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CONFORMAL SYMMETRIES AND KAĆ-MOODY ALGEBRA IN SIGMA MODELS

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A THESIS SUBMITTED FOR THE DEGREE OF
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GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

SİGMA MODELLERDE KONFORMAL SİMETRİLER VE KAĆ-MOODY CEBİRİ

MEHMET FATİH MİNTAŞ

YÜKSEK LİSANS TEZİ

FİZİK ANABİLİM DALI

DANIŞMANI

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SUMMARY

We discuss the conformal and Kać-Moody symmetries in two dimensional sigma models. We perform our analysis regarding the classical and quantum aspects of the sigma models whose fields are taking values on a target space manifold and living in a two dimensional world-sheet. We explicitly point out that existence of classical symmetries requires the on-shell holomorphic stress energy tensors. However, the quantum conformal symmetries are provided with the presence of conformal ward identities. We clearly demonstrate that ward identities are also split into holomorphic forms. While classical symmetries are presented by de-Witt algebra, conformal symmetries are configured by Virasoro algebra. We slightly touch the operator formalism with the construction of Hilbert space of sigma models. In the last chapter we work out the Kać-Moody symmetries of sigma models in a profound way by employing the hidden symmetry formalism. In this respect we establish the complex version of Lax-pair equations and investigate the hidden symmetries. We show that hidden symmetry formalism gives rise to Kać-Moody symmetries.

Key Words: Sigma Models, Kać-Moody Symmetry, Conformal Symmetry, Hidden Symmetry Mechanism, Ward Identity.

ÖZET

Bu çalışmada iki boyutlu sigma modellerindeki konformal ve Kać-Moody simetrilerini ele aldık. Analizimizi, bir hedef uzay katmanı ile iki boyutlu bir yer tabakasında değerlerini alan sigma model alanlarının klasik ve kuantum özelliklerini göz önünde bulundurarak yaptık. Açıkça ifade edebiliriz ki, klasik simetrilerin varlığı holomorfik stres-enerji tensörünün varlığını gerekli kılar. Halbuki kuantum konformal simetrilerin varlığı ise konformal Ward biriminin varlığı ile sağlanmış olur. Ward birimlerinin holomorfik olarak ayrıldığını açık bir şekilde gösterdik. Klasik simetriler de-Witt cebiri ile gösterilirken konformal simetriler Virasoro cebiri ile gösterilir. Sigma modellerin Hilbert uzayının oluşturulmasında operatör formalizmine çok az temas ettik. Son bölümde, gizli simetri formalizmini etkili bir şekilde kullanarak sigma modellerdeki Kać-Moody simetrilerini çalıştık. Bu itibarla, Lax-pair denklemlerinin kompleks versiyonun oluşturduk ve gizli simetrileri araştırdık. Ve gizli simetri formalizminin Kać-Moody simetrilerini verdiğini gösterdik.

Anahtar Kelimeler: Sigma Modeller, Konformal Simetri, Kać-Moody Simetri, Gizli Simetri Mekanizması, Ward Birimi.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
S	:	Sigma model action
Φ	:	Collection of fields
$\mathcal{L}$	:	Lagrangian
$G_{\boldsymbol{a}}$	:	Symmetry transformation generator
$\det$	:	Determinant
j_a^μ	:	Current
$T_{z\bar{z}}$	:	Stress-energy tensor
$h_{\mu\nu}$	:	Metric tensor
$\wedge$	:	Wedge product
$[\,,]$	:	Commutator
z and $\bar{z}$	:	Complex coordinates
$*$	:	Hodge star product

LIST of FIGURES

1. INTRODUCTION

It is our current understanding that physical theories can be best formulated in terms of their symmetries in which they are embodied. Thus, symmetry is the key leading term behind all theories. Since its first debut in 1960, sigma models [Gell-mann and Lévy, 1960] are the basic framework at which almost all physically realizable theories are rep- resented. Currently they have applications in many fields of physics varying from string theories, supersymmetric models and gravity theories to solid state, condensed matter and optical physics. Apparently understanding symmetries in sigma models leads many other theories to be comprehended in various features.

It is widely believed that an ultimate theory in unification attempts should contain all symmetries at hand. Therefore, one needs to inquire as much symmetries to extend to infinite in quantity, namely Kać-Moody symmetries. In order to better understand the character of Kać-Moody symmetry one further needs to consider its lie algebraic structure. In this thesis, we try to achieve this goal and obtain conformal symmetries in sigma models thereof consistently extent to Kać-Moody algebra by means of lie algebraic methods developed by Ellié Cartan[1].

We first outline some field theoretical methods and introduce sigma model in the following chapter. We then start investigating classical conformal symmetries in two dimensional sigma models, and then smoothly go on to the analysis of quantum conformal symmetries to obtain the corresponding Ward Identities. We successively get operator product formalism (OPF) by means of Hilbert space constructions built by Ward Identities. Using the standard approach, we show that OPF gives rise to canonical commutation relations, which is an indication of symmetries. It is well-known that local conformal symmetry in two dimensions yields infinite number of symmetries but globally they reduce to finite number. Inspired by this result it is quite reasonable to consider Kać-Moody symmetries in sigma models. In the last part of our work we do this.

We first construct the complex version of sigma model action and figure out

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[1] A French Mathematician lived at the beginning of twentieth century. He contributed differential geometry considerably, especially in the field of differential forms and lie algebra using his famous moving frame methods, see [2]-[5] for details of Cartan's approach.

the corresponding equations of motion. Together with the Cartan structural equations, we are able to build our Lax-pair equations. Equations we find are distinct from one usually expressed in literature since our equations contain coupled terms. We could express these equations compactly in differential form approach by getting rid of couplings. We then move on the hidden symmetry formalism on the steps of solutions of lax-pair equation. Analytic continuation provides the key factor in the hidden symmetry formalism and leads to Kać-Moody symmetries.

2. OUTLINE OF BACKGROUND MATERIAL

2.1. Review of Field Theory Methods

In this section we will review some field theory basics emphasizing the importance of symmetries. Consider a collection of fields which are collectively denoted by Φ. The action functional depending on Φ and its first derivatives and defined on a d-dimensional space-time, denoted by x is given by

$$S = \int d^d x \mathcal{L}(\Phi, \partial_\mu \Phi) \tag{2.1}$$

A transformation in space-time affects both space-time coordinates and fields by the rules:

$$x \to x \quad \text{and} \quad \Phi(x) \to \Phi'(x') \tag{2.2}$$

In this transformation the new position x' is a function of x', and the new field Φ' at x' is expressed as a function of Φ at x

$$\Phi'(x') \to \mathcal{F}(\Phi) \tag{2.3}$$

It is crucial to realize that the field Φ, considered as a mapping from space-time to some target space $\mathcal{M}(\Phi: \mathbb{R}^d \to \mathcal{M})$, is affected by the transformation (2.2) in two ways: first by the functional change $\Phi' = \mathcal{F}(\Phi)$, and the second by the change of argument $x \to x'$. The change of the action functional under the transformation (2.2) is obtained by substituting the new function $\Phi'(x)$ for the function $\Phi(x)$ (we note that the argument x is the same in both case). In other words, the new action is

$$S' = \int d^d x \mathcal{L}\left(\Phi'(x), \partial_\mu \Phi'(x)\right) = \int d^d x' \mathcal{L}\left(\Phi'(x'), \partial'_\mu \Phi'(x')\right)$$

$$= \int d^d x' \mathcal{L}\left(\mathcal{F}\big(\Phi(x)\big), \partial'_\mu \mathcal{F}\big(\Phi(x)\big)\right) \tag{2.4}$$

$$= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}\big(\Phi(x)\big), \left(\partial x^\nu / \partial x'^\mu\right) \partial_\nu \mathcal{F}\big(\Phi(x)\big)\right)$$

This shows the new transformed action in generalized form. To study the effect of infinitesimal transformation on the action we make use of the following transformations

$$x'^{\mu} = x^{\mu}\omega_a \frac{\delta x^{\mu}}{\delta\omega_a} \quad (2.5a)$$

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta\mathcal{F}}{\delta\omega_a}(x) \quad (2.5b)$$

Here $\{\omega_a\}$ is a set of infinitesimal parameters, which is kept to the first order. We de- fine the generator G_a of a symmetry transformation by the following expression for the infinitesimal transformation at the same point:

$$\delta\Phi(x) \equiv \Phi'^{(x)} - \Phi(x) \equiv -i\omega_a G_a \Phi(x) \quad (2.6)$$

We may relate this definition to Eq. (2.5a) and (2.5b) by noting that, to first order in ω_a

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta\mathcal{F}}{\delta\omega_a}(x) = \Phi(x') - \omega_a \frac{\delta x^{\mu}}{\delta\omega_a}\,\partial_\mu\Phi(x') + \omega_a \frac{\delta\mathcal{F}}{\delta\omega_a}(x') \quad (2.7)$$

The explicit expression for the generator therefore

$$iG_a\Phi = \frac{\delta x^{\mu}}{\delta\omega_a}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\omega_a} \quad (2.8)$$

From the last equation (2.4), we may write the effect on the action of the infinitesimal transformation (2.5). To first order, the Jacobian matrix is

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \delta_\mu^\nu + \partial_\mu(\omega_a \frac{\delta x^{\nu}}{\delta\omega_a}) \quad (2.9)$$

The determinant of this matrix may be calculated to first order from the formula

$$\det(1 + \mathbf{E}) \approx 1 + Tr\mathbf{E} \qquad \text{(for any matrix } \mathbf{E} \text{ which is small)} \quad (2.10)$$

$$\left|\frac{\partial x\prime}{\partial x}\right| \approx 1 + \partial_\mu(\omega_a \frac{\delta x^{\mu}}{\delta\omega_a}) \quad (2.11)$$

The inverse Jacobian matrix may be obtained to the first order simply by reversion the sign of the transformation parameter:

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta_\mu^\nu - \partial_\mu(\omega_a \frac{\delta x^{\nu}}{\delta\omega_a}) \quad (2.12)$$

With the help of these preliminary steps, the transformed action S' may be written as

$$S' = \int d^d x \left[1 + \partial_\mu \left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right] \times \\ \mathcal{L}\left[\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta_\mu^\nu - \partial_\mu(\omega_a \frac{\delta x^\mu}{\delta \omega_a})\right]\left[\partial_\nu \Phi + \partial_\nu \left(\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right)\right]\right] \tag{2.13}$$

The variation $\delta S = S' - S$ of the action contains terms with no derivatives of ω_a. These sum up to zero if the action is symmetric under rigid transformations. Then δS involves the first derivatives of ω_a, obtained by expanding the Lagrangian. We write

$$\delta S = -\int dx\, j_a^\mu \partial_\mu \omega_a \tag{2.14}$$

where

$$j_a^\mu = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta_\nu^\mu \mathcal{L}\right] \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{2.15}$$

The quantity j_a^μ is called the current associated with the infinitesimal transformation (2.5). Integration by parts yields

$$\delta S = \int d^d x\, \partial_\mu j_a^\mu \omega_a \tag{2.16}$$

Now comes Noether's theorem: If the field configuration obeys the classical equations of motion, the action is stationary against any variation of the fields. In other words, δS should vanish for any position-dependent parameters $\omega_a(x)$. This implies the conservation law

$$\partial_\mu j_a^\mu = 0 \tag{2.17}$$

In words, every continuous symmetry implies the existence of a current given by (2.15), which is classically conserved.

