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GEBZE TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON CYCLES AND BIPARTITE DIVISOR GRAPH RELEATED TO THE SET OF CONJUGACY CLASS SIZES

SERKAN DORU

A THESIS SUBMITTED FOR THE DEGREE OF

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GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

EŞLENİK SINIFI BOYUTLARIYLA İLİŞKİLİ DÖNGÜ VE İKİLİ BÖLEN GRAFİKLERİ

SERKAN DORU

YÜKSEK LİSANS TEZİ

MATEMATİK ANABİLİM DALI

DANIŞMANI

YRD. DOÇ. DR. ROGHAYEH HAFEZIEH

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GEBZE TEKNİK ÜNİVERSİTESİ	**YÜKSEK LİSANS JÜRİ ONAY FORMU**

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SUMMARY

Graphs associated with various algebraic structures have been actively investigated and many interesting results have been obtained. Let X be a set of positive integers. We associate three undirected graphs, which are called the prime vertex graph, the common divisor graph and the bipartite divisor graph, to the set X. Let G be a finite group and $cs(G)$ the set of the conjugacy class sizes of elements in G. Assume that $X = cs(G)$. By using combinatorial properties of the associated graphs, we give some information about the structure of the group. One of the main questions that naturally arises in this area is classifying the groups whose bipartite divisor graphs have special graphical shapes. In this thesis, we consider the case where the bipartite divisor graph of a finite group is a cycle. Bijan Taeri classified those groups with this property [Taeri, 2010]. In this thesis, we will write his proof in detail.

Key Words: Prime Vertex Graph, Common Divisor Graph, Bipartite Divisor Graph.

ÖZET

Çeşitli cebirsel yapılarla ilişkili grafikler incelenmiş ve birçok enteresan sonuçlar elde edilmiştir. X pozitif tamsayıların bir alt kümesi olsun. Biz üç yönsüz grafik olan asal köşe grafiği, ortak bölen grafiği ve ikili bölen grafiğini X kümesi ile ilişkilendirdik. G sonlu bir grup ve $cs(G)$ G'nin elemanlarının eşlenik sınıfı boyutlarının kümesi olsun. $X = cs(G)$ olsun. İlişkili grafiklerin kombinatoryal özelliklerini kullanarak, grubun yapısı hakkında bazı bilgiler verdik. Doğal olarak bu alanda ortaya çıkan başlıca sorulardan biri olan ikili bölen grafikleri özel grafik şekiller olan gruplarını sınıflandırılmasıdır. Bu tezde, sonlu grup ikili böleni grafiği bir döngü olan durum incelendi. Amacımız bu özelliği ile bu grupları sınıflandırmaktır. Bunun için biz Bijan Taeri tarfından yazılan bir makaleyi takip edeceğiz [Taeri, 2010].

Anahtar Kelimeler: Asal Köşe Grafiği, Ortak Bölen Grafiği, İkili Bölen Grafiği.

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I am grateful to my parents for their love and support.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
$\rho(X)$	:	The set of all prime divisors of the elements of a nonempty subset X of integers
X^*	:	$X \setminus \{1\}$ for a nonempty subset X of integers
$\Delta(X)$	:	Prime vertex graph of a nonempty subset X of integers
$\Gamma(X)$	:	Common divisor graph of a nonempty subset X of integers
$B(X)$	:	Bipartite divisor graph of a nonempty subset X of integers
$\Delta(G)$	:	Prime vertex graph of a group G
$\Gamma(G)$	:	Common divisor graph of a group G
$B(G)$	:	Bipartite divisor graph of a group G
$cs(G)$	:	The set of the conjugacy class sizes of a group G
$\pi(G)$	:	The set of the prime divisors of the order of a group G
G_p	:	A Sylow p-subgroup of a group G
$gcd(a, b)$	:	The greatest common divisor of the integers a and b

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1. INTRODUCTION

1.1. History

Over the last 30 years there have been many papers on the influence of the sizes of conjugacy classes on finite groups. We may ask the following question:

How much information can one expect to obtain from the sizes of conjugacy classes?

Sylow in 1872 examined what happened if there was information about the sizes of all conjugacy classes, whereas in 1904 Burnside showed that strong results could be obtained if there was particular information about the size of just one conjugacy class. Landau in 1903 bounded the order of the group in terms of the number of conjugacy classes whilst in 1919 Miller gave a detailed analysis of groups with very few conjugacy classes. Very little then seems to have been done until 1953 when both Baer and Itô published papers on this topic but with different conditions on the sizes.

By looking at these early results it can been seen that much will depend on how much information is given and it is important to be explicit. For example if one knows that there is only one conjugacy class size then the group is abelian, but this can be any abelian group. However if you know the collection of conjugacy class sizes, that is the multiplicities, then the order of the group is also known. However it would still not be possible to identify the group.

Various graphs can be constructed from the sets of conjugacy class sizes. The properties of the graphs and the relation to the structure of the groups are the main questions in this field of study. This has been a very active area in recent years. Recently, Lewis in [Lewis, 2008] discussed many remarkable connections among these graphs by analysing analogous of these graphs for arbitrary positive integer subsets. Then, inspired by the survey of Lewis, in [Iranmanesh and Praeger, 2010] introduced the bipartite divisor graph $B(X)$ for a finite set X of positive integers and studied some basic invariants of this graph (such as the diameter, girth, number of connected components). One of the main questions that naturally arises in this area is classifying the groups whose bipartite divisor graphs have special graphical shapes. For instance, in [Hafezieh and Iranmanesh, 2013], the writers have classified the groups whose bipartite divisor graphs are paths.

In this thesis, we will investigate the classification of the groups whose bipartite divisor group are cycles which are classified by Taeri [Taeri, 2010].

In chapter 2, we introduce three undirected graphs for a set of positive integers and discuss some of the combinatorial properties of these graphs. We will follow the paper of Iranmanesh and Praeger [Iranmanesh and Praeger, 2010].

In chapter 3, we give a structure theorem for the finite groups with three conjugacy class sizes which is proved by Dolfi and Jabara [Dolfi and Jabara, 2009]. In particular, we also see that they are either solvable groups with derived length at most three or nilpotent groups. We will follow the paper of Dolfi and Jabara [Dolfi and Jabara, 2009].

In chapter 4, we discuss the connections between prime divisors of conjugacy classes and prime divisors of a finite group G. We will investigate a theorem of Ferguson that says for a finite solvable group G, if each conjugacy class size has at most two prime divisors and there exists a conjugacy class size with two distinct prime divisors, then the set of all primes dividing the conjugacy class sizes of G has at most four elements. We will follow the paper of Ferguson [Ferguson, 1991].

Finally, in chapter 5, we consider the case that the bipartite divisor graph is a cycle. We investigate the theorem of Taeri that the bipartite divisor graph is a cycle if and only if it is a cycle of length six [Taeri, 2010]. Also he classified those groups with this property which covers the main idea of this thesis. We will follow the paper of Taeri [Taeri, 2010].

In the rest of this chapter, we give preliminary definitions and results in group theory and graph theory which will be used in other sections.

1.2. Preliminary Definitions and Results in Group Theory

Note that we work on finite groups.

Definition 1.1: Let π be a set of primes. A finite group G is called a π-group if for every prime dividing the order of G lies in π and a π-subgroup of a group G is a *subgroup which itself is a π-group.*

Definition 1.2: A Hall π-subgroup of a group G is a π-subgroup with index involving no prime of π.

Definition 1.3: Let G be a group and $K \trianglelefteq G$. A subgroup H of G is a complement for K in G if $G = KH$ and $K \cap H = 1$.

A Hall p'-subgroup of G is called a p-complement in G. Note that $\pi(G)$ denotes the set of prime divisors of $|G|$.

Theorem 1.1: (Schur-Zassenhaus Theorem) Let G be a group and K a normal subgroup of G such that $gcd(|K|, |G/K|) = 1$. Then K has a complement in G.

Definition 1.4: A subgroup H of a group G is a characteristic subgroup of G, if $\alpha(H) = H$ for all $\alpha \in Aut(G)$, where $Aut(G)$ denotes the group of automorphism of G, and we write $H\ char\ G$.

Note that a characteristic subgroup is normal. It is easy to see that if H, K are two subgroups of G such that $H\ char\ K$, and $K \trianglelefteq G$, then $H \trianglelefteq G$.

Definition 1.5: The largest normal p-subgroup of a group G is denoted by $O_p(G)$ and we can easily see that $O_p(G)$ lies in every Sylow p-subgroup of G. The Fitting subgroup of G is the largest nilpotent normal subgroup of G and it is denoted by $F(G)$, and it is well known that $F(G) = \times_{p\in\pi(G)} O_p(G)$.

Definition 1.6: The Frattini subgroup of a group G is the intersection of all maximal subgroups of G and it is denoted by $\Phi(G)$.

It is a well-known fact that the subgroups $O_p(G)$, $F(G)$ and $\Phi(G)$ of the group G are characteristic subgroups of G, and $F((G/\Phi(G)) = F(G)/\Phi(G)$ [Kurzweil and Stellmacher, 2004].

Theorem 1.2: (6.1.4 in [Kurzweil and Stellmacher,2004]) Let G be a solvable group. Then $C_G(F(G)) \leq F(G)$.

Definition 1.7: Let G be a group and let Ω be a nonempty set. A map "$\cdot$" from $\Omega \times G$ to Ω is an action of G on Ω if the following two conditions hold:

- *$\alpha \cdot 1 = \alpha$, for all $\alpha \in \Omega$, and*
- *$(\alpha \cdot g) \cdot h = \alpha \cdot (gh)$ for all $\alpha \in \Omega$ and all group elements $g, h \in G$.*

Definition 1.8: Let G be a group which acts on the set Ω. This action is called regular if for each pair of elements $\alpha, \beta \in \Omega$, there exists precisely one $g \in G$ such that $\alpha \cdot g = \beta$. The action of G on Ω is called faithful if $\alpha \cdot g = 1$ for all $\alpha \in \Omega$, then $g = 1$.

Definition 1.9: Let A be a group that acts on the group G. The action of A on G is coprime if $gcd(|A|, |G|) = 1$. A subgroup H of the group G is A-invariant if for all $a \in A$, $H^a := \{h \cdot a | h \in H\} = H$.

Suppose that A is a group that acts on the group G and H is a subgroup of G. For any $g \in G$, by $(Hg)^A$, we mean the set $\{(hg) \cdot a | h \in H, a \in A\}$.

Theorem 1.3: (8.2.1 in [Kurzweil and Stellmacher,2004]) Suppose that the action of A on the group G is coprime. Let U be A-invariant subgroup of G and $g \in G$ such that $(Ug)^A = Ug$. Then there exists $c \in C_G(A)$ such that $Ug = Uc$.

Definition 1.10: For groups A and G, we say that A acts via automorphisms on G if A acts on G, and $(xy) \cdot a = (x \cdot a)(y \cdot a)$ for all $x,y \in G$ and $a \in A$. Let A act via automorphisms on G, then we define $[G, A] = \langle g^{-1} g^a \, | g \in G, a \in A \rangle$ as a subgroup of $G \rtimes A$ where g^a is the action of a on g.

Theorem 1.4: (Lemma 4.28 in [Isaacs, 2008]) Let A and G be finite groups. Let A act via automorphisms on G and suppose that $gcd(|G|, |A|) = 1$ and that one of A or G

is solvable. Then $G = C_G(A)[G, A]$.

Theorem 1.5: (Theorem 4.34 in [Isaacs, 2008]) Let A *act via automorphisms on an abelian group* G. *Assume that* A *and* G *are finite groups and* $gcd(|A|, |G|) = 1$. *Then* $G = C_G(A) \times [G, A]$.

Definition 1.11: Suppose that the group A *acts on the group* G *via automorphisms. Assume that* $a \in A$, $x \in G$. *By* $C_G(a)$ *and* $C_A(x)$ *we mean* $\{g \in G | g.a = g\}$ *and* $\{a \in A | x.a = x\}$, *respectively.*

Definition 1.12: Let $G^{\#}$ *denote the set of nonidentity elements of a group* G. *The action of* A *on* G *is said to be Frobenius if* $x \cdot a \neq x$ *whenever* $x \in G^{\#}$ *and* $a \in A^{\#}$.

Equivalently, the action of A on G is Frobenius if and only if $C_G(a) = 1$ for all $a \in A^{\#}$, and also if and only if $C_A(x) = 1$ for all $x \in G^{\#}$.

Theorem 1.6: (Lemma 6.1 in [Isaacs, 2008]) Let A *and* G *be finite groups, and suppose that there is a Frobenius action of* A *on* G. *Then* $|G| \equiv 1 \ (mod|A|)$, *and hence* $|G|$ *and* $|A|$ *are coprime.*

Definition 1.13: A group A *is called a Frobenius complement if it has a Frobenius action on some nonidentity group* G, *and similarly, a group* G *is called a Frobenius kernel if it admits a Frobenius action by some nonidentity group* A.

Theorem 1.7: (Theorem 6.4 in [Isaacs, 2008]) Let N *be a normal subgroup of a finite group* G, *and suppose that* A *is a complement for* N *in* G. *Then the followings are equivalent.*

i) The conjugation action of A *on* N *is Frobenius.*
ii) $A \cap A^g = 1$ *for all elements* $G \setminus A$.
iii) $C_G(a) \leq A$ *for all* $a \in A^{\#}$.
iv) $C_G(n) \leq N$ *for all* $n \in N^{\#}$.

If both N and A are nontrivial in the above theorem, we say that G is a Frobenius group and that A and N are Frobenius complement and Frobenius kernel of G, respectively.

Theorem 1.8: (Theorem 6.7 in [Isaacs, 2008]) Let N be a normal subgroup of a finite group G and suppose that $C_G(n) \leq N$ for every $n \in N^{\#}$. Then N is complemented in G and if $1 < N < G$, then G is a Frobenius group with kernel N.

Theorem 1.9: (Corollary 6.17 in [Isaacs, 2008]) Suppose that A is a Frobenius complement. Then each Sylow subgroup of A is cyclic or generalized quaternion.

Definition 1.14: A group G is said to act on the n-dimensional vector space V over the field K if G acts on the additive group V and $(\lambda v) \cdot g = \lambda(v \cdot g)$ for any $\lambda \in K$, $v \in V$, $g \in G$. The action of G on V is callad irreducible if $V \neq 0$ and 0 and V are the only G-invariant subspaces of V.

Theorem 1.10: (Theorem 8 in [Huppert and Manz, 1990]) Let P be a nontrivial p-group, which acts irreducibly and faithfully on a finite vector space V over $GF(q)$, where q is a prime number different from p. If P contains no section isomorphic to $C_p \wr C_p$, then P has a regular orbit in its action on V.

Definition 1.15: Let G be a group and H be a subgroup of G. Assume that G acts on H and H is an F-module, where F is a field. Then H is called an FG-module.

Definition 1.16: Let G be a group. The exponent of G is a number $e \in \mathbb{N}$ which is minimal with respect to the propety that $g^e = 1$ for all $g \in G$. Then we write $exp(G) = e$.

Definition 1.17: A group G is said to be π-seperable, where π is some set of primes, if there exists a normal series $1 = N_0 \leq N_1 \leq \ldots \leq N_r = G$ such that each factor N_i/N_{i-1}, $i = 1, \ldots, r$, is either a π-group or a π'-group. The group G is π-solvable if it has a normal series where each factor is either a π'-group or is a solvable π-group. Clearly, a solvable group is π-solvable for every set π of primes. If G is π-seperable, *the π-length of G is the minimum possible number of factors that are π-groups in any normal series for G in which each factor is either a π-group or a π'-group.*

Theorem 1.11: (Corollary 3.19 in [Isaacs, 2008]) Let G be a finite solvable group. Then G is π-seperable for every set π of primes.

Theorem 1.12: (3.1.9 in [Kurzweil and Stellmacher, 2004]) Let H be a subgroup of a group G. Then $N_G(H)$ is the largest subgroup of G in which H is normal. In addition, the mapping $\varphi : N_G(H) \to Aut(H)$ via $x \mapsto (h \mapsto h^x)$ is a homomorphism and $Ker(\varphi) = C_G(H)$.

Lemma 1.13: (Frattini Argument) Let N be a normal subgroup of a finite group G. Assume that $P \in Syl_p(N)$. Then $G = N_G(P)N$.

Theorem 1.14: Let G be a solvable group. Then $F(G/\Phi(G)) = F(G)/\Phi(G)$ is a completely reducible and faithful $\frac{G}{F(G)}$-module.

Definition 1.18: For a group G the subgroup $O_{\pi',\pi}(G)$ is defined by

$$\frac{O_{\pi',\pi}(G)}{O_{\pi'}(G)} = O_\pi\left(\frac{G}{O_{\pi'}(G)}\right). \tag{1.1}$$

Clearly, $O_{\pi',\pi}(G)$ is a characteristic subgroup of G.

Theorem 1.15: (6.4.3 in [Kurzweil and Stellmacher, 2004]) Let G be p-seperable for $p \in \pi(G)$ and P a Sylow p-subgroup of $O_{\pi',\pi}(G)$. Then $C_G(P\ ) \leq O_{\pi',\pi}(G)$.

1.3. Preliminary Definitions and Results in Graph Theory

Definition 1.19: A graph G consists of a nonempty finite set $V(G)$ of elements called vertices, and a finite family $E(G)$ of unordered pairs of elements of $V(G)$ called edges. An edge $\{v, w\}$ is said to join the vertices v and w, and sometimes it is *abbreviated to $v - w$. Two vertices in a graph are called adjacent if there is an edge joining them.*

Definition 1.20: A walk in a graph G is a finite sequence of vertices $v_0, v_1, \dots, v_n$ and edges $a_1, a_2, \dots, a_n$ of G; $v_0, a_1, v_1, a_2, \dots, a_n, v_n$ where the endpoints of a_i are v_{i-1} and v_i for each $i = 1, 2, \dots, n$, and the number n is the length of the walk. A path is a walk in which no vertex is repeated. A walk is closed when the first and last vertices, v_0 and v_n, are the same. Two vertices v and w in a graph G are connected if there is a path joining them, and their distance $d(v, w)$ is the length of the shortest path joining them.

Definition 1.21: A cycle of length n (or an n-cycle) is a closed walk of length n, $n \geq 3$, in which the vertices $v_0, v_1, \dots, v_{n-1}$ are all different, and it is denoted by C_n. A graph is called acylic if it contains no cycles.

Definition 1.22: If the vertex set of a graph G can be split into two disjoints sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is called a bipartite graph.

Theorem 1.16: (Theorem 2.2 in [Wallis, 2007]) A graph is bipartite if and only if it contains no cycle of odd length.

Definition 1.23: Two graphs G_1 and G_2 are isomorphic if there is a one to one correspondence between vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

Definition 1.24: If G is a graph, it is possible to choose some of the vertices and some of the edges of G in such a way that these vertices and edges again form a graph, say H, then H is called a subgraph of G; one writes $H \leq G$. If U is any set of vertices of G, then the subgraph consisting of U and all the edges of G that join two vertices of U is called an induced subgraph, the subgraph induced by U.

Definiton 1.25: If the two graphs are $G_1 = (V(G_1), E(G_1))$ *and* $G_2 = (V(G_2), E(G_2))$ *where* $V(G_1)$ *and* $V(G_2)$ *are disjoint, then their union* $G_1 \cup G_2$ *is the graph with vertex set* $V(G_1) \cup V(G_2)$ *and edge family* $E(G_1) \cup E(G_2)$*. A graph* G *is connected if it cannot be expressed as the union of two graphs, and disconnected otherwise. Clearly any disconnected graph* G *can be expressed as the union of connected graphs, each of which is a component of* G*, and the number of the components of* G *is denoted by* $n(G)$*.*

Definition 1.26: A graph with n*-vertices is said to be a complete graph if any two vertices in the graph are adjacent and it is denoted by* K_n*. A complete bipartite graph is a bipartite graph with disjoint two vertices sets* V_1 *and* V_2 *such that any two vertices in different sets are adjacent and we write* $K_{m,n}$ *to mean a complete bipartite graph with* m *vertices in* V_1 *and* n *vertices in* V_2*.*

2. BIPARTITE DIVISOR GRAPHS FOR INTEGERS SUBSETS

In this chapter we follow the paper of Iranmanesh and Praeger [Iranmanesh and Praeger, 2010].

