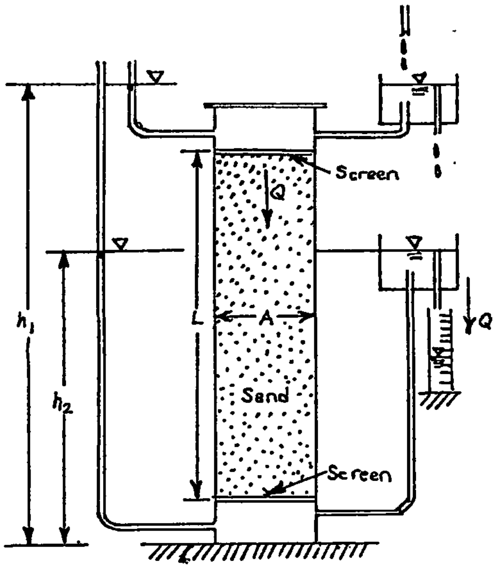


Figure 2.3. Darcy's experiment

Figure 2.3 shows the experimental set up he employed. From his experiment he concluded that the rate of flow (e.g., volume of water per unit time), Q, is (a) proportional to the cross sectional area A, (b) proportional to (h_1-h_2), and (c) inversely proportional to the length L, these conclusions give the famous Darcy formula (or law)

$$Q = KA(h_1-h_2)/L \qquad (2.1)$$

The lengths h_1 and h_2 are measured with respect to some arbitrary (horizontal) datum level. Here, h is the piezometric head and h_1-h_2 is the difference in piezometric head across the filter length L. In the formula $(h_1-h_2)/L$ is treated as hydraulic gradient. Denoting this gradient by $I(=(h_1-h_2)/L)$ and defining the specific discharge, q, as the volume of water flowing per unit time through a unit cross-sectional area normal to the direction of flow, we obtain

$$q = KI \qquad (2.2)$$

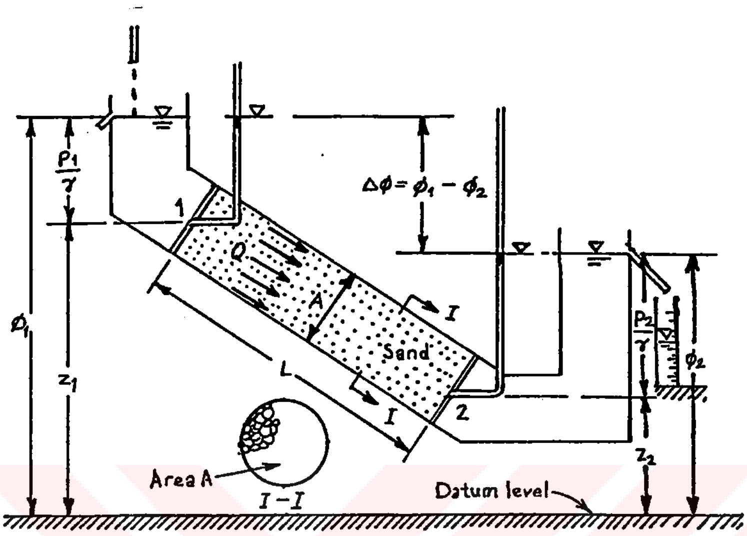


Figure 2.4. Seepage through an inclined sand filter

Figure 2.4 shows how Darcy's law (2.1) may be extended to flow through an inclined homogeneous porous medium column

$$Q=KA(\phi_1-\phi_2)/L \quad ; \quad q=K(\phi_1-\phi_2)/L = KI \quad ; \quad \phi=z+p/\gamma \qquad (2.3)$$

where p is pressure and γ is specific weight of water and p/γ is called pressure head.

The sum of the pressure head and the elevation head is the piezometric head ϕ.

The experimentally derived equation of Darcy's law (2.3) is limited to one dimensional flow of a homogeneous incompressible fluid.

When the flow is three-dimensional, the generalization of (2.3) is

$$q = KI = -K \text{ grad } \phi \qquad (2.4)$$

q is the specific discharge vector with components q_x, q_y, q_z in the directions of the cartesian coordinates, and $I = -\text{grad } \phi = \nabla\phi$ is the hydraulic gradient, with components $I_x = -\partial\phi/\partial x$, $I_y = -\partial\phi/\partial y$, $I_z = -\partial\phi/\partial z$. If the medium is homogeneous and isotropic, the coefficient of permeability K is a constant scalar, and (2.4) may be written as

$$q_x = KI_x = -K\partial\phi/\partial x \qquad q_y = KI_y = -K\,\partial\phi/\partial y \qquad (2.5)$$

$$q_z = KI_z = -K\partial\phi/\partial z$$

the vector q is everywhere normal to the equipotential surface ϕ= constant.

2.1.2. Dupuit Assumption for a Phreatic Aquifer

In a phreatic aquifer, water table serves as its upper boundary.

Both ϕ and q vary from point to point within a phreatic aquifer. In order to obtain the specific discharge $q = q(x,y,z,t)$ at every point, piezometric head $\phi = \phi(x,y,z,t)$ should be known.

The Dupuit assumptions are the most powerful tool for treating unconfined flows.

In most ground-water flows, the slope of the phreatic surface is very small.

In steady flow, (Fig. 2.5) the phreatic surface is a streamline.

At every point P along this streamline, the specific discharge is in a direction tangent to the streamline and is given by Darcy's law

$$q_s = -K\, d\phi/ds = -K dz/ds = -K \sin\theta \qquad (2.6)$$

since along the phreatic surface $p = 0$ and $\phi = z$. As θ is very small, Dupuit suggested that $\sin\theta$ be replaced by the slope $\tan\theta = dh/dx$. The assumption of small θ is equivalent to assuming that equipotential surfaces are vertical (that is, $\phi = \phi(x)$ rather than $\phi = \phi(x,z)$) and the flow is horizontal. Then, the Dupuit assumptions lead to the specific discharge expressed by

$$q_x = -K dh/dx \qquad h = h(x) \qquad (2.7)$$

In general, $h = h(x,y)$ and we have

$$q_x = -K\partial h/\partial x, \quad q_y = -K\partial h/\partial y; \quad h = h(x,y) \qquad (2.8)$$

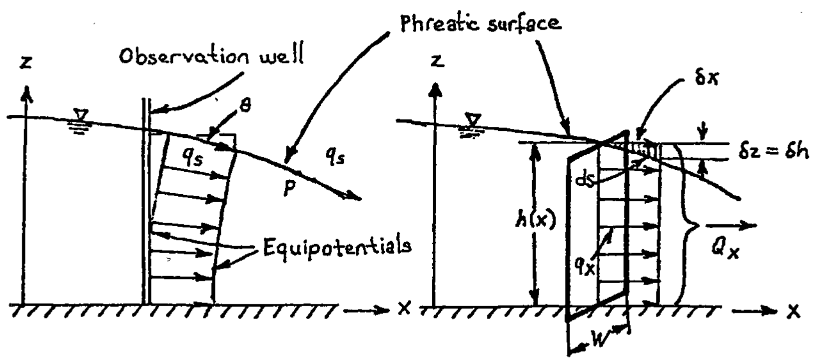


Figure 2.5. The Dupuit assumptions.

Dupuit assumptions may be considered as a good approximation in regions where θ is small and/or the flow is essentially horizontal.

The advantage of Dupuit assumptions is that $\phi = \phi(x,y,z)$ has been replaced by $h = h(x,y)$, that is, z does not appear as an independent variable. Also, since at a point on the free surface, $p = 0$ and $\phi = h$, it is assumed that the vertical line through the point is also an equipotential line on which $\phi = h = \text{const.}$

The Dupuit assumptions cannot be applied in regions where the vertical flow component is not negligible. Such flow conditions occur as a seepage face is approached (Fig.2.6a) or at a crest (water divide) in a phreatic aquifer with accretion (Fig.2.6b). Another example is the region close to the impervious vertical boundary of Fig. 2.6a.

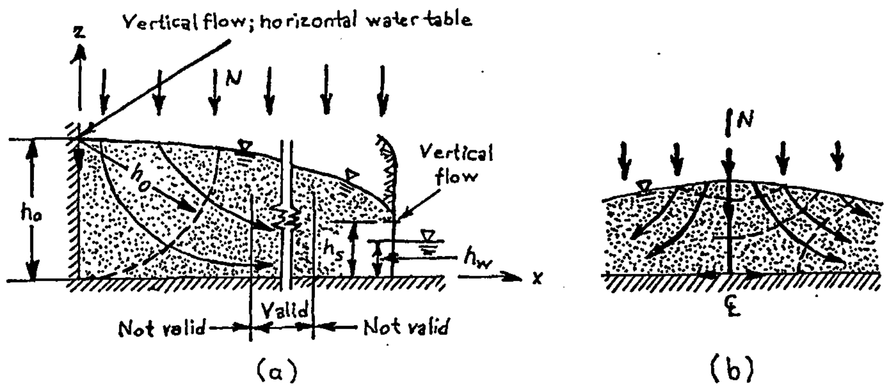


Figure 2.6. Regions where Dupuit assumptions are not valid.

2.2. Mathematical Statement of G.W. Problem

The basic laws governing the flow of water in phreatic and confined aquifers are introduced in the last three sections. It is apparent that, as in equation (2.4), there is one equation with two dependent variables: $q(x,y,z)$ and $\phi(x,y,z)$. This means that one additional equation is required in order to obtain a complete description of the flow in an aquifer. The additional basic law is the conservation of mass, which here takes the form of a continuity equation.

$$-\text{div } (\rho q) = \rho S_0 \ \partial\phi/\partial t \qquad (2.9)$$

The following treatment for describing the steady flow in phreatic aquifers is used (Bear, 1972).

Equation (2.9) is the continuity equation which relates the specific volume storativity S_0, to the elastic properties of the medium and the water. It gives the mass of water added to the storage (or released from it) in a unit volume of porous medium per unit rise (or decline) of potential ϕ. When the flow is steady (that is, $\partial\phi/\partial t = 0$) and/or when both fluid and solid matrix are incompressible (that is, $S_0 = 0$ and ρ = const.), or assumed so(as in an unconfined aquifer), Equation (2.9) reduces to

$$\text{div } q = 0 \qquad (2.10)$$

The next step is to introduce an equation of motion (e.g., an expression for q) into the continuity equation (2.9). Darcy's law gives the motion of ground-water with respect to the solid matrix, but in (2.9) q is with respect to the fixed coordinate system. Then, it should be taken account of the fact that here, a consolidating medium is considered and also, it should be accounted that the movement of the solid matrix with respect to the fixed coordinate system.

But, for practical purposes, for the derivation of the following equation, it is assumed that:

(a) The velocity of the solids is so small that q in (2.9) and (2.10) may be expressed by Darcy's law (2.4).

(b) K is constant.

(c) S_0 and K are unaffected by variations in porosity due to matrix deformability.

With these assumptions, (2.9) can be written in terms of ϕ.

$$-\text{div } q = \text{div}(K.\text{grad }\phi) = S_0 \, \partial\phi/\partial t \qquad (2.11)$$

For a homogeneous isotropic medium, (2.11) reduces to

$$K\nabla^2\phi = K \text{ div}(\text{grad } \phi) = K\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = S_0 \frac{\partial\phi}{\partial t} \qquad (2.12)$$

If the flow is steady, (2.12) reduces to the Laplace equation.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (2.13)$$

In the continuity equation (2.9), it is seen that for the considered domain, no sources or/and sinks are included. If sources and/or sinks are present, they should be represented by an additional term on the left-hand side of (2.9) expressing the rate at which the mass of water is added or substracted per unit time and unit volume of porous medium.

If the sources and/or sinks are considered, (2.12) may be written as

$$K\nabla^2\phi \pm \bar{Q} = S_o \frac{\partial\phi}{\partial t} \qquad (2.14)$$

in (2.14) $\bar{Q}$ represents sources and/or sinks.

2.3. Fundamentals of Flow Through Porous Medium

Equations (2.11) to (2.14) are partial differential equations with no information (e.g., the shape of the flow domain) related to any specific case of flow through a porous medium.

Therefore, each equation has an infinite number of solutions, corresponding to a particular case of flow through a porous medium domain.

To obtain one particular solution corresponding to a certain specific problem of interest, supplementery information is required. This supplementary information should include the following specifications:

(a) The geometry of the domain in which the considered flow takes place.

(b) Values of all physical coefficients (e.g., K, S_0, $\bar{Q}$).

(c) Initial conditions which correspond the initial state in the flow domain.

(d) Conditions on the boundaries of the considered flow domain.

Those requirements are necessary for the analytic solution and as well as for the numerical solution of the problem of ground-water flow through a porous medium domain.

In the equations (2.11) to (2.14), the dependent variable is ϕ for which a solution is sought in the form of $\phi = \phi(x,y,z,t)$. Hence, initial and boundary conditions should be specified in terms of ϕ.

For the steady flow, boundary conditions become essential. The various types of boundary conditions encountered in flow through porous medium domain.

(a) Boundary of prescribed potential: The potential, ϕ , is prescribed for all points of this boundary. A boundary of this kind is an equipotential surface. Since the piezometric head is the same at all points on this surface.

In the theory of partial differential equations, a problem with this type of boundary conditions is called boundary value problem.

(b) Boundary of prescribed flux: The flux normal to the boundary is prescribed for all points.

A special case of this type of boundary is the impervious boundary, where the flux normal to the boundary vanishes every-where.

2.3.1. Steady State Flow in Phreatic Aquifers

The partial differential equations are so far developed for the general case of flow through a porous medium. These equations are applicable whenever the flow is three dimensional or two dimensional in vertical plane. When the flow in the aquifer is treated as two dimensional flow in horizontal plane, the governing equations should be modified. To derive the necessary equations, a control box should be considered in a phreatic aquifer (Fig. 2.7). A rate of externally applied flux $\bar{Q}=\bar{Q}(x,y)$ positive should be added when vertically downward. This flux may be the net effect of natural replenishment, artifical recharge, and pumping. All these inputs, or outputs can be introduced as distributed sources and sinks or as point ones.

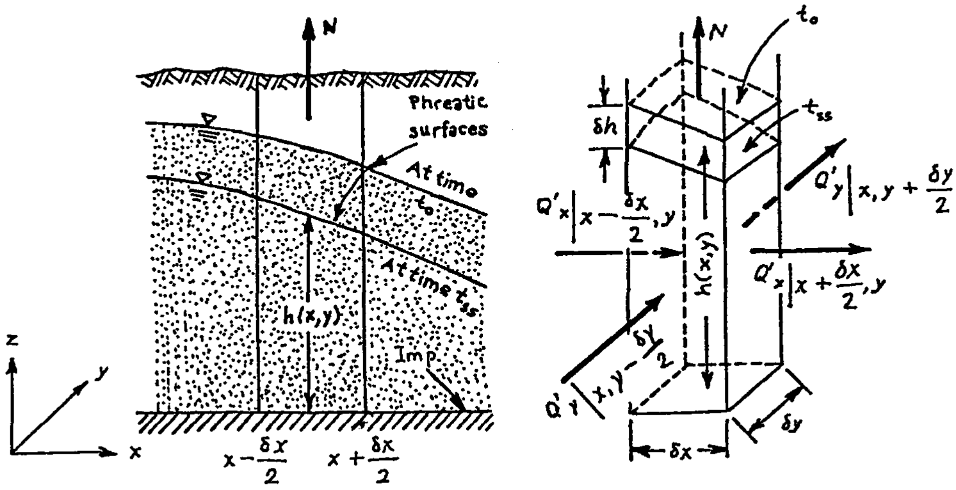


Figure 2.7. Flow in a phreatic aquifer

If the sources and sinks are introduced as point ones, the Dirac delta function is used to describe them

$$\bar{Q}(x,y) = \Sigma\bar{Q}(x_i,y_i)\ \delta\ (x-x_i,\ y-y_i) \qquad (2.15)$$

The balance equation based on the Dupuit assumption of horizantal flow is

$$\delta y\ [Q'_x(x-\frac{\delta x}{2},y)-Q'_x(x+\frac{\delta x}{2},y)] + \delta x\ [Q'_y(x,y-\frac{\delta y}{2})-Q'_y(x,y+\frac{\delta y}{2})]$$

$$+\ \bar{Q}\ \delta x\delta y = 0 \qquad (2.16a)$$

Expressing Q' (=T.grad ϕ) and dividing both sides of (2.15) by $\delta x\delta y$ and letting $\delta x,\delta y \to 0$, the equation for an inhomogeneous isotropic aquifer, in which $K = K(x,y)$, is obtained that

$$\frac{\partial}{\partial x}\ (K_x h\ \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}\ (K_y h\ \frac{\partial h}{\partial y}) + \bar{Q} \doteq 0 \qquad (2.16b)$$

For a homogeneous aquifer, K constant, it is obtained that

$$K\ [\frac{\partial}{\partial x}\ (h\ \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}\ (h\ \frac{\partial h}{\partial y})] + \bar{Q} = 0 \qquad (2.17)$$

This is the basic continuity equation for steady state ground-water flow in a phreatic aquifer with a horizontal base. It is called the Boussinesq equation.

Equation (2.16b) and (2.17), are non-linear (because of the product $h\,\partial h/\partial x$). The product Kh in (2.16) and in (2.17) represents the transmissivity, T, of the phreatic aquifer. However, here it may vary in space, as $h = h(x,y)$.

In order to have a solution, the methods of linearization can be applied to (2.16).

(i) Assume that $T = \bar{T} + \mathring{T}$; $\bar{T}(>> \mathring{T})$ is the average constant transmissivity of phreatic flow and $\mathring{T}$ is a deviation from the average. Then (2.16) reduces to the linear equation in h.

$$\bar{T}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) + \bar{Q} = 0 \qquad \bar{T} = K\bar{h} \qquad (2.18)$$

(ii) From (2.17), also the following equation can be written

$$K\left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2}\right) + 2\bar{Q} = 0 \qquad (2.19)$$

which is linear in h^2.

The approximation in the linearization is justified according to the relatively small changes in h(with respect to the total thickness h) in most phreatic aquifers. Whenever the situation is different, equations (2.16) and (2.17) should be used for the problems.

In equations (2.16) and (2.17), h can be replaced by ϕ(measured from the same datum level as h). Therefore, the general continuity equation describing the steady flow in porous medium is reached.

2.4. Methods for Solving Ground-water Flow Problems

In the previous sections, the general continuity equation of ground-water flow is put and the requirements in order to have a solution is shown.

Then, by the aid of necessary information that is given, equations, expressing the case (that is the steady ground-water flow in phreatic aquifers as a result of externally applied flux) which is interested, are developed.

To obtain $h(x,y)$ or $\phi(x,y)$, the partial differential equation should be solved for the specified initial and boundary conditions.

In principle, there are three methods for solving this kind of problems.

(a) Analytical methods

(b) Methods based on the use of models and analogs.

(c) Numerical methods

In order to choose the method desirable for the problem, all facts (e.g., time, cost) have to be considered.

2.4.1. Analytical Method

Analytical methods are superior to any of the other ones. Because, the influence of each parameter can be clearly observed. For one-dimensional cases, it is easier to derive analytical solutions.

In this part, the analytical solution of the problem of two-dimensional horizontal steady flow will be shown in a phreatic aquifer with the externally applied flux at a point.

Step I involves the understanding of the physical system. In place, a infinitely extending phreatic aquifer, with the horizontal impervious layer underlying it, is considered.

The porous medium domain is rectangular, and, there is a discharging well at a point in the prescribed domain.

Step II is related with expressing the physical system in mathematical terms, that is, in such a case, it is a partial differential equation similar to (2.18).

This partial differential equation is classified as an elliptic partial differential equation in mathematics. It is identical to the Poisson equation that has the form(Pinder,G.F.,Gray,W.G.,1977).

$$(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = f(x,y)$$

Classification of a partial differential equation is necessary in order to choose the way of using theorems effectively in the field of mathematics. Hence; such a case can be described by the following equation as

$$-K\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + f = 0 \qquad (2.20)$$

in this equation, ϕ is related to the piezometric head (measured from the bottom of the aquifer), f is rate of pumping and, K is the coefficient of permeability (or hydraulic conductivity). For the sake of simplicity, K value in x-direction and in y-direction is choosen as being the same.

Step III is related to initial and boundary conditions of the prescribed physical system. In (2.20) dependent variable is ϕ, and obviously, boundary conditions have to be given in terms of ϕ. In (Fig. 2.8), two dimensional horizantal flow domain is shown.

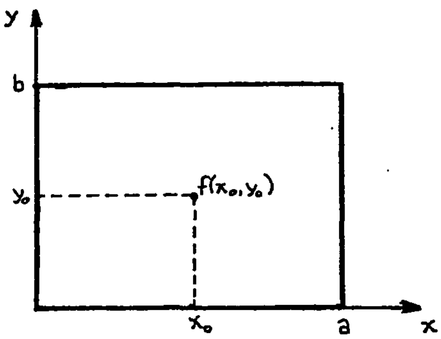


Figure 2.8. Two dimensional flow domain.

The boundary conditions are:

for $x = 0$, $0 \leqslant y \leqslant b$; $\phi = \phi_0$

for $x = a$, $0 \leqslant y \leqslant b$; $\phi = \phi_0$

for $y = 0$, $0 \leqslant x \leqslant a$; $\phi = \phi_0$

for $y = b$, $0 \leqslant x \leqslant a$; $\phi = \phi_0$

which describes an infinitely extent aquifer.

Step IV is the solution; in this case the method which is called FINDING GREEN FUNCTION BY THE METHOD OF EIGEN FUNCTION EXPANSION is used (Jackson, 1962), as can be seen in the following treatment:

In the equation (2.20), $f(= f(x_0,y_0))$ is describing the source term. If it is dealt with a point source or sink, Dirac delta function should be used, because in this way, a point source or sink may be included in description of the source term.

The Dirac delta function has a property that;

$$\int_D \int_D f(x,y)\ \delta\ (x-x_o)\ \delta\ (y-y_o)\ dxdy = f(x_o,y_o)$$

Therefore, (2.20) can be written as,

$$-K\nabla^2\phi = f_o\ \delta\ (x-x_o)\ \delta\ (y-y_o) \qquad (2.21a)$$

by dividing the both sides of (2.21a) with f_o, it is obtained that,

$$-\hat{K}\nabla^2\phi = \delta(x-x_o)\ \delta\ (y-y_o) \qquad (2.21b)$$

where $\hat{K}$ is obtained by dividing the original permeability K by source strength f_o.

