T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INITIAL TIME DIFFERENCE STABILITY ANALYSIS OF NONLINEAR FRACTIONAL DYNAMIC SYSTEMS

MUHAMMED ÇİÇEK

A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

GEBZE
2015

T.R.

GEBZE TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INITIAL TIME DIFFERENCE STABILITY ANALYSIS OF NONLINEAR FRACTIONAL DYNAMIC SYSTEMS

MUHAMMED ÇİÇEK

A THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

THESIS SUPERVISOR

ASSOC. PROF. DR. COŞKUN YAKAR

GEBZE

2015

T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

LİNEER OLMAYAN KESİRLİ TÜREVLİ DİNAMİK SİSTEMLERİN BAŞLANGIÇ ZAMAN FARKLI KARARLILIK ANALİZİ

MUHAMMED ÇİÇEK
DOKTORA TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMANI
DOÇ. DR. COŞKUN YAKAR

GEBZE
2015

GEBZE TEKNİK ÜNİVERSİTESİ	**DOKTORA JÜRİ ONAY FORMU**

GTÜ Fen Bilimleri Enstitüsü Yönetim Kurulu'nun 17/11/2015 tarih ve 2015/66 sayılı kararıyla oluşturulan jüri tarafından 28/12/2015 tarihinde tez savunma sınavı yapılan Muhammed ÇİÇEK'in tez çalışması Matematik Anabilim Dalında DOKTORA tezi olarak kabul edilmiştir.

JÜRİ

ÜYE
(TEZ DANIŞMANI) : Doç. Dr. Coşkun YAKAR

ÜYE : Prof. Dr. Mansur İSGENDEROĞLU

ÜYE : Prof. Dr. Fatih TAŞÇI

ÜYE : Prof. Dr. Gamze TANOĞLU

ÜYE : Yrd. Doç. Dr. Orhan Özgür AYBAR

ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun
……./……./…… tarih ve ……./……. sayılı kararı.

İMZA/MÜHÜR

SUMMARY

In the present thesis, some stability and boundedness properties for fractional differential equation (FDE) are considered. In the second chapter, some necessary definitions and facts on fractional calculus, on initial value problem of FDE are given. A comparison theorem demand only continuity as an assumption instead of local Hölder continuity or C^q continuity used in the literature is presented and Caputo fractional Dini derivative is given in the third chapter. Then, fractional extension of comparison method via Lyapunov function and scalar FDE is applied to obtain sufficient conditions on some stability, boundedness for FDE. However, some stability and boundedness with initial time difference for FDE are introduced and studied since it is not possible to keep measurements with the expected initial time in real world applications. Comparison results for scalar FDE with parameter relative to ITD are obtained in the fourth chapter. In these framework, sufficient conditions on some stability and boundedness are obtained. The behavior of solution of perturbed system that differs in initial position and initial time with respect to original unperturbed system are investigated in fifth chapter. Finally, obtained results on some stability and boundedness in previous chapter are generalized by using the notion of two measures.

Key Words: Stability, Boundedness , Initial Time Difference (ITD), Fractional Differential Equation (FDE), Lyapunov Function, Comporison Method.

ÖZET

Bu tezde, kesirli türevli diferansiyel denklemler için bazı kararlılık ve sınırlılık özellikleri ele alınmıştır. İkinci bölümde, fraksiyonel analiz ve kesirli türevli diferansiyel denklemler'in başlangıç değer problemi hakkında gerekli tanımlar verildi. Üçüncü bölümde literatürdeki Hölder süreklilik ya da C^q süreklilik yerine koşul olarak sadece süreklilik gerektiren mukayese teoremi ve Caputo fraksiyonel Dini türev verildi. Kesirli türevli diferansiyel denklemler'in bazı kararlılık ve sınırlılılığı üzerine yeterli koşullar belirlemek için mukayese metodunun fraksiyonel genişlemesi Lyapunov fonksiyonu ve skaler kesirli türevli diferansiyel denklemler aracılığıyla uygulandı. Bununla birlikte gerçek hayat uygulamalarında ölçümleri beklenen zamanda tutmak mümkün olmadığından başlangıç zaman farkı ile birlikte bazı kararlılık ve sınırlılık kesirli türevli diferansiyel denklemler için tanıtıldı ve çalışıldı. Dördüncü bölümde parametre içeren skaler kesirli türevli diferansiyel denklemler için mukayese sonuçları başlangıç zaman farkına göre elde edildi. Bazı kararlılık ve sınırlılık için yeterli koşullar bu sonuçlar çerçevesinde elde edildi. Orijinal pertörb olmayan sisteme göre hem başlangıç pozisyonu hem de başlangıç zamanı farklı olan pertörb sistemin çözümünün davranışı beşinci bölümde incelendi. Son olarak bir önceki bölümde bazı kararlılık ve sınırlılık için elde edilen sonuçlar iki ölçü kavramı kullanılarak genelleştirildi.

Anahtar Kelimeler: Kararlılık, Sınırlılık, Başlangıç Zaman Farkı, Fraksiyonel Diferansiyel Denklemler, Lyapunov Fonksiyonu, Karşılaştırma Metodu.

ACKNOWLEDGEMENTS

I would like to express my special appreciation and thanks to my supervisor Assoc. Prof. Dr. Coşkun YAKAR for all his guidance, help and advice. And i would like to thank to Prof. Dr. Mansur İSGENDEROĞLU who trust and encouraged me in academic life.

Besides my advisor, i would like to thank the rest of my thesis comittee: Prof. Dr. Gamze TANOĞLU, Prof. Dr. Fatih TAŞÇI, Asst. Prof. Dr. O. Özgür AYBAR.

A special thanks to my beloved wife for her incredible support at every step of my academic life, her encouragement, patience and help. Also i would like to express my very special thanks to my beloved daughter.

Lastly, i would like to thank to TÜBİTAK for their valuable financial support to me and many researchers with the National Scholarship Programme for PhD Students.

TABLE of CONTENTS

	Page
SUMMARY	v
ÖZET	vi
ACKNOWLEDGEMENTS	vii
TABLE of CONTENTS	viii
LIST of ABBREVIATIONS and ACRONYMS	x
LIST of FIGURES	xi
1. INTRODUCTION	1
2. FRACTIONAL CALCULUS: BASIC THEORY AND RESULTS	6
2.1. Special Functions	7
2.2. Definitions and Properties of Fractional Integral and Derivatives	11
2.2.1. Riemann-Liouville Fractional Integral and Derivative	11
2.2.2. Caputo Fractional Derivative	19
2.2.3. Other Approach	22
3. SOME STABILITY AND BOUNDEDNESS PROPERTIES FOR FRACTIONAL DIFFERENTIAL EQUATIONS (FDE)	26
3.1. Introduction	26
3.2. Preliminary Notes and Definitions	27
3.2.1. Definitions of Some Stability and Boundedness	27
3.3. Lyapunov Functions and its Caputo Fractional Dini Derivative	33
3.4. Fractional Order Extension of Differential Inequalities and Comparison Results	37
3.5. Main Results	42
3.6. Applications	52
4. SOME STABILITY AND BOUNDEDNESS CRITERIA FOR FDE WITH INITIAL TIME DIFFERENCE (ITD)	55
4.1. Introduction	55
4.2. Statement of The Problem	56
4.2.1. Main Definitions and Concepts with ITD for FDE	56
4.2.2. Stability Versus Stability with ITD	65

