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ON THE ENDOMORPHISM RINGS OF SOME MODULE CLASSES

ARDA KÖR

A THESIS SUBMITTED FOR THE DEGREE OF
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GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

BAZI MODÜL SINIFLARININ ENDOMORFİZMA HALKALARI

ARDA KÖR

DOKTORA TEZİ

MATEMATİK ANABİLİM DALI

DANIŞMANI

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SUMMARY

The endomorphism rings of modules and connections between a module and its endomorphism ring have long been of interest. In recent years, it has been discovered that various classes of modules (such as, couniformly presented modules [Facchini and Girardi, 2010], cyclically presented modules over local rings [Amini and Facchini, 2008], kernels of non-zero morphisms between indecomposable injective modules [Facchini et al., 2010], artinian modules whose socle is isomorphic to the direct sum of two fixed simple modules [Facchini and Prihoda, 2010], and so on.) have similar behaviors as having at most two maximal ideals in the endomorphism ring and the validity of a weak form of the Krull-Schmidt Theorem. In the first chapter of the thesis, we summarize the recent developments and some important ones. In the second chapter, we give basic concepts and definitions related to this topic. We add the results of the previous studies to chapter three. In the last chapter of the thesis we study the behavior of endomorphism ring of a cyclic, finitely presented module of projective dimension ≤ 1 over a local ring. This class of modules extends the class of couniformly presented modules over local rings to arbitrary rings.

Key Words: Couniformly presented module, Semilocal ring, Epigeny class, Monogeny class.

ÖZET

Modüllerin endomorfizma halkaları ve modüllerle endomorfizma halkaları arasındaki ilişki uzun zamandır literatürde ilgi çeken bir konu. Son yıllarda, bir çok değişik modül sınıfının (eş-düzgün tanımlı modüller [Facchini and Girardi, 2010], yerel halka üzerindeki yinelemeli tanımlı modüller [Amini and Facchini, 2008], sıfırdan farklı indirgenemeyen injektif modüller arasındaki morfizmaların çekirdeği [Facchini et al., 2010], socle'ı iki belirli basit modülün dik toplamına izomorf olan artin modüller [Facchini and Prihoda, 2010], vb.) endomorfizma halkalarının en fazla iki maksimal ideale sahip olmaları ve Krull-Schmidt Teoreminin zayıf formunun geçerli olması durumlarında benzer davranışlara sahip oldukları keşfedildi. Biz bu tezin ilk kısmında, yakın zamanda yapılan ve önemli gördüğümüz gelişmelerden bahsettik. İkinci kısımda ise, konuyla alakalı temel tanımlar ve içerikleri verdik. Üçüncü bölüme, daha önceden yapılmış olan çalışmalardan elde edilen sonuçları ekledik. Tezin son bölümünde, yerel halka üzerindeki, projektif boyutu ≤ 1 olan yinelemeli, sonlu tanımlı modüllerin endomorfizma halkalarının davranışını çalıştık. Bu modül sınıfı, yerel halka üzerinde eş-düzgün tanımlı modüller sınıfını keyfi halkaya genişletir.

Anahtar Kelimeler: Eş-düzgün tanımlı modüller, Yarı yerel halka, Epijen sınıf, Monojen sınıf.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		Explanations
R	:	Associative ring with identity
Hom$_R(M, N)$	:	The class of R-module homomorphisms from M to N
End(M)	:	Endomorphism ring of a module M
Ker(f)	:	The kernel of the homomorphism f
coker(f)	:	The cokernel of the homomorphism f
Im(f)	:	The image of the homomorphism f
Soc(M)	:	Socle of a module M
E(M)	:	Injective envelope of a module M
$J(R)$	:	Jacobson radical of R
codim(M_R)	:	Dual goldie dimension of M_R
$\dim(M_R)$	:	Goldie dimension of M_R
$\mathbb{M}_n R$	:	The set of all $n \times n$ matrices over an arbitrary ring R
$[M]_e$	:	Epigeny class of a module M
$[M]_m$	:	Monogeny class of a module M
$\mid X \mid$	:	The cardinality of a set X
$M^{(X)}$	:	Direct sum of $\mid X \mid$ copies of a module M

1. INTRODUCTION

In [Warfield, 1975], Warfield described the structure of serial rings and proved that every finitely presented module over a serial ring is a direct sum of uniserial modules. When he mentioned about problems that remained open, he said the outstanding open problem is the uniqueness question for decompositions of a finitely presented module into uniserial summands (which proved in the commutative case and in one noncommutative case by Kaplansky [Kaplansky, 1949]). Facchini solved Warfiled's problem completely in [Facchini, 1996].

The two main opinions in his paper were the epigeny class and monogeny class of a module. Two modules U and V are said to be in the same monogeny class and write $[U]_m = [V]_m$, if there exist a module monomorphism $U \to V$ and a module monomorphism $V \to U$ and U and V are said to be in the same epigeny class, written $[U]_e = [V]_e$, if there exist a module epimorphism $U \to V$ and a module epimorphism $V \to U$. Clearly, these are two equivalence relations. The significance of these definitions is that uniserial modules U, V are isomorphic if and only if $[U]_m = [V]_m$ and $[U]_e = [V]_e$ (see Proposition 1.6 in [Facchini, 1996]). He started with the endomorphism ring of a uniserial module has at most two maximal ideals and modulo those ideals it becomes a division ring (see Theorem 1.2 in [Facchini, 1996]).

He showed (see Theorem 1.9 in [Facchini, 1996]) that if $U_1, \ldots, U_n$, $V_1, \ldots, V_t$ are non-zero uniserial modules, then $U_1 \oplus \ldots \oplus U_n \cong V_1 \oplus \ldots \oplus V_t$ if and only if $n = t$ and there are two permutations σ, τ of $1, 2, \ldots, n$ such that $[U_{\sigma(i)}]_m = [V_i]_m$ and $[U_\tau(i)]_e = [V_i]_e$ for every $i = 1, 2, \ldots, n$. And he proved that for every $n \geq 2$ there exist $2n$ pairwise non-isomorphic finitely presented uniserial modules $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n$ over a suitable serial ring such that $U_1 \oplus U_2 \oplus \ldots \oplus U_n \cong V_1 \oplus V_2 \oplus \ldots \oplus V_n$ (see Example 2.2 in [Facchini, 1996]).

As author mentioned in [Facchini, 1996] the weakened form of the Krull-Schmidt Theorem that serial modules satisfy (see Theorem 1.9 in [Facchini, 1996]) is sufficient to allow one to compute the Grothendieck group of the class of serial modules of finite Goldie dimension over a fixed ring R. If the Krull-Schmidt Theorem holds for a certain of modules, its Grothendieck group is a free abelian group. Though the Krull-Schmidt Theorem does not hold for the class of serial modules of finite Goldie dimension, its Grothendieck group is a free abelian group. Krull-Schmidt Theorem fails because the Grothendieck group is free as an abelian group, but it is not order isomorphic to a free abelian group with the pointwise order. (see Section 3.2 in [Facchini, 1996]).

There is a uncertain similarity between the behavior of serial modules and that of artinian modules. For instance, in ([Facchini, 1996], Section 3) Facchini showed that endomorphism rings of serial modules of finite Goldie dimension are semilocal rings, that is, they are semisimple artinian modulo their Jacobson radical. Camps and Dicks proved that endomorphism rings of artinian modules also are semilocal [Camps and Dicks, 1993]. In [Facchini, 1996], he proved that Krull-Schmidt fails for serial modules. In [Facchini et al., 1995] Facchini, Herbera, Levy and Vamos proved that Krull-Schmidt fails for artinian modules, thus answering a question posed by Krull in 1932.

In [Corisello and Facchini, 2001], authors showed how properties of local rings extend to homogeneous semilocal rings. Like local rings, a homogeneous semilocal ring has a unique maximal two-sided ideal (its Jacobson radical) and a unique simple module S_R (up to isomorphism). Another resemblance is a homogeneous semilocal ring has only one indecomposable projective module P_R (up to isomorphism) and all projective modules are direct sums of copies of this P_R like in the case of local rings. Also they showed that whenever one can localize a right noetherian ring R at a right localizable prime ideal P, the ring R_P one obtain is homogeneous semilocal ring. Here R_P denotes the right quotient ring of R with respect to the set $\mathcal{C}_R(P) = \{x \in R : x+P$ is not a zero divisor in $R/P\}$ of all elements of R regular modulo P.

Since homogeneous semilocal rings generalize local rings and the Krull-Schmidt Theorem concerns modules whose endomorphism ring is local, Barioli, Facchini, Raggi and Rios studied whether the Krull-Schmidt Theorem holds for modules whose endomorphism ring is homogeneous semilocal. Clearly, the Krull-Schmidt Theorem says that if $M = M_1 \oplus \cdots \oplus M_n$ is a direct sum of modules M_i with local endomorphism ring, then any two direct sum decompositions of M into indecomposable direct summands are isomorphic, so that is natural to ask whether the theorem remains true if one substitute the condition of having local endomorphism ring with the condition of having homogeneous semilocal endomorphism ring. This leads naturally to the class of almost semiperfect rings, that is, the rings R with a complete set $e_1, ..., e_n$ of orthogonal idempotents with e_iRe_i homogeneous semilocal for every $i = 1, ..., n$. From that way of thinking, in [Barioli et al., 2001], authors firstly showed that every almost semiperfect ring is semilocal and if $M = M_1 \oplus \cdots \oplus M_n$ is direct sum of modules M_i with homogeneous semilocal endomorphism ring End(M_i), then the almost semiperfect endomorphism ring End(M) is necessarily semilocal, so that M has only finitely many direct sum decompositions (up to isomorphism). Finally they studied the modules which has a finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ such that all endomorphism rings End(M_i) are homogeneous semilocal and found complete results about uniqueness of such decompositions (Krull-Schmidt Theorem). Moreover they showed that such a module M can have different direct sum decompositions (up to isomorphism).

In [Amini and Facchini, 2008] Babak Amini, Afshin Amini and Alberto Facchini studied the uniqueness of the diagonal form when it exists. Obviously, the study of diagonal matrices over a local ring R up to matrix equivalence is the same as the study of finite direct sums of cyclically presented right R-modules (up to isomorphism). The case of R commutative local is particularly simple and follows from the Krull-Schmidt-Azumaya Theorem. So that, assume $a_1, ..., a_n, b_1, ..., b_n$ are elements of a commutative local ring R, $diag(a_1, ..., a_n)$ and $diag(b_1, ..., b_n)$ are equivalent matrices. Then $R/a_1R \oplus ... \oplus R/a_nR \cong R/b_1R \oplus ... \oplus R/b_nR$ and the modules R/a_iR, R/b_jR are either zero (when a_i, b_j are invertible), or have local endomorphism rings. By the Krull-Schmidt-Azumaya Theorem, there is a permutation σ of $1, ..., n$ such that $R/a_iR \cong R/b_{\sigma(i)}R$ for every $i = 1, ..., n$, so that $a_iR = b_{\sigma(i)}R$ for every i, that is, a_i and $b_{\sigma(i)}$ are associates, i.e., there exists invertible elements u_i in R with $a_iu_i = b_{\sigma(i)}$ for every i.

Their main result is weak form of a Krull-Schmidt type theorem that holds for finite direct sums of cyclically presented modules over a local ring and it is unexpectedly similar to the solution given in (Theorem 1.9 in [Facchini, 1996]) to the problem posed by Warfield (see p.189 in [Warfield, 1975]) of characterizing (up to isomorphism) the decomposition of module into uniserial summands.

They proved that the endomorphism ring of a non-zero cyclically presented module over a local ring has one or two maximal ideals, like the endomorphism of non-zero uniserial modules. For a uniserial module, the two maximal ideals roughly correspond to the monogeny class and the epigeny class, respectively. Likewise, for a cyclically presented module, the two maximal ideals correspond to the epigeny class and the lower part (see Remark 4.4 in [Amini and Facchini, 2008]). Also they extended the notion of having the same lower part from cyclically presented modules to arbitrary finitely presented modules over a local ring. Actually, beside extending it to the class of finitely presented modules, which is usually properly contains finite direct sums of cyclically presented modules, they also modified the definition of having the same lower part a little for the exceptional case of the regular module R_R.

Several classes of modules and modules described in (Theorem 2.5 and 4.3 of [Facchini and Girardi, 2010]) act similarly about having at most two maximal ideals in the endomorphism ring and the validity of a weak form of the Krull-Schmidt Theorem (such as cyclically presented modules over local rings [Amini and Facchini, 2008], kernels of non-zero morphisms between indecomposable injective modules [Facchini et al., 2010], artinian modules whose socle is isomorphic to the direct sum of two fixed simple modules [Facchini and Prihoda, 2010] and so on.

In [Facchini and Girardi, 2010] Facchini and Girardi introduced and studied the notion of couniformly presented modules, which extend to arbitrary rings the class of cyclically presented modules over local rings. Direct sums of modules all mentioned above are described by a pair of invariants: lower part and epigeny class for cyclically presented modules over local rings, upper part and monogeny class for kernels of non-zero morphisms between indecomposable injective modules, monogeny class and epigeny class for uniserial modules, or, more generally, biuniform modules.

The following theorem ([Facchini and Girardi, 2010], Theorem 2.5) of them describes the endomorphism ring of a couniformly presented module: Let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of a couniformly presented module M_R. Let $K := \{f \in \text{End}(M_R) \mid f \text{ is not surjective}\}$ and $I := \{f \in \text{End}(M_R) \mid f_1 : C_R \to C_R \text{ is not surjective }\}$. Then K and I are completely prime two-sided ideals of $\text{End}(M_R)$, the union $K \cup I$ is the set of all non-invertible elements of $\text{End}(M_R)$ and any proper right ideal of $\text{End}(M_R)$ and every proper left ideal of $\text{End}(M_R)$ is contained either in K or in I. Moreover, one of the following two conditions holds:

i) Either the ideals K and I are comparable, so that $\text{End}(M_R)$ is a local ring with maximal ideal the greatest ideal among K and I, or

ii) K and I are not comparable, $J(\text{End}(M_R)) = K \cap I$ and $\text{End}(M_R)/J(\text{End}(M_R))$ is canonically isomorphic to the direct product of the two division rings $\text{End}(M_R)/K$ and $\text{End}(M_R)/I$.

In this thesis, we study the behaviour of endomorphism rings of cyclic, finitely presented module of projective dimension ≤ 1. This class of modules extends the class of couniformly presented modules over local rings to arbitrary rings.

2. BASIC CONCEPTS AND DEFINITIONS

2.1. Rings

Notations in this chapter are quite standard and may found in many books on Algebra and Ring Theory. To keep the reader on track, we will introduce them as required. The following four books are our main references:

i) Rings and category of modules [Anderson and Fuller, 1992].
ii) Module Theory: Endomorphism rings and direct sum decompositions in some classes of modules [Facchini, 1998].
iii) Continuous and discrete modules [Mohamed and Müller, 1990].
iv) An introduction to homological algebra [Rotman, 1979].

We will use the results in these books whenever we have such a demand. All the rings considered will be associative rings and assumed to have an identity element. Rings will be denoted by R or as a triple $(R, +, \cdot)$.

Definition 2.1: An element r of a ring R is said to be:

i) a right zero-divisor if $r \neq 0$ and there exists $s \in R$ such that $s \neq 0$ and $sr = 0$;
ii) a left zero-divisor if $r \neq 0$ and there exists $s \in R$ such that $s \neq 0$ and $rs = 0$;
iii) a zero-divisor if it is either a right zero-divisor or a left zero-divisor;
iv) right invertible if there exists $s \in R$ such that $rs = 1_R$;
v) left invertible if there exists $s \in R$ such that $sr = 1_R$;
vi) invertible if it is both right invertible and left invertible.

As we called in previous definition if $r \in R$ is invertible, it is both right invertible and left invertible. This means that there exist elements $s \in R$ such that $rs = 1_R$ and $s' \in R$ such that $s'r = 1_R$. But then $s' = s' \cdot 1_R = s'(rs) = (s'r)s = 1_R \cdot s = s$. It follows that an invertible element r has a unique right inverse, a unique left inverse and the unique right inverse is equal to the unique left inverse, denoted by r^{-1}.

Definition 2.2: A ring R is a division ring if every non-zero element of R is invertible.

Lemma 2.1: The following conditions are equivalent for a ring R:

i) *R is a division ring.*

ii) *Every non-zero element of R is right invertible.*

iii) *Every non-zero element of R is left invertible.*

iv) *The only right ideals of R are 0_R and R.*

v) *The only left ideals of R are 0_R and R.*

Definition 2.3: Let A be an abelian group written additively, endomorphism of A means a group homomorphism $f : A \to A$; in other words, if we write our function on the left, $f(a+b) = f(a) + f(b)$ where $a, b \in A$. The set E of all such endomorphisms of A forms and abelian group with respect to the addition $(f, g) \to f + g$ defined by, $(f + g)(a) = f(a) + g(a)$ where $a \in A$.The identity and the inverse (negative) are given by, $0(a) = 0$ and $(-f)(a) = -(f)(a)$.