In this chapter we associate three distinct graphs to a set of positive integers and discuss some of the combinaorial properties of these graphs such as the number of connected components, the diameter and the girth.

2.1. Representing the Bipartite Graph as $B(X)$

Definition 2.1: Let X be a nonempty subset of positive integers. Then we define the following two graphs with respect to the set X:

i) Prime vertex graph: This is an undirected graph whose vertex set is $\rho(X)$ which is the set of all primes dividing some element of X, and two such primes p, q are joined by an edge if and only if pq divides some $x \in X$. We denote this graph by $\Delta(X)$.

ii) Common divisor graph: This is an undirected graph whose vertex set is $X^* = X \setminus \{1\}$ and two elements like x, y of X^* form an edge if and only if $gcd(x, y) > 1$.

iii) Bipartite divisor graph: This is an undirected graph whose vertex set is the disjoint union $\rho(X) \cup X^*$ and its edges are the pairs $\{p, x\}$ where $p \in \rho(X)$, $x \in X^*$ and p divides x.

Theorem 2.1: A bipartite graph G is isomorphic to $B(X)$, for some nonempty set of positive integers X, if and only if G is nonempty and has no isolated vertices, where by an isolated vertex we mean a vertex which lies on no edge.

Proof 2.1: Suppose that G is a bipartite graph with vertex bipartition $\{V_1|V_2\}$. Let $V_1 = \{v_1, v_2, \dots, v_m\}$ and $V_2 = \{u_1, u_2, \dots, u_n\}$ where $m \geq 1, n \geq 1$. First of all, suppose that G has no isolated vertices. Let $p_1, p_2, \dots, p_m$ be pairwise distinct primes, and let $M = \{p_1, p_2, \dots, p_m\}$. Define a bijection $f : V_1 \to M$ by $f(v_i) = p_i$ for each

$i = 1, 2, \ldots, m$. For $1 \le j \le n$ define $I_j = \{l | \{v_l, u_j\} \in E(G)\}$ and then set $x_j = \prod_{l \in I_j} p_l^j$. Since there are no isolated vertices in G, it is clear that $I_j \neq \emptyset$. Let $X = \{x_j | 1 \le j \le n\}$. We claim that $\rho(X) = M$, if not, then there exists a prime number $p_i \in M \setminus \rho(X)$ and $v_i \in V$ such that $f(v_i) = p_i$. Then for each $1 \le j \le n$, $p_i \nmid x_j$ and hence $i \notin I_j$, which implies that v_i is isolated. Now $\{p_i, x_j\} \in E(B)$, the edge set of $B = B(X)$, if and only if p_i divides $x_j = \prod_{l \in I_j} p_l^j$, that is, if and only if $i \in I_j$, and this holds if and only if $\{v_i, u_j\} \in E(G)$. Thus extending f to a map $V(G) \to V(B)$ by $f(u_j) = x_j$, for each j, defines an isomorphism from G to $B(X)$.

Conversely, suppose that $G \cong B(X)$ for some X. By the definition of a bipartite graph we deduce that G has at least one vertex, so $X \neq \{1\}$. The fact that $B(X)$ has no isolated vertices now follows from its definition. ■

Corollary 2.2: For a nonempty set X of positive integers such that $X \neq 1$, there exists a second nonempty set Y of positive integers, and a graph isomorphism $\varphi : B(X) \to B(Y)$ that induces isomorphisms $\Delta(X) \cong \Gamma(Y)$ and $\Gamma(X) \cong \Delta(Y)$.

Proof 2.2: Let $G = B(X)$ with vertex bipartition $\{\rho(X) | X^*\}$. By the definition, G is nonempty and has no isolated vertices. We can apply the proof of Theorem 2.1 to the reverse bipartition $\{X^* | \rho(X)\}$. This produces a nonempty subset Y of positive integers and a graph isomorphism $\varphi : G \to B(Y)$. Therefore it induces a graph isomorphism from $\Delta(Y)$ to the graph induced on the first part X^* of the bipartition (which by the definition of G is $\Gamma(X)$) and a graph isomorphism from $\Gamma(Y)$ to the graph induced on the second part $\rho(X)$ of the bipartition (which by the definition of G is $\Delta(X)$). ■

Thus if we wish to prove that a certain relationship holds between $B(X)$ and $\Delta(X)$, for all X and also between $B(X)$ and $\Gamma(X)$, for all X, it is often sufficient to prove only one of these assertions.

2.2. Relating the Parameters of $\boldsymbol{B}, \Delta, \boldsymbol{\Gamma}$

Throughout the section let X denote a nonempty subset of positive integers with $X \neq \{1\}$, so that $X^* \neq \emptyset$. As mentioned above we simplify our notation and write $B = B(X)$, $\Delta = \Delta(X)$, $\Gamma = \Gamma(X)$.

For $u \in V(B)$, let $[u]_B$ denote the connected component of B containing u, and similarly define $[u]_\Delta$, $[u]_\Gamma$ if $u \in V(\Delta)$ or $u \in V(\Gamma)$ respectively. By diamater of G, we mean the maximum distance between vertices in the same component of G. We show this by $diam(G)$.

Lemma 2.3: Let $p, q \in \rho(X)$ and $x, y \in X^*$ such that $[p]_B = [q]_B$ and $[x]_B = [y]_B$. Then,

i) $d_B(p,q) = 2d_\Delta(p,q)$, $d_B(x,y) = 2d_\Gamma(x,y)$;
ii) if p divides x and q divides y, then $[p]_B = [x]_B = [p]_\Delta \cup [x]_\Gamma$ and $d_B(p,q) - d_B(x,y) \in \{-2, 0, 2\}$;
iii) $n(B) = n(\Delta) = n(\Gamma)$;
iv) either

- *$diam(B) = 2max\{diam(\Delta), diam(\Gamma)\}$, and $|diam(\Delta) - diam(\Gamma)| \leq 1$, or*
- *$diam(\Delta) = diam(\Gamma) = \frac{1}{2}(diam(B) - 1)$.*

Proof 2.3: Let $p, q \in \rho(X)$ and $x, y \in X^*$ such that $[p]_B = [q]_B$ and $[x]_B = [y]_B$.

i) Suppose that $d_\Delta(p,q) = k$. There exists a shortest path $P_\Delta = (p_0, p_1, \dots, p_k)$ in Δ with $p = p_0$ and $q = p_k$. Now $\{p_i, p_{i+1}\}$ is an edge of Δ if and only if $d_B(p_i, p_{i+1}) = 2$, and so there exists a path $P_B = (p_0, x_1, p_1, \dots, x_{k-1}, p_k)$ in B of length $2k$. Thus $d_B(p,q) \leq 2k$, and as p and q are in the same part of the bipartition of B, we have $d_B(p,q) = 2l \leq 2k$. If $P'_B = (p'_0, x'_1, p'_1, \dots, x'_l, p'_l)$ is a shortest path in B with $p = p'_0$ and $q = p'_l$, then $P'_\Delta = (p'_0, p'_1, \dots, p'_l)$ is a path of length l in Δ, so $k = d_\Delta(p,q) \leq l$. Therefore $d_B(p,q) = 2k = 2d_\Delta(p,q)$. We can write a similar proof to show that $d_B(x,y) = 2d_\Gamma(x,y)$.
ii) Suppose that p divides x and q divides y. Let $P_B = (p_0, x_1, p_1, \dots, x_l, p_l)$ be the shortest path in B with $p = p_0$ and $q = p_l$. Then we have three cases:

- *Case 1: If P_B can be chosen with $x_1 = x$ and $x_l = y$, then it is clear that $d_B(p,q) - d_B(x,y) \geq 2$. Let $P'_B = (x'_0, p'_1, \dots, p'_k, x'_k)$ be a shortest path in*

B with $x = x'_0$ and $y = x'_k$. We must have $p'_1 \neq p$ and $p'_k \neq q$ otherwise $d_B(p,q) \leq d_B(x,y)$ which contradicts the minimality of P_B. Now $P''_B = (p, x'_0, p'_1, \dots, p'_k, x'_k, q)$ is a path of lenght $k+2$ between p and q in B. This implies that $d_B(p,q) \leq d_B(x,y) + 2$. Thus $d_B(p,q) - d_B(x,y) \leq 2$. Hence we have the equality occurs.

- *Case 2: Suppose that only one of these equalities holds. We may assume that $x_1 = x$ and $x_l \neq y$. Now it is clear that $d_B(x,y) \leq d_B(p,q)$. Let $P'_B = (x'_0, p'_1, \dots, p'_k, x'_k)$ be a shortest path in B with $x = x'_0$ and $y = x'_l$, and assume that $d_B(x,y) < d_B(p,q)$. So none of p and q can be in P'_B. It follows that $d_B(p,q) \leq d_B(x,y) + 2$, hence $d_B(p,q) = d_B(x,y) + 2$. So we can choose P_B with $x_1 = x$ and $x_l = y$, but it contradicts with our assumption, thus $d_B(p,q) = d_B(x,y)$.*
- *Case 3: If P_B can be only chosen with $x_1 \neq x$ and $x_l \neq y$, then $d_B(x,y) - d_B(p,q) = 2$. This proves that $d_B(p,q) - d_B(x,y) \in \{-2, 0, 2\}$, and all cases are possible. Moreover in this case, by assumption, the component $[p]_B$ contains all of p, q, x, y and the fact that $[p]_B = [p]_\Delta \cup [x]_\Gamma$ follows from the proof of part i).*

iii) It follows from the statement in part ii) about components.

iv) Let $m = max\{diam(\Delta), diam(\Gamma)\}$. It follows from part i) that $diam(B) \geq 2m$. Let $M := diam(B)$. Assume $v, w \in V(B)$ such that $d_B(v,w) = M$. If v and w are both in $\rho(X)$ (or both in X^*), then $M = d_B(v,w) = 2d_\Delta(v,w) \leq 2diam(\Delta) \leq 2m$ (respectively, $M \leq 2diam(\Gamma) \leq 2m$) and in either case we conclude that $M = 2m$. Now assume that $v \in \rho(X)$ and $w \in X^*$. Since they are in distinct partitions, we conclude that M is odd and hence $M \geq 2m+1$. Let $p \in \rho(X)$ be the vertex adjacent to w and P_B be a path of length M between v and w. By the definition of M it is clear that a path from p to v inside P_B is the shortest path between these two vertices, hence by part i) we deduce that $d_B(v,p) = 2d_\Delta(v,p) = M-1$ which is at most $2diam(\Delta)$. By a similar discussion we can see that $M \leq 2diam(\Gamma) + 1$. Thus $diam(\Delta) = diam(\Gamma) = \frac{M-1}{2}$.

Now assume that $diam(B) = 2m$. Let $j := diam(\Delta)$ and let p_0, $p_j \in \rho(X)$ be *such that* $d_\Delta(p_0, p_j) = j$. *Then by part i), there exists a path* P'_B *of length* $2j$ *in* B *from* p_0 *to* p_j. *Let* x_0, x_j *be vertices on* P'_B *adjacent to* p_0 *and* p_j, *respectively. Then the path inside* P'_B *from* x_0 *to* x_j *of length* $2j - 2$ *is a shortest path in* B *between two vertices. Thus* $d_B(x_0, x_j) = 2j - 2$. *By part i),* $d_\Gamma(x_0, x_j) = j - 1$ *and therefore we have* $diam(\Gamma) \geq diam(\Delta) - 1$. *A similar argument shows that* $diam(\Delta) \geq diam(\Gamma) - 1$. *Hence* $|diam(\Delta) - diam(\Gamma)| \leq 1$ *which completes the proof of the last assertion of the first part.* ■

For any nonempty subset $K \subseteq \{B, \Delta, \Gamma\}$ there exists a nonempty set X of positive integers such that the graphs in K are acyclic. Examples of subsets are provided in Table 2.1 for the seven nonempty subsets of K of $\{B, \Delta, \Gamma\}$, and if $X = X_2 \cup X_3 \cup X_4$, with the X_i as in Table 2.1, then all three graphs contain cycles. Note that $K_{\overrightarrow{m,n}}$ denotes the complete bipartite graph $B = B(X)$ with $|\rho(X)| = m$ and $|X^*| = n$ and if G and H are graphs, we use the notation $G + H$ to show the graph with connected components G and H.

Table 2.1: Illustration of acyclic possibilities for B, Δ, Γ.

i	X_i	B	Δ	Γ
1	$\{2\}$	K_2	K_1	K_1
2	$\{2, 4, 8\}$	$K_{\overrightarrow{1,3}}$	K_1	K_3
3	$\{105\}$	$K_{\overrightarrow{3,1}}$	K_3	K_1
4	$\{11.13, 11^2.13\}$	C_4	K_2	K_2
5	$X_2 \cup X_3$	$K_{\overrightarrow{1,3}} + K_{\overrightarrow{3,1}}$	$K_1 + K_3$	$K_1 + K_3$
6	$X_2 \cup X_4$	$K_{\overrightarrow{1,3}} + C_4$	$K_1 + K_2$	$K_2 + K_3$
7	$X_3 \cup X_4$	$K_{\overrightarrow{3,1}} + C_4$	$K_2 + K_3$	$K_1 + K_2$
8	$X_2 \cup X_3 \cup X_4$	$K_{\overrightarrow{1,3}} + K_{\overrightarrow{3,1}} + C_4$	$K_1 + K_2 + K_3$	$K_1 + K_2 + K_3$

The girth of a graph G is the length of its shortest cycle and is denoted by $g(G)$.

Lemma 2.4: Suppose that B contains a cycle of length greater than 4. Then each of Δ and Γ also contains a cycle. Moreover, for $\Phi \in \{\Delta, \Gamma\}$, either $g(\Phi) = 3$ or $g(\Phi) = \frac{1}{2}\, g'(B)$, where $g'(B)$ is the minimum length of cycles of B with more than four vertices.

Proof 2.4: Since B is bipartite and by Theorem 1.16 we have $g'(B) = 2k$ for some $k \geq 3$. Let $P_B = (p_1, x_1, \ldots, p_k, x_k)$ be a closed path of length $2k$ in B such that $p_i \in \rho(X)$ and $x_i \in X^*$. By the definition of B, p_i divides x_i and x_{i-1}, for $i = 1, \ldots, k$, reading the subscripts modulo k. Hence there exist closed paths of length k in both Δ and Γ. This implies that both Δ and Γ contain cycles and $g(\Delta) \leq k$, $g(\Gamma) \leq k$.

If $g(\Delta) = l < k$, then there exist a closed path $P_\Delta = (p'_1, p'_2, \ldots, p'_l)$ in Δ. By the definition of Δ, for all i, there exists $x'_i \in X^*$ that is divisible by p'_i and p'_{i+1}, reading subscripts modulo l. If the x'_i's are pairwise distinct, then $P'_B = (p'_1, x'_1, \ldots, p'_l, x'_l)$ is a closed path in B of length $2l$. Since $6 \leq 2l < 2k = g(B)$, we obtain a contradiction. Hence x'_i's are not all distinct. Let i, j be such that $1 \leq i < j \leq l$ and $x'_i = x'_j$. Then we have in Δ the induced subgraph on the set $\{p'_i, p'_{i+1}, p'_j, p'_{j+1}\}$ is a complete graph (of order 3 and 4) and hence $l = g(\Delta) = 3$. So either $g(\Delta) = 3$ or $g(\Delta) = k$. A similar proof shows that either $g(\Gamma) = 3$ or $g(\Gamma) = k$ (also one may use Corollary 2.2). ■

2.3. Subgraphs of B, Δ, Γ

Theorem 2.5: At least one of Δ, Γ contains a triangle that is 3-cycles if and only if B contains C_6 or $K_{1,3}$ as an induced subgraph.

Proof 2.5: First suppose that $g(\Gamma) = 3$ and let $P_\Gamma = (x_1, x_2, x_3)$ be a cycle of length three in Γ. If there exists a prime p which divides x_i, for all $i = 1, 2, 3$, then the set $\{p, x_1, x_2, x_3\}$ induces a subgraph $K_{1,3}$ of B. So we may assume that no such prime exists. Since P_Γ is a cycle in Γ, there are distinct primes p_1, p_2, p_3 such that, for each i, p_i divides x_{i-1} and x_i, writing subscripts modulo 3. Now it is clear that $\{p_1, x_1, p_2, x_2, p_3, x_3\}$ induces a subgraph C_6 of B. Thus $g(\Gamma) = 3$, we deduce that B contains an induced subgraph isomorphic to either C_6 or $K_{1,3}$. By Corollary 2.2, if

$g(\Delta) = 3$ verifies that B contains an induced subgraph isomorphic to either C_6 or $K_{1,3}$.

Conversely, if $\{p_1, x_1, p_2, x_2, p_3, x_3\}$ induces a subgraph C_6 in B, where $p_i \in \rho(X)$ and $x_i \in X^*$, then (p_1, p_2, p_3) and (x_1, x_2, x_3) are 3-cycles in Δ and Γ respectively, so $g(\Delta) = g(\Gamma) = 3$. Similarly if B contains an induced subgraph $K_{1,3}$, then at least one of Δ, Γ contains a triangle. This completes the proof. ■

Theorem 2.6: Both the graphs Δ and Γ are acyclic if and only if each connected component of B is a path or a cycle of length 4.

Proof 2.6: First suppose that Δ, Γ are both acyclic. If some vertex of B lies on at least three edges, then one of Δ, Γ contains a 3-cycle, which is a contradiction. Thus each vertex of B lies on at most two edges in B. Since B is bipartite, we conclude that each connected component of B is a path, or a cycle C_{2k} of even length $2k \geq 4$. Furthermore, in the case of a component C_{2k}, it follows from Lemma 2.4, we conclude that $k = 2$.

Conversely, suppose that each component of B is a path or isomorphic to C_4. For a component C_4 of B, the corresponding component of Δ, Γ is isomorphic to K_2. Consider a component B' of B which is a path. Suppose that $P_\Delta = (p_1, p_2, \ldots, p_l)$ is a cycle in the corresponding component of Δ of length $l \geq 3$. By the definition of Δ, for each i, there exists $x_i \in X^*$ which is divisible by both p_i and p_{i+1}, reading subscripts modulo l. If x_i's are pairwise distinct, then $P_{B'} = (p_1, x_1, \ldots, p_l, x_l)$ is a cycle in B', which is a contradiction. Hence there exist i, j such that $1 \leq i < j \leq l$ and $x_i = x_j$. This verifies that x_i is joined to at least three vertices in B', which contradicts the fact that B' is a path. Hence the component of Δ corresponding to B' is acyclic. A similar proof shows that the component of Γ corresponding to B' is also acyclic (also one may use Corollary 2.2). ■

Corollary 2.7: Both graphs Δ and Γ are trees (i.e connected acyclic graph) if and only if either B is a path or $B \cong C_4$.

Proof 2.7: The result follows from Theorem 2.6 and part iii) of Lemma 2.3. ■

3. THE STRUCTURE of FINITE GROUPS of CONJUGATE RANK 2

In this chapter we will follow the paper of Dolfi and Jabara [Dolfi and Jabara, 2009].

Let G be a finite group. If g is an element of a group G, then we denote by g^G the conjugacy class of g in G and $|g^G|$ which shows the size of this conjugacy class, is the positive integer $[G : C_G(g)]$. Let $cs(G)$ denote the set of the sizes of the conjugacy classes of a finite group G. The number of the distinct sizes of the noncentral conjugacy classes of G is $|cs(G)| - 1$ and is called the conjugate rank of G. Clearly, a group has conjugate rank zero when it is abelian. In this chapter, we discuss the structure of the groups of conjugacte rank two.

If n is a positive integer and p is a prime, then by n_p and p' we mean the largest power of p dividing n and the set of the primes different from p, respectively. Now it is clear that $n_{p'} = \prod_{q \in p'} n_q$ the integer n/n_p.

3.1. A Normal p-Complement of G when G/F is a p-Group

Lemma 3.1: Let G and H be groups. We have the following properties:

i) $cs(G \times H) = \{ab | a \in cs(G), b \in cs(H)\}$.
ii) If $x, y \in G$ commute and $gcd(|x|, |y|) = 1$, then $C_G(xy) = C_G(x) \cap C_G(y)$ and so $|x^G|$ and $|y|^G$ divide $|(xy)^G|$.
iii) If N is a normal subgroup of G and $x \in N$, then $|x^N|$ divides $|x^G|$. Also if $y \in G$, then $|(yN)^{G/N}|$ divides $|y^G|$.