The conserved charge associated with j_a^μ is

$$\mathcal{Q}_a = \int d^{d-1} x\, j_a^0 \tag{2.18}$$

where j_a^0 is the time component of j_a^μ, and a $d^{d-1}x$ stands for the purely spatial integration measure.

Classically, the invariance of the action under a continuous symmetry implies the existence of a conserved current. At the quantum level, correlation functions are the main object of study, and a continuous symmetry leads to constraints relating different correlation functions.

Consider again a theory involving a collection of fields Φ with an action $S(\Phi)$ invariant under a transformation of the type (2.2). Consider then the general correlation function[2]

$$\langle \Phi(x_1) \cdots \Phi(x_n) \rangle = \frac{1}{Z} \int [d\Phi]\, \Phi(x_1) \cdots \Phi(x_n) e^{-S[\Phi]} \tag{2.19}$$

where Z is the vacuum functional. The consequence of the symmetry of the action and of the invariance of the functional integration measure under the transformation (2.2) is

$$\langle \Phi(x_1') \cdots \Phi(x_n') \rangle = \langle \mathcal{F}(\Phi(x_1)) \cdots \mathcal{F}(\Phi(x_n)) \rangle \tag{2.20}$$

where the mapping $\mathcal{F}$ describes the functional change of the field under the transformation, as in Eq. (2.3). The demonstration of this identity is straightforward:

$$\langle \Phi(x_1') \cdots \Phi(x_n') \rangle = \frac{1}{Z} \int [d\Phi]\, \Phi(x_1') \cdots \Phi(x_n') e^{-S[\Phi]}$$

$$= \frac{1}{Z} \int [d\Phi']\, \Phi'(x_1) \cdots \Phi'(x_n') e^{-S[\Phi']} \tag{2.21}$$

$$= \frac{1}{Z} \int [d\Phi]\, \mathcal{F}(\Phi(x_1)) \cdots \mathcal{F}(\Phi(x_n))\, e^{-S[\Phi]} = \langle \mathcal{F}(\Phi(x_1)) \cdots \mathcal{F}(\Phi(x_n)) \rangle$$

The consequence of a symmetry of the action and the measure on the correlation functions may also be expressed via the so-called Ward identities, which we shall now demonstrate. An infinitesimal transformation may be written in terms of the generators as

$$\Phi'(x) = \Phi(x) - \, i\omega_a G_a \Phi(x) \tag{2.22}$$

where ω_a is a collection of infinitesimal, constant parameters. Note that the positions show up on both sides of this expression. We make a change of functional integration

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[2] We emphasize that fields inside the correlation functions are time-ordered so that an implicit time-ordering operator is intended. Time ordering requires that the relatively smaller times come to the right.

variables in the correlation function (2.19), in the form of the above infinitesimal transformation with $\omega_a$now a function of x. The action is not invariant under such a local transformation; its variation being given by (2.16). Denoting by X the collection $\Phi(x_1)) \cdots (\Phi(x_n))$ of the fields in the correlation function and by δX its variation under the transformation, we can write

$$\langle X \rangle = \frac{1}{Z} \int [d\Phi'] \, (X + \delta X) \, e^{-S[\Phi] - \int dx \partial_\mu j_a^\mu \omega_a(x)} \tag{2.23}$$

We again assume that the functional integration measure is invariant under the local transformation (i.e., $[d\Phi'] = [d\Phi]$) When expanded to first order in $\omega_a(x)$, the above yields

$$\langle \delta X \rangle = \int dx \, \partial_\mu \langle j_a^\mu(x) X \rangle \omega_a(x) \tag{2.24}$$

The variation δX explicitly given by

$$\delta X = -i \sum_{i=0}^{n} \big(\Phi(x_1) \cdots G_a \Phi(x_i) \cdots \; \Phi(x_n) \big) \, \omega_a(x_i) \tag{2.25a}$$

$$= -i \int dx \, \omega_a(x) \sum_{i=1}^{n} \{ \Phi(x_1) \cdots G_a \Phi(x_i) \cdots \; \Phi(x_n) \} \, \delta(x - x_i) \tag{2.25b}$$

Since (2.24) holds for any infinitesimal function $\omega_a(x)$, we may write the following local relation:

$$\frac{\partial}{\partial x^\mu} \langle j_a^\mu(x) \Phi(x_1) \cdots \Phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \, \langle \Phi(x_1) \cdots G_a \Phi(x_i) \cdots \; \Phi(x_n) \rangle \tag{2.26}$$

This is the Ward identity for the current j_a^μ. Note that the form of the current may be modified from the canonical definition (2.15) without affecting the Ward identity, if one adds to j_a^μ a quantity that is divergenceless identically.

We integrate the Ward identity (2.25) over a region of space-time that includes all the points x_i. On the left-hand side (l.h.s.), we obtain a surface integral

$$\int_\Sigma ds_\mu \langle j_a^\mu(x) \Phi(x_1) \cdots \Phi(x_n) \rangle \tag{2.27}$$

which vanishes, since the hyper-surface Σ may be sent to infinity without affecting the integral, indeed the divergence $\partial_\mu \langle j_a^\mu X \rangle$ vanishes away from the points x_i and the correlator $\langle j_a^\mu X \rangle$ goes to zero sufficiently fast as $x \to \infty$, by hypothesis. For the right hand side (r.h.s.) of Eq. (2.26), this implies

$$\delta\langle \Phi(x_1) \cdots \Phi(x_n) \rangle \coloneqq -i\omega_a \sum_{i=1}^{n} \langle \Phi(x_1) G_a \Phi(x_i) \cdots \Phi(x_n) \rangle = 0 \qquad (2.28)$$

In other words, the variation of the correlator under an infinitesimal transformation vanishes. This is simply the infinitesimal version of Eq. (2.21).

The Ward identity allows us to identify the conserved charge

$$Q_a = \int d^{d-1}x \, j_a^0(x) \qquad (2.29)$$

as the generator of the symmetry transformation in the Hilbert space of quantum states.

The fact that a correlation function is the vacuum expectation value of a time-ordered product in the operator formalism, we conclude that

$$[Q_a, \Phi] = -iG_a \Phi \qquad (2.30)$$

In other words, the conserved charge Q_a is the generator of the infinitesimal symmetry transformation in the operator formalism. These identities are of course usually obtained in the Euclidean formalism. An easy way to retrieve the Minkowski space-time is to replace the charge Q by $-iQ$, since it is the outcome of an integration of the time-like component of a vector.

2.1.1. The Energy-Momentum Tensor

In this subsection, we apply the general results of the previous section to the invariance of a theory with respect to the translations and rotations (or Lorentz transformations). The conserved current associated with translation invariance is the energy-momentum tensor, whose components are the density and flux density of energy and momentum. In this thesis, the consequences of conformal symmetry will be expressed in terms of the Ward identities associated with the energy-momentum tensor.

The infinitesimal translation $x'^\mu \to x^\mu + \epsilon^\mu$includes the following variations in the coordinates and fields:

$$\frac{\delta x^{\mu}}{\delta \epsilon^{\nu}} = \delta^{\mu}_{\nu}, \qquad \frac{\delta \Phi}{\delta \epsilon^{\nu}} = 0 \tag{2.31}$$

Consequently, the corresponding canonically conserved current is

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial^{\nu}\Phi \tag{2.32}$$

and the conservation law is $\partial_\mu T_c^{\mu\nu} = 0$. The conserved charge is the four-momentum

$$P^{\nu} = \int d^{d-1}x\, T_c^{0\nu} \tag{2.33}$$

In particular, the energy is

$$P^0 = d^{d-1}x \left\{\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\dot{\Phi} - \mathcal{L}\right\} \tag{2.34}$$

which is the usual definition of the Hamiltonian. As an operator, the conserved charge P_μ has therefore the following effect in Euclidean time, according to Eq. (2.30):

$$[P_\mu, \Phi] = -\partial_\mu \Phi \tag{2.35}$$

In real time, this relation becomes $[P_\mu, \Phi] = -i\partial_\mu \Phi$, which is the well-known commutator of an $x-$dependent operator with momentum in ordinary quantum mechanics.

2.2. Sigma Model Preliminaries

The two dimensional non-linear sigma model [Hull et al., 2009] has the action

$$S = -\frac{1}{4}\int_\Sigma d^2\sigma\sqrt{h}\left[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij}(\phi) + \epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j b_{ij}(\phi)\right] \tag{2.36}$$

for maps $\{\phi\}$ from a two dimensional manifold Σ to a n-dimensional target space $\mathcal{M}$:

$$\phi: \Sigma \to \mathcal{M} \tag{2.37}$$

specified locally by functions $\phi^i(\sigma)$ giving the dependence of the real coordinates ϕ^i of $\mathcal{M}$ on the real coordinates σ^μ of Σ, where σ^μ is identified by (τ, σ). For notational convenience, we employ the Greek letters for world-sheet coordinates while letters in

Latin alphabet are reserved for target space. The index i is running up to n which is the dimension of $\mathcal{M}$, n = dim($\mathcal{M}$). The target manifold $\mathcal{M}$ has a field-dependent metric g and 2-form potential field b, while Σ has a metric $h_{\mu\nu}$ with $h = |\det(h_{\mu\nu})|$. The potential b needs only be locally defined, but there is a globally-defined closed 3-form field strength H such that locally $H = db$. One shows that the equations of motion depend on b only through the 3-form field strength H and are well-defined as shown by the form

$$\partial^2\phi^k = \left[\epsilon^{\mu\nu}H^k_{ij} - h^{\mu\nu}\Gamma^k_{ij}\right]\partial_\mu\phi^i\partial_\nu\phi^j \tag{2.38}$$

with $\partial^2 = \partial^\mu\partial_\mu = h^{\mu\nu}\partial_\nu\partial_\mu$. Notice that we ignore the boundary terms. In the usual case, the metric $h_{\mu\nu}$ has Lorentzian signature[3] and $g_{ij}(\phi)$ and $b_{ij}(\phi)$ have real components. The Euclidean version of this action used in the path integral (given by a Wick rotation) is

$$S = -\frac{1}{4}\int_{\Sigma_E} d^2\sigma\sqrt{h}\left[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij}(\phi) + i\epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j b_{ij}(\phi)\right] \tag{2.39}$$

with $h_{\mu\nu}$ an Euclidean signature metric, and $\Sigma_E$represents the Euclidean world-sheet. Note that the term involving b is now pure imaginary, so that the action is complex. For both Lorentzian and Wick-rotated case, the quantum theory is well-defined if H is a globally-defined 3-form that represents an integral cohomology class, $H \in H^3(\mathbb{Z})$ Geometrically this means that there is a gerbe with curvature H and connection b_α in each coordinate patch $\mathcal{O}_\alpha$. For the path integral, if $H_2(\mathcal{M})$ is non-trivial, it is not sufficient to specify H, and a choice of b must be made. Then the term containing the b-field

$$e^{2i\pi\int\phi^*(b)} \tag{2.40}$$

defines the holonomy of a gerbe over the embedding of the world sheet.