Let's call that L is a operator as,

$$L = -\hat{K}\nabla^2$$

and with the knowledge of eigen functions, where

$$Lu = \lambda u \quad , \qquad u\,\Big|_{\Gamma} = 0$$

it can be written as,

$$-\hat{K}\ u_{xx}\ \ -\hat{K}\ u_{yy} = \lambda u \qquad (2.22a)$$

by the method of separation of variables,

$$u(x,y) = f(x)\ g\ (y) \qquad (2.22b)$$

substuting it in (2.22a), it is obtained that,

$$-\hat{K}f''g - \hat{K}fg'' = \lambda fg \qquad (2.22c)$$

by dividing both sides of (2.22c) with fg, it is obtained that,

$$-\hat{K}\frac{f''}{f} - \hat{K}\frac{g''}{g} = \lambda \qquad (2.22d)$$

it is known that, on the boundaries of the interested domain u(x,y) has a value 0. Therefore it can be written as:

$$u(x,0) = f(x)\, g(0) = 0 \qquad \therefore \quad g(0) = 0$$
$$u(x,b) = f(x)\, g(b) = 0 \qquad \therefore \quad g(b) = 0$$
$$u(0,y) = f(0)\, g(y) = 0 \qquad \therefore \quad f(0) = 0$$
$$u(a,y) = f(a)\, g(y) = 0 \qquad \therefore \quad f(a) = 0$$

in the equation (2.22d), in order to get λ value, $-\hat{K}f''/f$ should have a constant value, and as well as $-\hat{K}g''/g$.

If the first term is analysed, in which;

$$-\hat{K}\frac{f''}{f} = C, \qquad f'' = -\frac{C}{\hat{K}}f, \qquad f'' + \frac{C}{\hat{K}}f = 0$$

There is three possibilities for the constant C:

i) If $C = 0 \quad \therefore \quad f(x) = Ax + B$

$f(0) = 0 \quad \therefore \quad B = 0$

$f(a) = 0 \quad \therefore \quad Aa = 0 \quad \therefore \quad A = 0$

Therefore $f(x) = 0$, it is a trivial solution.

ii) If $C < 0$

Saying $C = -\eta^2$, $\eta \neq 0$

$\therefore \quad f'' - \frac{\eta^2}{\hat{K}} f = 0$ and $f(x) = Ae^{\frac{\eta}{\sqrt{\hat{K}}}x} + Be^{-\frac{\eta}{\sqrt{\hat{K}}}x}$

$$\begin{bmatrix} 1 & 1 \\ e^{\frac{\eta a}{\sqrt{\hat{K}}}} & e^{-\frac{\eta b}{\sqrt{\hat{K}}}} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \because \quad \begin{matrix} A = 0 \\ B = 0 \end{matrix}$$

It is trivial also.

iii) If $C > 0$

Saying $C = \eta^2$, $\eta \neq 0$

$\therefore \quad f'' + \frac{\eta^2}{\sqrt{\hat{K}}} f = 0$ and $f(x) = A \sin(\frac{\eta}{\sqrt{\hat{K}}}x) + B \cos(\frac{\eta}{\sqrt{\hat{K}}} x)$

$f(0) = 0 \quad \therefore \quad B = 0$

$f(a) = 0 \quad \therefore \quad A \sin(\frac{\eta a}{\sqrt{\hat{K}}}) = 0$

Assuming $A \neq 0$, then $\frac{\eta a}{\sqrt{\hat{K}}} = n\pi$, $n = \pm 1, \pm 2 \ldots.$

Any, $\eta_n = \frac{n\pi\sqrt{\hat{K}}}{a}$, $n = 1,2,3$

Therefore, $f_n(x) = \sin(\frac{\eta_n}{\sqrt{\hat{K}}} x) = \sin(\frac{n\pi}{a} x)$, $n = 1,2,3$ (2.23)

In a similar way, for g(y), we can find a constant,

$$\eta_m = \frac{m\pi\sqrt{\hat{K}}}{b} \quad , m = 1,2,3\ldots..$$

Therefore,

$$g_m(y) = \sin(\frac{m\pi}{b} y), \; m = 1,2,3\ldots \qquad (2.24)$$

And finally, the equation for $u_{nm}(x,y)$ is obtained, by substutiting (2.23) and (2.24) in (2.22b).

$$u_{nm}(x,y) = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right) \qquad \begin{array}{l} n = 1,2,3 \ldots\ldots \\ m = 1,2,3 \ldots\ldots \end{array} \quad (2.25)$$

At this stage, it is known that,

$$Lu = \lambda u = \delta(x-x_o)\,\delta(y-y_o) \qquad (2.26)$$

hence, an expression is obtained for the right hand side as,

$$\delta(x-x_o)\,\delta(y-y_o) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right) \quad (2.27)$$

by using the property of Dirac delta function, where,

$$\int_0^a\int_0^b \delta(x-x_o)\,\delta(y-y_o)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dxdy =$$

$$\sin\left(\frac{n\pi}{a}x_o\right)\sin\left(\frac{m\pi}{b}y_o\right)$$

if those integrals are solved,

$$\int_0^a \sin^2\frac{n\pi}{a}x\,dx = \frac{1}{2}\int_0^a\left(1-\cos\frac{2n\pi}{a}x\right)dx = \frac{a}{2}$$

$$\int_0^b \sin^2\frac{m\pi}{b}y\,dy = \frac{b}{2}$$

therefore,

$$a_{nm}\frac{ab}{4} = \sin\left(\frac{n\pi}{a}x_o\right)\sin\left(\frac{m\pi}{b}y_o\right)$$

and,

$$a_{nm} = \frac{\sin\left(\frac{n\pi}{a}x_o\right)\sin\left(\frac{m\pi}{b}y_o\right)}{ab/4} \qquad (2.28)$$

by substuting (2.28), in (2.27), it is obtained that;

$$\delta(x-x_o)\delta(y-y_o) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(\frac{n\pi}{a}x)\sin(\frac{n\pi}{a}x_o)\sin(\frac{m\pi}{b}y)\sin(\frac{m\pi}{b}y_o) \qquad (2.29)$$

As stated, in equation (2.21b), ϕ represents the piezometric head (measured from the bottom of the aquifer), but for the use of eigen function expansion, u(=0) on the boundaries of the interested domain is defined. So, there is a difference between the datum levels. In order to eliminate this difference, ϕ in(2.21b) is called as $G(x,y,x_o,y_o)$ (which represents the drawdown, measured from the top of the water table) that is the green function which is looked for. Hence, (2.21b) becomes,

$$-\hat{K}\nabla^2 G = \delta(x-x_o)\ \delta\ (y-y_o) \qquad (2.30)$$

in which, G is a variational expression that is equal to the product of a constant with $u_{nm}(x,y)$.

$$G(x,y,x_o,y_o) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm}\sin(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y) \qquad (2.31)$$

If the $L(=-\hat{K}\nabla^2)$operator is applied to the G that is defined in (2.31), the left hand side of equation (2.30)is obtained. After putting the right hand side of (2.30) which is defined as (2.29), the following equation is obtained,

$$-\hat{K} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \{-(\frac{n\pi}{a})^2\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)+(-)(\frac{m\pi}{b})^2 \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\} =$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4}{ab} \sin(\frac{n\pi}{a}x_o)\ \sin(\frac{m\pi}{b}y_o)\ \sin(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y) \qquad (2.32)$$

the simplified form of (2.32) is,

$$\hat{K}A_{nm}\ [(\frac{n\ \pi}{a})^2 + (\frac{m\ \pi}{b})^2] = \frac{4}{ab}\ \sin(\frac{n\ \pi}{a}\ x_o)\ \sin(\frac{m\ \pi}{b}\ yo) \qquad (2.33)$$

therefore, the constant A_{nm} can be written as,

$$A_{nm} = \frac{4}{ab} \quad \frac{\sin(\frac{n\ \pi}{a}x_o)\ \sin(\frac{m\ \pi}{b}y_o)}{\hat{K}\ [(\frac{n\ \pi}{a})^2 + (\frac{m\ \pi}{b})^2]} \qquad (2.34)$$

after, obtaining the constant A_{nm}, the final equation showing the G (or, in other words, drawdown) can be written as,

$$G(x,y,x_o,y_o) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{ab} \quad \frac{\sin(\frac{n\ \pi}{a}x_o)\sin(\frac{m\ \pi}{b}y_o)}{\hat{K}\ [(\frac{n\ \pi}{a})^2 + (\frac{m\ \pi}{b})^2]} \quad \sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y) \qquad (2.35)$$

in which, $\hat{K}$ is equal to K/f_o.

In order to find the piezometric head values over the domain, the following form of equation can be written,

$$\phi(x,y,x_o,y_o) = \phi_o \pm G(x,y,x_o,y_o) \qquad (2.36)$$

by using the equations (2.35) and (2.36), the exact location of phreatic surface can be found on a point with the coordinates (x,y), as a result of point sink or source located on (x_o,y_o).

It is even possible to solve analytically certain steady state problems in the vertical plane involving a phreatic surface.

Such a problem is classified as a free boundary problem, and it requires a deep knowledge in mathematics.

Unfortunately, in most regional studies of practical interest an analytical solution is not possible, mainly because of the irregularity of the shape of aquifer boundaries. Also, in most cases, the considered flow domain is inhomogeneous.

As a consequence, analytical methods are seldom applied in the practice of solution of regional problems.

2.4.2. Numerical Methods

Computer based numerical methods are practically the major elements for solving large scale ground-water problems encountered in practice. In recent years, parallel to the advance in computer technology, much effort has been devoted to the development of the methodology and techniques for numerical solution of partial differential equations that govern the flow of water in aquifers of various types.

There are mainly two methods of numerical solution which are commonly used for ground-water flow problems.

(1) Finite Difference Method.

(2) Finite Element Method.

In the following part, comments will be made on the Finite Element Method, and the method of application will be shown on the specific problem which is interested.

2.4.2.1. Finite Element Method

This paragraph is partly summarized from Christian, J.T., 1980. Since Zienkiewicz, Mayer, and Cheung(1966) and Taylor and Brown(1967) first demonstrated the application of the finite element method to steady-state flow of incompressible fluids in saturated porous media, it has been clear that this powerful tool permits one to solve a very large variety of practical problems. Its popularity is due to many advantages it offers compared with other solution procedures. For example, a finite difference procedure yields solutions at only fixed number of points in the domain of interest, and may require additional interpolation for solutions at other points. Also, the finite difference method becomes cumbersome for handling irregular boundaries and nonhomogeneities. In contrast, the FEM recognizes the multidimensional continuity of geologic masses, and does not require separate interpolation for extension to other points. The use of separate approximating models for each finite element permits greater flexibility in taking masses with extensive nonhomogeneities and complex geometries.

In the finite element method, the objective is to transform the partial differential equation into an integral equation which includes derivatives of the first order only. Then the integration is performed numerically over elements into which the considered domain is divided.

The method is often presented as an application of the calculus of variations. The starting point is an integral (= a functional) (Bear,J., 1972).

$$I = \iint_D F(x, y, u_1, u_2, \partial u_1/\partial x, \partial u_1/\partial y, \partial u_2/\partial x, \partial u_2/\partial y)\,dx\,dy \quad (2.37)$$

where D nenotes the considered domain, x,y, are two independent variables and $u_1 = u_1(x,y)$, $u_2 = u_2(x,y)$ are two dependent ones. It is tried to make I stationary, i.e., to determine u_1 and u_2 which will make I an extremum. This is done by requiring that the variation (or differential) of I vanishes, i.e., $\delta I = 0$.

Hildebrand, 1962 and; Gelfand and Fomin, 1963 have shown that this requirement holds if the following partial differential equations are satisfied (Bear,J.,1972).

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{1x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{1y}}\right) - \frac{\partial F}{\partial u_1} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{2x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{2y}}\right) - \frac{\partial F}{\partial u_2} = 0 \qquad (2.38)$$

where subscripts x and y denote differentiation with respect to x and y, respectively ($u_{ix} = \partial u_1 / x$, etc.)

Equations (2-38) are called the Euler equations associated with (2.37).

In these equations, u_1, u_{2x}, u_2, u_{2x}, x and y are treated as independent variables. These equations are the necessary conditions for I to be stationary.

The finite element technique is based on the solution of variational problem in its original form (2.37). Once the differential equations describing the problem have been formulated, the functional is sought for which they are the Euler equations. Then, instead of solving the differential equation, the minimization problem can be solved.

In our case, satisfying the partial differential equation.

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) = \bar{Q} \qquad \text{in domain D} \qquad (2.39)$$

that describes steady two-dimensional flow in a nonhomogeneous anisotropic porous medium (x,y principal directions), can be shown to be equivalent to minimizing the functional

$$I = \frac{1}{2} \int \left[K_x\left(\frac{\partial \phi}{\partial x}\right)^2 + K_y\left(\frac{\partial \phi}{\partial y}\right)^2 - (\bar{Q})\phi \right] dxdy \qquad (2.40)$$

Next, the solution domain, D, is divided into elements. It is assumed that the value of the dependent variable varies in some manner, for example linearly, over each element. This means that the value of the dependent variable at any point within the element is uniquely determined by the values of the variable at the nodes of the element and the position of the point under consideration inside the element.

The contribution of each element to the integral (2.37) can be expressed in terms of the values of the dependent variables at the nodes of the element and its geometry. By differentiating this expression with respect to the dependent variable at each node, and adding up the resulting equations for all the elements in the field, a set of simultaneous equations are obtained in which the unknowns are the values of the dependent variables at the nodes, and the coefficients are functions of the coordinates of the nodes. The right-hand side includes the source term.

Boundary conditions are transposed from conditions along sides of an element to conditions at its nodes.

The finite element technique uses the following procedure:

(a) For the partial differential equation which governs the considered flow, derive the associated variational problem.

(b) Divide the field into elements.

(c) Formulate the variational functional within an element.

(d) Take derivatives with respect to the dependent variable at all nodes of the element.

(e) Assemble the equations for all elements.

(f) Express the boundary conditions in terms of nodal values.

(e) Incorporate the boundary conditions into the equations and solve.

(h) The shape and size of the elements is arbitrary. Different shapes (triangles, rectangles,etc.) can be used simultaneously. Smaller elements can be chosen in regions where the rapid variations in the properties of the materials, or in the values of the dependent variables.

The above steps are examplified by considering the flow described by (2.39) and (2.40) * . The flow domain is divided into elements, for example, triangular element as shown in Fig. 2.9.

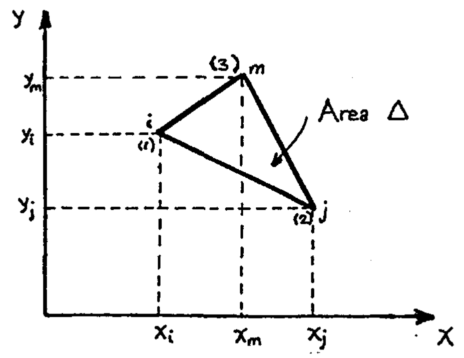


Figure 2.9. A triangular element.

*The following concepts and derivations are written by using the references Cheung,Y.K. and Yeo, M.F.,1979; Brebbia,C.A. and Ferrante,A.J.,1986.

ϕ can be approximated by the expression

$$\phi = \sum_{k=1} \phi_k N_k \qquad (2.41)$$

where ϕ_k are the values of ϕ at the point (x_k, y_k) and N_k are the linear interpolation functions with the property

$$N_k(x_\ell, y_\ell) = \delta_{k\ell} \qquad (2.42)$$

The linear interpolation functions are polynomials which are piecewise continuous over subdomains called finite elements.There is a correspondence between both the number and location of nodal points and the number of primary unknowns per node in a finite element and the number of terms used in the polynomial approximations of a dependent variable over an element. In two-dimensional second-order problems, the correspondence between the number of nodes (which is equal to the number of terms in the approximating polynomial) and the degree of the polynomial is not unique. For example, the polynomial

$$\phi(x,y) = a + bx + cy \qquad (2.43)$$

contains three (linearly independent) terms, and it is linear in both x and y. On the other hand, the polynomial

$$\phi(x,y) = a + bx + cy + dxy \qquad (2.44)$$

contains four (linearly independent) terms, but it is also linear in both x and y. The former requires an element with three nodes (with one primary unknown per node) as in our case, the latter requires an element with four nodes.

In order to derive the linear interpolation functions for a three-node triangular element. Consider the linear approximation (2.43). The approximation (2.43) should be rewritten as it satisfies the conditions

$$\phi(x_k, y_k) = \phi_k \qquad k = 1,2,3 \tag{2.45}$$

where (x_k, y_k) $(k = 1,2,3)$ are the (global) coordinates of three nodes of the triangle. The three constants can be determined in equation (2.43) in terms of ϕ_k from equation (2.45):

$$\begin{aligned} \phi_1 &= \phi(x_1,y_1) = a + b\,x_1 + c\,y_1 \\ \phi_2 &= \phi(x_2,y_2) = a + b\,x_2 + c\,y_2 \\ \phi_3 &= \phi(x_3,y_3) = a + b\,x_3 + c\,y_3 \end{aligned} \tag{2.46}$$

In matrix form

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2.47}$$

Note that the nodes are numbered counterclockwise. Solving equation (2-47) for a, b, and c, we obtain

$$\begin{aligned} a &= \frac{1}{2\Delta}\left[\phi_1(x_2y_3 - x_3y_2) + \phi_2(x_3y_1 - x_1y_3) + \phi_3(x_1y_2 - x_2y_1)\right] \\ b &= \frac{1}{2\Delta}\left[\phi_1(y_2 - y_3) + \phi_2(y_3 - y_1) + \phi_1(y_1 - y_2)\right] \\ c &= \frac{1}{2\Delta}\left[\phi_1(x_3 - x_2) + \phi_2(x_1 - x_3) + \phi_3(x_2 - x_1)\right] \end{aligned} \tag{2.48}$$

where Δ is the area of the triangle,

$$2\Delta = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$= (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1) \qquad (2.49)$$

Substituting for a, b, and c from equations (2.48) into equation (2.43), we obtain

$$\phi(x,y) = \phi_1 N_1(x,y) + \phi_2 N_2(x,y) + \phi_3 N_3(x,y)$$

$$= \sum_{k=1}^{3} \phi_k N_k \qquad (2.50)$$

where N_i are the linear interpolation functions for the triangular element,

$$N_k = \frac{1}{2\Delta}(a_k + b_k x + c_k y) \qquad k = 1,2,3 \qquad (2.51)$$

and for the node 1 (or i), N = $1/2\Delta(a_1+b_1x+c_1y)(=1/2\Delta(a_i+b_ix+c_iy))$ and the constants a_i, b_i, and c_i are the following

$$a_i = x_jy_m - x_my_j$$

$$b_i = y_j - y_m \qquad (2.52)$$

$$c_i = x_m - x_j$$

For example, a_2 is given by setting $i = 2$, $j = 3$ and $m = 1$ in equation (2.52):

$$a_2 = x_3y_1 - x_1y_3$$

The linear interpolation functions N_i are shown in Fig. 2.10. Also N_i has the property

$$N_k(x_\ell, y_\ell) = \delta_{k\ell} \qquad k,\ell = 1,2,3$$

$$\sum_{k=1}^{3} N_k = 1$$

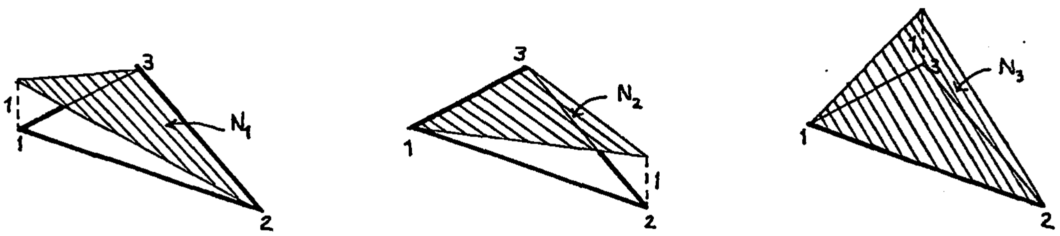


Figure 2.10. Linear interpolation functions

Equation (2.50) determines a plane surface passing through ϕ_1, ϕ_2 and ϕ_3. Hence, use of the linear interpolation functions N_k of a triangle will result in the approximation of the curved surface $\phi(x,y)$ by a planar function $\sum_{k=1}^{3} \phi_k N_k$ (see Fig. 2.11).

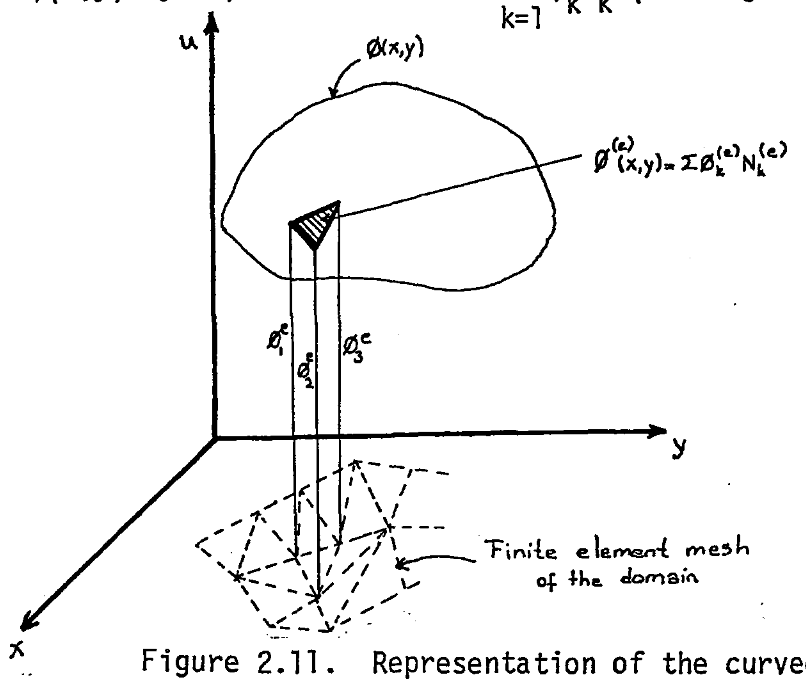


Figure 2.11. Representation of the curved surface $\phi(x,y)$ by linear interpolation functions of three-node triangular elements.

After, deriving the linear interpolation functions. The derivatives in equation (2.40) can be carried out.

$$\frac{\partial \phi}{\partial x} = \left[\frac{\partial N_k}{\partial x} \right]_{k=1}^{3} \{\phi_e\} = \frac{1}{2\Delta} \left[b_k \right]_{k=1}^{3} \{\phi_e\}$$

$$\frac{\partial \phi}{\partial y} \quad \left[\frac{\partial N_k}{\partial y} \right]_{k=1}^{3} \{\phi_e\} = \frac{1}{2\Delta} \left[c_k \right]_{k=1}^{3} \{\phi_e\} \qquad (2.53)$$

$$\{\phi_e\} = \left[\phi_k \right]_{k=1}^{3}$$

Hence (2.40) may be written as

$$\frac{\partial I}{\partial \phi_i} = K_x \frac{1}{4\Delta^2} [b_i^2, b_i b_j, b_i b_m] \{\phi_e\} \iint_\Delta dxdy + K_y \frac{1}{4\Delta^2} [c_i^2, c_i c_j, c_i c_m] \{\phi_e\} \iint_\Delta dxdy$$

$$= \frac{\Delta}{4\Delta^2} \left(K_x [b_i^2, b_i b_j, b_i b_m] + K_y [c_i^2, c_i c_j, c_i c_m] \right) \{\phi_e\} \qquad (2.54)$$

In (2.54), the term corresponding to the source doesn't appear. If the source term is included, the following equation is obtained

$$\left[\frac{\Delta}{\Delta^2} \left(K_x [b_i^2, b_i b_j, b_i b_m] + K_y [c_i^2, c_i c_j, c_i c_m] \right) \right] \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} = \begin{bmatrix} \bar{Q}_i \\ \bar{Q}_j \\ \bar{Q}_m \end{bmatrix} \qquad (2.55)$$

where $\bar{Q}$ values correspond to the point source and/or sink values at the nodes of the element. If these are inputs to the aquifer they have a plus sign, otherwise they have a minus sign. Also, if they act on the nodes, their values are applied directly, otherwise the source should be distributed to the nodes of the element by interpolation. (see Example 2-1).

