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GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

2-FACTORIZATION OF COMPLETE EQUIPARTITE GRAPHS WITH FOUR AND EIGHT CYCLES

ZEHRA NUR ÖZBAY
A THESIS SUBMITTED FOR THE DEGREE OF
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GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

EŞ PARÇALI TAM ÇİZGELERİN 4 VE 8 DÖNGÜLERİYLE 2-FAKTÖRİZASYONU

ZEHRA NUR ÖZBAY

YÜKSEK LİSANS TEZİ

MATEMATİK ANABİLİM DALI

DANIŞMANI

DOÇ. DR. SİBEL ÖZKAN

GEBZE

2015

GEBZE TEKNİK ÜNİVERSİTESİ	**YÜKSEK LİSANS JÜRİ ONAY FORMU**

GTÜ Fen Bilimleri Enstitüsü Yönetim Kurulu'nun 24/06/2015 tarih ve 2015/30 sayılı kararıyla oluşturulan jüri tarafından 09/07/2015 tarihinde tez savunma sınavı yapılan Zehra Nur ÖZBAY'ın tez çalışması Matematik Anabilim Dalında YÜKSEK LİSANS tezi olarak kabul edilmiştir.

JÜRİ

ÜYE
(TEZ DANIŞMANI) : Doç. Dr. Sibel Özkan

ÜYE : Yrd. Doç. Dr. Didem GÖZÜPEK KOCAMAN

ÜYE : Yrd. Doç. Dr. Roghayeh HAFEZIEH

ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun
....../....../...... tarih ve ....../...... sayılı kararı.

İMZA/MÜHÜR

SUMMARY

A k-regular graph is a graph in which all the degrees are k. A spanning 2-regular subgraph of G is called a 2-factor in G. A 2-factorization of G is a decomposition of all the edges of G into edge-disjoint 2-factors. An equipartite graph is a graph whose vertex set can be partitioned into subsets of the same size such that no two vertices from the same subset are connected by an edge. The complete equipartite graph with u subsets of size m is denoted by $K(m : u)$ and it contains every edge between vertices of different subsets. In this thesis we will find a 2-factorization of complete equipartite graph $K(m : u)$ with four and eight cycles. In fact, this is a Hamilton-Waterloo problem for $K(m : u)$.

Key Words: Complete multipartite graphs, Resolvable cycle decomposition, Hamilton-Waterloo problem, Oberwolfach problem.

ÖZET

K-düzenli bir çizge bütün derecelerin k olduğu bir çizgedir. 2-faktör ise G çizgesinin 2-düzenli kapsayıcı bir altçizgesidir. G'nin bir 2-faktorizasyonu, G'nin bütün kenarlarının 2- faktörlere parçalanışıdır. Eş parçalı bir çizge, köşe seti aynı kümedeki herhangi iki köşe bir kenar ile bağlı olmayacak şekilde eşit büyüklükte parçalara ayrılabilen bir çizgedir. u tane m elemanlı parçaya sahip tam eş parçalı çizge $K(m : u)$ ile gösterilir ve farklı parçalardaki noktaların arasındaki bütün kenarları içerir. Bu tezde tam eş parçalı $K(m : u)$ çizgesinin 4 ve 8 döngüleriyle 2-faktörizasyonunu incelenecektir. Aslında bu $K(m : u)$ için bir Hamilton-Waterloo problemidir.

Anahtar Kelimeler: Çok parçalı tam graflar, Yeniden çözülebilir döngü parçalanışı, Hamilton-Waterloo problemi, Oberwolfach problemi.

ACKNOWLEDGEMENTS

It was a great opportunity and an honor to work with my supervisor, Assoc. Prof. Dr. Sibel Özkan. Her mathematical experience and research insight guided me a lot through my studies. Thank you.

I would like to thank TÜBİTAK for providing the 113F033-numbered project scholarship throughout my studies.

Finally, I am grateful to my family for their permanent unconditional love and support throughout my life.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		Explanations
$V(G)$	:	Vertex set of a graph G
$E(G)$	:	Edge set of a graph G
$(V(G), E(G))$	:	Representation of a graph G with the vertex set $V(G)$ and the edge set $E(G)$
$d(v)$	:	Degree of a vertex v
C_n	:	Cycle of length n
K_n	:	Complete graph with n vertices
$K(m : n)$	:	Complete n-partite graph with m vertices in each part
(m,n)-HWP($v; r, s$)	:	Hamilton-Waterloo Problem with v vertices; r of the 2-factors are cycle of length m and s of the 2-factors are cycle of length n
(4,8)-HWP($4t : m; r, s$)	:	Hamilton-Waterloo Problem applied to $K(4t : m)$; r of the 2-factors are cycle of length 4 and s of the 2-factors are cycle of length 8
GD[K, λ, M, v]	:	Group divisible design where K is the set of blocks, λ is the number of blocks which each pair from distinct groups contained in, M is the set of groups and v is the set of points
3-RGDD	:	Resolvable group divisible design with k = 3
4-RGDD	:	Resolvable group divisible design with k = 4
STS(v)	:	Steiner Triple System of order v
KTS(v)	:	Kirkman Triple System of order v
$\eta(v)$	:	Amalgamation number of v

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1. INTRODUCTION

We start this section with some introductory basics for graph theory. The following two subsections are about two well-known problems in graph theory. Then in the last two subsections, we give some definitions from design theory which will be used in the proof of the main theorem.

A *graph* G is an ordered pair $G = (V(G), E(G))$ where $V(G)$ is called the vertex (node) set, $E(G)$ is called the edge set and each edge is associated with two vertices (not necessarily different) which are called as the *endpoints* of this edge. A *loop* is an edge whose endpoints are the same. Multiple edges are edges having the same pair of endpoints. A *simple graph* is a graph which has no loops or multiple edges. For convenience, we use (V, E) instead of $(V(G), E(G))$ if it is not obligatory to indicate the graph. $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. According to Wilson's definition [1], a *walk* is a "way of getting from one vertex to another", and consists of a sequence of edges, one following after another. A u, v*-path* is a walk between the vertices u and v in which no vertex appears more than once. A graph G is *connected* if for every pair of vertices u, v of G, there is a u, v*-path* in G. If u and v are the endpoints of an edge, then we say u is *adjacent* to v and vice versa. The number of edges adjacent to a vertex v in a graph G is called the *degree of* v and it is denoted by $d_G(v)$ or $d(v)$ in short. Two subgraphs are said to be *edge disjoint* if they have no edges in common. Likewise, two subgraphs are *vertex disjoint* if they have no vertices in common. The *union* of two graphs G_1 and G_2, with disjoint vertex sets $V(G_1)$, $V(G_2)$ and edge sets $E(G_1)$, $E(G_2)$ respectively, is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

A *k-regular graph* is a graph in which each vertex has degree k. A spanning (i.e. including each vertex of a graph) k-regular subgraph of G is called a *k-factor* in G. Let G be a graph and H be a subgraph of G. If all edges of G can be decomposed into edge disjoint copies of H, then this decomposition is called an *H-decomposition of G*. If all edges of G can be decomposed into edge disjoint copies of k-factors, then this decomposition is called a *k-factorization* and G is called *k-factorable*. A *parallel class* (or *resolution class*) of a decomposition of G is a subset of vertex disjoint graphs whose union partitions the vertex set of G.

Example 1.1: Let G be the graph shown in Figure 1.1 with the following vertex set: $V(G) =$

$\{v_0, v_1, v_2, v_3, v_4, v_5\}$.

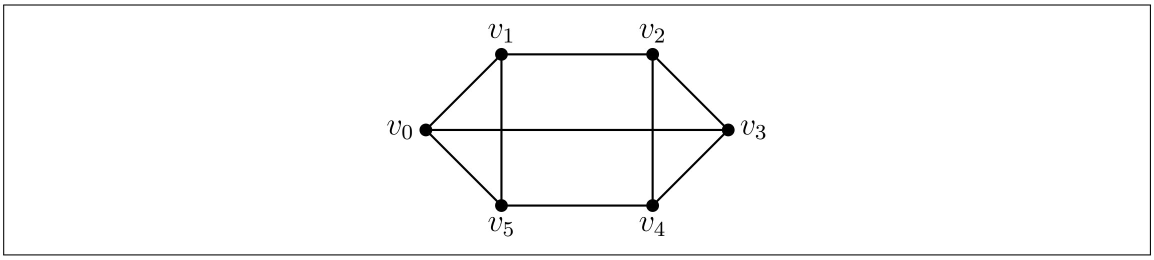


Figure 1.1: Representation of a graph G.

Then G has the following 1-factors F_1, F_2 and F_3 as given in Figure 1.2.

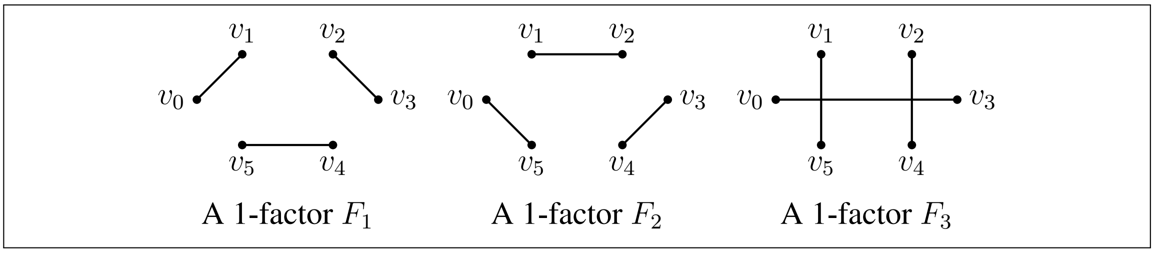


Figure 1.2: A 1-factorization of G.

We see that F_1, F_2 and F_3 are edge disjoint. Furthermore, $F_1 \cup F_2 \cup F_3 = G$. Hence, G is 1-factorable.

Example 1.2: Let D be the graph shown in Figure 1.3 with the following vertex set: $V(D) = \{v_0, v_1, v_2, v_3, v_4\}$.

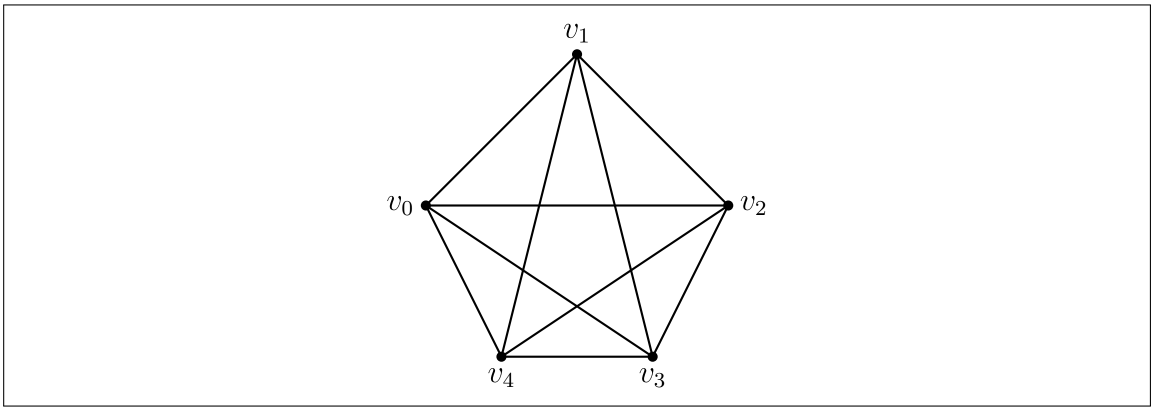


Figure 1.3: Representation of a graph D.

Then D has the following 2-factors S_1 and S_2, as given in Figure 1.4.

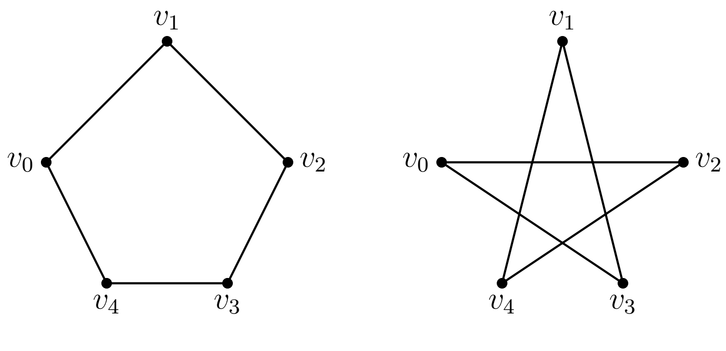


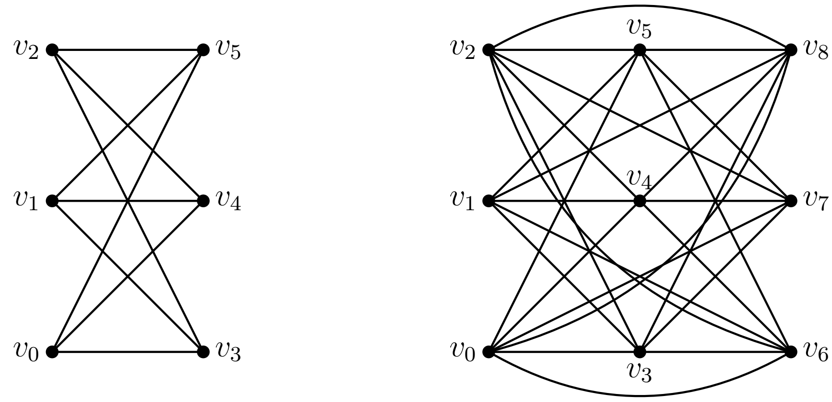
Figure 1.4: A 2-factorization of D.

We see that S_1 and S_2 are edge disjoint. Furthermore, $S_1 \cup S_2 = D$. Hence, D is 2-factorable.