Now on E it also happens that composition of functions is an associative operation that distributes over the additive operation on E. So if $A \neq 0$ (i.e, if E has at least two elements), then E is actually a ring whose identity is the identity map $1_A : A \to A$. But note that if $f, g \in E$, then in general, the product fg in E depends on whether we consider these as functions operating on the left or on the right: $(fg)(a) = f(g(a)); (a)(fg) = (a(f))g$.

In other words, there arise naturally for every (non-zero) abelian group A two endomorphism rings, a ring of left endomorphisms and a ring of right endomorphisms, denoted by $\text{End}^l(A)$ and $\text{End}^r(A)$, respectively.

2.2. Exact Sequences

From now on, all modules will be right modules over a fixed ring R. We will denote the zero module with one element by 0.

We will now consider sequences of modules, where by a sequence of modules $\ldots \to M_{i-1} \stackrel{f_{i-1}}{\to} M_i \stackrel{f_i}{\to} M_{i+1} \stackrel{f_{i+1}}{\to} \ldots$, we mean a family of modules M_i indexed by integer numbers and a set of module morphisms $f_i \colon M_i \to M_{i+1}$. Sequences can be either finite or infinite on one side or both sides.

A sequence of modules is called a 0-sequence (or a complex of modules, or a chain complex of modules) if $f_i(M_i) \subseteq \text{Ker} f_{i+1}$ for every index i. Equivalently, if

$f_{i+1}f_i = 0$ for every i.

Definition 2.4: A sequence is called exact in M_i if $f_{i-1}(M_{i-1}) = \mathrm{Ker} f_i$. And a sequence is called exact if it is exact in M_i for every i.

Lemma 2.2: Let $f\colon M \to N$ be any module homomorphism.

i) The sequence $0 \to M \xrightarrow{f} N$ is exact if and only if f is injective.
ii) The sequence $M \xrightarrow{f} N \to 0$ is exact if and only if f is surjective.
iii) A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if f is injective, g is surjective and $f(A) = \mathrm{Ker} g$.

Definition 2.5: Exact sequences of the form $0 \to A \to B \to C \to 0$ are called short exact sequences (s.e.s., for short).

The followings are some examples,

i) For every submodule N of M, there is a short exact sequence $0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \to 0$, where ι denotes the embedding of N into M and π denotes the canonical projection of M onto M/N.
ii) For every pair of modules M and N, there is a short exact sequence $0 \to M \xrightarrow{\varepsilon} M \oplus N \xrightarrow{\pi} N \to 0$, where $\varepsilon(m) = (m, 0)$ and $\pi(m, n) = n$ for every $m \in M$ and $n \in M$.

Lemma 2.3: Let $f\colon M \to N$ and $g\colon N \to M$ be homomorphisms with $gf = \iota_M$. Then $N = f(M) \oplus \mathrm{Ker} g$.

Proposition 2.1: The following conditions are equivalent for a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$:

i) There exists a homomorphism $f'\colon B \to A$ such that $f'f = \iota_A$ (i.e., f is left invertible).
ii) There exists a homomorphism $g'\colon C \to B$ such that $gg' = \iota_C$ (i.e., g is right invertible).
iii) $f(A) = \mathrm{Ker} g$ is a direct summand of B.

Moreover, if these three equivalent conditions hold, then $B \cong A \oplus C$.

A short exact sequence satisfying the three equivalent conditions in the statement of Proposition 2.1 is called a split exact sequence.

2.3. Categories and (Exact) Functors

Definition 2.6: A category $\mathcal{C}$ consists of:

i) *a class* Ob$\mathcal{C}$*, whose elements called the objects of* $\mathcal{C}$*;*

ii) *for each pair* (A, B) *of objects of* $\mathcal{C}$*, a set* $\mathrm{Hom}_{\mathcal{C}}(A, B)$*, whose elements called morphisms of* A *into* B*;*

iii) *for each triple* (A, B, C) *of objects of* $\mathcal{C}$*, a mapping*

$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C) \tag{2.1}$$

called composition.

Before stating the axioms for categories, we introduce some notation. Instead of writing $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we will often write $f\colon A \to B$. For the composition $\circ\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ and morphisms $f\colon A \to B$, $g\colon B \to C$, we will denote the composite morphism by gf or $g\circ f$. The axioms which a category must satisfy are:

i) If A, B, C, D are objects of $\mathcal{C}$ and $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ are morphisms, then $(hg)f = h(gf)$ (associativity of composition).

ii) For every $A \in$ Ob$\mathcal{C}$, there exists an element of $\mathrm{Hom}_{\mathcal{C}}(A, A)$, which we will denote 1_A, such that $f\circ 1_A = f$ and $1_A\circ g = g$ for every $B \in$ Ob$\mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Examples of some categories;

i) The category **Set**: The objects of **Set** are all sets. If A, B are sets, the morphisms $f\colon A \to B$ are all mappings $f\colon A \to B$, that is, $\mathrm{Hom}_{\mathbf{Set}}(A, B) := B^A$. The composition is the composition of mappings. Then **Set** is a category, in which $1_A\colon A \to A$ is the identity mapping defined by $1_A(a) = a$ for every $a \in A$.

ii) The category **Grp**: The objects of **Grp** are all groups. If G, H are groups, the morphisms $f\colon G \to H$ are the “usual” group morphisms of G into H, that is, the mappings $f\colon G \to H$ such that $f(xy) = f(x)f(y)$ for every $x, y \in G$. The composition is the composition of mappings. Then **Grp** turns out to be a category.

iii) The category **Rng**: The objects of **Rng** are all rings with identity. The morphisms $f\colon R \to S$ are the ring morphisms of R into S. The composition is the composition of mappings.

iv) The category **Ab**: The objects of **Ab** are all abelian additive groups. The morphisms $G \to H$ are the group morphisms $G \to H$. The composition is the composition of mappings.

Definition 2.7: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor (or a covariant functor) $F\colon \mathcal{C} \to \mathcal{D}$ assigns to every object $C \in \mathrm{Ob}\mathcal{C}$ an object $F(C) \in \mathrm{Ob}\mathcal{D}$ and to every morphism $f\colon C \to C'$ in $\mathcal{C}$ a morphism $F(f)\colon F(C) \to F(C')$ in $\mathcal{D}$ and the following axioms are satisfied:

i) for every morphism $f\colon C \to C'$ and $g\colon C' \to C''$ in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$;

ii) $F(1_C) = 1_{F(C)}$ for every $C \in \mathrm{Ob}\mathcal{C}$.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The composite functor $GF\colon \mathcal{C} \to \mathcal{E}$ is defined in the obvious way: $GF(C) = G(F(C))$ for every object C in $\mathcal{C}$; $GF(f) = G(F(f))$ for every morphism $f\colon C \to C'$ in $\mathcal{C}$. Notice that if F and G are both covariant or both contravariant, the composite functor GF is covariant. If one is covariant and the other is contravariant, the composite functor GF is contravariant.

An isomorphism of a category $\mathcal{C}$ into a category $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ with GF the identity functor $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and FG the identity functor $1_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$. If there is an isomorphism $F\colon \mathcal{C} \to \mathcal{D}$ we will say that the two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

In category theory, a commutative diagram is a diagram of objects (also known as vertices) and morphisms (also known as arrows or edges) such that all directed paths in the diagram with the same start and endpoints lead to the same result by composition. Commutative diagrams play the role in category theory that equations play in algebra.

Recall that a morphism $f\colon C \to C'$ in a category $\mathcal{C}$ is an isomorphism if there exists a morphism $g\colon C' \to C$ such that $g \circ f = 1_C$ and $f \circ g = 1_{C'}$.

Definition 2.8: Let $\mathcal{C}$ be a category and $f\colon C \to C'$ a morphism. We say that:

i) f is a monomorphism if, for any object B in $\mathcal{C}$ and any two morphisms $g, h\colon B \to C$, $f \circ g = f \circ h$ implies $g = h$.

ii) f is an epimorphism if, for any object D in $\mathcal{C}$ and any two morphisms $g, h\colon C' \to D$, $g \circ f = h \circ f$ implies $g = h$.

Definition 2.9: Let $\mathcal{C}$ be a category with a zero-object Z. For every objects A and B in $\mathcal{C}$ the zero morphism $Z_{A,B} \in \mathrm{Hom}_{\mathcal{C}(A,B)}$ is the unique composition morphism $A \to Z \to B$. The kernel of a morphism $f\colon A \to B$ is the equalizer of f and $Z_{A,B}$. The cokernel of f is the coequalizer of f and $Z_{A,B}$.

Clearly when a kernel of a morphism exists, it is unique up to isomorphisms.

Definition 2.10: Let R and let S be rings and $F : M_R \to M_S$ be a covariant functor. F is said to be exact (right-left) if for every exact sequence

$$0 \longrightarrow A_R \xrightarrow{\alpha} B_R \xrightarrow{\beta} C_R \longrightarrow 0 \tag{2.2}$$

in M_R, the sequence

$$0 \longrightarrow F(A_R) \xrightarrow{F(\alpha)} F(B_R) \xrightarrow{F(\beta)} F(C_R) \longrightarrow 0 \tag{2.3}$$

is exact (right-left).

Lemma 2.4: (Snake Lemma) Given a commutative diagram of modules with exact rows.

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \end{array} \tag{2.4}$$

there is an exact sequence $0 \longrightarrow \mathrm{Ker} f \longrightarrow \mathrm{Ker} g \longrightarrow \mathrm{Ker} h \longrightarrow \mathrm{coker} f \longrightarrow \mathrm{coker} g \longrightarrow \mathrm{coker} h \longrightarrow 0$.

Lemma 2.5: (Five Lemma) Let

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 \\ N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 & \xrightarrow{g_4} & N_5 \end{array} \tag{2.5}$$

be a commutative diagram with exact rows and isomorphisms φ_i, $i = 1, 2, 4, 5$. Then φ_3 is also an isomorphism.

2.4. Projective and Injective Modules and Resolutions

Definition 2.11: A right R-module P_R is projective if, for every epimorphism $f\colon M_R \to N_R$ and every homomorphism $g\colon P_R \to N_R$, there exists a morphism $h\colon P_R \to M_R$ with $f \circ h = g$.

The situation in the previous definition is described by the following commutative diagram in which the row is exact:

$$\begin{array}{ccccc} P_R & & & & \\ h\downarrow & \overset{g}{\searrow} & & & \\ M_R & \xrightarrow{f} & N_R & \to & 0 \end{array} \tag{2.6}$$

Definition 2.12: A right R-module M is called free if it has a basis, $\{m_i \mid i \in I\}$, $m_i \in M$ such that every element of M can be written uniquely in the form;

$$m = \sum_{i \in I} m_i r_i \tag{2.7}$$

where $r_i \in R$ and all but a finite number of r_i are 0.

Proposition 2.2: Let M be a right R-module.

i) A right R module M is free if and only if it is isomorphic to a direct sum of copies of R_R.

ii) Every module M is homomorphic image of a free module.

Lemma 2.6: The followings are some properties of projective modules.

i) Every free module is projective.

ii) Every direct summand of a projective module is projective.

iii) Every direct sum of projective modules is projective.

Proposition 2.3: The following conditions are equivalent for a right R-module P_R:

i) The module P_R is projective.

ii) Every short exact sequence of the form $0 \to M_R \to N_R \to P_R \to 0$ splits.

iii) The module P_R is isomorphic to a direct summand of a free module.

Definition 2.13: A right R-module M is said to be finitely generated if there exist elements $m_1, m_2, ...m_n \in M$ such that every element of M can be written $m = \sum_{j=1}^{n} m_j r_j$. In this case, we say that $\{m_1, m_2, ..., m_n\}$ is a set of generators of M.

Thus projective modules are exactly the modules isomorphic to direct summands of free modules. Since every free module is projective and every module is a homomorphic image of a free module, we get that every module is a homomorphic image of a projective module. Similarly, every finitely generated module is a homomorphic image of a finitely generated projective module.

Corollary 2.1: A module P_R is a finitely generated projective module if and only if it is isomorphic to a direct summand of R_R^n for some $n \geq 0$.

Theorem 2.1: Every projective module is a direct sum of countably generated projective modules.

Fix two modules M_R and N_R. We already know that there is a covariant functor Hom$(M_R, -)\colon$ Mod$R \to$ Ab and a contravariant functor Hom$(-, N_R)\colon$ Mod$R \to$ Ab. Also, we have already seen that these functors Hom are "left exact", in the sense that, for every fixed module M_R, if $0 \to N'_R \to N_R \to N''_R$ is exact, then so is $0 \to$ Hom$(M_R, N'_R) \to$ Hom$(M_R, N_R) \to$ Hom(M_R, N''_R) and for every fixed module N_R, if $M'_R \to M_R \to M''_R \to 0$ is exact, then so is $0 \to$ Hom$(M''_R, N_R) \to$ Hom$(M_R, N_R) \to$ Hom(M'_R, N_R).

In general, these functors Hom$(M_R, -)$ and Hom$(-, N_R)$ are not exact, that is, it is not always true that, for every fixed module M_R, if $0 \to N'_R \to N_R \to N''_R \to 0$ is a short exact sequence, then $0 \to$ Hom$(M_R, N'_R) \to$ Hom$(M_R, N_R) \to$ Hom$(M_R, N''_R) \to 0$ is necessarily exact and for every fixed module N_R, if $0 \to$

$M'_R \to M_R \to M''_R \to 0$ is exact, then $0 \to \mathrm{Hom}(M''_R, N_R) \to \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M'_R, N_R) \to 0$ is necessarily exact. It is easily seen that a module M_R is projective if and only if the functor $\mathrm{Hom}(M_R, -)$ is exact, that is, for every exact sequence $0 \to N'_R \to N_R \to N''_R \to 0$, the sequence of abelian groups $0 \to \mathrm{Hom}(M_R, N'_R) \to \mathrm{Hom}(M_R, N_R) \to \mathrm{Hom}(M_R, N''_R) \to 0$ is exact.

Proposition 2.4: The following conditions are equivalent for a right R-module E_R:

i) *The functor* $\mathrm{Hom}(-, E_R)\colon \mathrm{Mod}R \to \mathrm{Ab}$ *is exact, that is, for every exact sequence* $0 \to M'_R \to M_R \to M''_R \to 0$ *of right R-modules, the sequence of abelian groups* $0 \to \mathrm{Hom}(M''_R, E_R) \to \mathrm{Hom}(M_R, E_R) \to \mathrm{Hom}(M'_R, E_R) \to 0$ *is exact.*

ii) *For every monomorphism $M'_R \to M_R$ of right R-modules,* $\mathrm{Hom}(M_R, E_R) \to \mathrm{Hom}(M'_R, E_R)$ *is an epimorphism of abelian groups.*

iii) *For every submodule M'_R of a right R-module M_R, every morphism $M'_R \to E_R$ can be extended to a morphism $M_R \to E_R$.*

iv) *For every monomorphism $f\colon M'_R \to M_R$ and every homomorphism $g\colon M'_R \to E_R$, there exists a morphism $h\colon M_R \to E_R$ with $h \circ f = g$.*

A module E_R is injective if it satisfies the equivalent conditions of Proposition 2.4. Condition (iv) is described by the following commutative diagram, in which the row is exact:

$$\begin{array}{ccccc} 0 & \to & M'_R & \xrightarrow{f} & M_R \\ & & & {\scriptstyle g}\searrow & \downarrow h \\ & & & & E_R \end{array} \tag{2.8}$$

Essentially, the unique characterization of projective modules that cannot be immediately dualized to injective modules is the characterization of projective modules as direct summands of free modules. In the next proposition we give a further criterion to recognize injective modules, that is, a further characterization of injective modules, which does not have an analog for projective modules.

Proposition 2.5: (Baer's Criterion) A right module M over a ring R is injective if and only if for every right ideal I of R, every morphism $\sigma\colon I \to M$ can be extended to a morphism $\sigma^*\colon R \to M$.

Definition 2.14: An abelian group G is divisible if $nG = G$ for every non-zero integer n. Equivalently, for every positive integer n. Thus G is divisible if and only if for every $g \in G$ and $n > 0$ there exists $h \in G$ such that $nh = g$.

For instance, the abelian group $\mathbb{Z}$ is not divisible and the abelian group $\mathbb{Q}$ is divisible. Homomorphic images of a divisible abelian groups are divisible.

Proposition 2.6: A $\mathbb{Z}$-module G is injective if and only if it is a divisible abelian group.

Proposition 2.7: Direct summand of an injective module is injective.

Proposition 2.8: Direct product of injective modules is injective.

In particular, a direct sum of finitely many injective modules is an injective module.