Example 2.1 (Ground-water flow or seepage; Reddy, 1984). The governing differential equation for a homogeneous aquifer with flow in the xy plane is given by (2.39), which is

$$-(K_x \frac{\partial^2 \phi}{\partial x^2} + K_y \frac{\partial^2 \phi}{\partial y^2}) + \bar{Q} = 0 \quad \text{in domain D}$$

it is apparent that K_x and K_y are the coefficients of permeability (meters per day) along the x and y directions, ϕ is the piezometric head, measured from a reference level (usually the bottom of the aquifer), and $\bar{Q}$ is the rate of pumping ($m^3/(day.m^3)$) .

Consider the problem of finding the lines of constant potential ϕ in a 3000 m x 1500 m rectangular aquifer D(see Fig. 2.12) bounded on the long sides by an impermeable material (e.g., $\partial\phi/\partial n = 0$) and on the short sides by a constant piezometric head of 200 m(ϕ_0= 200 m). Further, suppose that a river is passing through the aquifer, infiltrating the aquifer at a rate of 0.24 m^3/day per unit length (meters), and two pumps are located at (1000,670) and (1900,900), pumping at a rate of $\bar{Q}_1 = 1200\ m^3/(day.m^3)$ and $\bar{Q}_2 = 2400\ m^3/(day.m^3)$ respectively.

A mesh of 64 triangular elements and 45 nodes is used to model the domain (see Fig. 2.13a).

The river forms the interelement boundary between the sets (33,35,37,39) and (26,28,30,32) as elements are shown in Fig.2.13(a). Note that neither pump is located at a node. This is done intentionally for the purpose of illustrating the calculation of the generalized forces due to a point source within an element. It should be calculated that the generalized force components due to the distributed line source (e.g., the river) and the point sources (e.g., the pumps). Calculation of the element coefficient matrices should be a routine task by now; Let us concentrate on the calculation of the generalized forces from the given information.

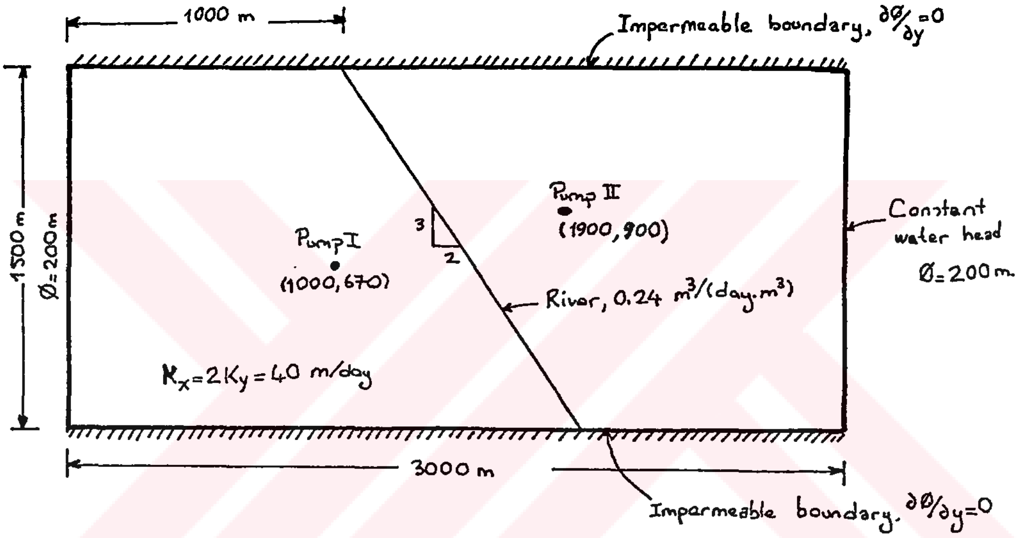


Figure 2.12. Geometry and boundary conditions for the ground-water flow problem of Example 2.1.

First, consider the line source. The river can be viewed as a source of constant intensity, 0.24 $m^3/(day.m^3)$. Since the length of the river is equally divided by nodes 21 through 25 (into four parts), the contribution of the infiltration of the river can be computed at each of the nodes 21 through 25 by evaluating the integrals (see Fig. 2.13(b)):

$$\text{Node 21} \quad : \quad \int_0^h (0.24)\,\Psi_1^{(1)}\,ds$$

$$\text{Node 22} \quad : \quad \int_0^h (0.24)\,\Psi_2^{(1)}\,ds + \int_0^h (0.24)\,\Psi_1^{(2)}\,ds$$

$$\text{Node 23} \quad : \quad \int_0^h (0.24)\,\Psi_2^{(2)}\,ds + \int_0^h (0.24)\,\Psi_1^{(3)}\,ds \qquad (2.56)$$

$$\text{Node 24} \quad : \quad \int_0^h (0.24)\,\Psi_2^{(3)}\,ds + \int_0^h (0.24)\,\Psi_1^{(4)}\,ds$$

$$\text{Node 25} \quad : \quad \int_0^h (0.24)\,\Psi_2^{(4)}\,ds$$

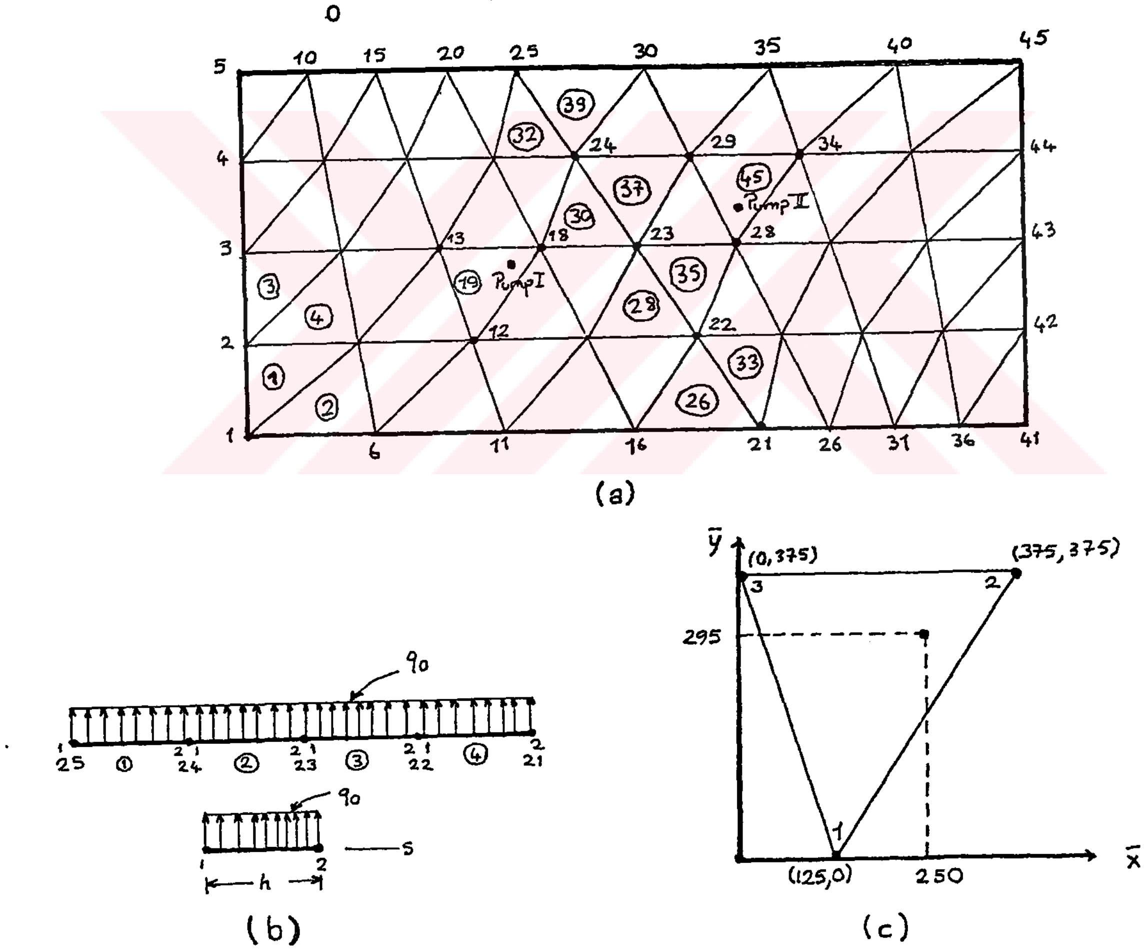


Figure 2.13. Finite-element mesh, and computation of force components for the ground-water flow.(a) Finite-element mesh of triangular elements.(b) Computation of global forces due to infiltration of the river.(c) Computation of global forces for Pump 1, located inside element 19.

For constant intensity q_o and the linear interpolation functions $\Psi_1(s) = 1 - s/h$ and $\Psi_2(s) = s/h$, the contribution of these integrals is well known:

$$\int_o^h q_o \ \Psi_1 \ ds = \frac{q_o h}{2} \qquad h = \frac{1}{4} \{(1000)^2 + (1500)^2\}^{\frac{1}{2}} q_o = 0.24 \tag{2.57}$$

Next, the contribution of the point sources is considered. Since the point sources are located inside an element, the source is distributed to the nodes of the element by interpolation. For example, the source at pump 1 (located in element 19) gives

$$\bar{Q}_1(x,y) = -1200\delta(x-1000)\ \delta\ (y-670) \tag{2.58a}$$

where $\delta(.)$ is the Dirac delta function as given by equation (2.15). We have

$$\bar{Q}_k^e = \int_{area} \bar{Q}(x,y)\ N_k dxdy = -1200\ N_k^e(1000,670) \tag{2.58b}$$

The interpolation functions for element 19 are (in terms of the coordinates $\bar{x}$ and $\bar{y}$; see Fig. 2.13 (c)):

$$N_k\ (\bar{x},\bar{y}) = \frac{1}{2\Delta} (a_k + b_k\ \bar{x} + c_k\bar{y})$$

$$A = \frac{1}{2} (375)^2 \qquad a_1 = x_2y_3 - x_3y_2 = (375)^2$$

$$a_2 = x_3y_1 - x_1y_3 = -375(125)$$

$$a_3 = x_1y_2 - x_2y_1 = 375(125)$$

$$b_1 = y_2 - y_3 = 0 \tag{2.59a}$$

$$b_2 = y_3 - y_1 = 375$$
$$b_3 = y_1 - y_2 = -375$$
$$c_1 = x_3 - x_2 = -375$$
$$c_2 = x_1 - x_3 = 125$$
$$c_3 = x_2 - x_1 = 250$$

Where $\bar{x} = x - 750$ and $y = \bar{y} - 375$, and therefore,

$$N_1\ (250,295) = 0.2133 \quad N_2(250,295) = 0.595 \quad N_3(250,295) = 0.1911 \tag{2.59b}$$

Similar computations can be done for pump 2. Thus, the known generalized displacements (meters) and the nonzero forces (m^3/(day.m^3) are given by

$$U_1 = U_2 = U_3 = U_4 = U_5 = U_{41} = U_{42} = U_{43} = U_{44} = U_{45} = 200.0$$

$F_{21} = 54.08$ $F_{22} = F_{23} = F_{24} = 108.17$ $F_{25} = 54.08$

$F_{12} = -255.6$ $F_{13} = -299.2$ $F_{18} = -715.2$ $F_{28} = -1440.0$

$F_{29} = -410.4$ $F_{34} = -549.6$

Global forces at nodes 6 through 11, 14 through 17, 19, 20, 26, 27, 30 through 33, and 35 through 40 are zero. This completes the data generation for the finite-element modeling of the problem.

The solution of the equations (on a computer) for the unknown U_i(piezometric heads at the nodes) is shown in Fig. 2.14. The greatest drawdown (of water) occurs at node 28, which has the largest portion of the discharge from pump 2.

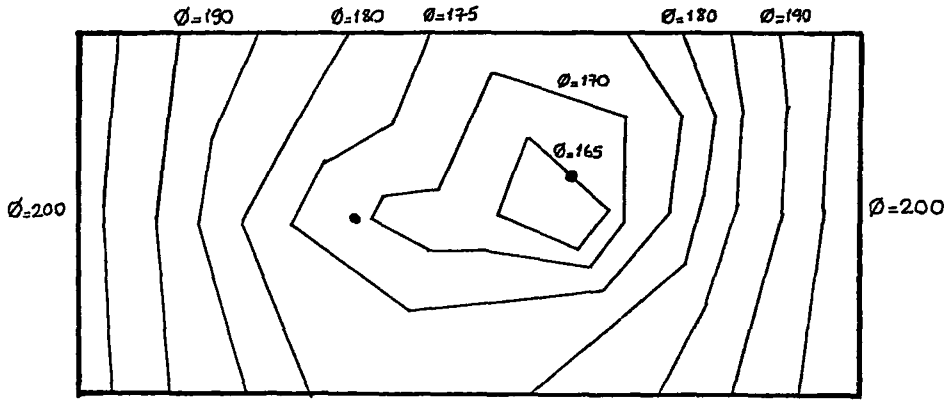


Figure 2.14. Plots of constant piezometric head for the ground-water flow.

Of course, in a standard finite-element analysis one should see that any point sources in the problem are located at a node point. In such a case, it's pumping rate can be used directly in the generation of the global force matrix in coincidence with the node number which it is located.

3. CASE STUDIES AND COMPARISONS BETWEEN THE NUMERICAL AND THE ANALYTICAL SOLUTION

In the previous section, the comments are made on the physical system of a phreatic aquifer, and the derivations of the partial differential equations describing the steady flow through the porous medium domain are shown.

Also, the finite element method is explained in the last part. Essentially, the continuous problem is divided into a number of discrete elements and approximate solutions for the unknown potentials are found at the nodes of the elements. To reach the solution, we have to solve the following system of equations(Desai,C.S.,1972).

$$[C]_{nxn} \; [\phi]_{1xn} = [f]_{1xn} \tag{3.1}$$

[C] : global coefficient matrix which is obtained from the relations between the coordinates of the elements, and material properties.

[ϕ] : unknown potential matrix

[f] : load matrix including the boundary conditions of prescribed potentials, and the source strength values of the corresponding nodes.

[n] : number of nodes

The physical system of an aquifer is usually simplified for the numerical analysis. For example, the aquifer bottom is taken as approximately horizontal in many cases, also, for the permeability of the aquifer, some assumptions are made to include the effect of clay packs etc.

Where the two dimensional flow is considered, the system can be analyzed horizontally or vertically. So, the following three types of approaches will be adequate to handle two dimensional flow.

(a) Horizontal plane flow analysis
(b) Axisymmetric flow analysis
(c) Vertical plane flow analysis

3.1. Horizontal Plane Flow Analysis

This type of analysis is desirable for a homogeneous and isotropic aquifer in which the aquifer properties don't change along the direction vertical to the analyzed plane, and the saturated thickness of it is approximated as constant.

It is remembered that, there is an analytical solution describing the steady flow in such a aquifer (See, Section 2.4.1). Therefore, the reliability of the numerical solution can be illustrated, by comparing with the analytical solution.

Example 3.1.

A well is pumped from an unconfined aquifer surrounded by a constant head boundary as shown in Fig.(3.1). The values of aquifer properties are given in the Figure.

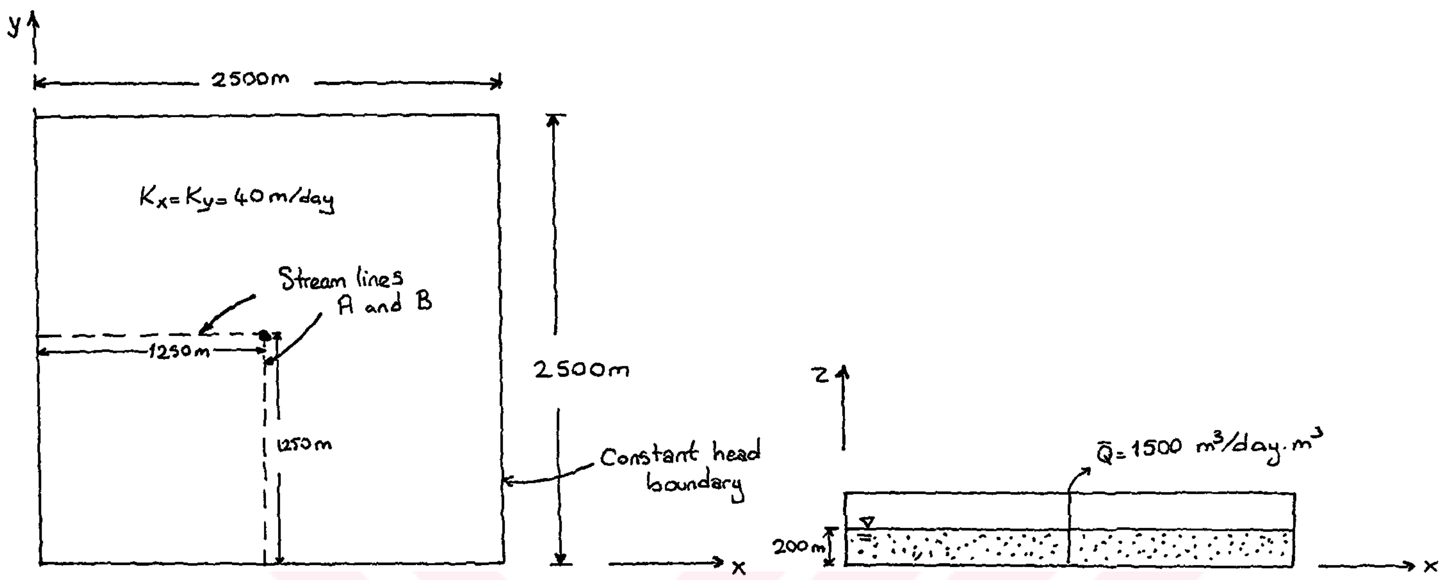


Figure 3.1. Definition sketch for Example 3.1.

To take the advantage of symmetry, a quadrant of the flow is analyzed rather than the entire flow field. There is no flow across the streamlines A and B, and therefore, the stream lines A and B can be represented by impermeable boundaries (see Fig. 3.2)(McWhorter,D. and Sunada, D.K., 1977).

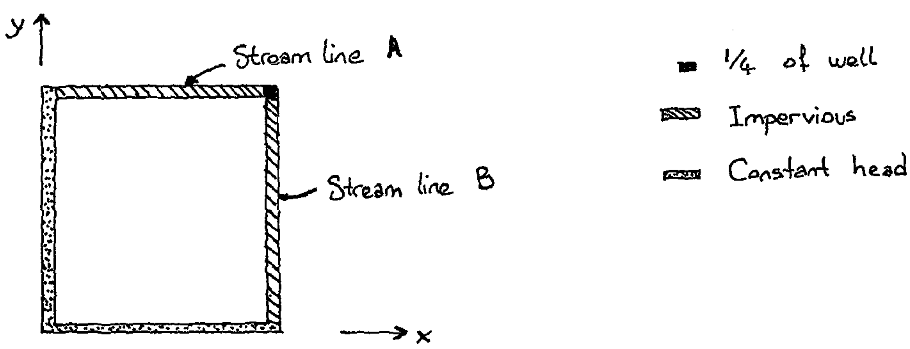


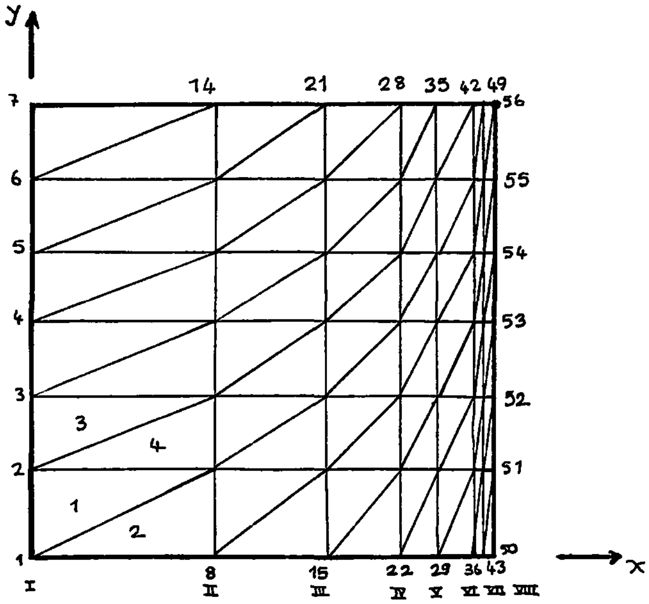
Figure 3.2. Schematic of boundaries of equivalent domain for Example 3.1.

The mesh used for the problem is shown in Fig.(3.3). The end coordinates of the generating lines, the number of intervals which the each one is divided, and the weighting factors that adjust the interval lengths, are given in Table (3.1a).

The aquifer properties, corresponding to the boundary nodes of the grid (Fig.3.3), are given in Table (3.1b). Equation (2.55) should be used to develop the matrices in (3.1). For this processes, the computer program in the Appendix A is used. After the program run is completed, the potentials or piezometric heads on the nodes of the flow domain are known.

By using equation (2.35), we can calculate the drawdowns, on the points of interest, analytically. Figure (3.4) shows the radial distribution of drawdown due to numerical and analytical solutions, with the properties of the domain which is given and when the steady state is reached.

As can be seen, excellent results were obtained. Only, at the well face and near to it, there is a difference. This occurs due to Dupuit Assumptions, and the errors involved during numerical solutions. You can find the detailed explanations on these subjects in the Discussion and Conclusion part. Another example dealing with this type of analysis is given in the Appendix B.



Total # of nodes : 56
Total # of elements : 84

Figure 3.3. Schematic of grid overlay for Example 3.1.

G.L. Num:	I	II	III	IV	V	VI	VII	VIII
Beginning coord	(0,0)	(500,0)	(800,0)	(1000,0)	(1100,0)	(1200,0)	(1225,0)	(1250,0)
End coord.	(0,1250)	(500,1250)	(800,1250)	(1000,1250)	(1100,1250)	(1200,1250)	(1225,1250)	(1250,1250)
Num. of int.	6	6	6	6	6	6	6	6
Weighting fact.	1	1	1	1	1	1	1	1

(a)

Impermeable boundaries where $\partial\phi/\partial y=0$		14	21	28	35	42	49
$\partial\phi/\partial x=0$		51	52	53	54	55	
Constant head boundaries where $\phi_o = 200$ m	1	2	3	4	5	6	7
	8	15	22	29	36	43	50
Pumping well	Rate : 375 m³/day.m³ Location : Node 56						
Permeabilities	$K_x = 40$ m/day $K_y = 40$ m/day						

(b)

Table 3.1. Data for Example 3.1, (a) the mesh data*, (b) the data about aquifer characteristics.