A *cycle* is a connected graph which is 2-regular. A cycle with n vertices is denoted by C_n. A spanning cycle is called an *Hamilton cycle*. Clearly, a 2-factor consists of vertex disjoint union of cycles. *Cycle decomposition* of a graph G is an H-decomposition in which all H's are cycles. A *resolvable cycle decomposition* is a cycle decomposition which forms a 2-factorization, in other words, it is a cycle decomposition which can be partitioned into parallel classes.

A *complete graph* is a simple graph whose vertices are pairwise adjacent. The complete graph with n vertices is denoted by K_n. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets A and B such that every edge connects a vertex in A to one in B. A *complete bipartite graph* is a simple bipartite graph in which each vertex in A is joined to each vertex in B. We denote the complete bipartite graph with $|A| = m$ and $|B| = n$ by $K_{m,n}$. A *k-partite graph* (*multipartite graph*) is a graph whose vertices can be partitioned into k disjoint sets such that every edge connects a vertex in a set to one in another set. An *equipartite graph* is a multipartite graph in which all sets have the same number of vertices. *Complete multipartite graph* is a multipartite graph such that each vertex in any set is joined to each vertex in any other set. We will denote the complete n-partite graph with m vertices in each part by $K(m : n)$.

Example 1.3: A complete bipartite graph $K_{3,3}$ and a complete multipartite graph $K_{3,3,3}$ are given in Figure 1.5. $K_{3,3}$ has two disjoint vertex sets, one set is A with the vertices $\{v_0, v_1, v_2\}$, and the other set is B with the vertices $\{v_3, v_4, v_5\}$. $K_{3,3,3}$ has vertices which are partitioned into three disjoint sets.



A complete bipartite graph $K_{3,3}$ A complete multipartite graph $K_{3,3,3}$

Figure 1.5: A complete bipartite and a complete multipartite graph.

The following definitions are taken from the doctoral thesis of Özkan [2]: An amalgamation H of a graph G is formed by a graph homomorphism $f : V(G) \to V(H)$, where each vertex v of H represents $\eta(v) = |f^{-1}(v)|$ vertices of G. $\eta(v)$ is called the amalgamation number of v, and f is called the amalgamation function of G.

Informally, an amalgamation of a graph G is a new graph H, obtained by partitioning the vertices of G and replacing each element p of this partition, say P, by a single vertex in H, where edges incident with this single vertex are in one-to-one correspondence with the edges incident with original vertices of G in P. If there is any multiple edges, we ignore them and regard as one edge. Disentanglement of vertices is the reverse process of amalgamation. That is, G is a disentanglement of H.

We do not use amalgamation number in this thesis. The aim of using amalgamations is to reduce a graph a simpler graph in the proof of the main theorem.

Example 1.4: For example, if we amalgamate the vertices in each part of the complete bipartite graph $K_{4,4}$ into groups of two, we obtain the graph $K_{2,2}$ as shown in Figure 1.6.

Among the decompositions of graphs, cycle decompositions have attracted most of

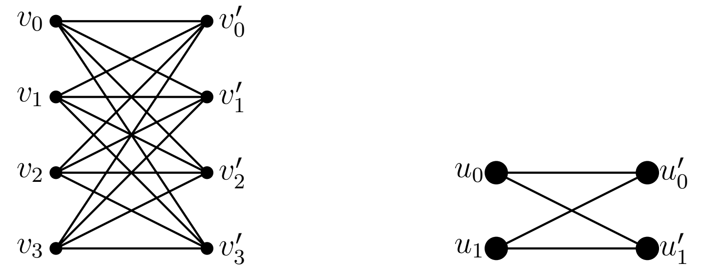


Figure 1.6: Amalgamation of vertices in $K_{4,4}$.

the attention. The two well-known resolvable cycle decomposition problems are the Oberwolfach problem and the Hamilton-Waterloo problem.

1.1. The Oberwolfach Problem

The Oberwolfach problem was first stated by Ringel in 1967 at a conference in Oberwolfach, Germany:

" *Is it possible to seat an odd number v of people at s round tables $T_1, T_2, \ldots, T_s$ (where each T_i can accommodate $t_i \geq 3$ people and $\sum_{i=1}^{s} t_i = v$) for $\frac{v-1}{2}$ different meals so that each person has every other person for a neighbor exactly once?*"

In graph theory language, this problem is equivalent to finding a 2-factorization for K_v in which each 2-factor consists of cycles of lengths $t_1, t_2, \ldots, t_s$. Here, v must be odd so that each vertex has an even degree. Note that, 2-factorization of a graph G exists if and only if G is even regular [3]. This is because each 2-factor attributes two degrees to a vertex. In total, degree of a vertex equals two times the number of factors in a 2-factorization. In the problem, the number of factors correspond to the number of nights that the meal takes place : $\frac{v-1}{2}$. Even if we say v must be odd, this problem is also applied to the cases where v is even by substracting a 1-factor from the given graph. This is called the "spouse-avoiding" version of the problem. In this case, the number of factors is $\frac{v-2}{2}$. The Oberwolfach Problem is completely solved for fixed table sizes in [4] and [5]. And also, Liu [6] solved the Oberwolfach problem for the complete equipartite graphs with uniform cycle lengths.

1.2. The Hamilton-Waterloo Problem

Another resolvable cycle decomposition problem is the Hamilton-Waterloo problem. In the Oberwolfach problem we have a conference taking place in only one dining hall with table sizes uniform or not. Now, in the Hamilton-Waterloo problem, people can use two dining halls (one in Hamilton and the other in Waterloo) by choosing one of them in one sitting. Again table sizes can be uniform or not. Obviously, total number of people (v) is equal to the summation of table sizes at one night. Since there are two dining halls, this sum is the same in both of the dining halls. For example, 48 people can be placed in a dining hall which has 12 uniform tables with 4 seats and in another dining hall which has 6 uniform tables with 8 seats. By using these two dining halls, we want to arrange conference dinners such that each person sits next to another person exactly once. In terms of graph theory, this is a Hamilton-Waterloo Problem applied to K_{48} with cycle lengths 4 and 8. The number of factors is the same as in the Oberwolfach Problem. Therefore, we have $\frac{v-2}{2} = \frac{48-2}{2} = 23$ cycle factors. Since the cycles have uniform lengths, we can name the 2-factors as C_4-factors and C_8-factors. In general, a 2-factorization of K_v (or $K_v - I$) where r of the 2-factors are C_m-factors and s of the 2-factors are C_n-factors corresponds to the solution of the Hamilton-Waterloo problem with uniform cycle sizes. It is denoted by (m, n)-HWP($v; r, s$).

In 2002, Adams et al. [7] solved the Hamilton-Waterloo problem for the cases $(m, n) \in \{(4, 6), (4, 8), (4, 16), (8, 16), (3, 5), (3, 15), (5, 15)\}$ and settled the problem for all $v \leq 16$. Danziger et al. [8] solved the problem for the case $(m, n) = (3, 4)$ with a few exceptions. Horak et al [9], Dinitz and Ling [10, 11] worked on the case $m = 3$ and $n = v$, that is, triangle factors and Hamilton cycles. Bryant et al. settled the Hamilton-Waterloo problem for bipartite 2-factors [12].

In 2008, the case of 4-cycles and n-cycles for even n is solved by Fu and Huang [13] and they also settled all cases where $n = 2t$ and t is even. Then, in 2013, Keranen and Özkan solved the case of 4-cycles and a single factor of n-cycles where n is odd [14].

Although the generalization of the Oberwolfach problem to the complete multipartite graphs have been studied [6], there is no such generalization is known for the Hamilton-Waterloo problem up to date. In this thesis, unlike in [6], we study the Hamilton-Waterloo problem on a complete equipartite graph. Within the parts of a complete multipartite graph there is no edge which makes the problem harder. We want to study C_4 and C_8 factors, so we choose $4t$ for the number of vertices in each part and we work on $K(4t : m)$. If we worked on a complete graph K_{4tm}, we would use the notation (4,8)-HWP($4tm; r, s$) for the problem. Therefore, we need different notation for $K(4t : m)$ that we will use (4,8)-HWP($4t : m; r, s$) to denote a $\{C_4^r, C_8^s\}$-factorization of $K(4t : m)$ such that r of the 2-factors are cycle of length 4, s of the 2-factors are cycle of length 8 where r and s satisfy $0 \leq r, s, \leq 2t(m-1)$ and $2t(m-1)$ is the total number of factors. Since Liu [6] solved the Oberwolfach problem for the complete equipartite graphs with uniform cycle lengths, we have already the cases $r = 0$, $s = 2t(m-1)$ and $s = 0$, $r = 2t(m-1)$.

We have those obvious necessary conditions for the complete multipartite graph $K(n : m)$ to have a $\{C_4^r, C_8^s\}$-factorization:

Theorem 1.1: If $K(n : m)$ has a $\{C_4^r, C_8^s\}$-factorization for non-negative integers r, s, then it satisfies:

i) $8 \mid nm$,

ii) $r + s = \frac{n(m-1)}{2}$.

Proof 1.1: The number of cycle factors in $K(n : m)$ is $\frac{n(m-1)}{2}$. So, $r + s = \frac{n(m-1)}{2}$. Since a C_4-factor is a spanning subgraph, 4 divides the total number of vertices, that is nm. In the same way, $8 \mid nm$, which implies that $4 \mid nm$, so we only need $8 \mid nm$. ■

$K(4t : m)$ satisfies the conditions of Theorem 1.1 if at least one of t or m is even. Since, $8 \mid 4tm$ only when at least one of t or m is even. Indeed, if one of n or m were a multiple of 8 this problem would be easier. On the other hand, if one of n or m is a multiple of 6, a $\{C_6^r, C_{12}^s\}$-factorization can be studied. In general, whether there exists a $\{C_d^r, C_{2d}^s\}$-factorization of $K(n : m)$ or not for some positive integer d is a challenging problem. It can be studied as a future work. In our main theorem, we show that the necessary conditions are also sufficient for $K(4t : m)$ with a few exceptions.

Theorem 1.2: The complete multipartite graph $K(4t : m)$ for $m \geq 2$, $t \geq 1$ has a $\{C_4^r, C_8^s\}$-factorization for any non-negative integers r, s with $0 \leq r, s \leq 2t(m-1)$ if and only if it satisfies the following conditions:

i) $8 \mid 4tm$,

ii) $r + s = 2t(m-1)$

possibly except $m \equiv 5$ (*mod* 12) *when* $t \equiv 2, 10$ (*mod* 12), *and* $m \equiv 2$ (*mod* 24) *when* $t \equiv 1, 5$ (*mod* 6).

1.3. Resolvable Group Divisible Designs

In the proof of Theorem 1.2, we use some results from the design theory. Here, we shortly define resolvable group divisible designs.

Definition 1.1: Let v *>2 be a positive integer. A group divisible design (which is abbreviated as GDD)* $GD[K, \lambda, M, v]$ *is a triple (X,*$\mathcal{G}$*,B) where* X *is a set of points,* $\mathcal{G} = \{G_1, G_2, \dots\}$ *is a partition of* X *and* B *is a class of subsets of* X *with the following properties:*

i) $|X| = v$,

ii) Cardinality of each $\mathcal{G}_i$ *is a member of* M,

iii) Cardinality of each block is a member of K,

iv) Every pair from distinct groups is contained exactly in λ *blocks,*

v) No pair from the same group is contained in a block.

A GDD becomes a *resolvable group divisible design* if its blocks can be partitioned into parallel classes.

Let $k \in K$ be a fixed scalar, if $\lambda = 1$, we denote $GD[K, \lambda, M, v]$ for fixed M and v as k-GDD of type m^u, where m is the group size and u is the number of groups. We use only the case where $\lambda = 1$ so that there is no edge repetition. In this thesis, we need the cases where $\lambda = 1$, $k = 3$ and $\lambda = 1$, $k = 4$. In the third section we use group divisible designs which allow us to find out which triples or quadruples we should use for the right partition of the graph.

We have the following known results for 3-RGDDs and 4-RGDDs:

Theorem 1.3: [15] A (3,λ*)-RGDD of type* t^m *exists if and only if* $m \geq 3$, $\lambda t(m-1)$ *is even,* $tm \equiv 0$ (*mod* 3), *and* $(\lambda, t, m) \notin \{(1,2,6), (1,6,3)\} \cup \{(2j+1, 2, 3), (4j+2, 1, 6) : j \geq 0\}$.

Theorem 1.4: [16]-[26] The necessary conditions for the existence of a 4-RGDD of type h^u, namely, $u \geq 4$, $hu \equiv 0$ (mod 4) and $h(u-1) \equiv 0$ (mod 3), are also sufficient except for $(h,u) \in \{(2,4),(2,10),(3,4),(6,4)\}$ and possibly except:

i) $h \equiv 2, 10$ (*mod* 12) *:*
$h = 2$ *and* $u \in \{34,46,52,70,82,94,100,118,130,178,184,202,214,238,250,334\}$*;* $h = 10$ *and* $u \in \{4,34,52,94\}$*;* $h \in [14,454] \cup \{478,502,514,526,614,626,686\}$ *and* $u \in \{10,70,82\}$.

ii) $h \equiv 6$ (*mod* 12) *:* $h = 6$ *and* $u \in \{6,68\}$*;* $h = 18$ *and* $u \in \{18,38,62\}$.

iii) $h \equiv 9$ (*mod* 12) *:* $h = 9$ *and* $u = 44$.

iv) $h \equiv 0$ (*mod* 12) *:* $h = 36$ *and* $u \in \{11,14,15,18,23\}$.

In [26], Wei H. and Ge G. solved the 4-RGDD of types $2^{184}, 9^{44}, 14^{10}, 18^{18}, 22^{10}$ and 36^{11}.