Proposition 2.9: If R is a ring and G is a divisible abelian group, then $\mathrm{Hom}(R_{\mathbb{Z}}, G_{\mathbb{Z}})$ *is an injective right R-module.*

Here the right R-module structure on $\mathrm{Hom}(R_{\mathbb{Z}}, G_{\mathbb{Z}})$ is that induced by the bimodule structure on ${}_R R_{\mathbb{Z}}$. Hence, for every $f \in \mathrm{Hom}(R_{\mathbb{Z}}, G_{\mathbb{Z}})$ and every $r \in R$, it is defined by $fr := f \circ \lambda_r$, where $\lambda_r \colon R \to R$ denotes left multiplication by r.

Theorem 2.2: Every right R-module can be embedded in an injective right R-module.

Definition 2.15: A projective resolution of a module M_R is a chain complex P such that $P_i = 0$ for every $i < 0$, P_i is projective for every $i \in \mathbb{Z}$ and there is an epimorphism $\varepsilon\colon P_0 \to M$ such that the "augmented chain complex"

$$... \to P_2 \to P_1 \to P_0 \stackrel{\varepsilon}{\to} M \to P_{-1} \to P_{-2} \to P_{-3} \to ... \tag{2.9}$$

is exact.

Definition 2.16: A projective resolution $0 \to P_n \to ... \to P_0 \to M \to 0$ of the R-module M is said to be of lenght n. The smallest such n is called the projective dimension of M.

Derived functors of $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$: Let N_R be fixed. Then there is a contravariant functor $h_N := \text{Hom}_R(-, N)\colon \text{Mod}R \to \text{Ab}$. Let h_N^i denote the i-th derived functor of h_N (obtained starting from a projective resolution). Any short exact sequence of right R-modules $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence $0 \to h_N^0(M'') \to h_N^0(M) \to h_N^0(M') \to h_N^1(M'') \to h_N^1(M) \to h_N^1(M') \to h_N^2(M'') \to ...$

Moreover h_N is left exact, so $h_N^0 \cong h_N$. Fix M_R. There is a covariant functor $H_M := \text{Hom}_R(M, -)\colon \text{Mod}R \to \text{Ab}$. Consider an injective resolution of N_R, and apply to it the functor H_M, getting a cochain complex whose cohomology groups are by definition $H_M^i(N)$. As for the tensor product, $H_M^i(N) \cong h_N^i(M)$ in a natural way. Denote them by $\text{Ext}_R^1(M, N)$. Any short exact sequence $0 \to N' \to N \to N'' \to 0$ induces a long exact sequence $0 \to \text{Ext}_R^0(M, N') \to \text{Ext}_R^0(M, N) \to \text{Ext}_R^0(M, N'') \to \text{Ext}_R^1(M, N') \to \text{Ext}_R^1(M, N) \to \text{Ext}_R^1(M, N'') \to ....$ Since $H_M = \text{Hom}(M, -)$ is left exact, $\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N)$, as for the tensor product.
The following are equivalent for M_R:

i) M_R is projective.
ii) $\text{Ext}_R^i(M, N_R) = 0$ for every $i > 0$ and N_R.
iii) $\text{Ext}_R^1(M, N_R) = 0$ for every N_R.

2.5. Noetherian, Artinian Modules and Rings

Definition 2.17: M_R is a module that has a maximal submodule M_1 and either $M_1 = 0$ or it has a maximal submodule M_2. Then every such process leads to an infinite descending chain $M > M_1 > M_2 > ....$ of submodules, each maximal in its predecessor, or there is finite chain $M > M_1 > M_2 > ... > M_n = 0$ with each term maximal in its predecessor. Observe that if in addition M is artinian, then only the latter option can occur. Similarly, if M is a non-zero module with the property that every non-zero factor module has a simple submodule (e.g if M is artinian), then there is an ascending chain $0 < L_1 < L_2 < ...$ of submodules of M each maximal in its successor. Again if M is noetherian the chain terminates at M after finitely many terms; i.e $L_n = M$ for some n.

Corollary 2.2: If R is a right artinian ring (right noetherian ring) and I is a two-sided *ideal of R, then R/I is a right artinian ring (right noetherian ring).*

The followings are some examples:

i) Simple modules are both noetherian and artinian.

ii) The abelian group $\mathbb{Z}(p^\infty)$, where p is a prime, is a $\mathbb{Z}$-module that is artinian, but not noetherian.

iii) The $\mathbb{Z}$-module $\mathbb{Q}$ is neither noetherian nor artinian.

iv) The abelian groups $\mathbb{Z}$ is a $\mathbb{Z}$-module that is noetherian, but not artinian.

Corollary 2.3: Let $A_1, \ldots, A_n$ be right R-modules. Then $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is a noetherian (artinian) module if and only if A_i is a noetherian (artinian) module for every $i = 1, \ldots, n$.

Corollary 2.4: If R is a right noetherian (right artinian) ring and M_R is a finitely generated right R-module, then M_R is a noetherian (artinian) module.

Let M_R be a module. A series for M_R is a finite chain

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M_R \tag{2.10}$$

of submodules of M_R. The factors of the series are the modules M_i/M_{i-1} for $i = 1, \ldots, n$. The length of the series is n. Series (2.10) is called a composition series for M_R if, for every $i = 1, \ldots, n$, the factor M_i/M_{i-1} is a simple module (equivalently, M_{i-1} is a maximal submodules of M_i). Two series $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M_R$ and $0 = M_0' \subseteq M_1' \subseteq \cdots \subseteq M_m' = M_R$ of M_R are equivalent if $n = m$ and there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $M_i/M_{i-1} \cong M_{\sigma(i)}/M_{\sigma(i)-1}$ for every $i = 1, \ldots, n$. That is, two series are equivalent if and only if they have the same length and the same factors up to the order.

The first series is a refinement of the second if, for every $j = 1, \ldots, m$, there exists $i = 1, \ldots, n$ such that $M_j' = M_i$.

Thus a series (2.10) is a composition series if and only if it is a series without proper refinements, that is, all the refinements of (2.10) are obtain from (2.10) inserting submodules that are already in (2.10).

i) Simple modules have a composition series.

ii) For a fixed prime p, the Prüfer group $\mathbb{Z}(p^\infty)$ has no composition series.

Theorem 2.3: (Artin-Schreier) Any two composition series of a module have equivalent refinements.

Theorem 2.4: (Jordan-Hölder) If a module M_R has a composition series of length n, then all its composition series are equivalent and every series $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M_R$ of M_R can be refined to a composition series. In particular, its length m is $\leq n$.

By the Jordan-Hölder Theorem, if a module M_R has a composition series, then the length and the (simple) factors of the composition series do not depend on the composition series itself, but are determined uniquely by the module. They are called the length (or composition length) and the composition factors of the module, respectively. Modules that have a composition series are also called modules of finite composition length (or of finite length).

For instance, the $\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z}$ is a module of finite composition length (it is a module with finitely many elements) Both $0 = 6\mathbb{Z}/6\mathbb{Z} \subsetneq 2\mathbb{Z}/6\mathbb{Z} \subsetneq \mathbb{Z}/6\mathbb{Z}$ and $0 = 6\mathbb{Z}/6\mathbb{Z} \subsetneq 3\mathbb{Z}/6\mathbb{Z} \subsetneq \mathbb{Z}/6\mathbb{Z}$ are composition series, so that $\mathbb{Z}/6\mathbb{Z}$ is a module of composition length 2. The two compositions factors of $\mathbb{Z}/6\mathbb{Z}$ are isomorphic one to $\mathbb{Z}/2\mathbb{Z}$ and the other to $\mathbb{Z}/3\mathbb{Z}$ and they do not appear in the same order in the two composition series written above.

Proposition 2.10: Let R be a ring. A module M_R is of finite length if and only if it is both artinian and noetherian.

2.6. The Radical of a Module and a Ring

A submodule N of a module M_R is small (or superfluous , or inessential) in M_R if for every submodule L of M_R, $N + L = M_R$ implies $L = M_R$. To denote that N is small in M_R we will write $N \leq_s M_R$.

The followings are some examples:

i) The only small submodule of $\mathbb{Z}_\mathbb{Z}$ is 0.

ii) In $\mathbb{Z}(p^\infty)$ all proper submodules are small, because the sum of any two proper submodules is a proper submodule.

Lemma 2.7: Followings are some properties of small submodule.

i) If $K \leq N \leq M_R$, then $N \leq_s M$ if and only if $K \leq_s M$ and $N/K \leq_s M/K$.

ii) If $N, N' \leq M_R$, then $N + N' \leq_s M$ if and only if $N \leq_s M$ and $N' \leq_s M$.

iii) The zero submodule is always a small submodule of any module M_R, also when $M_R = 0$.

iv) If $f\colon M \to M'$ is an R-module homomorphism and $N \leq_s M$, then $f(N) \leq_s M'$.

v) $K \leq_s M \leq N$ implies $K \leq_s N$.

vi) Assume $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \leq_s M_1 \oplus M_2$ if and only if $K_1 \leq_s M_1$ and $K_2 \leq_s M_2$.

We will say that an epimorphism $g\colon M_R \to N_R$ is small if Ker$g$ is a small submodule of M_R.

Lemma 2.8: For any module M_R, the submodule Rad(M_R) *is the sum of all small submodules of M_R.*

Proposition 2.11: For every right module M_R over a ring R, Rad$(M_R/\text{Rad}(M_R)) = 0$.

From Lemma 2.7(iv) and Lemma 2.8 we immediately get that:

Corollary 2.5: If $f\colon M_R \to M'_R$ is a homomorphism of R-modules, then $f(\text{Rad}(M_R)) \leq \text{Rad}(M'_R)$. In particular, Rad$(M_R)$ *is a subbimodule of the bimodule ${}_{\text{End}(M_R)}M_R$.*

The radical of the right R-module R_R is called the Jacobson radical of the ring R. It is denoted $J(R)$. Thus $J(R) := \text{Rad}(R_R)$ is the intersection of all maximal right ideals of R. Clearly, $J(R)$ is a right ideal of R, because it is defined an intersection of right ideals. It is a two-sided ideal, as can be seen applying Corollary 2.5 to the endomorphism $f := \lambda_r$ of R_R given by left multiplication by r ($\lambda_r(\text{Rad}(R_R)) \subseteq \text{Rad}(R_R)$ simply means that $rJ(R) \subseteq J(R)$.

To be more precise, we should call Rad(R_R) the right Jacobson radical of R, but we will see as a corollary to Proposition 2.13(3) that $\text{Rad}(R_R) = \text{Rad}({}_RR)$ for any ring R. For every right R-module M, the right annihilator $\mathrm{r}_R(M)$ of M is the set of all $r \in R$ such that $Mr = 0$. The right annihilator of any right R-module is a two-sided ideal of R.

Lemma 2.9: The Jacobson radical $J(R)$ of any ring R is the intersection of the right annihilators $\mathrm{r}_R(S_R)$ of all simple right R-modules S_R.

Proposition 2.12: (Nakayama's Lemma) Let M_R be a finitely generated right module and let N be a submodule of M_R. Then $N + MJ(R) = M$ implies $N = M$.

Notice that Nakayama's Lemma can also be stated as "If M_R is finitely generated, then its submodule $MJ(R)$ is small."

Proposition 2.13: The Jacobson radical $J(R)$ of a ring R can also be described as:

i) The unique largest small right ideal of R.
ii) The set of all $x \in R$ such that $1 - xr$ is right invertible for every $r \in R$.
iii) The set of all $x \in R$ such that $1 - rx$ is left invertible for every $r \in R$.
iv) The set of all $x \in R$ such that $1 - rxs$ is invertible for every $r, s \in R$.

Notice that condition (iii) is right/left symmetric, so that $J(R)$ can also be described as the intersection of all maximal left ideals of R (i.e., $\mathrm{Rad}(R_R) = \mathrm{Rad}({}_RR)$), or the unique largest small left ideal of R, or the set of all $x \in R$ such that $1 - rx$ is left invertible for every $r \in R$.

Definition 2.18: Let R be a ring and let $x \in R$. x is said to be right (resp. left) quasi-regular if $1 - x$ is right (resp. left) invertible in R. x is quasi-regular if $1 - x$ is invertible in R. A subset $S \subseteq R$ is (right/left) quasi-regular if every element of S is (right/left) quasi-regular.

Proposition 2.14: The following are equivalent for a right ideal I of a ring R

i) I is right quasi-regular.
ii) I is quasi-regular.
iii) $I \leq_s R_R$.

It immediately follows that the Jacobson radical of a ring R is the sum of all (right) quasi-regular right ideals of R.

Proposition 2.15: Let R be a ring and I be a two-sided ideal of R and $I \subseteq J(R)$. Then $J(R/I) = J(R)/I$.

2.7. (Semi)simple, (Semi)local, (Almost) Semiperfect Rings

Definition 2.19: A simple ring is a non-zero ring that has no two-sided ideal besides the zero ideal and itself.

Proposition 2.16: The following conditions on a ring R are equivalent.

i) R is semisimple ring.
ii) Every left (or right) R-module M is a semisimple module.
iii) Every left (or right) R-module M is injective.
iv) Every short exact sequence of left (or right) R-modules splits.
v) Every left (or right) R-module M is projective.

Lemma 2.10: Let I be a minimal right ideal of a ring R. Then either $I^2 = 0$ or $I = eR$ for some idempotent $e \in R$.

Proposition 2.17: The following conditions are equivalent for a ring R:

i) The ring R has a unique maximal right ideal.
ii) The Jacobson radical $J(R)$ is a maximal right ideal.
iii) The set of elements of R without right inverses is closed under addition.
iv) $J(R) = \{\, r \in R \mid rR \neq R \,\}$.
v) $R/J(R)$ is a division ring.
vi) $J(R) = \{\, r \in R \mid r$ is not invertible in $R \,\}$.
vii) For every $r \in R$, either r is invertible or $1 - r$ is invertible.

Notice that some of these conditions are right/left symmetric, so that right can be substituted with left in the other conditions. The rings with identity that satisfy the equivalent conditions of Proposition 2.17 are called local rings.

Theorem 2.5: Every projective module over a local ring is free.

Proposition 2.18: Let M_R be a module over an arbitrary ring R and assume that

$\mathrm{End}(M_R)$ *is a local ring. Then the right R-module M_R is indecomposable.*

Lemma 2.11: Let M be a module and f an endomorphism of M.

i) If n is a positive integer such that $f^n(M) = f^{n+1}(M)$, then $\mathrm{Ker}(f^n) + f^n(M) = M$.

ii) If M is an artinian module, then f is an automorphism if and only if f is injective.

Lemma 2.12: Let M be a module and f an endomorphism of M.

i) If n is a positive integer such that $\mathrm{Ker} f^n = \mathrm{Ker} f^{n+1}$, then $\mathrm{Ker}(f^n) \cap f^n(M) = 0$.

ii) If M is a noetherian module, then f is an automorphism if and only if f is surjective.

Lemma 2.13: (Fitting's Lemma) Let M be a module of finite composition length n. Then;

i) $M = \mathrm{Ker}(f^n) \oplus f^n(M)$ for every endomorphism f of M.

ii) If M is indecomposable, the ring $\mathrm{End}(M_R)$ is local.

Proposition 2.19: If a module M is a direct sum of modules with local endomorphism rings, then every indecomposable direct summand of M has local endomorphism ring.

Theorem 2.6: (Krull-Schmidt-Remak-Azumaya Theorem) Let M be a module that is a direct sum of modules with local endomorphism rings. Then any two direct sum decompositions of M into indecomposable direct summands are isomorphic, that is, if $M = \oplus_{i \in I} M_i = \oplus_{j \in J} N_j$ for suitable indecomposable submodules M_i, N_j, there exists a one-to-one correspondence $\varphi\colon I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$.

Corollary 2.6: Let R be a ring. The following conditions are equivalent.

i) R is semisimple artinian.

ii) R^{op} is semisimple artinian.

iii) R is left artinian and does not have any non-zero nilpotent left ideals.

If $R/J(R)$ is semisimple artinian then R is called semilocal. This is a finiteness condition on the ring R. For instance, in a semilocal ring R every set of orthogonal idempotents is finite and R has only finitely many simple modules up to isomorphism and only finitely many indecomposable finitely generated projective modules up to isomorphism. Semilocal rings are exactly the rings of finite dual Goldie dimension codim(R_R) and codim$(R_R) =$ codim$(_RR)$.

A ring R is said to be a homogeneous semilocal ring if $R/J(R)$ is simple artinian, that is, if $R/J(R) \cong \mathbb{M}_n(D)$ for some $n > 0$ and some division ring D. The class of homogeneous semilocal rings contains all local rings S and, more generally, all rings of matrices $\mathbb{M}_n(S)$ over a local ring S. For example, all local rings are homogeneous semilocal, all simple artinian rings are homogeneous semilocal and trivially, all homogeneous semilocal rings are semilocal. Since a simple artinian ring has only one simple module up to isomorphism, a homogeneous semilocal ring R has only one simple module S_R up to isomorphism and the annihilator of S_R in R is $J(R)$. The notion of homogeneous semilocal ring is a most natural extension of the notion of local ring. For instance, homogeneous semilocal rings are exactly the semilocal rings with a unique maximal two-sided ideal (see Proposition 2.1 in [Corisello and Facchini, 2001]).