*The presented data is prepaired according to the needs of the computer program included in Appendix A. Because the program makes automatic mesh generation (See, the description of it in Appendix A).

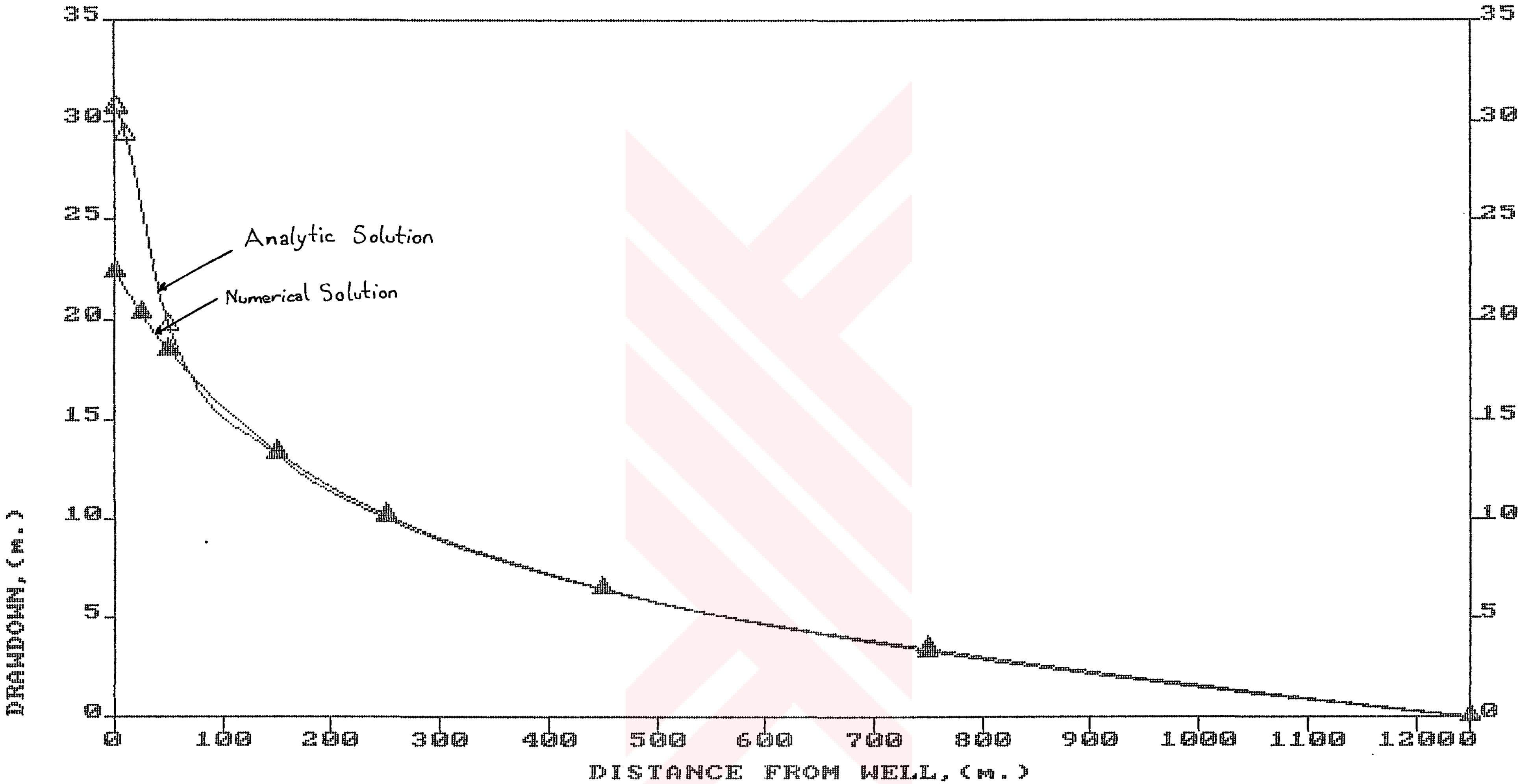


Figure 3.4. Distribution of Drawdown along streamline A for Example 3.1.

3.2. Axisymmetric Flow Analysis

With the finite element method, problems of axially symmetric flow in the (r,z) plane can be solved as easily as problems of plane flow. The minimized functional corresponding to axisymmetric case (e.g. flow towards a well) can be written as (Bear, 1972).

$$\frac{1}{2} \int_D \left[r\left(K_r \frac{\partial^2 \phi}{\partial r^2} + K_z \frac{\partial^2 \phi}{\partial z^2}\right) - \bar{Q}\phi \right] drdz \qquad (3.2)$$

Equation (3.2) is identical to equation (2.40) describing the horizontal plane flow. If r axis is assumed as x, and z axis as y, the coefficient matrix calculation shown by equation (2.55) can be used, but obviously, the effect of extra term r in (3.2) should be accounted. This is simple, because by multiplying the coefficient matrix which is obtained by (2.55), with r, the modified coefficient matrix that is used to solve axisymmetric flow is obtained. The term r describes the distance from the point under consideration to the well. When more than one well acting in the system, the multiple effect of wells should be accounted on the points which are considered. For this process, by using the superposition principles, an equivalent radius, r_e, should be calculated.

In order to get a solution, it is apparent that, the boundary conditions of the flow domain should be described. . In this type of analysis, a free surface which makes the upper boundary of the aquifer is involved.

The condition to find the free surface is simply that at any point on it, the potential head ϕ is equal to the free surface elevation head H from a reference plane. Therefore, when the steady flow is considered either axisymmetrically or vertically, the iteration technique is used (Connor and Brebbia, 1980).

For this, a top flow line is initially guessed and the flow domain is divided into elements. After each iteration the values of ϕ at the free surface are compared with the elevation head (in our case, this process is simply done, by directly comparing the elevation of generating lines with the potentials obtained at the upper end nodes of the generating lines), if they are different the mesh is moved to satisfy the condition $\phi = H$. The solution for the free surface is reasonably accurate after one or two iterations.

For the the iteration technique, the boundary conditions at the well face is necessary. Since the porous media stops at the well face, the aquifer not only has a boundary around its perimeter, but each well is also considered a boundary to the aquifer. The boundary conditions at wells are treated as constant or variable specified flux, or constant head, depending on which best describes the actual physical conditions. The program which is available in the Appendix A, accept both of them.

Example 3.2

Radial flow towards a well in a layered aquifer is considered. The well completely penetrates the aquifer (Figure 3.5a). The water level in the well is kept constant at 30 m above the impermeable base by pumping at a constant rate.

The aquifer consists of two layers, the lower layer has permeabilities as $K_x = K_y = 30$ m/day, and the upper layer has permeabilities as $K_x = K_y = 60$ m/day. The flow through the aquifer confined at the bottom by an impermeable bed, while the top water surface remains free. The radius of influence of the well was extended up to 500 m where the ground-water level was taken at 80 m above the impervious boundary and the flow was assumed uniform and horizontal.

The problem was analysed by using cylindrical coordinates with the axisymmetric equation (3.2).

In Fig.(3.5a) the initial guess of phreatic line is shown, together with the grid overlay. Also, the generating line elevations after the 1^{st} iteration and 2^{nd} iteration is shown in Table 3.2a. Throughout the iterations, the aquifer characteristics, which is shown in Table 3.2b, is used.

As it is seen, two iterations are enough to find the location of phreatic line. Radial distribution of drawdown is shown in Fig. 3.6.

It was experienced that, when the location of phreatic line due to pumping of one well is considered, both of the boundary conditions at the well face (e.g., specified flux or constant head) can be used. But, if more than one well pumping in the system, one has to make

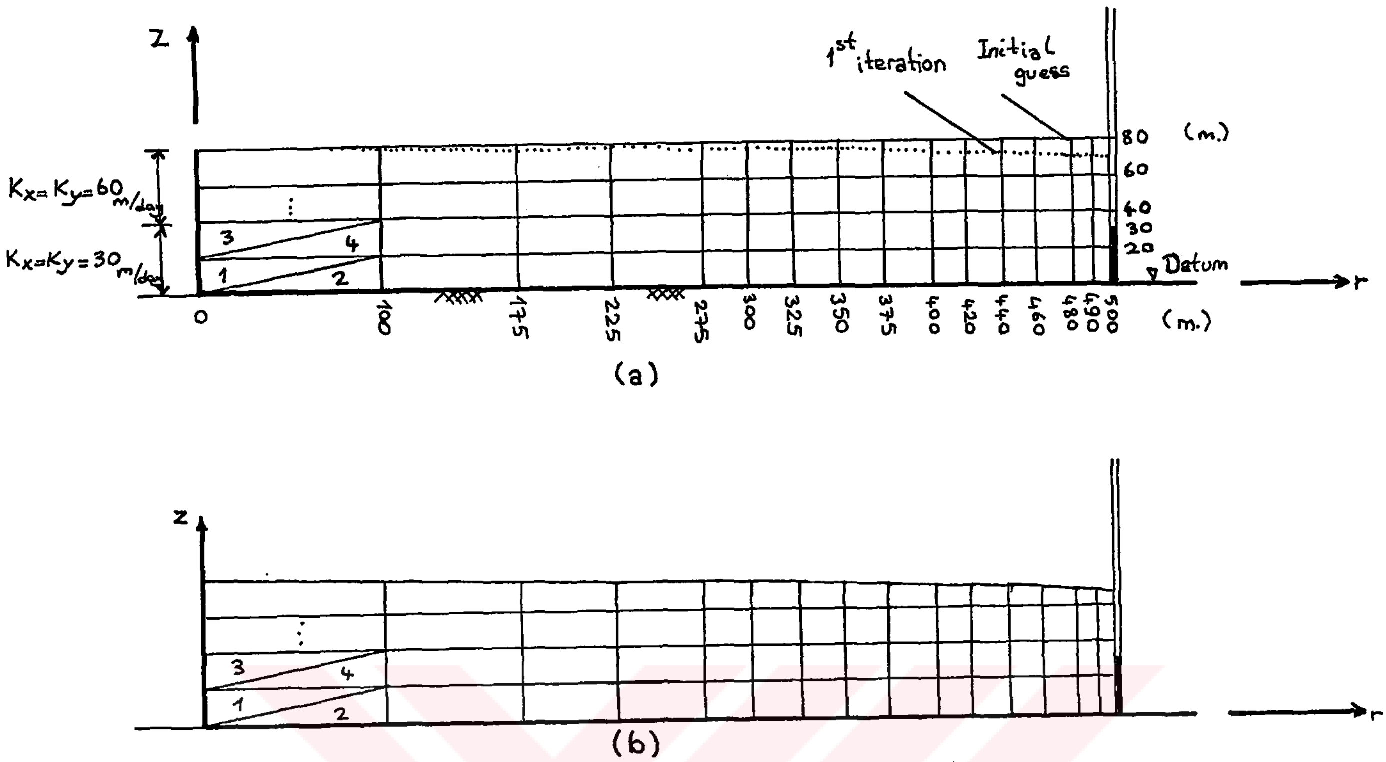


Figure 3.5. Grid overlay for Example 3.2,(a) Initial mesh configuration, (b) Final mesh configuration, that is $\phi = H$.

Gen. Line Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Initial guess	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80
1st iteration	80	79.26	78.58	78.02	77.36	76.97	76.53	76.03	75.44	74.74	74.09	73.35	72.56	71.82	7159	71.49
2nd iteration	80	79.31	78.67	78.15	77.52	77.14	76.72	76.22	75.65	74.95	74.28	73.51	72.62	71.71	71.41	71.27

(a)

MAXNOD=80	MAXLOD=0	MAXFIX=7	NUMLOC=1	XLOC(1)=500 m
Bound. Conditions	$\varnothing_1=80m$ $\varnothing_2=80m$ $\varnothing_3=80m$ $\varnothing_4=80m$ $\varnothing_5=80m$ $\varnothing_{76}=30m$ $\varnothing_{77}=30m$			

(b)

Table 3.2 Data for Example 3.2,(a) The mesh data(b) Data* about aquifer characteristics.

*Permeability values of the layered aquifer are given within a loop added to the program.

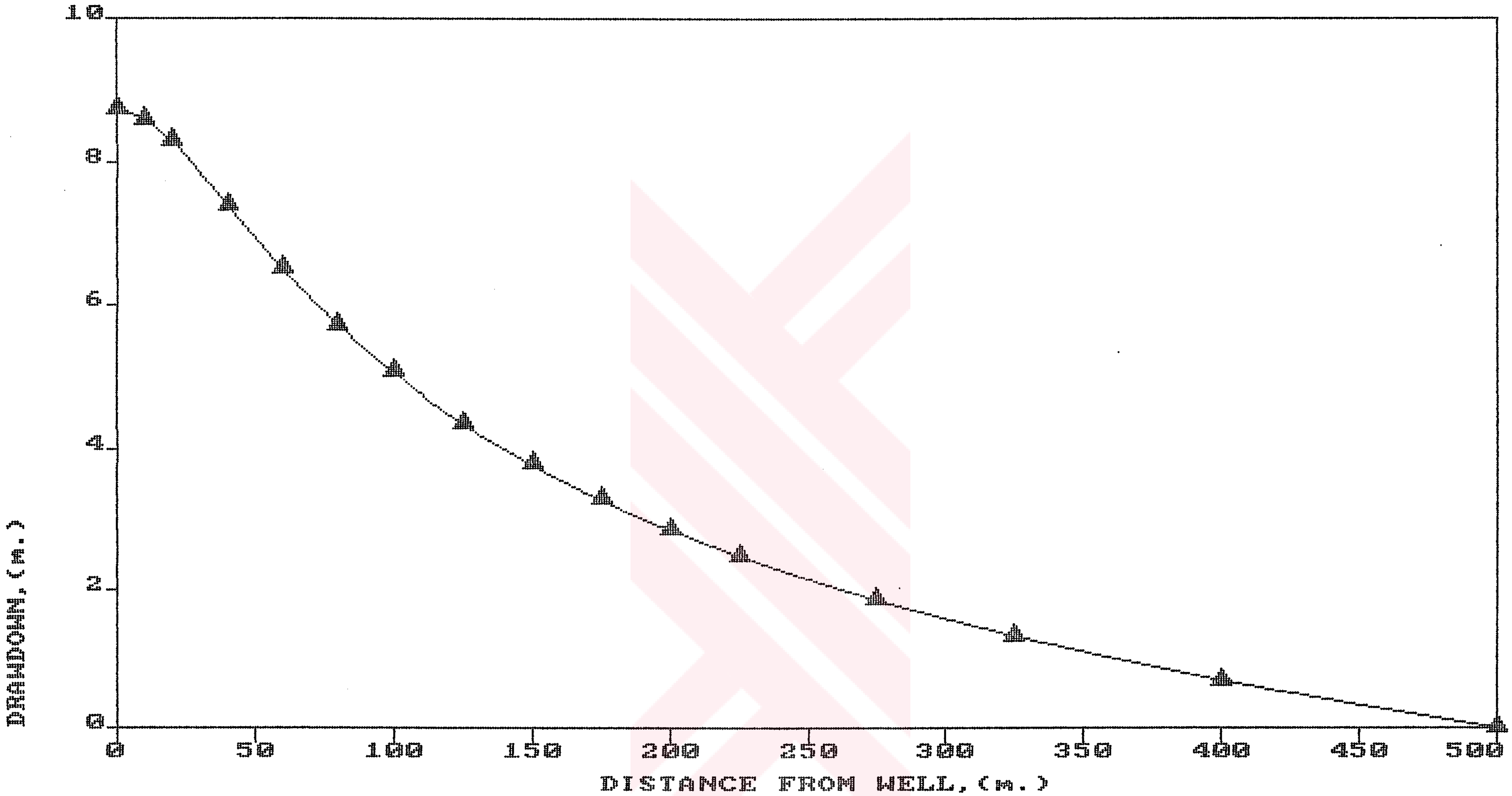


Figure 3.6. Radial Distribution of Drawdown for Example 3.2.

sure not to give same piezometric heads (e.g. in reference to the same datum level) to the adjacent wells. Because, the iteration trend adjusts the generating line heights between such wells to the same piezometric head values. So one never can find the actual location of phreatic line between such wells. However, by giving the prescribed flux at those well faces, this problem can be eliminated.

3.3. Vertical Plane Flow Analysis

This type of analysis is used to handle cases in which no approximation is made for the non-horizontal bottom.

Actually, it is not used for the problems of flow towards a well. Because, the solutions of the same system with the axisymmetric analysis and vertical plane flow analysis are different. But, at the same time, it was seen that, it can give a good idea. Therefore, if a system with pumping wells is not symmetric according to the radial axes, also not approximated so, this type of analysis is available.

Again, a free surface in involved like in Section 3.2. Therefore, the iteration technique can handle such cases. Explanations about this technique have been made in Section 3.2.

For this analysis, the functional shown by the equation (2.40) can be used. Therefore the equation (2.55) is reached which is used for the horizontal plane flow analysis, but for this case, the horizontal plane, where the equation (2.55) is derived according to, is considered as a vertical plane with the same notations (e.g. x,y).

Example 3.3

Vertical plane flow in a homogeneous, isotropic aquifer with two pumping wells is considered. The shape of domain, grid overlay and pump locations together with the datum level is shown in Fig.(3.7.). Permeabilities of the aquifer are $K_x = 35$ m/day and $K_y = 40$ m/day. The water levels in the wells are kept constant at 20 m above the impermeable base by pumping at a constant rate. The flow through the aquifer is confined at the bottom by an impermeable bed, while the top water surface remains free. The length of the area under consideration is 640 m.

The problem was analysed by using the coordinate system shown in Fig.(3.7), together with the functional (2.40).

The generating line elevations throughout the each iteration is shown in Table (3.3a). Also the aquifer properties shown in Table (3.3b) are used in each run.

As it is seen , after four iterations, the phreatic line locations are obtained (see Fig.(3.8)). Radial distribution of drawdowns is shown in Fig.(3.9).

As it was said , it doesn't describe the actual flow towards a well. On the other hand, it is very convenient for the variety of the plane flow problems involving flow beneath a structure, flow through an earth dam and flow towards a trench.

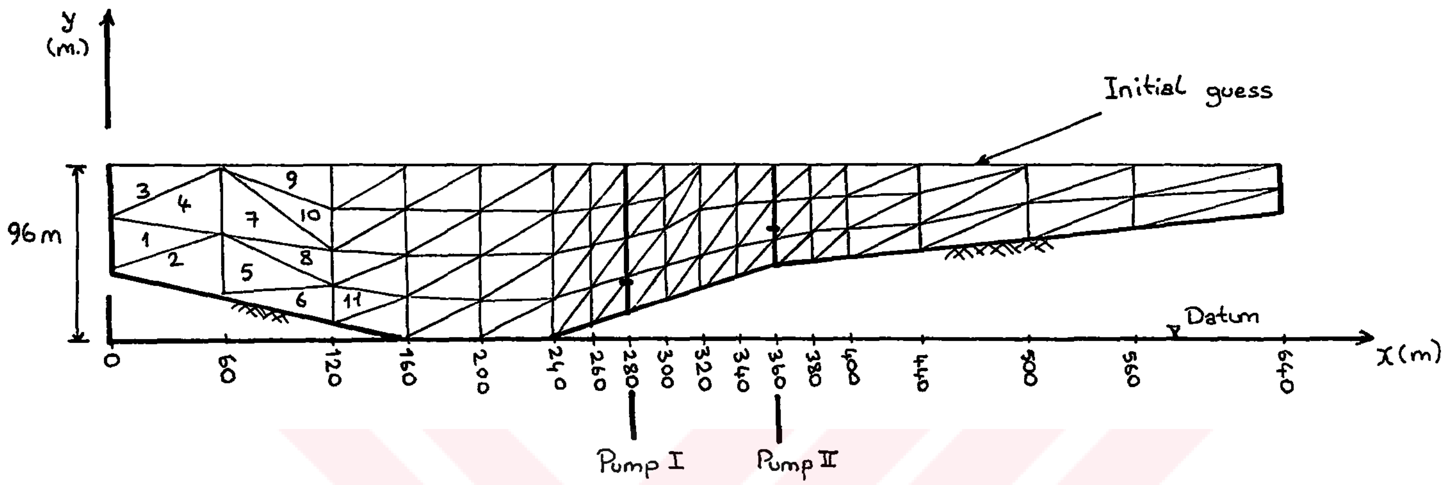


Figure 3.7. Initial grid overlay for Example 3.3.

G.L. Num.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
B. y-coord (fixed)	40	24	10	0	0	0	8	14	22	28	34	40	44	46	50	56	60	72
E. y-coord	96	96	96	96	96	96	96	96	96	96	96	96	96	96	96	96	96	96
1st iteration	96	82.51	71.12	64.45	58.93	53.23	51.55	50.96	51.89	53.85	56.41	59.13	61.37	63.47	67.7	74.71	82.68	96
2nd iteration	96	85.01	73.30	65.73	58.10	49.92	45.74	42.85	45.25	49.46	54.52	59.89	62.83	65.75	71.56	79.97	87.41	96
3rd iteration	96	85.3	74.16	66.87	59.37	50.89	46.0	41.62	44.86	49.81	55.15	60.2	63.31	66.38	72.15	79.99	86.94	96
4th iteration	96	85.22	74.10	66.87	59.47	51.12	46.27	41.59	45.07	50.07	55.28	60.19	63.27	66.29	71.39	79.71	86.77	96

(a)

MAXNOD=74	MAXLOD=0	MAXFIX=9	PX= 35m/day Py= 40 m/day
Bound. Conditions	$\emptyset_1=\emptyset_2=\emptyset_3=\emptyset_{72}=\emptyset_{73}=\emptyset_{74}=96$ m. $\emptyset_{32}=34$ m. $\emptyset_{50}=\emptyset_{51}=60$ m.		

(b)

Table 3.3. Data for Example 3.3,(a)The mesh data,(b) The data about aquifer characteristics.