4.3. Lyapunov functions and its Caputo fractional Dini Derivative with ITD 67
4.4. Comparison Results with ITD for Scalar FDE 70
4.5. Main Results 74
4.5.1. Stability and Practical Stability Criteria with ITD 74
4.5.2. Boundedness and Lagrange Stability Criteria with ITD 83
4.6. Applications 88
5. STABILITY and BOUNDEDNESS of PERTURBED FDE with ITD 92
5.1. Introduction 92
5.2. Statement of the Problem 93
5.3. Main Results 98
5.3.1. Comparison Results with ITD for Scalar FDE 99
5.3.2. Some Stability and Boundedness Criteria 103
6. SOME STABILITY AND BOUNDEDNESS of PERTURBED FDE with ITD in TERMS of TWO MEASURES 110
6.1. Introduction 110
6.2. Main Definitions and Concepts with Two Measures 110
6.3. Main Results 114

REFERENCES 121
BIOGRAPHY 126
APPENDICES 127

LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
FDE	:	Fractional Differential Equation
GL	:	Grünwald-Letnikov
ITD	:	Initial Time Difference
RL	:	Riemann-Liouville
$E_{q,q}$	:	Mittag-Leffler function with two parameter
AC	:	Absolutely Continuous
IVP	:	Initial Value Problem
ODE	:	Ordinary Differential Equation

LIST of FIGURES

Figure No:		**Page**
2.1:	Gamma function.	9
3.1:	Approximate solutions with $\mu = 0.5$, $\alpha = 3$ and $q = 0.5$, $q = 0.8$, $q = 0.9$.	31
3.2:	Approximate solutions with $q = 0.5, 0.8$ and $q = 0.9$.	53
4.1:	Approximate solutions with $h(t) = sin(t)$, $t_0 = 0$, $\tau_0 = 0.2$, $x_0 = 2$, $y_0 = 2.3$ and $L = 1$.	62
4.2:	Approximate solutions with $h_1(t) = sin(t)\,, h_2(t) = cos(t)$ $t_0 = 0, \tau_0 = 0.2$ and $L = 1$.	89
4.3:	Graph of $M(t, \eta)$ for various η.	91
4.4:	Graph of $M(t, \tau_0)$ for various τ_0.	91

1. INTRODUCTION

One of the main problems in the qualitative theory of differential equations is stability of the solutions. It is well known that the theory of stability in the sense of Lyapunov has been successfully investigated to understand qualitative properties of dynamic systems for many years [1]-[6]. However, the concept of boundedness has played a significant role in the existence of periodic and almost periodic solutions which has many applications in biological population management and control [7]-[8]. In real world processes, it is realized that a system may be stable or asymptotically stable in theory, but it is actually unstable in practice because of the stable domain or domain of attraction may be so small that desired deviation to cancel out is not allowed. Whereas, the desired state of a system may be mathematically unstable, but the system may oscillate sufficiently near this state so that the performance is considered acceptable. To deal with such phenomena, the concept of practical stability is introduced in [9] and a systematic study is presented in [10]. Briefly stated, practical stability is essentially based on the pre-specified bounds for the perturbation of initial conditions response and the allowable perturbation of the system response [10]. In nonlinear systems, Lyapunov's direct method (also called the Lyapunov's second method) allows us to obtain sufficient conditions for the stability, practical stability and boundedness of a system without explicitly solving given equations. The method generalizes the idea which shows that the system is stable if there are some Lyapunov function candidates for the system [4]-[6].

Fractional calculus is the theory of integral and derivative of arbitrary non-integer order, which unifies and generalizes the concepts of integer order derivative and integral. The subject is as old as the classical calculus and goes back to the 17th century. Although there are several possible generalizations of ordinary differentiation and integration of a function, the most commonly used definitions are Riemann-Liouville (RL), Grünwald-Letnikov (GL) and Caputo fractional derivatives. For more details on the basic theory of fractional calculus, one can see the monographs [11]-[16]. In chapter 2, we present some basic definitions and facts, available in literature [11]-[16], about fractional calculus and fractional differential equations (FDE) for further development of the work.

Only a few decades ago, it was realized that the fractional differential equations (FDE) which involves fractional derivative provides an attractive tool for modelling properly the anomalous dynamics of various processes [17] related to complex systems in a variety of disciplines from physics, chemistry, biology and engineering such as viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos and fractals [11], [15], [18]-[20]. Therefore, the qualitative theory of FDE has received increasingly attentions [21]. The existence and uniqueness of solutions of initial and boundary value problems for nonlinear FDE have been extensively studied by monotone iterative technique or fixed point method [22]-[28] and the references therein. Among them [16], [22]-[23] have investigated the basic theory of initial value problems (IVP), including fractional differential and integral inequalities, comparison result, local and global existence of solutions, for FDE involving RL and the Caputo derivatives of order $0 < q < 1$. They followed the classical approach of the theory of differential equations of integer order, in order to compare and contrast the differences as well as the intricacies that might result in development [3]-[4]. In addition to existence and uniqueness result, the authors have investigated in particular the dependence of the solution for RL type FDE on the order of the equation and on the initial condition in [27]. Generalization of Gronwall inequality for the RL type of fractional integral equation are obtained in [28]. As an application of this result, uniqueness and continuous dependence of the solution of the RL type FDE are proved.

Recently, fractional calculus was introduced to the stability analysis of FDE and stability of FDE has attracted increasing interest. In 1996, the author in [29] firstly studied the stability of linear FDE with the Caputo derivative. Since then, further studies on the stability of linear FDE have been done [30]-[31]. Whereas, the stability analysis of the nonlinear FDE is much more difficult and only a few are available. In the base of Lyapunov's second method, sufficient conditions on stability for nonlinear FDE and nonlinear time-delayed FDE has been discussed in several papers [32]-[38]. Among them in [32]-[33], the authors proposed fractional Lyapunov's second method and firstly extended the exponential stability of ordinary differential equations (ODE) to the Mittag-Leffler stability and generalized Mittag-Leffler stability of FDE, respectively. The authors in [35]-[36] have applied the fractional comparison principle to discussing the asymptotic stability and Mittag-Leffler stability of FDE with RL derivative, respectively. Very recently, a stability criterion for autonomous and non-autonomous nonlinear fractional differential system with Caputo derivative is derived in [37]-[38], respectively. In aforementioned papers, sufficient conditions are established by application of the Caputo derivative of continuously differentiable Lyapunov function. Besides, there is another approach in which the authors [39]-[40] have defined Caputo fractional Dini derivative of continuous Lyapunov function in an appropriate way. Motivated by the known fact that the stability or asymptotic stability are neither necessary nor sufficient to assure practical stability [10] and the concept of boundedness is valid even the studied FDE has no zero solution, we have investigated practical stability, boundedness and Lagrange stability for FDE by using fractional comparison method via Caputo fractional Dini derivative of continuous Lyapunov function in Chapter 3. The statement of the problem and relation between stability and boundedness is presented in section 3.2. Then, in section 3.3 natural relationship between the Dini derivative of Lyapunov function for classical case ($q = 1$) and Caputo fractional Dini derivative is given by appropriate examples. Some comparison results demand only continuity as an assumption instead of local Hölder continuity or C^q continuity used in the literature [16], [23], [41] is presented in section 3.4. Then, comparison method via Lyapunov function and a scalar FDE is applied to obtain sufficient conditions on some stability and boundedness for system of FDE in section 3.5. Finally some examples are given as an application of the obtained results.