Proof 3.1: Let G and H be groups.

i) Let (g, h) be an arbitrary element of $G \times H$. By the definition we have

$$\begin{aligned}(g,h)^{G\times H} &= \{(g'^{-1}, h'^{-1})(g,h)(g',h') \big| (g',h') \in G \times H\} \\ &= \{(g'^{-1}gg', h'^{-1}hh') \big| (g',h') \in G \times H\} = (g^G, h^H).\end{aligned} \tag{3.1}$$

Now it is clear that $cs(G \times H) = \{ab | a \in cs(G), b \in cs(H)\}$.

ii) If $x, y \in G$ *commute and* $gcd(|x|, |y|) = 1$, *then* $\langle xy \rangle = \langle x \rangle \langle y \rangle$. *Thus it follows that* $C_G(xy) \leqslant C_G(x) \cap C_G(y)$. *The other inclusion is trivial.*

iii) Let N *be a normal subgroup of the group* G. *Assume that* $x \in N$. *It is clear that* $C_G(x) \cap N = C_N(x)$. *As* N *is normal, we conclude that* $NC_G(x)$ *is a subgroup of* G *and we have:*

$$|NC_G(x)| = \frac{|N||C_G(x)|}{|N \cap C_G(x)|} = \frac{|N||C_G(x)|}{|C_N(x)|}. \tag{3.2}$$

Thus $|x^N| = \frac{|N|}{|C_N(x)|} = \frac{|NC_G(x)|}{|C_G(x)|}$. *On the other hand* $C_G(x) \leq NC_G(x) \leq G$. *This verifies that*

$$\begin{aligned} |x^G| &= [G : C_G(x)] \\ &= [G : NC_G(x)][NC_G(x) : C_G(x)] = [G : NC_G(x)]|x^N|. \end{aligned} \tag{3.3}$$

Hence $|x^N|$ *divides* $|x^G|$. *Let* $\bar{G} := G/N$, *and for each subgroup* K *of* G, *let* $\overline{K} = KN/N$. *Assume that* $y \in G$. *Since* $\overline{C_G(y)} \subseteq C_{\bar{G}}(\bar{y})$, *we have* $|\bar{y}^{\bar{G}}| = [\overline{G} : C_{\bar{G}}(\bar{y})]$ *divides* $[\overline{G} : \overline{C_G(y)}] = [G : NC_G(y)]$. *It is clear that* $[G : NC_G(y)]$ *divides* $[G : C_G(y)] = |y^G|$. *Thus* $|(yN)^{G/N}|$ *divides* $|y^G|$. ■

Theorem 3.2: Suppose G *is a group and* H *is a subgroup of* G. *Then we have*

$$\left| \bigcup_{g \in G} H^g \right| \leq |G| - [G : H] + 1 \tag{3.4}$$

where $H^g = \{g^{-1}hg | h \in H\}$ *for any* $g \in G$.

Proof 3.2: The number of different conjugates of H *in* G *is* $[G : N_G(H)]$, *so*

$$\begin{aligned} \left| \bigcup_{g \in G} H^g \right| &\leq (|H| - 1)[G : N_G(H)] + 1 \\ &\leq (|H| - 1)[G : H] + 1 = |G| - [G : H] + 1 \end{aligned} \tag{3.5}$$

where the second inequality is obtained from the fact $[G : N_G(H)] \leq [G : H]$. ■

Lemma 3.3: Let p be a prime and let P be a Sylow p-subgroup of the group G. Then p does not divide any $n \in cs(G)$ if and only if P is abelian and $G = P \times H$ for some $H \leq G$.

Proof 3.3: Suppose that p does not divide any conjugacy class sizes of G, that is $p \nmid |g^G|$ for each $g \in G$. Thus $|G|_p = |C_G(g)|_p$ for any $g \in G$, so every element of G centralizes some Sylow p-subgroup of G. Hence $G = \bigcup_{g \in G} C_G(P^g) = \bigcup_{g \in G} C_G(P)^g$ and by Theorem 3.2 we have

$$|G| = \left|\bigcup_{g \in G} C_G(P)^g\right| \leq |G| - [G : C_G(P)] + 1. \tag{3.6}$$

It follows that $C_G(P) = G$ so P is abelian and $P \trianglelefteq G$ since $C_G(P) \leq N_G(P)$. By Theorem 1.1, P has a p-complement in G, say H, therefore $G = P \times H$. Conversely, $P \in Syl_p(G)$ and by part i) of Lemma 3.1 we have $cs(G) = cs(H)$ as P is abelian, so the result follows. ■

Theorem 3.4: Let G be a finite group and H a p-complement subgroup of G with $H \leq Z(G)$ where p is a prime divisor of the order of G. Then $n_p(G) = 1$.

Proof 3.4: Take any $P \in Syl_p(G)$, then $G = HP$ since H is a p-complement. As $H \leq Z(G)$, H is also in $N_G(P)$, hence $G = N_G(P)$. That is $P \trianglelefteq G$, so $n_p(G) = 1$. ■

Lemma 3.5: Let G be a group and $F = F(G)$. Assume that G/F is a p-group for some prime p. Let L be a normal p-complement of G. Then the following properties hold:

i) $C_G(L) \leq F$;
ii) if $exp(G/F) = p$, then there exists a $g \in L$ such that $C_G(g) \nleqslant F$;
iii) assume that $exp(G/F) = p$ and that there exists a normal subgroup Z of G, such that $Z < L$ and $C_G(y) \leq F$ for every $y \in L \setminus Z$. Then $|G/F| = p$.

Proof 3.5: Let G be a group. Assume that G/F is a p-group for some prime p where $F = F(G)$.

i) Since $L \trianglelefteq G$*, we have* $N_G(L) = G$ *and Theorem 1.12 verifies that* $N := C_G(L) \trianglelefteq G$*. It is clear that* $Z(L) = L \cap C_G(L) = L \cap N$*. As* L *is a normal* p*-complement of* G*, we deduce that* $Z(L) = L \cap N \trianglelefteq N$ *and* $p \nmid |Z(L)|$*. On the other hand,* $\frac{NL}{L} \cong \frac{N}{N \cap L}$*. This implies that* $\frac{N}{Z(L)} \cong \frac{N}{N \cap L}$ *is a* p*-group, so* $gcd([N : Z(L)], |Z(L)|) = 1$*. Hence* $Z(L)$ *is a normal* p*-complement of* N*. As* $Z(L) \leq Z(N)$*, by Theorem 3.4 we deduce that* N *has a unique Sylow* p*-subgroup* P*, so* $P\ char\ N$*. Since* N *is a normal subgroup of* G*, we conclude that* $P \trianglelefteq G$*, so* $P \leq F$*. It is obvious that* $Z(L) \leq F$*. Thus* $N = Z(L)P \leq F$*.*

ii) We work by induction on $|F|$*.*

First suppose that $\Phi(G) \neq 1$ *and let* $\bar{G} := G/\Phi(G)$*. As* $F(\bar{G}) = F/\Phi(G)$*, we have* $|F(\bar{G})| < |F|$*. On the other hand,* $\frac{\bar{G}}{F(\bar{G})} \cong \frac{G}{F}$ *is a* p*-group. If* $exp(G/F) = p$*, then* $exp(\bar{G}/F(\bar{G})) = p$*. Now by induction we deduce that there exists an element* $\bar{g}$ *of* $\bar{L}$ *such that* $C_{\bar{G}}(\bar{g}) \leq F(\bar{G})$ *and this will happen if and only if* $C_G(g) \leq F$*. So without loss of generality we may assume that* $\Phi(G) = 1$*. As* $\Phi(F) \leq \Phi(G)$*, we have* $\Phi(F) = 1$ *which implies that* $F' = 1$ *(since* $F' \leq \Phi(F) = 1$*). Since* F *is abelian, it has a* $\mathbb{Z}$*-module structure. Also, it is easy to see that* $P := G/F$ *acts on* F *by conjugation, so* F *is a* $\mathbb{Z}P$*-module. From now on instead of a* $\mathbb{Z}P$*-module, we simply write a* P*-module. Since* $\Phi(G) = 1$*, by Theorem 1.14 we deduce that* F *is completely reducible and faithful* P*-module. Further,* P *acts faithfully on* L *because* $C_G(L) \leq F$ *by part i). Now we claim that there exists an element* g *of* L *such that* $C_P(g) = C_P(L) = 1$*. If* L *is reducible, then there exist nontrivial* P*-modules* L_1 *and* L_2 *such that* $L = L_1 \times L_2$*. Since* L *is a* p*-compement of* G*, it is clear that* $LF = F$*, which implies that* $L \leq F$*. We have* $G = LQ$ *where* $Q \in Syl_p(G)$*, so* $G = LQ = FQ$*. Now we deduce that* $O_p(G) = Q \cap F$ *and* $G/F \cong Q/(Q \cap F) \cong Q/O_p(G)$*. Let* $G_i = L_iQ$*,* $i = 1, 2$*. We claim that* G_i *has the properties of* G*.*

Similar to the above part, we can see that $O_p(G_i) = Q \cap F(G_i)$ *and* $G_i/F(G_i) \cong Q/(Q \cap F(G_i)) \cong Q/O_p(G_i)$ *for each* $i \in \{1,2\}$*. Now we have:*

$$\frac{G_i}{F(G_i)} \cong \frac{Q}{Q \cap F(G_i)} = \frac{Q}{O_p(G_i)} \cong \frac{\frac{Q}{O_p(G)}}{\frac{O_p(G_i)}{O_p(G)}}. \tag{3.7}$$

Thus $\frac{G_i}{F(G_i)}$ *has exponent* p. *Since* $|F(G_i)| < |F(G)|$, *by inductive hypothesis we conclude that there exists* $g_i \in L_i$ *for* $i = 1,2$ *such that* $C_P(g_i) = C_P(L_i)$. *Considering* $g = g_1 g_2$, *then*

$$C_P(g) = C_P(g_1) \cap C_P(g_2) = C_P(L_1) \cap C_P(L_2) = C_P(L) = 1. \tag{3.8}$$

Therefore we can assume that L *is an irreducible* P*-module. Observe that* P *does not have any* $C_p \wr C_p$ *section, because* $exp(P) = p$. *Hence by Theorem 1.10,* P *has a regular orbit in its action on* L, *that is there is a* $g \in L$ *such that* $C_P(g) = 1$.
iii) We first show that $P := G/F$ *acts fixed point freely on* L/Z, *which means* P *acts on* L/Z *and* $C_{L/Z}(P) = 1$. *Let* $yZ \in L/Z$ *such that* $A = C_P(yZ) \neq 1$. $A \leq P$, *thus the action of* A *on* L *is coprime and* Z *is an* A*-invariant subgroup of* L *such that* $(yZ)^A = yZ$. *Hence by Theorem 1.3 there exists an element* $y_0 \in C_L(A)$ *such that* $yZ = y_0 Z$, *so* $C_G(y_0) \nleq F$ *and hence* $y_0 \in Z$ *and* $yZ = Z$. *Consequently,* P *acts fixed point freely on* L/Z. *This implies that* $M = \frac{L}{Z} \rtimes P$ *is a Frobenius group with the Frobenius kernel* L/Z *and the Frobenius complement* P, *from Theorem 1.9,* P *is either cyclic or generalized quaternion. However,* P *has exponent* p, *so* $|P| = |G/F| = p$. ■

3.2. Maximal p-Defect

A group G is metabelian if it has a normal abelian subgroup K with G/K abelian.

Theorem 3.6: (Theorem 1 in [Higman, 1957]) Let G *be a solvable group all of whose elements have prime power order. Let* p *be the prime such that* G *has a normal*

p-subgroup greater than 1, and let P be the greatest normal p-subgroup of G. Then G/P is one of the following groups:

i) A cyclic group whose order is a power of a prime other than p.
ii) A generalized quaternion group , p being odd.
iii) A group of order $p^a q^b$ with cyclic Sylow subgroups, q being a prime of the form $kp^a + 1$.

Thus G has order divisible by at most two primes, and G/P is metabelian.

Theorem 3.7: (Chapter IX Theorem 4.3 in [Huppert and Blackburn, 1982]) Suppose that G is a p-solvable group of p-length l and that p^e is the exponent of a Sylow p-subgroup of G.

i) $l \leq e$, provided that one of the following three conditions hold:

- *p is odd and p is not a Fermat prime.*
- *p is a Fermat prime and the Sylow 2-subgroups of G are abelian.*
- *$p = 2$ and the Sylow q-subgroups of G are abelian for every Mersenne prime q.*

ii) If p is a Fermat prime, $l \leq 2e$.

Proposition 3.8: Assume that every element of the solvable group G has prime order. Then there exists a normal p-subgroup P of G such that $exp(P) = p$ and either $G = P$ or $[G : P] = q$, where p and q are distinct prime numbers. Moreover, if $P < G$, then G is a Frobenius group with kernel P.

Proof 3.8: It is well known that a minimal normal subgroup of a solvable group is an elementary abelian p-group for some prime p. Thus there exists a prime p such that $P := O_p(G) > 1$. Since every element of G has prime order, we deduce that $exp(P) = p$. If $P < G$, then by Theorem 3.6 it follows that either $|G/P| = q$ or $|G/P| = pq$ for a prime $p \neq q$. If $|G/P| = pq$ and G/P is abelian, then G/P is a product of normal cyclic Sylow subgroups which means G/P is cyclic. As every element of G has prime order, we have $|G/P|$ is a prime number which contradicts *our assumption. Thus G/P is nonabelian. Now Theorem 1.11 verifies that $l_p(G) = 2$ where $l_p(G)$ is the p-length of G. As the Sylow p-subgroups of G has exponent p and the Sylow q-subgroups of G are abelian, by Theorem 3.7 we have $l_p(G) = 1$ which is a contradiction. Thus, $|G/P| = q$. Therefore there exists an element $x \in G$ such that $G = P \rtimes \langle x \rangle$. Since G does not have an element of order pq, by Theorem 1.8 we conclude that G is a Frobenius group with kernel P.* ■

It turns out to be useful to have a notation for the set of the elements of ‘‘maximal p-defect’’ of the group.

Let G be a group and let p be a prime number. We define

- $m_p(g) = \max\{|C_G(x)|_p | x \in G \setminus Z(G)\}$ and
- $M_p(G) = \{g \in G \setminus Z(G) | |C_G(g)|_p = m_p(G)\}$.

Lemma 3.9: Let N be a normal subgroup of G and assume that p does not divide $|N|$. If $g \in M_p(G)$ and $gN \notin Z(G/N)$, then $gN \in M_p(G/N)$.

Proof 3.9: Write $\bar{G} := G/N$ and use the bar convention. We claim that, for every $x \in G \setminus Z(G)$, there exists a $y \in G \setminus Z(G)$ such that $\bar{y} = \bar{x}$ and $|C_G(y)|_p = |C_{\bar{G}}(\bar{x})|_p$. Let $D \leq G$ such that $D = C_G(x)$ and consider a Sylow p-subgroup P of D. Clearly P acts coprimely on N by conjugation as $gcd(|P|, |N|) = 1$. For each $y \in P$ it is clear that $y \in C_G(x)$, so we have $(Nx)^y = Nx^y = Nx$. By Theorem 1.3 there exists an element $y_0 \in C_G(P)$ such that $\overline{y_0} = \bar{x}$. Hence $P \leq C_G(y_0)$. Let P_1 be a Sylow p-subgroup of $C_G(y_0)$ such that $P \leq P_1$. As $\overline{P_1}$ centralizes $\overline{y_0} = \bar{x}$, we have $\bar{P} \leq \overline{P_1} \leq \bar{D}$. Since $\bar{P}$ is a Sylow p-subgroup of $\bar{D}$, it follows that $\overline{P_1} = \bar{P}$. We have $|\overline{P_1}| = |P_1N/N| = |P_1|$ since $P_1 \cap N = 1$, thus $|C_G(y_0)|_p = |P_1| = |\bar{P}| = |C_{\bar{G}}(\bar{x})|_p$. Finally, $y_0 \notin Z(G)$, as $\overline{y_0} = \bar{x} \notin Z(\bar{G})$.

In particular, $\bar{x} \in M_p(\bar{G})$. So there exists a $y \in G \setminus Z(G)$ such that $\bar{y} = \bar{x}$ and $|C_G(y)|_p = |C_{\bar{G}}(\bar{x})|_p$. Thus $m_p(\bar{G}) \leq m_p(G)$. Let $g \in M_p(G)$ and assume that $\bar{g} \notin Z(\bar{G})$. Then $|C_{\bar{G}}(\bar{g})|_p \leq m_p(\bar{G})$. If P is a Sylow p-subgroup of $C_G(g)$, then $m_p(G) = |P| = |\bar{P}|$ because $N \cap P = 1$. As $\bar{P} \leq C_{\bar{G}}(\bar{g})$, we conclude that

$$m_p(G) \leq |C_{\bar{G}}(\bar{g})|_p \leq m_p(\bar{G}) \leq m_p(G). \tag{3.9}$$

Hence, $m_p(\bar{G}) = m_p(G)$ *and* $\bar{g} \in M_p(\bar{G})$. ■

Lemma 3.10: Let G *be a group and* P_0 *be a Sylow* p*-subgroup of* $C_G(g)$ *where* $g \in M_p(G)$. *If* P *is a* p*-subgroup of* G *such that* $P_0 \leq P$, *then* $C_P(P_0) \leq P_0$.

Proof 3.10: Let $x \in C_P(P_0)$. *If* $x \in Z(G)$, *then* $x \in O_p(Z(G)) \leq P_0$. *So we can assume* $x \notin Z(G)$. *As* $\langle P_0, x\rangle \leq C_G(x)$, *then* $|\langle P_0, x\rangle| \leq m_p(G) = |P_0|$ *and hence* $x \in P_0$. ■

To prove next proposition we need Thompson's $P \times Q$-Lemma.

Theorem 3.11: Let $A = P \times Q$ *be the direct product of a* p*-group* P *and a* p'*-group* Q *and* A *acts via automorphism on a* p*-group* G. *Suppose that* $C_G(P) \leq C_G(Q)$. *Then* Q *acts trivially on* G.

Proposition 3.12: Let G *be a* p*-solvable group and* g *a* p'*-element of* G. *If* $g \in M_p(G)$, *then* $g \in O_{p'}(G)$.

Proof 3.12: We proceed by induction on $|G|$. *If* $O_{p'}(G) \neq 1$, *then let* $\bar{G} \coloneqq G/O_{p'}(G)$ *which is a* p*-solvable group. Assume that* $g \in M_p(G)$. *If* $\bar{g} \in Z(\bar{G})$, *then* $\langle \bar{g}\rangle$ *is a normal subgroup of* $\bar{G}$. *This implies that* $\langle g\rangle$ *is a normal* p'*-subgroup of* G. *Thus* $g \in O_{p'}(G)$. *So we may assume that* $\bar{g} \notin Z(\bar{G})$. *Now by Lemma 3.9 we conclude that* $O_{p'}(G/O_{p'}(G)) = 1$. *The inductive hypothesis implies that* $\bar{g} \in O_{p'}(\bar{G}) = 1$, *so* $g \in O_{p'}(G)$. *Thus without loss of generality we may assume that* $O_{p'}(G) = 1$. *Let* P_0 *be a Sylow* p*-subgroup of* $C_G(g)$ *and let* $P = P_0L$, *where* $L = O_p(G)$. *Then* $P_0 \times \langle g\rangle$ *acts on* P *and by Lemma 3.10 it is clear that* $C_P(P_0) \leq P_0$. *By Theorem 3.11* $\langle g\rangle$ *acts trivially on* P, *so* g *centralizes* P. *In particular* g *centralizes* L. *Since* $O_{p'}(G) = 1$, *we have* $C_G(L) \leq L$. *Hence* $g = 1$. ■

3.3. Groups of Conjugate Rank 2

Proposition 3.13: Let G be a solvable group. Assume that there is an element $m \in cs(G) \setminus \{1\}$ such that m divides every $n \in cs(G) \setminus \{1\}$. Then every $g \in G$ such that $|g^G| = m$ belongs to $F(G)$.