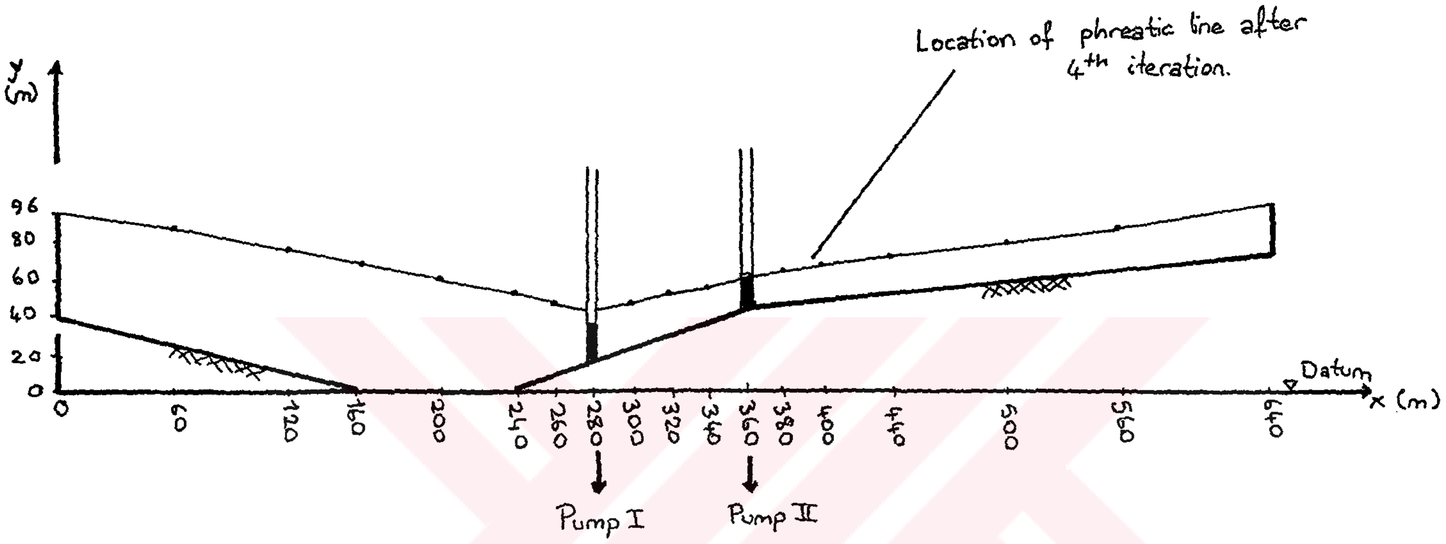


Figure 3.8. Phreatic line location for Example 3.3 at the end of 4^{th} iteration.

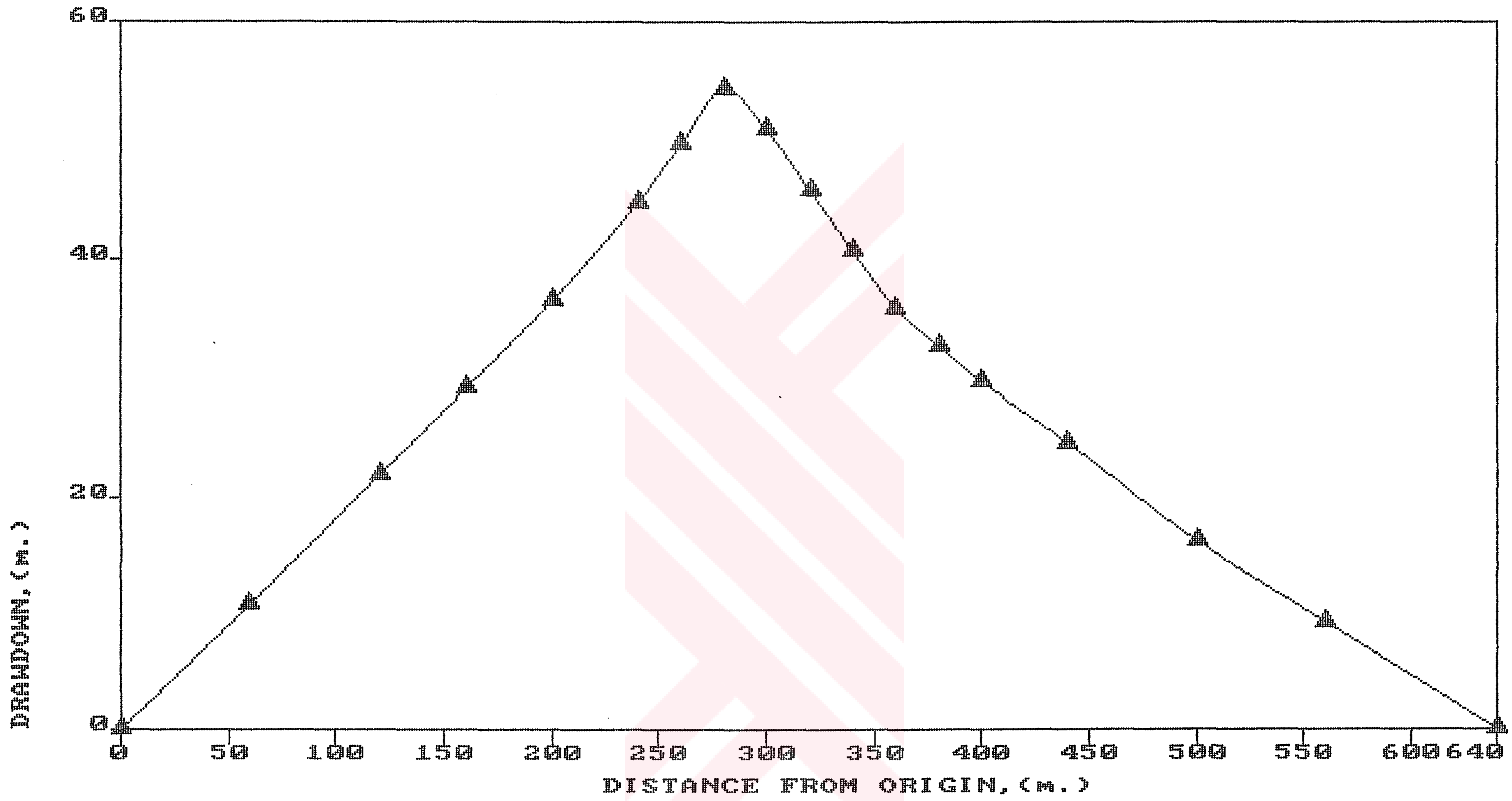


Figure 3.9. Distribution of drawdown along x-axis for Example 3.3.

4. DISCUSSION and CONCLUSION

The discussions and conclusions are partly summarized from Bear,J.,1972; Desai,C.S.,1972 and; Faust,C.R. and Mercer,J.W.,1980.

It seems evident that the active pursuit of the finite element method is a forceful tool in solving the problem at hand. Nevertheless, the present limitations of the method should be recognized and it should be avoided for those problems for which the other methods are more suitable and economical. A number of finite element scheme and subscheme may be available for a class of problems. Each one might have established its validity with respect to a subclass of problems.

The choice of the most suitable scheme is governed by a number of factors. In Fig. 4.1 the important factors in this choice are depicted. Up to now, the development of the finite element solution is explained for the problems of the two-dimensional steady flow in a phreatic aquifer.

In this part, it will be tried to show how one chooses the appropriate model and which parameters affect the solution in the light of knowledge that has become evident.

Modelling plays a great part in a numerical solution. Because in this way a set of equations which are derived, may be incorporated, to a system in which we are interested.

For general problems involving aquifers having irregular boundaries, heterogeneities, or highly variable pumping rates; no ways of solution other then the numerical solutions exist. Thus; the finite element method is handy to handle such cases. In this process, a satisfactory model is a must.

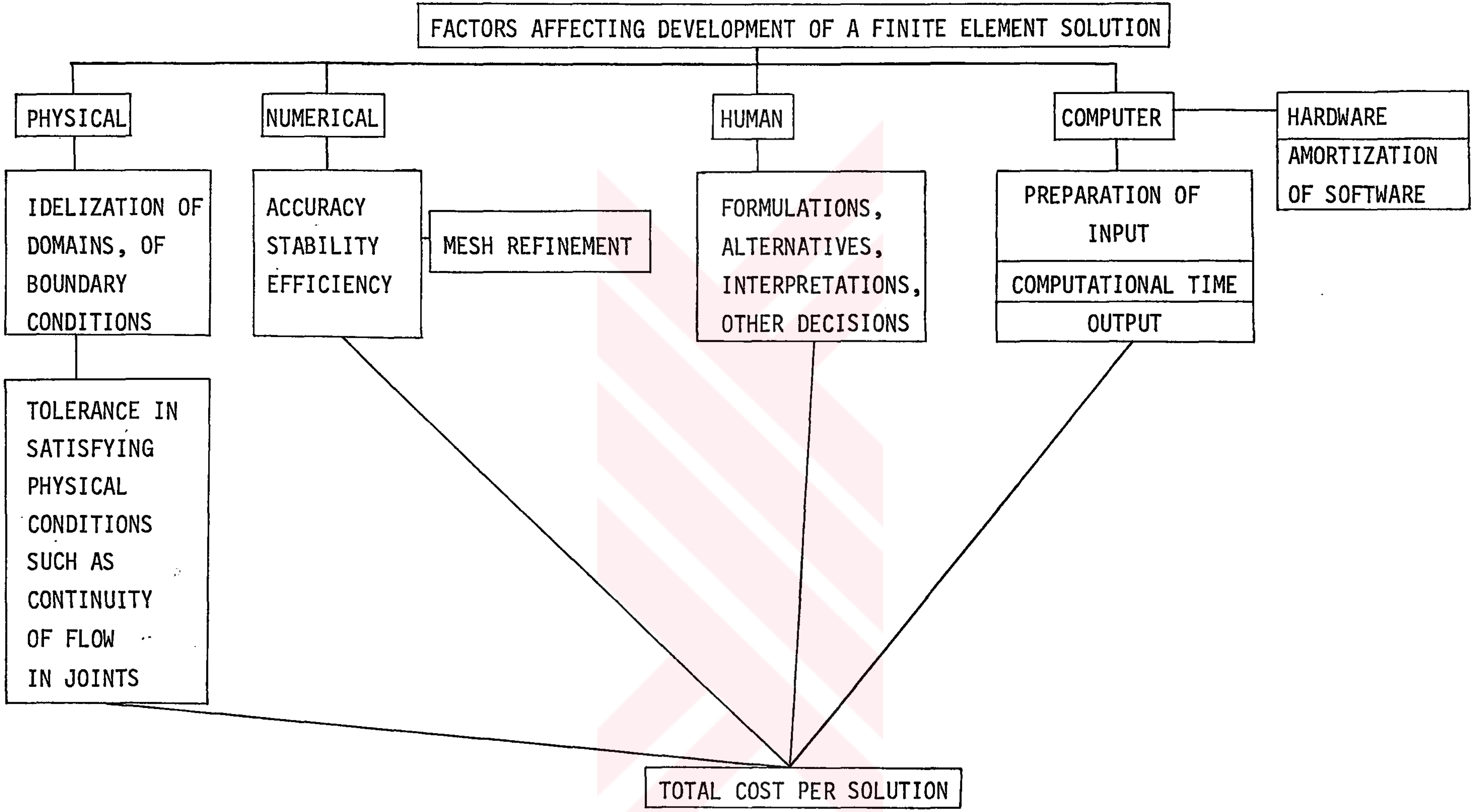


Figure 4.1. Development of economical numerical technique

Initially the following two questions should be asked:

(1) What are the study objectives?

(2) How much is known about the aquifer system; that is, what data are available?

The model used will depend on the study objectives. For example, the thesis author has experienced that, if one is interested in the drawdown near a well, then a regional model, where the local effects are lost due to the large spacing between nodes, should not be used. Instead, a radial flow model with small grid spacing would be sufficient.

Data preparation for the ground-water model first involves determining the boundaries of the region to be modelled. The boundaries may be physical (impermeable or no flow, recharge or specified flux, and constant head) or merely chosen for convenience to solve a situation (small subregions of a large aquifer).

Once the boundaries of the aquifer are determined it is necessary to discretize the region, that is, subdivide it into a grid. Depending on the numerical procedure used, the grid may have any shape (herein, triangular subdivisions are used in the problems).

One of the critical steps in applying a ground-water model is designing the grid. Intuitively it is expected that the finer the grid the more accurate the solution. Numerical analysis confirms this intuition; therefore, fine grids should be used where we want accurate solutions, and coarse grids can be used where details are not important.

The following considerations have to be taken into account while designing a grid:

(1) Locate "well" nodes near the physical location of the pumping well or center of the well field.

(2) Locate boundaries accurately. For distant boundaries the grid spaces may be expanded, but avoid large spacings next to small ones.

(3) Nodes should be placed closer together in areas where there are large spatial changes in transmissivity or hydraulic head.

(4) Align axes of grid with the major directions of anisotropy as much as possible (that is, orient grid with major trends).

As can be seen in the examples, given in this work, the grid spacing closer to the wells is finer, because of the considerable changes in potential near that region.

The effect of mesh refinement can be seen in Fig.4.2. In the Figure, it is seen that as the mesh gets finer closer values, to the analytical solution have been obtained for the phreatic line. The discrepancy between the values obtained by the numerical and the analytical solution near to well face occur due to Dupuit Assumptions. This is the main disadvantage of Dupuit's approximation that is fails to take into account a free seepage surface of the type shown in Fig. 4.3. In other words it is assumed that $h = h_w$ for the well shown in the figure. The error involved in this assumption is generally small and confined to a short distance from the well.

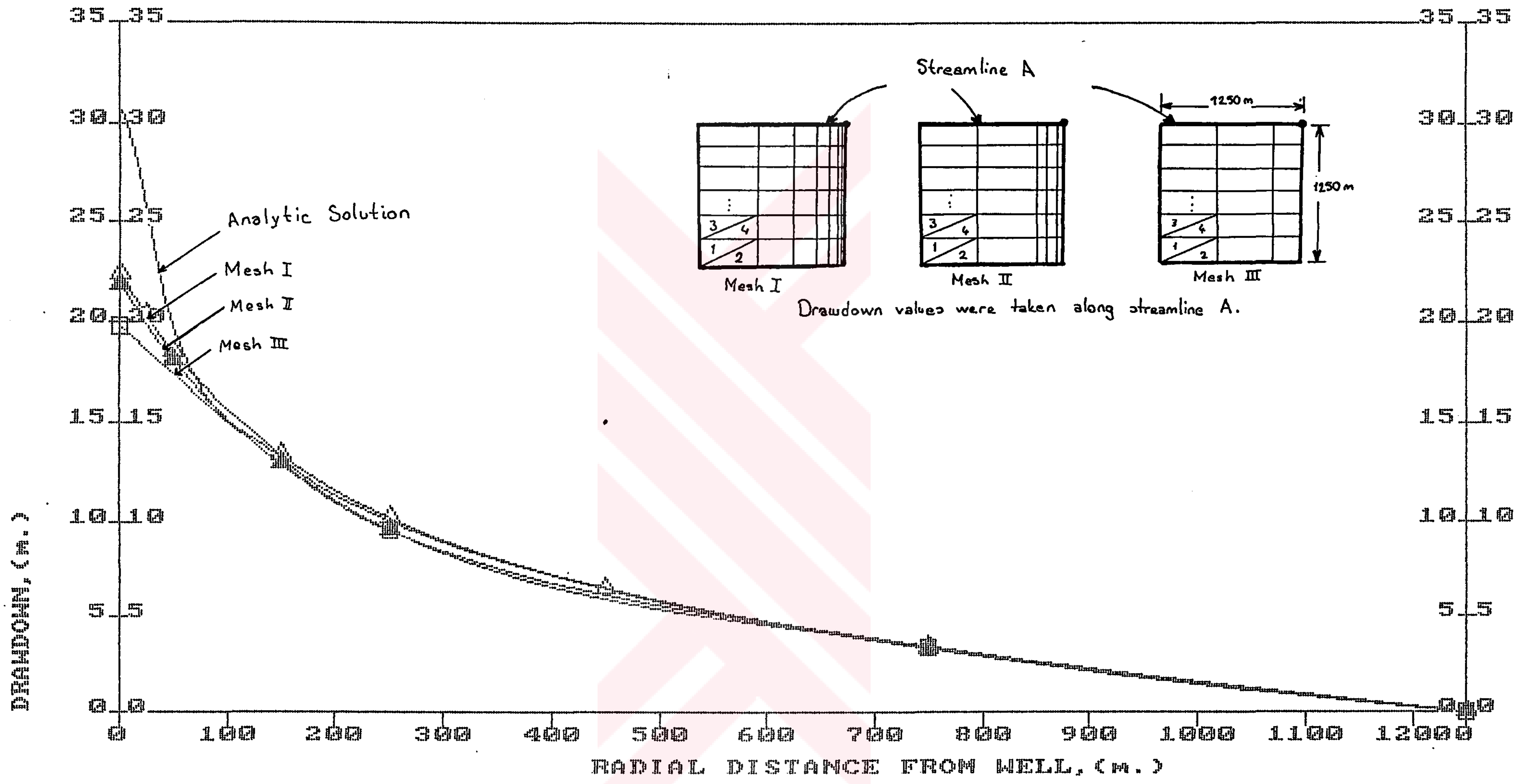


Figure 4.2. The effect ofmesh refinement in the numerical solution of Example 3.1.

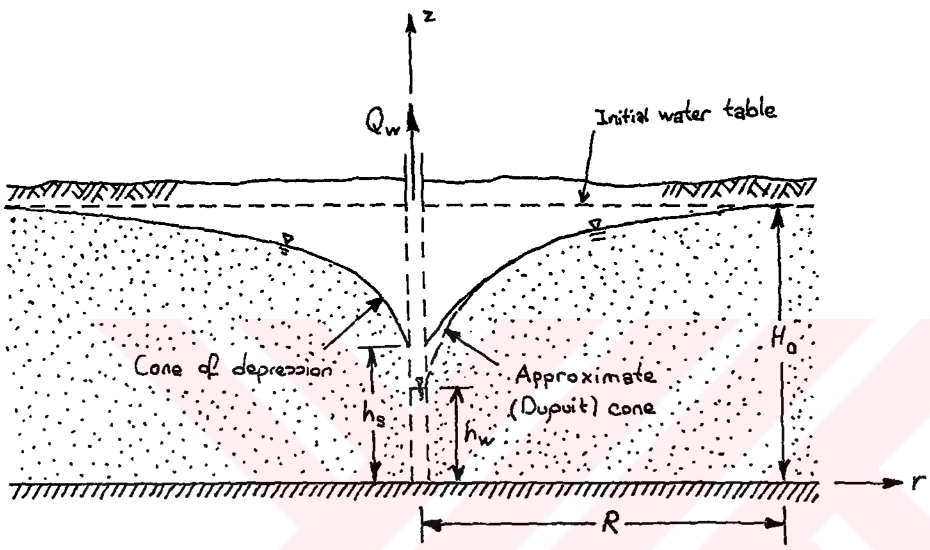


Figure 4.3. Free seepage surface

Once the grid is designed, it is necessary to specify aquifer parameters and initial data for the grid. Required program input data include aquifer properties for each grid node such as the hydraulic conductivity. Computed results generally consist of hydraulic heads at each of the grid nodes throughout the aquifer. For the required input data, see Table 4.1.

I. Pysical Framework

1. Hydrogeologic map showing areal extent boundaries, and boundary conditions of all aquifers.
2. Topographic map showing surface-water bodies.
3. Water table, bedrock-configuration, and saturated thickness maps.
4. Transmissivity map showing aquifer and boundaries.
5. Relation of stream and aquifer (hydraulic connection).

II. Stresses on System

1. Type and extent of recharge areas (irrigated areas, recharge basins, recharge wells, etc.).
2. Ground-water pumpage.
3. Stream flow.
4. Precipitation.

III. Other Factors

1. Economic information of water supply.
2. Legal and administrative rules.
3. Environmental factors.
4. Planned changes in water and land use.

Table 4.1. Data Requirement for a Ground-water Flow Model

Ground-water modelling allows estimates of:

(a) the effects of boundaries and boundary conditions,

(b) the effects of well locations and spacings, and

(c) the effect of various withdrawal rates.

In addition to these, the feasibility of certain proposed mechanisms for observed behavior can be tested. Parameters may be changed to learn what effect they may have on the over-all process. This is sometimes referred to as a sensitivity analysis, since results from these runs will indicate what parameters the computed hydraulic heads are most sensitive to. In Fig. 4.4, the effect of hydraulic conductivity can be seen.

In order to prevent the misusage of models a general rule might be to start with the simplest possible model and a course aquifer description then refine the model and data until the desired estimation of aquifer performance is obtained. Because ground-water models deal with the subsurface, there are always unknown factors that could effect results. It is important to know and understand the limitations and possible sources of error in numerical models. All numerical models are based on a set of simplifying assumptions, which limit their use for certain problems.

To avoid applying an otherwise valid model to an inappropriate field situation, it is not only important to understand the field behaviour but also to understand assumptions thatform the basis of the model. For example, the model results may not be indicative of the field's behaviour. Errors of this type are considered conceptual errors.

In addition to these limitations; replacement of the model differential equations by a set of algebraic equations introduces truncation errors, that is, the exact solution of the algebraic equations differs somewhat from the solution of the original differential equations. Also the exact solutions of the algebraic equations are not obtained due to the round-off error, as a result of the finite accuracy of computer calculations. Finally, and most importantly, aquifer description data (e.g. hydraulic conductivity and the distribution of heads within the aquifer) are seldom known completely, thus producing data error.

The level of truncation error in computed results may be estimated by repeating runs or portions of runs with smaller space increments. Significant sensitivity of computed results to changes in these increment sizes indicates a significant level of truncation error and the corresponding need for smaller space increments. Compared to the other error sources, round-off error is generally negligible.

Error caused by aquifer description data is difficult to assess since the true aquifer description is almost never known. A combination of core analysis and geological studies often give valuable insight into the nature of transmissivity and aquifer geometry. However, much of this information many be very local in extent and should be regarded carefully when used in a model of a large area.

Upon meeting modelling criteria, a succesful model study will not only improve the understanding of the particular hydrologic

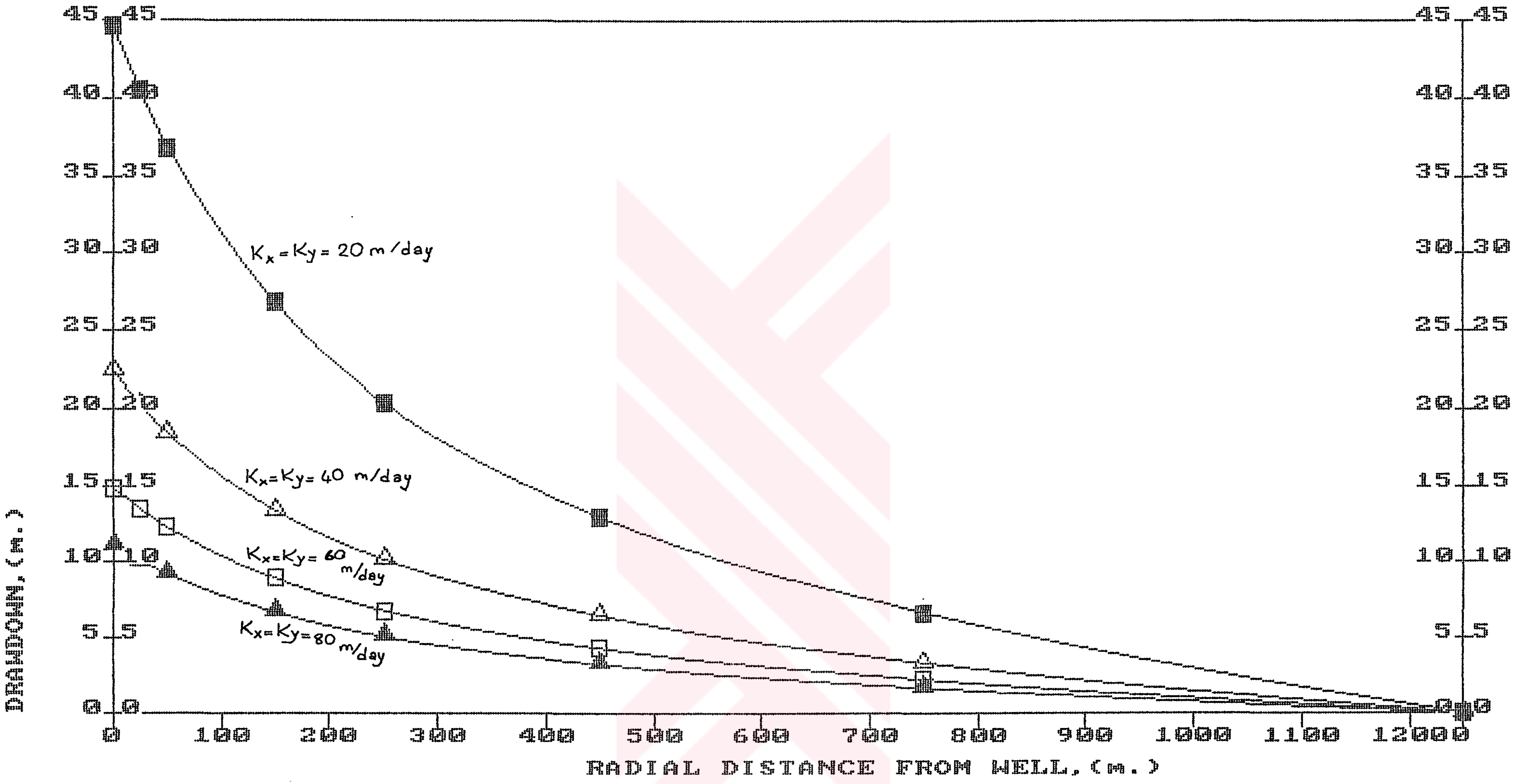


Figure 4.4. The effect ofhydraulic conductivity on the drawdown values of Example 3.1.

system, but should also provide appropriate prediction and analysis of the problem under study.