- Kirkman Triple Systems

First known cycle decompositions are triple systems. We will make use of them in the proof main theorem. Wesley S. B. Woolhouse was the first person who defined the Steiner Triple Systems (STS(v) in short for a Steiner Triple System on v points). Existence problem of STS(v) was posed by W. S. B. Woolhouse in The Lady's and Gentleman's Diary. Later in 1847, Rev. T. P. Kirkman solved this problem [27]. Before giving the definition of a Kirman triple system, first we define the Steiner Triple Systems:

Definition 1.2: A STS(v) is an ordered pair (S,T), where S is a finite set of v points or symbols, and T is a set of 3-element subsets of S called triples, such that each pair of distinct elements of S occurs in precisely one triple of T.

The *order* of a STS is the cardinality of the set S. If the triples in T can be partitioned into parallel classes, then STS is *resolvable*.

Definition 1.3: A resolvable Steiner triple system of order v is called a Kirman triple system. It is denoted by KTS(v).

Kirkman Triple System with $v = 15$, that is KTS(15), is known as the solution for the Kirkman's schoolgirl problem:

There are 15 schoolgirls. Is it possible to take them for a walk each day of the 7 days of a week, walking with 5 rows of 3 girls in each, in such a way that each pair of girls walks together in the same row on exactly one day? In 1850, this problem was posed by Rev. T. P. Kirkman and solved in 1851. Let the girls are numbered from 1 to 15. In Table 1, a solution of the problem is given.

Table 1: Resolution classes of a KTS(15).

Day 1	Day 2	Day 3	Day 4
{01, 02, 05}	{02, 03, 06}	{05, 06, 09}	{08, 10, 01}
{15, 13, 06}	{14, 01, 07}	{02, 04, 10}	{13, 14, 02}
{04, 03, 07}	{04, 05, 08}	{07, 08, 11}	{03, 05, 11}
{12, 09, 08}	{09, 10, 13}	{12, 13, 01}	{04, 06, 12}
{11, 10, 14}	{11, 12, 15}	{14, 15, 03}	{07, 09, 15}

Day 5	Day 6	Day 7
{15, 01, 04}	{01, 03, 09}	{01, 06, 11}
{09, 11, 02}	{15, 02, 08}	{02, 07, 12}
{10, 12, 03}	{11, 13, 04}	{03, 08, 13}
{05, 07, 13}	{12, 14, 05}	{04, 09, 14}
{06, 08, 14}	{06, 07, 10}	{05, 10, 15}

It is proved that a STS(v) exists if and only if $v \equiv 1 \pmod 6$ or $v \equiv 3 \pmod 6$ [27]. The following theorem, which is a generalization of the Kirkman's schoolgirl problem, was published by Ray-Chaudhuri D. K. and Wilson R. M. in 1971.

Theorem 1.5: [28] A Kirkman triple system of order v exists if and only if $v \equiv 3$ $(mod\ 6)$.

2. PRELIMINARY RESULTS

Most of the time in the proof of the main theorem depending on the graph we work on, say G, we amalgamate vertices in groups of four or eight. Then, we ask for a 1-factorization of the amalgamated graph, say H. If there exists a 1-factorization of H, then each 1-factor of H corresponds to a $K_{4,4}$ factor or a $K_{8,8}$ factor in G when the vertices are disentangled. In this section, we will show that $K_{8,8}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for each $0 \leq r', s' \leq 4$ which verifies that G has also a $\{C_4^r, C_8^s\}$-factorization for each possible r and s. However, $K_{4,4}$ has $\{C_4^{r'}, C_8^{s'}\}$-factorization only for $r' = 0$, $s' = 2$ and $r' = 2$, $s' = 0$. Because, $K_{4,4}$ has two 2-factors in total, this forces $K_{4,4}$ to have a $\{C_4^{r'}, C_8^{s'}\}$-factorization where r' and s' are even. So, odd cases are open. On the other hand, we showed that $K_{4,4,4,4}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for all $0 \leq r', s' \leq 6$ and $K_{8,8,8}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for all $0 \leq r', s' \leq 8$. This lead us to consider resolvable group divisible designs. We again amalgamate vertices in groups of four or eight. So, if there exists a 3-RGDD or a 4-RGDD for the amalgamated graph H, then we can use the $\{C_4^r, C_8^s\}$-factorizations of $K_{4,4,4,4}$ and $K_{8,8,8}$.

Now, we begin with the 1-factorable graphs. Among the 1-factorization of graphs, one of the most important is complete graphs. In 1992, Wallis W. D. solved the following theorem.

Theorem 2.1: [29] K_n has a 1-factorization for even n.

The following famous theorem was obtained by König in 1916.

Theorem 2.2: [30] Every regular bipartite graph has a 1-factorization.

The following theorem was obtained by Auerbach and Laskar in 1976.

Theorem 2.3: [31] Let $s \geq 2$ and $2|(s-1)n$. Then the complete s-partite graph $K(n : s)$ can be decomposed into Hamilton cycles.

With the help of the theorems above, we can now prove the following:

Lemma 2.1: There exists a 1-factorization of the complete multipartite graph $K(t : m)$ for *positive integers $m \geq 2$ and t except t and m are both odd.*

Proof 2.1: Let $K(t:m)$ be a complete multipartite graph where $m \geq 2$ and t are positive integers. There are two cases depending on the parity of t:

- *Case 1:*

  Assume t is odd. If m is also odd, then total number of vertices is odd. So we can not pair vertices of $K(t:m)$ and there is no 1-factor. Hence, there is no 1-factorization. If m is even, we amalgamate all t vertices in each part to represent each part by one vertex and obtain a complete graph K_m. Since m is even K_m has a 1-factorization by Theorem 2.1. Each 1-factor of K_m corresponds to a complete bipartite graph $K_{t,t}$ when vertices are disentangled. So we get a $K_{t,t}$ factorization of $K(t:m)$. Using Theorem 2.2, we obtain 1-factorizations of $K_{t,t}$'s. This gives the 1-factorization of $K(t:m)$.

- *Case 2:*

  Assume t is even. So, $(m-1)t$ is even for all m. Thus, by Theorem 2.3, $K(t:m)$ can be decomposed into Hamilton cycles. Since there are even number of vertices, tm, each Hamilton cycle can be decomposed into 1-factors and we can get a 1-factorization out of this Hamilton decomposition. This implies that $K(t:m)$ has a 1-factorization. ■

Now we find a $\{C_4^r, C_8^s\}$-factorization of $K_{4,4,4,4}$, $K_{8,8}$ and $K_{8,8,8}$ for all possible r, s satisfying the necessary conditions to make use of them in the proof of the main theorem.

Lemma 2.2: There exists a $\{C_4^r, C_8^s\}$-factorization of $K_{4,4,4,4}$ for each $0 \leq r, s \leq 6$.

Proof 2.2: Let the four partite sets of $K_{4,4,4,4}$ be $U = \{u_i \mid 0 \leq i \leq 3\}$, $U' = \{u'_i \mid 0 \leq i \leq 3\}$, $V = \{v_i \mid 0 \leq i \leq 3\}$, and $V' = \{v'_i \mid 0 \leq i \leq 3\}$ respectively. Since each 2-factor attributes two degrees to a vertex, total number of factors equals to the degree of a vertex over 2. Each vertex in $K_{4,4,4,4}$ has degree 12. So there are six factors in the 2-factorization of $K_{4,4,4,4}$. That is $r + s = 6$. We will first analyse the cases where r and s are odd.

A $\{C_4^3, C_8^3\}$-factorization of $K_{4,4,4,4}$ is given by the following factors:

- $\{(u_0, u'_3, u_1, u'_2, u_2, u'_1, u_3, u'_0), (v_0, v'_3, v_1, v'_2, v_2, v'_1, v_3, v'_0)\}$ *is a C_8-factor.*
- $\{(u_0, v'_0, u_3, v'_1, u_2, v'_2, u_1, v'_3), (u'_0, v_0, u'_3, v_1, u'_2, v_2, u'_1, v_3)\}$ *is a C_8-factor.*

- $\{(u_0, v'_2, u'_1, v_1, u_2, v'_0, u'_3, v_3), (u'_0, v'_3, u_3, v_0, u'_2, v'_1, u_1, v_2)\}$ *is a* C_8*-factor.*

- $\{(u_0, v_0, u'_1, v'_1), (u'_0, v'_0, u_1, v_1), (u_2, v_2, u'_3, v'_3), (u'_2, v'_2, u_3, v_3)\}$ *is a* C_4*-factor.*

- $\{(u_0, u'_1, v'_0, v_1), (u_1, u'_0, v'_1, v_0), (u_2, u'_3, v'_2, v_3), (u_3, u'_2, v'_3, v_2)\}$ *is a* C_4*-factor.*

- $\{(u_0, u'_2, v'_0, v_2), (u_2, u'_0, v'_2, v_0), (u_1, u'_1, v'_3, v_3), (u_3, u'_3, v'_1, v_1)\}$ *is a* C_4*-factor.*

These are shown in Figure 2.1.

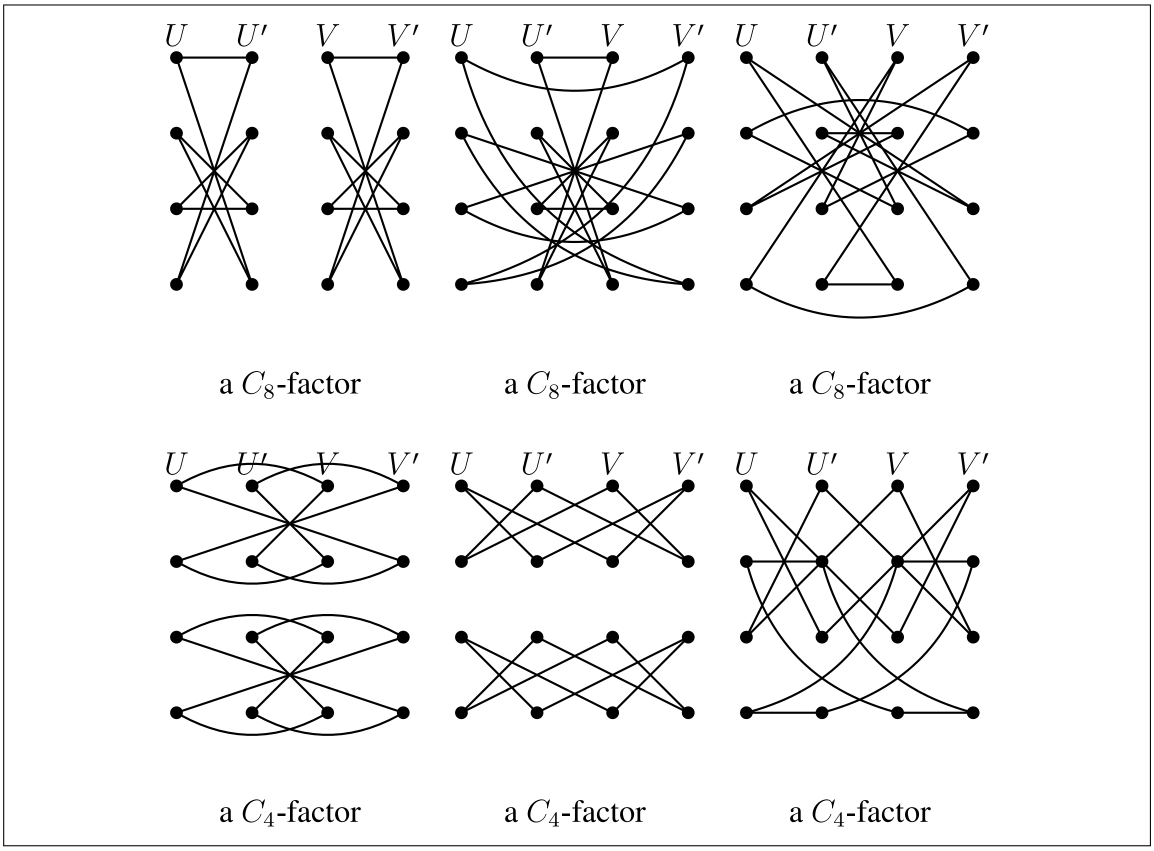


Figure 2.1: A $\{C_4^3, C_8^3\}$-factorization of $K_{4,4,4,4}$.

A $\{C_4^1, C_8^5\}$*-factorization is given by the following factors:*

- $\{(u_0, v'_0, u_1, v_0, u'_1, v_1, u'_0, v'_1), (u_2, v'_3, u_3, v'_2, u'_3, v_2, u'_2, v_3)\}$ *is a* C_8*-factor.*

- $\{(u_0, u'_3, u_1, u'_2, u_2, u'_1, u_3, u'_0), (v_0, v'_3, v_1, v'_2, v_2, v'_1, v_3, v'_0)\}$ *is a* C_8*-factor.*

- $\{(u_0, u'_1, u_1, u'_0, u_2, u'_3, u_3, u'_2), (v_0, v'_1, v_1, v'_0, v_2, v'_3, v_3, v'_2)\}$ *is a* C_8*-factor.*

- $\{(u_0, v'_2, u_2, v'_0, u_3, v_0, u'_0, v'_3), (u_1, v'_1, u'_2, v_1, u'_3, v_3, u'_1, v_2)\}$ *is a* C_8*-factor.*

- $\{(u_0, v_0, u_2, v_2, u_3, v_3, u_1, v_1), (u'_0, v'_0, u'_1, v'_1, u'_3, v'_3, u'_2, v'_2)\}$ *is a* C_8*-factor.*
- $\{(u_0, v_2, u'_0, v_3), (v_0, u'_2, v'_0, u'_3), (u_2, v_1, u_3, v'_1), (u'_1, v'_2, u_1, v'_3)\}$ *is a* C_4*-factor.*

These are shown in Figure 2.2.