Proof 3.13: Let $g \in G$ such that $|g^G| = [G : C_G(g)] = m$. By assumption $|C_G(x)|$ divides $|C_G(g)|$ for each $x \in G \setminus Z(G)$. Hence $g \in M_p(G)$ for every $p \in \pi := \pi(G)$. If $|\pi| = 1$, then G is a p-group, which implies that G is nilpotent and $G = F(G)$. So we can assume that $|\pi| \geq 2$. We write $g = \prod_{q\in\pi} g_q$, where each g_q is a q-element and a power of g; this can be done as $\langle g \rangle$ is the product of its Sylow subgroups. We prove that $g_q \in O_q(G)$ for each $q \in \pi$. If $g_q \in Z(G)$, then $\langle g_q \rangle$ is a normal nilpotent q-subgroup of G, so $\langle g_q \rangle \leq O_q(G)$. If $g_q \notin Z(G)$, then by Lemma 3.1 $C_G(g) = \bigcap_{q\in\pi} C_G(g_q)$. Hence $C_G(g) = C_G(g_q)$ since $|C_G(g_q)|$ divides $|C_G(g)|$. Thus $g_q \in M_p(G)$ for all $p \in \pi$. Now Proposition 3.12 verifies that $g_q \in \bigcap_{p\neq q} O_{p'}(G) = O_q(G)$. Therefore $g \in \prod_{q\in\pi} O_q(G) = F(G)$. ■

For a group G, G is abelian if $|cs(G)| = 1$. If we have $|cs(G)| = 2$, then G is nilpotent [Itô, 1953]. In addition, if $|cs(G)| = 3$, then G is solvable [Itô, 1970]. Thus we have the following theorem in general we have the following remark.

Remark 3.14: If G is a group with $|cs(G)| \leq 3$, then G is solvable.

Now the following theorem is proved by Isaacs [Isaacs, 1970].

Theorem 3.15: Let N be a normal subgroup of a group G. Assume that $|x^G| = |y^G|$ for all $x, y \in G \setminus N$. Then one of the following occurs:

i) G/N is cyclic and G has an abelian Hall π-subgroup and a normal π-complement, where $\pi = \pi(G/N)$ is the set of the prime divisors of $|G/N|$;
ii) Every nonidentity element in G/N has prime order.

Theorem 3.16: Let G be a group such that $cs(G) = \{1, m, n\}$. If m divides n, then either $[G : Z(G)]$ is a prime power or there exists an abelian normal subgroup of prime index in G.

Proof 3.16: Assume that $[G : Z(G)]$ is not a prime power. To prove that there exists an abelian normal subgroup of prime index in G, we work by induction on $|G|$. By Remark 3.14 we have G is a solvable group. Let $Z := Z(G)$ and $F := F(G)$.

Let $a = \frac{|G|}{n}$ and $b = \frac{|G|}{m}$. Then, for every $g \in G \setminus Z$, $|C_G(g)|$ is either a or b. By assumption, we conclude that a divides b. We can assume that $a_r < |G|_r$ for every prime divisor r of $|G|$. If not, there exists a prime divisor r of the order of G such that $|G|_r = a_r = b_r$ and hence r does not divide either m or n. Hence by Lemma 3.3, we have $G = H \times R$, where R is the normal abelian Sylow r-subgroup of G. Now Lemma 3.1 implies that $cs(G) = cs(H)$. Since $Z(G) = Z(H) \times R$, we deduce that $[H : Z(H)] = [G : Z]$ is not a prime power. Because of $|H| < |G|$, by inductive hypothesis, there exists an abelian normal subgroup B of prime index in H. Thus $A = B \times R$ is an abelian normal subgroup of prime index in G as $[G : A] = [H : B]$.

Let $a(G) := \{g \in G \| C_G(g)| = a\}$ and $b(G) := \{g \in G \| C_G(g)| = b\}$. We proceed by the following series of steps:

i) $\emptyset \neq G \setminus F \subseteq a(G)$:

By the assumption, G is a disjoint union of the subsets Z, $a(G)$ and $b(G)$. As G is solvable and $m \mid n$, Proposition 3.13 implies that $b(G) \subseteq F$. Since $Z \leq F$ and G is the disjoint union of Z, $a(G)$ and $b(G)$, we have $G \setminus F \subseteq a(G)$. Finally we claim that $F < G$. In fact, if G is nilpotent, then $G = \prod_{i=1}^{h} P_i$, where $P_1, P_2, \ldots, P_h$ are the distinct Sylow subgroups of G and therefore we have the set $cs(G) = \{\prod k_i \,|k_i \in cs(P_i), i = 1,2, \ldots, h\}$. Since $cs(G) < 4$, it follows that G has just one noncentral Sylow subgroup, otherwise if $i \neq j$ and P_i, P_j are two distinct noncentral Sylow subgroup of G, then there exists $x \in P_i$ and $y \in P_j$ such that $l = |x^{P_i}| \neq 1$ and $k = |y^{P_j}| \neq 1$ so $l, k \in cs(G) \setminus \{1\}$ and $gcd(l, k) = 1$ which contradicts our assumption. Thus G has just one noncentral Sylow subgroup.

Hence $[G : Z]$ *is a prime power against our assumption. Thus* $F < G$ *and so* $\emptyset \neq G \setminus F \subseteq a(G)$.

ii) There exists $K \trianglelefteq G$ *such that* $F < K$ *and* K/F *is a* p*-group of exponent* p*, for a suitable prime divisor* p *of* $|G/F|$*. Further, if* $K < G$*, then* $[G : K] = q$*, with* q *is a prime,* $q \neq p$*, and* $C_G(x) \leq K$ *for every* $x \in K \setminus F$*:*

By part i), it is clear that $|g^G| = n$ *for every* $g \in G \setminus F$*. Now we can apply Theorem 3.15. First suppose that is* G/F *is cyclic and* G *has an abelian Hall* π*-subgroup and a normal* π*-complement, with* $\pi = \pi(G/F)$*. Then the Sylow* r*-subgroups of* G *are abelian for every prime divisor* r *of* $|G/F|$*. Let* x *be an* r*-element of* $G \setminus F$ *for such a prime* r*, so there exists an abelian Sylow* r*-subgroup of* G *like* R *such that* $x \in R$*. This implies that* $R \leq C_G(x)$*. Also by part i) we have* $a = |C_G(x)|$*. Thus* $a_r = |C_G(x)|_r = |R| = |G|_r$ *which is a contradiction (since we assumed that* $a_r < |G|_r$*). Hence by Theorem 3.15, we deduce that every nonidentity element of* G/F *has prime order. So by Proposition 3.8, there exists* $K/F \trianglelefteq G/F$ *such that* K/F *is a* p*-group of exponent* p *for some prime divisor* p *of* $|G/F|$*. Further, if* $K/F < G/F$*, then* G/F *is a Frobenius group with kernel* K/F *and complement of prime order* $q \neq p$*. Thus, if* $K < G$*, then* $[G : K] = q$ *and* $C_G(x) \leq K$ *for every* $x \in K \setminus F$ *since* $C_{G/F}(xF) \leq K/F$*.*

iii) We write $F = P \times L$*, where* P *is a Sylow* p*-subgroup of* F *and* L *is a* p*-complement of* F*. Then* $a_p \leq |P|$*:*

Note that F *is nilpotent, so the direct product* $F = P \times L$ *has meaning. Let* K *be the normal subgroup as in part ii). Since* $F(K)$ *is characteristic in* K*, by normality of* K *in* G*, we deduce that* $F(K) \trianglelefteq G$ *so* $F(K) \leq F$*. Therefore* $F = F(K)$*. As* K/F *is a* p*-group of exponent* p *and* L *a* p*-complement of* F*, we conclude that* L *is the normal* p*-complement of* K*. Hence by part ii) of Lemma 3.5, there exists an element* $g \in L$ *such that* $C_K(g) \leq F$*. Now* $[G : K] = 1$ *or* $[G : K] = q$*, where* $q \neq p$ *is a prime. Since* L *is a normal* p*-complement of* K*, we conclude that for any* $Q \in Syl_p(K)$*,* $Q \leq C_K(g) \leq F$*, so* Q *is a Sylow subgroup of* F*. Hence* $|Q| = |P|$*. Since* $g \in a(G) \cup b(G)$ *and* $a \mid b$*, we conclude that* $|C_G(g)|_p = |P| \geq a_p$*.*

iv) $\emptyset \neq P \setminus Z_p \subseteq b(G)$*, where* $Z_p = Z \cap P$*:*

Since K/F is a nontrivial p-group, we can fix a p-element $x \in K \setminus F$. First we show that $P \setminus Z_p \neq \emptyset$. If $P \setminus Z_p = \emptyset$, then $P \leq Z$. This implies that $\langle P, x\rangle$ is a p-subgroup of $C_G(x)$. However by part i), we know that $|C_G(x)| = a$. On the other hand, by part iii) we have $|\langle P, x\rangle| = |\langle P, x\rangle|_p \leq |C_G(x)|_p = a_p \leq |P|$. This implies that $\langle P, x\rangle = P$ which contradicts our assumption. Therefore $P \setminus Z_p \neq \emptyset$. Let x_0 be an arbitray element of $P \setminus Z_p$. Assume that $x_0 \in a(G)$. As $F = P \times L$ and $|C_G(x_0)| = a$, we conclude that $L \leq C_G(x_0)$ and $|L| = |L|_{p'} \leq a_{p'}$. As $x \in K \setminus F$, part i) verifies that $a_{p'} = |C_G(x)|_{p'}$ and by part ii) we have $C_G(x) \leq K$. So $C_K(x) = K \cap C_G(x) = C_G(x)$. Now we deduce that $a_{p'} = |C_G(x)|_{p'} = |C_K(x)|_{p'}$. Since L is the normal p-complement of K, we obtain $|L| \leq a_{p'} = |C_K(x)|_{p'} = |C_K(x) \cap L| \leq |L|$. Thus $C_K(x) \cap L = L$, which implies that x centralizes L. Now part i) of Lemma 3.5 verifies that $C_K(L) \leq F$, so $x \in F$ which is a contradiction. Therefore $P \setminus Z_p \subseteq b(G)$.

v) $b_{p'} = |G|_{p'}$ and then m is a power of p:

By part iv), we know that $P \setminus Z_p \neq \emptyset$, so there exists a nontrivial element $x \in P \setminus Z_p$. Since L centralizes P, we deduce that $L \leq C_G(x)$. Thus $|L|$ divides $|C_G(x)| = b$. Since $gcd(p, |L|) = 1$, we conclude that $|L| = |L|_{p'}$ divides $b_{p'}$. Assume that $b_{p'} < |G|_{p'}$. Let K be the normal subgroup in part ii). Then either $[G : K] = q$, where q is a prime and $q \neq p$, or $G = K$. First suppose that $[G : K] = q$. Since $[K : L] = [K : F][F : L]$ is a power of p, we have $|G|_{p'} = q|L|$. On the other hand, if $G = K$, then $|G|_{p'} = |L|$ so $b_{p'} = |C_G(x)|_{p'}$ divides $|G|_{p'} = |L|$. Hence in both cases $b_{p'} = |C_G(x)|_{p'}$ divides $|L|$. So we deduce that $|L| = b_{p'}$. Therefore, for every $x_0 \in P \setminus Z_p$, L is a p-complement of $C_G(x_0)$. If $G = K$, then clearly $C_G(x_0) \leq K = G$. So assume that $[G : K] = q$. Then $[C_G(x_0)K : K]$ is either 1 or q. If it is one, then we have $C_G(x_0) \leq K$. Now assume that $[C_G(x_0)K : K] = q$. This implies that $C_G(x_0)K = G$. Since $L \leq C_G(x_0)$ is a p-complement and $x_0 \in b(G)$, we can easily see that $q = 1$ which is impossible. So all together, we have $C_G(x_0) \leq K$. Now consider $u \in G \setminus K$. By part ii) it is clear that $C_K(u) \leq F$. Also by part i) we have $u \in a(G)$ since $u \in G \setminus K$. Since $Syl_p(G) = Syl_p(K)$, we deduce that $|C_G(u)|_p = |C_K(u)|_p = |C_K(u) \cap P|$ as $C_K(u) \leq F$. On the other hand, if $|C_P(u)| \neq |Z_p|$, then there *exists* $x \in P \setminus Z_p$ *such that* $x \in C_P(u)$. *This implies that* $u \in C_G(x) \leq K$ *which contradicts our hypothesis, so* $|C_P(u)| = |Z_p|$ *and we have* $a_p = |C_G(u)|_p = |C_P(u)| = |Z_p|$. *Finally, choose a* p*-element* $x \in K \setminus F$. *Then* $x \in a(G)$ *and* $a_p = |C_G(x)|_p \geq |\langle Z_p, x\rangle| > |Z_p|$, *which is a contradiction. Thus* $b_{p'} = |G|_{p'}$.

vi) For every $y \in L \setminus Z(L)$, *we have* $C_K(y) \leq F$ *and* $y \in a(G)$:

Let $y \in L \setminus Z(L)$. *Clearly,* $y \notin Z$. *If* $y \in b(G)$, *then by part v), it is clear that* y *centralizes some* p*-complement* H *of* G, *and hence* y *centralizes* $L \leq H$, *which contradicts our hypothesis that* $y \notin Z(L)$. *Thus we have* $y \in a(G)$. *As* $P \leq C_G(y)$, *then part iii) yields that* P *is a Sylow* p*-subgroup of* $C_G(y)$. *Since* K/F *is a* p*-group and* $C_K(y)F/F$ *is a subgroup of* K/F *such that*

$$\left|\frac{C_K(y)F}{F}\right|_p = \frac{|C_K(y)|_p}{|C_F(y)|_p} = \frac{|P|}{|P|} = 1 \qquad (3.10)$$

we conclude that $C_K(y)F/F$ *is the trivial subgroup, so* $C_K(y) \leq F$.

vii) L *is abelian:*

Assume that $Z(L) < L$. *Since* $F = F(K)$, *by part ii) and part iii) we conclude that* K/F *is a* p*-group of exponent* p, L *is a normal* p*-complement of* K *such that* $C_K(y) \leq F$ *for each* $y \in L \setminus Z(L)$. *Now Lemma 3.5 implies that* $|K/F| = p$.

We have $L \setminus Z(L)$ *is nonempty. Also by part vi), we conclude that* $L \setminus Z(L) \subseteq a(G)$. *Therefore part iii) (or* $a_p < |G|_p$*) yields that* $a_p = |P|$. *Since* $a \mid b$, *we have* $a_p b_{p'} = |P||G|_{p'} = \frac{|G|}{p}$ *divides* b. *As* $b < |G|$ *we deduce that* $b_p = a_p = |P|$.

Let $Z_{p'} = Z \cap L$. *Assume that* $w \in L \setminus Z_{p'}$. *Then we have* $|C_G(w)|_p = |P|$. *Since* P *centralizes* w *and* K/F *is a* p*-group, we deduce that* $C_K(w) \leq F$.

Now let $x \in K \setminus F$. *We claim that* $C_L(x) = Z_{p'}$, *otherwise there exists an element* $a \in C_L(x)$ *such that* $a \in L \setminus Z_{p'}$. *Then by the previous paragraph,* $C_K(a) \leq F$ *and so* $x \in F$ *which is impossible. By part ii) we have* $C_G(x) \leq K$ *since* $x \in K \setminus F$. *As* L *is the normal* p*-complement of* K, *we obtain* $|C_G(x)|_{p'} = |C_K(x)|_{p'} = |C_L(x)| = |Z_{p'}|$. *As, by part i),* $x \in a(G)$, *we conclude that* $a_{p'} = |Z_{p'}|$. *Finally, consider* $y \in L \setminus Z(L)$. *By part vi), we can observe that* $y \in a(G)$. *Since* $y \notin Z_{p'}$, *we obtain that* $a_{p'} = |C_G(y)|_{p'} \geq |\langle Z_{p'}, y\rangle| > |Z_{p'}|$ *which is a* *contradiction.*

viii) $F \cap a(G) = \emptyset$*:*

Assume that there exists a $g \in F \cap a(G)$*. By part vii),* L *is abelian. As* $F = P \times L$*, we have* $L \leq Z(F)$*. Since* $Z(F) \leq C_G(g)$*, we deduce that* $|L|$ *divides* $a_{p'}$*. Let* $x \in K \setminus F$*. Now by part i) we have* $x \in a(G)$ *and by part ii) we have* $C_G(x) \leq K$*. Therefore* $a_{p'} = |C_K(x)|_{p'} = |C_L(x)|$*, because* L *is the normal* p*-complement of* K*. It follows that* $x \in C_K(L)$*, which contradicts part i) of Lemma 3.5.*

ix) Conclusion: F *is abelian and* $[G : F] = p$*:*

By part i) and part viii), we know that $F \setminus Z \subseteq b(G)$*. It should be mentioned that* L *is not contained in* Z*, otherwise there exists an element* $g \in L$ *such that* $C_K(g) = K$*. Now by part ii) of Lemma 3.5, we have* $C_K(g) \leq F$*, so* $K = F$ *which is a contradiction. By part iv),* P *is also not contained in* Z*. Let* $x \in P \setminus Z$ *and* $y \in L \setminus Z$*. Then* $C = C_G(xy) = C_G(x) \cap C_G(y)$ *since* x *and* y *commute with* $gcd(|x|, |y|) = 1$*. As* $x, y \in b(G)$*,* xy *must be in* $F \setminus Z$*. This implies that* $xy \in b(G)$*, so* $C = C_G(x) = C_G(y)$*. Let us fix* x*, then we see that* $C = C_G(y)$ *for every* $y \in L \setminus Z$*. Similarly, by keeping* y *fixed, we get* $C = C_G(x)$ *for every* $x \in P \setminus Z$*. It follows that* $C = C_G(g)$ *for every element* $g \in F \setminus Z$ *because* $F = P \times L$*. Furthermore,* $P \leq C$ *as* $P \leq C_G(y)$ *for every* $y \in L \setminus Z$ *and* $L \leq C$ *as* $L \leq C_G(x)$ *for every* $x \in P \setminus Z$*. So* $F \leq C = C_G(F)$ *since* $C_G(F) = \bigcap_{g \in F} C_G(g)$*.*

Conversely, $C \leq F$ *because* G *is solvable. Thus we have* $F = C = C_G(g)$ *for every* $g \in F \setminus Z$*. In particular, this implies that* F *is abelian.*

Furthermore, for any element $y \in F \setminus Z$*, we have* $|C_G(y)| = b = |F|$*. Thus by part v) we see that* $|G|_{p'} = |F|_{p'}$*. Hence* $G = K$ *otherwise* $[G : K] = q$*,* $q \neq p$ *prime. Finally, by part iii) of Lemma 3.5, we conclude that* $[G : F] = p$*. ■*

Definition 3.1: A nonabelian group G *is an* F*-group if, for every* $x, y \in G \setminus Z(G)$*, we have that* $C_G(x) \leq C_G(y)$ *implies* $C_G(x) = C_G(y)$*.*

Definition 3.2: A nonabelian group G *is a* CA*-group if all centralizers of noncentral elements are abelian. Clearly,* CA*-groups are* F*-groups.*

Theorem 3.17: Let G be a finite group of conjugate rank 2. Then G is either an F-group or the direct product of an abelian group and a group of prime power order.

Proof 3.17: Let G be a finite group of conjugate rank 2 and assume that G is not an F-group. Then there exist elements $x, y \in G \setminus Z(G)$ such that $C_G(y) < C_G(x)$. Let $m = |x^G|$ and $n = |y^G|$ be the sizes of the conjugacy classes of x and y in G, respectively. Then m divides n, and since G has conjugate rank 2, we can apply Theorem 3.16. Suppose that there exists an abelian normal subgroup A of prime index in G. As $A \leq AZ(G) \leq G$ and $[G : A] = p$, for a prime p, either $Z(G) < A$ or $G = AZ(G)$. If $G = AZ$, then G is abelian which contradicts our hypothesis about rank of G. Thus we have $Z(G) < A$. We claim that for any $g \in A \setminus Z(G)$, we have $C_G(g) = A$. As $A \leq C_G(g) \leq G$ and $[G : A] = p$, we have either $C_G(g) = A$, or $C_G(g) = G$ which is impossible since g is noncentral. Thus for every $h \in G \setminus A$, we have $C_G(h) = Z(G)\langle h \rangle$. Hence G is a CA-group, and in particular G is an F-group, which contradicts our assumption. Thus, $[G : Z(G)]$ is a power of some prime p by Theorem 3.16 and therefore we get $G = P \times A$ where P is a Sylow p-subgroup of G and A is a subgroup of $Z(G)$. ■

The structure of the groups with conjugate rank 2 is determined by the following theorem. Let p denote a suitable prime number.