In applying numerical methods (in our case FEM), it is concerned with three general characteristics of the solution procedure:

(1) Accuracy.

(2) Efficiency.

(3) Stability.

Accuracy deals with how well the discretized solution approximates the solution to the continuous problem it represents (For example, reliable results can not be obtained, describing the flow towards a well, with the vertical plane flow analysis). Efficiency is a measure of how much computational work and computer resources are required to obtain a solution. In Fig.(4.5) some of data-input and computational times related to the corresponding studies are shown.

Stability addresses the question of whether or not a solution is possible et all. These definitions are simplified ones, but for practical purposes, sufficient.

Finally, it can be said that; for any given class of problems the choice of the best approach depends on the processes being modelled, the accuracy desired, and the effort that can be expanded on obtaining a solution.

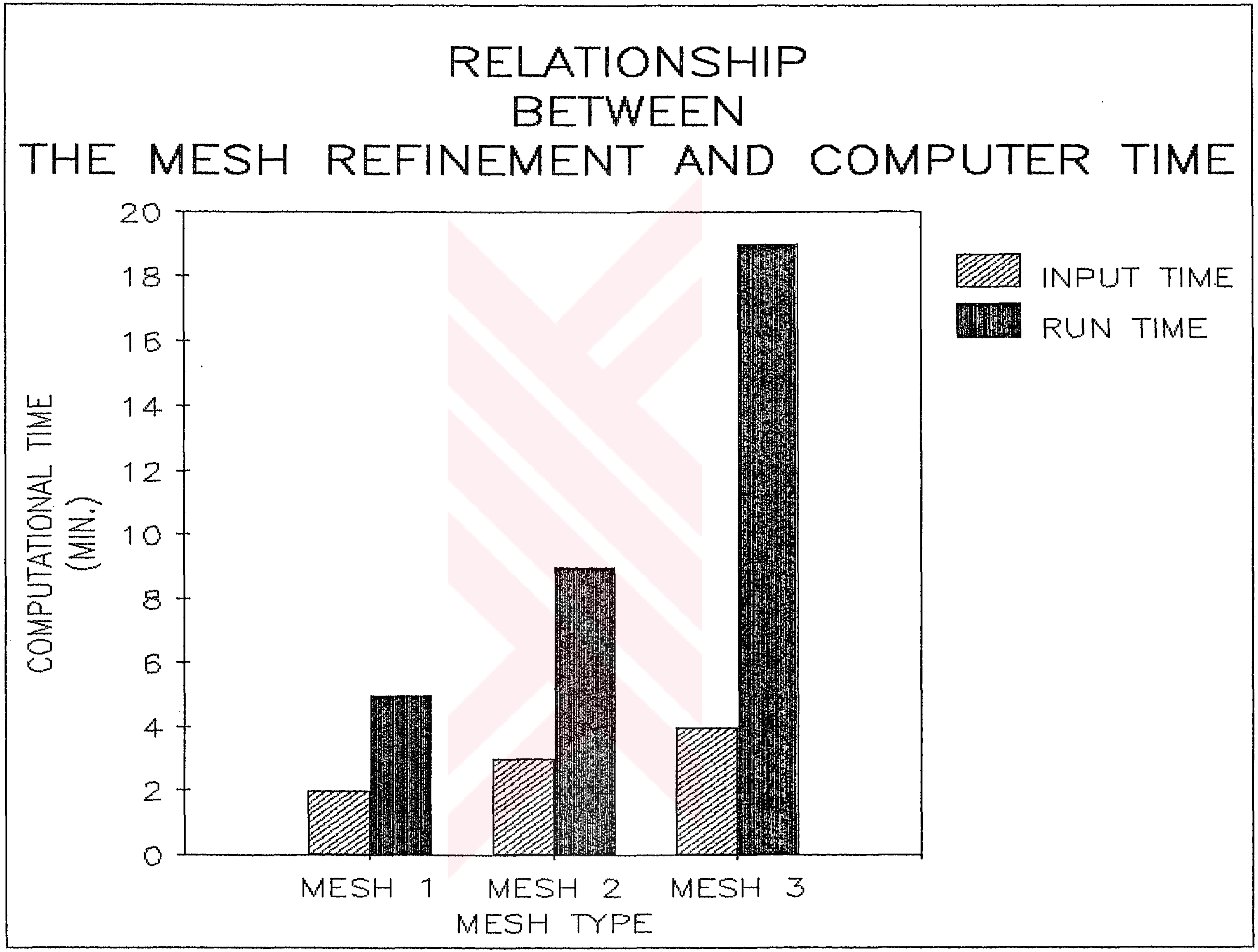


Figure 4.5. Relationship between the mesh refinement and computer time.

Before finishing this chapter, some words may be said for ones who will be dealt with unsteady-flow. In section (2.2), equations are derived from the continuity equation together with the equation describing flow of water by Darcy's law. As can be seen in section (2.2), if the time factor is considered. Specific volume storativity, S_0, becomes essential. Understanding the aquifer storativity begins with understanding the effective stress concept. If the vicinity of a point is considered in an aquifer where water pressure is reduced by pumping, this results in an increase in the intergranular stress transmitted by the solid skeleton of the aquifer. This, in turn causes the aquifer to be compacted, reducing its porosity. At the same time, as a result of pressure reduction, the water will expand. Together, the two effects-the slight expansion of water and the small reduction in porosity - cause a certain amount of water to be released from storage in an aquifer. Based on the above considerations, specific storativity, S_0, can be defined as the volume of water released from storage (or added to it) in a unit volume of aquifer, per unit change in the piezometric head. Therefore, in the case of an unsteady flow analysis, specific volume storativity, S_0, should be identified together with the time factor.

The above paragraph is summarized from Bear,J.,1972. Detailed knowledge about unsteady flow can be found in the following references which are given in this thesis,

Bear, J.,1972.
Brebbia, C.A. and Ferrante, A.J, 1986.
Desai, C.S.,1972.
McWhorter, D. and Sunada, D.K.,1977.
Pinder, G.F. and Gray, W.G.,1977.
Reddy, J.N.,1984.

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BEAR,J., 1972, "Hydraulics of Ground-Water", McGraw-Hill, Haifa.

BREBBIA, C.A. and FERRANTE, A.J.,1986, "Computational Methods for the Solution of Engineering Problems", Pentech Press, London.

CHEUNG, Y.K. and YEO, M.F., 1979, "A Practical Introduction to Finite Element Analysis", Pitman, London.

CHRISTIAN, J.T.,1980,"Flow Nets by the Finite Element Method", Ground-Water, Vol.18, No.2,pp. 178-181.

CONNOR, J.J. and BREBBIA, C.A.,1980, "Finite Element Techniques for Fluid Flow", Newnes-Butterworths, London.

DESAI, C.S.,1972,"Application of FEM in Geotechnical Engineering", Proceedings of the symposium held at Vicksburg, Mississippi, 1-4 May 1972; Vol.1,2.3.

FAUST, C.R. and MERCER J.W., 1980, "Ground-Water Modeling: Numerical Models", Ground-Water, Vol.18, No.4, pp. 385-406.

JACKSON, J.D.,1962, "Classical Elektro Dynamics", Wiley, Newyork.

MCWHORTER, D. and SUNADA, D.K.,1977, "Ground-Water Hydrology and Hydraulics", Water Resources Publications, Fort Collins, Colarado.

MERCER, J.W., and FAUST, C.R.,1980, "Ground-Water Modeling: An Overview", Ground-Wäter, Vol.18, No.2, pp,108-115.

MERCER, J.W. and FAUST, C.R., 1980, "Ground-Water Modeling: Mathematical Models", Ground-Water, Vol.18, No.3, pp.212-227.

PINDER, G.F. and GRAY, W.G.,1977 "Finite Element Simulation in Surface and Subsurface Hydrology", Academic Press, Newyork.

REDDY, J.N., 1984, "Introduction to the Finite Element Method", McGraw-Hill, Newyork.

APPENDIX A

Finite Element Ground-Water Model
prepaired by the thesis author

This appendix presents a two-dimensional ground-water model which may be used for analysis of the areal distribution and the vertical plane distribution of heads in ground-water aquifers when the steady flow is considered. This model will treat unconfined ground-water flow problems. The model is based upon the finite element scheme (triangular elements) using the Gauss elimination procedure for solving the matrix which was described previously by(3.1).

A flow diagram of the model is presented in Fig. A-1. A complete listing ofthe program is given in the last part of this appendix. The model can use the system of units which is wanted.

<u>Procedure for Analysis</u>

The area to be studied is overlied with a grid system. The total number of elements selected is dependent on the storage capacity of the computer being used (in our case, the personal computer IBM PC, 256 k.byte RAM is used, and it can handle 120 elements as maximum).

The grid system selected should be oriented to allow for easy boundary approximation,provide for easy adaptation of hydrologic and geologic data, and provide for the desired model accuracy. In areas where detailed values of water level or piezometric head are desired,

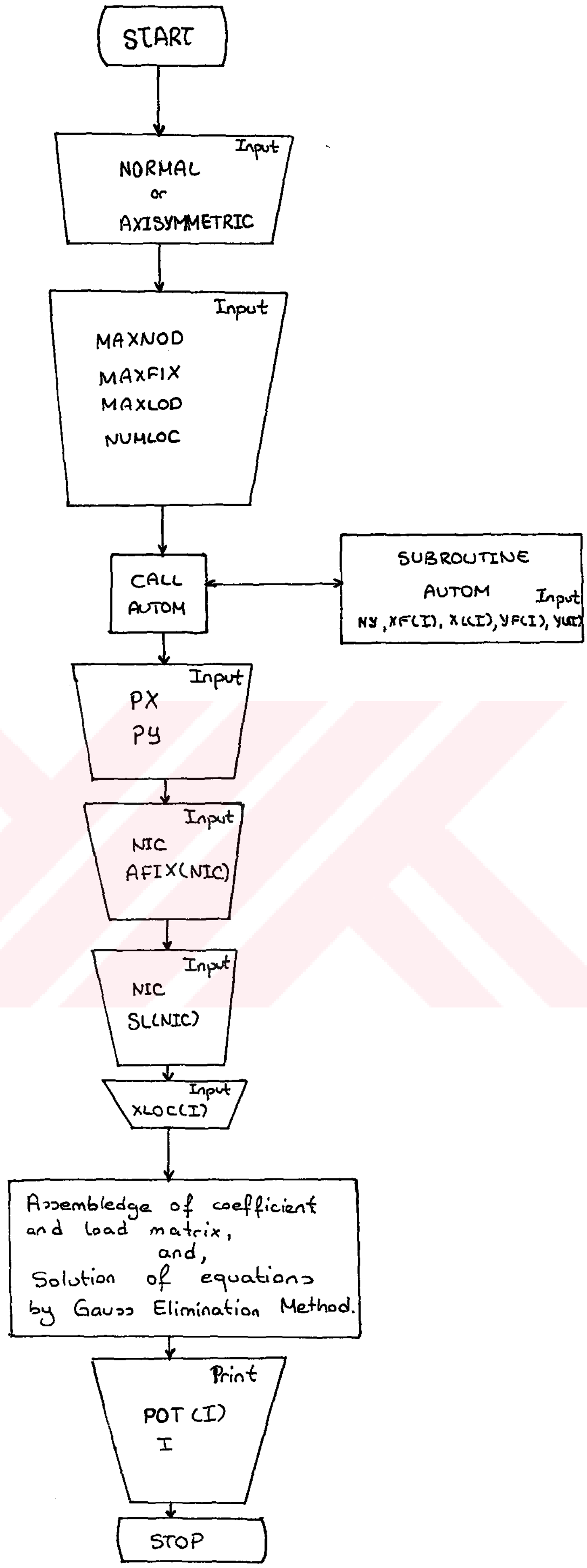

Figure A.1. Flow chart for program KOSGAN smaller intervals for grid spacings should be taken. Hydraulically connected lakes and rivers should be specified as constant head grids.

Boundary conditions due to geologic and hydrologic influences include (1) impermeable or no flow boundaries, (2) constant head or hydraulic boundaries. Program KOSGAN automatically considers the boundaries as impermeable when no data about potential is given. It is a basic coded, intelligent program in which it asks questions about the required data as it proceeds.

Data Input and Definition of Parameters

Input is by screen as outlined below. Note that, there is no relation between parameter names and values which are defined for them (e.g. In Fortran, If a parameter name begins with the letters I,J,K,L,M,N; it refers to an integer value). Also the questions should be answered one at a time.

(1) Choose "N", if a problem involves horizontal plane or vertical plane flow analysis.
Choose "A", if a problem involves radial flow towards a well.

(2) Give,
MAXNOD : Max. number of nodes in a domain under consideration.
MAXFIX : Max. number of nodes where constant heads are defined.
MAXLOD : Max. number of nodes where sources are acted (If there is no source, then we can give 0). Wells, rivers, lakes and precipitation should be included.

NUMLOC: Max. number of wells. It is necessary for axisymmetric flow analysis.

XLOC(I):Distance from origin to the wells

I :Number of wells.

(3) Give,

NY : Number of generating lines(e.g. lines crossing to x axis in designed mesh system).

NX(I) : Number of intervals which we want to divide a generating line which is seen on the screen.

XF(I) : Starting x-coordinate of a generating line.

YF(I) : Starting y-coordinate of a generating line.

XL(I) : Ending x-coordinate of a generating line.

YL(I) : Ending y-coordinate of a generating line.

CON : The weighting factor which is < 1, =1 or > 1, the intervals along a generating line will become progressively shorter, stay equal, or become progressively longer.

(4) Give,

PX : Permeability in X-direction.

PY : Permeability in y-direction.

(5) Give,

AFIX(I): The matrix including constant heads at boundaries.

I : Node number corresponding to a boundary node with constant head.

NFIX(I): Control matrix which is used to understand whether it is a constant head boundary node or not.

(6) Give,

SL(NIC) : Matrix including source terms.

NIC : Node number where a source is defined.

(7) ELSTIF(3,3): Element stiffness matrix.

SS(MAXNOD,MAXNOD): Global stiffness matrix.

COORDS(MAXNOD,2) : Matrix storing the coordinates of nodes.

LNODS(MAXNEL,3) : Matrix storing the nodes of each element.

POT (I) : Matrix storing the potential values at the nodes.

It is very easy to run the program, because it will make necessary reminders as it proceeds, and it will give a chance to replace wrong inputs with the true ones during data input.

For output, there are two choices, (1) Screen output, (2) Printer output. One can choose any of them.

If larger capacity is required, there is a copy of the same program which is Fortran coded.