For Euclidean signature one can also consider the real action (2.36) with $h_{\mu\nu}$ a Euclidean signature metric. For the action to be well-defined, b should be a globally-defined 2-form. However, the field equations are well-defined provided only that H is a well-defined 3-form, so that a classical theory exists for any closed 3-form H.

[3] By convention, Lorentzian signature is (-1, 1), while Euclidean signature is the usual one (1, 1).

One way that non-linear σ-models differ from other quantum field theories often considered in high energy physics is that the coefficient in front of their kinetic term depends on the fields themselves. This feature also makes these models interesting, since, at least naively, it might alter the expected short-distance behavior of n-point correlation functions or operator product expansions.

It is manifest in literature that sigma models are diffeomorphism invariant together with the Weyl invariance [Meessen].

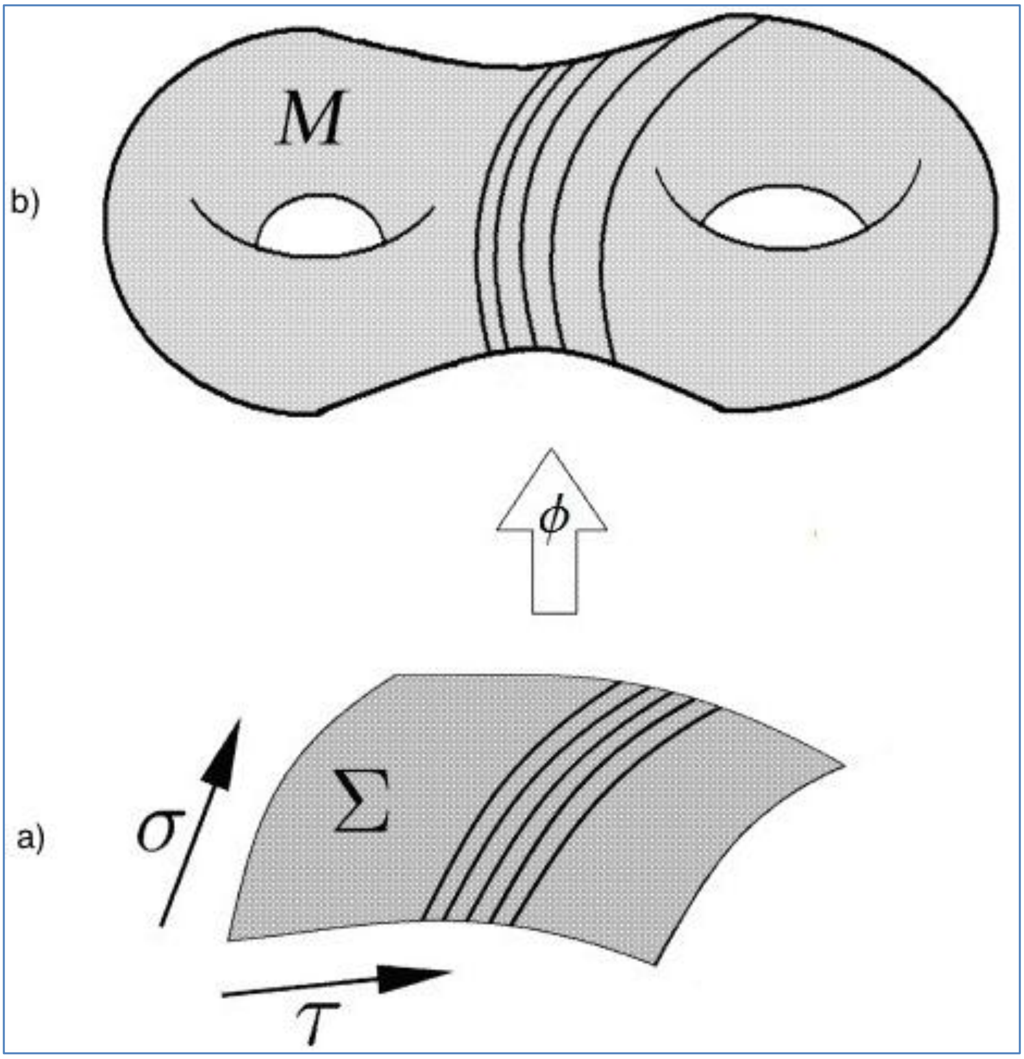

Figure 2.1: Graph configures the sigma model map. The map ϕ embeds world-sheet a) Σ into the target space $\mathcal{M}$, b) and represents the local coordinates on $\mathcal{M}$. Notice that τ and σ are local coordinates of Σ.

Most often one encounters sigma models defined on a group manifold G. In this case sigma model turns out to be Wess-Zumino-Witten Model (WZW Model) [Witten, 1984] and can be expressed by the following action

$$S = S_0 + \Gamma \tag{2.41}$$

where

$$S_0 = -\frac{1}{4}\int d^2\sigma\, Tr(g^{-1}\partial^{\mu} g g^{-1}\partial_{\mu} g) \tag{2.42}$$

and Γ represents the WZ term:

$$\Gamma = \int d^3y\, \epsilon_{\alpha\beta\gamma} Tr(\partial^{\alpha} g^{-1} g \partial^{\beta} g^{-1} \partial^{\gamma} g) \tag{2.43}$$

In this representation Tr stands for the trace, the field g is given by the map $g: \Sigma \rightarrow G$. We take Σ to be two dimensional Minkowski space, and G be group valued target space manifold which usually takes values in a compact Lie group. We emphasize that $S_0$is expressed on the boundary of the region at which WZW Model is defined, and Γ-term describes the corresponding bulk-space counterpart, which is known as Wess-Zumino term (WZ-term). Without WZ-term, this model is not scale invariant at the quantum level, and therefore remains missing. WZ-term is necessary to guarantee the scale invariance, and as we will see below needed for conformal invariance. This term is relevant to the geometry of the model and inherits the topology of the space in which the theory lies.

2.3. Simple Lie Groups

Since WZW Models are based on Lie groups, we briefly review basics of simple Lie groups and lie algebra in this section. A lie algebra **g** is a vector space equipped with an antisymmetric binary operation $[\,,]$, called a commutator, mapping **g** × **g** into **g**, and further constrained to satisfy the Jacobi identity

$$\left[X, [Y, Z]\right] + \left[Z, [X, Y]\right] + \left[Y, [Z, X]\right] = 0 \qquad \text{for } X, Y, Z \in \mathbf{g} \tag{2.44}$$

Lie group g is the exponential of **g**, i.e. to $X \in$ **g** there corresponds the group elements e^{iaX} where a is some parameter and the exponential is defined from its power expansion. Hence the algebra describes the group in the vicinity of the identity.

A Lie algebra can be specified by a set of generators J^a and their commutation relations

$$[J^a, J^b] = \sum_c i f_c^{ab} J^c \quad (2.45)$$

The number of generators is the dimension of the algebra. The constants f_c^{ab} are called c the structure constants. Simple Lie algebras are the Lie algebras that do not contain any proper subset of generators $\{L^a\}$ such that $[L^a, J^b] \in \{L^a\}$ for any J^b. A direct sum of simple algebras is said to be semisimple. Based on the basis constructed to define algebra, generators take corresponding values.

In the standard Cartan-Weyl basis, the generators are constructed as follows. We first find the maximal set of commuting Hermitian generators H^i, $i = 1, \cdots, r$ where r is the rank of the algebra:

$$\left[H^i, H^j\right] = 0 \quad (2.46)$$

The set of generators from the Cartan subalgebra h. The generators of the Cartan subalgebra can all be diagonalized simultaneously. The remaining generators are chosen to be those particular combinations of the J^a's that satisfy the following eigenvalue equation:

$$\left[H^i, E^\alpha\right] = \alpha^i E^\alpha \quad (2.47)$$

The vector $\alpha = (\alpha^1, \cdots, \alpha^r)$ is called a root and E^α is the corresponding ladder operator. Because h is the maximal Abelian subalgebra of g, the roots are non-degenerate. The root α naturally maps an element $H^i \in h$ to the number α^i by $\alpha\left(H^i\right) = \alpha^i$. Hence, the roots are elements to the dual of the Cartan subalgebra, $\alpha \in h^*$.

Since $E^{-\alpha} = (E^\alpha)^\dagger$ $(\mathrm{E}^\alpha)^\dagger$, one can show that $-\alpha$ is necessarily a root whenever α is. In this thesis Δ will denote the set of all roots. Root components can be regarded as the nonzero eigenvalues of H^i in the adjoint representation, for which the Lie algebra itself serves as the vector space on which the generator acts. In the adjoint representation the action of a generator X is represented by $\mathrm{ad}(X)$, defined as

$$\mathrm{ad}(X)Y \coloneqq [X, Y] \quad (2.48)$$

In view of specifying the remaining commutators, the full set of commutation relations in the Cartan-Weyl basis is i) $[H^i, H^j] = 0$, ii) $[H^i, E^\alpha] = \alpha^i E^\alpha$, iii) $[E^\alpha, E^\beta] = \mathcal{N}_{\alpha,\beta} E^{\alpha+\beta}$ if $\alpha + \beta \in \Delta$ it is $\frac{2}{\alpha^2}\alpha.H$ if $\alpha = -\beta$ and it is 0 otherwise.

Commutators are usually fixed by means of the renormalized version of the Killing form defined as

$$K(X,Y) \coloneqq \frac{1}{2\mathrm{g}}\mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y) \tag{2.49}$$

where g is a constant and the dual Coxeter number of the algebra **g**. Killing form defines a scalar product and it can be used to lower or raise indices. Standard basis $\{J^\alpha\}$ is understood to be orthonormal with respect to K:

$$K(J^a, J^b) = \delta^{a,b} \tag{2.50}$$

The same normalization holds for the generators of the Cartan subalgebra

$$K(H^i, H^j) = \delta^{i,j} \tag{2.51}$$

Killing form of E^α and $E^{-\alpha}$ turns out to be

$$K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha^2|} \tag{2.52}$$

However, the fundamental role of the Killing form is to establish an isomorphism between the Cartan subalgebra h and its dual h^*: the form $K(H^i,\cdot)$ (i fixed) maps every element of the Cartan subalgebra onto a number. Hence, to every element $\gamma \in h^*$, there corresponds a $H^\gamma \in h$ through

$$\gamma(H^i) = K(H^i, H^j) \tag{2.53}$$

With this isomorphism, the Killing form can be transferred into a positive definite scalar product in the dual space

$$(\gamma, \beta) = K(H^\beta, H^\gamma) \tag{2.54}$$

- Weights

Up to this point, we have analyzed the structure of the algebra from the point of view of a particular representation (the adjoint), that for which the algebra itself plays the role of the vector space. In this representation, the eigenvalues of the Cartan generators are called the roots and the scalar product between roots is induced by the Killing form. Since the essential structure of the algebra is coded in this representation, it needs to be studied in more detail. For this, it is useful to first recast the problem in the general context of a finite-dimensional representation.