Theorem 3.18: A finite group G has conjugate rank 2 if and only if, up to an abelian factor, either of the following cases hold:

- *G is a p-group of conjugate rank 2; or*
- *$G = KL$ with $K \trianglelefteq G$, $gcd(|K|, |L|) = 1$ and one of the following occurs:*
  i) both K and L are abelian, $Z(G) \leq L$ and $G/Z(G)$ is a Frobenius group;
  ii) K is abelian, L is nonabelian p-group, $M = O_p(G)$ is an abelian subgroup of index p in L and G/M is a Frobenius group;
  iii) K is a p-group of conjugate rank 1, L is abelian, $Z(K) = Z(G) \cap K$ and $G/Z(G)$ is a Frobenius group.

Now we will prove a lemma which helps us to prove the above theorem. Note that Dedekind's Modular Law states that if H, K and U are subgroups of G such that $H \leq U \leq G$, then $HK \cap U = H(K \cap U)$.

Lemma 3.19: According to the notation of Theorem 3.18, we have the following sets of conjugacy class sizes:

i) $cs(G) = \{1, |L/Z(G)|, |K|\}$;
ii) $cs(G) = \{1, p, p^a|K|\}$, *where* $p^a = [M : Z(L)]$;
iii) $cs(G) = \{1, p^a, p^b|L/Z(G)|\}$, *where* $p^a = [K : Z(K)]$ *and* $cs(K) = \{1, p^b\}$.

Proof 3.19: Let $Z = Z(G)$.

i) K *is abelian, so* $KZ \leq C_G(x)$ *for all* $x \in KZ \setminus Z$. G/Z *is a Frobenius group, clearly* KZ/Z *and* L/Z *are the Frobenius kernel and the Frobenius complement of* G/Z, *respectively. Hence if* $x \in KZ \setminus Z$, *then* $\bar{x} \in \overline{KZ}$, *so* $C_{\bar{G}}(\bar{x}) \subseteq \overline{KZ}$. *Assume that* $KZ < C_G(x)$, *then there exists an element* $a \in C_G(x) \setminus KZ$, *so* $\bar{a}$ *and* $\bar{x}$ *are nontrivial such that* $\bar{a} \in C_{\bar{G}}(\bar{x}) \subseteq \overline{KZ}$.*This verifies that* $a \in KZ$ *which is a contradiction. Thus in this case, we have* $C_G(x) = KZ$ *for all* $x \in KZ \setminus Z$. *If* $x \in G \setminus KZ$, *then* xZ *is a nonidentity element of some Frobenius complement of* G/Z. *Hence* $x \in L^g Z = L^g$ *for some* $g \in G$, *and* $C_G(x) = L^g$ *because* L^g *is abelian and* G/Z *is a Frobenius group with complement* L^g/Z. *Thus we have the following three cases:*

- *If* $x \in Z$, *then* $|x^G| = 1$;
- *If* $x \in KZ \setminus Z$, *then* $|x^G| = \frac{|G|}{|KZ|} = \frac{|K||L|}{|K||Z|} = \frac{|L|}{|Z|}$;
- *If* $x \in G \setminus KZ$, *then* $|x^G| = \frac{|G|}{|L^g|} = |K|$

So we get $cs(G) = \{1, |L/Z|, |K|\}$.

ii) Let $M = O_p(G)$, *then* M *centralizes the normal* p*-complement* K. G/M *is a Frobenius group with the Fobenius kernel* KM/M *and the Frobenius complement* L/M, *hence if there exists a* $z \in Z \setminus L$, *then* $\bar{z} \in C_{\bar{G}}(\bar{l}) = \bar{L}$ *where* $\bar{l}$ *is a* *nonidentity element of* $\bar{L}$. *So* $z \in L$, *a contradiction. Thus* $Z \leq L$, *in particular* $Z \leq M$ *by the definition of* M. *Since* $Z \leq Z(L) \leq M$ *and* M *centralizes* K, *we have* $Z(L) = Z$. *Let* $A = KM$. *Then* A *is an abelian normal subgroup of index* p *in* G. *If* $x \in A \setminus Z$, *then* $A \leq C_G(x)$ *and if there exists an element* $a \in C_G(x) \setminus A$, *then* $A \leq \langle a \rangle A \leq G$ *so*

$$[G : A] = [G : \langle a \rangle A][\langle a \rangle A : A] = [\langle a \rangle A : A]. \tag{3.11}$$

This means that $x \in Z$, *so* $C_G(x) = A$. *If* $x \in G \setminus A$, *then* x *centralizes no element in* $A \setminus Z$ *otherwise as* $[G : A] = p$, *we can see that* $a \in Z$, *and* $C_G(x) = Z\langle x \rangle$, *with* $x^p \in Z$ *as* $[G : A] = p$. *Hence*

$$|x^G| = \frac{|G|}{|Z\langle x \rangle|} = \frac{|G|}{p|Z|} = \frac{|K||L|}{|L/M||Z|} = |K|\frac{|M|}{|Z|}. \tag{3.12}$$

In this case $cs(G) = \{1, p, p^a|K|\}$, *where* $p^a = [M : Z(L)]$.

iii) We have the Frobenius group G/Z *with the Frobenius kernel* KZ/Z *and the Frobenius complement* LZ/Z. *If* $x \in G \setminus KZ$, *then* x *is a nonidentity element of some Frobenius complement of* G/Z. *Hence* $x \in L^gZ$, *for some* $g \in G$ *and* $C_G(x) = L^gZ$ *because* L *is abelian. If* $x \in KZ \setminus Z$, *then* xZ *is a nonidentity element of the Frobenius kernel* KZ/Z. *It follows that* $C_G(x) \leq KZ$ *since* $C_{\bar{G}}(\bar{x}) \leq \overline{KZ}$. *There exist* $y \in K$ *and* $z \in Z$ *such that* $x = yz$. *We claim that* $C_G(x) = C_K(y)Z$. *It is clear that* $C_K(y)Z \leq C_G(x)$. *If* $a \in C_G(x)$, *then* $yz = x = a^{-1}xa = a^{-1}yaz$. *This implies that* $a \in C_G(y)$, *thus* $a \in KZ \cap C_G(y) = C_K(y)Z$ *by Dedekind's Modular Law. Therefore* $C_G(x) = C_K(y)Z$.

In order to find the conjugacy class size of x, *we need to prove* $Z = (Z \cap K)(Z \cap L) = Z(K)(Z \cap L)$. *Assume that* $Z \nleq Z(K)(Z \cap L) = Z \cap Z(K)L$. *This means that* $Z \nleq Z(K)L$. *Hence there exists a* $z \in Z \setminus Z(K)L$. *As* $G = KL$ *we have* $z = kl$ *for some* $k \in K$, $l \in L$, *so* $l = k^{-1}z \in KZ$. $l \notin Z$ *as* $z \notin Z(K)L$. *Thus* $l \in KZ \setminus Z$ *and so* $C_G(l) \leq KZ$. *As* L *is abelian,* $L \leq C_G(l)$, *hence* $LZ \leq KZ$, *which is a contradiction. Consequently,* $Z = Z(K)(Z \cap L)$. *It follows that if* $x \in KZ \setminus Z$, *then*

$$|x^G| = \frac{|G|}{|C_K(y)Z|} = \frac{|K||L||Z \cap C_K(y)|}{|C_K(y)||Z|} = \frac{p^b|L||Z(K)|}{|Z(K)||Z \cap L|} = \frac{p^b|L|}{|Z \cap L|} \quad (3.13)$$

where p^b *is the size of conjugacy class of any noncentral element of* K. *If* $x \in G \setminus KZ$, *then*

$$|x^G| = \frac{|G|}{|L^gZ|} = \frac{|K||L||Z \cap L|}{|L||Z|} = [K : Z(K)] = p^a \quad (3.14)$$

since K *is a* p*-group. So we obtained that* $cs(G) = \{1, p^a, p^b|L/L \cap Z|\}$, *where* $p^a = [K : Z(K)]$ *and* $cs(K) = \{1, p^b\}$. ■

It should be mentioned that, in part iii) of Lemma 3.19, we have $p^a > p^b$ since for every $x \in K$ such that $x \notin Z$, clearly $C_K(x) > Z$. Further, LZ/Z is a Frobenius complement, and hence $L/L \cap Z \neq 1$. Thus none of the nontrivial class sizes of G divides any other.

The following theorem which shows the classification of F-groups is form [Rebmann, 1971]. It is easy to see that the F-groups in i) - v) are solvable while the F-groups in vi) - vii) are nonsolvable groups.

Theorem 3.20: Let G *be a nonabelian group. Then* G *is an* F*-group if and only if it is one of the following types:*

i) $G = P \times A$ *where* P *is an* F*-group of prime power order and* A *is abelian.*

ii) G *has a normal abelian subgroup of prime index.*

iii) $G/Z(G)$ *is a Frobenius group with Frobenius kernel* $K/Z(G)$ *and Frobenius complement* $L/Z(G)$ *with* K *and* L *are abelian.*

iv) $G/Z(G)$ *is a Frobenius group with Frobenius kernel* $K/Z(G)$ *and Frobenius complement* $L/Z(G)$ *where* L *is abelian,* $Z(K) = Z(G)$, $K/Z(G)$ *has prime power order and* K *is an* F*-group.*

v) $G/Z(G) \cong S_4$ *and if* $V/Z(G)$ *is the Klein* 4*-group in* $G/Z(G)$, *then* V *is not abelian.*

vi) $G/Z(G) \cong PSL_2(p^n)$ *or* $PGL_2(p^n)$, $G' \cong SL_2(p^n)$, *where* G' *is the derived* *subgroup of G, p a prime, $p^n > 3$.*

vii) $G/Z(G) \cong PSL_2(9)$ ($\cong A_6$) or $PGL_2(9)$, and G' is isomorphic to the Schur cover of $PSL_2(9)$.

CA-groups are classified by Schmidt [Schmidt, 1994]. Now we write a part of this classifacition.

Theorem 3.21: Let G be a finite group. If $G/Z(G) \cong S_4$ and V is not abelian if $V/Z(G)$ is the Klein 4-group, then G is a CA-group and $cs(G/Z(G)) = \{1, 6, 8, 12\}$.

Now we can give the proof of the main Theorem 3.18.

Proof 3.18: Assume that G is a group of conjugate rank 2. By Theorem 3.17, it is clear that G is either an F-group or it is the product of an abelian group and a group of prime power order, that is a group of type first case. By part i) of Lemma 3.1, we can assume G has no nontrivial abelian factor. Let $Z = Z(G)$ and suppose G is an F-group. By applying Theorem 3.20, and we have the following types of groups:

i) G is a group of prime power order. Thus, we again have the first case.

ii) G is nonabelian and has an abelian normal subgroup B of prime index p. If G is nilpotent, since we assumed that G has no nontrivial abelian direct factor, then we have the first case. Otherwise, let L be a Sylow p-subgroup of G, let K be a p-complement of B. Let $M = L \cap B$. Then K is an abelian normal subgroup of G. Since $|K|$ and $|L|$ are relatively prime, by Theorem 1.5 we conclude that $K = [K, L] \times C_K(L)$. Now we have:

$$G = BL = (K \times M)L = \big[\big([K, L] \times C_K(L)\big) \times M\big]L = N \times C_K(L) \qquad (3.15)$$

where $N = ([K, L] \times M)L$. Thus $C_K(L)$ is an abelian direct factor of G. Hence by the assumption, we have $C_K(L) = 1$. Since $|L/M| = p$, it follows that G/M is a Frobenius group with L/M as its Frobenius complement and KM/M as the Frobenius kernel. Since $M, K \leq B$ and B is abelian, we have M centralizes K. Also

$M = O_p(G)$ since $p = [L : M] = [L : O_p(G)][O_p(G) : M]$ and L/M is the Frobenius complement of G/M. If L is nonabelian, then we have case ii). If L is abelian, then $M \leq Z$ as M centralizes K. As $C_K(L) = 1$, it follows that $M = Z(G)$ otherwise if $M < Z$, then either $L \leq Z$ or $K \cap Z \neq 1$ which contradicts the fact that $C_K(L) = 1$. Hence G is a group of type i).

iii) G/Z is a Frobenius group with kernel K_0/Z and complement L/Z, with K_0 and L abelian groups. Let $\pi = \pi(L)$ and let K be the π-complement of K_0. It is clear that $K \trianglelefteq G$. Let A be a π-Hall subgroup of K_0, so $K_0 = KA$ and we have:

$$\frac{|G|}{|K|} = \frac{|G|}{|Z|}\frac{|Z|}{|K|} = \frac{|K_0|}{|Z|}\frac{|L|}{|Z|}\frac{|Z|}{|K|} = \frac{|K|\,|A|\,|L|}{|Z|}\frac{1}{|K|} = \frac{|A|\,|L|}{|Z|}. \tag{3.16}$$

Thus K is a normal π-complement of G. Since $|K|$ and $|L|$ are relatively prime, by Theorem 1.5 we conclude that $K = [K, L] \times C_K(L)$. It is clear that $K \cap Z \leq C_K(L)$. Assume that there exists an element $x \in C_K(L) \setminus (K \cap Z)$ then $x \notin Z$ and so xZ is a nontrivial element in K_0/Z. This implies that $C_{\bar{G}}(\bar{x}) \leq \overline{K_0}$, so $\bar{L} \leq \overline{K_0}$, which is a contradiction. Therefore $C_K(L) = K \cap Z$. Similar to part ii), we can see that $C_K(L)$ is an abelian direct factor of G. Hence Z is a π-group and $Z \leq L$. Therefore $G = KL$, with $K \trianglelefteq G$ and $gcd(|L|, |K|) = 1$. Thus, we get case i).

iv) G/Z is a Frobenius group with kernel K_0/Z and complement L_0/Z where L_0 is abelian, K_0 is an F-group such that $Z(K_0) = Z$ and K_0/Z is a p-group, for a prime p.

Let K be the Sylow p-subgroup of K_0. Then $K_0 = K \times Z_0$, with $Z_0 \leq Z$. As K is a characteristic subgroup of K_0 and $K_0 \trianglelefteq G$, we have $K \trianglelefteq G$. Now we have $Z(K) = Z(K_0) \cap K = Z \cap K$. Let L be p-complement of L_0. As

$$\frac{G}{Z} \cong \frac{K_0}{Z} \rtimes \frac{L_0}{Z} \cong \frac{K}{Z(K)} \rtimes \frac{L_0}{Z} \tag{3.17}$$

is a Frobenius group, it follows that p does not divide $[L_0 : Z]$. Hence, K is a Sylow p-subgroup of G and $G = KL$ because

$$|G|_{p'} = \frac{|K_0|_{p'}|L_0|_{p'}}{|Z|_{p'}} = |L_0|_{p'} = |L|. \tag{3.18}$$

Similar to the argument of the proof of case iii) of Lemma 3.19, we can see that $cs(G) = \{1,\ [K : Z(K)],\ k[L : L \cap Z] |\ k \in cs(K), k \neq 1\}$*. Since* G *has conjugate rank* 2*, it follows that* K *has conjugate rank* 1*.*

v) $G/Z \cong S_4$ *and, if* V/Z *is the Klein* 4*-subgroup of* G/Z*, then* V *is nonabelian. By Theorem 3.21, this case will not occur when* G *is assumed to be a group of conjugate rank 2.*

Conversely, if G *is one of the groups which are listed in Theorem 3.20, then* G *has conjugate rank* 2 *by Lemma 3.19.* ■

Corollary 3.22: Let $cs(G) = \{1, m, n\}$*. If* m *and* n *are not coprime, then either* m *or* n *is a prime power.*

Proof 3.22: Let $cs(G) = \{1, m, n\}$ *with* $gcd(m, n) \neq 1$*. If* $m = n$*, then we conclude that* G *is nilpotent. So* G *can be written as a direct product of the normal Sylow subgroups and* $cs(G) = \{1, m\}$ *where* m *is a prime power. If* $m \neq n$*, then* G *is a group of conjugate rank* 2 *hence* G *is one of the groups described in first part, i), ii) or iii) of the second case in Theorem 3.18. In case first part, both* m *and* n *are powers of the some prime. By applying Lemma 3.19 we can observe that* G *cannot be a group as in case i), because* m *and* n *are not coprime, and that both in cases ii) and iii) either* m *or* n *is a prime power.* ■

Theorem 3.23: (Main Theorem in [Ishikawa, 2002]) Let G *be a finite* p*-group for a prime* p *such that* $cs(G) = \{1, p^n\}$ $(n \geq 1)$*. Then* G' *is an elementary abelian* p*-group.*

Note that $dl(G)$ denotes the derived length of the solvable group G.

Corollary 3.24: If $|cs(G)| = 3$ *and* G *is nilpotent, then* $dl(G) \leq 3$*.*

Proof 3.24: First note that G *is solvable by Remark 3.14. Let* G *be a nonnilpotent group of conjugate rank* 2*. Then by Theorem 3.18 it is clear that* G *is, up to an abelian direct factor, one of the groups described in i), ii) or iii) of the second case. Hence,* $dl(G) \leq dl(K) + dl(L)$*. In case i) we have* $dl(K) = dl(L) = 1$ *since* K *and* L *are abelian. In case ii) we have* $dl(K) = 1$ *and* $dl(L) = 2$*, since* L *has an abelian normal subgroup with cyclic factor group. In case iii) we have* $dl(L) = 1$*, and* $dl(K) = 2$ *by Theorem 3.23.* ■

4. CONNECTIONS BETWEEN PRIME DIVISORS of CONJUGACY CLASSES and PRIME DIVISORS of $|G|$

In this chapter we follow a paper of Ferguson [Ferguson, 1991].

Assume that n is a positive integer, then $n = \prod_{i=1}^{k} {p_i}^{a_i}$ is the factorization of n into distinct prime powers. Let $w(n) := k$. For a finite group G, let $\alpha(G) = \max\{w(|x^G|) | x \in G\}$ and $\rho(G) = \{p | p\ is\ a\ prime\ and\ p \mid |x^G|, for\ some\ x \in G\}$.

Proposition 4.1: A prime p divides $|G/Z(G)|$ if and only if $p \mid |x^G|$ for some $x \in G$.

Proof 4.1: G is a finite group and $Z(G) \leq C_G(x)$ for each $x \in G$, therefore $[G : Z(G)] = [G : C_G(x)][C_G(x) : Z(G)]$. If $p \mid |x^G|$, then we have $p \mid |G/Z(G)|$.

Conversely, assume that p divides $|G/Z(G)|$ but p does not divide $|x^G|$ for all $x \in G$. Let $P \in Syl_p(G)$, then $P \leq C_G(x)$ for some $x \in G$, thus $G = \bigcup_{g\in G} C_G(P^g) = \bigcup_{g\in G} C_G(P)^g$. By using Theorem 3.2, we have:

$$|G| = \left|\bigcup_{g\in G} C_G(P)^g\right| \leq |G| - [G : C_G(P)] + 1. \tag{4.1}$$

So $[G : C_G(P)] = 1$. Therefore $P \leq Z(G)$, which contradicts our hypothesis. ■

Lemma 4.2: Assume that G is a solvable group and r is a prime divisor of $|G|$.

i) If g is an r'-element which normalizes a nontrivial r-subgroup R, then either $r \mid |g^G|$ or $[R, g] = 1$.
ii) If R is a minimal normal r-subgroup of G and v is a prime such that $O_v(G/C_G(R)) \neq 1$, then $v \mid |x^G|$ for all $x \in R^\#$.
iii) If H, K are subgroups such that $R \leq C_G(H) \cup C_G(K)$, where R is an r-group, then $R \leq C_G(H)$ or $R \leq C_G(K)$.