Proposition 2.20: ([Corisello and Facchini, 2001], Proposition 3.1) A ring R is homogeneous semilocal of dual Goldie dimension n if and only if there exists a division ring D such that $R/J(R) \cong \mathbb{M}_n(D)$.

Proof Let R be a ring and D be a division ring such that $R/J(R) \cong \mathbb{M}_n(D)$. Then R is homogeneous semilocal and codim$(R) = \dim(R/J(R))$*by (Proposition 2.43 in [Facchini, 1998]) so that* codim$(R) = \dim(\mathbb{M}_n(D)) = n$*. Conversely, if R is a homogeneous semilocal ring of finite dual Goldie dimension n, then $R/J(R) \cong \mathbb{M}_m(D)$ for a suitable $m \geq 0$ and a suitable division ring D. Moreover $n =$* codim$(R) = \dim(R/J(R))$ *by (Proposition 2.43 in [Facchini, 1998]). It follows that $n = \dim(\mathbb{M}_m(D)) = m$.* □

Theorem 2.7: The following conditions are equivalent for a ring R.

i) Every right R-module is projective.

ii) Every right R-module is semisimple.

iii) The ring R is semisimple artinian.

Theorem 2.8: (Artin-Wedderburn Theorem) A ring R is semisimple artinian if and only *if there exist integers $t, n_1, \ldots, n_t \geq 1$ and division rings $k_1, \ldots, k_t$ such that $R \cong \mathbb{M}_{n_1}(k_1) \times \cdots \times \mathbb{M}_{n_t}(k_t)$.*

Moreover, if R is semisimple artinian, the integers $t, n_1, \ldots, n_t$ in the decomposition are uniquely determined by R and $k_1, \ldots, k_t$ are determined by R up to ring isomorphism.

Proposition 2.21: Let R be a semisimple artinian ring. Then $J(R) = 0$.

Definition 2.20: A ring R is semiperfect in case $R/J(R)$ is semisimple and idempotents lift modulo J(R).

Local rings and left (or right) artinian rings are semiperfect. It is worthy of note that in a semiperfect ring the radical is the unique largest ideal containing no non-zero idempotents.

In ([Facchini, 1998], Proposition 3.6), it is proved that a ring R is semiperfect if and only if it has a complete set $e_1, ..., e_n$ of orthogonal idempotents for which every e_iRe_i is a local ring.

Definition 2.21: In ([Facchini, 1998], Proposition 3.14) A ring R is called almost semiperfect if it has a complete set $e_1, ..., e_n$ of orthogonal idempotents for which every e_iRe_i is a homogeneous semilocal ring.

Thus every semiperfect ring is almost semiperfect.

2.8. (Semi)simple, Cyclic Modules

Definition 2.22: A right R-module M is said to be cyclic if there is an element $m_0 \in M$ such that every $m \in M$ is of the form $m = m_o r$, where $r \in R$. Also m_0 is called the generator of M and we write $M =< m_o >$. In other words, a module with a simple element spannig set.

Definition 2.23: A simple right module is a non-zero right module M_R whose submodules are only M_R and 0 in other words a simple module has exactly two submodules.

Lemma 2.14: A right module M_R is simple if and only if it is isomorphic to R_R/I for some maximal right ideal I of R.

Lemma 2.15: (Schur's Lemma) The endomorphism ring of a simple module is a division ring.

Definition 2.24: A module M_R is semisimple if every submodule of M_R is a direct summand of M_R.

Every simple module is semisimple. If R is a division ring, every module over R is semisimple.

Lemma 2.16: Submodules and homomorphic images of semisimple modules are semisimple modules.

Definition 2.25: Let M_R be a right R-module. The socle of M_R ($\mathrm{Soc}(M_R)$) is the sum of all simple submodules of M_R.

Thus $\mathrm{Soc}(M) = 0$ if and only if M has no simple submodules.

Theorem 2.9: The following conditions are equivalent for a right R-module M:

i) M is a sum of simple submodules, that is, M is equal to its socle.
ii) M is a direct sum of finitely many simple submodules.
iii) M is semisimple.
iv) S is of finite composition length.
v) S is artinian.
vi) S is noetherian.

2.8.1. Projective Cover and Injective Envelope of a Module

Recall that every module is a homomorphic image of a projective module. Now we look for the smallest possible representation of M_R as a homomorphic image of a projective module.

Definition 2.26: A projective cover of a module M_R is a pair (P_R, p) where P_R is a *projective right R-module and $p\colon P \to M$ is a small epimorphism.*

Lemma 2.17: (Fundamental lemma for projective covers and uniqueness of projective covers up to isomorphism)

i) Let (P, p) be a projective cover of a right R-module M. If Q is a projective module and $q\colon Q \to M$ is an epimorphism, then Q has a direct-sum decomposition $Q = P' \oplus P''$ where $P' \cong P$, $P'' \subseteq \mathrm{Ker}(q)$ and $(P', q_{|P'}\colon P' \to M)$ is a projective cover.

ii) Projective covers, when they exist, are unique up to isomorphism in the following sense. If (P, p), (Q, q) are two projective covers of a right R-module M, there is an isomorphism $h\colon Q \to P$ such that $p \circ h = q$.

A submodule N of a module M_R is essential (or large) in M_R if, for every submodule L of M_R, $N \cap L = 0$ implies $L = 0$. In this case, we will write $N \leq_e M_R$.

A monomorphism $f\colon N_R \to M_R$ is said to be essential if its image $f(N_R)$ is an essential submodule of M_R.

i) A monomorphism $f\colon N \to M$ is essential if and only if for every module L and every homomorphism $g\colon M \to L$, if gf is injective, then g is injective.

ii) Let $f\colon N \to M$ and $g\colon M \to P$ be two monomorphisms. The composite mapping gf is an essential monomorphisms if and only if f and g are both essential monomorphisms.

Let M be a right R-module. An extension of M is a pair (N, f), where N is a right R-module and $f\colon M \to N$ is a monomorphism. An essential extension of M is an extension (N, f) where $f\colon M \to N$ is an essential monomorphism. An extension (N, f) is proper if f is not an isomorphism.

Proposition 2.22: A module M_R is injective if and only if it does not have proper essential extensions.

Definition 2.27: An injective envelope of a module M_R is a pair (E_R, i) where E_R is an injective right R-module and $i\colon M_R \to E_R$ is an essential monomorphism.

For example, if i is the inclusion of $\mathbb{Z}_\mathbb{Z}$ into $\mathbb{Q}_\mathbb{Z}$, then $(\mathbb{Q}_\mathbb{Z}, i)$ is an injective envelope of $\mathbb{Z}_\mathbb{Z}$. Dualizing the proof of the fundamental theorem of projective covers we get the following

Lemma 2.18: (Fundamental lemma for injective envelopes and uniqueness of projective covers up to isomorphism)

i) Let (E, i) be an injective envelope of a right R-module M. If F is an injective module and $j\colon M \to F$ is a monomorphism, then F has a direct-sum decomposition $F = F' \oplus F''$ where $F' \cong E$, $j(M) \subseteq F'$ and if $j'\colon M_R \to F'$ is the mapping obtained from j restricting the codomain to F', then (F', j') is an injective envelope of M.

ii) Every right R-module has an injective envelope, which is unique up to isomorphism in the following sense: if (E, i) and (E', i) are both injective envelopes of M, then there exists an isomorphism $h\colon E \to E'$ such that $hi = i'$.

Like for the tensor product of two modules, the injective envelope of a module M_R, which is unique up to isomorphism, will not be usually denoted as a pair. We will usually omit to indicate the embedding of M_R into the injective module. The injective envelope of a module M_R will be usually denoted by $\mathrm{E}(M_R)$.

Lemma 2.19: If $N_R \leq_e M_R$, then $\mathrm{E}(N_R) = \mathrm{E}(M_R)$. More precisely, if (M, f) is an essential extension of N and (E, ε) is an injective envelope of M, then $(E, \varepsilon \circ f)$ is an injective envelope of N.

Proposition 2.23: An extension (E, ε) of a module M is an injective envelope of M if and only if it is a maximal essential extension of M. More precisely, let $\varepsilon\colon M \to E$ be a right R-module monomorphism. Then (E, ε) is an injective envelope of M if and only if it is an essential extension and, for every monomorphism $f\colon E \to N$, if $(N, f \circ \varepsilon)$ is an essential extension of M, then f is an isomorphism.

Proposition 2.24: If $M = M_1 \oplus \cdots \oplus M_n$ for suitable submodules $M_1, \ldots, M_n$ of M, then $\mathrm{E}(M) = \mathrm{E}(M_1) \oplus \cdots \oplus \mathrm{E}(M_n)$.

Proposition 2.25: If $N \leq M$, then $\mathrm{E}(N)$ is a direct summand of $\mathrm{E}(M)$.

Recall that the radical of a module is the sum of all small submodules. Next proposition shows the dual result. Recall that the socle is the sum of all simple submodules, that is, the sum of all minimal submodules.

Proposition 2.26: For every module M, $\mathrm{Soc}(M)$ *is the intersection of all essential submodules of M.*

2.8.2. (Co)uniform Modules and (Dual) Goldie Dimension

Definition 2.28: A module U_R is couniform if it has dual Goldie dimension 1, that is, it is non-zero and the sum of any two proper submodules of U_R is a proper submodule of U_R.

Lemma 2.20: (Lemma 8.7 in [Amini and Facchini, 2008]) The following conditions are equivalent for a projective right module P_R over an arbitrary ring R:

i) P_R *is couniform.*

ii) P_R *is the projective cover of a simple module.*

iii) The endomorphism ring $\mathrm{End}(P_R)$ *of* P_R *is local.*

iv) There exists an idempotent $e \in R$ with $P_R \cong eR$ and eRe a local ring.

v) P_R *is a finitely generated module with a unique maximal submodule.*

vi) P_R *has a greatest proper submodule.*

Moreover, if these equivalent conditions hold, then $\mathrm{Hom}(P_R, R)$ is a couniform projective left R-module.

Lemma 2.21: The following conditions are equivalent for a non-zero module M:

i) If $N, N' \leq M$ and $N \cap N' = 0$, then $N = 0$ or $N' = 0$.

ii) The intersection of two non-zero submodules of M is non-zero.

iii) Every non-zero submodule of M is essential in M.

iv) Every non-zero submodule of M is indecomposable.

v) The injective envelope $\mathrm{E}(M)$ *of M is indecomposable.*

The modules that satisfy the equivalent conditions of Lemma 2.21 are called uniform modules.

Proposition 2.27: The following conditions are equivalent for an injective module E_R:

i) E_R is indecomposable.

ii) E_R is uniform.

iii) The endomorphism ring of E_R is local.

Lemma 2.22: Let M_R be a module without uniform submodules. Then M_R has an infinite independent set of non-zero submodules.

Theorem 2.10: The following conditions are equivalent for a module M_R.

i) M_R does not have an infinite independent set of non-zero submodules.

ii) M_R has a finite independent set $\{A_1, A_2, \ldots, A_n\}$ of uniform submodules and $A_1 \oplus \cdots \oplus A_n$ is essential in M_R.

iii) There exists a non-negative integer m such that the cardinalities of all the independent sets of non-zero submodules of M_R are $\leq m$.

iv) If $A_0 \leq A_1 \leq A_2 \leq \ldots$ is an ascending chain of submodules of M_R, then there exists $i \geq 0$ such that A_i is essential in A_j for every $j \geq i$.

Moreover, if these equivalent conditions hold and $\{A_1, A_2, \ldots, A_n\}$ is a finite independent set of uniform submodules and $A_1 \oplus \cdots \oplus A_n$ is essential in M_R, then any other independent set of non-zero submodules of M_R has cardinality $\leq n$.

Thus, for a module M_R, either there is a finite independent set $\{A_1, A_2, \ldots, A_n\}$ of uniform submodules with $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ essential in M_R and in this case n is said to be the Goldie dimension of M_R denoted by $\dim(M_R)$, or M_R contains infinite independent sets of non-zero submodules, in which case M_R is said to have infinite Goldie dimension. For instance, uniform modules are exactly the modules of Goldie dimension one. Since a module M is essential in its injective envelope E(M), $\dim(M) = \dim(\mathrm{E}(M))$. If a module M has finite Goldie dimension n, it contains an essential submodule that is the finite direct sum of n uniform submodules $U_1, \ldots, U_n$ and in this case E(M) = E$(U_1) \oplus$ E$(U_2) \oplus \cdots \oplus$ E(U_n) is the finite direct sum of n indecomposable modules. By the Krull-Schmidt-Azuyama Theorem, if E(M) is a finite direct sum of indecomposable modules, then the number of direct summands in any indecomposable decomposition of E(M) does not depend on the decomposition.

Hence a module M has finite Goldie dimension n if and only if its injective envelope $\mathrm{E}(M)$ is the direct sum of n indecomposable modules.

In the next proposition we collect the most important arithmetical properties of the Goldie dimension of modules. Some of these properties have already been noticed. Their proof is elementary.

Proposition 2.28: Let M be module.

- *i)* $\dim(M) = 0$ *if and only if* $M = 0$.
- *ii)* $\dim(M) = 1$ *if and only if* M *is uniform.*
- *iii) If* $N \leq M$ *and* M *has finite Goldie dimension, then* N *has finite Goldie dimension and* $\dim(N) \leq \dim(M)$.
- *iv) If* $N \leq M$ *and* M *has finite Goldie dimension, then* $\dim(N) = \dim(M)$ *if and only if* N *is essential in* M.
- *v) If* M *and* M' *are modules of finite Goldie dimension, then* $M \oplus M'$ *is a module of finite Goldie dimension and* $\dim(M \oplus N) = \dim(M) + \dim(N)$.

Note that artinian modules and noetherian modules have finite Goldie dimension. For an artinian module M, the Goldie dimension of M is equal to the composition length of its socle ($\mathrm{Soc}(M)$). In particular, an artinian module M has Goldie dimension 1 if and only if it has a simple socle.

2.8.3. Finitely, Couniformly, Cyclically Presented Modules

Definition 2.29: A module M is finitely presented if it is finitely generated and for every epimorphism $\varphi\colon F_R \to M_R$ with F_R a finitely generated free R-module, the kernel of φ is finitely generated.

Example 2.1: A ring R is right noetherian if and only if every finitely generated right R-module is finitely presented.

Example 2.2: Every finitely generated projective module is finitely presented.

Lemma 2.23: (Schanuel's Lemma) Let $0 \longrightarrow K \xrightarrow{f} E \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow K \xrightarrow{f'} E' \longrightarrow M' \longrightarrow 0$ be two short exact sequences of right R-modules with E, E'

injective modules. Then $E \oplus M' \cong E' \oplus M$.

Corollary 2.7: A module M_R is finitely presented if and only if it is isomorphic to P_R/S where P_R is finitely generated and projective and S is a finitely generated submodule of P_R.

Theorem 2.11: Every right module is a direct limit of finitely presented modules.

Definition 2.30: [Facchini and Girardi, 2010] A module M_R is couniformly presented if it is non-zero and there exists short exact sequence $0 \to C_R \to P_R \to M_R \to 0$ with P_R projective and both P_R and C_R are couniform modules.

In this case, we will say that $0 \to C_R \to P_R \to M_R \to 0$ is a couniform presentation of M_R. Notice that $P_R \to M_R$ is necessarily a projective cover of M_R, because every proper submodule of P_R is small. Without loss of generality, we can suppose that the monomorphism $\iota : C_R \to P_R$ is the inclusion. Clearly, every couniformly presented module is cyclic. Every cyclically presented module over a local ring R is either zero, or isomorphic to R, or couniformly presented [Facchini and Girardi, 2010].

Definition 2.31: [Amini and Facchini, 2008] A right module over a ring R is said to be cyclically presented if it is isomorphic to R/aR for some $a \in R$.

The endomorphism ring of a non-zero cyclically presented module R/aR is canonically isomorphic to E/aR, where $E := \{r \in R \mid ra \in aR\}$ is the idealizer of aR and the right ideal aR of R is a two-sided ideal in the subring E of R [Amini and Facchini, 2008].

3. PRELIMINARY RESULTS

Various classes of modules with a behavior close to the behavior described in ([Facchini and Girardi, 2010], Theorem 2.5 and 4.3) have been studied: cyclically presented modules over local rings [Amini and Facchini, 2008], kernels of non-zero morphisms between indecomposable injective modules [Facchini et al., 2010], artinian modules whose socle is isomorphic to the direct sum of two fixed simple modules [Facchini and Prihoda, 2010] and so on.

In [Facchini and Girardi, 2010] Facchini and Girardi came up with notion of couniformly presented modules, which extend to arbitrary rings the class of cyclically presented modules over local rings. Now we will give some important results of them.