For an arbitrary representation, a basis $\{|\lambda\rangle\}$ can always be found such that

$$H^i|\lambda\rangle = \lambda^i|\lambda\rangle \tag{2.55}$$

The eigenvalues build the vector $\lambda = (\lambda^1, \cdots, \lambda^r)$ = (λ^1 , · · · , λ^r), called a weight. Weights live in the space h^*: $\lambda(H^i) = \lambda^i$. Hence, the scalar product between weights is also fixed by the Killing form. In the adjoint representation, the weights deserve the special name of roots. The commutator (2.3.4) shows that E^α changes the eigenvalue of a state by

$$H^i E^\alpha|\lambda\rangle = [H^i, E^\alpha]|\lambda\rangle + E^\alpha H^i|\lambda\rangle = (\lambda^i + \alpha^i)E^\alpha|\lambda\rangle \tag{2.56}$$

so that $E^\alpha|\lambda\rangle$ if nonzero, must be proportional to state $|\lambda + \alpha\rangle$ This justifies the name ladder (or step) operator for E^α. Representations of interest are the finite-dimensional ones. For these, we will derive an important relation, to be used shortly for the adjoint representation. For any state $|\lambda\rangle$ in a finite-dimensional representation, there are necessarily two positive integers p and q, such that

$$(E^\alpha)^{p+1}|\lambda\rangle \sim E^\alpha|\lambda + p\alpha\rangle = 0 \tag{2.57a}$$

$$(E^{-\alpha})^{q+1}|\lambda\rangle \sim E^\alpha|\lambda - p\alpha\rangle = 0 \tag{2.57b}$$

for any root α. Indeed, notice that the triplet of generators E^α, $E^{-\alpha}$ and $\alpha.H/|\alpha^2|$ forms an $SU(2)$ subalgebra to the set $\{J^+, J^-, J^3\}$, with commutation relations

$$[J^+, J^-] = 2J^3, \qquad [J^3, J^\pm] = \pm J^\pm \tag{2.58}$$

Therefore, $|\lambda\rangle$ belongs to a finite-dimensional representation, its projection onto the $SU(2)$ subalgebra associated with the root α must also be finite dimensional.

- Simple Roots and Cartan Matrix

The number of roots is equal to the dimension of the algebra minus its rank, and this number is in general much larger than the rank itself. This means that the roots are linearly dependent. We then fix a basis $\{\beta_1\beta_2 \cdots \beta_3\}$in the space h^*, so that any root can be expanded as

$$\alpha = \sum_{i=1}^{r} n_i \, \beta_i \qquad (2.59)$$

In this basis, an ordering can be defined as follows: α is said to be positive if the first number in the sequence $(n_1, n_2, \cdots, n_r)$ is positive. We denote by Δ_+ the set of positive roots. The set of negative roots Δ_+ is defined in the obvious way. We have already observed that whenever α is a root, $-\alpha$ is also a root, hence $\Delta_- = \Delta_+ = \Delta_+$. A simple root, α_i is defined to be a root that cannot be written as the sum of two positive roots. There are necessarily r simple roots, and their set $\{\alpha_1, \cdots, \alpha_r\}$ provides the most convenient basis for the r-dimensional space of roots. Notice that the subindex is a labeling index: it does not refer to a root component. Two immediate consequences of the definition of simple roots are : (i) $\alpha_i - \alpha_j \notin \Delta$; (ii) any positive root is a sum of positive roots (indeed, if a positive root is not simple, it can be written as a sum of two positive roots, which, if not simple, can also be written as the sum of two positive roots, and so on). The scalar products of simple roots define the Cartan matrix

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_j^2} \qquad (2.60)$$

the entries of this matrix are necessarily integers. Its diagonal elements are equal to 2 and it is not symmetric in general. The Schwarz inequality implies that $A_{ij}A_{ji} > 0$ for $i \neq j$. Since $\alpha_i - \alpha_j$ is not a root, $E_j^{\alpha}|\alpha_i\rangle = 0$, and $q = 0$ in Eq. (2.57) for $\lambda = \alpha_i$ and $\alpha = \alpha_j$.

It is convenient for us to introduce a special notation for the quantity $2\alpha_i/|\alpha_i|^2$:

$$\breve{\alpha}_i = \frac{2\alpha_i}{|\alpha_i|^2} \qquad (2.61)$$

$\breve{\alpha}_i$ is called coroot associated with the root α_i. The scalar product between roots and coroots is thus always an integer. The Cartan matrix now takes the compact form

$$A_{ij} = (\alpha_i, \breve{\alpha}_\iota) \quad (2.62)$$

The dual Coxeter number is defined as

$$g = \sum_{i=1}^{r} \breve{\alpha}_\iota + 1 \quad (2.63)$$

which is used to define the normalization constant of Killing form above. The theory of Simple Lie groups is in fact vast, and requires a much tedious analysis. But for our purpose, we stop here and go on to their applications by means of conformal symmetries which is the topic of next chapter.

3. CONFORMAL INVARIANCE IN SIGMA MODELS

In this chapter we construct the classical and quantum conformal symmetry of sigma models in two dimensions, and generalize them to Lie algebra valued fields by using WZW Sigma Models. Classical conformal symmetries are embodied in stress-energy tensor, which has a form such that only holomorphic and antiholomorphic components exist, while quantum conformal symmetries are demonstrated by the presence of ward identities, which are derived through correlation (green) functions.

3.1. Conformal Invariance in Classical Sigma Models

Classical symmetries are displayed via action functional, and in turn leads to the classical conserved quantities by means of currents [8]-[11]. It is well-known that conformal symmetry in two dimensions implies the Cauchy-Riemann conditions and therefore makes us use of separability of holomorphic and antiholomorphic coordinates [11], [12].

We consider coordinates $\pi = (\tau, \sigma)$ on the world-sheet plane with metric tensor $h_{\mu\nu}$ and we perform a coordinate transformation $\pi^\mu \to \xi^\mu(\pi)$. It is known that metric tensor transforms as

$$h^{\mu\nu} \to \left(\frac{\partial \xi^\mu}{\partial \pi^\alpha}\right)\left(\frac{\partial \xi^\nu}{\partial \pi^\beta}\right) h^{\alpha\beta} \qquad (3.1a)$$

and yields conformal symmetry via the relation for a conformal mapping

$$h'_{\mu\nu}(\xi) = \Lambda(\pi) h_{\mu\nu}(\pi) \qquad (3.1b)$$

Therefore, it is quite easy to show that conformal symmetry condition produces the result, which is already known as complex Cauchy-Riemann condition for analyticity

$$\partial \xi(z, \bar{z}) = 0 \qquad (3.2)$$

where we introduce the complex coordinates

$$z \coloneqq \tau + i\sigma \qquad \bar{z} \coloneqq \tau - i\sigma \qquad (3.3a)$$

$$\partial_z := \frac{1}{2}(\partial_\tau - i\partial_\sigma) \qquad \partial_{\bar{z}} := \frac{1}{2}(\partial_\tau + i\partial_\sigma) \tag{3.3b}$$

and $\partial := \partial_z$ and $\bar{\partial} := \partial_{\bar{z}}$ are used for notational convenience. Notice that Cauchy-Riemann condition (3.2) leads to the conformal symmetry and further investigation produces a solution that conformal symmetry is a holomorphic mapping

$$z \to \xi(z) \tag{3.4}$$

Infinitesimally, the conformal transformation gives rise to

$$\xi = z + \epsilon(z), \quad \bar{\xi} = \bar{z} + \bar{\epsilon}(\bar{z}) \tag{3.5}$$

This infinitesimal mapping admits a Laurent expansion around $z = 0$ and $\bar{z} = 0$ respectively

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \qquad \bar{\epsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{c}_n \bar{z}^{n+1} \tag{3.6}$$

which leads to the specifications of the generators

$$\ell_n := -z^{n+1}\partial, \qquad \bar{\ell}_n := -\bar{z}^{n+1}\bar{\partial} \tag{3.7}$$

These generators obey the following commutation relations, known as the Witt algebra

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}, \quad [\ell_n, \bar{\ell}_m] = 0, \quad [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m} \tag{3.8}$$

It is obvious that there are infinite number of conformal transformations locally, but only three of them contribute to global conformal symmetries, which are $n = (-1, 0, 1)$. $n = -1$ is the familiar translations on the plane, $n = 0$ yields Lorentz (rotation) and scale transformations commonly, and finally $n = 1$ produces what is known as special conformal transformations. These four transformations are the subgroup of conformal group, and indications of global conformal symmetries.

Now we come to the transformation of a field under conformal symmetry. How a field is affected by a conformal transformation depends on the type of the field. In general a field may be assigned a conformal dimension u and $\bar{u}$, which are defined by

$$u := \frac{1}{2}(\Delta + s), \qquad \bar{u} := \frac{1}{2}(\Delta - s) \tag{3.9}$$

where Δ is the scaling dimension of the field[4] and s is its planar spin. A (quasi)-primary field, at which we are interested in the sigma models, transforms as

$$\phi'(\xi,\bar{\xi}) = \left(\frac{d\xi}{dz}\right)^{-u}\left(\frac{d\bar{\xi}}{d\bar{z}}\right)^{-\bar{u}}\phi(z,\bar{z}) \tag{3.10}$$

according to the conformal map $z \to \xi(z)$ and $\bar{z} \to \bar{\xi}(\bar{z})$.

In case of infinitesimal conformal transformations (3.5) this expression turns out to be

$$\delta_{\epsilon,\bar{\epsilon}}\phi \coloneqq \phi'(z,\bar{z}) - \phi(z,\bar{z}) = -(u\phi\partial\epsilon + \epsilon\partial\phi) - (\bar{u}\phi\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\phi) \tag{3.11}$$

In the view of all these constructions, the sigma model action in complex coordinates defined in (3.3) can be established in a simple form[5] as follows

$$S = -\int d^2z\,(g_{ij} + b_{ij})\partial\phi^i\bar{\partial}\phi^j \tag{3.12}$$

Classically, conformal invariance requires the holomorphism of currents, and hence the stress-energy tensor associated to this action [11], [13]. The currents can be derived from the expression (2.1.15), where $\mathcal{L} = (g_{ij} + b_{ij})\partial\phi^i\bar{\partial}\phi^j$ is the Lagrangian of the sigma model action.

The energy-momentum tensor components associated with this Lagrangian[67] is found by (2.32)

$$T_{z\bar{z}} = T_{\bar{z}z} = 0, \quad T_{zz} = \frac{1}{2}(g_{ij} + b_{ij})\partial\phi^i\partial\phi^j, \quad T_{\bar{z}\bar{z}} = \frac{1}{2}(g_{ij} + b_{ij})\bar{\partial}\phi^i\bar{\partial}\phi^j \tag{3.13}$$

The form of (3.13) verifies that stress-energy tensor only has holomorphic components and therefore traceless, hence proves the conformal symmetry of sigma model action.