KOSGAN

```
DIM LNODS(120,3),COORDS(80,2),ELSTIF(3,3),SS(80,80),SUM1(20),XL(20)
DIM NFIX(80),SL(80), POT (80),AFIX(80),NX(20),YL(20),XF(20),YF(20)
DIM NOD(4),XLOC(5),R(5)
NNODZ=3
NVABZ=1
NDIM=2
CLS
PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT
PRINT"                              THIS PROGRAM IS PREPAIRED"
 PRINT
 PRINT"                                   FOR TO SOLVE"
 PRINT
 PRINT"                                     PROBLEMS "
 PRINT
 PRINT"                    INVOLVING STEADY FLOW IN PHREATIC AQUIFERS"
 PRINT
 PRINT:PRINT:PRINT:PRINT:PRINT
 INPUT "IF YOU WANT TO SEE THE NEXT PAGE (Y/N):";A$
 IF ((A$="y") OR (A$="Y")) THEN GOTO 210
 STOP




 CLS
 PRINT "  (A) MAKE SURE THAT,THE MESH WHICH YOU PREPAIRED IS COMPATIBLE"
 PRINT "      FOR THE SUBROUTINE AUTOM.BECAUSE IT AUTOMATICALLY DECIDES"
 PRINT "      ON THE ELEMENT DIVISIONS WITH THE GIVEN DIRECTIONS (IT ASKS"
 PRINT "      ABOUT THE $ OF INTERVALS,END POINT COORDINATES OF THE GENERATIN
G"
 PRINT "      LINES AND WEIGHTING FACTOR).IT MAKES GENERATION FROM BOTTOM TO"
 PRINT "      TOP,SO PLEASE GIVE END COORDINATES LIKE (10,0) TO (15,180),NOT"
 PRINT "      (15,180) TO (10,0)."
 PRINT
 PRINT "  (B) PERMEABILITIES IS ASKED FOR THE X-DIRECTION AND Y-DIRECTION."
 PRINT "      THEREFORE, IF THE SITUATION IS DIFFERENT ,YOU SHOULD ADD A "
 PRINT "      PART IN THE BEGINNING OF THE PROGRAM IN ORDER TO STORE PERMEABI
LITY "
 PRINT "      VALUES FOR EACH NODE TO A MATRIX,AND ALSO YOU HAVE TO CHANGE"
 PRINT "      RELATED PARTS IN THE PROGRAM (LIKE DEFINING THIS MATRIX IN "
 PRINT "      DIMENSION STATEMENT AND CONVERTING SINGLE VARIABLES PX AND PY"
 PRINT "      TO MATRIX VARIABLES PX(I) AND PY(I))."
 PRINT
 PRINT "  (C) YOU CAN SEE THE DIMENSIONS OF THE MATRICES IN THE DIMENSION"
 PRINT "      STATEMENTS AT THE BEGINNING OF THE PROGRAM.IF YOU NEED LARGER"
```

```
PRINT "       DIMENSIONS ,PLEASE APPLY TO GULGUN KOSGAN ,BECAUSE THERE IS A
PRINT "       COPY OF THE SAME PROGRAM IN FORTRAN CODE AND AVALIABLE IN THE"
PRINT "       BURROUGHS SYSTEM IN METU . "
PRINT
INPUT" DO YOU WANT TO CONTINUE  (Y/N):";A$
IF ((A$="Y") OR (A$="y")) THEN GOTO 470
STOP
CLS
PRINT " PLEASE CHOOSE ONE OF THE FOLLOWING ,THAT IF YOU DEALING WITH A "
PRINT " HORIZANTAL OR VERTICAL PLANE FLOW ANALYSIS ,CHOOSE (N).IF YOU"
PRINT " DEALING WITH A AXISYMMETRIC FLOW ANALYSIS ,CHOOSE (A)."
PRINT
INPUT "NORMAL/AXISYMMETRIC (N/A):";CE$
CLS
PRINT
INPUT "MAXNOD ( MAX. $ OF ELEMENTS IN THE DOMAIN ):";MAXNOD
PRINT
INPUT "MAXFIX ( MAX. $ OF NODES WITH CONS. BOUND. HEAD ):";MAXFIX
PRINT
INPUT "MAXLOD ( MAX. $ OF NODES WHERE THERE IS A SOURCE):";MAXLOD
PRINT
IF ((CE$<>"A") AND (CE$<>"a")) THEN GOTO 630
INPUT "NUMLOC ( MAX. $ OF NODES WHERE THERE IS A WELL  ):";NUMLOC
PRINT
IF ((CE$<>"A") AND (CE$<>"a")) THEN NUMLOC=0
INPUT " DO YOU WANT A CHANGE IN THIS PART (Y/N):";A$
IF ((A$="Y") OR (A$="y")) THEN GOTO 540
CLS
MAXVAR=MAXNOD*NVABZ
FOR I=1 TO MAXVAR
SL(I)=0!
POT  (I)=0!
NFIX(I)=0
FOR J=1 TO MAXVAR
SS(I,J)=0!
NEXT J
NEXT I
GOSUB 2210
CLS
PRINT
INPUT "PX (PERMEABILITY IN X-DIRECTION):";PX
```

```
PRINT
INPUT "PY (PERMEABILITY IN Y-DIRECTION):";PY
PRINT
INPUT " DO YOU WANT A CHANGE IN THIS PART (Y/N):";A$
IF ((A$="Y") OR (A$="y")) THEN GOTO 790
CLS
IF (NUMLOC=0) THEN GOTO 940
PRINT " PLEASE ,GIVE THE DISTANCES FROM THE ORIGIN TO THE WELLS EACH AT A"
PRINT " TIME."
PRINT
FOR I=1 TO NUMLOC
INPUT "XLOC(I) ";XLOC(I)
NEXT I
IF (MAXFIX=0) THEN GOTO 1070
CLS
PRINT "PLEASE GIVE THE CONSTANT HEADS TO THE CORRESPONDING BOUND. NODES."
PRINT "PLEASE PAY ATTENSION NOT TO GIVE WRONG VALUES TO THE FOLLOWING"
PRINT "QUESTIONS BECAUSE THERE IS NO TURN BACK."
PRINT:PRINT:PRINT
 FOR J=1 TO MAXFIX
```

```
 PRINT
 INPUT "NIC (NODE NUMBER):";NIC
 INPUT "AFIX(NIC) (CONSTANT HEAD VALUE):";AFIX(NIC)
 NFIX(NIC)=1
 NEXT J
 IF (MAXLOD=0) THEN GOTO 1160
 PRINT:PRINT:PRINT
 PRINT "NOW YOU WILL GIVE THE SOURCE STRENGTH VALUES ,PLEASE DON'T FORGET"
 PRINT "TO PUT (-) FOR DISCHARGES ,IN FRONT OF VALUE THAT YOU WILL GIVE."
 PRINT:PRINT:PRINT
 FOR J=1 TO MAXLOD
 PRINT
 INPUT "NIC (NODE NUMBER):";NIC
 INPUT "SL(NIC) (SOURCE STRENGTH):";SL(NIC)
 NEXT J
 CLS
 PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT:PRINT
 PRINT"                                PLEASE WAIT !"
 FOR NEL=1 TO MAXNEL
 NIC1=LNODS(NEL,1)
```

```
NIC2=LNODS(NEL,2)
NIC3=LNODS(NEL,3)
X1=COORDS(NIC1,1)
Y1=COORDS(NIC1,2)
X2=COORDS(NIC2,1)
Y2=COORDS(NIC2,2)
X3=COORDS(NIC3,1)
Y3=COORDS(NIC3,2)
AREA=(X2*Y3-X3*Y2-X1*(Y3-Y2)+Y1*(X3-X2))/2!
AI=X2*Y3-X3*Y2
BI=Y2-Y3
CI=X3-X2
AJ=X3*Y1-X1*Y3
BJ=Y3-Y1
CJ=X1-X3
AM=X1*Y2-X2*Y1
BM=Y1-Y2
CM=X2-X1
IF ((CE$<>"a") AND (CE$<>"A")) THEN GOTO 1490
MULKAT=1




FOR NUM=1 TO NUMLOC
R(NUM)=(XLOC(NUM)-((X1+X2+X3)/3))
IF (R(NUM)<0) THEN  R(NUM)=(-1)*R(NUM)
NEXT NUM
FOR NUM=1 TO NUMLOC
MULKAT=MULKAT*R(NUM)
NEXT NUM
KAT=MULKAT^(1/NUMLOC)
ELSTIF(1,1)=PX*BI*BI+PY*CI*CI
ELSTIF(2,1)=PX*BI*BJ+PY*CI*CJ
ELSTIF(2,2)=PX*BJ*BJ+PY*CJ*CJ
ELSTIF(3,1)=PX*BM*BI+PY*CM*CI
ELSTIF(3,2)=PX*BM*BJ+PY*CM*CJ
ELSTIF(3,3)=PX*BM*BM+PY*CM*CM
FOR I=1 TO 2
IP1=I+1
FOR J=IP1 TO 3
ELSTIF(I,J)=ELSTIF(J,I)
NEXT J
NEXT I
```

```
FOR I=1 TO 3
FOR J=1 TO 3
IF ((CE$<>"A") AND (CE$<>"a")) THEN KAT=1
ELSTIF(I,J)=ELSTIF(I,J)*KAT/(4!*AREA)
NEXT J
NEXT I
FOR I=1 TO NNODZ
ISTRST=LNODS(NEL,I)
FOR J=1 TO NNODZ
JSTRST=LNODS(NEL,J)
SS(ISTRST,JSTRST)=SS(ISTRST,JSTRST)+ELSTIF(I,J)
NEXT J
NEXT I
NEXT NEL
FOR I=1 TO MAXNOD
IF (NFIX(I)=0) THEN GOTO 1790
SS(I,I)=SS(I,I)+1E+15
SL(I)=SL(I)+SS(I,I)*AFIX(I)
NEXT I
 M1=MAXNOD*NVABZ

M2=M1-1
FOR I=1 TO M2
II=I+1
FOR K=II TO M1
FACT=SS(K,I)/SS(I,I)
FOR J=II TO M1
SS(K,J)=SS(K,J)-FACT*SS(I,J)
NEXT J
SS(K,I)=0!
SL(K)=SL(K)-FACT*SL(I)
NEXT K
NEXT I
FOR I=1 TO M1
II=M1-I+1
PIVOT=SS(II,II)
SS(II,II)=0!
FOR J=II TO M1
SL(II)=SL(II)-SS(II,J)* POT (J)
NEXT J
POT  (II)=SL(II)/PIVOT
```

```
NEXT I
CLS
MM=0
PRINT "   NODE NUMBER   ";"    POTENTIAL    "
PRINT "   -----------   ";"    ---------    "
FOR I=1 TO MAXNOD
MM=MM+1
B=MM/15
IF ((B=1) OR (B=2) OR (B=3) OR (B=4) OR (B=5) OR (B=6) OR (B=7) OR (B=8)) T
EN GOSUB 2140
PRINT USING"      SSS           SSSSS.SS ";I; POT (I)
NEXT I
STOP
END
PRINT
INPUT "DO YOU WANT TO CONTINUE (Y/N):";A$
IF ((A$<>"Y") AND (A$<>"y")) THEN GOTO 2120
PRINT "   NODE NUMBER    ";"   POTENTIAL"
PRINT "   -----------    ";"   ---------"
RETURN
END
```

```
INPUT "NY (NUMBER OF GENERATING LINES):";NY
PRINT:PRINT:PRINT
PRINT "PLEASE GIVE THE DATA ABOUT THE GENERATING LINES FROM LEFT TO RIGHT"
PRINT "(THAT STARTING WITH THE GENERATING LINE CROSSING THE ORIGIN TO"
PRINT "THE GENERATING LINE RIGHT MOST IT."
PRINT:PRINT:PRINT
FOR I=1 TO NY
PRINT
PRINT "PLEASE GIVE THE DATA ABOUT THE GEN. LINE ";I
PRINT "-----------------------------------------"
PRINT
INPUT "NX(I) (NUMBER OF INTERVALS):";NX(I)
INPUT "XF(I) (BEGIN. X-COORDINATE):";XF(I)
INPUT "YF(I) (BEGIN. Y-COORDINATE):";YF(I)
INPUT "XL(I) (END    X-COORDINATE):";XL(I)
INPUT "YL(I) (END    Y-COORDINATE):";YL(I)
PRINT
INPUT "DO YOU WANT A CHANGE ABOUT GEN. LINE DATA  (Y/N):";A$
IF ((A$="Y") OR (A$="y")) THEN GOTO 2310
NEXT I
```

```
CLS
N=0
FOR I=1 TO NY
PRINT:PRINT
PRINT "NOW THE PROGRAM WILL DIVIDE THE GEN. LINE ";I
PRINT
NXI=NX(I)+1
SUM1(1)=0!
SUM1(2)=1!
SUM=1!
INPUT "CON (WEIGHTING FACTOR ):";CON
PRINT
PRINT "X-COORD.   ";"Y-COORD.   ";"NODE NUM."
PRINT "--------   ";"--------   ";"---------"
IF (((NXI-2)=-1) OR ((NXI-2)=1)) THEN GOTO 2570
IF ((NXI-2)=0) THEN GOTO 2610
FOR K=3 TO NXI
SUM1(K)=SUM1(K-1)*CON
SUM=SUM+SUM1(K)
NEXT K




PRINT
X=XF(I)
Y=YF(I)
FOR J=1 TO NXI
N=N+1
X=(XL(I)-XF(I))*SUM1(J)/SUM+X
Y=(YL(I)-YF(I))*SUM1(J)/SUM+Y
PRINT USING "SSSSSS.SS SSSSSS.SS SSSSSS    ";X;Y;N
COORDS(N,1)=X
COORDS(N,2)=Y
NEXT J
NEXT I
PRINT:PRINT:PRINT
PRINT "EL.NO";"NODE1";"NODE2";"NODE3"
PRINT "-----";"-----";"-----";"-----"
N=0
NSUM=0
NYI=NY-1
FOR I=1 TO NYI
NXI=NX(I)
```

```
FOR J=1 TO NXI
IF (((J-NXI)=-1) OR ((J-NXI)=1)) THEN GOTO 2870
IF ((J-NXI)=0) THEN GOTO 2840
IF ((NX(I+1)-NX(I))=-1) THEN GOTO 2920
IF ((NX(I+1)-NX(I))=0) THEN GOTO 2870
IF ((NX(I+1)-NX(I))=1) THEN GOTO 2960
NOD(1)=J+NSUM
NOD(2)=NOD(1)+1
NOD(3)=NOD(2)+NXI+1
NOD(4)=NOD(3)-1
GOTO 3130
NOD(1)=NOD(2)
NOD(2)=NOD(1)+1
NOD(4)=0
GOTO 3130
NOD(1)=J+NSUM
NOD(2)=NOD(1)+1
NOD(3)=NOD(2)+NXI+1
NOD(4)=NOD(3)-1
N=N+1



PRINT USING" $$$$ $$$$ $$$$ $$$$";N;NOD(3);NOD(2);NOD(1)
LNODS(N,1)=NOD(3)
LNODS(N,2)=NOD(2)
LNODS(N,3)=NOD(1)
N=N+1
PRINT USING" $$$$ $$$$ $$$$ $$$$";N;NOD(4);NOD(3);NOD(1)
LNODS(N,1)=NOD(4)
LNODS(N,2)=NOD(3)
LNODS(N,3)=NOD(1)
NOD(1)=NOD(2)
NOD(2)=NOD(3)+1
NOD(4)=0
N=N+1
PRINT USING" $$$$ $$$$ $$$$ $$$$";N;NOD(3);NOD(2);NOD(1)
LNODS(N,1)=NOD(3)
LNODS(N,2)=NOD(2)
LNODS(N,3)=NOD(1)
IF ((NOD(4)=-1) OR (NOD(4)=1)) THEN GOTO 3200
IF (NOD(4)=0) THEN GOTO 3250
N=N+1
```

```
PRINT USING" SSSS SSSS SSSS SSSS";N;NOD(4);NOD(3);NOD(1)
LNODS(N,1)=NOD(4)
LNODS(N,2)=NOD(3)
LNODS(N,3)=NOD(1)
PRINT
NEXT J
NSUM=NSUM+NXI+1
NEXT I
MAXNEL=N
RETURN
END
```

APPENDIX B

In this part, an example will be introducted. It involves areal flow due to multiple well system.Example

Example B.1.

Seventeen wells are pumped from an unconfined aquifer surrounded by a constant head boundary as shown in Fig.B.1. The values of aquifer properties are given in the Figure.

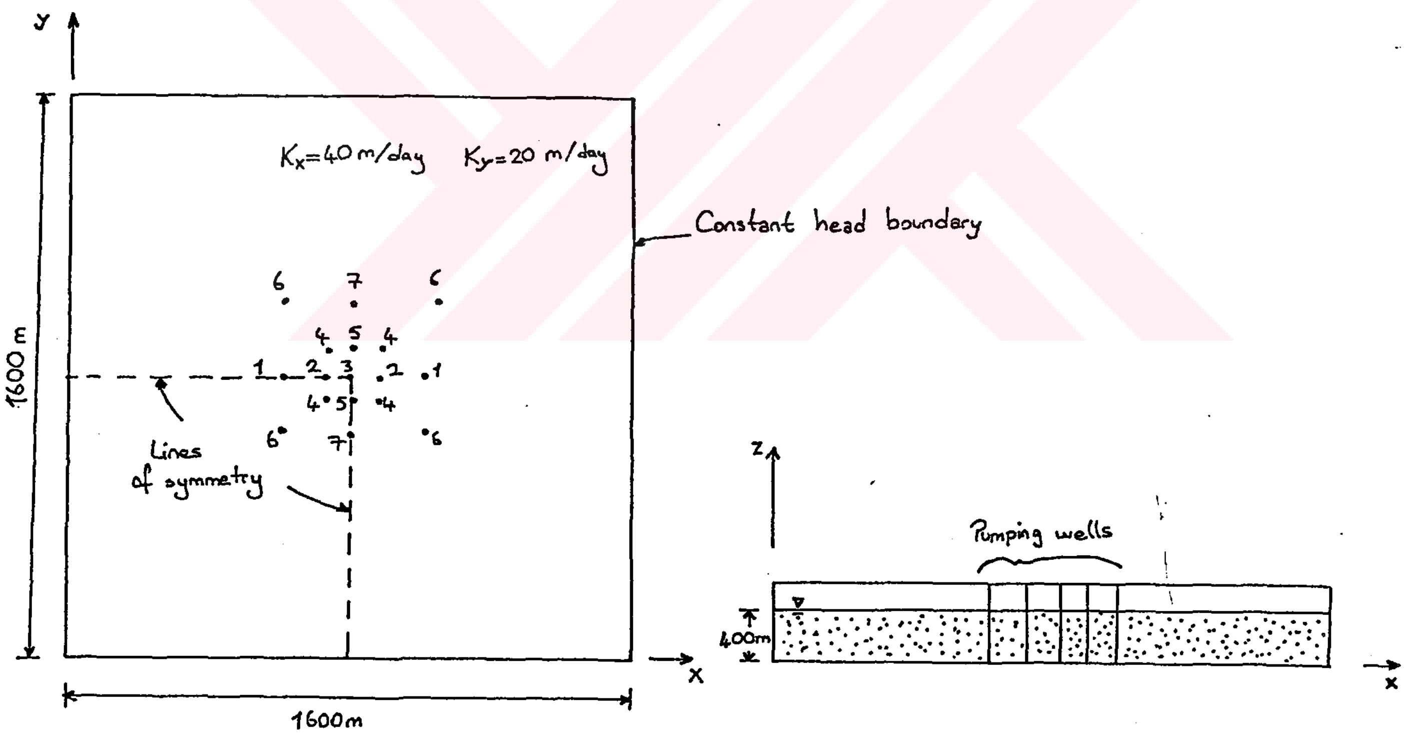

Figure B.1. Definition sketch for Example B.1.

To take the advantage of symmetry a quandrant of flow is analysed. The grid overlay is shown in Fig.B.2. The physical data about the aquifer is given in Table B.1. It was known previously, that along the stream lines (lines of symmetry), pumping rates divided accordingly, in this case, $\bar{Q}_1 = \bar{Q}_2 = \bar{Q}_5 = \bar{Q}_7 = 1500/2\ m^3/day.m^3$ $\bar{Q}_3 = 1500/4\ m^3/day.m^3$ is taken.

The potential distribution (piezometric head distribution) is shown in a number of graphs (Fig.B.3, Fig.B.4, Fig. B.5, Fig.B.6, Fig. B.7, Fig. B.8, Fig. B.9).

1		2		3		4		5		6		7		8		9		G.L. Num.	
(0,0)		(254.68,0)		(432.96,0)		(557.75,0)		(645.11,0)		(706.26,0)		(749.06,0)		(779.03,0)		(800,0)		Begin. coor.	
(0,800)		(254.68,800)		(432.96,800)		(557.75,800)		(645.11,800)		(706.26,800)		(749.06,800)		(779.03,800)		(800,800)		End coor.	
8	0.7	8	0.7	8	0.7	8	0.7	8	0.7	8	0.7	8	0.7	8	0.7	8	0.7	Int.	Weight. fact.

$\bar{Q}^{45}_{I}=-750$	$\bar{Q}^{63}_{II}=-750$	$\bar{Q}^{81}_{III}=-375$	$\bar{Q}^{61}_{IV}=-2400$	$\bar{Q}^{79}_{V}=-750$	$\bar{Q}^{41}_{VI}=-1200$	$\bar{Q}^{77}_{VII}=-750$	Pumping rates $m^3/day.m^3$

MAXNOD= 81	MAXFIX=17	MAXLOD=7
$\varnothing_1=\varnothing_2=\varnothing_3=\varnothing_4=\varnothing_5=\varnothing_6=\varnothing_7=\varnothing_8=\varnothing_9=\varnothing_{10}=\varnothing_{19}=\varnothing_{28}=\varnothing_{37}=\varnothing_{46}=\varnothing_{55}=\varnothing_{64}=\varnothing_{73}=400\ m$		

Table B.1. Physical data for Example B.1.

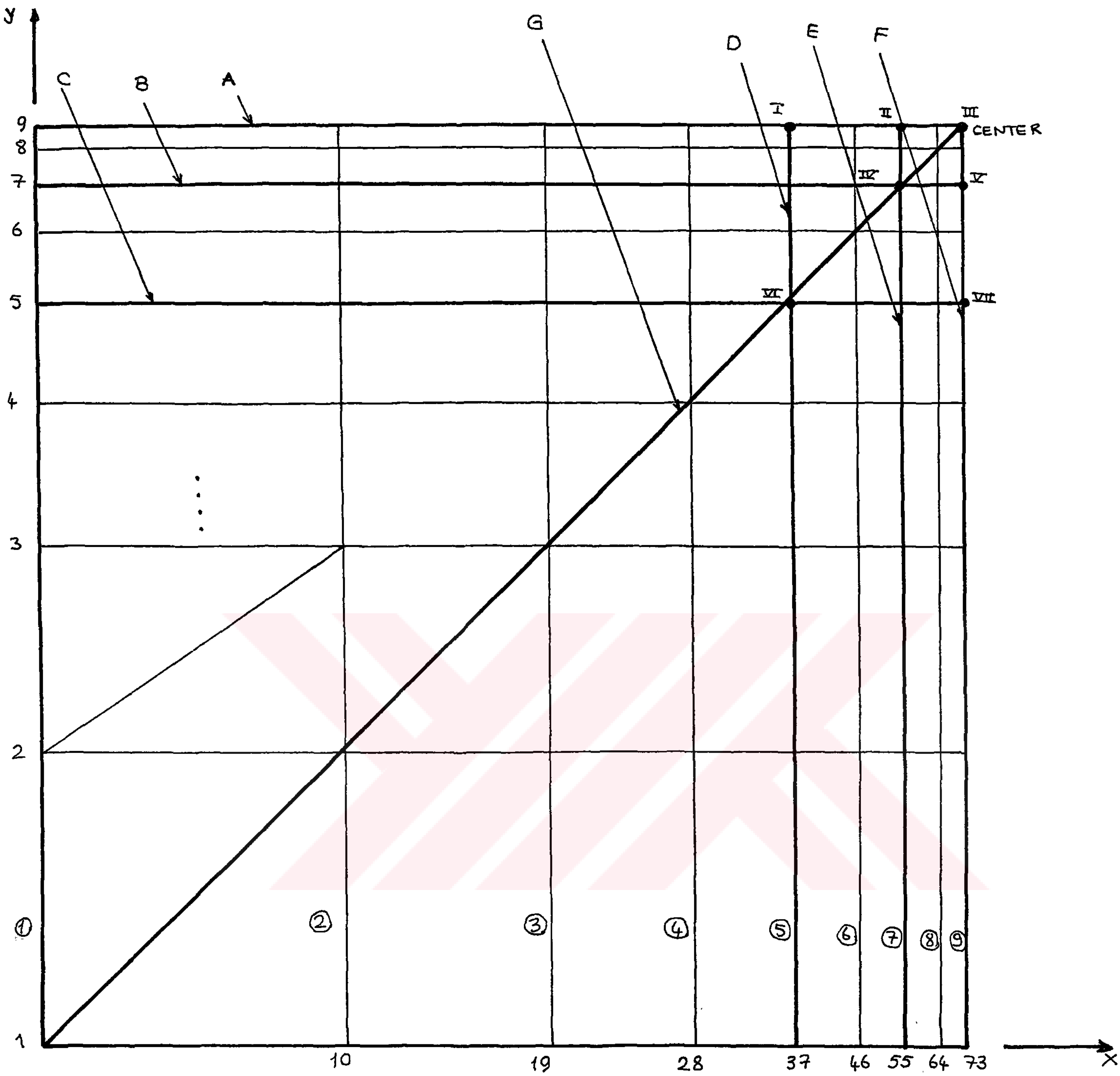

Figure B.2. Grid overlay and areal distributions of potentials of Example B.1.

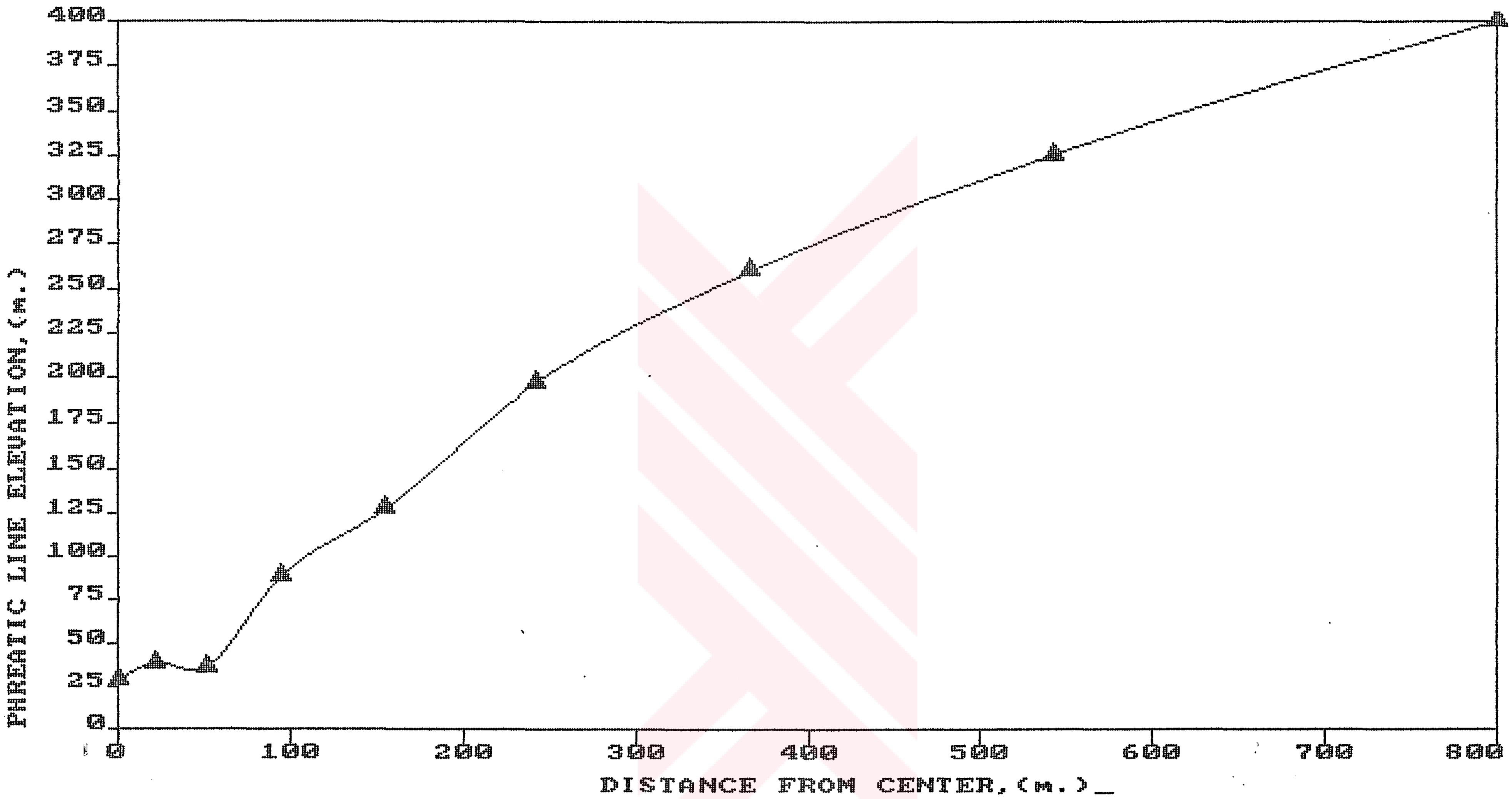

Figure B.3. Distribution of phreatic line elevations along stream line A.