Let R be an arbitrary ring. Given any couniformly presented right module M_R with couniform presentation $0 \to C_R \overset{\iota}{\to} P_R \to M_R \to 0$, every endomorphism f of M_R lifts to an endomorphism f_0 of the projective cover P_R and f_1 is the restriction of f_0 to C_R. Hence one has a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & C_R & \overset{\iota}{\longrightarrow} & P_R & \to & M_R & \to & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & C_R & \overset{\iota}{\longrightarrow} & P_R & \to & M_R & \to & 0 \end{array} \tag{3.1}$$

The morphisms f_0 and f_1 that complete above diagram are not uniquely determined by f. Nevertheless, it is easily seen that $f : M_R \to M_R$ is an epimorphism if and only if $f_0 : P_R \to P_R$ is an epimorphism, if and only if f_0 is an automorphism. It follows that if one substitutes f_0 and f_1 with two other morphisms f_0' and f_1' making the diagram analogous to diagram commute, then $f_0 : P_R \to P_R$ is an epimorphism if and only if $f_0' : P_R \to P_R$ is an epimorphism. In this notation, they showed the same holds for C_R, i.e., that

Lemma 3.1: ([Facchini and Girardi, 2010], Lemma 2.3) $f_1 : C_R \to C_R$ is an epimorphism if and only if $f_1' : C_R \to C_R$ is an epimorphism.

Proof The commutativity of the two diagrams (one relative to f_0, f_1, the other relative to f_0', f_1') gives, by subtraction, a commutative diagram,

$$
\begin{array}{ccccccccc}
0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0 \\
 & & \downarrow f_1-f_1' & & \downarrow f_0-f_0' & & \downarrow 0 & & \\
0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0
\end{array} \tag{3.2}
$$

Hence $(f_0 - f_0')(P_R) \subseteq C_R$. Since C_R is superfluous in P_R, it follows that $(f_0 - f_0')(C_R)$ is superfluous in $(f_0 - f_0')(P_R)$, so that $(f_0 - f_0')(C_R) = (f_1 - f_1')(C_R)$ is a proper submodule of C_R. Thus $f_1 - f_1'$ is not an epimorphism. This and the fact that C_R is couniform yields that $f_1 : C_R \to C_R$ is an epimorphism if and only if $f_1' : C_R \to C_R$ is an epimorphism. □

Recall that, P is called prime if it is proper ideal of an arbitrary ring R and for any ideal A and B of R the relation $AB \subset P$ implies $A \subset P$ or $B \subset P$.

P' is called completely prime if it is proper ideal of an arbitrary ring R and for any element a and b of R the relation $ab \subset P'$ implies $a \subset P'$ or $b \subset P'$. Trivially, every completely prime ideal is prime.

For every couniform module U_R, the endomorphism ring End(U_R) has a proper completely prime two-sided ideal K_{U_R} consisting of all the endomorphisms of U_R that are not surjective (see Lemma 6.26 in [Facchini, 1998]). The ring End$(U_R)/K_{U_R}$ is an integral domain, but it is not a division ring in general (for instance, take as U_R the Prüfer group $\mathbb{Z}(p^\infty)$ viewed as a $\mathbb{Z}$-module). The previous lemma also shows that for every couniformly presented right module M_R with couniform presentation $0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0$, there is a well-defined ring morphism End$(M_R) \to$ End$(C_R)/K_{C_R}$, defined by $f \longmapsto f_1 + K_{C_R}$.

By previous lemma the ring morphism Φ : End$(M_R) \to$ End$(M_R)/K_{M_R} \times$ End$(C_R)/K_{C_R}$ defined by $\Phi(f) = (f + K_{M_R}, f_1 + K_{C_R})$ for every $f \in$ End(M_R). Recall that a ring morphism $\varphi : S \to S'$ is said to be a local morphism if, for every $s \in S$, $\varphi(s) \in U(S')$ implies $s \in U(S)$.

Lemma 3.2: ([Facchini and Girardi, 2010], Lemma 2.4) Let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of a couniformly presented module M_R. Then the ring morphism Φ is local.

Proof Let $f \in$ End(M_R) be an endomorphism with $\Phi(f)$ invertible. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & C_R & \stackrel{\iota}{\longrightarrow} & P_R & \rightarrow & M_R & \rightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ 0 & \rightarrow & C_R & \stackrel{\iota}{\longrightarrow} & P_R & \rightarrow & M_R & \rightarrow & 0. \end{array} \tag{3.3}$$

Then $f + K_{M_R}$ and $f_1 + K_{C_R}$ are invertible in $\text{End}(M_R)/K_{M_R}$ and $(C_R)/K_{C_R}$ respectively, so that, in particular, $f \notin K_{M_R}$ and $f_1 \notin K_{C_R}$, that is, the morphisms f and f_1 are epimorphisms. Thus f_0 is also an epimorphism, hence an automorphism of P_R because P_R is projective and indecomposable. By the Snake Lemma applied to above diagram , f_0 isomorphism and f_1 epimorphism imply f monomorphism. □

The following result describe the endomorphism ring of a couniformly presented module.

Theorem 3.1: ([Facchini and Girardi, 2010], Theorem 2.5) Let $0 \rightarrow C_R \rightarrow P_R \rightarrow M_R \rightarrow 0$ be a couniform presentation of a couniformly presented module M_R. Let $K := \{f \in \text{End}(M_R) \mid f \text{ is not surjective}\}$ and $I := \{f \in \text{End}(M_R) \mid f_1 : C_R \rightarrow C_R \text{ is not surjective }\}$. Then K and I are completely prime two-sided ideals of $\text{End}(M_R)$, the union $K \cup I$ is the set of all non-invertible elements of $\text{End}(M_R)$ and any proper right ideal of $\text{End}(M_R)$ and every proper left ideal of $\text{End}(M_R)$ is contained either in K or in I. Moreover, one of the following two conditions holds:

i) Either the ideals K and I are comparable, so that $\text{End}(M_R)$ is a local ring with maximal ideal the greatest ideal among K and I, or

ii) K and I are not comparable, $J(\text{End}(M_R)) = K \cap I$ and $\text{End}(M_R)/J(\text{End}(M_R))$ is canonically isomorphic to the direct product of the two division rings $\text{End}(M_R)/K$ and $\text{End}(M_R)/I$.

Proof Let π_1 and π_2 be the canonical projections of $\text{End}(M_R)/K_{M_R} \times \text{End}(C_R)/K_{C_R}$ onto $\text{End}(M_R)/K_{M_R}$ and $\text{End}(C_R)/K_{C_R}$, respectively. It is known that $K = K_{M_R}$ is a completely prime ideal of $\text{End}(M_R)$. Notice that I is the kernel of the composite morphism $\pi_2\Phi$: $\text{End}(M_R) \rightarrow \text{End}(C_R)/K_{C_R}$. As $\text{End}(C_R)/K_{C_R}$ is an integral domain, it follows that I is a completely prime ideal of $\text{End}(M_R)$.

As the ideals K and I are proper, it follows that $K \cup I \subseteq \text{End}(M_R)(\text{End}(M_R))$. Conversely, if $f \in \text{End}(M_R)$ is non-invertible, it is not an auto-morphism, so that it is either non-surjective or non-injective. If f is not surjective, then $f \in K$. If f is surjective but not injective, then in diagram from previous f_0 is surjective, so that f_0

is an automorphism of P_R. By the Snake Lemma applied to the same diagram, f_0 automorphism of P_R and f non-injective imply f_1 non-surjective. Thus $f \in I$.

Every proper right or left ideal L of $\text{End}(M_R)$ *is contained in $K \cup I$. If there exist $x \in L$ and $y \in L$, then $x + y \in L$, $x \in I$ and $y \in K$. Hence $x + y \notin K$ and $x + y \notin I$. Thus $x + y \notin K \cup I$, so that $x + y \in L$ and is an invertible element of* $\text{End}(M_R)$*, a contradiction. This proves that L is contained either in K or in I. In particular, the unique maximal right ideals of* $\text{End}(M_R)$ *are at most K and I. Similarly, the unique maximal left ideals of* $\text{End}(M_R)$ *are at most K and I.*

If K and I are comparable, then (i) clearly holds. If K and I are not comparable, the ring $\text{End}(M_R)$ *has exactly two maximal right ideals K and I, so that* $J(\text{End}(M_R)) = K \cap I$, $\text{End}(M_R)/K$ *and* $\text{End}(M_R)/I$ *are division rings and there is a canonical injective ring homomorphism* $\pi : \text{End}(M_R)/J(\text{End}(M_R)) \to \text{End}(M_R)/K \times \text{End}(M_R)/I$. *But* $K + I = \text{End}(M_R)$ *because K and I are indecomposable maximal right ideals of* $\text{End}(M_R)$*, hence π is surjective by the Chinese Remainder Theorem.* □

If M_R and M'_R are two couniformly presented modules with couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$, M_R and M'_R have the same lower part and denoted by $[M_R]_l = [M'_R]_l$, if there are two homomorphisms $f_0 : P_R \to P'_R$ and $f'_0 : P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$. In particular, if M_R and M'_R have the same lower part, then C_R and C'_R have the same epigeny class.

If M_R and M'_R are two couniformly presented modules with couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$, then there are idempotents $e, e' \in R$ with $P_R \cong eR$ and $P'_R \cong e'R$. If one assumes $P_R = eR$ and $P'_R = eR$, C, C' right ideals of R contained in eR, $e'R$ respectively and $M_R = eR/C$, $M'_R = e'R/C'$, then M_R and M'_R have the same lower part if and only if there exists $r, s \in R$ such that $rC = C'$ and $sC' = C$. Also their definition of having the same lower part for arbitrary couniformly presented modules over arbitrary rings extends the definition of having the same lower part given in [Facchini et al., 2010] for cyclically presented modules over local rings.

Remark 3.1: ([Facchini and Girardi, 2010], Remark 3.1) Let M_R and M'_R be couniformly presented modules. It is easily seen that M_R and M'_R have the same *lower part if and only if there exists an endomorphism* $f \in \text{End}(M_R) \setminus I$ *of* M_R *that factors through* M'_R*. Similarly,* M_R *and* M'_R *have the same epigeny class if and only if there exists an endomorphism* $f \in \text{End}(M_R) \setminus K$ *of* M_R *that factors through* M'_R*. Here* I *and* K *are the completely prime ideals of* $\text{End}(M_R)$ *defined in the statement of previous theorem.*

Lemma 3.3: ([Facchini and Girardi, 2010], Lemma 3.2) Let M_R *and* M'_R *be couniformly presented right modules over a ring* R*. Then* $M_R \cong M'_R$ *if and only if* $[M_R]_l = [M'_R]_l$ *and* $[M_R]_e = [M'_R]_e$*.*

Proof Let $E := \text{End}(M_R)$ *and let* I *and* K *be the ideals of* E *as in previous theorem. Assume that* M_R *and* M'_R *have the same epigeny class and the same lower part. Then there exists* $f \in E \setminus I$ *and* $g \in E \setminus K$ *such that both* f *and* g *factor through* M'_R*. If either* f *or* g *is an automorphism, it follows that* M_R *is isomorphic to a non-zero direct summand of* M'_R*, which is indecomposable, thus* $M_R \cong M'_R$*. Assuming that* f *and* g *are not automorphisms, since* $f \in I$ *and* $g \in K$*, hence* $f + g$ *is an automorphism of* M_R *that factors through* $M'_R \oplus M'_R$*. By ([Dung and Facchini, 1998], Lemma 2.3), it follows that* M_R *is isomorphic to a direct summand of* M'_R*, thus also in this case* $M_R \cong M'_R$*.The converse is obvious.* □

Theorem 3.2: [Facchini and Girardi, 2010] Let $M_1, ..., M_n, N_1, ..., N_t$ *be couniformly presented right* R*-modules. Then* $M_1 \oplus ... \oplus M_n \cong N_1 \oplus ... \oplus N_t$ *if and only if* $n = t$ *and there are two permutations* σ, τ *of* $1, 2, ..., n$ *such that* $[N_{\sigma(i)}]_l = [M_i]_l$ *and* $[N_{\tau(i)}]_e = [M_i]_e$ *for every* $i = 1, 2, ..., n$*.*

In [Corisello and Facchini, 2001], Corisello and Facchini showed how properties of local rings extend to homogeneous semilocal rings.

Proposition 3.1: ([Corisello and Facchini, 2001], Proposition 2.1) In a homogeneous semilocal ring R *the Jacobson radical* $J(R)$ *is the unique maximal proper two-sided ideal of* R*, that is,* $J(R)$ *contains all proper two-sided ideals of* R*. Conversely, if a semilocal ring* R *has a unique maximal proper two-sided ideal, then* R *is homogeneous semilocal.*

Proof If a proper two-sided ideal of a homogeneous semilocal ring R*, then* $I +$

$J(R)/J(R)$ is a two-sided ideal of the simple ring $R/J(R)$, so that either $I + J(R) = R$ or $I \subseteq J(R)$. But $I + J(R) = R$ implies that $x + y = 1$ for some $x \in I$ and some $y \in J(R)$, so that $x = 1 - y$ is invertible in R. Thus $I = R$, a contradiction.

Conversely, if a semilocal ring R has a unique maximal two-sided ideal I, then $I \supseteq J(R)$, so that $I/J(R)$ is the unique maximal two-sided ideal of the semisimple artinian ring $R/J(R)$. A semisimple artinian ring is a direct product of matrices over division rings, so that if such a ring has a unique maximal two-sided ideal, then the ring must be simple. Thus $R/J(R)$ is simple and R is homogeneous semilocal. □

In particular, every homomorphic image of a homogeneous semilocal ring is a homogeneous semilocal ring (see Example(6), p.7 in [Facchini, 1998]). A commutative ring is homogeneous semilocal if and only if it is local.

Proposition 3.2: ([Corisello and Facchini, 2001], Proposition 2.2)

i) If R is a homogeneous semilocal ring and p is a positive integer, then the ring $\mathbb{M}_p(R)$ of $p \times p$ matrices with entries in R is a homogeneous semilocal ring.
ii) If R is a homogeneous semilocal ring and e is a non-zero idempotent element of R, then the ring eRe is homogeneous semilocal.

Proof If R is homogeneous semilocal, then $R/J(R) \cong \mathbb{M}_n(D)$ for some $n > 0$ and some division ring D. Therefore, $M_p(R)/J(\mathbb{M}_p(R)) = \mathbb{M}_p(R)/\mathbb{M}_p(J(R)) \cong \mathbb{M}_p(R/J(R)) \cong \mathbb{M}_{pn}(D)$ is simple artinian. Similarly, if $0 \neq e = e^2 \in R$, then $eRe/J(eRe) = eRe/eJ(R)e \cong (e + J(R))(R/J(R))(e + J(R))$. Hence it suffices to show that S is a simple artinian ring and $\overline{e}$ is a non-zero idempotent of S, then $\overline{e}S\overline{e}$ is simple artinian. Now $S \cong \text{End}(V_D)$ for a finite dimensional vector space V_D over a division ring D and $\overline{e}$ corresponds to an idempotent endomorphism of V_D. If U_D and W_D ate the image and the kernel of this idempotent endomorphism, then $V_D = U_D \oplus W_D$ and $\overline{e}S\overline{e} \cong \text{End}(U_D)$ is a simple artinian ring. □

Theorem 3.3: ([Corisello and Facchini, 2001], Theorem 2.3) Let R be homogeneous semilocal ring. Then

i) There exists a unique indecomposable finitely generated projective R-module P up to isomorphism.
ii) Every projective R-module is isomorphic to direct sum $P^{(X)}$ for some set X.

iii) If X, Y are sets, then $P^{(X)}$ and $P^{(Y)}$ are isomorphic if and only if X and Y have the same cardinality.

Proof Let S be the unique simple R-module up to isomorphism. For every finitely generated projective R-module Q, the module $Q/QJ(R)$ is semisimple because it is a module over the simple artinian ring $R/J(R)$ and therefore there exists a unique nonnegative integer n_Q such that $Q/QJ(R) \cong S^{n_Q}$. By Nakayama's Lemma, $n_Q = 0$ if and only if $Q = 0$. Let P be a non-zero finitely generated projective R-module with n_P minimal and let Q be an arbitrary non-zero finitely generated projective R-module. Then $n_Q \geq n_P$, so that there is an epimorphism $S^{n_Q} \to S^{n_P} \cong P/PJ(R)$. Since there exists also an epimorphism $Q \to Q/QJ(R) \cong S^{n_Q}$, the composite mapping is an epimorphism $\varphi : Q \to P/PJ(R)$. As Q is projective, φ lifts to a homomorphism $\psi : Q \to P$ such that $\psi(Q) + PJ(R) = P$. Again from Nakayama's Lemma it follows that $\psi(Q) = P$, i.e., ψ is surjective. Since P is projective, P is isomorphic to a direct summand of Q. This shows that every non-zero finitely generated projective R-module has a direct summand isomorphic to P. In particular, (i) holds.