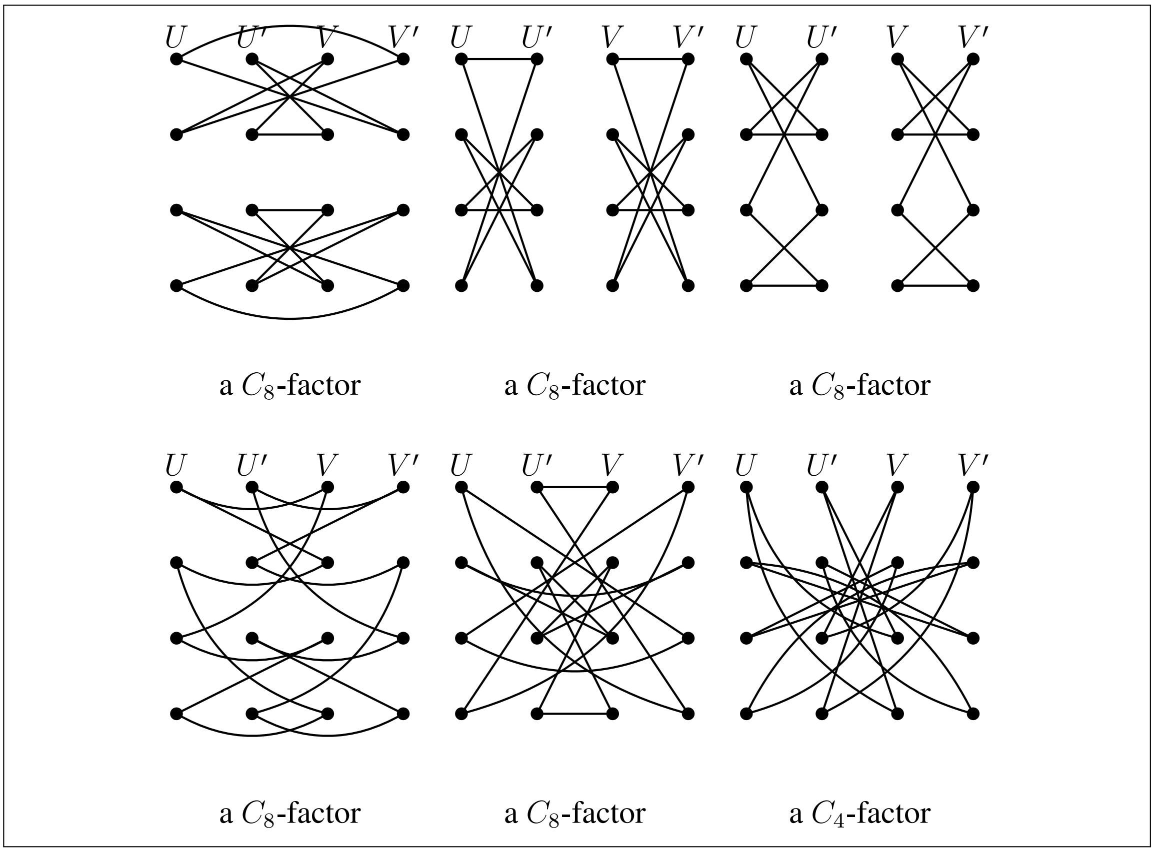


Figure 2.2: A $\{C_4^1, C_8^5\}$-factorization of $K_{4,4,4,4}$.

A $\{C_4^5, C_8^1\}$*-factorization is given by the following factors:*

- $\{(u_0, u'_0, u_1, u'_1), (u_2, u'_2, u_3, u'_3), (v_0, v'_0, v_1, v'_1), (v_2, v'_2, v_3, v'_3)\ \}$ *is a* C_4*-factor.*
- $\{(u_0, u'_2, u_1, u'_3), (u_2, u'_0, u_3, u'_1), (v_0, v'_2, v_1, v'_3), (v_2, v'_0, v_3, v'_1)\ \}$ *is a* C_4*-factor.*
- $\{(u_0, v'_0, u'_3, v_3), (u'_0, v_0, u_3, v'_3), (u_1, v'_1, u'_2, v_2), (u'_1, v_1, u_2, v'_2)\}$ *is a* C_4*-factor.*
- $\{(u_0, v_1, u_3, v_2), (u'_1, v'_0, u'_2, v'_3), (u'_0, v'_2, u_1, v_3), (u_2, v_0, u'_3, v'_1)\ \}$ *is a* C_4*-factor.*
- $\{(u_0, v'_1, u_3, v'_2), (v'_0, u_1, v'_3, u_2), (u'_0, v_1, u'_3, v_2), (v_0, u'_1, v_3, u'_2)\}$ *is a* C_4*-factor.*
- $\{(u_0, v_0, u_1, v_1, u'_2, v'_2, u'_3, v'_3), (u'_0, v'_0, u_3, v_3, u_2, v_2, u'_1, v'_1)\}$ *is a* C_8*-factor.*

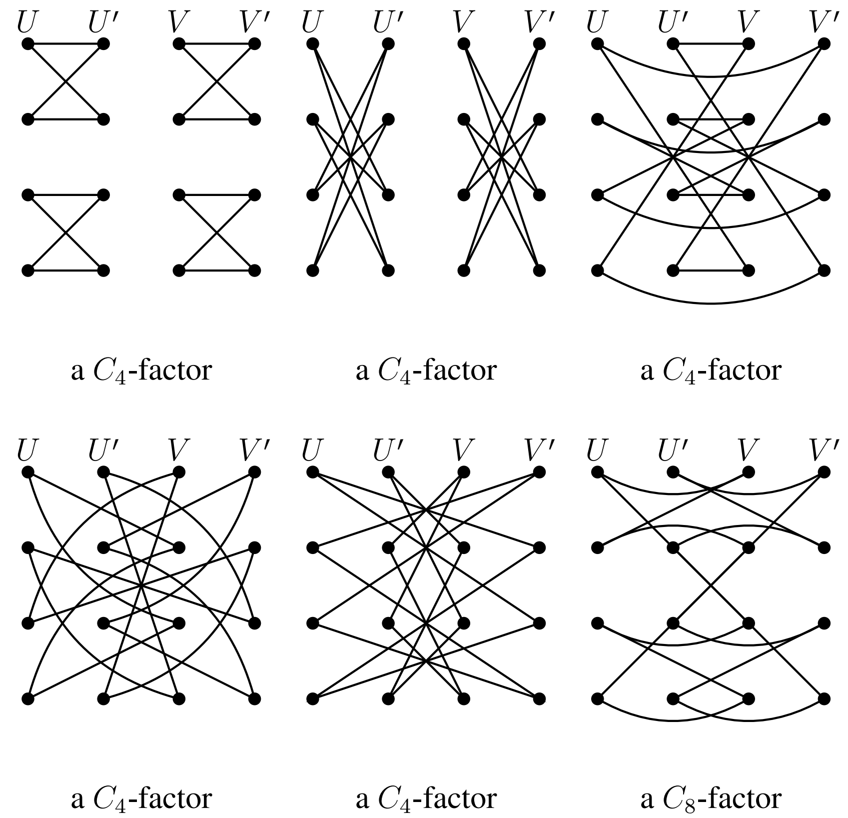


Figure 2.3: A $\{C_4^5, C_8^1\}$-factorization of $K_{4,4,4,4}$.

These are shown in Figure 2.3.

We can find the even cases easily via the following construction: By amalgamating all the vertices in each part, we obtain the complete graph K_4. By Theorem 2.1, there exists a 1-factorization of K_4. Each 1-factor in K_4 turns into a $K_{4,4}$ factor in $K_{4,4,4,4}$ when we disentagle the vertices. By [6], we know that $K_{4,4}$ has a C_4-factorization and also has a C_8-factorization. For a $\{C_4^2, C_8^4\}$-factorization of $K_{4,4,4,4}$, we use a C_4-factorization of $K_{4,4}$ for one of the $K_{4,4}$ factors and a C_8-factorization of $K_{4,4}$ for the other two $K_{4,4}$ factors. For a $\{C_4^4, C_8^2\}$-factorization of $K_{4,4,4,4}$, we use a C_8-factorization of $K_{4,4}$ for one of the $K_{4,4}$ factors K_4 and a C_4-factorization for the other two $K_{4,4}$ factors. C_4-factorization and C_8-factorization of $K_{4,4,4,4}$ are already known by [6]. ■

Lemma 2.3: There exists a $\{C_4^r, C_8^s\}$-factorization of $K_{8,8}$ where $0 \le r, s \le 4$.

Proof 2.3: Let the two parts of $K_{8,8}$ be $V = \{v_i \mid 0 \le i \le 7\}$ and $V' = \{v_i' \mid 0 \le i \le 7\}$. There are four 2-factors of $K_{8,8}$. That is $r + s = 4$. Now, we will list all possible cases for r and s.

A $\{C_4^3, C_8^1\}$-factorization is given by the following factors:

- $\{(v_0, v'_2, v_6, v'_4), (v_1, v'_3, v_7, v'_5), (v_2, v'_0, v_4, v'_6), (v_3, v'_1, v_5, v'_7)\}$ *is a C_4-factor.*
- $\{(v_0, v'_1, v_6, v'_7), (v_1, v'_2, v_5, v'_6), (v_2, v'_3, v_4, v'_5), (v_3, v'_0, v_7, v'_4)\}$ *is a C_4-factor.*
- $\{(v_0, v'_5, v_3, v'_6), (v_1, v'_4, v_2, v'_7), (v_4, v'_2, v_7, v'_1), (v_5, v'_3, v_6, v'_0)\}$ *is a C_4-factor.*
- $\{(v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3), (v_4, v'_4, v_5, v'_5, v_6, v'_6, v_7, v'_7)\}$ *is a C_8-factor.*

These are shown in Figure 2.4.

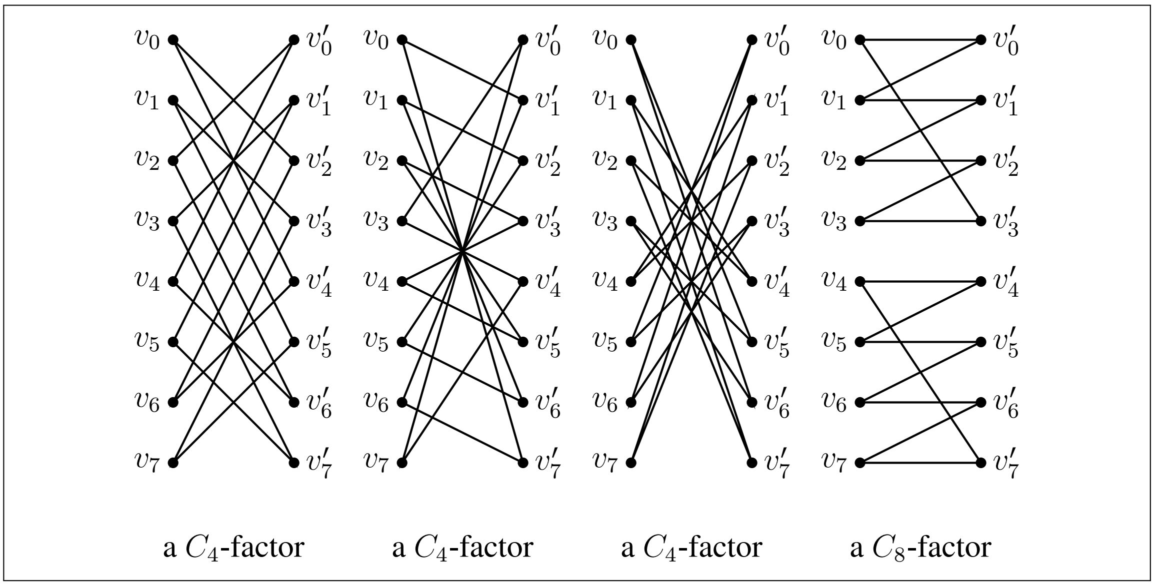


Figure 2.4: A $\{C_4^3, C_8^1\}$-factorization of $K_{8,8}$.

A $\{C_4^1, C_8^3\}$-factorization is given by the following factors:

- $\{(v_0, v'_2, v_6, v'_4), (v_1, v'_3, v_7, v'_5), (v_2, v'_0, v_4, v'_6), (v_3, v'_1, v_5, v'_7)\}$ *is a C_4-factor.*
- $\{(v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3), (v_4, v'_4, v_5, v'_5, v_6, v'_6, v_7, v'_7)\}$ *is a C_8-factor.*
- $\{(v_0, v'_1, v_6, v'_7, v_2, v'_3, v_4, v'_5), (v_1, v'_2, v_7, v'_4, v_3, v'_0, v_5, v'_6)\}$ *is a C_8-factor.*
- $\{(v_0, v'_6, v_3, v'_5, v_2, v'_4, v_1, v'_7), (v_4, v'_1, v_7, v'_0, v_6, v'_4, v_5, v'_2)\}$ *is a C_8-factor.*

These are shown in Figure 2.5.