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[4] Here, an abuse of notation is encountered. In the previous chapter, Δ is used for the set of roots but here it denotes the scaling dimension of the field.

[5] d^2z is given by $d^2z = -\frac{1}{2}dz \wedge d\bar{z}$.

[6] Note that the metric tensor in complex coordinates is $\eta_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ in covariant basis $\eta^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ in contravariant basis.

[7] Using $T^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}T_{\alpha\beta}$, one can readily show that $T_{\bar{z}\bar{z}} = \frac{1}{4}T^{zz}$and $T_{zz} = \frac{1}{4}T^{\bar{z}\bar{z}}$.

3.2. Quantum Conformal Symmetries in Sigma Models

It is known that conformal symmetries at quantum level are described by correlation (or namely green) functions (2.19). Corresponding quantity of conserved currents in correlation functions is the ward identity (2.26). Conformal transformation in correlation functions gives rise to the ward identities. Therefore, presence of conformal ward identities is the evidence of quantum conformal symmetries in sigma models.

Ward identities corresponding to global conformal symmetries [Francesco et al., 1997] are found to be

$$\frac{\partial}{\partial \pi^{\mu}} \langle T^{\mu}_{\nu}(\pi) X \rangle = -\sum_{i=1}^{n} \delta(\pi - \pi_i) \frac{\partial}{\partial \pi_i^{\nu}} \langle X \rangle$$

$$\varepsilon_{\mu\nu} \langle T^{\mu\nu}(\pi) X \rangle = -i \sum_{i=1}^{n} s_i \delta(\pi - \pi_i) \langle X \rangle \tag{3.14}$$

$$\langle T^{\mu}_{\mu}(\pi) X \rangle = -i \sum_{i=1}^{n} \delta(\pi - \pi_i) \Delta_i \langle X \rangle$$

where X is the collection of fields, $\varepsilon_{\mu\nu}$ is the antisymmetric tensor and s_i is the spin of the field ϕ_i. In terms of complex coordinates, these identities are reduced to following forms

$$\langle T(z) X \rangle = \sum_{i=1}^{n} \left\{ \frac{1}{z - \xi_i} \partial_{\xi_i} \langle X \rangle + \frac{u_i}{(z - \xi_i)^2} \langle X \rangle \right\} + reg. \tag{3.15}$$

$$\langle \bar{T}(\bar{z}) X \rangle = \sum_{i=1}^{n} \left\{ \frac{1}{\bar{z} - \bar{\xi}_i} \partial_{\bar{\xi}_i} \langle X \rangle + \frac{\bar{u}_i}{(\bar{z} - \bar{\xi}_i)^2} \langle X \rangle \right\} + reg. \tag{3.16}$$

where $T(z)$ and $\bar{T}(\bar{z})$ are holomorphic and antiholomorphic normalized energy-momentum tensors respectively defined as follows

$$T \coloneqq -2\pi T_{zz}, \qquad \bar{T} \coloneqq -2\pi \bar{T}_{\bar{z}\bar{z}} \tag{3.17}$$

and reg shows that the function of z is analytic at ω_i. These ward identities give rise to single so-called conformal ward identities, which is the variation of correlation function under conformal symmetry

$$\delta_{\epsilon,\bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz\, \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z}\, \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \tag{3.18}$$

where $T(z)$ and $\bar{T}(\bar{z})$ are found in (3.13). Notice that (3.15) and (3.16) are the expansions of the operator valued fields with stress-energy tensor. It is of curiosity to find out all the expansions of fields with operator products. Ward identities gives this expectation. One distinct expansion is the product of two stress-energy tensors, which are given by

$$T(z)T(\xi) \sim \frac{c/2}{(z-\xi)^4} + \frac{2T(\xi)}{(z-\xi)^2} + \frac{\partial T(\xi)}{z-\xi} \tag{3.19}$$

Here c is the central charge and depends on the model under study (it's 1 for a free boson, $\frac{1}{2}$ for a free fermion, and -2 for a simple ghost system, etc.). Operator product expansion (OPE) of stress-energy tensor shows that conformal dimension of T is $u = 2$. OPE of fields are important because they give rise to the construction of Hilbert space.

3.3. Hilbert Space and Virasoro Algebra

We know that a Hilbert space $\mathcal{H}$ is a complete inner product space [Prugovećki, 2007]. Obviously this leads to a proper definition of hermitian product on state functions. Together with the conformal symmetry, one is required to introduce a complex valued parameter, which is defined by a radial quantization. In this scheme, a cylindrical world- sheet with a spacial period L is mapped to a complex plane with $t = -\infty$ denoted at the origin and $t = \infty$ by the infinite distant point on the complex plane. Radial quantization is represented by mapping

$$z = e^{2\pi\xi/L} \tag{3.20}$$

where $\xi \coloneqq t + ix$ with an initial world-sheet a Minkowski space, on which the points $(0, t)$ and (L, t) are identified, yielding a cylindrical geometry.

In order to establish a Hilbert space, we need a vacuum state upon which the Hilbert space is to be constructed. Usually, vacuum state is obtained by using a field at far distance so that interaction could be eliminated. This field could be demonstrated by means of the asymptotic field $\phi_{in} \approx \lim_{t\to-\infty} \phi(x,t)$ at $t = -\infty$. For the Hilbert space requirements, there will be a corresponding field $\phi_{out} \approx \lim_{t\to\infty} \phi(x,t)$ at $t = \infty$, which is connected to ϕ_{in} by a hermitian conjugation. In radial quantization these fields correspond to the fields

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle$$

(3.21a)

$$|\phi_{out}\rangle = \lim_{z,\bar{z}\to \infty} \phi(z,\bar{z})\,|0\rangle \qquad (3.21b)$$

Hilbert space is built up by virtue of mode expansions, defined for a conformal field of dimensions $(u, \bar{u})$ as fallows

$$\phi(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} z^{(-m-u)}\,\bar{z}^{(-n-\bar{u})}\phi_{m,n} \qquad (3.22)$$

where the modes are expressed in terms of the original fields as

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz\, z^{(m+u-1)}\,\frac{1}{2\pi i}\oint d\bar{z}\,\bar{z}^{(n+\bar{u}-1)}\phi(z,\bar{z}) \qquad (3.23)$$

We realize that Hermitian conjugation produces

$$\phi^{\dagger}_{m,n} = \phi_{-m,-n} \qquad (3.24)$$

Now we come to the general consequence of OPE obtained in previous section by using stress-energy tensor. We figured out that conformal ward identity is expressed in terms of OPE such that it contains conformal symmetry within its own structure. To reveal this symmetry, we convert integrals of type (3.18) to commutation relations

$$[A,B] = \oint d\xi \oint dz\, a(z)b(\xi) \qquad (3.25a)$$

where

$$A = \oint a(z)dz \qquad\qquad B = \oint b(z)dz \qquad (3.25b)$$

Commutation relations are the main objects that reflect the symmetry properties behind the theory. Combining all these results, one can explicitly translate conformal ward identity into

$$\delta_{\epsilon}\phi(\xi) = -[Q_{\epsilon}, \phi(\xi)] \qquad (3.26)$$

where we introduced the conformal charge

$$Q_\epsilon \coloneqq \frac{1}{2\pi i}\oint dz\, \epsilon(z)T(z) \tag{3.27}$$

The energy-momentum tensors can be mode-expanded as follows

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\, L_n \qquad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z) \tag{3.28a}$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\, \bar{L}_n \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1}\bar{T}(\bar{z}) \tag{3.28b}$$

where a conformal dimension of 2 is assumed. Here L_n and $\bar{L}_n$ are the mode operators of energy-momentum tensors and they generate the local conformal transformations. These mode operators have commutation relations like classical counterparts $\ell_n$and $\bar{\ell}_n$ (3.8)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

$$[L_n, \bar{L}_m] = 0 \tag{3.29}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

except for the terms containing c, where c is the central charge of the theory. These commutation relations constitute the Virasoro algebra.

All these constructions can be combined to give rise to the Hilbert space. We first guarantee the existence of ground state by inserting the vacuum state $|0\rangle$ by the condition $L_n|0\rangle = 0$ and $\bar{L}_n|0\rangle = 0$ if $n \geq -1$ which is due to the fact that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well-defined as $z, \bar{z} \to 0$. Primary fields create asymptotic states, the eigenstates of the Hamiltonian, when acting on the vacuum. Therefore, we can define the asymptotic state for the eigenstate of the Hamiltonian $|u, \bar{u}\rangle \coloneqq \phi(0,0)|0\rangle$. Thus $L_0|u, \bar{u}\rangle = u|u, \bar{u}\rangle$ and $\bar{L}_0|u, \bar{u}\rangle = \bar{u}|u, \bar{u}\rangle$. Likewise $L_n|u, \bar{u}\rangle = 0$ and $\bar{L}_n|u, \bar{u}\rangle = 0$ if $n > 0$. Other than the ground state, i.e. excited states can be obtained by acting the ladder operators to the ground state. The commutation relation could be shown to satisfy

$$[L_0, \phi_m] = [n(u-1) - m]\phi_{n+m} \tag{3.30}$$

We understand that the operator ϕ_m raises or lowers the eigenstates of $\mathrm{ad}(L_0)$. Likewise, the generator L_{-m} raise the conformal dimension with respect to the

Virasoro algebra

$$[L_0, L_{-m}] = mL_{-m} \tag{3.31}$$

So, we conclude that excited states of holomorphic Hilbert space can be obtained by direct application of the operators on the ground state $|u\rangle$[8]

$$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|u\rangle \qquad (1 \le k_1 \le \cdots \le k_n) \tag{3.32}$$

As a consequence, we observe that conformal symmetry indeed yields the same quantum states within a given formalism with the shifted quantum states. This is consistent with the classical interpretation of the symmetry, and proves the validity of conformal ward identity in constructing the Hilbert space.

3.4. Conformal Symmetries in WZW Model

The resulting outcome of last section brings forth the use of Lie algebra valued model, WZW model (2.3). It is easy to derive the classical conserved currents yielding the conformal symmetry as follows[9]

$$J_z = \partial_z g g^{-1} \qquad J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g \tag{3.33}$$

Likewise, the definition of currents separately implies the relation

$$\left[J_\mu, J_\nu\right] + \partial_\mu J_\nu - \partial_\nu J_\mu = 0 \tag{3.34}$$

We notice that these currents are invariant under $g(z,\bar{z}) = f(z)h(\bar{z})$ where f and h are Lie group valued fields [11], [15]. We then observe that the classical conformal symmetry of these currents imply the equations of motion in the form

$$\partial_{\bar{z}}(\partial_z g g^{-1}) = 0 \qquad \partial_z(g^{-1}\partial_{\bar{z}} g) = 0 \tag{3.35}$$

It is of special importance that these currents have $G(z) \times G(\bar{z})$ symmetry given by $g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\Omega^{-1}(\bar{z})$ where $\Omega(z)$ ($\Omega^{-1}(\bar{z})$) is a Lie algebra valued fields with parameter z ($\bar{z}$).