It shall be proven now that every non-zero projective right R-module Q is a generator (recall that a module Q_R is a generator if and only if R_R is isomorphic to a direct summand of $Q_R^{(X)}$ for some set X). Let Q be a non-zero projective module. By (Proposition 2.7 in [Bass, 1960]), $Q/QJ(R) \neq 0$. As $R/J(R)$ is simple artinian, every $R/J(R)$-module is isomorphic to a direct sum of copies of the unique simple $R/J(R)$-module S. Thus $Q/QJ(R) \cong S^(X)$ for some nonempty set X. Hence there is an epimorphism $(Q/QJ(R))^{n_R} \to S^{n_R} \cong R/J(R)$. As in the proof of (i) the module R_R must be isomorphic to a direct summand of Q^{n_R}. This shows that every non-zero projective right R-module is a generator, that is, R is (right)p-connected in the terminology of Bass [Bass, 1963] and of Fuller and Shutters [Fuller and Shutter, 1975]. By (Theorem 1 in [Fuller and Shutter, 1975]) every projective right R-module is isomorphic to a direct sum of copies of a projective indecomposable cyclic R-module and from this (ii) follows trivially.

If $P^{(X)} \cong P^{(Y)}$, then $P^{(X)}/P^{(X)}J(R) \cong P^{(Y)}/P^{(Y)}J(R)$, that is, $(S^{n_P})^{(X)} \cong (S^{n_P})^{(Y)}$. By (Theorem 2.14 in [Facchini, 1998]) $n_P \mid X \mid = n_P \mid Y \mid$, from which $\mid X \mid = \mid Y \mid$.

□

Corollary 3.1: ([Corisello and Facchini, 2001], Corrollary 2.4) Let $M_R \neq 0$ *be a module over an arbitrary ring* R *and suppose that the endomorphism ring* $\text{End}(M_R)$ *is homogeneous semilocal. Then there exists an indecomposable submodule* N_R *of* M_R *with* $\text{End}(N_R)$ *homogeneous semilocal and a nonnegative integer* t *such that* $M_R \cong N_R^t$*. Moreover, if* $M_R = M_1 \oplus \cdots \oplus M_n$ *is another direct sum decomposition of* M_R *into indecomposable direct summands, then* $n = t$ *and* $M_1 \cong ... \cong M_n \cong N_R$*.*

Proof There is a category equivalence induced by the functors $\text{Hom}_R(M_R, -) : \text{Mod}-R \to \text{Mod} - \text{End}(M_R)$ *and* $- \otimes_{\text{End}(M_R)} M_R : \text{Mod} - \text{End}(M_R) \to \text{Mod} - R$ *between the category* $\text{add}(M_R)$ *of direct summands of finite direct sums* M_R^n *of copies of* M_R *and the category of finitely generated projective right* $\text{End}(M_R)$*-modules (see Theorem 4 in [Facchini, 1998]). Now apply , (Theorem 2.3 in [Corisello and Facchini, 2001]).* □

Next theorem shows that whenever one can localize a right noetherian ring R at a right localizable prime ideal P, the ring R_P is a homogeneous semilocal ring. Here R_P denotes the right quotient ring R with respect to the set $C_R(P) = \{x \in R : x + P \text{is not a zero divisor in} R/P\}$ of all elements of R regular modulo P.

Theorem 3.4: ([Corisello and Facchini, 2001], Theorem 4.1) Let R *be a right noetherian ring,* P *a right localizable prime ideal and* R_P *the localization. Then* R_P *is a homogeneous semilocal ring.*

Proof The right R*-module* R/P *is* $C_R(P)$*-torsionfree. Let* $P^e = \{px^{-1} : p \in P, x \in C_R(P)\}$ *be the extension of* P *in* R_P*. From (Theorems 9.17 and 9.20 in [Goodearl and Warfield, 1989]) it is known that* $P^e = PR_P$ *is an ideal in* R_P *and* $P^e c = P$*. It is easliy checked that* R/P *is a right order in* R_p/PR_P *via the canonical mapping* $R/P \to R_P/PR_P$*. As* R *is right noetherian,* R/P *is a prime right Goldie ring. Since* R/P *is a right order in* R_P/PR_P*, the ring,* R_P/PR_P *is simple artinian.*

In order to conclude the proof it suffices to show that $PR_P = J(R_P)$*. As* R_P/PR_P *is simple artinian, it has a unique simple module, which is faithful. Thus* R_P *has a simple module whose annihilator is* PR_P*. Since* $J(R_P)$ *is the intersection of the annihilators of all simple* R_P- *modules, it follows that* $PR_P \supseteq J(R_P)$*. In order* *to prove the opposite inclusion $PR_P \subseteq J(R_P)$, it shall be shown that every maximal right ideal I of R_P contains PR_P. Assume that I is a maximal right ideal I of R_P contains PR_P. Then $I + PR_P = R_P$, so that there exist $a \in I^c$, $b \in C_R(P)$ and $x \in PR_P$ such that $ab^{-1} + x = 1$. Then $a + xb = b$, i.e., $a - b = xb \in (PR_P)^c = P$. As b is regular modulo P, $a \in I^c$ must be regular modulo P. Therefore $I = R_P$, contradiction.* □

In [Barioli et al., 2001], Barioli, Facchini, Raggi and Rios firstly, showed that every almost semiperfect ring is semilocal and if $M = M_1 \oplus \cdots \oplus M_n$ is direct sum of modules M_i with homogeneous semilocal endomorphism ring End(M_i), then the almost semiperfect endomorphism ring End(M) is necessarily semilocal, so that M has only finitely many direct sum decompositions up to isomorphism. And finally they studied the modules M with a finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ such that all endomorphism rings End(M_i) are homogeneous semilocal and found complete results about uniqueness of such decompositions (Krull-Schmidt Theorem). Moreover they showed that such a module M can have different direct sum decompositions up to isomorphism.

Theorem 3.5: ([Barioli et al., 2001], Theorem 2.1) Let $e_1, \ldots, e_n$ be a complete set of orthogonal idempotents of a ring R. If e_iRe_i is semilocal for every $i = 1, \ldots, n$, then R is semilocal.

Proof It is denoted by codim the dual Goldie dimension (see [Facchini, 1998]). For a projective module M_R with endomorphism ring $S =$ End(M_R), it is known that codim$(M_R) =$ codim(S_S) *(see Corrollary 4.3 in [Garcia Hernandez and Gomez Pardo, 1987]). In particular, for any idempotent $e \in R$,* codim$(eR_R) =$ codim(eRe_eRe).

As semilocal rings are exactly the rings of finite dual Goldie dimension (see Proposition 2.43 in[Facchini, 1998]), if e_iRe_i is semilocal for every $i = 1, \ldots, n$, then codim$(e_iR_R) =$ codim$(e_iRe_i) < \infty$ *for every $i = 1, \ldots, n$, so that* codim$(R_R) =$ codim$(e_1R_R) + \cdots +$ codim$(e_nR_R) < \infty$ *and therefore R is semilocal.* □

Corollary 3.2: ([Barioli et al., 2001], Corollary 2.2) Let $M_1, \ldots, M_n$ be right modules over a ring R. Set $M = M_1 \oplus \cdots \oplus M_n$. Suppose that End$_R(M_i)$ *is a semilocal ring* *for every $i = 1, \dots, n$. Then* $\mathrm{End}_R(M)$ *is semilocal. In particular, M has only finitely many direct sum decomposition up to isomorphism.*

Proof *The first part of the statement follows immediately from (Theorem 2.1 in [Barioli et al., 2001]). The second part follows from the first one and (see [Facchini, 1998], p.107).* □

Theorem 3.6: ([Barioli et al., 2001], Theorem 3.5) Let $\overline{M}_R$ be a module over a ring R and $\overline{M}_R = N \oplus N'$ a direct sum decomposition of $\overline{M}_R$. Suppose that $\overline{M}_R = M_1 \oplus \cdots \oplus M_t$ and $N = N_1 \oplus \cdots \oplus N_m$ are two direct sum decomposition into indecomposable direct summands of $\overline{M}_R$ and N respectively and that all the endomorphism rings $\mathrm{End}(M_k)$ *and* $\mathrm{End}(N_l)$ *are homogeneous semilocal. Then also N' has a direct sum decomposition $N' = N'_1 \oplus \cdots \oplus N'_r$ into indecomposable direct summands N'_j such that all the endomorphism rings* $\mathrm{End}(N'_j)$ *are homogeneous semilocal and there is an injective mapping $\varphi : 1, \dots, m \to 1, \dots, t$ such that $N_l \cong M_{\varphi(l)}$ for every $l = 1, \dots, m$.*

Theorem 3.7: ([Barioli et al., 2001], Theorem 3.6) (Krull-Schmidt Theorem for direct sums of modules with homogeneous semilocal endomorphism rings) Let $\overline{M}_R$ be a module over a ring R. Suppose $\overline{M}_R = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_m$ are two direct sum decompositions of $\overline{M}_R$ into indecomposable direct summands and that all the endomorphism rings $\mathrm{End}(M_k)$ *and* $\mathrm{End}(N_l)$ *are homogeneous semilocal. Then the two given direct sum decompositions of $\overline{M}_R$ are isomorphic.*

Proof *Apply ([Barioli et al., 2001], Theorem 3.5) with $N' = 0$, so that there exists an injective mapping $\varphi : 1, \dots, m \to 1, \dots, t$ such that $N_l \cong M_{\varphi(l)}$ for every $l = 1, \dots, m$. By symmetry there exists an injective mapping $\psi : 1, \dots, t \to 1, \dots, m$ such that $M_k \cong N_{\psi(k)}$ for every $k = 1, \dots, t$. The conclusion follows immediately.* □

Corollary 3.3: ([Barioli et al., 2001], Corollary 3.7) Let $\overline{M}_R = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_m$ be two direct sum decompositions of a module $\overline{M}_R$ over an arbitrary ring R. If all the endomorphism rings $\mathrm{End}(M_k)$ *and* $\mathrm{End}(N_l)$ *are almost semiperfect, then the two given direct sum decompositions of $\overline{M}_R$ have isomorphic refinements.*

In [Amini and Facchini, 2008] Authors proved that two matrices $diag(a_1, ..., a_n)$

and $diag(b_1, ..., b_n)$ over a local ring R are equivalent if and only if there are two permutations σ, τ of $\{1, 2, ..., n\}$ such that $[R/a_iR]_l = [R/b_\sigma iR]_l$ and $[R/a_iR]_e = [R/b_\tau iR]_e$ for every $i = 1, 2, ..., n$. Following results were obtained studying the direct-sum decompositions of finite direct sums of cyclically presented modules over local rings. The theory of these decompositions turned out to be incredibly similar to the theory of direct-sum decompositions of finite direct sums of uniserial modules over arbitrary rings.

Theorem 3.8: ([Amini and Facchini, 2008], Theorem 2.1) Let a be a non-zero non-invertible element of a local ring R, let E be the idealizer of aR and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Let $I = \{r \in R : ra \in aJ(R)\}$ and $K = J(R) \cap E$. Then I and K are completely prime two-sided ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of E/aR and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR. Moreover, exactly one of the following two conditions hold:

i) Either the ideals I and K are comparable, so that E/aR is a local ring with maximal ideal $(I/aR) \cup (K/aR)$, or

ii) I and K are not comparable, $J(E/aR) = (I \cap K)/aR$ and $(E/aR)/J(E/aR)$ is canonically isomorphic to the direct product of the two division rings E/I and E/K.

In particular, cyclically presented modules over local rings have semilocal endomorphism ring, hence cancel from direct sums (see Corollary 4.6 in [Facchini, 1998]).

Remark 3.2: ([Amini and Facchini, 2008], Remark 2.2) If a is non-zero non-invertible element of a commutative local ring R, then the idealizer E coincides with R and $I = K$. To see it, notice that if $r \in R$, then $ra = aj$ for some $j \in J(R)$, so that $a \neq 0$ annihilates $r - j$. Thus $r - j \in J(R)$ and from this $r \in K$. Conversely, if $r \in K$, then $r \in J(R)$, hence $ra \in aJ(R)$ and $r \in I$. Thus the endomorphism ring E/aR of R/aR is the local commutative ring R/aR with maximal ideal $I/aR = K/aR = J(R)/aR$.

Recall the following elementary lemma, which was proved in (Lemma 2.1 in

[Amini and Facchini, 2008]).

Lemma 3.4: ([Amini and Facchini, 2008], Lemma 2.3) Let R be a ring in which all right zero-divisors are in the Jacobson radical, e.g., let R be a local ring and let $r, s \in R$. Then $rR = sR$ if and only if there exists an invertible element u of R such that $r = su$.

Lemma 3.5: ([Amini and Facchini, 2008], Lemma 2.4) Let R be a local ring and let $r, s \in R$. Then $R/rR \cong R/sR$ if and only if there are invertible elements $u, v \in R$ such that $urv = s$.

From previous lemma, one can immediately get:

Corollary 3.4: ([Amini and Facchini, 2008], Corollary 2.5) Let R be a local ring and let $r, s \in R$. Then $R/rR \cong R/sR$ if and only if $R/Rr \cong R/Rs$.

Remark 3.3: ([Amini and Facchini, 2008], Remark 2.6) If R is a local ring and $r, s \in R$, then $rR \cong sR$ if and only if there exists $u \in U(R)$ such that $r.ann_R(r) = r.ann_R(su)$. In fact, if u is an invertible element of R such that $r.ann_R(r) = r.ann_R(su)$, then $rx \to sux$ is a well defined isomorphism $rR \to suR = sR$. Conversely, if $f : rR \to sR$ is an isomorphism, then $f(r)R = sR$, so that $f(r) = su$ for some unit $u \in R$ Lemma 3.4. The condition $\mathrm{r}_R(r) = \mathrm{r}_R(su)$ is now obvious.

The followings are some known results necessary for the sequel. Their aim was the study of the modules that are finite direct sums of cyclically presented modules over local rings. The following result, due to Warfield (Theorem 1.4 in [Warfield, 1975]), is therefore fundamental.

Theorem 3.9: ([Amini and Facchini, 2008], Theorem 3.1) Let M be a finitely generated module over a semiperfect ring R. Then there is a decompositon $M = N \oplus P$, where P is projective and N has no non-zero projective summands. Further, if $M = N' \oplus P'$ is another such decomposition, then $N \cong N'$ and $P \cong P'$.

Therefore every finitely presented module M over a local ring R has a decomposition $M = N_1 \oplus ... \oplus N_t \oplus R_R^n$ for suitable indecomposable modules $N_1, ..., N_t$ not isomophic to R_R and a suitable integer $n \geq 0$. Further, if $N_1' \oplus ... \oplus N_s' \oplus R_R^m$ is another such decomposition, then $N_1 \oplus ... \oplus N_t \cong N'_1 \oplus ... \oplus N'_s \oplus$ and $n = m$.

Moreover, isomorphism of finitely presented modules is related to equivalence of matrices. Namely, recall that two (possibly rectangular) $m \times n$ matrices A and B over a ring R are said to be equivalent if there exist an $m \times n$ matrix P and an $n \times n$ matrix Q, both with two-sided inverses, such that $A = PBQ$. For an arbitrary ring R, not necessarily local, the following result due to Fitting holds (see Proposition 4 in [Fitting, 1936]).

Proposition 3.3: ([Amini and Facchini, 2008], Proposition 3.2) Let R be a ring and let $r, s \in R$. Then:

i) $R/rR \cong R/sR$ if and only if the 2×4 matrices $A' = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $B' = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ are equivalent.

ii) $R/Rr \cong R/Rs$ if and only if the 4×2 matrices $A'' = \begin{pmatrix} r & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B'' = \begin{pmatrix} s & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ are equivalent.

iii) If r and s are either right or left zero-divisor, then $R/Rr \cong R/Rs$ if and only if $R/rR \cong R/sR$, if and only if 2×2 matrices $A^* = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ and $B^* = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ are equivalent.

Recall that any finitely presented R-module can be presented via a matrix. Namely, let M be a module with a presentation with n generators and m relations. Then there is an exact sequence $R_R^m \to R_R^n \to M \to 0$, that is, M is the cokernel of a morphism $\varphi : R_R^m \to R_R^n$, which must be necessarily left multiplication by an $n \times m$ matrix A (the matrix associated to φ relative to the canonical bases). The morphism φ is denoted by the same symbol A used for the matrix. In particular, $M \cong \text{coker}(A)$.

Theorem 3.10: ([Levy and Robson, 1974], Theorem 4.3), Let R be a semiperfect ring.Then two $m \times n$ matrices A and B with entries in R are equivalent if and only if the finitely presented right R-modules $\text{coker}(A)$ and $\text{coker}(B)$ are isomorphic.

Thus two modules M_R and N_R over a semiperfect ring, both with n generators and m relations, described by A and B respectively, are isomorphic if and only if the matrices A and B are equivalent. This holds, in particular, for local rings and our Lemma 2.4 is a particular case (for $n = m = 1$) of Theorem 3.3.