A $\{C_4^2, C_8^2\}$-factorization is given by the following factors:

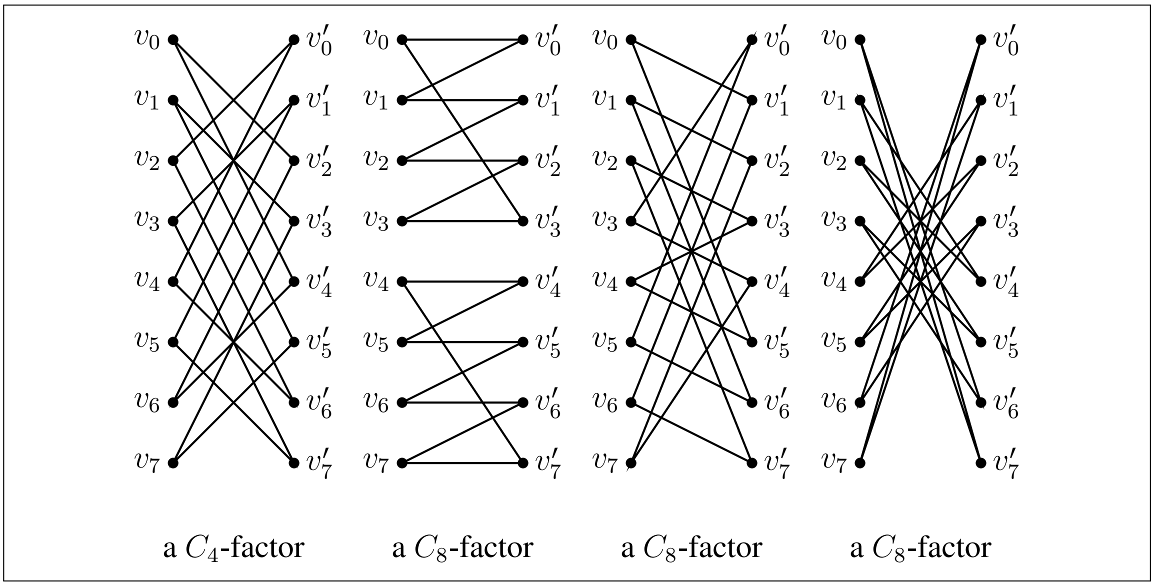


Figure 2.5: A $\{C_4^1, C_8^3\}$-factorization of $K_{8,8}$.

- $\{(v_0, v_0', v_1, v_1'), (v_2, v_2', v_3, v_3'), (v_4, v_4', v_5, v_5'), (v_6, v_6', v_7, v_7')\}$ *is a* C_4*-factor.*
- $\{(v_0, v_2', v_1, v_3'), (v_2, v_0', v_3, v_1'), (v_4, v_6', v_5, v_7'), (v_6, v_4', v_7, v_5')\}$ *is a* C_4*-factor.*
- $\{(v_0, v_4', v_1, v_5', v_2, v_6', v_3, v_7'), (v_4, v_0', v_7, v_3', v_6, v_2', v_5, v_1')\}$ *is a* C_8*-factor.*
- $\{(v_0, v_5', v_3, v_4', v_2, v_7', v_1, v_6'), (v_4, v_2', v_7, v_1', v_6, v_0', v_5, v_3')\}$ *is a* C_8*-factor.*

These are shown in Figure 2.6.

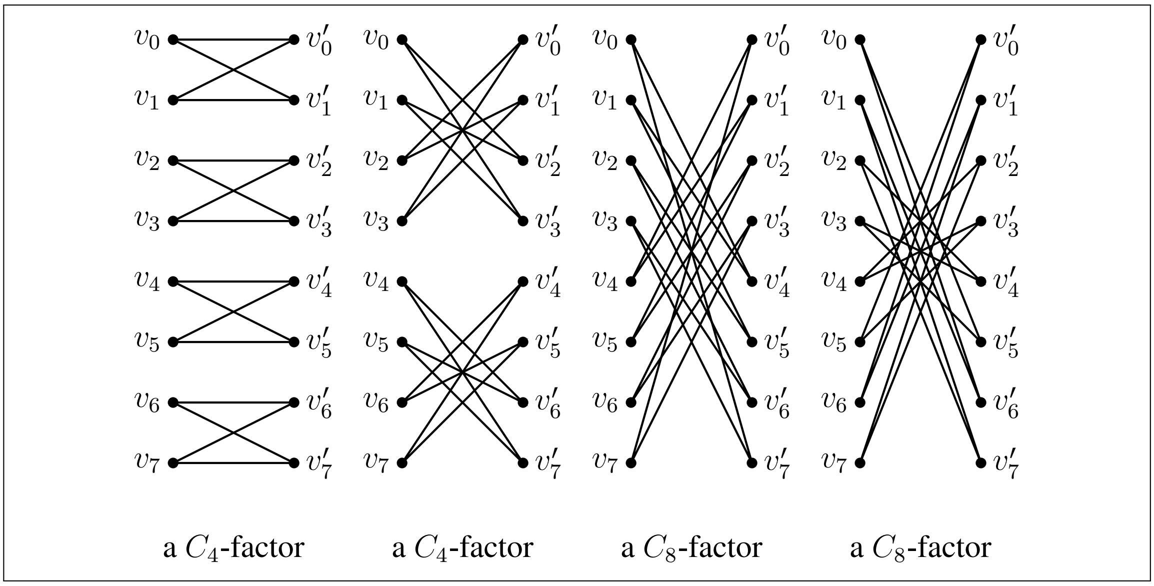


Figure 2.6: A $\{C_4^2, C_8^2\}$-factorization of $K_{8,8}$.

$\{C_4^0, C_8^4\}$-factorization and $\{C_4^4, C_8^0\}$-factorization of $K_{8,8}$ are already known by [6]. ■

Lemma 2.4: There exists a $\{C_4^r, C_8^s\}$-factorization of $K_{8,8,8}$ where $0 \le r, s \le 8$.

Proof 2.4: For the even cases of r and s we amalgamate each group of 4 vertices in each part of $K_{8,8,8}$ and get $K_{2,2,2}$. By Lemma 2.1, $K_{2,2,2}$ has a 1-factorization; it can be decomposed into four 1-factors. When we disentangle the vertices, each 1-factor of $K_{2,2,2}$ turns into a $K_{4,4}$. For each $K_{4,4}$ factor we can use a C_4-factorization or a C_8-factorization of $K_{4,4}$. Now, we follow the same method as in Lemma 2.2. When $r = 2$ and $s = 6$, we use a C_4-factorization of $K_{4,4}$ for one of the $K_{4,4}$ factors and a C_8-factorization of $K_{4,4}$ for the other three $K_{4,4}$ factors. Similarly, we can solve the cases $r = 4$ and $s = 4$, $r = 6$ and $s = 2$, $r = 8$ and $s = 0$, and $r = 0$ and $s = 8$. It only remains to discuss the odd cases.

Let U=$\{u_i \mid 0 \le i \le 7\}$, U'=$\{u'_i \mid 0 \le i \le 7\}$, and U''=$\{u''_i \mid 0 \le i \le 7\}$ be the three parts of $K_{8,8,8}$ respectively.

A $\{C_4^3, C_8^5\}$-factorization is given by the following factors:

- $\{(u_0, u'_0, u_1, u'_1, u_2, u'_2, u_3, u'_3)$, $(u_4, u''_0, u_5, u''_1, u_6, u''_2, u_7, u''_3)$, $(u'_4, u''_4, u'_5, u''_5, u'_6, u''_6, u'_7, u''_7)\}$ *is a C_8-factor.*
- $\{(u_0, u''_1, u_1, u''_2, u_2, u''_3, u_3, u''_0)$, $(u_4, u'_5, u_5, u'_6, u_6, u'_7, u_7, u'_4)$, $(u'_0, u''_5, u'_1, u''_6, u'_2, u''_7, u_3, u''_4)\}$ *is a C_8-factor.*
- $\{(u_0, u''_5, u_1, u''_6, u_2, u''_7, u_3, u''_4)$, $(u_4, u'_1, u_5, u'_2, u_6, u'_3, u_7, u'_0)$, $(u'_4, u''_1, u'_5, u''_2, u'_6, u''_3, u'_7, u''_0)\}$ *is a C_8-factor.*
- $\{(u_0, u'_4, u_5, u'_7, u_3, u'_6, u_1, u'_5)$,$(u_2, u''_4, u_6, u''_7, u_4, u''_6, u_3, u''_5)$, $(u'_0, u''_0, u_7, u''_1, u'_1, u''_2, u'_2, u''_3)\}$ *is a C_8-factor.*
- $\{(u_0, u'_6, u_7, u'_5, u_2, u'_4, u_1, u'_7)$, $(u_3, u''_1, u'_2, u''_0, u'_3, u''_3, u_5, u''_2)$, $(u'_0, u''_6, u_6, u''_5, u_4, u''_4, u'_1, u''_7)\}$ *is a C_8-factor.*
- $\{(u_0, u'_1, u_7, u'_2)$, $(u_1, u''_4, u'_6, u''_7)$, (u_2, u'_0, u_5, u'_3), $(u_3, u''_5, u'_4, u''_6)$, $(u_4, u''_1, u'_7, u''_2)$, $(u_6, u''_3, u'_5, u''_0)\}$ *is a C_4-factor.*
- $\{(u_0, u''_6, u_5, u''_7)$, $(u_1, u''_0, u'_1, u''_3)$, (u_2, u'_6, u_4, u'_7), (u_3, u'_4, u_6, u'_5), $(u_7, u''_5, u'_2, u''_4)$, $(u'_0 u''_1, u'_3, u''_2)\}$ *is a C_4-factor.*

- $\{(u_0, u_2'', u_4', u_3''), (u_1, u_2', u_4, u_3'), (u_2, u_0'', u_0', u_1''), (u_3, u_0', u_6, u_1'),$
  $(u_5, u_4'', u_7', u_5''), (u_5', u_6'', u_7, u_7'')\}$ *is a C_4-factor.*

  These are shown in Figure 2.7.

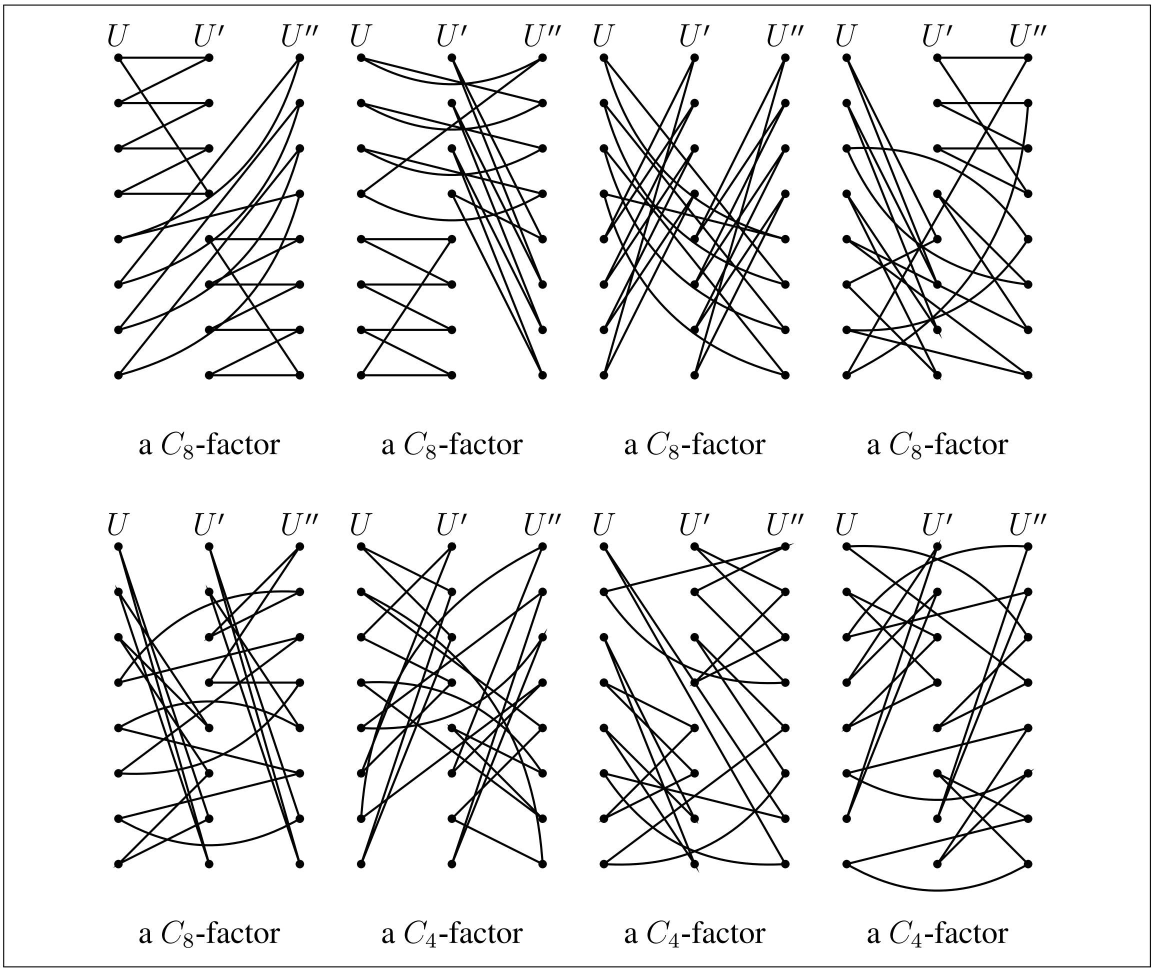


Figure 2.7: A $\{C_4^3, C_8^5\}$-factorization of $K_{8,8,8}$.