In practical situations, it is possible to have not only a change in initial position but also in initial time because of all kinds of disturbed factors. So it is reasonable to study the solutions of the differential equation with variation in the initial time. When we do consider such a deviation in initial time, it causes measuring the difference between any two different solutions starting with different initial times. We call this type of stability analysis, initial time difference stability analysis. An investigation of IVP of differential equations where the initial time changes with each solution in addition to the initial position was initiated by [42]-[43]. There are two ways of comparing and measuring the difference of two solutions. In [44]-[45], the method of variation of parameters is used to discuss such situations in one direction. In [46]-[49], the authors have obtained various types of some stability and boundedness results relative to initial time difference (ITD) for ODE by employing the construction of various types of Lyapunov functions with differential inequalities technique.

In Chapter 4, we have investigated stability, practical stability, boundedness and Lagrange stability with ITD for nonlinear FDE by using fractional extension of comparison principle. In section 4.2, main definitions and concepts with ITD for FDE with Caputo derivative are introduced and the differences between classical notion of stability and the notion of stability with ITD are discussed, respectively. Then, in section 4.3 natural relationship between the Dini derivative of Lyapunov function with ITD for classical case $(q = 1)$ and the introduced Caputo fractional Dini derivative with ITD is shown by appropriate examples. Then, comparison results relative to ITD are obtained in section 4.4. Comparison method via Lyapunov function and scalar FDE with parameter is applied to obtain several sufficient conditions on stability, practical stability, boundedness and Lagrange stability with ITD for system of FDE in section 4.5. Finally some examples are given as an application of the obtained results.

Determining which stability properties of a particular differential system are preserved under sufficiently small perturbations is another important problem in stability theory. This problem was investigated in several ways in [1]-[6]. The author in [50] investigated the problem of determining the behavior of the solutions of a perturbed differential equation with respect to the solutions of the original unperturbed differential equation. The principal mathematical technique employed is a modification of Lyapunov's direct method which is applied to the difference of the solutions of perturbed and unperturbed system where the initial positions are sufficiently close. In [51], the authors applied variational Lyapunov method (VLM), combines the method of variation of parameters and the method of Lyapunov, to connect the solutions of perturbed and unperturbed system with initial time unchanged. However, the possibility of making error in initial time as well as in initial position when we deal with real world problems needs to be considered. So far, several studies have been made on this problem for ODE to explore the ITD stability, boundedness, etc. criteria by using generalized variation of parameters and comparison method via Lyapunov functions in [52], [53] and references therein. However, there are a few results for FDE. In [52], VLM is applied to connect between the solutions of system of perturbed and unperturbed FDE that have the same initial time. On the other hand sufficient conditions on stability with ITD are obtained in [54]. In chapter 5, we have investigated stability, practical stability, boundedness and Lagrange stability for system of nonlinear perturbed FDE with ITD by using fractional comparison method via Lyapunov function and scalar FDE with parameter. We begin with section 5.2 which includes the necessary some stability and boundedness definitions of system of perturbed FDE relative to unperturbed FDE with ITD and Caputo fractional Dini derivative of Lyapunov function with respect to the system of perturbed FDE and unperturbed FDE in relation with definition in [55]-[56]. In section 5.3, firstly we present a comparison result which uses Lyapunov function to connect the solutions of the perturbed and the unperturbed systems in terms of solution of a scalar FDE. We have obtained some sufficient conditions for ITD stability, boundedness and Lagrange stability of nonlinear system of perturbed FDE.

There are many stability concepts are presented in the literature such as the partial stability, eventual stability, conditional stability, Lipschitz stability, relative stability and so on. In 1960, [57] introduced the concept of stability in terms of two measures which unified the foregoing stability concepts. Then, the theories of the stability in terms of two measures have been successfully developed in [58] and some stability and boundedness results are obtained by means of various types of Lyapunov functions for several kinds of differential equations in [59]-[61] and references therein. We have investigated some stability and boundedness in terms of two measures for system of perturbed FDE with ITD in chapter 6. We begin with section 6.2 which includes the necessary definitions of stability, practical stability, boundedness and Lagrange stability in terms of two measures with ITD. Then, we have generalized the main results obtained in previous chapter 5 by using the notion of two measures.

2. FRACTIONAL CALCULUS: BASIC THEORY AND RESULTS

Fractional calculus is the branch of mathematics that generalizes the derivative and the integral of a function to a non-integer (arbitrary) order. The subject is as old as the classical calculus and goes back to the 17th century. For the first time in a letter dated 30th september 1695, Leibniz proposed the following question to L'Hospital: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?". Since then, several mathematicians studied this question, using their own notation and methodology, among them RL, Caputo, Weyl, GL and Erdelyi-Kober, etc. The most commonly used definitions are RL, GL and Caputo fractional derivatives. Fractional derivatives possess memorizing properties, which makes fractional derivative more suitable than integer-order to describe the properties of various materials. The physical and geometric interpretation of fractional integral and derivative was discussed in [62].

Only a few decades ago, it was realized that fractional calculus provides an attractive tool for modelling the real world problems. The idea of fractional calculus has been a subject of interest not only among mathematicians, but also among physicists and engineers. There are many books that provide broad and deep understanding of the theory and applications of fractional calculus and FDE [11]-[15], [19]-[21].

In this chapter, we present some basic definitions and facts, available in literature [12], [14]-[15], [63]-[65], about fractional calculus and FDE for further development of the work. One can see more detailed information about this chapter in [12], [14]-[15], [63]-[65]. The organization of this chapter as follows. In 2.1, we give definitions and the properties of the special functions that are important in fractional calculus. In 2.2, we give the necessary definitions, some properties of RL, Caputo and GL fractional derivatives. Then, the initial value problem for FDE and some basic results on existence and uniqueness from literature are given.

2.1. Special Functions

Here, we give definitions and the simplest properties of some special symbols and related special functions that are important in fractional calculus. One can see more detailed information about this part in the book [12], [14].

• The Pochhammer symbol $(z)_n$ with integer n is defined by

$$(z)_n = z(z+1)\ldots(z+n-1), \qquad n = 1,2,\ldots, \qquad (z)_0 = 1. \tag{2.1}$$

It is easy to see that

$$(z)_n = (-1)^n(1-n-z)_n, \;\; (1)_n = n!, \tag{2.2}$$

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} \tag{2.3}$$

where $\Gamma(z)$ is given by (2.7). Equation (2.3) can be used for introducing the symbol $(\mathrm{z})_{\mathrm{n}}$ with complex n.