[8] Similar states for antiholomorphic states could be obtained in the same manner.
[9] For convenience, we adopt the notation $J(z)$ for J_z and $\bar{J}(\bar{z})$ for $J_{\bar{z}}$.

Quantum conformal symmetry is written down via the conformal ward identity. First we realize that

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = \langle(\delta_{\epsilon,\bar{\epsilon}}S)X\rangle \tag{3.36}$$

and

$$\delta_{\epsilon,\bar{\epsilon}}S = \frac{i}{4\pi}\oint dz\, Tr[\epsilon(z)J(z)] - \frac{i}{4\pi}\oint d\bar{z}\, Tr[\bar{\epsilon}(\bar{z})\bar{J}(\bar{z})] \tag{3.37a}$$

With Lie algebra valued fields J and ϵ expressed in basis $\{\mathbf{t}^a\}$[10] $J = \sum_c J^a\, t^a$ and $\epsilon = \sum_c \epsilon^a\, t^a$. **(**3.36) can be turned into the desired form

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint dz \sum_a \epsilon^a \langle J^a X\rangle + \frac{1}{2\pi i}\oint d\bar{z} \sum_a \bar{\epsilon}^a \langle \bar{J}^a X\rangle \tag{3.37b}$$

The variation of J in the basis ϵ is expressed as[11]

$$\delta_\epsilon J = [\epsilon, J] - \partial\epsilon \tag{3.38}$$

Therefore, we can construct the OPE of the currents with above expressions

$$J^a(z)J^a(\xi) \sim \frac{\delta_{ab}}{(z-\xi)^2} + \sum_c i\, f_c^{ab}\, \frac{J^c(\xi)}{z-\xi} \tag{3.39}$$

Then the commutation relation of the currents is

$$[J_n^a, J_m^b] = \sum_c i\, f_c^{ab} J_{n+m}^c + n\delta_{ab}\delta_{n+m,0} \tag{3.40}$$

Likewise, there is a similar expression fort the $\bar{J}$'s (with a bar placed on each item) and the commutation relation of J and $\bar{J}$ is 0 as expected

$$[J_n^a, \bar{J}_m^b] = 0 \tag{3.41}$$

One would naturally construct Hilbert space based on these commutation relations following the same ideas as in previous section. Notice that $n = 0$ causes the 2^{nd} term to vanish in (3.40) which gets similar result in usual sigma model. The interesting fact is that when $n \neq 0$ and $n + m = 0$, which yields $n = -m$, there is a

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[10] We realize that $Tr(\mathbf{t}^a\mathbf{t}^b) = 2\delta_{a,b}$ and $[\mathbf{t}^a, \mathbf{t}^b] = \sum_c i f_{abc}\, \mathbf{t}^c$.

[11] This follows from $\delta_\epsilon g = \epsilon g$ and $\delta_{\bar{\epsilon}} g = -g\bar{\epsilon}$.

central-term contribution to the algebra. This results in an additional analysis in the context of Kać-Moody algebra, which we will discuss in the next chapter.

4. KAĆ-MOODY SYMMETRY IN SIGMA MODELS

4.1. Motivation

We first generally review basics of Kać-Moody algebra in an affine Lie algebra. This is actually an extension of simple Lie algebra with a central term. Let **g** be a simple Lie algebra. We consider the loop algebra $\tilde{\mathbf{g}}$ which is the generalization of **g** in which the elements of the algebra are also Laurent polynomials in some variable t. The set of such polynomials are denoted by $\mathbb{C}[t, t^{-1}]$. The loop algebra[12] can be written by $\tilde{g} = g \otimes \mathbb{C}[t, t^{-1}]$ with the generators $J^a \otimes t^n$. We use the shorthand notation J_n^a to stand for this generator, $J_n^a \coloneqq J^a \otimes t^n$. Therefore, commutation relation is described with a central term due to loop algebras:

$$[J_n^a, J_m^b] = \sum_c i\, f_c^{ab} J_{n+m}^c + \hat{k} n \delta_{ab} \delta_{n+m,0} \tag{4.1}$$

where the generators J^a are orthogonal with respect to Killing form K (2.50) and $\hat{k}$ commutes with all J^a with $\left[J_n^a, \hat{k}\right] = 0$.

One can Show that only one central element exists in the loop extension of a simple Lie algebra [Francesco, 1997]. Commutation relations of this new algebra in the Cartan-Weyl basis are

$$\begin{aligned}
\left[H_n^i, H_m^j\right] &= \hat{k} n \delta^{ij} \delta_{n+m,0}, \\
\left[H_n^i, E_m^\beta\right] &= \alpha^i E_{n+m}^\alpha, \\
\left[E_n^\alpha, E_m^\beta\right] &= \frac{2}{\alpha^2}\{\alpha. H_{n+m} + \hat{k} n \delta_{n+m,0}\} && \alpha = -\beta, \\
&= \mathcal{N}_{\alpha,\beta} E_{n+m}^{\alpha+\beta} && \alpha + \beta \in \Delta, \\
&= 0 && \text{otherwise.}
\end{aligned} \tag{4.2}$$

The set of generators $\{H_0^i, \cdots, H_0^r, \hat{k}\}$ is manifestly Abelian. In the adjoint

[12] Expressing $t = e^{i\gamma}$ with γ real, this yields a map from the circle S^1 to **g**, hence the name loop.

representation, in which the action of a generator X is represented by $\mathrm{ad}(X)$, the eigenvalues of $\mathrm{ad}(H_0^i)$ and $\mathrm{ad}(\hat{k})$ on the generator E_n^α are respectively α^i and 0. Being independent of n, the eigenvector $(\alpha^1, \cdots, \alpha^r, 0)$ is thus infinitely degenerate. Hence, $\{H_0^i, \cdots, H_0^r, \hat{k}\}$ is not a maximal Abelian subalgebra. It must be augmented by the addition of a new grading operator L_0, whose eigenvalues in the adjoint representation depend upon n; it is defined as follows $L_0 \coloneqq -t\frac{d}{dt}$. Its action on the generator $[L_0, J_n^a] = -n\, J_n^a$. The maximal Cartan subalgebra is generated by $\{H_0^i, \cdots, H_0^r, \hat{k}, L_0\}$. The other genera- tors, E_n^α for any n and H_n^i for $n \neq 0$, play the role of ladder operators. With the addition of the operator L_0, the resulting algebra is denoted by $\hat{g}$

$$\tilde{g} = g \oplus \mathbb{C}\,\hat{k} \oplus \mathbb{C}L_0 \tag{4.3}$$

It will be referred to as an affine Lie algebra. It is clearly an infinite dimensional algebra, given that it has an infinite number of generators $\{J_n^a\}$, $g \in \mathbb{Z}$.

4.2. Hidden Symmetries in Sigma Models

In this section we make use of the mechanism of hidden symmetry in two dimensions and point out that they are relevant to Kać-Moody symmetries. This is in fact a long-standing approach and after Schwarz's seminal works [16], [17] in showing that infinite dimensional extensions of classical symmetries are the hidden symmetries and they turn out to be Kać-Moody type symmetries, it became somehow popular in recent years [18], [19]. We follow his approach and investigate the hidden symmetries arised from Lax pair relations which are obtained via the use of equations of motion and Cartan structural equations in our case. We start with our previous action (3.12) and find out the equations of motion associated with it as follows

$$\phi_{zz}^i + \phi_{\bar{z}\bar{z}}^i = -\Gamma_{jk}^i\left(\phi_z^j\phi_z^k + \phi_{\bar{z}}^j\phi_{\bar{z}}^k\right) + 2iH_{jk}^i\phi_{\bar{z}}^j\phi_z^k \tag{4.4}$$

where we use $\phi_z = \partial\phi$ and $\phi_{\bar{z}} = \bar{\partial}\phi$, Γ_{ijk} is the Riemannian connection and is expressed by $\Gamma_{jk}^i = g^{il}\Gamma_{ljk}$ and $\Gamma_{ljk} = \frac{1}{2}(\partial_k g_{lj} + \partial_j g_{kl} - \partial_l g_{jk})$, and $H_{jk}^i = g^{il}H_{ljk}$ is globally defined torsion and is given by $H_{ljk} = \frac{1}{2}(\partial_k b_{lj} + \partial_j b_{kl} + \partial_l b_{jk})$. Since the use of orthonormal coframe bundle approach[13] is more efficient in our analysis, we would like to use the power of Cartan's differential geometric methods [2], [4], [20], [21], and carry our equations to orthonormal coframe. It is already known that orthonormal coframe bundle is defined by a orthonormal coframing θ^i and the corresponding connection one form ω^i_j The orthonormal coframing θ^i is given by

$$\theta^i \coloneqq \phi^i_a d\pi^a = \phi^i_z dz + \phi^i_{\bar{z}} d\bar{z} \tag{4.5}$$

where the beginning of the alphabet represents the world-sheet indices whereas the middle of the alphabet does the target space coordinates, and $\pi \coloneqq (z, \bar{z})$ is used for convenience. Notice that the exterior derivative[14] of θ^i is computed to give

$$d\theta^i = \phi^i_{z\bar{z}} d\bar{z} \wedge dz + \phi^i_{\bar{z}z} dz \wedge d\bar{z} \tag{4.6a}$$

which can be associated with (4.3). To do that, we use the Hodge star product (namely, Hodge duality operator) $*$, specified by[15]

$$* dz = i d\bar{z}, \qquad * d\bar{z} = -i dz \tag{4.6b}$$

This in turn leads to $* \theta^i = i\phi^i_z d\bar{z} + \phi^i_{\bar{z}} dz$ and $d * \theta^i = i(\phi^i_{zz} + \phi^i_{\bar{z}\bar{z}}) dz \wedge d\bar{z}$. It is obvious that $d * \theta^i$ yields the left-hand side of the equations of motion (4.3). Right-hand side could easily be derived in terms of the orthonormal coordinates and can be found out to be

$$d * \theta^i = \Gamma^i_{jk} * \theta^j \wedge \theta^k + H^i_{jk} \theta^j \wedge \theta^k \tag{4.7}$$

If we introduce the connection one form as $\omega^i_k \coloneqq H^{+i}_{jk} \theta^j$ with the shorthand notation $H^{+i}_{jk} = H^i_{jk} + \Gamma^i_{jk}$ the equations of motion is reduced to an elegant familiar compact form as

$$d * \theta^i + \omega^i_k \wedge * \theta^k = \omega^i_k \wedge \theta^k \tag{4.8}$$

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[13] Orthonormal coframe bundle is obtained by n-dimensional rotation of manifold $\mathcal{M}$ in which sigma model is defined, and is represented by $SO(\mathcal{M}) = \mathcal{M} \times SO(n)$, where $SO(n)$ is n-dimensional rotation matrix.

[14] If α is a p-form and β is a q-form, then exterior derivative of their wedge product is $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{pq} d\beta \wedge \alpha$.

[15] In Minkowski space, Hodge duality operator acts on basis one forms as $* d\tau = d\sigma$ and $* d\sigma = d\tau$.