A $\{C_4^5, C_8^3\}$-factorization is given by the following factors:

- $\{(u_0, u_1'', u_1, u_2'', u_2, u_3'', u_3, u_0''), (u_4, u_5', u_5, u_6', u_6, u_7', u_7, u_4'),$
  $(u_0', u_5'', u_1', u_6'', u_2', u_7'', u_3, u_4'')\}$ *is a C_8-factor.*

- $\{(u_0, u_4', u_5, u_7', u_3, u_6', u_1, u_5'),(u_2, u_4'', u_6, u_7'', u_4, u_6'', u_3, u_5''),$
  $(u_0', u_0'', u_7, u_1'', u_1', u_2'', u_2', u_3'')\}$ *is a C_8-factor.*

- $\{(u_0, u_6', u_7, u_5', u_2, u_4', u_1, u_7'), (u_3, u_1'', u_2', u_0'', u_3', u_3'', u_5, u_2''),$
  $(u_0', u_6'', u_6, u_5'', u_4, u_4'', u_1', u_7'')\}$ *is a C_8-factor.*

- $\{(u_0, u_0', u_7, u_3'), (u_1, u_5'', u_6', u_6''), (u_2, u_1', u_5, u_2'), (u_3, u_4'', u_4', u_7''),$
  $(u_4, u_0'', u_7', u_3''), (u_6, u_1'', u_5', u_2'')\}$ *is a C_4-factor.*

- $\{(u_0, u_4'', u_5', u_5''), (u_1, u_0', u_4, u_1'), (u_2, u_6'', u_7', u_7''), (u_3, u_2', u_6, u_3'),$
  $(u_5, u_0'', u_4', u_1''), (u_7, u_2'', u_6', u_3'')\}$ *is a C_4-factor.*

- $\{(u_0, u_1', u_7, u_2'), (u_1, u_4'', u_6', u_7''), (u_2, u_0', u_5, u_3'), (u_3, u_5'', u_4', u_6''),$
  $(u_4, u_1'', u_7', u_2''), (u_6, u_3'', u_5', u_0'')\}$ *is a C_4-factor.*

- $\{(u_0, u_6'', u_5, u_7''), (u_1, u_0'', u_1', u_3''), (u_2, u_6', u_4, u_7'), (u_3, u_4', u_6, u_5'),$
  $(u_7, u_5'', u_2', u_4''), (u_0' u_1'', u_3', u_2'')\}$ *is a C_4-factor.*

- $\{(u_0, u_2'', u_4', u_3''), (u_1, u_2', u_4, u_3'), (u_2, u_0'', u_0', u_1''), (u_3, u_0', u_6, u_1'),$
  $(u_5, u_4'', u_7', u_5''), (u_5', u_6'', u_7, u_7'')\}$ *is a C_4-factor.*

These are shown in Figure 2.8.
A $\{C_4^1, C_8^7\}$-factorization is given by the following factors:

- $\{(u_0, u_0', u_1, u_1', u_2, u_2', u_3, u_3'), (u_4, u_0'', u_5, u_1'', u_6, u_2'', u_7, u_3''),$
  $(u_4', u_4'', u_5', u_5'', u_6', u_6'', u_7', u_7'')\}$ *is a C_8-factor.*

- $\{(u_0, u_1'', u_1, u_2'', u_2, u_3'', u_3, u_0''), (u_4, u_5', u_5, u_6', u_6, u_7', u_7, u_4'),$
  $(u_0', u_5'', u_1', u_6'', u_2', u_7'', u_3, u_4'')\}$ *is a C_8-factor.*

- $\{(u_0, u_5'', u_1, u_6'', u_2, u_7'', u_3, u_4''), (u_4, u_1', u_5, u_2', u_6, u_3', u_7, u_0'),$
  $(u_4', u_1'', u_5', u_2'', u_6', u_3'', u_7', u_0'')\}$ *is a C_8-factor.*

- $\{(u_0, u_4', u_5, u_7', u_3, u_6', u_1, u_5'),(u_2, u_4'', u_6, u_7'', u_4, u_6'', u_3, u_5''),$
  $(u_0', u_0'', u_7, u_1'', u_1', u_2'', u_2', u_3'')\}$ *is a C_8-factor.*

- $\{(u_0, u_6', u_7, u_5', u_2, u_4', u_1, u_7'), (u_3, u_1'', u_2', u_0'', u_3', u_3'', u_5, u_2''),$
  $(u_0', u_6'', u_6, u_5'', u_4, u_4'', u_1', u_7'')\}$ *is a C_8-factor.*

- $\{(u_0, u_1', u_6, u_0', u_2, u_3', u_4, u_2'), (u_1, u_4'', u_5, u_5'', u_3, u_6'', u_7, u_7'')$
  $(u_4', u_2'', u_7', u_1'', u_6', u_0'', u_5', u_3'')\}$ *is a C_8-factor.*

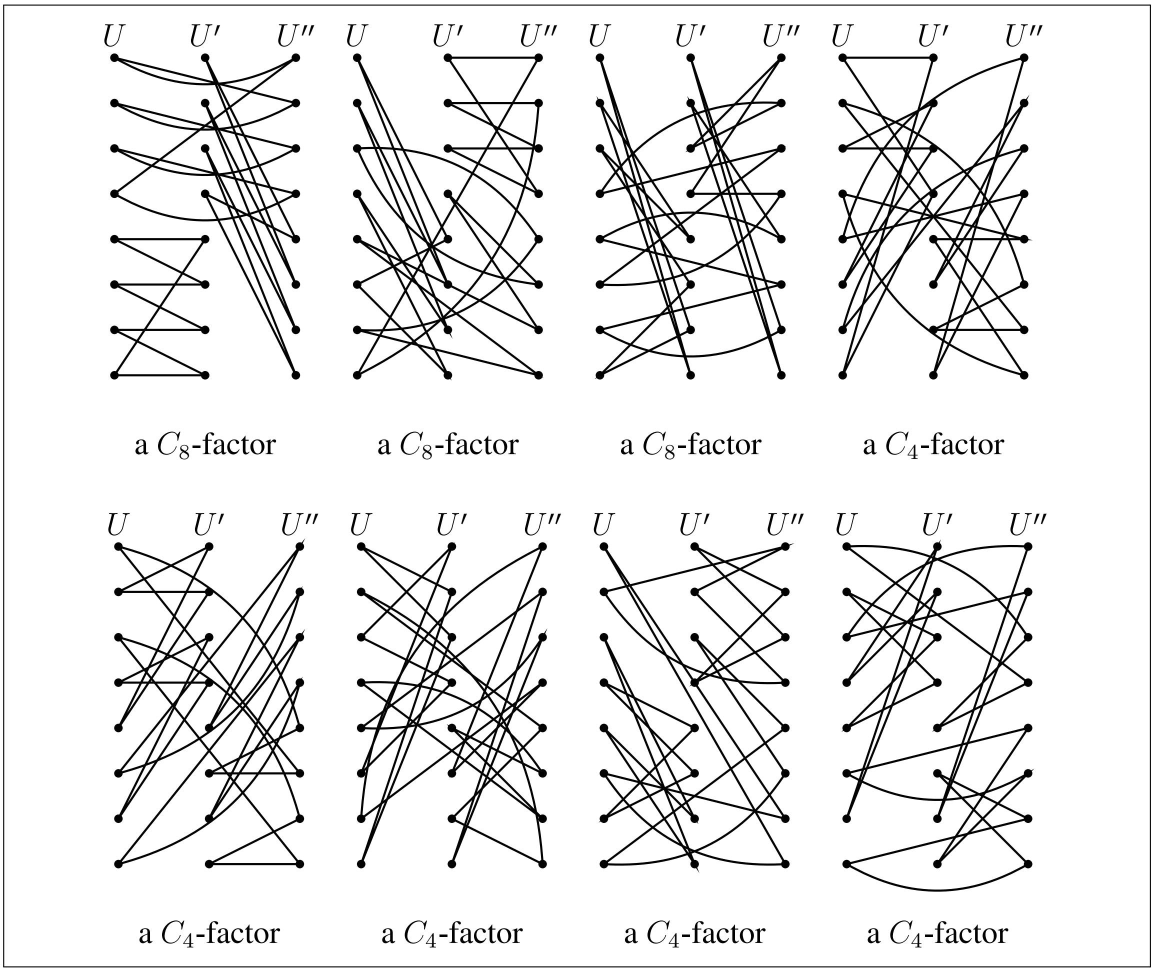


Figure 2.8: A $\{C_4^5, C_8^3\}$-factorization of $K_{8,8,8}$.

- $\{(u_0, u''_2, u_4, u''_1, u_2, u''_0, u_6, u''_3), (u_1, u'_2, u_7, u'_1, u_3, u'_0, u_5, u'_3)$ $(u'_4, u''_5, u'_7, u''_4, u'_6, u''_7, u'_5, u''_6)\}$ *is a* C_8*-factor.*

- $\{(u_0, u''_6, u_5, u''_7), (u_1, u''_0, u'_1, u''_3), (u_2, u'_6, u_4, u'_7), (u_3, u'_4, u_6, u'_5),$ $(u_7, u''_5, u'_2, u''_4), (u'_0, u''_1, u'_3, u''_2)\}$ *is a* C_4*-factor.*

These are shown in Figure 2.9.

A $\{C_4^7, C_8^1\}$*-factorization is given by the following factors:*

- $\{(u_0, u'_0, u_1, u'_1, u_2, u'_2, u_3, u'_3), (u_4, u''_0, u_5, u''_1, u_6, u''_2, u_7, u''_3),$ $(u'_4, u''_4, u'_5, u''_5, u'_6, u''_6, u'_7, u''_7)\}$ *is a* C_8*-factor.*

- $\{(u_0, u''_6, u'_5, u''_7), (u_1, u''_2, u'_4, u'_3), (u_2, u''_4, u'_7, u''_5), (u_3, u'_0, u_6, u'_1),$ $(u_5, u''_2, u'_4, u''_3), (u'_7, u''_0, u'_6, u''_1)\}$ *is a* C_4*-factor.*

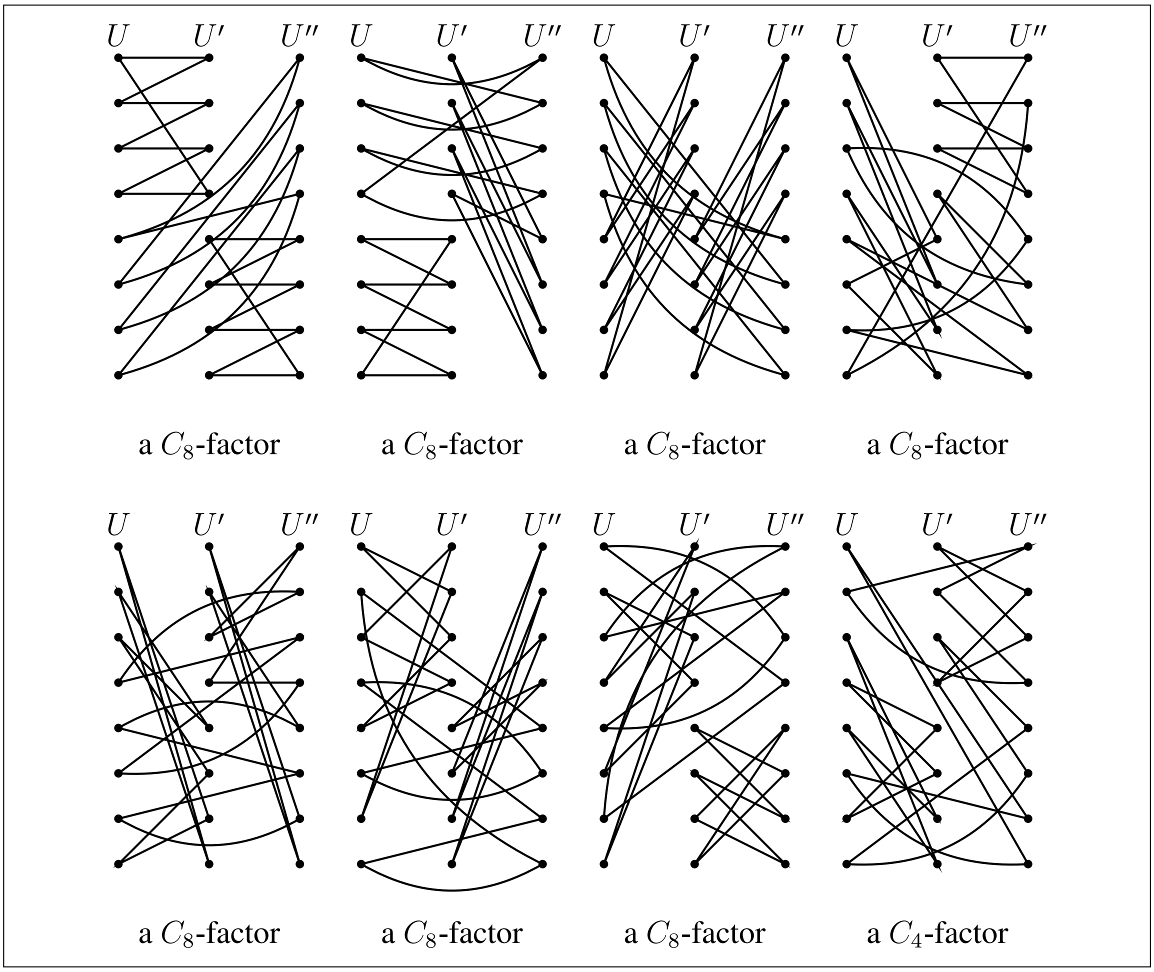


Figure 2.9: A $\{C_4^1, C_8^7\}$-factorization of $K_{8,8,8}$.