• Binomial coefficients are defined by the formula

$$\binom{\alpha}{n} = \frac{(-1)^n(-\alpha)_n}{n!} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{2.4}$$

In particular when $\alpha = m,\; m = 1,2,...,$ we have $\binom{m}{n} = \frac{m!}{n!(m-n)!}\; if\; m \geq n$ and $\binom{m}{n} = 0\; if\; 0 \leq m < n$. We also have the following relations

$$(-1)^j\binom{\alpha}{j} = \binom{j-\alpha-1}{j}, \tag{2.5}$$

and

$$\sum_{j=0}^{k} \binom{\alpha}{j} \binom{\beta}{k-j} = \binom{\alpha+\beta}{k}. \tag{2.6}$$

• The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt \tag{2.7}$$

which converges in the right half of the complex plane $Re(z) > 0$. Indeed, we have

$$\begin{aligned} \Gamma(x+iy) = \int_{0}^{\infty} e^{-t} t^{x-1+iy} dt &= \int_{0}^{\infty} e^{-t} t^{x-1} e^{iyln(t)} dt \\ &= \int_{0}^{\infty} e^{-t} t^{x-1} \left[cos\left(yln(t)\right) + sin\left(yln(t)\right)\right] dt. \end{aligned} \tag{2.8}$$

The expression in the square brackets in (2.8) is bounded for all t and convergence at infinity. It is provided by e^{-t} and we must have $x = Re(z) > 0$ for the convergence at $t = 0$. The 'beauty' of the gamma function can be found in its properties. An integration by parts yields the functional equation for $\Gamma(z)$ as follow

$$\Gamma(z+1) = z\Gamma(z), \;\; Re(z) > 0. \tag{2.9}$$

More generally, when n is a positive integer,

$$\Gamma(z+n) = (z+n-1)(z+n-2) \ldots z\Gamma(z), \quad Re(z) > 0. \tag{2.10}$$

By putting $z = 1$ in (2.10), $\Gamma(n+1) = n!$ is obtained. Therefore, this function is a generalization of the factorial. The Gamma function is extended to the half-plane

$Re(z) \leq 0,\ z \neq 0, -1, -2, \ldots,$ by analytic continuation of the integral (2.7). Then (2.10) yields the equality

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n} = \frac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)} \tag{2.11}$$

where $Re(z) > -n$, $n = 1,2, \ldots$, $z \neq 0, -1, -2, \ldots$, which allows to carry out such an analytic continuation into the half-plane $Re(z) > -n$ with any n. Figure 2.1 demonstrates the Gamma function at and around zero.

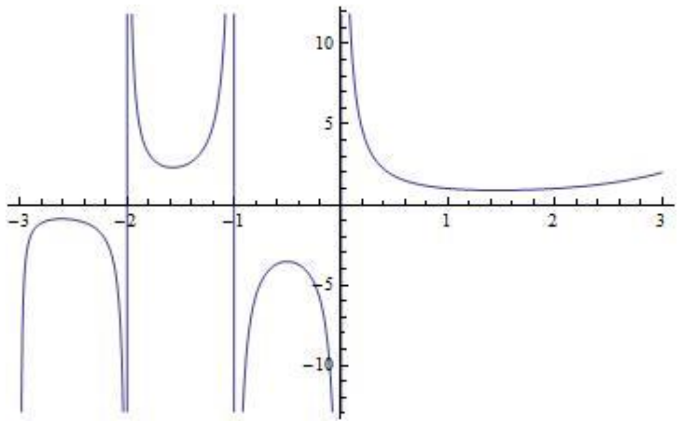

Figure 2.1: Gamma function.

- The Beta function, Euler integral of the first kind, is defined by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt\ ,\quad (Re(z) > 0,\ \ Re(w) > 0). \tag{2.12}$$

The Beta function is important in relationship in fractional calculus. In many cases, it is more convenient to use instead of certain combination of values of the Gamma function. It is connected with the Gamma function by the relation

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(w+z)} = B(w,z). \tag{2.13}$$

• The standard definition of the Mittag-Leffler is given in [12]-[15]

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (\alpha > 0). \tag{2.14}$$

The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. The function $E_{\alpha}(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential series i.e. for $\alpha = 1$, $E_1(z) = e^z$.

A two parameter function of the Mittag-Leffler type is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0, \quad \beta > 0). \tag{2.15}$$

The function defined by (2.15) is a generalization of (2.14).

Now, we give the following propositions which include Laplace transform and monotonicity of the Mittag-Leffler functions. The proofs of the following results can be found in [14]-[15].

Proposition 2.1: For $\lambda > 0$, $q > 0$, *we have for* $t > 0$

$$\frac{d}{dt} E_{q,1}(-\lambda t^q) = -\lambda t^{q-1} E_{q,q}(-\lambda t^q). \tag{2.16}$$

Proposition 2.2: For any $q \in (0,1)$ *the function* $E_{q,1}(-t)$ *is completely monotone for* $t \geq 0$, *i.e.* $(-1)^n \frac{d^n}{dt} E_{q,1}(-t) \geq 0$. *In particular, it holds that for any* $t \geq 0$, $E_{q,1}(-t) > 0$ *and* $E_{q,q}(-t) \geq 0$.

Proposition 2.3: Laplace transform of the function $t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})$ is given

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)}, \quad Re(s) > |\lambda|^{1/\alpha}. \tag{2.17}$$

2.2. Definitions and Properties of Fractional Integral and Derivatives

Although there are several possible generalizations of $\frac{d^n f}{dt^n}$, the most commonly used definitions are RL, GL and Caputo fractional derivatives. The concept of RL fractional derivative is historically the first and the theory about this concept has been studied comprehensively very well in [11]-[15]. But applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a), f'(a),\ etc.$ Unfortunately, the the initial conditions for FDE with RL derivative contain the limit values of the RL fractional derivative at the lower terminal. In order to overcome this difficulty Caputo fractional derivative is defined. The main advantage of Caputo's approach is that the initial conditions for FDE with Caputo derivative take on the same form as for ODE.

2.2.1. Riemann-Liouville Fractional Integral and Derivative

The idea of fractional integration is closely connected with Abel's integral equation. Thus, it is reasonable to start from the solution of this equation. In this context, fractional differentiation is constructed as an operation inverse to fractional integration. Proceeding from this idea, the corresponding definitions and results are given from [12]. The integral equation

$$\frac{1}{\Gamma(q)}\int_a^t \frac{\varphi(s)dt}{(t-s)^{1-q}} = f(t), \qquad t > a \tag{2.18}$$

where $0 < q < 1$, is called Abel's equation. (2.18) may be solved by changing x to t and t to s; respectively, multiplying both sides $(x-t)^{-q}$ and integrating, it follows

$$\int_a^t \varphi(s)ds = \frac{1}{\Gamma(1-q)} \int_a^t \frac{f(s)ds}{(t-s)^q}. \tag{2.19}$$

One can see more detailed information and calculations about this part from chapter 1 in [12]. Hence after differentiation (2.20) is obtained

$$\varphi(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t \frac{f(s)dt}{(t-s)^q}. \tag{2.20}$$

So if (2.18) has a solution, this solution is necessarily given by (2.20) and therefore it is unique.

• On the solvability of the Abel's equation in the space of integrable function

Under what conditions on $f(t)$ the Abel's equation is solvable is needed to be clarified. In order to formulate the main result, the following notation is introduced

$$f_{1-q}(t) = \frac{1}{\Gamma(1-q)} \int_a^t \frac{f(s)ds}{(t-s)^q}. \tag{2.21}$$

Then $f(t) \in L_1(a,b)$ implies that $f_{1-q} \in L_1(a,b)$, see in [12].