Proof 4.2: Let G be a solvable group and r be a prime divisor of $|G|$.

i) By Frattini Argument we have $G = N_G(R_1)\, O_{r'}(G)$, where R_1 is a Sylow

r-subgroup of $O_{r',r}(G)$. Since $\frac{O_{r',r}(G)}{O_{r'}(G)} = O_r(G/O_{r'}(G))$ is an r-group, we conclude that $O_{r'}(G)$ is a Hall r'-subgroup of $O_{r',r}(G)$. Since R_1 is a Sylow r-subgroup of $O_{r',r}(G)$, we have $O_{r',r}(G) = R_1 O_{r'}(G)$. As G is solvable, Theorem 1.15 İmplies that $C_G(R_1) \leq R_1 O_{r'}(G)$. Let $\bar{G} := G/O_{r'}(G)$ and $\bar{A}$ denote the image of a set A in G. If $\bar{g} = 1$, then $g \in O_{r'}(G)$ and $[g, R] = 1$ since g normalizes R and $O_{r'}(G) \cap R = 1$. If $\bar{g} \neq 1$, then $\bar{g} \in N_{\bar{G}}(\overline{R_1}) \setminus C_{\bar{G}}(\overline{R_1})$ as $G = N_G(R_1)O_{r'}(G)$. Therefore, $\overline{R_1} \trianglelefteq \bar{G}$ yields $\bar{S} \nleq C_{\bar{G}}(\bar{g})$ for any $S \in Syl_r(G)$. Hence, $r \mid |\bar{g}^{\bar{G}}|$ yields $r \mid |g^G|$ by Lemma 3.1.

ii) Let $V \in Syl_v(K)$ where $K/C_G(R) = O_v(G/C_G(R))$, then we have $\frac{VC_G(R)}{C_G(R)} \in Syl_v\left(\frac{K}{C_G(R)}\right) = \frac{K}{C_G(R)}$, therefore $K = VC_G(R)$ and by Frattini Argument we can see that $G = N_G(V)C_G(R)$. It follows that $C_R(V) \trianglelefteq G$ since $xy \in G$ where $x \in N_G(V)$ and $y \in C_G(R)$, then $C_R(V)^{xy} = C_R(V^x)^y = C_R(V)$. If $C_R(V) = R$, then $V \leq C_G(R)$ hence $K = C_G(R)$ and so $O_v(G/C_G(R)) = 1$, a contradiction. Thus $C_R(V) < R$ and the minimality of R yields $C_R(V) = 1$. Therefore, $v \neq r$ otherwise $R \leq Z(V)$ and so $v \mid |x^G|$ for all $x \in R^\#$.

iii) For any group G if $G = A \cup B$, where A and B are subgroups of G, it is well-known that $G \subseteq A$ or $G \subseteq B$. We have $R \leq C_G(H) \cup C_G(K)$, then we can say that $R = (C_G(H) \cap R) \cup (C_G(K) \cap R)$, it follows $R \leq C_G(H)$ or $R \leq C_G(K)$. ■

Chillag and Herzog have shown that for a finite solvable group G, we have $|\rho(G)| \leq 2$ if $\alpha(G) = 1$ [Chillag and Herzog, 1990]. The following theorem is the main theorem of this section.

Theorem 4.3: Assume G is a finite solvable group, then $|\rho(G)| \leq 4$ if $\alpha(G) = 2$.

By Proposition 4.1 we have the following corollary.

Corollary 4.4: Assume G is a finite solvable group and $|x^G|$ is divisible by at most two distinct primes for all $x \in G$, then $|G/Z(G)|$ has at most four distinct prime divisors.

Note that in Lemma 3.1 we saw that if $x, y \in G$ commute and $gcd(|x|, |y|) = 1$, then $C_G(xy) = C_G(x) \cap C_G(y)$. So $|x^G|$ and $|y^G|$ divide $|(xy)^G|$.

We will say G satisfies Hypothesis A, if G is a counterexample of minimal order to Theorem 4.3.

Lemma 4.5: Assume G satisfies Hypothesis A, then $Z(G) = 1$.

Proof 4.5: Suppose $p \mid |Z(G)|$ and let P_1 be a nontrivial minimal normal subgroup of $Z(G)$. Let $H \geq P_1$ be the subgroup of G such that $H/P_1 = Z(G/P_1)$. Since $|G/P_1| < |G|$ and G/P_1 satisfies the hypothesis of Theorem 4.3, G/H is divisible by at most four primes. Let g be a p'-element of H, then $\langle g \rangle \trianglelefteq G$ since $\langle g \rangle P_1 \trianglelefteq G$ and $P_1 \leq Z(G)$. Now $[g, G] \leq \langle g \rangle \cap P_1$, as $H/P_1 = Z(G/P_1)$, implies that $g \in Z(G)_{p'}$. As we saw if $g \in H_{p'}$, then $g \in Z(G)_{p'}$, so $H_{p'} \leq Z(G)_{p'}$. Thus $H_p = H_{p'} \times H_p \leq Z(G)_{p'} \times H_p$. On the other hand $P_1 \leq Z(H)$ and H/P_1 is nilpotent, hence $Z(G) \leq H$, Now we can conclude that $H = Z(G)_{p'} \times H_p$. Since $p \mid |x^G|$ for some $x \in G$ if and only if $p \mid |G/Z(G)|$, we may assume $H_p = G_p$ but $G_p = H_p \nleq Z(G)$. Therefore $G_p \trianglelefteq G$ as H_p is a characteristic subgroup of the normal subgroup H of G. We have $[\overline{G_p}, \overline{G_{p'}}] = [\frac{G_p P_1}{P_1}, \frac{G_{p'} P_1}{P_1}]$. Since $G_p = H_p$ $H = Z(G)_{p'} \times H_p$, we deduce that $P_1 \leq H_p$. Now we can see that $[\overline{G_p}, \overline{G_{p'}}] = 1$ which implies that $[G_p, G_{p'}] \leq P_1$ and $[P_1, G_{p'}] = 1$. Then $\bar{G} = G/P_1$, $\bar{G} = \overline{G_{p'}} \times \overline{G_p}$, hence $G_{p'} P_1 \trianglelefteq G$ so $G_{p'} \trianglelefteq G$ which yields $G = G_{p'} \times G_p$. Since $Z(G_p) \neq G_p$ by $G_p \nleq Z(G)$ and $Z(G) = Z(G_{p'}) \times Z(G_p)$, we deduce that G_p is not abelian, so there exists an element $x \in G$ such that $G_p \nleq C_G(x)$. If for each $x \in G_p$, p does not divide the conjugacy class size of x, then $G_p \leq \bigcap_{x \in G_p} C_G(x) = Z(G_p) \leq Z(G)$ which is a contradiction. Hence there is an $x \in G_p$ such that $p \mid |x^G|$. If $y \in G_{p'}$, then $C_G(y) = G_p(C_G(y) \cap G_{p'})$ hence $|y^G| = |G_{p'}/C_{G_{p'}}(y)|$. Therefore, $p|y^G|$ divides $|(xy)^G|$ and $|y^G|$ is a prime power since $\alpha(G) = 2$. Thus $\alpha(G_{p'}) = 1$, so $|G_{p'}/Z(G_{p'})|$ has at most two prime divisors. Now $|G/Z(G)| = |G_p/Z(G_p)||G_{p'}/Z(G_{p'})|$ yields that $w(|G/Z(G)|) \leq 3$, which is a contradiction. ■

Theorem 4.6: (3.2.8 in [Kurzweil and Stellmacher, 2004]) Let $N \trianglelefteq G$ *with* $\bar{G} = G/N$, *and let* P *be a* p*-subgroup of* G. *Assume that* $gcd(|N|, p) = 1$. *Then* $N_{\bar{G}}(\bar{P}) = \overline{N_G(P)}$ *and* $C_{\bar{G}}(\bar{P}) = \overline{C_G(P)}$.

Lemma 4.7: Assume that G *satisfies Hypothesis A and* P *is a nontrivial minimal normal* p*-subgroup of* G, *then* $G_p = P$.

Proof 4.7: Since G *satisfies Hypothesis A, Lemma 4.5 implies that* $Z(G) = 1$. *Let* $\bar{G} = G/P$ *and* $\bar{A}$ *denote the image of the set* A *in* $\bar{G}$. *Since* $|\bar{G}| < |G|$, $|\bar{G}/Z(\bar{G})|$ *is divisible by at most four primes. Since* G *satisfies Hypothesis A, we conclude that* $|\pi(G/Z(G))| \geq 5$. *Thus there exists a prime* $t \in \pi(G/Z(G))$ *such that* $\bar{G}_t = Z(\bar{G})_t$, *which implies that the Sylow* t*-subgroup of* $\bar{G}$ *is central. Therefore* $\bar{G} = \bar{G}_{t'} \times \bar{G}_t$. *Now we have the following two cases:*

- *Case 1:* $t = p$. *Without loss of generality, we may assume that* $P \leq G_p$, *then* $G_p \trianglelefteq G$ *and* $\bar{G} = \bar{G}_{p'} \times \bar{G}_p$. *Since* $1 \neq P \trianglelefteq G_p$, *we conclude that* $Z(G_p) \cap P \neq 1$. *As* P *is a minimal normal* p*-subgroup of* G, *we deduce that* $P \leq Z(G_p)$. *It is easy to see that* $[G_{p'}, G_p] \trianglelefteq G$. *Now minimality of* P *verifies that* $\left[G_{p'}, G_p\right] = P$. *Now by Theorem 1.4 we conclude that* $G_p = C_{G_p}(G_{p'}) \times P$. *Hence* $Z(C_{G_p}(G_{p'})) \leq Z(G) = 1$ *yields* $C_{G_p}(G_{p'}) = 1$ *and so* $P = G_p$.
- *Case 2:* $t \neq p$. *We claim that* $G = C_G(P)G_t$. *We saw that* $\bar{G} = \bar{G}_{t'} \times \bar{G}_t$ *and* $\bar{G}_t$ *is central, therefore by Theorem 4.6 we deduce that* $\bar{G} = C_{\bar{G}}(\bar{G}_t) = \overline{C_G(G_t)}$. *This implies that* $G = C_G(G_t)P$. *If* t *divides* $|C_G(P)|$, *then there exists a* t*-element* $x \in C_G(P)$. *Without loss of generality, we may assume that* $x \in G_t$. *Let* g *be an element of* G, *so there exist* $c \in C_G(G_t)$ *and* $y \in P$ *such that* $g = cy$. *It is easy to see that* $[g, x] = 1$. *Since* g *is an arbitrary element of* G, *we conclude that* $x \in Z(G) = 1$. *Hence* $gcd(|C_G(P)|, t) = 1$. *Since*

$$\frac{G_t C_G(P)}{C_G(P)} = Z\left(\frac{G}{C_G(P)}\right) \tag{4.2}$$

if $G \neq C_G(P)\, G_t$, *then there is a prime* $v \neq t$ *such that* $O_v(G/C_G(P)) \neq 1$, *because G is solvable. If* $t \nmid |x_p^G|$ *for some* $x_p \in P^{\#}$*,then* $G_t \leq C_G(x_p)$*. By part ii) of Lemma 4.2, we conclude that* $tv \mid |x_p^G|$ *for* $x_p \in P^{\#}$*. Thus* $gcd(tv, p) = 1$*. Let* r *be any prime, such that* $r \notin \{t, v, p\}$*. Since* $tv \mid |x_p^G|$*,* $\alpha(G) = 2$ *and* $t \neq v$*, we conclude that* r *does not divide* $|x_p^G|$*, so* $G_r \leq C_G(x_p)$*. If* $g \in G_r^{\#}$*, as* x_p *and* g *commute and* $p \neq r$*, by Lemma 3.1, we conclude that* $|x_p^G| \mid |(x_p g)^G|$*. Since* $|(x_p g)^G|$ *has two prime divisors, we deduce that* $|g^G| = v^b t^a$*, for some positive integers* a *and* b*. We saw that* $G = C_G(G_t)P$*, so* $a = 0$ *and* $|g^G| = v^b > 1$*.*

It follows that $P \leq G_p \leq C_G(G_r)$*, so* $G_r \leq C_G(P)$*. On the other hand we have* $|\pi(G/Z(G))| \geq 5$*, thus there exists a prime* $s \neq r$ *in* $\pi(G/Z(G))$ *such that* $s \notin \{v, p, t\}$*. Similarly we can see that* $G_s \leq C_G(P)$*. Therefore,* $C_G(P) \geq G_{\{r,s\}}$*.*

If y *is any* p'*-element of* $C_G(P)$*, then* $|x_p^G| \mid |(x_p y)^G|$ *for* $x_p \in P^{\#}$ *implies* $|y^G| = v^b t^a$*, for some positive integers* a *and* b*. Now* $G = C_G(G_t)P$ *again implies* $|y^G| = v^b > 1$*. Hence, if* $v \mid |C_G(P)|$*, let* V *be a Sylow* v*-subgroup of* $C_G(P)$*. Now* $P \trianglelefteq G$ *implies* $C_G(P) \trianglelefteq G$*. Hence* V *is a normal subgroup of some* G_v *as* $V = C_G(P) \cap G_v$ *for some* $G_v \in Syl_v(G)$*. Thus, there is an element* $1 \neq y \in Z(G_v) \cap V$*, so* $G_v \leq C_G(y)$*. This happens if and only if* v *does not divide* $|y^G|$*, which is a contradiction. Hence* $gcd(|C_G(P)|, v) = 1$ *and* $G_v \cap C_G(P) = 1$*. Now* $G_{\{r,s\}} \leq C_G(P) \trianglelefteq G$*, so by Frattini Argument* $G = N_G(G_r)C_G(P) = N_G(G_s)C_G(P)$*, and it yields* $G_v \leq N_G(G_r) \cup N_G(G_s)$ *for some* G_r, G_s *of* G*. Since* $G_v \cap C_G(P) = 1$*,* $p \mid |g^G|$ *for* $g \in G_v^{\#}$*. Since* $\alpha(G) = 2$*, by part i) of Lemma 4.2 we deduce that* $g \in C_G(G_r) \cup C_G(G_s)$*. Now part iii) of Lemma 4.2 implies that either* $G_v \leq C_G(G_r)$ *or* $G_v \leq C_G(G_s)$*. Without loss of generality, we may assume that* $G_v \leq C_G(G_r)$*. Hence,* $v^b = |y^G|$ *for* $y \in G_r^{\#}$*. If* $b > 0$*, then* v *divides* $[G : C_G(y)]$ *and* $G_v \nleq C_G(y)$*. This implies that* $G_v \nleq C_G(G_r)$ *which is a contradiction. Thus* $b = 0$ *and* $1 \neq y \in Z(G) = 1$ *which is impossible. Therefore,* $G = C_G(P)G_t$*.*

We have $\bar{G}_t = Z(\bar{G})_t$*, so* $G_tP \trianglelefteq G$ *and therefore* $C_G(G_tP) \trianglelefteq G$*. Since* $G = C_G(P)G_t = C_G(G_t)P$ *and* $C_G(G_tP) = C_G(G_t) \cap C_G(P)$*, by Dedekind's Modular Law we have* $G = C_G(G_tP)PG_t$*. It follows that* $G = C_G(G_tP) \times PG_t$*. Let* u *be any prime dividing* $C_G(G_tP)$*, then every Sylow* u*-subgroup of* G *centralizes* P *as* $C_G(G_tP) \leq C_G(P)$*. It follows that if* u *divides* $|C_G(G_tP)/C_{C_G(G_tP)}(g)|$ *for* $g \in$

$C_G(G_tP)$, then $u \mid |(gx)^G|$ for $x \in P^\#$ otherwise there exists a Sylow u-subgroup G_u of G such that $G_u \leq C_G(gx)$ for some $x \in P^\#$, then $G_u \leq C_G(g)$, but it contradicts our assumption. Now g centralizes all Sylow t-subgroups of G so $ut \mid |(gx)^G|$ otherwise $G_t \leq C_G(x)$ for some $x \in P^\#$, hence $x \in Z(G)$ as $G = C_G(P)G_t$, a contradiction. Thus, the number of elements in any $C_G(G_tP)$-class of $C_G(G_tP)^\#$ is a prime power because $\alpha(G) = 2$. Hence $\alpha(C_G(G_tP)) = 1$, so at most two primes divide $|C_G(G_tP)/Z(C_G(G_tP))|$. But $Z(C_G(G_tP)) \leq Z(G) = 1$ yields a contradiction. ■

Now we can prove Theorem 4.3.

Proof 4.3: Let γ denote the set of prime divisors of $|F(G)|$. Since $F(G) = \prod_{p\in\gamma} O_p(G)$, Lemma 4.7 implies that $F(G) = \prod_{p\in\gamma} G_p$ is an abelian Hall γ-subgroup of G. Let $\bar{G} := G/F(G)$ and $\bar{A}$ denote the image of A in G. We first show $w(|\bar{G}|) \leq 2$. G is solvable, therefore $C_G(F(G)) \leq F(G)$. As $F(G)$ is abelian, we have $C_G\big(F(G)\big) = F(G)$. If $\bar{x} \in \bar{G}^\#$, then $x \notin C_G\big(F(G)\big) = F(G)$ implies that $p \mid |x^G|$ for some $p \in \gamma$. Now $gcd(|\bar{G}|, |F(G)|) = 1$,0 so $p|\bar{x}^{\bar{G}}| \mid |x^G|$. It follows that for each nontrivial element $\bar{x} \in \bar{G}$, $|\bar{x}^{\bar{G}}|$ is a prime power. This implies that $\alpha(\bar{G}) = 1$ whence $w(\bar{G}/Z(\bar{G})) \leq 2$. Assume that $w(\bar{G}) \geq 3$. We claim that there are distinct primes u, t such that $u, t \in \pi(|F(\bar{G})|)$. Since $w(\bar{G}) \geq 3$ and $w(\bar{G}/Z(\bar{G})) \leq 2$, there exists a prime $u \in \pi(|F(\bar{G})|)$. If $\pi(|F(\bar{G})|) = \{u\}$, then let $\bar{U}$ be a Sylow u-subgroup of $\bar{G}$. Since $u \in w(\bar{G})\backslash\rho(\bar{G})$, we conclude that $\bar{U}$ is an abelian normal direct factor of $\bar{G}$. This implies that $\bar{U} \leq Z(\bar{G})$ and so $F(\bar{G}) = Z(\bar{G})$, which is a contradiction. Thus there are distinct primes u, t such that $u, t \in \pi(|F(\bar{G})|)$. Since $t, u \notin \gamma$, there are primes p, q (possibly $p = q$) in γ such that $O_t(G/C_G(G_p)) \neq 1$ and $O_u(G/C_G(G_q)) \neq 1$. Since G_p and G_q are minimal normal subgroups of G, by Lemma 4.7, and part ii) of Lemma 4.2, we deduce that $t \mid |x_p^G|$ for every $x_p \in G_p^\#$ and $u \mid |x_q^G|$ for every $x_q \in G_q^\#$. If $p = q$, then $tu \mid |x_p^G|$. If $p \neq q$, then $tu \mid |(x_px_q)^G|$. In either case, there is an $x \in F(G)$ such that $tu \mid |x_p^G|$. Let v be a prime divisor of $|\bar{G}|$, where $v \notin \{t, u\}$. Since $tu \mid |x_p^G|$, $G_v \leq C_G(x)$ for some Sylow v-subgroup G_v. Let $g \in G_v$. Now $gcd(|\langle x\rangle|, v) = 1$ implies that every prime divisor $|g^G|$ and $|x^G|$ divides $|(xg)^G|$.

Thus, $tu \mid |(xg)^G|$ implies that $|g^G| = t^a u^b$. Therefore, $|F(G)|$ divides $|C_G(g)|$. As $F(G)$ is the Hall γ-subgroup, we conclude that $F(G) \leq C_G(g)$, a contradiction because $\bar{g} \in \bar{G}^{\#}$ and so $g \notin C_G(F(G)) = F(G)$. Hence $w(|\bar{G}|) \leq 2$ and $|F(G)|$ is divisible by at least three primes.

Let u be a prime divisor of $|\bar{G}|$, then there is a prime $p \mid |F(G)|$ such that $G_u \nleq C_G(G_p)$. Let $r \in \gamma \setminus \{p\}$, if $G_u \nleq C_G(G_p) \cup C_G(G_r)$, then by Lemma 4.2 i), iii) verify that there is a $g \in G_u^{\#}$ with $pr \mid |g^G|$. Therefore, $G_s \leq C_G(g)$, where $s \in \gamma \setminus \{p, r\}$. Since $gcd(s,u) = 1$, $pr \mid |(gx)^G$ and $|x^G| \mid |(gx)|^G$ for $x \in G_s^{\#}$ imply that $|x^G|$ is a $\{p,r\}$-number. However, $x \in F(G)$ and $F(G)$ abelian already yield $G_p G_r \leq C_G(x)$. Thus, $x \in Z(G) = 1$, which is a contradiction. Hence, $G_u \leq C_G(G_p) \cup C_G(G_r)$, and part iii) of Lemma 4.2 implies $G_u \leq C_G(G_r)$. Since r was an arbitrary prime in $\gamma \setminus \{p\}$, $G_u \leq C_G(F(G)_{p'})$.