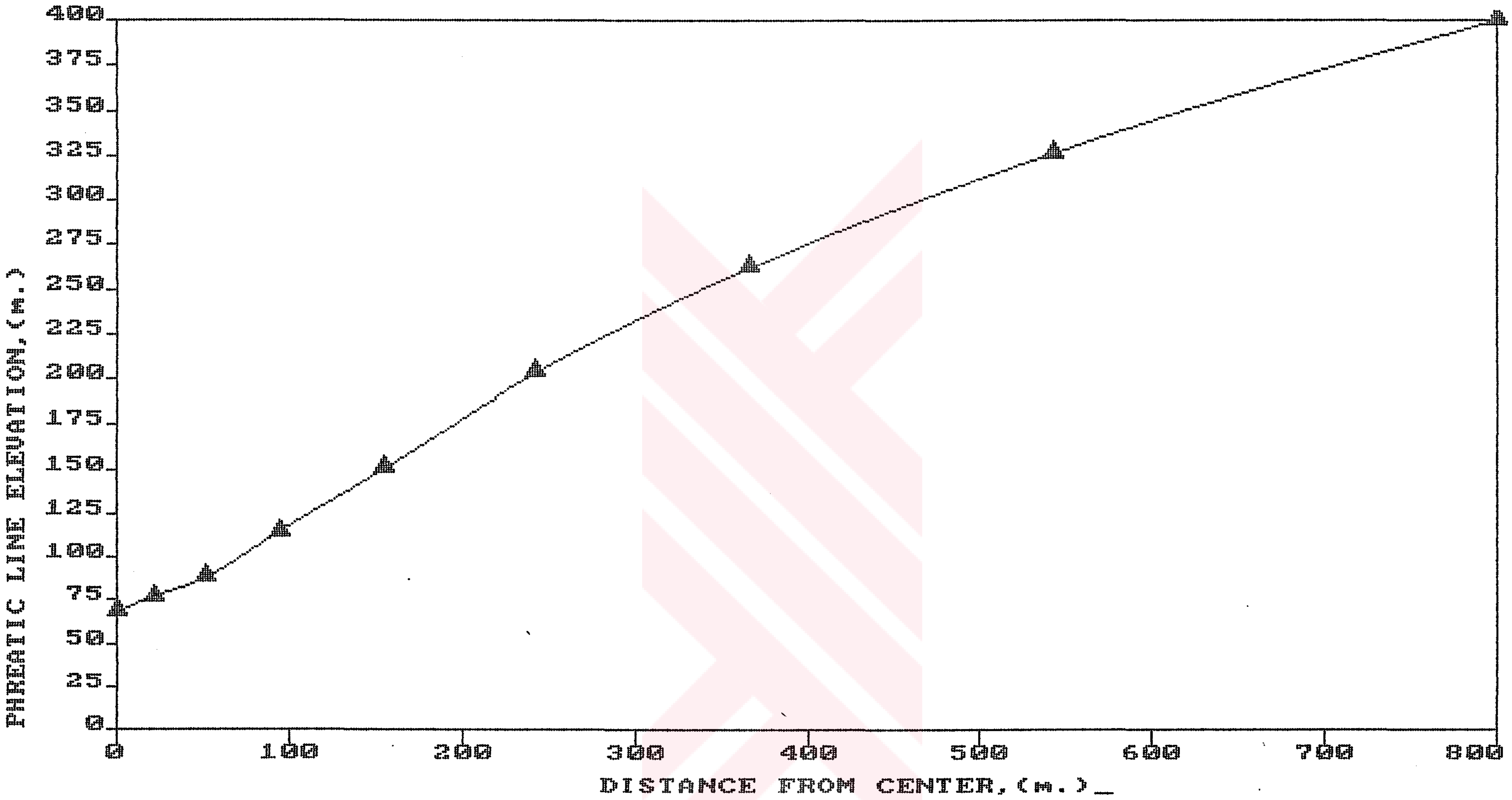

Figure B.4. Distribution of phreatic line elevations along stream line B.

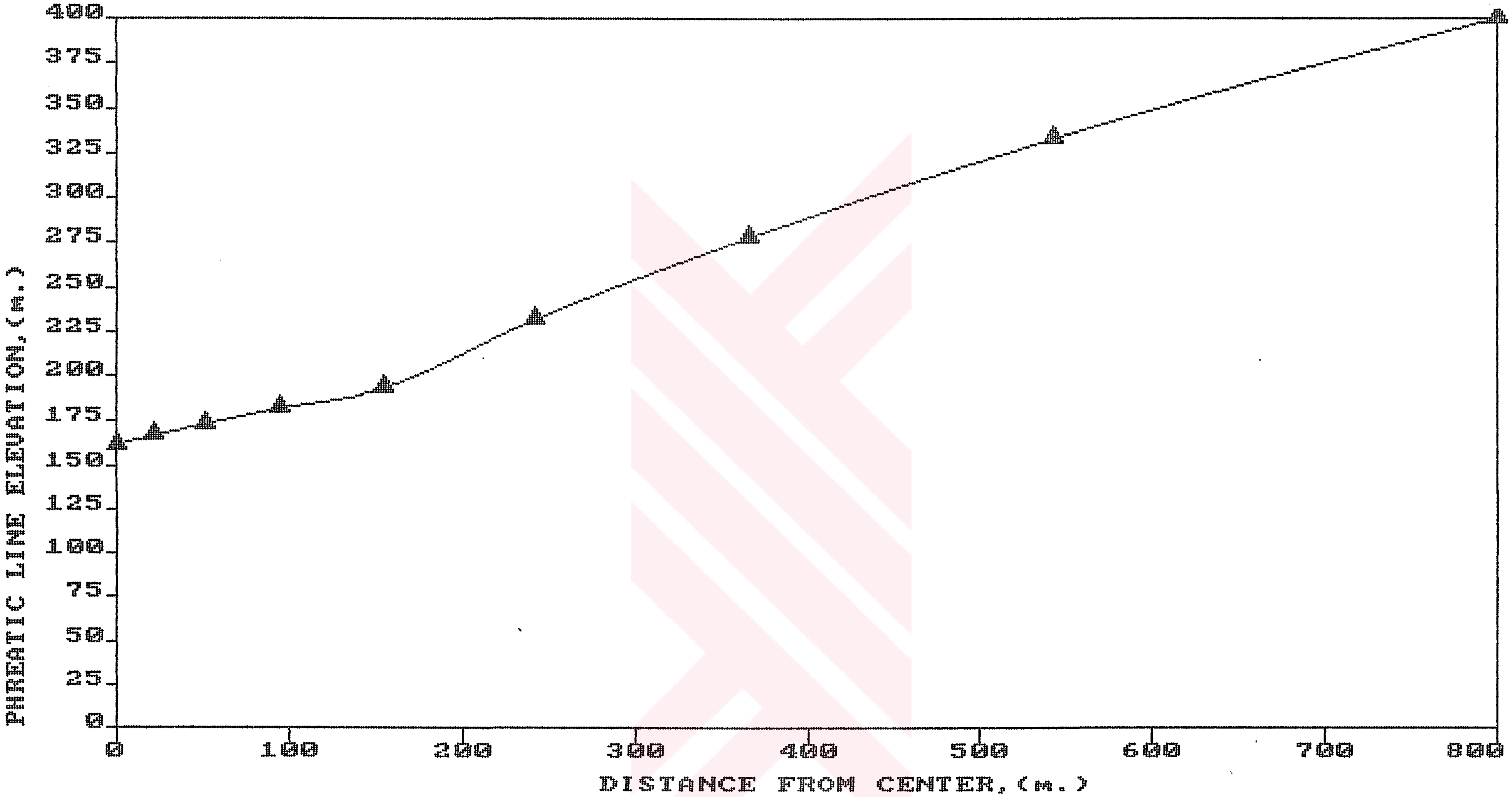

Figure B.5. Distribution of phreatic line elevations along stream line C.

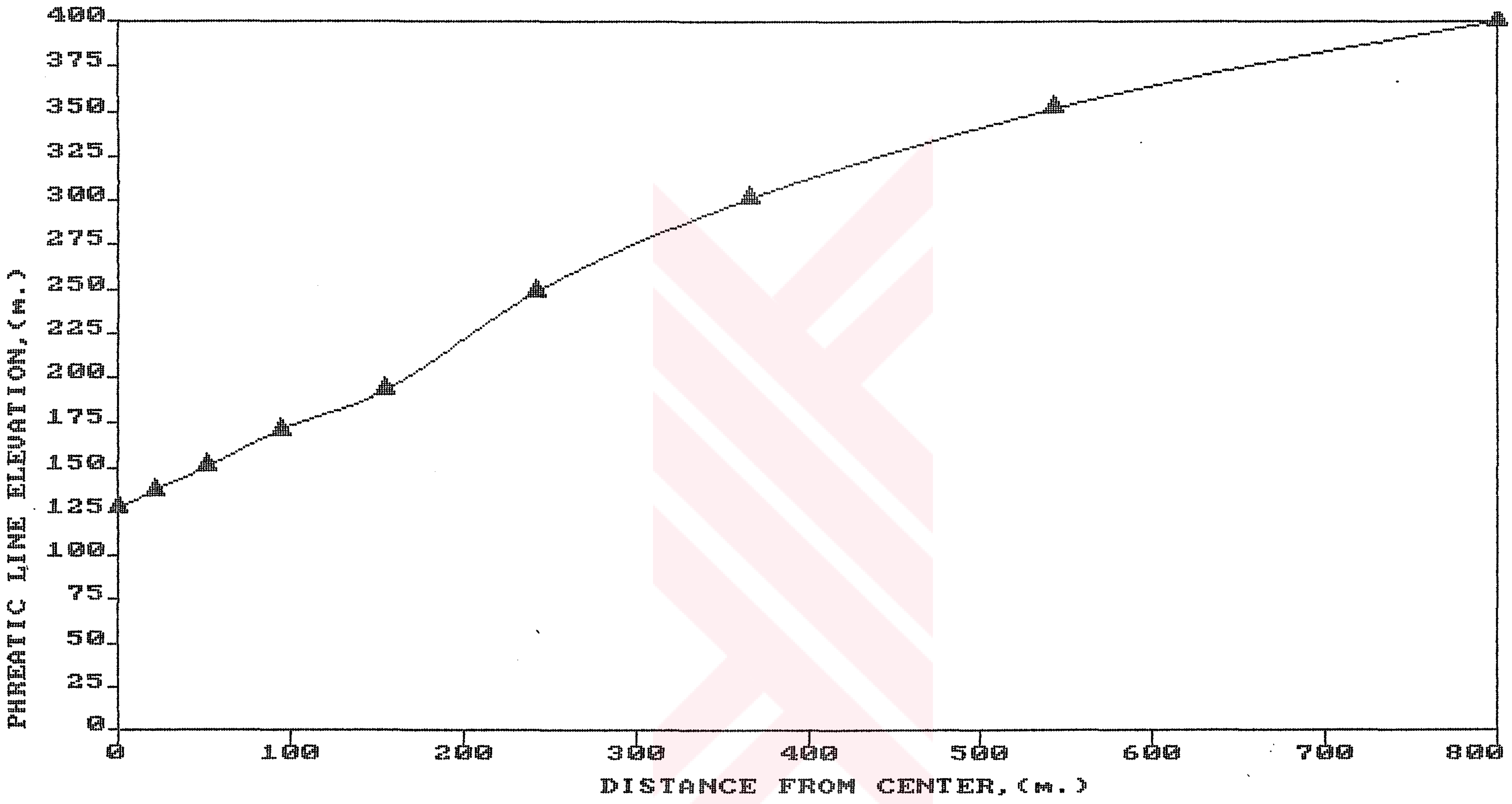

Figure B.6. Distribution of phreatic line elevations along stream line D.

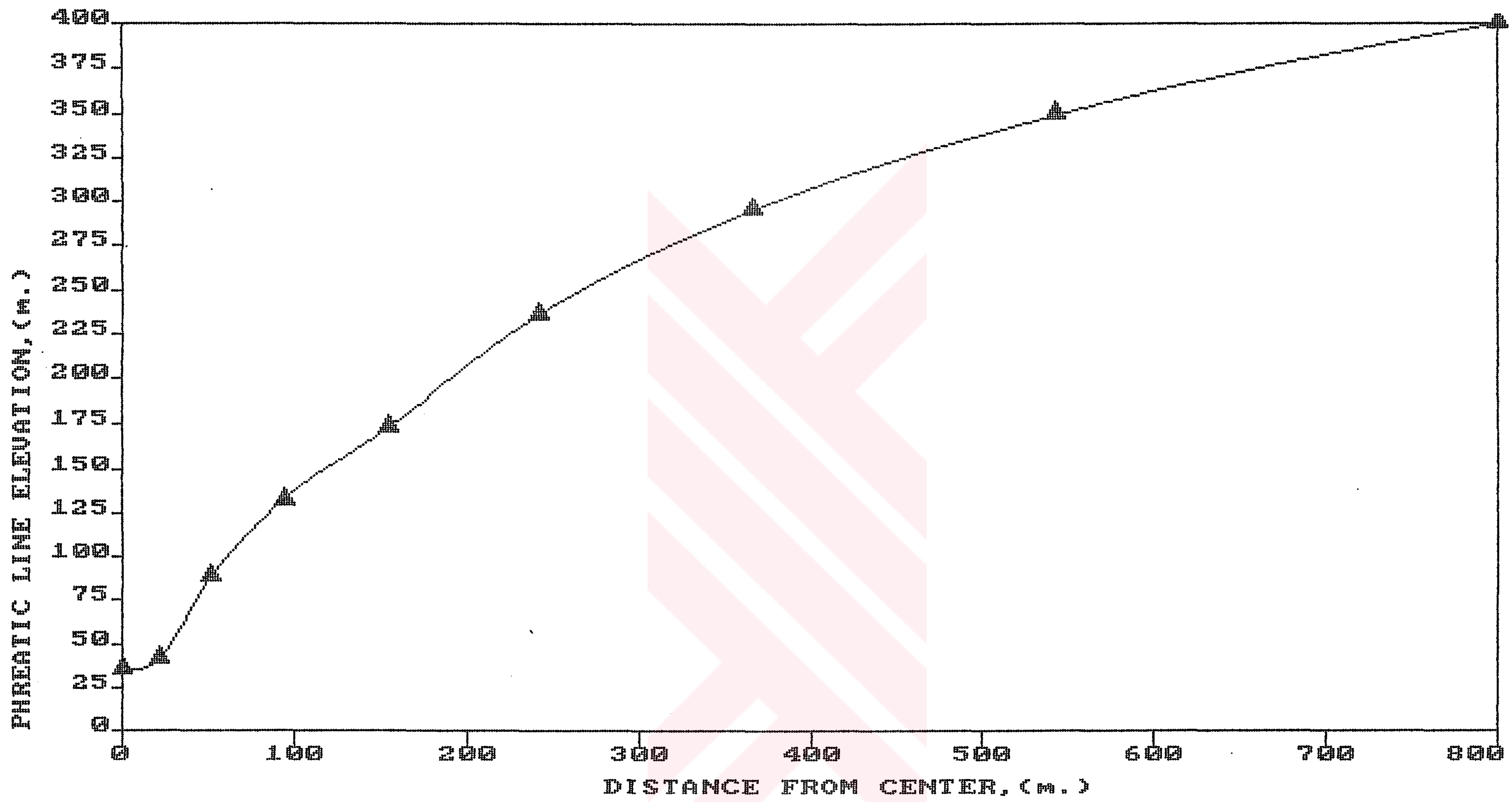

Figure B.7. Distribution of phreatic line elevations along stream line E.

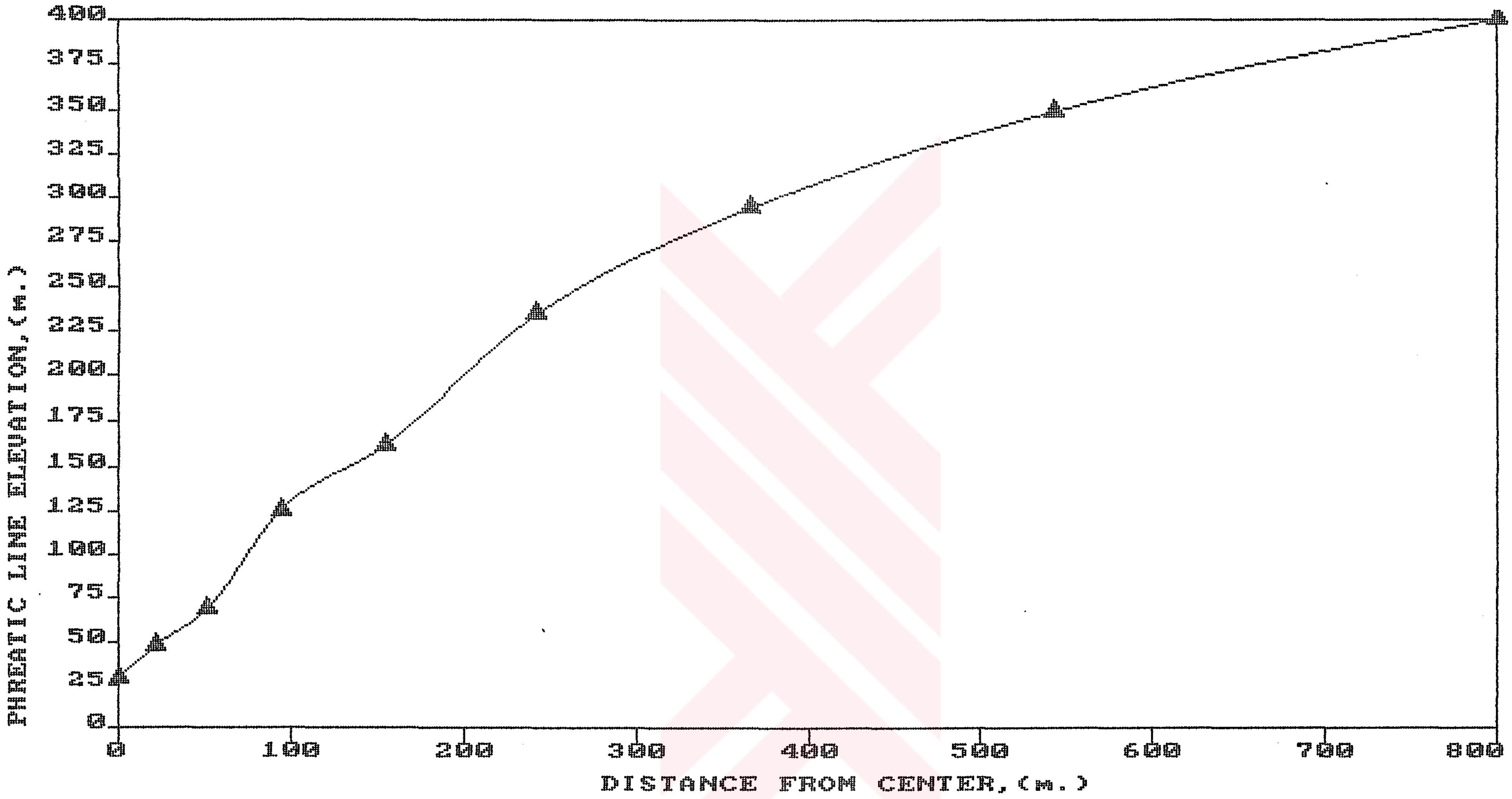

Figure B.8. Distribution of phreatic line elevations along stream line F.

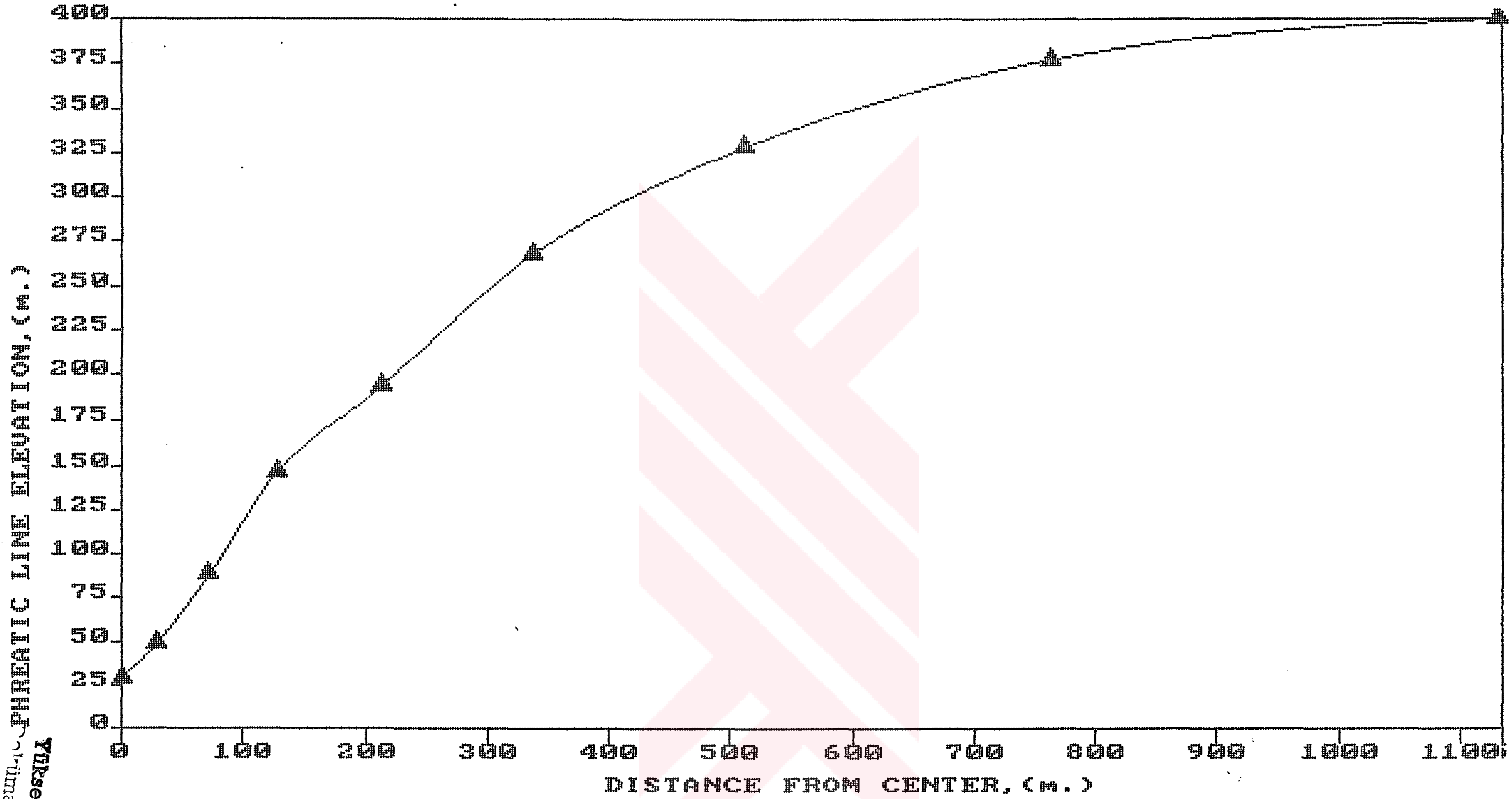

Figure B.9. Distribution of phreatic line elevations along stream line G.