It is already seen that every finitely presented modules over a local ring decomposes in a unique way as a direct sum of a free part and a part with no non-zero free summands.

Recall that A and B are two modules, it is said that A and B have the same epigeny class and write $[A]_e = [B]_e$, if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$ (see [Facchini, 1996]). If R/aR and R/bR are two cyclically presented right modules over a local ring R, R/aR and R/bR have the same lower part, denoted by $[R/aR]_l = [R/bR]_l$, if there exist $r, s \in R$ such that $raR = bR$ and $sbR = aR$. Note that, by Lemma 2.4, this definition does not depend on the presentation of the cyclically presented module; that is, $R/aR \cong R/bR$ implies $[R/aR]_l = [R/bR]_l$. Clearly, having the same epigeny class and having the same lower part are two equivalence relations in the class of all cyclically presented right R-modules. For cyclically presented left modules, it is said that R/Ra and R/Rb have the same lower part and write $[R/Ra]_l = [R/Rb]_l$ if there exist $r, s \in R$ such that $Rar = Rb$ and $Rbs = Ra$.

Remark 3.4: ([Amini and Facchini, 2008], Remark 4.1) The unique cyclically presented module, up to isomrphism, with the same epigeny class as 0 is 0, and the unique cyclically presented module, up to isomorphism, with the same epigeny class as R_R is R_R. Similarly for the lower part.

Notice that, if a, b are elements of a local ring, then $[R/aR]_e = [R/bR]_e$ if and only if there exist $u, v \in U(R)$ with $ua \in bR$ and $vb \in aR$, if and only if there exist $u, v \in U(R)$ and $r, s \in R$ with $ua = br$ and $vb = as$. Also, for $a, b \in R$, $[R/aR]_l = [R/bR]_l$ if and only if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$.

Lemma 3.6: ([Amini and Facchini, 2008], Lemma 4.2) Let a, b be elements of a local ring R. Then $R/aR \cong R/bR$ if and only if $[R/aR]_l = [R/bR]_l$ and $[R/aR]_e = [R/bR]_e$.

Another consequence of Lemma 3.4 is:

Corollary 3.5: ([Amini and Facchini, 2008], Corollary 4.3) Let R be a local ring and let $a, b \in R$. Then:

i) $[R/aR]_l = [R/bR]_l$ *if and only if* $[R/aR]_e = [R/bR]_e$

ii) $[R/aR]_e = [R/bR]_e$ *if and only if* $[R/aR]_l = [R/bR]_l$

Proposition 3.4: ([Amini and Facchini, 2008], Proposition 5.1) Let $a, c_1, ..., c_n$ $(n \geq 2)$ be non-invertible elements of a local ring R. Suppose that R/aR is a direct summand of $R/c_1R \oplus ... \oplus R/c_nR$ and R/aR is not isomorphic to R/c_iR for every $i = 1, 2, ..., n$. Then there are two distinct indices $i, j = 1, ..., n$ such that $[R/aR]_l = [R/c_iR]_l$ and $[R/aR]_e = [R/c_jR]_e$.

Lemma 3.7: ([Amini and Facchini, 2008], Lemma 5.2) Let a, b, c be non-invertible elements of a local ring R and assume $[R/aR]_l = [R/bR]_l$ and $[R/aR]_e = [R/cR]_e$. Then:

i) $R/aR \oplus D \cong R/bR \oplus R/cR$ *for some R-module D;*

ii) the module D in (i) is unique up to isomorphism and is cyclically presented;

iii) $[D]_l = [R/cR]_l$ *and* $[D]_e = [R/bR]_e$.

Theorem 3.11: ([Amini and Facchini, 2008], Theorem 5.3) Let $a_1, ..., a_n, b_1, ..., b_n$ be non-invertible elements of a local ring R. Then $R/a_1R \oplus ... \oplus R/a_nR$ and $R/b_1R \oplus ... \oplus R/b_tR$ are isomorphic right R-modules if and only if $n = t$ and there are permutations σ, τ of $\{1, 2, ..., n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $1, 2, ..., n$.

It must be noticed that the two permutations and always preserve the free part and the part without free summands. That is, let $a_1, ..., a_n, b_1, ..., b_t$ be non-invertible elements and assume that $a_i = 0$ for $i = 1, ..., m$ and $a_i \neq 0$ for $i = m + 1, ..., n$. Similarly, suppose that $b_j = 0$ for $j = 1, ..., s$ and $b_j \neq 0$ for $j = s+1, ..., t$. If $R/a_1R \oplus ... \oplus R/a_nR \cong R/b_1R \oplus ... \oplus R/b_tR$, then $n = t$ and there are two permutations σ, τ of $\{1, 2, ..., n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, ..., n$. Then $m = s$ and both σ and τ send $\{1, 2, ..., m\}$ onto $\{1, 2, ..., m\}$ (hence $\{m + 1, m + 2, ..., n\}$ onto $\{m + 1, m + 2, ..., n\}$ also). This follows from previous remark.

Corollary 3.6: ([Amini and Facchini, 2008], Corollary 5.4) Let $a_1, ..., a_n, b_1, ..., b_n$ be elements of a local ring R. The $diag(a_1, ..., a_n)$ and $diag(b_1, ..., b_n)$ are equivalent if and only if there are two permutations σ, τ of $\{1, 2, ..., n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, ..., n$.

In [Warfield, 1975] Warfield said the uniqueness question for decompositions of a finitely presented module into uniserial summands is an open problem which demonstrated in the commutative case and in one noncommutative case by Kaplansky in [Kaplansky, 1949]. Then in [Facchini, 1996], Facchini solved Warfiled's problem completely.

Recall that a module is uniserial if its lattice submodules is linearly ordered under inclusion and is a serial module if it is a direct sum of uniserial modules. A ring is serial if it is a serial module both as a right modules and as a left module over itself. The symbol $\subset$ will denote proper inclusion and if S is a ring, $J(S)$ will denote the Jacobson radical of S.

A serial module is of finite Goldie dimension if and only if it is the direct sum of a finite number of uniserial modules. More precisely, a serial module M has finite Goldie dimension n if and only if it is the direct sum of n non-zero uniserial modules, so that the number n of direct summands of M that appear in any decomposition of M as a direct sum of non-zero uniserial modules does not depend on the decomposition.

Lemma 3.8: ([Facchini, 1996], Lemma 1.1) Let A, C be non-zero right modules over an arbitrary ring R, B a uniserial right R-module and $\alpha : A \to B$, $\beta : B \to C$ homomorphisms. Then

i) $\beta\alpha$ is a monomorphism if and only if β and α are both monomorphisms;

ii) $\beta\alpha$ is an epimorphism if and only if β and α are both epimorphisms.

Proof It must be proven that if $\beta\alpha$ is a monomorphism, β also is a monomorphism. Now if $\beta\alpha$ is a monomorphism, then $\alpha(A) \cap \text{Ker}(\beta) = 0$. Since B is uniserial, either $\alpha(A) = 0$ or $\text{Ker}(\beta) = 0$. Now $\alpha(A) = 0$ implies $\beta\alpha = 0$ and this is not a monomorphism because $A \neq 0$. Hence $\text{Ker}(\beta) = 0$.

It must be proven that if $\beta\alpha$ is an epimorphism, α also is an epimorphism. Now if $\beta\alpha$ is an epimorphism and $C \neq 0$, then $\beta\alpha \neq 0$, so that $\beta \neq 0$. Hence $\text{Ker}(\beta) \subset B$. If $\alpha(A) \subset B$, then $\text{Ker}(\beta) + \alpha(A) \subset B$. Now β induces a one-to-one order preserving *mapping between the set of all submodules of B containing $\mathrm{Ker}(\beta)$ and the set of all submodules of $\beta(B)$. Hence $\mathrm{Ker}(\beta)+\alpha(A) \subset B$ implies $\beta(\mathrm{Ker}(\beta)+\alpha(A)) \subset \beta(B)$, that is, $\beta\alpha(A) \subset \beta(B) \subseteq C$. Hence $\beta\alpha$ is not an epimorphism, a contradiction. This proves that $\alpha(A) = B$ and α is an epimorphism.* □

Theorem 3.12: ([Facchini, 1996], Theorem 1.2) Let A_R be a non-zero uniserial module and $E = \mathrm{End}(A_R)$ its endomorphism ring. Let I be the subset of E consisting of all the endomorphisms of A_R that are not monomorphisms and J be the subset of E consisting of all the endomorphisms of A_R that are no epimorphisms. Then I and J are completely prime two-sided ideals of E, every right (or left) proper ideal of E is contained either in I or in J and either

i) the ideals I and J are comparable, so that E is a local ring with maximal ideal $I \cup J$, or

ii) the ideals I and J are comparable, $I \cap J$ is the Jacobson radical $J(E)$ of E and $E/J(E)$ is canonically isomorphic to the direct product $E/I \times E/J$ of the two division rings E/I and E/J.

Proof Obviously I and J are additively closed. They are two-sided completely prime ideals of E by previous lemma.

Let K be an arbitrary proper right or left ideal of E. Since $I \cup J$ is exactly the set of non-invertible elements of E, it follows that $K \subseteq I \cup J$. But then either $K \subseteq I$ or $K \subseteq J$.

(Otherwise there exist $x \in K \backslash I$ and $y \in K \backslash J$. Then $x + y \in K$, $x \in J$ and $y \in I$. Thus $x + y \notin I$ and $x + y \notin J$. Hence $x + y \notin I \cup J$. This is a contradiction because $K \subseteq I \cup J$.)

Thus every proper right or left ideal of E is contained either in I or in J. Therefore the unique maximal right ideals of E are at most I and J and similarly for left ideals. If $I \subseteq J$ or $J \subseteq I$, then E is local ring with maximal ideal $I \cup J$ and case (i) holds. Otherwise I and J are the two unique maximal right ideals of E. Therefore $I \cap J$ is the Jacobson radical of E and hence there is a canonical injective ring morphism $E/J(E) \to E/I \times E/J$. Since $I + J = R$, this ring morphism is onto by the Chinese Remainder Theorem. □

Corollary 3.7: ([Facchini, 1996], Corollary 1.3) Uniserial modules cancel from direct *sums, that is, if A is a serial module of finite Goldie dimension and B, C are arbitrary modules, then $A \oplus B \cong A \oplus C$ implies $B \cong C$.*

By previous theorem, if N is a non-zero uniserial module and $\text{End}(N)$ is its endomorphism ring, then either $\text{End}(N)/J(\text{End}(N))$ is a division ring (that is, $\text{End}(N)$ is a local ring) or $\text{End}(N)/J(\text{End}(N))$ is the direct product of two division rings. It shall be said that a non-zero uniserial module is of type 1 if its endomorphism ring is local and of type 2 otherwise. Hence a non-zero uniserial module N is of type d if and only if $\text{End}(N)/J(\text{End}(N))$ is the direct product of d division rings (and only $d = 1$ or $d = 2$ can occur).

For instance, every commutative valuation ring is a uniserial module of type 1 as a module over itself.

Lemma 3.9: ([Facchini, 1996], Lemma 1.4) Let A, B be non-zero uniserial modules over an arbitrary ring R.

i) If $f, g : A \to B$ are two homomorphisms, f is injective and non-surjective and g is surjective and non-injective, then $f + g$ is an isomorphism.

ii) Conversely, suppose that $f_1, \ldots, f_n : A \to B$ are n homomorphisms none of which is an isomorphism. If $f_1 + \ldots + f_n$ is an isomorphism, then there exist two indices $i, j = 1, 2, ..., n$ such that f_i is injective and non-surjective and f_j is surjective and non-injective.

Proof The proof of (i) is elementary. For the proof of (ii), consider the n elements $(f_1 + \ldots + f_n)^{-1} f_i$ of $\text{End}(A_R)$*. Their sum is 1_A and none of them is invertible in* $\text{End}(A_R)$*. Hence* $\text{End}(A_R)$ *is not local ring. By Theorem 1.2 the ring* $\text{End}(A_R)/J(\text{End}(A_R))$ *is canonically isomorphic to the direct product of two division rings* $\text{End}(A_R)/I$ *and* $\text{End}(A_R)/J$*. Now the conclusion follows easily.* □

The next proposition reduces the study of the Krull-Schmidt property for serial modules to the case of a direct sum of two uniserial modules. Its proof was inspired by the proof in [Stenström, 1975].

Proposition 3.5: ([Facchini, 1996], Proposition 1.5) Suppose $A \oplus B = C_1 \oplus \ldots \oplus C_n$, with $n \geq 2$ and A is uniserial. Then there are two distinct indices i and j and a direct *decomposition* $A' \oplus B' = C_i \oplus C_j$ *of* $C_i \oplus C_j$ *such that* $A \cong A'$ *and* $B \cong B' \oplus (\bigoplus_k \neq i, jC_k)$.

The relationship between isomorphism, monogeny and epigeny classes is described in the next proposition.

Proposition 3.6: ([Facchini, 1996], Proposition 1.6) Let A *and* B *be uniserial modules. Then* $A \cong B$ *if and only if* $[A]_m = [B]_m$ *and* $[A]_e = [B]_e$.

Proposition 3.7: ([Facchini, 1996], Proposition 1.7) Let $A, U_1, ..., U_n$ *be uniserial modules,* $n \geq 2$ *and* $A \neq 0$. *Suppose that* A *is isomorphic to a direct summand of* $U_1 \oplus ... \oplus U_n$ *and* A *is a non isomorphic to* U_i *for every i. Then there are two distinct indices* $i, j = 1, 2, ..., n$ *such that* $[A]_m = [U_i]_m$ *and* $[A]_e = [U_j]_e$.

Conversely, let A, U, V *be uniserial modules such that* $[A]_m = [U]_m$ *and* $[A]_e = [V]_e$. *then* $A \oplus X \cong U \oplus V$ *for some module X, necessarily uniserial, that is unique up to isomorphism.*

Lemma 3.10: ([Facchini, 1996], Lemma 1.8) Let U_1, U_2, V_1, V_2 *be non-zero uniserial modules and suppose that* $U_1 \oplus U_2 \cong V_1 \oplus V_2$. *Then* $\{[U_1]_m, [U_2]_m\} = \{[V_1]_m, [V_2]_m\}$ *and* $\{[U_1]_e, [U_2]_e\} = \{[V_1]_e, [V_2]_e\}$.

Theorem 3.13: ([Facchini, 1996], Theorem 1.9) Let $U_1, ..., U_n, V_1, ..., V_t$ *be non-zero uniserial modules. Then* $U_1 \oplus ... \oplus U_n \cong V_1 \oplus ... \oplus V_t$ *if and only if* $n = t$ *and there are two permutations* σ, τ *of* $1, 2, ..., n$ *such that* $[U_{\sigma(i)}]_m = [V_i]_m$ *and* $[U_{\tau(i)}]_e = [V_i]_e$ *for every* $i = 1, 2, ..., n$.

4. ENDOMORPHISM RINGS OF SOME MODULE CLASSES

This chapter of the thesis contains the results of [Şahinkaya et al., 2014]. Throughout this chapter all rings will be consider associative with identity and modules will be unital right modules. And we will assume $M_R \neq 0$ and we describe the endomorphism ring of cyclic, finitely presented module of projective dimension ≤ 1 over local ring.

Theorem 4.1: Let R be a local ring and let $M_R := R_R/I$ be a cyclic, finitely presented module of projective dimension ≤ 1. Suppose $\mathrm{Ext}^1_R(M_R, R_R) = 0$. Assume $0 \neq I \neq R$ and let E be the idealizer of the right ideal I of R, that is, the set of all $r \in R$ with $rI \subseteq I$, so that $\mathrm{End}(M_R) \cong E/I$. Set $L := \{r \in R \mid rI \subseteq IJ(R)\}$ and $K := E \cap J(R)$. Let $\psi : E \to \mathrm{End}_R(I/IJ(R))$ be the ring morphism defined by $\psi(e)(x + IJ(R)) = ex + IJ(R)$, for every $e \in E$ and $x \in I$. Let n be the dimension of the right vector space $I/IJ(R)$ over the division ring $R/J(R)$. Then:

i) L and K are prime two-sided ideals of E containing I and K is a completely prime ideal of E.

ii) For every $e \in E$, the element $e + I$ of E/I is invertible in E/I if and only if $e + J(R)$ is invertible in $R/J(R)$ and $\psi(e)$ is invertible in $\mathrm{End}_R(I/IJ(R))$.

iii) The quotient ring E/L is isomorphic to the ring $M_n(R/J(R))$ of all $n \times n$ matrices over the division ring $R/J(R)$.

iv) Exactly one of the following two conditions holds:

a) Either $K \subseteq L$, in which case E/I is a homogeneous semilocal ring with Jacobson radical L/I,or

b) L and K are not comparable.