If $w(|\bar{G}|) = 2$, there is a prime $v \neq u$ dividing $|\bar{G}|$. The same argument yields a prime $q \in \gamma$ such that $G_v \leq C_G(F(G)_{q'})$. Since $|\gamma| \geq 3$, there is a prime r such that $G_r \leq F(G)_{q'} \cap F(G)_{p'}$. Now $F(G)$ abelian and $w(\bar{G}) \leq 2$ yield $G_r \leq Z(G)$, a contradiction. If $w(|\bar{G}|) = 1$, then $F(G)_{q'} \leq Z(G)$ by the same argument, which is again a contradiction. ■

5. CYCLES and BIPARTITE GRAPH ON CONJUGACY CLASS of GROUPS

In this chapter we will follow the paper of Bijan Taeri [Taeri, 2010].

Let G be a finite nonabelian group and $B(G)$ be the bipartite divisor graph of a finite group releated to the set of conjugacy class sizes of G. Let $cs^*(G) = cs(G) \setminus \{1\}$ be the set of sizes of the noncentral classes of G. In this chapter, we consider the case where $B(G)$ is a cycle. We prove that this case will happen if and only if $B(G)$ is a cycle of length six. Further, we classify those groups whose bipartite divisor graphs are cycles which is proved by Taeri [Taeri, 2010].

5.1. A Group G with the Cycle $B(G)$

It is obvious that each cycle is a 2-regular graph where by 2-regular graph we mean a graph with this property that every vertex is endpoints of two distinct edges. It is easy to see that bipartite 2-regular graphs are cycles. If $B(G)$ is 2-regular then every noncentral conjugacy class has exactly 2 prime divisors which implies that $\alpha(G) = 2$. For the case where G is nonsolvable, Casolo has proved the following lemma:

Lemma 5.1: (Proposition 3.3 in [Casolo, 1994]) Let G be a nonsolvable group with $\alpha(G) = 2$. Then $G = A \times S$, where A is abelian and S is isomorphic to either $PSL_2(4)$ or $PSL_2(8)$.

Definition 5.1: A group G is called a quasi-Frobenius group if $G/Z(G)$ is a Frobenius group. The inverse images in G of the Frobenius kernel and complement of $G/Z(G)$ are called the kernel and complement of G, respectively.

Now we state a theorem about quasi-Frobenius group with abelian kernel and abelian complement [Fang and Zhang, 2003].

Lemma 5.2: Let G be a quasi-Frobenius group with abelian kernel N and abelian *complement H. Let $|Z(G)| = r$, $|H/Z(G)| = s+1$, and $N/Z(G)$ be the disjoint union of $t+1$ conjugacy classes of $G/Z(G)$. Then we have*

i) Let C be a G-conjugacy class. Then $|C| = 1$, $s+1$, or $(s+1)t+1$.

ii) The numbers of all different conjugacy classes of G with length 1, $s+1$, $(s+1)t+1$ are respectively r, rt, rs.

Theorem 5.3: Let G be a finite group such that $B(G)$ is a cycle. Then $G \cong A \times S$, where A is abelian, and $S \cong SL_2(q)$, $q = 4, 8$. Consequently $B(G)$ is a cycle if and only if $B(G)$ is the 6-cycle $2-12-3-15-5-20-2$ and $G \cong A \times SL_2(4)$, or is the 6-cycle $2-72-3-63-7-56-2$ and $G \cong A \times SL_2(8)$, where A is abelian.

Proof 5.3: Since $B(G)$ is a cycle, we have $\alpha(G) = 2$. If G is nonsolvable, then by Lemma 5.1 $G \cong A \times SL_2(q)$, where $i = 4, 8$, and A is abelian ($PSL_2(4) \cong SL_2(4) \cong A_5$ and $PSL_2(8) \cong SL_2(8)$). If $G \cong A \times SL_2(4) \cong A \times A_5$, then since $cs^*(A_5) = \{12, 15, 20\}$, $B(G)$ is a cycle of length 6 as follows:

$$2-12-3-15-5-20-2. \tag{5.1}$$

If $G \cong A \times SL_2(8)$, then since $cs^*\big(SL_2(8)\big) = \{56, 63, 72\}$, $B(G)$ is the following cycle of length 6:

$$2-72-3-63-7-56-2. \tag{5.2}$$

Now we claim that G cannot be solvable. By the way of contradiction, suppose that G is solvable and $B(G)$ is a cycle. Thus, for all noncentral $g \in G$, we have $|\pi(g^G)| = 2$. By Corollary 3.22, it is clear that if $cs^*(G) = \{m, n\}$, and $gcd(m, n) \neq 1$, then either m or n is a prime power and so $B(G)$ cannot be a cycle. In particular, if $B(G)$ is a cycle, it cannot be a cycle of length four. On the other hand, by Theorem 4.3, we have $|V(\Delta(G))| \leq 4$. If $|V(\Delta(G))| = 2$, then since $B(G)$ is 2-regular, $B(G)$ is a cycle of length 4, which is impossible. It is obvious that

$|V(\Delta(G))| \neq 1$, so $|V(\Delta(G))| \geq 3$. On the other hand, by Proposition 4.1, we know that $\pi(G/Z(G)) = V(\Delta(G))$, so $3 \leq |\pi(G/Z(G))| \leq 4$. Since $B(G)$ cannot be a cycle of length 4, we have $|cs^*(G)| \geq 3$.

We claim that G is an F-group. First note that if $c = p^a q^b \in cs^*(G)$, with a and b positive, then for all $x \in G$ with $p \in \pi(x^G) \subseteq \{p, q\}$ we must have $|x^G| = p^a q^b$, since $B(G)$ is 2-regular. Let x, y be two elements of G such that $C_G(x) \leq C_G(y)$. We have to prove $C_G(x) = C_G(y)$. If $C_G(x) < C_G(y)$, then $|x^G|$ and $|y^G|$ are distinct and $|y^G|$ divides $|x^G|$, which is impossible. Hence G is an F-group and so is one of the groups listed in Theorem 3.20. The groups listed in Theorem 3.20 vi) and vii) are nonsolvable. Thus G is one of the group listed in i) – v).

If i) holds, then $cs^*(G) = cs^*(P)$, a contradiction.

Suppose ii) holds. Then G has an abelian normal subgroup N of prime index p. We show that $cs^*(G) = \{p, m\}$ for some positive integer m. Let x be any noncentral element of G. If $x \in N$, then since $N \leq C_G(x) < G$ and $|G/N| = p$, we have $|x^G| = p$ as $|G/N| = [G : C_G(x)][N : C_G(x)]$. If $x \notin N$, then $G = N\langle x \rangle$ and so

$$m = |x^G| = \frac{|N\langle x \rangle|}{|C_G(x)|} = \frac{|NC_G(x)|}{|C_G(x)|} = \frac{|N|}{|N \cap C_G(x)|} = \frac{|N|}{|C_N(x)|}. \tag{5.3}$$

Let $y \in G$ be a noncentral element. As $G = N\langle x \rangle$, without loss of generality, we can see that there exists $a \in N$ such that $y = ax$. Since N is abelian, we can see that $C_N(x) = C_N(y)$ and this implies that $|y^G| = m$. Thus $cs^*(G) = \{p, m\}$, which is a contradiction.

If iii) holds, then by Lemma 5.2 we have $|cs^*(G)| = 2$, which is a contradiction.

Suppose iv) holds. Let $x \in L \setminus Z(G)$. Since $\frac{G}{Z(G)} = \frac{K}{Z(G)} \rtimes \frac{L}{Z(G)}$, we deduce that $C_{G/Z(G)}(xZ(G)) \leq L/Z(G)$. As L is abelian, we have $C_G(x) = L$ hence

$$|x^G| = \frac{|G|}{|L|} = \frac{|L|\,|K|}{|Z(G)|\,|K|} = \frac{|K|}{|Z(G)|} \tag{5.4}$$

which is a prime power, which is a contradiction.

If v) holds, then $\pi(G/Z(G)) = \pi(S_4) = \{2, 3\}$, which is a contradiction.

Consequently, G is nonsolvable. Now the first part of the proof implies that $B(G)$ is a cycle if and only if it is a cycle of length six. ■

5.2. Groups with Satisfying the One-Prime Power Hypothesis

We say that G satisfies the one-prime power hypothesis if $m, n \in cs^*(G)$, then either $gcd(m, n) = 1$ or $gcd(m, n)$ is a power of a prime. We can easily see that the graph $B(G)$ has no cycle of length 4 if and only if G satisfies one-prime power hypothesis. Suppose that $B(G)$ has no cycle of length 4 and let $m, n \in cs^*(G)$. If pq divides $gcd(m, n)$, then $p - m - q - n - p$ is a cycle of length 4 in $B(G)$, which is a contradiction. Thus $gcd(m, n) = p^a$, for some prime p and integer $a \geq 0$. Conversely, if for all $m, n \in cs^*(G)$, $gcd(m, n)$ is a prime power, then G has no cycle of length 4. In fact if $p - m - q - n - p$ is a cycle of length 4 in $B(G)$, then pq divides $gcd(m, n)$ and so $gcd(m, n)$ is not a prime power.

Definition 5.2: A central extension of a group G is an exact sequence $1 \to Z \to H \to G \to 1$ such that the image of Z in H is a subgroup of $Z(H)$. A central extension determined by pair (H, Z) consisting of a group H and a subgroup Z of the center of H such that $H/Z \cong G$. Such a central extension is often denoted by (H, Z).

Definition 5.3: A central extension (H, Z) of a finite group G is said to be irreducible if there is no proper subgroup L having the property $H = ZL$. A central extension H of G is called a representation group G if H satisfies the following conditions:

i) H is irreducible,
ii) $|M(G)| = |H' \cap Z|$,
iii) $|H| = |G||M(G)|$,

where $M(G)$ is the Schur multiplier of the group G.

Theorem 5.4: (Theorem 9.18 in [Suzuki, 1982]) Let G be a finite group which satisfies the property $G' = G$, and let (H, Z) be a central extension of G.

i) Set $H_1 = H'$ *and* $Z_1 = Z \cap H'$. *Then* (H_1, Z_1) *is an irreducible central extension of* G.

ii) If a noncentral extension (H, Z) *of* G *is irreducible, then the representation group of* H *is (isomorphic to) the representation group of* G. *Hence, we have* $|M(H)| = |M(G)|/|Z|$.

Suppose that $G/Z(G) \cong S$, where S is a group such that $S = S'$. Let x be a noncentral element of G. We claim that $|x^G| = ca$, where $c = |(xZ(G))^{G/Z(G)}|$ and a is a divisor of $|M(S)|$. To see this let $D = \{g \in G | [g, x] \in Z(G)\}$, where $[g, x] = g^{-1}g^x$, be the preimage of $C_{G/Z(G)}(xZ(G))$ in G. Since $C_G(x)$ is the kernel of the homomorphism $D \to G' \cap Z(G)$ with $d \mapsto [d, x]$, it follows that $|D/C_G(x)|$ divides $|G' \cap Z(G)|$. We have the central extension $(G, Z(G))$ for the group S. By Theorem 5.4, we obtain that $(G', G' \cap Z(G))$ is an irreducible central extension of S and hence $|M(G')| = |M(S)|/|G' \cap Z(G)|$, so $|G' \cap Z(G)|$ divides $|M(S)|$ and so $a = |D/C_G(x)|$ divides $|M(S)|$. Now

$$|x^G| = [G : C_G(x)] = \left[\frac{G}{Z(G)} : \frac{D}{Z(G)}\right]\left[\frac{D}{Z(G)} : \frac{C_G(x)}{Z(G)}\right] = ca \qquad (5.5)$$

as claimed. Also we have that the order of the Schur mutiplier of $PSL_2(q)$ by the following theorem [Huppert, 1967].

Theorem 5.5: For $p^f \neq 2^2, 3^2$ $SL_2(p^f)$ *is a representation group of* $PSL_2(p^f)$. *For* $p^f > 2^2$ *and* $p^f \neq 3^2$ $SL_2(p^f)$ *is the only representation group of* $PSL_2(p^f)$*; furthermore,* $SL_2(5)$ *is the only representation of* $PSL_2(4)$. *It is*

$$\left|M\left(PSL_2(p^f)\right)\right| = \begin{cases} 2, & for\ p > 2\ and\ p^f \neq 3^2; \\ 1, & for\ p = 2\ and\ p^f \neq 3^2; \\ 2, & for\ p^f = 2^2; \\ 6, & for\ p^f = 3^2. \end{cases} \qquad (5.6)$$

In 1970, Itô proved the following theorem for a simple group with three nonidentity conjugacy class sizes [Itô, 1970].

Theorem 5.6: If G is a simple group with $cs^*(G) = 3$, then G is isomorphic with some $SL_2(2^m)$, $m \geq 2$.

In the following lemma we use the well known Burnside's p^α-Lemma which states that a finite group which has a conjugacy class of a prime power size is not simple.

Lemma 5.7: Let G be a finite nonsolvable F-group. If G satisfies one-prime power hypothesis, then $G/Z(G) \cong SL_2(4)$ or $SL_2(8)$.

Proof 5.7: Note that in Theorem 3.20 all groups satisfying i) − v) are solvable. So a nonsolvable F-group satisfying the one-prime power hypothesis is either a group of type vi) or vii). Suppose that $G/Z(G) \cong PSL_2(q)$ and q is odd. It is well-known that if $q \equiv 1\ (mod 4)$, we have $cs^*(G/Z(G)) = \{q(q+1), q(q-1), \frac{1}{2}q(q+1), \frac{1}{2}(q-1)(q+1)\}$, and if $q \equiv 3\ (mod 4)$, we have $cs^*(G/Z(G)) = \{q(q+1), \frac{1}{2}q(q+1), \frac{1}{2}(q-1)(q+1), \frac{1}{2}q(q-1)\}$. If $q = 9$, then $q \equiv 1\ (mod\ 4)$, so $cs^*\big(PSL_2(9)\big) = \{9 \times 10, 9 \times 8, 9 \times 5, 8 \times 5\}$ which does not satisfy the one-prime-power hypothesis. So we may assume that $q \neq 9$.

Suppose that we have the first case. By the above discussion, for any $x \in G \setminus Z(G)$ we have $|x^G| = |(xZ(G))^{G/Z(G)}|a$, where $a \in \{1,2\}$. As $G/Z(G) \cong PSL_2(q)$ is simple, by Burnside's p^α-Lemma we conclude that there is no nontrivial conjugacy class of $G/Z(G)$ of prime power size. If $b_1 = a_1c \in cs^*(G)$ and $b_2 = a_2c \in cs^*(G)$, where $c \in cs^*(G/Z(G))$, then c divides $gcd(b_1, b_2)$ and so $gcd(b_1, b_2)$ is not a prime power, since $c \in cs^*(G/Z(G))$. Thus $b_1 = b_2$ since G satisfies the one-prime power hypothesis. So for any $c \in cs^*(G/Z(G))$ at most one conjugacy class size of G is a multiple of c, hence $cs^*(G) = \{q(q+1)a_1, q(q-1)a_2, \frac{1}{2}q(q+1)a_3, \frac{1}{2}(q-1)(q+1)a_4\}$, where $a_1, a_2, a_3, a_4 \in \{1, 2\}$, and $|cs^*(G)| \leq 4$. If $b_1 = q(q+1)a_1 \in cs^*(G)$ and $b_2 = q(q-1)a_2 \in cs^*(G)$, are distinct, then $2q$ divides $gcd(b_1, b_2)$ and so $gcd(b_1, b_2)$ not a prime power. So $b_1 = b_2$. Therefore there exist at most one conjugacy class size which is multiple of $q(q+1)$ and $q(q-1)$. So $cs^*(G) = \{q(q+1)a_1, \frac{1}{2}q(q+1)a_3, \frac{1}{2}(q-1)(q+1)a_4\}$ and $|cs^*(G)| \leq 3$. Now we consider

$b_1 = q(q+1)a_1 \in cs^*(G)$ and $b_4 = \frac{1}{2}(q-1)(q+1)a_4 \in cs^*(G)$, then $q+1$ divides $gcd(b_1, b_4)$. If $q+1$ is a prime power, then $q+1 = 2^k$ as q is odd. As we assumed $q \equiv 1\ (mod 4)$, there exists an integer s such that $q = 4s+1$, so we have $2^k = 2(2s+1)$, which is a contradiction. Thus $q+1$ is not a prime power and this implies that $gcd(b_1, b_4)$ is not a prime power. Hence $b_1 = b_4$ and there exist at most one conjugacy class size which is multiple of $q(q+1)$ and $\frac{1}{2}(q-1)(q+1)$. Thus $cs^*(G) = \{q(q+1)a_1, \frac{1}{2}q(q+1)a_3\}$ and $|cs^*(G)| \leq 2$. Hence G is solvable which contradicts our hypothesis.

Similarly in the second case we can obtain a contradiction.

Suppose we have $cs^*(G) = \{q(q+1)a_1, \frac{1}{2}q(q+1)a_2, \frac{1}{2}(q-1)(q+1)a_3, \frac{1}{2}q(q-1)a_4\}$, where $a_1, a_2, a_3, a_4 \in \{1,2\}$. If $b_1 = q(q+1)a_1 \in cs^*(G)$ and $b_2 = \frac{1}{2}q(q+1)a_2 \in cs^*(G)$ are distinct, then $2q$ divides $gcd(b_1, b_2)$, so $gcd(b_1, b_2)$ is not a prime power. Hence $b_1 = b_2$. Thus there exist at most one conjugacy class size which is multiple of $q(q+1)$ and $\frac{1}{2}q(q+1)$. Therefore we have $cs^*(G) = \{q(q+1)a_1, \frac{1}{2}(q-1)(q+1)a_3, \frac{1}{2}q(q-1)a_4\}$ and $|cs^*(G)| = 3$. Now we consider $b_1 = q(q+1)a_1 \in cs^*(G)$ and $b_3 = \frac{1}{2}(q-1)(q+1)a_3 \in cs^*(G)$, then $2q \mid gcd(b_1, b_3)$ and so $gcd(b_1, b_3)$ is not a prime power. So $b_1 = b_3$. Hence there exist at most one conjugacy class size which is multiple of $q(q+1)$ and $\frac{1}{2}(q-1)(q+1)$. Thus $cs^*(G) = \{q(q+1)a_1, q(q-1)a_2\}$ and $|cs^*(G)| \leq 2$, which is a contradiction.

If $G/Z(G) \cong PGL_2(q)$, where q is odd, then it follows that $cs^*(G/Z(G)) = \{q(q+1), q(q-1), (q-1)(q+1), \frac{1}{2}q(q+1), \frac{1}{2}q(q-1)\}$. Similar to the above case, we can see that at most one conjugacy class size of G is a multiple of c, for every $c \in cs^*(G/Z(G))$, and $cs^*(G) = \{q(q+1)a_1, q(q-1)a_2, (q-1)(q+1)a_3, \frac{1}{2}q(q+1)a_4, \frac{1}{2}q(q-1)a_5\}$ and $|cs^*(G)| \leq 5$. As above there exist at most one conjugacy class size which is a multiple of $q(q+1)$ and $q(q-1)$. Thus $cs^*(G) = \{q(q+1)a_1, (q-1)(q+1)a_3, \frac{1}{2}q(q+1)a_4, \frac{1}{2}q(q-1)a_5\}$ and so $|cs^*(G)| \leq 4$. Now either $q = 4k+1$ or $q = 4k+3$. First suppose that $q = 4k+1$, then $q+1$ is not a prime power. Since $q+1$ divides $gcd(b_1, b_3)$, where $b_1 = q(q+1)a_1$ and

$b_3 = (q-1)(q+1)a_3$ we have $b_1 = b_3$ and $cs^*(G) = \{q(q+1)a_1, \frac{1}{2}q(q+1)a_4,$ $\frac{1}{2}q(q-1)a_5\}$. Hence by Theorem 5.6, $PGL_2(q) \cong G/Z(G) \cong PSL_2(2^m)$, which is a contradiction. Now suppose $q = 4k+3$. Then $2q$ divides $gcd(b_4, b_5)$, where $b_4 = \frac{1}{2}q(q+1)a_4$, and $b_5 = \frac{1}{2}q(q-1)a_5$. Thus $b_4 = b_5$ and $cs^*(G) = \{q(q+1)a_1,$ $(q-1)(q+1)a_3, \frac{1}{2}q(q-1)a_4\}$. Hence by Theorem 5.6, $PGL_2(q) \cong G/Z(G) \cong PSL_2(2^m)$, which is a contradiction.