Theorem 2.1: Abel's equation (2.18) with $0 < q < 1$ *is solvable in* $L_1(a,b)$ *if and only if* $f_{1-q}(t) \in AC([a,b])$ *and* $f_{1-q}(a) = 0$.

For proof of Theorem 2.1, please see in [12].

The criterion of solvability for Abel's equation is given in Theorem 2.1 in terms of the auxiliary function $f_{1-q}(t)$. However, the following lemma and corollary give a simple sufficient condition in terms of the function $f(t)$ itself.

Lemma 2.1: If $f(t) \in AC([a,b])$, *then* $f_{1-q}(t) \in AC([a,b])$.

For the proof Lemma 2.1 see in [12].

Corollary 2.1: If $f(t) \in AC([a,b])$, *then Abel's equation (2.18) is solvable in* $L_1(a,b)$ *and its solution (2.20) may herein be represented in the form*

$$\varphi(t) = \frac{1}{\Gamma(1-q)}\left[\frac{f(a)}{(t-a)^q} + \int_a^t \frac{f'(s)ds}{(t-s)^q}\right] \tag{2.22}$$

- Definition of Fractional Integrals and Derivatives

There is a well-known formula

$$\int_a^t dt_1 \int_a^{t_1} dt_2 \dots \int_a^{t_{n-1}} \varphi(t_n)dt_n = \frac{1}{(n-1)!}\int_a^t \frac{\varphi(s)ds}{(t-s)^{1-n}} \tag{2.23}$$

for an n-fold integral. The right-hand side of (2.23) may have a meaning for non-integer values of n since $(n-1)! = \Gamma(n)$. So it is natural to define the integration of a non-integer order as follows.

Definition 2.1: ([12]) Let $\varphi(t) \in L_1(a,b)$. *The* fractional integral *of order* $q > 0$ *is defined as*

$$_aI_t^q\varphi(t) = \frac{1}{\Gamma(q)}\int_a^t \frac{\varphi(s)ds}{(t-s)^{1-q}} \ , \ t > a \tag{2.24}$$

where a and t limit of the operation. The accepted name for the integral (2.24) is the RL fractional integral. In some other references [14], [16] the notation ${}_a\mathcal{D}_t^{-q}\varphi(t)$ can also be used for RL fractional integral. For $a = 0$, the fractional integral (2.24) can be written as ${}_0I_t^q\varphi(t) = \varphi(t) * \psi_q(t)$ where $\psi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$ for $t > 0$ and $\psi_q(t) = 0$ for $t \leq 0$. To simplify the notations we will use I_a^q instead of ${}_aI_t^q$.

Fractional integration has the semigroup property as follow:

- $\mathrm{I}_a^\alpha \mathrm{I}_a^\beta \varphi = \mathrm{I}_a^{\alpha+\beta}\varphi, \qquad \alpha > 0,\ \beta > 0.$

In view of the inversion of Abel's equation (2.20) which was obtained above, it is natural to introduce fractional differentiation as an operation inverse to fractional integration.

Definition 2.2: ([12]) The RL fractional derivative *of* order q, $0 < q < 1$, *for an integrable function* $f(t)$ *defined on the interval [a,b], is defined as*

$$ {}_a\mathcal{D}_t^q f(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_a^t \frac{f(s)ds}{(t-s)^q}. \tag{2.25}$$

Note that we have defined fractional integral for any $q > 0$, while fractional derivative are for now introduced only for order $0 < q < 1$. It should be noted that use of half-order derivative and integral lead to a formulation of certain real world problems in different areas [11]. Before passing to the case $q \geq 1$, we give a simple and sufficient condition for the existence of fractional derivatives.

Lemma 2.2: ([12]) Let $f(t) \in AC([a,b])$, *then* $\mathcal{D}_a^q f(t)$ *exist almost everywhere for* $0 < q < 1$. *Moreover* $D_a^q f \in L_r(a,b), 1 \leq r \leq \frac{1}{q}$, *and*

$$ {}_a\mathcal{D}_t^q f(t) = \frac{1}{\Gamma(1-q)}\left[\frac{f(a)}{(t-a)^q} + \int_a^t \frac{f'(s)ds}{(t-s)^q}\right]. \tag{2.26}$$

Now we can define fractional derivative for large order $q \geq 1$. We will use the notations: $[q]$ stands for the largest integer not greater than q and $\{q\}$ meaning “fractional’’ part, $0 \leq \{q\} < 1$, so that $q = [q] + \{q\}$.

Let $q > 0$ be any real number. It is natural to introduce ${}_a\mathcal{D}_t^q f(t)$ by the relation

$$ {}_a\mathcal{D}_t^q f(t) := \left(\frac{d}{dt}\right)^{[q]} {}_a\mathcal{D}_t^{\{q\}} f(t) = \left(\frac{d}{dt}\right)^{[q]+1} I_a^{1-\{q\}} f(t). \tag{2.27} $$

Definition 2.3: ([12]) The RL fractional derivative of order q, $n-1 \leq q < n$ *, for* $f(t)$ *defined on the interval [a,b], is given as*

$$ {}_a\mathcal{D}_t^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)ds}{(t-s)^{q-n+1}}, n = [q] + 1. \tag{2.28} $$

To simplify the notations we will use $\mathcal{D}_a^q$ instead of ${}_a\mathcal{D}_t^q$. Now we recall some of the properties of the RL fractional derivative of order $n-1 \leq q < n$ as follow [63]:

- $\mathcal{D}_a^q (t-a)^\vartheta = \frac{\Gamma(1+\vartheta)}{\Gamma(1+\vartheta-q)} (t-a)^{\vartheta-q}$ where $q \in \mathbb{R}_+, \vartheta > -1$.
- $\mathcal{D}_a^p \left(\mathcal{D}_a^q f(t)\right) = \mathcal{D}_a^{p+q} f(t) - \sum_{j=1}^n \left[\mathcal{D}_a^{q-j} f(t)\right]_{t=a} \frac{(x-a)^{-p-j}}{\Gamma(1-p-j)}$, where $p, q \in \mathbb{R}$.
- $\mathcal{D}_a^q (\mathrm{I}_a^q f(t)) = f(t)$.
- $\mathrm{I}_a^q (\mathcal{D}_a^q f(t)) = f(t) - \sum_{j=1}^n \left[\mathcal{D}_a^{q-j} f(t)\right]_{t=a} \frac{(t-a)^{q-j}}{\Gamma(q-j+1)}$.
- $\mathcal{D}_a^q C = \frac{C(t-a)^{-q}}{\Gamma(1-q)}$ where C is an any constant.

Also, the Laplace transform of the RL fractional derivative is

- $\mathcal{L}\left\{\mathcal{D}_0^q f(t)\right\} = s^q F(s) - \sum_{k=0}^{n-1} s^k \left[\mathcal{D}_0^{q-k-1} f(t)\right]_{t=0}$.