- $\{(u_0, u_1', u_5, u_5'), (u_1, u_6'', u_3, u_7''), (u_2, u_0', u_7, u_7'), (u_4, u_2'', u_6', u_4''),$
  $(u_6, u_0'', u_4', u_5''), (u_2', u_1'', u_3', u_3'')\}$ *is a C_4-factor.*

- $\{(u_0, u_2', u_5, u_7'), (u_1, u_4'', u_3, u_5''), (u_2, u_3', u_7, u_5'), (u_4, u_1'', u_4', u_6''),$
  $(u_6, u_3'', u_6', u_7''), (u_0', u_0'', u_1', u_2'')\}$ *is a C_4-factor.*

- $\{(u_0, u_4', u_2, u_6'), (u_1, u_2'', u_3, u_3''), (u_4, u_0', u_5, u_7''), (u_6, u_2', u_7, u_4''),$
  $(u_1', u_5'', u_3', u_6''), (u_5', u_0'', u_7', u_1'')\}$ *is a C_4-factor.*

- $\{(u_0, u_0'', u_3', u_2''), (u_1, u_4', u_3, u_6'), (u_2, u_1'', u_1', u_3''), (u_4, u_5', u_6, u_7'),$
  $(u_5, u_4'', u_2', u_5''), (u_7, u_6'', u_0', u_7'')\}$ *is a C_4-factor.*

- $\{(u_0, u_1'', u_0', u_3''), (u_1, u_5', u_3, u_7'), (u_2, u_0'', u_2', u_2''), (u_4, u_6', u_7, u_5''),$
  $(u_5, u_4', u_6, u_6''), (u_1', u_4'', u_3', u_7'')\}$ *is a C_4-factor.*

- $\{(u_0, u_4'', u_0', u_5''), (u_1, u_0'', u_3, u_1''), (u_2, u_6'', u_2', u_7''), (u_4, u_1', u_7, u_4'),$

(u_5, u'_3, u_6, u'_6), $(u'_5, u''_2, u'_7, u''_3)\}$ is a C_4-factor.

These are shown in Figure 2.10. ■

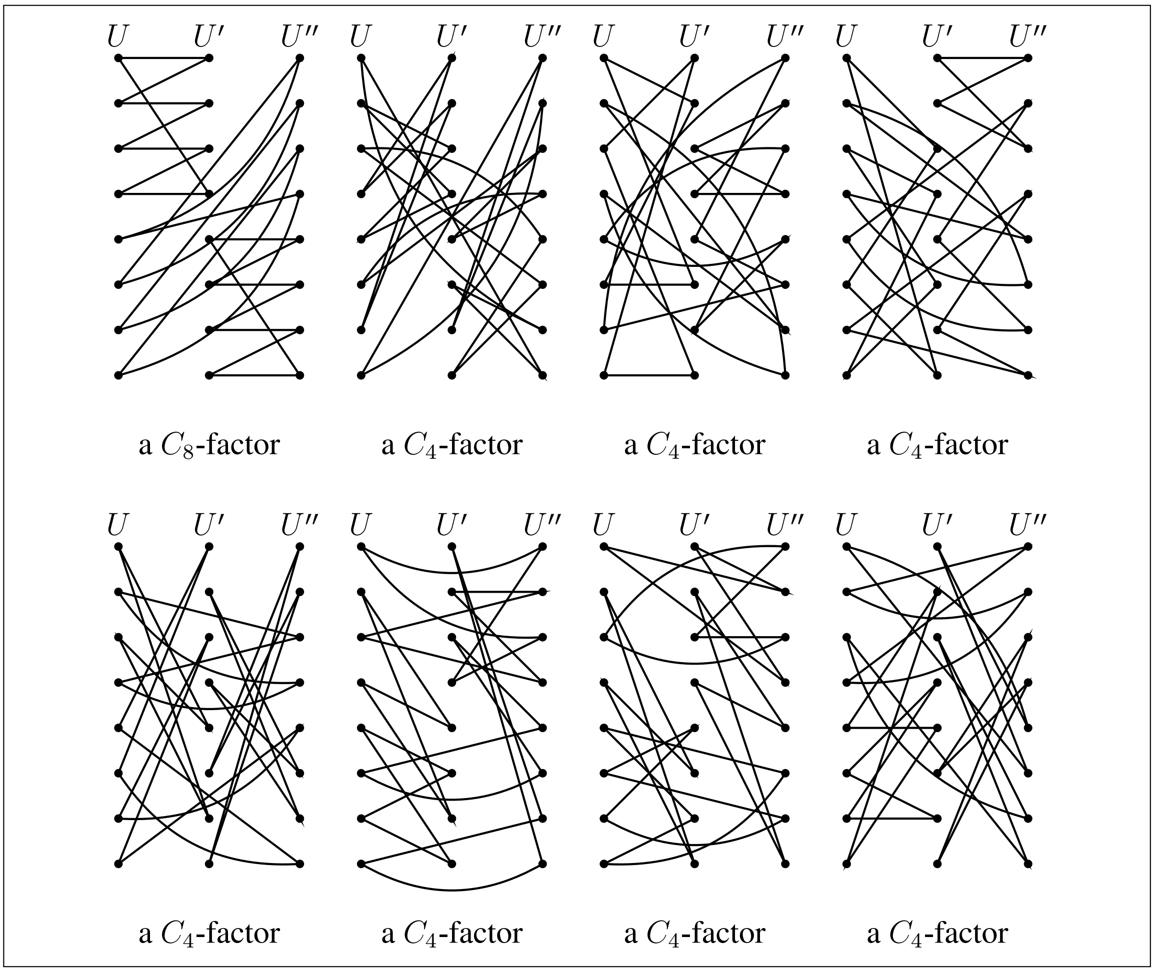


Figure 2.10: A $\{C_4^7, C_8^1\}$-factorization of $K_{8,8,8}$.

Lemma 2.5: There exists a $\{C_4^r, C_8^s\}$-factorization of $K_{16,16}$ where $0 \leq r, s \leq 8$.

Proof 2.5: Amalgamating vertices into groups of 8 in each part of $K_{16,16}$, we obtain the complete multipartite graph $K_{2,2}$. $K_{2,2}$ is 1-factorable. Each 1-factor in $K_{2,2}$ turns into a $K_{8,8}$ factor in $K_{16,16}$ when we disentagle the vertices. Then, we use a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{8,8}$ for each $K_{8,8}$ factor where $0 \leq r', s' \leq 4$. Since there are two $K_{8,8}$ factors, we have $r = r_1 + r_2$ and $s = s_1 + s_2$ for $0 \leq r_i, s_i \leq 4$, $i = 1, 2$, so that a $\{C_4^r, C_8^s\}$-factorization of $K_{16,16}$ exists for $0 \leq r, s \leq 8$. ■

We can genaralize this to a complete bipartite graph $K_{16h,16h}$ where h is a positive integer:

Corollary 2.1: There exists a $\{C_4^r, C_8^s\}$*-factorization of* $K_{16h,16h}$ *where* $0 \le r, s \le 8h$ *and* h *is a positive integer.*

Proof 2.1: We amalgamate vertices into groups of 8 in each part of $K_{16h,16h}$ *and obtain the complete bipartite graph* $K_{2h,2h}$*.* $K_{2h,2h}$ *is 1-factorable. Each 1-factor in* $K_{2h,2h}$ *turns into a* $K_{8,8}$ *factor in* $K_{16h,16h}$ *when we disentangle the vertices. Then, we use a* $\{C_4^{r'}, C_8^{s'}\}$*-factorization of* $K_{8,8}$ *for each* $K_{8,8}$ *factor where* $0 \le r', s' \le 4$*. Since there are* $2h$ $K_{8,8}$ *factors, we have* $r = r_1 + r_2 + \dots r_{2h}$ *and* $s = s_1 + s_2 + \dots + s_{2h}$ *for* $0 \le r_i, s_i \le 4$*,* $i = 1, \dots 2h$*, so that* $0 \le r, s \le 8h$ *as needed.* ■

Lemma 2.6: There exists a $\{C_4^r, C_8^s\}$*-factorization of* $K_{2,2,2,2}$ *for each* $0 \le r, s \le 3$*.*

Proof 2.6: Let the four partite sets of $K_{2,2,2,2}$ *be* $U = \{u_i \mid 0 \le i \le 1\}$*,* $U' = \{u'_i \mid 0 \le i \le 1\}$*,* $V = \{v_i \mid 0 \le i \le 1\}$*, and* $V' = \{v'_i \mid 0 \le i \le 1\}$ *respectively. Each vertex in* $K_{2,2,2,2}$ *has degree 6. So there are three factors in the 2-factorization of* $K_{2,2,2,2}$*. That is* $r + s = 3$*. We already know the cases* $r = 0$*,* $s = 3$ *and* $r = 3$*,* $s = 0$ *by Liu [6]. We analyse the cases where* r *and* s *are odd:*

A $\{C_4^1, C_8^2\}$*-factorization of* $K_{2,2,2,2}$ *is given by the following factors:*

- $\{(u_0, u'_1, v'_1, v_0), (u_1, u'_0, v'_0, v_1)\}$ *is a* C_4*-factor.*
- $\{(u_0, u'_0, v_0, v'_0, u_1, u'_1, v_1, v'_1)\}$ *is a* C_8*-factor.*
- $\{u_0, v'_0, u'_1, v_0, u_1, v'_1, u'_0, v_1\}$ *is a* C_8*-factor.*

These are shown in Figure 2.11.

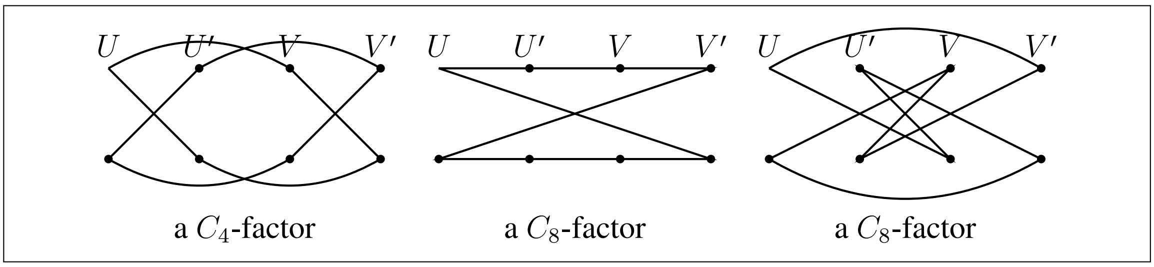


Figure 2.11: A $\{C_4^1, C_8^2\}$-factorization of $K_{2,2,2,2}$.

A $\{C_4^2, C_8^1\}$*-factorization of* $K_{2,2,2,2}$ *is given by the following factors:*

- $\{(u_0, u_1', v_1, v_0'), (u_1, u_0', v_0, v_1')\}$ *is a* C_4*-factor.*
- $\{(u_0, v_1, u_0', v_1'), (u_1, u_1', v_0, v_0')\}$ *is a* C_4*-factor.*
- $\{u_0, u_0', v_0', u_1', v_1', v_1, u_1, v_0\}$ *is a* C_8*-factor.*

These are shown in Figure 2.12. ■

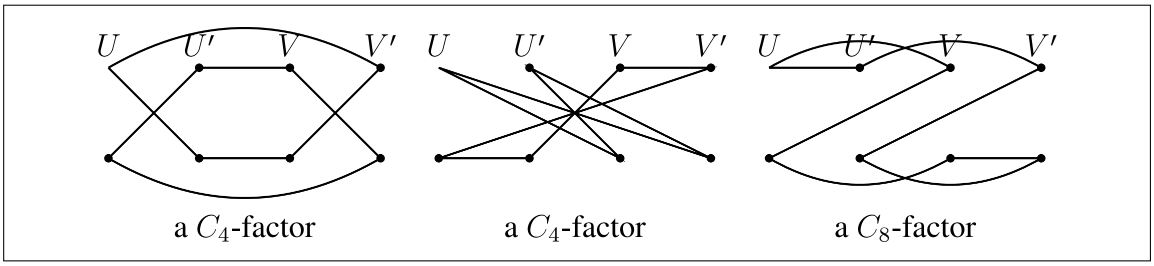


Figure 2.12: A $\{C_4^2, C_8^1\}$-factorization of $K_{2,2,2,2}$.

3. MAIN RESULT

The following theorem gives us the necessary and sufficient conditions for $K(4t : m)$ to have a $\{C_4^r, C_8^s\}$-factorization with a few exceptions.

Theorem 3.1: The complete multipartite graph $K(4t : m)$ for $m \geq 2$, $t \geq 1$ has a $\{C_4^r, C_8^s\}$-factorization for any non-negative integers r, s with $0 \leq r, s \leq 2t(m-1)$ if and only if it satisfies the following conditions:

i) $8 \mid 4tm$,

ii) $r + s = 2t(m-1)$

possibly except $m \equiv 5$ $(mod\ 12)$ *when* $t \equiv 2, 10$ $(mod\ 12)$, *and* $m \equiv 2$ $(mod\ 24)$ *when* $t \equiv 1, 5$ $(mod\ 6)$.

Proof 3.1: $(\Rightarrow)$ *First assume that $K(4t : m)$ for $m \geq 2$, $t \geq 1$ has a $\{C_4^r, C_8^s\}$-factorization for any non-negative integers r, s with $0 \leq r, s \leq 2t(m-1)$. Now, Theorem 1.1 implies that $8 \mid 4tm$ and $r + s = 2t(m-1)$.*

$(\Leftarrow)$ *Conversely, let $K(4t : m)$ be a complete multipartite graph where $t \geq 1$ and $m \geq 2$ are positive integers. Let r, s be non-negative integers satisfying $r + s = 2t(m-1)$ with $0 \leq r, s \leq 2t(m-1)$ and assume $8 \mid 4tm$. Therefore, at least one of t or m must be even. We have two cases depending on the parity of t.*

- *Case 1: t is even:*

There exists a positive integer k such that $t = 2k$. So, we can write $K(4t : m)$ as $G_1 = K(8k : m)$ and amalgamate the vertices in each part into groups of 8 and get $H_1 = K(k : m)$. If k is even, by Lemma 2.1, H_1 has a 1-factorization. Each 1-factor in H_1 gives us a $K_{8,8}$ factor in G_1 when we disentangle the vertices. For each $K_{8,8}$ factor, we can use a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{8,8}$ for any $0 \leq r', s' \leq 4$ by Lemma 2.3. Since there are $k(m-1)$ $K_{8,8}$ factors in G_1, we have $r = r_1 + \cdots + r_{k(m-1)}$ and $s = s_1 + \cdots + s_{k(m-1)}$ for $0 \leq r_i, s_i \leq 4$, $i = 1, \ldots, k(m-1)$. So, r and s covers all the integers in the range $0 \leq r, s \leq 4k(m-1) = 2t(m-1)$ as required.