Now we come to the constraints due to torsion-free Cartan's structural equations which are expressed by

$$d\theta^i + \omega^i_j \wedge \theta^j = 0 \tag{4.9}$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j \tag{4.10}$$

where Ω^i_j is the curvature two form and is defined by $\Omega^i_j \coloneqq \frac{1}{2} R^i_{jkl}\theta^k \wedge \theta^l$ and $R^i_{jkl}$is the Riemann curvature tensor. In case of curvature-free problem we just make use of the first equation (4.8) together with the equations of motions we just derived in (4.7). Now the problem is to figure out a pair of differential equations which satisfy both equations of motion (4.7) and so-called Maurer-Cartan equation (4.8). This is achieved by means of well-known Lax-pair relations. In literature, people developed lax-pair relations of principal chiral models based on a lie group, but in our case this requires a tedious analysis and in turn verification. Heuristically one can show that lax-pair equations are

$$\left[\bar{\partial} + t\left(\partial - A_i \phi^i_{\bar{z}}\right)\right]X = 0 \tag{4.11}$$

$$\left[\partial + t\left(\bar{\partial} - A_i \phi^i_{z}\right)\right]X = 0 \tag{4.12}$$

These lax-pair equations are unusual compared to ones encountered in literature. The reason is that derivative terms are coupled. This feature is peculiar to complex coordinates. When we switch the complex coordinates into the usual flat space (or light cone coordinates) it is manifest that these equations are uncoupled. In these equations t is a spectral parameter which is a constant and A_i is a vector valued quantity which may depend on ϕ. To show that these equations give rise to our constraint relations which are equations of motion and Cartan's first structural equation, we multiply the first equation (4.10) by idz and second equation (4.11) by $id\bar{z}$ and subtract from each other to obtain[16]

$$* dX = itdX + tA_i\theta^i * X \tag{4.13}$$

We take the star product of (4.12) to obtain

[16] One can show that $d = dz\partial + d\bar{z}\bar{\partial}$ and $* d = d\bar{z}\partial + dz\bar{\partial}$.

$$dX = it * dX + tA_i\theta^i X \tag{4.14}$$

Now replace (4.12) with (4.13) to give rise to

$$dXX^{-1} = \frac{it^2}{1+t^2}A_i * \theta^i + \frac{t}{1+t^2}A_i\theta^i \tag{4.15}$$

We next define a quantity M, which is given by $M = dXX^{-1}$, satisfying the Maurer-Cartan equation $dM + M \wedge M = 0$. This is the place that lax-pair equations come into play so that the ultimate Maurer-Cartan equation reveals the fact that our constraint relations are contained in it. Consequently, Maurer-Cartan equation yields[17]

$$t\left[d\theta^k + \omega_i^k \wedge \theta^i\right] + it^2[d * \theta^k + \omega_i^k \wedge * \theta^i + A_j\theta^k \wedge \theta^j] = 0 \tag{4.16}$$

where we introduced $A_k^{-1}A_i \coloneqq \delta_{ik}$, $A_k^{-1}dA_i \coloneqq \omega_i^k$ and $A_j\theta^k \coloneqq i\omega_j^k = iH_{lj}^{+k}\theta^l$. It is intriguing that the first term is our second constraint relation (4.8), Cartan's first structural equation, and the second one is the equations of motion (4.7), both of which guarantee the validity of Maurer-Cartan equation in an obvious way. We understand that we can safely use our lax-pair equations to find out hidden symmetries. To unravel the hidden symmetries associated with a symmetry relation, we figure out the solution of lax-pair relations (4.14) as

$$X(t) = Pe^{\int(\frac{t^2}{1-t^2}A_i*\theta^i + \frac{t}{1-t^2}A_i\theta^i)} \tag{4.17}$$

where P denotes an ordering operator which puts the terms in order. In [Ogura and Hosoya, 1985] it is pointed out that Sigma model fields are required to take values on a (Lie) group valued homogeneous space. Thus, right and left actions of the fields become significant. This is expressed by the lie group valued field $g = e^{-i\theta^i \mathbf{t}_k}$ with $\mathbf{t}_k$ represents the basis for the corresponding lie algebra as given by $\left[\mathbf{t}_i, \mathbf{t}_j\right] = f_{ij}^k \mathbf{t}_k$ and f_{ij}^k is the structure constant. Therefore, holomorphic and antiholomorphic conserved currents are given by $\bar{J} = (g^{-1}\bar{\partial}g)^k \mathbf{t}_k$ and $J = (\partial g g^{-1})^k \mathbf{t}_k$ with the conservation law $\partial\bar{J} = \bar{\partial}J = 0$. In this situation, our sigma model action turns out to

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[17] One can consistently write the exterior derivative of A_i by means of definitions right after (4.15), $dA_i + \omega_i^k A_k$ + = 0.

be equivalent to the WZW model defined on a Lie group G with $g \in G$ (2.41).

4.2.1. Kać-Moody Algebra

We make use of the hidden symmetry mechanism to obtain the Kać-Moody algebra and employ the symmetry relations as follows

$$\delta g = \epsilon g \tag{4.18}$$

$$\bar{\delta} g = -g\bar{\epsilon} \tag{4.19}$$

In these symmetry transformations one observes the automorphism $g \to g^{-1}$. It is manifest that $\epsilon, \epsilon \in$ g, the lie algebra of G with $\epsilon = \epsilon^k \mathbf{t}_k$ and $\bar{\epsilon} = \bar{\epsilon}^k \mathbf{t}_k$. We emphasize that these symmetry relations are just remnants of an infinite number of symmetry transformations which are extended by a spectral parameter t playing the role of time[18]. In this scenario, (4.17) and (4.18) are just the first of these symmetry transformations. Therefore, we express the extended symmetry transformations as follows

$$\delta(\epsilon, t) g = \eta(\epsilon, t) g \tag{4.20}$$

$$\bar{\delta}(\bar{\epsilon}, t) g = -g\bar{\eta}(\bar{\epsilon}, t) \tag{4.21}$$

where $\eta(\epsilon, t) \coloneqq X(t)\epsilon X(t)^{-1}$ and $\bar{\eta}(\bar{\epsilon}, t) \coloneqq X(t)\bar{\epsilon}X(t)^{-1}$. We again stress out that $\eta, \bar{\eta} \in$ g and $\eta = \eta^k \mathbf{t}_k$, $\bar{\eta} = \bar{\eta}^k \mathbf{t}_k$. In these representations, we can express the infinitesimal operators as follows

$$\delta(\epsilon, t) = \sum_{n=0}^{\infty} t^n \, \delta_n(\epsilon) \qquad \bar{\delta}(\bar{\epsilon}, t) = \sum_{n=1}^{\infty} t^n \, \bar{\delta}_n(\bar{\epsilon}) \tag{4.22}$$

The reason of why the expansion of $\bar{\delta}$ does not embody $n = 0$ term is explained below by the property of analytic continuation together with the corresponding symmetry of Lax-pair equation.

As a consequence, finding symmetry relations for the Kać-Moody algebra requires constructing the full set of commutation relations $[\delta_1, \delta_2]g$, $[\bar{\delta}_1, \bar{\delta}_2]g$ and

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[18] In this case time propagation of the operator ϵ is given by $X(t)\epsilon X(t)^{-1}$ and $\frac{t^2}{1-t^2} A_i * \theta^i + \frac{1}{1-t^2} A_i \theta^i$ plays the role Hamiltonian.

$[\delta_1, \bar{\delta}_2]g$, where subindex i with $i = 1, 2$ denotes that corresponding expression depends on parameters ϵ_i and t_i[19]. We remark that these commutation relations are all needed to see that the algebra constructed is a closed algebra. One of the main obstacles in computing the mentioned symmetry algebra is to figure out the transformation of $\delta_i X_i$. It can readily be shown by taking δ_i of Lax pair relation stated in parameters X_i to get the resulting expression

$$\delta_i X_i = \frac{t_j}{t_i - t_j}(\eta_i X_j - X_J \epsilon_i) \qquad (4.23)$$

To show the details of this result we start with (4.14) with $t = t_i$ and take the variation δ_i to obtain the result

$$d(\delta_i X_i) = \left[\frac{it_j^2}{1+t_j^2}\delta_i(A_k * \theta^k) + \frac{t_j}{1+t_j^2}\delta_i(A_k\theta^k)\right]X_j + [\frac{it_j^2}{1+t_j^2}A_k * \theta^k + \frac{t_j}{1+t_j^2}A_k\theta^k]\delta_i X_i \qquad (4.24a)$$

We next notice that the δ_i variation of $A_k^{-1}A_j = \delta_{jk}$ leads to $\delta_i A_k = 0$ To find the variations of θ^k and $* \theta^k$ we reexpress them as follows

i) $\theta^k = \eta \cdot \mathbf{\Delta}\phi^k = \mathbf{\Delta}^T\phi^k \cdot \eta^T$, ii) $* \theta^k = \eta \cdot \sigma_y \cdot \mathbf{\Delta}\phi^k = -\mathbf{\Delta}^T\phi^k \cdot \sigma_y \cdot \eta^T$

where we introduce $\eta \coloneqq (dz \quad d\bar{z})$ and $\mathbf{\Delta} \coloneqq \begin{pmatrix}\partial \\ \bar{\partial}\end{pmatrix}$. $\eta^T$and $\mathbf{\Delta}^T$ are transposes of η and $\mathbf{\Delta}$ respectively, and σ_y is the Pauli spin matrix in y-direction[20]. We notice the infinitesimal form of (4.19) as $\delta_i\phi^k = \eta^k$

Therefore, (4.23a) turns out to be

$$d(\delta_i X_j) = \left[\frac{it_j^2}{1+t_j^2}A_k\eta\sigma_y\mathbf{\Delta}\eta^k + \frac{t_j}{1+t_j^2}A_k\eta\mathbf{\Delta}\eta^k\right]X_j + dX_jX_j^{-1}(\delta_i X_j) \qquad (4.24b)$$

One can verify by substitution that (4.22) is indeed the desired solution. Let us

[19] At this point, we remark the abuse of notation t_i with the basis t_i used for the lie algebra of g, which might cause reader confuse.