• IVP for Fractional Differential Equation with RL Derivative

A fractional differential equation is an equation which contains fractional derivative. The IVP for FDE with RL derivative of order $n-1<q<n$ has the form

$$\begin{cases} \mathcal{D}_{t_0}^{q} x(t) = f(t, x(t)) \\ \mathcal{D}_{t_0}^{q-k} x(t)\big|_{t=t_0} = b_k, \ \ (k = 1,2, \dots, n) \end{cases} \tag{2.29}$$

where $n=[q]+1$ and b_k are given constants. The notation $\mathcal{D}_{t_0}^{q-k} x(t)\big|_{t=t_0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(t_0, t_0+\epsilon)$ of t_0 as follows

$$\begin{cases} \left[\mathcal{D}_{t_0}^{q-k} x(t)\right]_{t=t_0} = \lim\limits_{t\to t_0+} \mathcal{D}_{t_0}^{q-k} x(t) \ , \qquad (1 \le k \le n-1) \\ \left[\mathcal{D}_{t_0}^{q-n} x(t)\right]_{t=t_0} = \lim\limits_{t\to t_0+} I_{t_0}^{n-q} x(t) \end{cases} \tag{2.30}$$

where $I_{t_0}^{n-q}$ is the RL fractional integral of order $n-q$ defined by (2.22). IVP (2.29) is equivalent to the following fractional integral equation

$$x(t) = \sum_{k=1}^{n} \frac{b_k (t-t_0)^{q-k}}{\Gamma(q-k+1)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s, x(s)) ds}{(t-s)^{1-q}}. \tag{2.31}$$

In the case when $0<q<1$, the (2.29) takes the following form

$$\begin{cases} \mathcal{D}_{t_0}^{q} x(t) = f(t, x(t)) \\ \left[I_{t_0}^{1-q} x(t)\right]_{t=t_0} = b \end{cases} \tag{2.32}$$

And this problem can be rewritten as the weighted Cauchy type problem [11]-[16]

$$\begin{cases} \mathcal{D}_{t_0}^q x(t) = f(t, x(t)) \\ \lim_{t \to t_0 +} \Gamma(q) x(t)(t - t_0)^{1-q} = x^0. \end{cases} \tag{2.33}$$

The corresponding Volterra integral equation to IVP (2.33) is

$$x(t) = \frac{x^0(t - t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s, x(s))ds}{(t - s)^{1-q}} \tag{2.34}$$

and it is valid for the functions $x(t) \in C_p([t_0, T], \mathbb{R}),\ p = 1 - q$ where $C_p([t_0, T], \mathbb{R}) = \{x(t) \colon x(t) \in C[(t_0, T], \mathbb{R}]\ and\ x(t)(t - t_0)^{1-q} \in C([t_0, T], \mathbb{R})\}$

Definition 2.4: ([16], [41]) A function $x(t)$ *is called a* solution of *(2.33) if* $x(t) \in C_p([t_0, t_0 + a], \mathbb{R})$, $\mathcal{D}_{t_0}^q x(t)$ *exists and continuous on* $[t_0, t_0 + a]$ *and* $x(t)$ *satisfies (2.33).*

As an example we consider linear FDE including homogeneous and non-homogeneous, respectively [14].

Example 2.1: ([14]) A generalization of an equation solved in [11]

$$\begin{cases} \mathcal{D}_0^{1/2} x(t) + a x(t) = 0, \quad t > 0 \\ \left[I_0^{1/2} x(t) \right]_{t=0} = C \end{cases} \tag{2.35}$$

By applying the Laplace transform $X(s) = \frac{C}{s^{1/2} + a}$ *is obtained and the inverse transform with the help of (2.17) gives the solution of (2.35) as* $x(t) = C t^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(-a\sqrt{t})$.

Example 2.2: ([14]) Let us consider the following IVP for a non-homogeneous FDE under non-zero initial conditions

$$\begin{cases} \mathcal{D}_0^q x(t) + \lambda x(t) = h(t), \quad t > 0 \\ \left[\mathcal{D}_0^{q-k} x(t)\right]_{t=0} = b_k, \quad (k = 1,2,\dots,n) \end{cases} \tag{2.36}$$

where $\lambda \in \mathbb{R}$ is a constant.

Problem (2.36) was analytically solved in [12] by the iteration method. With the help of Laplace transform and formula the same solution is obtained directly and more quickly. The Laplace transform of equation (2.36) yields by taking into account initial conditions

$$s^q X(s) - \sum_{k=1}^{n} b_k s^{k-1} - \lambda X(s) = H(s)$$

$$X(s) = \frac{H(s)}{s^q - \lambda} + \sum_{k=1}^{n} b_k \frac{s^{k-1}}{s^q - \lambda}. \tag{2.37}$$

The inverse transform by using Proposition 2.3 gives the solution

$$x(t) = \sum_{k=1}^{n} b_k t^{q-k} E_{q,q-k+1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) h(s) ds. \tag{2.38}$$

In the case when $0 < q < 1$ the solution (2.38) has the following form $x(t) = b_1 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) h(s) ds$.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a), f'(a)$, etc. Unfortunately, the RL approach leads to initial conditions containing the limit values of the RL fractional derivatives at the lower terminal $t = t_0$. However, the IVP (2.29) can be successfully solved mathematically [12], [14], their solutions are practically useless, because there is no known physical interpretation for such types of initial conditions. This problem does not exist in the Caputo definition which is sometimes called smooth fractional derivative in literature.

2.2.2. Caputo Fractional Derivative

The definition (2.28) of the fractional differentiation of the RL type played an important role in the development of the theory of fractional derivatives and integrals, so called fractional calculus. However, the demands of modern technology require a certain revision of the well-established pure mathematical approach. A certain solution to this demand was proposed by Caputo.

The Caputo fractional derivative of order $q > 0$ can be written as (Caputo, 1967):

$$ {}_{t_0}^{C}\mathcal{D}^{q}\, x(t) = I_{t_0}^{n-q}\left[\frac{d^n}{dt^n}x(t)\right] = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}\frac{x^n(s)ds}{(t-s)^{q-n+1}} \tag{2.39}$$

where $n-1 < q < n$. In special case when $0 < q < 1$, (2.39) takes the form

- ${}_{t_0}^{C}\mathcal{D}^{q}\, x(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}\frac{x'(s)ds}{(t-s)^{q}}$.

Under the natural condition on the function $x(t)$ as $q \to n$, the Caputo derivative becomes the conventional n-th derivative of $x(t)$ [14]. The main advantage of Caputo's approach is that the initial conditions for FDE with Caputo derivative take on the same form as for ODE.

Now we recall some of the properties of the Caputo fractional derivative below [63]:

- ${}_{t_0}^{C}\mathcal{D}^{q}\, \mathrm{I}_{t_0}^{q} x(t) = x(t)$.
- $\mathrm{I}_{t_0}^{q}\, {}_{t_0}^{C}\mathcal{D}^{q}\, x(t) = x(t) - \sum_{k=0}^{n-1}\frac{(t-t_0)^k}{\Gamma(k+1)}x^{(k)}(t_0)$.
- $\mathcal{L}\{{}_{t_0}^{C}\mathcal{D}^{q}\, x(t)\} = s^q X(s) - \sum_{k=0}^{n-1} s^{q-k-1}\, x^{(k)}(t_0)$.
- ${}_{t_0}^{C}\mathcal{D}^{q}\, C = 0$ where C is any constant.