Finally suppose that $G/Z(G) \cong PSL_2(2^n) = PGL_2(2^n)$. Then we have $cs^*(G/Z(G)) = \{(2^n-1)(2^n+1), 2^n(2^n-1), 2^n(2^n+1)\}$. Therefore $cs^*(G) = \{(2^n-1)(2^n+1)a_1, 2^n(2^n-1)a_2, 2^n(2^n+1)a_3\}$. It is clear that 2^n-1 divides $gcd(b_1, b_2)$, where $b_1 = (2^n-1)(2^n+1)a_1$ and $b_2 = 2^n(2^n-1)a_2$. Also (2^n+1) divides $gcd(b_1, b_3)$, where $b_3 = 2^n(2^n+1)a_3$. Therefore we conclude that 2^n-1 and 2^n+1 are both prime power. Thus $n=2$ or $n=3$, which implies that $G/Z(G) \cong SL_2(4)$ or $SL_2(8)$. ■

Let G be finite group and $g \in G$. Then the subgroup of G which is generated by the set g^G is the smallest normal subgroup subgroup of G which contains g and we denote it by $\langle g^G \rangle$.

Lemma 5.8: (Lemma 6 in [Baer, 1953]) $\langle g^G \rangle$ is a p-group if and only if g is a p-element and there exists a normal subgroup N of G which contains g such that $[N : C_N(g)]$ is a power of p.

Proof 5.8: First suppose that $\langle g^G \rangle$ is a p-group, then g is a p-element and we can take $N = \langle g^G \rangle$, so $[N : C_N(g)]$ is a power of p.

Conversely assume that g is a p-element, that g is contained in the normal subgroup N of G, and that $[N : C_N(g)]$ is a power of p. Now we will show that there exists a p-Sylow subgroup P of N which contains g such that $N = C_N(g)P$. To show this we only prove that $C_N(g) \cap P$ is a p-Sylow subgroup of $C_N(g)$. If $[N : C_N(g)] = p^m$ and if $|C_N(g)| = p^n$, then $|N|_p = p^{n+m}$ so that $|P| = p^{n+m}$. Denote by p^k the order of $C_N(g) \cap P$. It is clear that p^k is a divisor of p^n. We can see that every right coset of P modulo $C_N(g) \cap P$ is contained in one and only one right coset of N

modulo $C_N(g)$ so that $[P : C_N(g) \cap P] \leq [N : C_N(g)]$. Since $[P : C_N(g) \cap P]$ is a divisor of p^m and thus $p^{n+m} = |P| = [P : C_N(g) \cap P]|C_N(g) \cap P|$, it follows that $[P : C_N(g) \cap P] = p^m$ and $|C_N(g) \cap P| = p^n$. Consequently $C_N(g) \cap P$ is a p-Sylow subgroup of $C_N(g)$. Hence $\langle g^N \rangle = \langle g^P \rangle \leq P$, so that the normal subgroup $\langle g^N \rangle$ of N is a p-group. Since every $\langle g^{Nx} \rangle = \langle g^N \rangle^x$ is part of N, $x \in G$, it follows that $\langle g^G \rangle$ is the product of normal subgroups of N which are p-groups. So $\langle g^G \rangle$ is a p-group. ■

Theorem 5.9: (Theorem 1 in [Camina and Camina, 1998]) Let G be a finite group. Then all elements of prime conjugacy class size are in $F_2(G)$, where $F_2(G)/F(G) = F(G/F(G))$.

Lemma 5.10: Let G be a finite group satisfying the one-prime power hypothesis. If $G/Z(G)$ has no solvable normal subgroup, then G is an F-group.

Proof 5.10: Let x, y be two noncentral elements of G such that $C_G(x) \leq C_G(y)$. We have to prove $C_G(x) = C_G(y)$. If $C_G(x) < C_G(y)$, then $|x^G|$ and $|y^G|$ are distinct and $|y^G|$ divides $|x^G|$, so $gcd(|x^G|, |y^G|) = |y^G|$. Since G satisfies the one-prime power hypothesis $|y^G|$ is a prime power. On the other hand $|(yZ(G))^{G/Z(G)}|$ divides $|y^G|$. Since y is noncentral, it follows that $|(yZ(G))^{G/Z(G)}|$ is also a prime power. Thus, by Theorem 5.9, the Fitting subgroup (which is solvable and normal) of $G/Z(G)$ is nontrivial, which contradicts our hypothesis. Therefore $C_G(x) = C_G(y)$ and so G is an F-group. ■

Corollary 5.11: Let G be a finite simple group. Then G satisfies the one-prime power hypothesis if and only if $G \cong SL_2(4)$ or $SL_2(8)$.

Proof 5.11: G is simple, so it is nonsolvable and $Z(G) = 1$. If G satisfies the one-prime power hypothesis, then by Lemma 5.10, G is a finite nonsolvable F-group. Now Lemma 5.7 verifies that $G \cong SL_2(4)$ or $SL_2(8)$. Conversely, it is clear that $SL_2(4)$ and $SL_2(8)$ have the one-prime power hypothesis. ■

The converse of Lemma 5.7 is not true. For example if $G = SL_2(5)$, then $G/Z(G) \cong SL_2(4)$ ($\cong A_5$) and $cs^*(G) = \{12, 20, 30\}$. It is well-known that in A_5

every element of order 3 or 5, is self centralizing, that is $C_{A_5}(x) = \langle x \rangle$, for all $x \in A_5$ of order 3 or 5. Also if x is any element of order 2 in A_5, then $C_{A_5}(x)$ is a Sylow 2-subgroup. Thus if $x \in P^{\#}$ is an arbitrary element of a Sylow p-subgroup P of A_5, for $p \in \{2, 3, 5\}$, then $C_G(x) = P$.

We claim that if x is a noncentral element of a group G such that $[C_G(x) : Z(G)] = p^2$, where p is a prime, then $C_G(x)$ is abelian and thus there is no centralizer of any noncentral element of G strictly contained in $C_G(x)$. We use this fact in the proof of the following lemma. On the contrary suppose that $Z(C_G(x)) < C_G(x)$. Then we have $Z(C_G(x))/Z(G) < C_G(x)/Z(G)$ and $\left[C_G(x) : Z\left(C_G(x)\right)\right] = [C_G(x)/Z(G) : Z(C_G(x))/Z(G)] = p$. Thus $C_G(x)$ is abelian, which contradicts our assumption that $Z(C_G(x)) < C_G(x)$. Thus $Z(C_G(x)) = C_G(x)$ and so $C_G(x)$ is abelian. Now let y be any noncentral element of G such that $C_G(y) \leq C_G(x)$. We want to show that $C_G(y) = C_G(x)$. Let $u \in C_G(x)$. Since $C_G(x)$ is abelian, $[u, v] = 1$ for all $v \in C_G(x)$. In particular, since $y \in C_G(y) \leq C_G(x)$, we have $[u, y] = 1$ and so $u \in C_G(y)$. Hence $C_G(y) = C_G(x)$, as required. This completes the proof of the claim.

Lemma 5.12: Let G be a finite group such that $G/Z(G) \cong A_5$. Then $cs^*(G) = \{12, 15, 20\}$, $G' \cong A_5$ or $cs^*(G) = \{12, 20, 30\}$, $G' \cong SL_2(5)$. Therefore G satisfies the one-prime power hypothesis if and only if $G' \cong A_5$.

Proof 5.12: Let x be a noncentral element of G. Then since $Z(G) < C_G(x) < G$, we have $[G : Z(G)] = |x^G|[C_G(x) : Z(G)]$ and so $|A_5| = 60 = |x^G|[C_G(x) : Z(G)]$. Since $|(xZ(G))^{G/Z(G)}|$ divides $|x^G|$, we have $|x^G| \in \{12a, 15b, 20c\}$, where a, b, c are positive integers. If $|x^G| = 12a$, then $5 = a[C_G(x) : Z(G)]$ and so $a = 1$, also $C_G(x)$ is abelian. If $|x^G| = 20c$, then $3 = c[C_G(x) : Z(G)]$ and so $c = 1$, also $C_G(x)$ is abelian. If $|x^G| = 15b$, then $4 = b[C_G(x) : Z(G)]$ and so $b = 1$ or 2. We consider the following two cases.

- *Case 1: Suppose that there exists $x \in G$ such that $|(xZ(G))^{G/Z(G)}| = 15$ with $C_G(x)/Z(G) = C_{G/Z(G)}(xZ(G))$. Therefore*

$$|x^G| = \left[\frac{G}{Z(G)} : \frac{C_G(x)}{Z(G)}\right] = \left[\frac{G}{Z(G)} : C_{G/Z(G)}(xZ(G))\right] = 15. \tag{5.7}$$

We will prove that for all $y \in G$ *with* $|(yZ(G))^{G/Z(G)}| = 15$, *we have* $C_G(y)/Z(G) = C_{G/Z(G)}(yZ(G))$ *and so* $|y^G| = 15$. *Firstly we know that* $C_{G/Z(G)}(xZ(G))$ *is a Sylow* 2*-subgroup of* $G/Z(G)$ *of order* 4. *Now it is clear that for each nontrivial* $yZ(G) \in C_{G/Z(G)}(xZ(G))$, $C_{G/Z(G)}\big(xZ(G)\big) = C_{G/Z(G)}\big(yZ(G)\big)$. *If* $yZ(G) \in C_{G/Z(G)}(xZ(G))$, *then*

$$\frac{C_G(y)}{Z(G)} \leq C_{G/Z(G)}\big(yZ(G)\big) = C_{G/Z(G)}\big(xZ(G)\big) = \frac{C_G(x)}{Z(G)} \tag{5.8}$$

since $|C_{G/Z(G)}\big(xZ(G)\big)| = \left|\frac{C_G(x)}{Z(G)}\right| = 4$, *by the previous note, we conclude that* $C_G(x) = C_G(y)$. *Therefore*

$$\frac{C_G(y)}{Z(G)} = C_{G/Z(G)}\big(yZ(G)\big) = C_{G/Z(G)}\big(xZ(G)\big) = \frac{C_G(x)}{Z(G)}. \tag{5.9}$$

Now suppose that $yZ(G) \notin C_{G/Z(G)}(xZ(G))$. *Therefore* $C_{G/Z(G)}(yZ(G))$ *is a Sylow* 2*-subgroup of* $G/Z(G)$ *different from* $C_{G/Z(G)}(xZ(G))$. *Since* A_5 *acts transitively, by conjugation, on the set of its Sylow* 2*-subgroup, there exists* $u \in G$ *such that* $u^{-1}xuZ(G) \in C_{G/Z(G)}(yZ(G))$. *It follows* $C_{G/Z(G)}(yZ(G)) = C_{G/Z(G)}(u^{-1}xuZ(G))$ *and*

$$\frac{C_G(y)}{Z(G)} \leq C_{\frac{G}{Z(G)}}\big(yZ(G)\big) = C_{\frac{G}{Z(G)}}\big(xZ(G)\big)^u = \left(\frac{C_G(x)}{Z(G)}\right)^u = \frac{C_G(x^u)}{Z(G)}. \tag{5.10}$$

Hence, by the previous note, we have $\frac{C_G(y)}{Z(G)} = \frac{C_G(x^u)}{Z(G)}$, *therefore* $\frac{C_G(y)}{Z(G)} = C_{G/Z(G)}\big(yZ(G)\big)$. *Thus we have proved that if* $|(xZ(G))^{G/Z(G)}| = 15$ *and* $\frac{C_G(x)}{Z(G)} = C_{G/Z(G)}(xZ(G))$, *then for all* $y \in G$ *with* $|(yZ(G))^{G/Z(G)}| = 15$, *we have* $\frac{C_G(y)}{Z(G)} = C_{G/Z(G)}(yZ(G))$ *and so* $|y^G| = 15$. *Hence in this case we have* $cs^*(G) = \{12, 15, 20\}$. *Note that if* $|x^G| = 12$ *or* 20, *then* $C_G(x)$ *is abelian since*

$60 = |x^G|[C_G(x) : Z(G)]$. *Also if* $|x^G| = 15$ *we saw that* $C_G(x)$ *is abelian. Hence there is no centralizer of any noncentral element of* G *strictly contained in* $C_G(x)$. *Thus* G *is an* F*-group. Therefore by Theorem 3.20, we have* $G/Z(G) \cong PSL_2(4) \cong A_5$ *and* $G' \cong SL_2(4) \cong A_5$. *It should be mentioned that in this case* $G/Z(G) \cong PSL_2(5) \cong A_5$ *but we cannot have* $G' \cong SL_2(5)$, *since otherwise* 30 *divides a conjugacy class size of* G *as* $cs^*(SL_2(5)) = \{12, 20, 30\}$.

• *Case 2: Now suppose that* $C_G(x)/Z(G) < C_{G/Z(G)}(xZ(G))$, *for all* $x \in G$ *with* $|(xZ(G))^{G/Z(G)}| = 15$. *Therefore*

$$|x^G| = \left[\frac{G}{Z(G)} : \frac{C_G(x)}{Z(G)}\right] = |(xZ(G))^{G/Z(G)}| \left[C_{G/Z(G)}\big(xZ(G)\big) : \frac{C_G(x)}{Z(G)}\right] \quad (5.11)$$

and so $|x^G| = 30$ *and also* $|C_G(x)/Z(G)| = 2$. *Thus in this case* $cs^*(G) = \{12, 20, 30\}$. *Note that in this case* $C_G(x)$ *is abelian, for all* $x \in G$. *Therefore* G *is an* F*-group and by Lemma 3.20,* $G/Z(G) \cong PSL_2(5) \cong A_5$ *and* $G' \cong SL_2(5)$. ■

Lemma 5.13: Let G *be a finite group such that* $G/Z(G) \cong S$, *where* S *is a simple group with trivial Schur multiplier. Then* $G \cong Z(G) \times S$. *Therefore if* $G/Z(G) \cong SL_2(2^m)$, *where* $m \geq 3$, *then* G *satisfies the one-prime power hypothesis if and only if* $G = Z(G) \times SL_2(8)$.

Proof 5.13: By Theorem 5.4, $|G' \cap Z(G)|$ *divides the order of the Schur multiplier of* S. *Thus* $G' \cap Z(G) = 1$ *as the Schur multiplier of* S *is trivial. Since* $G'Z(G)/Z(G)$ *is a normal subgroup of the simple group* $G/Z(G) \cong S$ *and* $G'Z(G)/Z(G) \cong G'$, *it follows that either* $G = Z(G)$ *if* $G' = 1$ *or* $G = Z(G) \times G'$ *if* $G' \neq 1$. *Hence* $G = Z(G) \times G' \cong Z(G) \times S$.

On the other hand, by Theorem 5.5, the Schur multiplier of $SL_2(2^m)$, $m \geq 3$, *is trivial. Also by the proof of Lemma 5.7,* $SL_2(2^m)$ *satisfies the one-prime power hypothesis if and only if* $m = 3$, *therefore the other assertion follows.* ■

Theorem 5.14: Let G *be a finite group such that* $G/Z(G)$ *is simple. Then* $B(G)$ *has no cycle of length* 4 *if and only if* $G \cong A \times S$, *where* A *is abelian, and* $S \cong SL_2(q)$, $q = 4, 8$.

Proof 5.14: Suppose that $B(G)$ *has no cycle of length* 4*. Therefore* G *satisfies the one-prime power hypothesis. Then, by Lemma 5.10,* G *is an* F*-group and so, by Lemma 5.7,* $G/Z(G) \cong SL_2(q)$*, where* $q \in \{4, 8\}$*. Hence by Lemma 5.12 and Lemma 5.13,* $G/Z(G) \cong SL_2(q) \cong G'$*, where* $q \in \{4, 8\}$*. Let* $H = Z(G)G'$*. Then* $H = Z(G) \times G'$ *is a normal subgroup of* G*. Therefore* $G' \cong H/Z(G)$ *is a normal subgroup of* $G/Z(G)$*, which implies that* $H = G$*. Hence* $G \cong A \times G'$*, where* A *is an abelian subgroup of* G*.*

The converse is obvious since $cs^*(SL_2(4)) = \{12, 15, 20\}$ *and* $cs^*(SL_2(8)) = \{56, 63, 72\}$*.* ■

REFERENCES

Baer R., (1953), "Group elements of prime power index", Transactions of the American Mathematical Society, 75 (1), 20-47.

Camina A. R., Camina R. D., (1998), "Implications of conjugacy class size", Journal of Group Theory, 1 (3), 257-269.

Casolo C., (1994), "Finite groups with small conjugacy classes", Manuscripta Mathematica, 82 (1), 171-189.

Chillag D., Herzog M., (1990), "On the length of the conjugacy classes in finite groups", Journal of Algebra, 131 (1), 110-125.

Conway J. H., Curtis R. T., Norton S. P., Wilson R. A., (1985), "ATLAS of Finite Groups", 1st Edition, Oxford University Press.

Dolfi S., Jabara E., (2009), "The structure of finite groups of conjugate rank 2", Bulletin of the London Mathematical Society, 41 (5), 916-926.

Fang M., Zhang P., (2003), "Finite groups with graphs containing no triangles", Journal of Algebra, 264 (2), 613-619.

Ferguson P. A., (1991), "Connections between prime divisors of conjugacy classes and prime divisors of $|G|$", Journal of Algebra, 143 (1), 25-28.

Gorenstein D., (1980), "Finite Groups", 2nd Edition, Chelsea Publishing Company.

Hafezieh R., Iranmanesh M.A., (2013), "Bipartite Divisor Graph for the Product of Subsets of Integers", Bulletin of the Australian Mathematical Society, 87 (2), 288-297.

Higman G., (1957), "Finite groups in which every element has prime power order", Journal of the London Mathematical Society, 32 (3), 335-342.

Huppert B., (1967), "Endliche Gruppen I", 1st Edition, Springer-Verlag.

Huppert B., Manz O., (1990), "Orbit sizes of p-groups", Archiv der Mathematik, 54 (2), 105-110.

Huppert B., Blackburn N., (1982), "Finite Goups II", 1st Edition, Springer-Verlag.

Iranmanesh M. A., Praeger C. E., (2010), "Bipartite divisor graphs for integer subsets", Graphs and Combinatorics, 26 (1), 95-105.

Isaacs I.M., (2008), “Finite Group Theory”, 1st Edition, American Mathematical Society.

Isaacs I. M., (1970), “Groups with many equal classes”, Duke Mathematical Journal, 37 (3), 501-506.

Ishikawa K., (2002), “On finite p-groups whichhave only two conjugacy lengths”, Israel Journal of Mathematics, 129 (1), 119-123.

Itô N., (1953), “On finite groups with given conjugate types I”, Nagoya Mathematical Journal, 6, 17-28.

Itô N., (1970), “On finite groups with given conjugate types II”, Osaka Journal of Mathematics, 7 (1), 231-251.

Itô N., (1970), “On finite groups with given conjugate types III”, Mathematische Zeitschrift, 117, 267-271.

Kazarin L. S., (1981), “On groups with isolated conjugacy classes”, Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 25 (7), 40-45.

Kurzweil H., Stellmacher B., (2004), “The Theory of Finite Groups-An Introduction”, 1st Edition, Springer.

Lewis M.L., (2008), “An Overview of Graphs Associated with Character Degrees and Conjugacy Class Sizes in Finite Groups”, Rocky Mountain Journal of Mathematics, 38 (1), 175-211.

Manz O., Wolf T.R., (1993), “Representations of Solvable Groups”, 1st Edition, Cambridge University Press.

Rebmann J., (1971), “F-Gruppen”, Archiv der Mathematik, 22 (1), 225-230.

Schmidt R., (1994), “Subgroup Lattices of Groups”, 1st Edition, Walter de Gruyter.

Suzuki M., (1982), “Grouph Theory I”, 1st Edition, Springer-Verlag.

Taeri B., (2010), “Cycles and bipartite graph on conjugacy class of groups”, Rendiconti del Seminario Matematico della Università di Padova, 233-247, Padova, Italy, 22-27 August.

Wallis W. D., (2007), “A Beginner’s Guide to Graph Theory”, 2nd Edition, Birkhäuser.

Wilson R. J., (1996), “Introduction to Graph Theory”, 4th Edition, Prentice Hall.

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