[20] $\sigma_y = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$

concentrate on the first symmetry relation, namely $[\delta_i, \delta_2]g$. Using (4.19) and the definition of $\eta(\epsilon, t)$ one gets

$$[\delta_1, \delta_2]g = (\delta_1\eta_2 - \delta_2\eta_1)g + [\eta_2, \eta_1]g \tag{4.25a}$$

First we notice that

$$\delta_i\eta_j = \left[(\delta_i X_j)X_j^{-1}, \eta_j\right] = \frac{t_j}{t_i - t_j}[\eta_i - X_j\epsilon_i X_j^{-1}, \eta_j]) \tag{4.25b}$$

Thus one finds out that

$$[\delta_1, \delta_2]g = \frac{t_1\delta_1(\epsilon_{12})g - t_2\delta_2(\epsilon_{12})g}{t_1 - t_2} \tag{4.25c}$$

where $\epsilon_{12} \coloneqq [\epsilon_1, \epsilon_2] = f_{ij}^k \epsilon_1^i \epsilon_2^j \mathbf{t}_k$ and $\delta_k(\epsilon_{ij}) \coloneqq \delta(\epsilon_{ij}, t_k)$. Right-hand side of (4.24c) can be expanded by means of (4.21) as follows

$$\frac{t_1\delta_1(\epsilon_{12}) - t_2\delta_2(\epsilon_{12})}{t_1 - t_2} = \sum_{n=0}^{\infty}\left(\frac{t_1^{n+1} - t_2^{n+1}}{t_1 - t_2}\right)\delta_n(\epsilon_{12}) \tag{4.25d}$$

$$= \sum_{p\geq 0}\sum_{q=0}^{p} t_1^q\, t_2^{p-q}\, \delta_p(\epsilon_{12}) = \sum_{m+n=0}^{\infty} t_1^m\, t_2^n \delta_{m+n}(\epsilon_{12}) \tag{4.25e}$$

Likewise, the left-hand side of (4.24c) has modes as given by (4.21). Therefore, one can easily obtain the resulting commutation relation of the modes

$$[\delta_m(\epsilon_1), \delta_n(\epsilon_2)] = \delta_{m+n}(\epsilon_{12}) \tag{4.26}$$

where the ranges of m and n are $m \geq 0$ and $n \geq 0$.

Now let us compute the commutator $[\bar{\delta}_1, \bar{\delta}_2]g$. In order to find this commutator we adopt the Schwartz approach in which our second symmetry transformation is obtained via the analytic continuation of the first one by means of the replacements $t \to \frac{1}{t}$ and $\bar{\delta}(\bar{\epsilon}, t) \to -\delta\left(\epsilon, \frac{1}{t}\right)$. In this situation one get $\epsilon\bar{\eta} \to g^{-1}\eta g$. Thus one can observe that (4.20) is the analytic continuation of (4.19). So with the guidance of these remarks it is easy to find that

$$[\bar{\delta}_1, \bar{\delta}_2] = g[\bar{\eta}_1, \bar{\eta}_2] + g(\bar{\delta}_2\bar{\eta}_1, \bar{\delta}_1\bar{\eta}_2) \tag{4.27}$$

Employing the identity

$$\bar{\delta}_i \bar{X}_j = \frac{t_i}{t_j - t_i}\{\bar{X}_j \bar{\epsilon}_i - \bar{\eta}_i \bar{X}_j\} \tag{4.28}$$

in the expressions of $\bar{\delta}_i \bar{\eta}_j$ one gets that

$$\bar{\delta}_i \bar{\eta}_j = \left[(\bar{\delta}_i \bar{X}_j)\bar{X}_j^{-1}, \bar{\eta}_j\right] = \frac{t_i}{t_j - t_i}\{\bar{X}_j \bar{\epsilon}_{ij} \bar{X}_j^{-1} - [\bar{\eta}_i, \bar{\eta}_j]\} \tag{4.29}$$

Therefore, we come up with

$$\left[\bar{\delta}_1, \bar{\delta}_2\right] g = \frac{t_2 \bar{\delta}_1(\bar{\epsilon}_{12}, t_1) g - t_1 \bar{\delta}_2(\bar{\epsilon}_{12}, t_2) g}{t_1 - t_2} \tag{4.30a}$$

We notice that expansion of $\bar{\delta}_i$ with $i = 1, 2$ does not include $n = 0$ term due to analytic continuation of the symmetry transformations and the form of (4.27). Expanding the right-hand side of (4.29a)

$$\frac{t_2 \bar{\delta}_1(\bar{\epsilon}_{12}, t_1) - t_1 \bar{\delta}_2(\bar{\epsilon}_{12}, t_2)}{t_1 - t_2} = \sum_{n=0}^{\infty} \left(\frac{t_2 t_1^{n+1} - t_1 t_2^{n+1}}{t_1 - t_2}\right) \bar{\delta}_{n+1}(\bar{\epsilon}_{12}) \tag{4.30b}$$

$$\bar{\epsilon}_{12} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} t_1^q \, t_2^{p-q-1} \, \bar{\delta}_{p+1}(\bar{\epsilon}_{12}) = \sum_{n,m=0}^{\infty} t_1^{m-1} \, t_2^{n-1} \bar{\delta}_{mn}(\bar{\epsilon}_{12}) \tag{4.31}$$

Therefore

$$\left[\bar{\delta}_m(\bar{\epsilon}_1), \bar{\delta}_n(\bar{\epsilon}_2)\right] g = \bar{\delta}_{m+n}(\bar{\epsilon}_{12}) \qquad m, n \geq 1 \tag{4.32}$$

Finally, we figure out the commutator $\left[\delta_1, \bar{\delta}_2\right] g$ as follows

$$\left[\delta_1, \bar{\delta}_2\right] g = -g(\delta_1 \bar{\eta}_2) - (\bar{\delta}_2 \bar{\eta}_1) g \tag{4.33}$$

Under the view of analytic continuation, one can find the following variations

$$\delta_1 \bar{\eta}_2 = \frac{\bar{\eta}_2(\bar{\epsilon}_{12}) - [\bar{\eta}_1, \bar{\eta}_2]}{1 - t_1 t_2} \tag{4.34}$$

$$\bar{\delta}_2 \eta_1 = \frac{t_1 t_2\{-\eta_1(\epsilon_{12}) + [\eta_1, \eta_2]\}}{1 - t_1 t_2} \tag{4.35}$$

where $\epsilon_{12} \coloneqq [\epsilon_1, \epsilon_2]$ and $\bar{\epsilon}_{12} \coloneqq [\bar{\epsilon}_1, \bar{\epsilon}_2]$. In obtaining these expressions we make use of the identities

$$\delta_i \bar{X}_j = \frac{1}{1 - t_i t_j}\{\bar{X}_j \epsilon_i - g^{-1} \eta_i g \bar{X}_j\} \tag{4.36a}$$

$$\bar{\delta}_i X_j = \frac{t_i t_j}{1-t_i t_j}\{X_j \bar{\epsilon}_i - g\bar{\eta}_i g^{-1} X_j\} \qquad (4.36b)$$

Indeed, these are the symmetries of Lax-pair equation, which can be shown to satisfy, but easy way to show these arguments follows from analytic continuation of (4.22). therefore, $[\delta_1, \bar{\delta}_2]g$ comes out to be

$$[\delta_1, \bar{\delta}_2]g = \frac{t_1 t_2 \delta_1(\epsilon_{12})g + \bar{\delta}_2(\bar{\epsilon}_{12})g}{1-t_1 t_2} \qquad (4.37a)$$

Expanding δ and $\bar{\delta}$ with respect to the corresponding spectral parameters and following analogous statements we used above we obtain

$$[\delta_m(\epsilon_1), \bar{\delta}_n(\bar{\epsilon}_2)] = \delta_{m-n}(\epsilon_{12}) + \bar{\delta}_{n-m}(\bar{\epsilon}_{12}) \qquad m \geq 0,\ n \geq 1 \qquad (4.37b)$$

As a consequence, we have obtained three sets of infinite symmetry relations. These symmetry relations can be combined into just one symmetry relation by introducing a new set of variations as

$$\Delta_n \coloneqq \delta_n \qquad n \geq 0 \qquad (4.38a)$$

$$\Delta_{-n} \coloneqq \bar{\delta}_n \qquad n \geq 1 \qquad -\infty < n < \infty \qquad (4.38b)$$

Therefore, our commutation relations (4.25), (4.32) and (4.36b) turn out to be

$$[\Delta_n(\epsilon_1), \Delta_m(\epsilon_2)] = \Delta_{n+m}(\epsilon_{12}) \qquad (4.39)$$

We notice that ϵ and $\bar{\epsilon}$ are not distinguishable in our framework and treated as the same in our notation. The reason is that $\bar{\epsilon}$ is expressed in terms of ϵ via the group element g_0 which is defined at the boundary of the integral (4.16) as $\bar{\epsilon} = g_0^{-1}\epsilon g_0$ and we take g to be identity which leads to $\epsilon = \bar{\epsilon}$ In order to observe the analogy of relation with the Kać-Moody algebra we perform the specification $\Delta_n(\epsilon^a) \rightarrow, J_n^a \coloneqq J_n \otimes t^a$and $t^a = e^{ia}$ where ϵ^a is the component of ϵ in the basis $\mathbf{t}_a$. In this association it is implied that currents $J^a(\sigma)$ are expressed on a circle as $J^a(\sigma) = \sum_{n=-\infty}^{\infty} e^{in\sigma} J_n^a$ which verifies the extension of our group G to the Kać-Moody algebra based on a loop group $\hat{G}$ with generators J_n^a. Thus, our final commutation relation will be

$$[J_m^a, J_n^b] = \sum i f_c^{ab} J_{m+n}^k \qquad (4.40)$$

This is centreless Kać-Moody algebra based on the group $\hat{G} \times \hat{G}$ as shown in (4.2). This is general characteristic feature of sigma models which give rise to centreless Kać-Moody algebra based on a loop group. The reason that center term does not appear in this expression is due to lack of curvature and conformal symmetry of spinless field. At this point it would be natural to consider the Virasoro symmetry investigation along the same track in the hidden symmetry mechanism. This could be achieved by means of $\eta(\epsilon, t) = \dot{X}(t)\epsilon X^{-1}(t)$ with the dot representing the derivative with respect to the spectral parameter t. This is beyond the scope of our primary aim and will not be discussed here.

5. CONCLUSION

In this thesis we have analyzed the classical and quantum conformal symmetries of the sigma model in a detailed fashion and looked for the conditions if the sigma model field gives rise to the Kać-Moody symmetries. For this purpose, we made use of the hidden symmetry mechanism raised by some previous authors [16]-[19]. The reason why we perform such an inquiry is that we would like to figure out the infinite number of symmetries comprising our model. In this way we highlight the claim of people using Kać-Moody formalism that uncovering the whole symmetries of a theory gives rise to a complete under- standing of that theory. Our construction serves this aim by means of finding Kać-Moody symmetries of the sigma model. Sigma models are important in the sense that almost all theories in physics can be reduced from a sigma model under appropriate conditions. Therefore, understanding all symmetries of this model also serves finding the unification scheme of the final theories.

In the first few chapters we focused on the classical and quantum symmetries of the sigma model and found that classically holomorphic configuration of the currents, and in turn stress-energy tensors, leads to a conformal symmetry. Quantum version requires that the presence of Ward identity is the indication of quantum conformal symmetries. In the last chapter we concentrate upon the hidden symmetry mechanism to display the Kać-Moody algebra. In this direction we established the lax-pair equations and showed that integrability of these equations produces the equations of motion and curvature-free Maurer-Cartan equation. We imposed the extension of symmetries on the symmetry of currents in association to our sigma model. We employed a spinless field for easiness. The key point in this way was to use the analytic continuation of our symmetry relations. In this way, we behave our second current arising from the non-holomorphic part as the analytic continuation of the first one. Consequently, the ultimate symmetry relation is the Kać-Moody algebra which is an extension of the usual currents on a circle, and thus a loop algebra.

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