Also there exist relations between RL and the Caputo fractional derivative:

- ${}_{t_0}^{C}\mathcal{D}^q\, x(t) = \mathcal{D}_{t_0}^q x(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{k-q}}{\Gamma(k-q+1)} x^{(k)}(t_0)$.
- If $x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0$, then ${}_{t_0}^{C}\mathcal{D}^q\, x(t) = \mathcal{D}_{t_0}^q x(t)$.
- In the case $0 < q < 1$, ${}_{t_0}^{C}\mathcal{D}^q\, x(t) = \mathcal{D}_{t_0}^q x(t) - \frac{x(t_0)}{\Gamma(1-q)}(t-t_0)^{-q}$.
- If $x(t_0) = 0$, then ${}_{t_0}^{C}\mathcal{D}^q\, x(t) = \mathcal{D}_{t_0}^q x(t)$.

Contrary to the Laplace transform of the RL fractional derivative, only integer order derivatives of function $x(t)$ appears in the Laplace transform of the Caputo fractional derivative.

- IVP for Fractional Differential Equation with Caputo Derivative

The IVP of Caputo FDE is given by

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x(t) = f(t, x(t)) \\ \quad x(t_0) = x_0 \end{cases} \tag{2.40}$$

where $0 < q < 1, f \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$. The corresponding Volterra integral equation to IVP (2.40) is

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s, x(s))ds}{(t-s)^{1-q}}. \tag{2.41}$$

The first result is an existence result on IVP (2.40) that corresponds to the classical Peano existence theorem for first order ordinary differential equations.

Theorem 2.2: ([64]) Let the function $f: G \to \mathbb{R}$ *be continuous and bounded by M where* $G = \{(t, x): t_0 \le t \le t_0 + a,\ |x - x_0| \le K\}$, $x_0 \in \mathbb{R}$, $K > 0$ *and* $a > 0$. *Then there exist a solution* $x(t) \in C[t_0, t_0 + \alpha]$ *where* $\alpha = min[a, (\frac{K\Gamma(q+1)}{M})^{\frac{1}{q}}]$.

Proof of Theorem 2.2, please see in [64].

Now we will give the basic existence and uniqueness result with the Lipschitz condition, extension of previous result to vector-valued functions, by using contraction mapping theorem.

Theorem 2.3: ([16]) Assume that

i) $f \in C[R, \mathbb{R}^n]$ *and bounded by* M *on* R *where* $R = \{(t, x): t_0 \leq t \leq t_0 + a, \|x - x_0\| \leq b\}$;

ii) $\|f(t, x) - f(t, y\| \leq L\|x - y\|, \ L > 0, \ (t, x) \in R$ *where the inequalities are componentwise.*

Then there exists a unique solution $x(t) = x(t, t_0, x_0)$ *on* $[t_0, t_0 + \alpha]$ *where* $\alpha = min[a, (\frac{b\Gamma(q+1)}{M})^{\frac{1}{q}}]$.

Proof of Theorem 2.3, please see in [16].

We give some sufficient conditions for global existence of solutions.

Theorem 2.4: ([16]) Assume that there exists the function $g \in C \in [\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ *and non-decreasing with respect to second argument such that*

$$|f(t, x)| \leq g(t, |x|). \tag{2.42}$$

If the maximal solution of the initial value problem

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q u(t) = g(t, u(t)) \\ \quad u(t_0) = u_0 \end{cases} \tag{2.43}$$

exists in $[t_0, \infty)$. *Then the largest existence of any solution of (2.40) such that* $|x_0| \leq u_0$ *is* $[t_0, \infty)$.

Proof of Theorem 2.4, please see in [16].

Theorem 2.5: ([65]) Assume that there exists a continuous function $F: [t_0, \infty) \to \mathbb{R}_+$ such that $|f(t,x) - f(t,y)| \leq F(t)|x - y|$ for all $t \geq t_0$, $x, y \in \mathbb{R}^n$. Then, (2.40) has a unique solution defined in $[t_0, \infty)$.

Proof of Theorem 2.5, please see in [64].

Example 2.3: ([14]) Let us consider non-homogeneous linear scalar FDE with Caputo derivative

$$\begin{cases} {}_0^C\mathcal{D}^q\, x(t) + \lambda x(t) = h(t), \quad (\, t > 0) \\ \qquad\qquad x(0) = x_0 \end{cases} \tag{2.44}$$

where $\lambda > 0$ is a constant. The solution of (2.44) was obtained by applying successive approximations in [16]. With the help of Laplace transform the same solution is obtained directly and more quickly. The Laplace transform of equation (2.44) yields by taking into account initial condition

$$s^q X(s) - s^{q-1}x_0 \; - \lambda X(s) = H(s)$$

$$X(s) = \frac{H(s)}{s^q - \lambda} + \frac{s^{q-1}}{s^q - \lambda}. \tag{2.45}$$

The inverse transform by using Proposition 2.3 gives the solution $x(t) = x_0 E_q(-\lambda t^q) + \int_0^t (t - s)^{q-1}\, E_{q,q}(-\lambda(t - s)^q)h(s)ds$.

2.2.3. Other Approach

Unlike the RL approach, which derives its definition from the repeated integral, the Grünwald-Letnikov formulation approaches the problem from the derivative side. For this, let us consider the continuous function $f(t)$. Its first derivative can be expressed as [14]

$$f'(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t) - f(t - h)}{h}. \tag{2.46}$$

Applying this definition twice gives the second order derivative,

$$f''(t) = \frac{d^2f}{dt^2} = \lim_{h\to 0}\frac{f'(t) - f'(t-h)}{h}$$
$$= \lim_{h\to 0}\frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \tag{2.47}$$

Using (2.46) and (2.47) it follows that

$$f'''(t) = \frac{d^3f}{dt^3} = \lim_{h\to 0}\frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3} \tag{2.48}$$

and by induction

$$f^{(n)}(t) = \frac{d^nf}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}f(t-rh) \tag{2.49}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \tag{2.50}$$

is the usual notation for the binomial coefficients. Now, consider the following expression generalizing the fractions in (2.46) - (2.49),

$$f_h^{(q)}(t) = \frac{1}{h^q}\sum_{r=0}^{n}(-1)^r\binom{q}{r}f(t-rh) \tag{2.51}$$

where q is an arbitrary integer number and n is also an integer. Because from (2.50) all the coefficients in the numerator after $\binom{q}{q}$ are equal to 0, the following

$\lim_{h \to 0} f_h^{(q)}(t) = f^{(q)} = \frac{d^q f}{dt^q}$ for $q \leq n$ is obviously satisfied. In the case of negative value q it follows that

$$\binom{-q}{r} = \frac{-q(-q-1)(-q-2)\ldots(-q-r+1)}{r!} = (-1)^r \begin{bmatrix} q \\ r \end{bmatrix} \tag{2.52}$$

where $\begin{bmatrix} q \\ r \end{bmatrix}$ is defined as

$$\begin{bmatrix} q \\ r \end{bmatrix} = \frac{q(q+1)(q+2)\ldots(q+r-1)}{r!}. \tag{2.53}$$