Now, assume k is odd (i.e: $t \equiv 2$ $(mod\ 4)$). If m is even, H_1 has a 1-factorization as before and the above construction still applies. So, assume m is odd. If $m \equiv 3$ $(mod\ 6)$, i.e. $m = 6n + 3$ for some positive integer n, we represent each part of G_1 with one vertex *in the amalgamation. Since amalgamated graph is also complete, we get a K_{6n+3}. By Theorem 1.5, there exists a KTS($6n+3$) on $6n+3$ points for K_{6n+3}. So, each triple of KTS($6n+3$) in K_{6n+3} corresponds to a $K_{8k,8k,8k}$ when the vertices are disentangled. We amalgamate the vertices in each part of $K_{8k,8k,8k}$ into groups of 8 and obtain the graph $K_{k,k,k}$. By Theorem 1.3, there is a 3-RGDD of type k^3 since k is odd. Each triple in the 3-RGDD corresponds to a $K_{8,8,8}$ in $K_{8k,8k,8k}$ when the vertices are disentangled. Then, we use a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{8,8,8}$ for $0 \le r', s' \le 8$ by Lemma 2.4. Therefore, $K_{8k,8k,8k}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for $0 \le r', s' \le 8k$. KTS($6n+3$) has $3n+1$ parallel classes and each of them corresponds to $8k$ 2-factors via $K_{8k,8k,8k}$'s. Hence, in total, there are $8k(3n+1)$ 2-factors for $m = 6n+3$ where $r = r_1 + \cdots + r_{3n+1}$, and $s = s_1 + \cdots + s_{3n+1}$ for $0 \le r_i, s_i \le 8k$, $i = 1, \ldots, 3n+1$ so that $0 \le r, s \le 8k(3n+1) = 4t(3n+1) = 2t(m-1)$ as required.*

If $m \not\equiv 3 \pmod 6$, (i.e: $m \equiv 1, 5 \pmod 6$), we fix one part of G_1, say P, and represent it with a vertex p and then represent every two parts of $G_1 \backslash P$ with one vertex in the amalgamation. We have $\frac{m-1}{2}$ pairs and vertex p so that the amalgamated graph has $\frac{m-1}{2} + 1 = \frac{m+1}{2}$ vertices. Since we have started with a complete multipartite graph, amalgamated graph must also be complete. Hence, we get a K_z where $z = \frac{m+1}{2}$. If z is even (i.e. if $m \equiv 3 \pmod 4$), K_z has a 1-factorization by Theorem 2.1. Each 1-factor of K_z corresponds to the union of one $K_{8k,8k,8k}$ and $\frac{m-3}{4}$ copies of $K_{16k,16k}$ when the vertices are disentangled. By Corollary 2.1, $K_{16k,16k}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for $0 \le r', s' \le 8k$. We know that $K_{8k,8k,8k}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for $0 \le r', s' \le 8k$. There are $\frac{m+1}{2} - 1 = \frac{m-1}{2}$ 1-factos in K_z and for each 1-factor there are $8k$ 2-factors coming from the union of $K_{8k,8k,8k}$ and $K_{16k,16k}$. In total, there are $8k(\frac{m-1}{2}) = 4k(m-1)$ factors in G_1 where $r = r_1 + \cdots + r_{\frac{m-1}{2}}$, and $s = s_1 + \cdots + s_{\frac{m-1}{2}}$ for $0 \le r_i, s_i \le 8k$, $i = 1, \ldots, \frac{m-1}{2}$ so that $0 \le r, s \le 4k(m-1) = 2t(m-1)$ as required. Hence there is also a $\{C_4^r, C_8^s\}$-factorization of G_1 when k is odd, $m \equiv 1, 5 \pmod 6$ and $m \equiv 3 \pmod 4$ meaning that $m \equiv 7, 11 \pmod{12}$.

Note that $m \equiv 3, 9 \pmod{12}$ cases are covered by the KTS($6n+3$). For the exceptions, that is $m \equiv 1, 5 \pmod{12}$ when $t \equiv 2 \pmod 4$, we use $\{C_4^r, C_8^s\}$-factorization of $K_{2,2,2,2}$ for $0 \le r, s \le 3$: We amalgamate the vertices in each part of $K(4t : m)$ into groups of two and get $H_2 = K(2t : m)$. By Theorem 1.4, there is a 4-RGDD of type $(2t)^m$ when $m \equiv 1 \pmod{12}$ and $t \equiv 2 \pmod 4$. However, because of the necessary conditions in Theorem 1.4, there is a 4-RGDD of type $(2t)^m$ when $m \equiv 5 \pmod{12}$ and $t \equiv 2 \pmod 4$

only if $t \equiv 0$ $(mod\ 3)$ additionally. Now, we have exceptions for $m \equiv 5$ $(mod\ 12)$ when $t \equiv 1, 2$ $(mod\ 3)$ and $t \equiv 2$ $(mod\ 4)$ (i.e. $t \equiv 2, 10$ $(mod\ 12)$) and also for $t = 18$, $m = 23$ (because of the exceptions in Theorem 1.4). Each block in H_2 corresponds to a $K_{2,2,2,2}$ in $K(4t : m)$ when the vertices are disentangled. Then, we use a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{2,2,2,2}$ for each block where $0 \le r', s' \le 3$ by Lemma 2.6 and get the $\{C_4^{r}, C_8^{s}\}$-factorization of $K(4t : m)$ for $m \equiv 1$ $(mod\ 12)$ when $t \equiv 2$ $(mod\ 4)$ and for $m \equiv 5$ $(mod\ 12)$ when $t \equiv 6$ $(mod\ 12)$.

- *Case 2: t is odd:*

Since t is odd and $8 \mid 4tm$, m must be even. So, there exists a positive integer l such that $m = 2l$. Since $4t$ vertices in each part of $K(4t : 2l)$ may not be divided into groups of 8, first we will combine every two parts and work on the graph $G_2 = K(8t : l)$. After that, we apply the same method as in Case 1. However, unlike Case 1, since G_2 does not include the edges between the amalgamated parts, later we need to work on the factorizations of these bipartite graphs additionally.

If l is even, we amalgamate the vertices in each part of G_2 into groups of 8 so that we have $H_3 = K(t : l)$. Since l is even, by Lemma 2.1, H_3 has a 1-factorization. As in Case 1, each 1-factor in H_3 turns into a $K_{8,8}$ factor in G_2 when we disentangle the vertices. Therefore, for each $K_{8,8}$ factor, we can use a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{8,8}$ for $0 \le r', s' \le 4$ which is given in Lemma 2.3. There are $(l-1)t$ $K_{8,8}$ factors of G_2 and each $K_{8,8}$ factor there are four 2-factors. So, in total, there are $4(l-1)t$ 2-factors in G_2. When we go back to $K(4t : m)$, the unused edges form l copies of $K_{4t,4t}$. Since we have considered every two parts as one. Now, we amalgamate each 4 vertices in $K_{4t,4t}$ and get $K_{t,t}$. By Theorem 2.2, $K_{t,t}$ has a 1-factorization. Each edge in a 1-factor of $K_{t,t}$ corresponds to a $K_{4,4}$ in $K_{4t,4t}$. And we know that $K_{4,4}$ has a C_4 and a C_8-factorization by [6]. So, we use a C_4 or a C_8-factorization for each of the $K_{4,4}$ factors of $K_{4t,4t}$. Hence, there exists a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{4t,4t}$ for r' and s' are even and $0 \le r', s' \le 2t$. Since $K_{4t,4t}$ has $2t$ 2-factors, there are $4(l-1)t + 2t = 2t(m-1)$ 2-factors in total in $K(4t : m)$ where $r = r_1 + \cdots + r_{(l-1)t} + r'$, and $s = s_1 + \cdots + s_{(l-1)t} + s'$ for $0 \le r_i, s_i \le 4$, $i = 1, \ldots, (l-1)t$ and $0 \le r', s' \le 2t$ for even r', s' so that $0 \le r, s \le 2t(m-1)$ as required.

When l is odd (i.e: $m \equiv 2, 6, 10$ $(mod\ 12)$), we can not use 1-factorization of H_3 directly since l is odd. If $l \equiv 3$ $(mod\ 6)$ where $l = 6d + 3$ for some positive integer d, we represent each part of $K(8t : l)$ with one vertex in the amalgamation and get a K_{6d+3}. By Theorem 1.5 there exists a KTS($6d + 3$) on $6d + 3$ points for K_{6d+3}. So, each triple of

KTS($6d$+3) in K_{6d+3} corresponds to a $K_{8t,8t,8t}$ when the vertices are disentangled as before. We amalgamate the vertices in each part of $K_{8t,8t,8t}$ into groups of 8 and obtain the graph $K_{t,t,t}$. By Theorem 1.3, there is a 3-RGDD of type t^3. Each block corresponds to a $K_{8,8,8}$ in $K_{8t,8t,8t}$ when the vertices are disentangled. Then, we use a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{8,8,8}$ for $0 \le r', s' \le 8$ by Lemma 2.4. Therefore, $K_{8t,8t,8t}$ has a $\{C_4^{r'}, C_8^{s'}\}$-factorization for $0 \le r', s' \le 8t$. There are $3d+1$ 2-factors (parallel classes) and each of them has $8t$ 2-factors. In addition, there are $2t$ 2-factors from the $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{4t,4t}$ for some even r' and s'. Therefore, in total, there are $8t(3d+1)+2t = 2t(12d+5)$ 2-factors where $r = r_1 + \cdots + r_{3d+1} + r'$, and $s = s_1 + \cdots + s_{3d+1} + s'$ for $0 \le r_i, s_i \le 8t$, $i = 1, \ldots, 3d+1$ and $0 \le r', s' \le 2t$ so that $0 \le r, s \le 2t(m-1)$ as required. Otherwise, $m \equiv 2, 10$ (mod 12) since $m = 2l$ and l is odd. Since G_2 has odd number of parts, we fix one part, say P, and represent every two parts of $G_2 \backslash P$ with one vertex in the amalgamation. Also we represent P with one vertex, say p. In the amalgamation, applying this to G_2, we get a K_q where $q = \frac{m+2}{4}$. If q is even, by Theorem 2.1 there is a 1-factorization of K_q. So, we have an exception for the cases where q is odd here, that is, $m \equiv 2, 10$ (mod 24). Since m is also 2,10 (mod 12). In each 1-factor of K_q, p is adjacent to a vertex in $G_2 \backslash P$. Thus, in each 1-factor of K_q, edges between P and a vertex which is adjacent to P form one copy of $K_{8t,8t,8t}$ when the vertices are disentangled. The remaining edges of 1-factor of K_q form $(m-6)/8$ copies of $K_{16t,16t}$. By Corollary 2.1, $K_{16t,16t}$ has $\{C_4^{r'}, C_8^{s'}\}$-factorization for $0 \le r', s' \le 8t$. We know that there is a $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{8t,8t,8t}$ for $0 \le r', s' \le 8t$. For each 1-factor of K_q we obtain $8t$ 2-factors from the union of $K_{8t,8t,8t}$ and $K_{16t,16t}$'s. There are $q-1 = \frac{m-2}{4}$ 1-factors in K_q so, this makes $8t\frac{m-2}{4} = 2t(m-2)$ 2-factors. In addition, there are $2t$ 2-factors from the $\{C_4^{r'}, C_8^{s'}\}$-factorization of $K_{4t,4t}$ for r' and s' are even and $0 \le r', s' \le 2t$. Because, we have considered every two parts of $K(4t:m)$ as one and edges between each of these pairwise parts constitue a graph $K_{4t,4t}$. All of this makes $2t(m-2)+2t = 2t(m-1)$ 2-factors where $r = r_1 + \cdots + r_{q-1} + r'$, and $s = s_1 + \cdots + s_{q-1} + s'$ for $0 \le r_i, s_i \le 8t$, $i = 1, \ldots, q-1$ and $0 \le r', s' \le 2t$ so that $0 \le r, s \le 8t(q-1) = 2t(m-1)$ as required. Although $K_{4t,4t}$ has a $\{C_4^r, C_8^s\}$-factorization for only even r and s, we can cover all the odd cases in G; we take even number of 2-factors from the 2-factorization of bipartite graphs, (i.e. $K_{4t,4t}$) and odd number of 2-factors from the 2-factorization of the remaining 2-factors.

For the exceptions, that is $m \equiv 2, 10$ (mod 24) when t is odd, we proceed as in Case 1: We amalgamate the vertices in each part of $K(4t:m)$ into groups of two and get

$H_4 = K(2t : m)$. *By Theorem 1.4, there is a 4-RGDD of type* $(2t)^m$ *when* $m \equiv 10 \ (mod\ 24)$ *for all odd* t. *But, there are some exceptions of type* $(2t)^m$ *for odd* t *given in Theorem 1.4:*

For $t = 1$ *and* $m \in \{10, 34, 82, 130, 178, 202, 250, 346)\}$, $(2t, m) \in (10, 34)$ *and* $t \in [7, 227] \cup \{251, 257, 263, 307, 313, 343\}$ *and* $m \in \{10, 82\}$. *In [26],* $(2t)^m$ *of types* 14^{10} *and* 22^{10} *are obtained.*

Also, because of the necessary conditions in Theorem 1.4, there is a 4-RGDD of type $(2t)^m$ *for* $m \equiv 2 \ (mod\ 24)$ *only when* $t \equiv 3 \ (mod\ 6)$ *except* $t \in [7, 227]$ *with* $t \equiv 3 \ (mod\ 6)$ *and* $m \in \{10, 82\}$. *Therefore, we have exceptions for* $m \equiv 2 \ (mod\ 24)$ *when* $t \equiv 1, 5 \ (mod\ 6)$ *now.* ■

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