Proof Notice that L is contained in E and is the kernel of ψ, so that L is a two-sided ideal of E. Trivially, I is contained in L. Let us prove that ψ is onto. Let $f : I/IJ(R) \to I/IJ(R)$ be a morphism. Since $M_R := R_R/I$ is of projective dimension ≤ 1, the ideal I_R is projective, so that f lifts to a morphism $f' : I_R \to I_R$. Apply functor $\mathrm{Hom}(-, R_R)$ to exact sequence $0 \to I_R \to R_R \to M_R \to 0$, getting *a short exact sequence* $0 \to \mathrm{Hom}(M_R, R_R) \to \mathrm{Hom}(R_R, R_R) \to \mathrm{Hom}(I_R, R_R) \to 0$ *because* $\mathrm{Ext}^1_R(M_R, R_R) = 0$. *Hence* f' *can be extended to a morphism* $f'' : R_R \to R_R$, *which is necessarily left multiplication by an element* $r \in R$. *Since* f'' *restricts to the endomorphism* f' *of* I_R, *we get that* $r \in E$ *and* $\psi(e) = f$. *This proves that* ψ *is an onto ring morphism, so that* $E/L = E/\mathrm{Ker}(\psi) \cong \mathrm{End}_R(I/IJ(R)) \cong M_n(R/J(R))$. *This proves (iii).*

As $\mathrm{End}_R(I/IJ(R)) \cong M_n(R/J(R))$ *is a simple ring, it follows that* L *is a prime ideal and a maximal two-sided ideal. Similarly,* K *is the kernel of the composite morphism* $\varphi : E \to R/J(R)$ *of the embedding* $E \to R$ *and the canonical projection* $R \to R/J(R)$. *Since* $R/J(R)$ *is a division ring, we get that* K *is completely prime, two sided ideal of* E *containing* I. *This concludes the proof of (i).*

(ii) $(:\Rightarrow)$ *Since* $\varphi(I) = 0$ *and* $\psi(I) = 0$, *the morphisms* φ *and* ψ *induce morphisms* $\overline{\varphi} : E/I \to R/J(R)$ *and* $\overline{\psi} : E/I \to \mathrm{End}(I/IJ(R))$, *respectively. Hence* $e + I$ *invertible implies* $\varphi(e) = e + J(R)$ *invertible in* $R/J(R)$ *and* $\psi(e)$ *is invertible in* $\mathrm{End}_R(I/IJ(R))$. $(\Leftarrow:)$ *Assume that* $e \in E$ *and that* $\varphi(e)$ *and* $\psi(e)$ *are invertible in* $R/J(R)$ *and* $\mathrm{End}_R(I/IJ(R))$, *respectively. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & I & \longrightarrow & R_R & \xrightarrow{\pi} & R_R/I & \to & 0 \\ & & \downarrow e & & \downarrow e & & \downarrow e & & \\ 0 & \to & I & \longrightarrow & R_R & \xrightarrow{\pi} & R_R/I & \to & 0 \end{array} \tag{4.1}$$

Now $\varphi(e) = e + J(R)$ *invertible implies that* $e \in R(R)$,*and so* e *is invertible in* R. *Hence the middle vertical arrow is an isomorphism. Since* $\psi(e)$ *is invertible, it is an automorphism of* $I/IJ(R)$ *and so* $e(I/IJ(R)) = I/IJ(R)$, *that is,* $eI + IJ(R) = I$. *By Nakayama's Lemma,* $eI = I$. *Hence the deft vertical arrow is an epimorphism. By the Snake Lemma, the right vertical arrow is a monomorphism, hence an isomorphism. That is,* $e + I$ *is invertible in* E/I.

(iv) We have the three cases $L \subseteq K$, $K \subseteq L$, $L \nsubseteq K$ *and* $K \nsubseteq L$.

Assume $L \subseteq K$. *In this case,* $L \subseteq K \subset E$ *implies that* $0 \subset K/L \subset E/L$, *so that* $E/L \cong M_n(R(R))$ *has a proper non-zero two-sided ideal. This is impossible, because* $M_n(R(R))$ *is a simple ring. Hence this case can not occur.*

Assume $K \subseteq L$. *From (ii), it follows that an element* $e + I$ *of* E/I *is invertible in* E/I *if and only if* $e + J(R)$ *is invertible in* $R/J(R)$ *and* $e + I$ *is invertible in* E/I.

Hence, in order to prove (iv) in this case $K \subseteq L$, it suffices to prove that $J(E/I) = L/I$.

($\subseteq$) *If $e + I \in J(E/I)$, then $1 - xey + I$ is invertible in E/I for every $x, y \in E$. Thus $1 - xey + I$ is invertible in E/L for all $x, y \in E$, so that $e + L \in J(E/L)$. But $E/L \cong M_n(R(R))$ has Jacobson radical 0 so that $e \in L$.*

($\supseteq$) *Take $l + I \in L/I$ with $l \in L$. Then $1 - xly + L = 1 + L$ in E/L for every $x, y \in E$. Hence $1 - xly + L$ is invertible in E/L. In particular, $1 - xly \notin L$. Thus $1 - xly \notin K$, so that $1 - xly \notin J(R)$. As $R/J(R)$ is a division ring, it follows that $1 - xly + J(R)$ is invertible in $R/J(R)$. Thus $1 - xly + I$ is invertible in E/I and $l \in J(E/I)$.* □

It is known that a finitely presented module over a semilocal ring always has a semilocal endomorphism ring. Then we have the following natural question.

Question 4.1: Characterize $J(E/I)$. This was done in [Amini and Facchini, 2008] for cyclically presented modules.

As far as Question 4.1 is concerned, notice that, in the proof of Theorem 4.1(ii), we have seen that the mapping $\overline{\varphi} \times \overline{\psi} : E/J \to R/J(R) \times \mathrm{End}(I/IJ(R))$ is local morphism, so that its kernel $K/I \cap L/I$ is contained in $J(E/I)$. In particular, when $K \subseteq L$, we have that $L/I = J(E/I)$ as we have seen in Theorem 4.1(iv)(a). We are not able to describe $J(E/I)$ when L and K are not comparable.

Remark 4.1: Let R be a local right self-injective ring. Let M_R be a cyclic and finitely presented module of projective dimension ≤ 1. Since R_R is injective, we have that $\mathrm{Ext}^1_R(M_R, R_R) = 0$. Thus Theorem 4.1 can be applied.

Theorem 4.2: Let R be a semiperfect ring and let R_R/L be a cyclic uniform right R-module with $L \neq 0$. Let E be the idealizer of the right ideal L of R, that is, the set of all $r \in R$ with $rL \subseteq L$, so that $\mathrm{End}(R_R/L) \cong E/L$. Similarly, let E' be the idealizer of the right ideal $L + J(R)$ of R, so that $\mathrm{End}(R_R/(L + J(R))) \cong E'/(L + J(R))$. Set $I := \{e \in E \mid$ left multiplication by $e + L$ is a non-injective endomorphism of $R_R/L\}$ and $K := E \cap (L + J(R))$. Then:

i) I and K are two-sided ideals of E containing L and I is completely prime in E.

ii) For every $e \in E$, the element $e + L$ of E/L is invertible in E/L if and only if $e + L + J(R)$ is invertible in $E'/L + J(R)$ and $e \notin I$.

iii) Moreover,

a) If $I \subseteq K$, then every epimorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L,

b) $K \nsubseteq I$ if and only if $[R_R/L]_m = [L + J(R)/L]_m$.

Proof *We know that* $\mathrm{End}(R_R/L) \cong E/L$. *Every endomorphism $e+L$ of R_R/L extends to an endomorphism e_1 of the injective envelope* $\mathrm{End}(R_R/L)$. *Define a ring morphism $\varphi : E \to \mathrm{End}(E(R_R/L))/J(\mathrm{End}(E(R_R/L)))$ by $\varphi(e) = e_1 + J(\mathrm{End}(E(R_R/L)))$ for every $e \in E$. Since R_R/L is uniform, the injective envelope $E(R_R/L)$ is indecomposable, the endomorphism ring $\mathrm{End}(E(R_R/L))$ is a local ring and the Jacobson radical $J(\mathrm{End}(E(R_R/L)))$ consists of all non-injective endomorphisms of $E(R_R/L)$. It follows that I, which is equal to the kernel of the ring morphism φ, whose range is the division ring $\mathrm{End}(E(R_R/L))/J(\mathrm{End}(E(R_R/L)))$, must be a completely prime two-sided ideal of E. The remaining part of statement (i) is easily checked.*

(ii) We have already seen that there is a ring morphism $\varphi : E \to$ $\mathrm{End}(E(R_R/L))/J(\mathrm{End}(E(R_R/L)))$ *whose kernel is I. Hence if $e \in E$ and $e + L$ is invertible in E/L, then $\varphi(e)$ must be invertible in the division ring* $\mathrm{End}(E(R_R/L))/J(\mathrm{End}(E(R_R/L)))$. *Thus $\varphi(e) \neq 0$, that is, $e \notin \mathrm{Ker}\varphi = I$. Similarly, we can consider the ring morphism $\psi : E \to \mathrm{End}(R_R/(L + J(R)))$ defined by $\psi(e)(r + L + J(R)) = er + L + J(R)$ for every $e \in E$ and every $r \in R$. Its kernel is K, which contains L. Hence $e + L$ invertible in E/L implies $\psi(e)$ invertible in* $\mathrm{End}(R_R/(L + J(R)))$. *But* $\mathrm{End}(R_R/(L + J(R))) \cong E'/(L + J(R))$, *so that $e + L + J(R)$ must be invertible in $E'/(L + J(R))$.*

Conversely, assume $e \in E$, $e + L + J(R)$ invertible in $E'/(L + J(R))$ and $e \notin I$. We want to show that $e + L$ is invertible in E/L. Since $E/L \cong \mathrm{End}(R_R/L)$, this is equivalent to showing that left multiplication $\mu_e : R_R/L \to R_R/L$ by e is an automorphism of R_R/L. Now $e \notin I$ is equivalent to μ_e is injective by definition of I. In order to show that μ_e is onto as well, it suffices to prove that μ_e induces an onto endomorphism $(R_R/L)/(R_R/L)J(R) \to (R_R/L)/(R_R/L)J(R)$ by Nakayama's Lemma. But $(R_R/L)J(R) = (L + J(R))/L$, so that $(R_R/L)/(R_R/L)J(R) \cong R_R/(L + J(R))$. Hence $e + L + J(R)$ invertible in $E'/(L + J(R)) \cong \mathrm{End}(R_R/(L +$

$J(R)))$ means that the endomorphism $\psi(e)$ of $R_R/(L+J(R))$ induced by μ_e is onto, as desired.

(iii) (a) Assume $I \subseteq K$. Let $e+L : R_R/L \to R_R/L$ be an epimorphism with $e \in E$. Then the induced morphism $\psi(e) : R_R/L+J(R) \to R_R/L+J(R)$ is also an epimorphism, so that it is an automorphism because $R_R/L+J(R)$ is a semisimple module of finite Goldie dimension. In this isomorphism $\mathrm{End}(R_R/L+J(R)) \cong E'/(L+J(R)$*, we obtain that $e+L+J(R)$ is invertible in the ring $E'/(L+J(R)$. Thus $e \notin K$. Hence $e \notin I$. It follows from (ii) that $e+L$ is invertible, that is, it is an automorphism of R_R/L.*

(b) Assume $K \nsubseteq I$. Then there is an element $f \in K$, $f \notin I$. Thus $f \in E$ induces an endomorphism f of R_R/L. Now $f \notin I$ means that f is injective and $f \in K$ means that the image of f is contained in $L+J(R)/L$. Hence $[R_R/L]_m = [L+J(R)/L]_m$. Conversely, if $[R_R/L]_m = [L+J(R)/L]_m$, then there is a monomorphism $f : R_R/L \to L+J(R)/L$. If we compose it with the inclusion $L+J(R)/L \to R_R/L$ we get an epimorphism of R_R/L which is in K but not in I. Hence $K \nsubseteq I$. □

We finish the results of this section with the following result.

Theorem 4.3: Let R be a semiperfect ring, let R_R/L, R_R/L' be two cyclic uniform modules with $L \neq 0$ and $L' \neq 0$ proper right ideals of R. Assume that either

i) every monomoprhism $R_R/L \to R_R/L$ is an automorphism of R_R/L, or

ii) every epimorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L, or

iii) $[R_R/L]_m = [L+J(R)/L]_m$.

The followings are equivalent.

a) $R_R/L \cong R_R/L'$,

b) $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_e = [R_R/L']_e$.

Proof Assume $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_e = [R_R/L']_e$. Then there are monomorphisms $\alpha : R_R/L \to R_R/L'$ and $\beta : R_R/L' \to R_R/L$ and epimorphisms $\alpha' : R_R/L \to R_R/L'$ and $\beta' : R_R/L' \to R_R/L$. Then $\beta\alpha$ is a monomorphism $R_R/L \to R_R/L$ and $\beta'\alpha'$ is an epimorphism $R_R/L \to R_R/L$. If hypothesis (i) holds, then $\beta\alpha$ is an automorphism of R_R/L that factors through R_R/L', so that R_R/L is isomorphic to a direct summand of R_R/L'. But $R_R/L \neq 0$ and R_R/L' is uniform, *so that $R_R/L \cong R_R/L'$. This proves our theorem under hypothesis (i). Dually, one proves that the theorem holds when hypothesis (ii) holds.*

Assume that hypothesis (iii) holds, i.e., $[R_R/L]_m = [L + J(R)/L]_m$. Equivalently, there exists a monomorphism $\gamma : R_R/L \to R_R/L$ whose image is contained in $L + J(R)/L$. Now if either α or α' are isomorphism, then the existence of α or α' shows that $R_R/L \cong R_R/L'$. This allows us to conclude. Thus we can assume that α is not an epimorphism and α' is not a monomorphism. Then $\alpha' + \alpha\gamma : R_R/L \to R_R/L'$ is an isomorphism, because:

i) It is injective, because it is the sum of the injective morphism $\alpha\gamma : R_R/L \to R_R/L'$ and the non-injective morphism $\alpha' : R_R/L \to R_R/L'$ and R_R/L is uniform.

ii) $J(R)$ is superfluous in R_R by Nakayama's Lemma. Considering the canonical projection $R_R \to R_R/L$, it follows that $L + J(R)/L$ is superfluous in R_R/L. Applying the morphism $\alpha : R_R/L \to R_R/L'$, we get the image of $\alpha\gamma$ is contained in $\alpha(L + J(R)/L)$, hence is superfluous submodule of R/L'. Thus the sum of $\alpha\gamma$ and the surjective morphism $\alpha' : R_R/L \to R_R/L'$ is a surjective morphism $\alpha' + \alpha\gamma : R_R/L \to R_R/L'$. Thus $\alpha + \alpha'\gamma$ is an isomorphism of R_R/L onto R_R/L'.

□

Remark 4.2: By Theorem 4.2, the only case in which we cannot apply Theorem 4.3 is when K is properly contained in I. Namely, if $K \nsubseteq I$, then $[R_R/L]_m = [L+J(R)/L]_m$ and we can apply Theorem 4.3; if $K \subseteq I$, then either K is properly contained in I, which is the case still unknown, or $K = I$, but in the latter case every epimorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L by Theorem 4.2(iii-a).

5. CONCLUSIONS

It is shown that how is the behavior of endomorphism ring of a cyclic, finitely presented module of projective dimension ≤ 1 over a local ring. This class of modules extends the class of couniformly presented modules over local rings to arbitrary rings. It is seen that we are able to characterize Jacobson radical of the endomorphism ring of this class of modules whenever $L := \{r \in R \mid rI \subseteq IJ(R)\}$ and $K := \{E \cap J(R)\}$ which are defined in the statement of Theorem 4.1, are comparable. Beside these we showed that one can also apply Theorem 4.1 to the cyclic, finitely presented module of projective dimension ≤ 1 over a local self-injective ring. Also description of endomorphism ring of a cyclic uniform module over a semiperfect ring is studied. Finally, we introduce an elementary property of having the same monogeny class and having the same epigeny class for cyclic uniform modules over a semiperfect ring.

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Şahinkaya S., Kör A., Koşan M. T., (2014), “A note on the endomorphism ring of finitely generated modules of the projective dimenson $\leq$ 1 ”, Hacettepe Journal of Mathematics and Statistics, 43(6), 985-991.

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Arda KÖR was born in İstanbul, Turkey, in 1985. She got a Bsc degree in Department of Mathematics from Abant İzzet Baysal University, in 2008. She received a M.Sc degree in Mathematics from Abant İzzet Baysal University, in 2011. She worked as research assistant at Piri Reis University between 2011 and 2012. She has been a doctorate student in Mathematics department of Graduate School of Engineering and Sciences since 2011. She has been working as a teaching assistant at Gebze Technical University, since 2012.

APPENDICES

Appendix A: Publications Based on the Thesis

Şahinkaya S., Kör A., Koşan M. T., (2014), “A note on the endomorphism ring of finitely generated modules of the projective dimenson ≤ 1 ”, Hacettepe Journal of Mathematics and Statistics, 43(6), 985-991.