Now replacing q in (2.51) with $-q$, then $f_h^{(-q)}(t) = \frac{1}{h^{-q}} \sum_{r=0}^{n} \begin{bmatrix} q \\ r \end{bmatrix} f(t - rh)$, where q is a positive integer number. If n is fixed, then $f_h^{(-q)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. To arrive at a non-zero limit, it is needed to suppose that $n \to \infty$ as $h \to 0$. Here $n = \frac{t - t_0}{h}$ can be taken where t_0 is a real constant, and consider the limit value, either finite or infinite, of $f_h^{(-q)}(t)$, which will be denoted as

$$\lim_{h \to 0} f_h^{(-q)}(t) = \mathcal{D}_{t_0}^{-q} f(t). \tag{2.54}$$

Here $\mathcal{D}_{t_0}^{-q} f(t)$ denotes a certain operation performed on the function $f(t)$ with t_0 and t are the limits relating to this operation. After observing the particular case $q = 1,2,3 \ldots$, following expression is followed

$$\mathcal{D}_{t_0}^{-q} f(t) = \lim_{h \to 0} h^q \sum_{r=0}^{n} \begin{bmatrix} q \\ r \end{bmatrix} f(t - rh) = \frac{1}{(q-1)!} \int_a^t \frac{f(t)}{(t-s)^{1-q}} ds. \tag{2.55}$$

As result [14] obtained general expression which represents the derivative of order m if $q = m$ and the m-fold integral if $q = -m$. This observation naturally leads to the idea of a generalization of the notions of differentiation and integration by allowing q in (2.48) to be an arbitrary real number.

Definition 2.3: ([14]) The Grünwald-Letnikov derivative *of the function $f(t)$ is given by*

$$ {}^{GL}\mathcal{D}_{t_0}^{q} f(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^r \binom{q}{r} f(t-rh) \tag{2.56} $$

where $\binom{q}{r} = \frac{\Gamma(1+q)}{r!\Gamma(1+q-m)}$.

Also the following formula was obtained in [14] for $m-1 < q < m$

$$ {}^{GL}\mathcal{D}_{t_0}^{q} f(t) = \sum_{k=0}^{m-1} \frac{(t-t_0)^{k-q}}{\Gamma(k-q+1)} f^{(k)}(t_0) + \frac{1}{\Gamma(-q+m)} \int_a^t \frac{f^{(m)}(s)ds}{(t-s)^{q-m+1}} \tag{2.57} $$

under the conditions that the function $f(t)$ is $m-1$- times continuously differentiable and $f^{(m)}(t)$ is integrable .

In the case when $0 < q < 1$ i.e. $x(t)$ is continuous and $x'(t)$ is integrable RL and GL derivatives have the following relation

$$ \mathcal{D}_{t_0}^{q} f(t) = {}^{GL}\mathcal{D}_{t_0}^{q} f(t) = \frac{x(t_0)(t-t_0)^{-q}}{\Gamma(1-q)} + \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{x'(s)ds}{(t-s)^q}. \tag{2.58} $$

Finally, Caputo, RL and GL fractional derivatives have the relation

$$ {}^{C}\mathcal{D}_{t_0}^{q} f(t) = \mathcal{D}_{t_0}^{q}[f(t) - f(t_0)] = {}^{GL}\mathcal{D}_{t_0}^{q}[f(t) - f(t_0)]. \tag{2.59} $$

3. SOME STABILITY AND BOUNDEDNESS PROPERTIES FOR FRACTIONAL DIFFERENTIAL EQUATIONS

3.1. Introduction

Fractional calculus deals with the generalization of differentiation and integration of non-integer order [11]. It is a mathematical field goes back to the 17th century almost as old as the calculus itself. It is realized and approved with experimental studies that various processes with anomalous dynamics in science and engineering can be more properly formulated by using fractional differential operators due to its memory and hereditary properties [11]-[15], [18]-[21]. For instance, the anomalous diffusion (subdiffusion, superdiffusion, non-Gaussian diffusion) phenomena show many different aspects from classical diffusion processes [17]-[18].

Stability analysis is one of the most fundamental and important issues for qualitative theory of differential equations, for instance control systems. Many different types of stability are defined and studied in the literature. However, the concept of boundedness has played a significant role in the existence of periodic [7] and almost periodic solutions [8] which has many applications in biological population management and control. In real world processes, there is a realization that a system may be stable or asymptotically stable in theory, but it is actually unstable in practice because of the stable domain or domain of attraction may be so small that desired deviation to cancel out is not allowed. Whereas, the desired state of a system may be mathematically unstable, but the system may oscillate sufficiently near this state so that the performance is considered admissible. To deal with such phenomena, the concept of practical stability is introduced in [9] and a systematic study is presented in [10]. Briefly stated, practical stability is essentially based on the pre-specified bounds for the perturbation of initial conditions response and the allowable perturbation of the system response.

Recently, fractional calculus was introduced to the stability analysis of FDE. However, Lyapunov method in dealing with ODE $(q = 1)$ cannot be simply extended to FDE since fractional differential operators are nonlocal and have weakly singular kernels. In literature there are some approaches to stability analysis of FDE without or with delay via application of differentiable or continuously differentiable Lyapunov functions, see [32]-[38]. Sufficient conditions on stability, asymptotical stability, Mittag-Leffler stability, generalized Mittag-Leffler stability are established by application of the Caputo derivative of Lyapunov functions in aforementioned papers. Besides, there is another approach in which the authors [39]-[40] have defined Caputo fractional Dini derivative of continuous Lyapunov function in an appropriate way. Motivated by the fact that the stability or asymptotic stability are neither necessary nor sufficient to assure practical stability [10] and the concept of practical stability and boundedness are valid even the studied FDE has no zero solution, we have investigated asymptotically stability, practical stability, boundedness and Lagrange stability for FDE by using fractional comparison method via Caputo fractional Dini derivative of continuous Lyapunov function. The statement of the problem and relation between stability and boundedness is presented in section 3.2. Then, in section 3.3 natural relationship between the Dini derivative of Lyapunov function for classical case ($q = 1$) and Caputo fractional Dini derivative is given by appropriate examples [39]-[40]. Some comparison results demand only continuity as an assumption instead of local Hölder continuity or C^q continuity used in the literature [16], [23], [41] is presented in section 3.4. Then, comparison method via Lyapunov function and a scalar FDE is applied to obtain sufficient conditions on some stability and boundedness for system of FDE in section 3.5. Finally some examples are given as an application of the obtained results.

3.2. Preliminary Notes and Definitions

3.2.1. Definitions of Some Stability and Boundedness

Consider the following IVP for the system of FDE for $0 < q < 1$

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q x(t) = f(t, x(t)) \\ \qquad x(t_0) = x_0 \end{cases} \tag{3.1}$$

where $t_0 \in \mathbb{R}_+,\ x,\ x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$. Denote the solution of (3.1) by $x(t) = x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$. Some sufficient conditions for global existence of solutions of (3.1) can be seen in [16], [64]-[65].

The main goal of this chapter is to study the practical stability, boundedness and Lagrange stability properties for the system of FDE (3.1) via fractional comparison principle in which Lyapunov function and scalar FDE is employed.
When (3.1) has a zero solution, i.e. $f(t, 0) = 0$ we shall use the following stability definition.

Definition 3.1: The zero solution $x(t) \equiv 0$ of (3.1) is said to be:

S1) stable *if given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(t_0, \epsilon) > 0$ such that for any initial position $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$, where $x(t; t_0, x_0)$ is any solution of (3.1);*
S2) uniformly stable, *if δ in S1) is independent of t_0;*
S3) attractive *if for given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that for any initial position $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta_0$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0 + T$;*
S4) uniformly attractive, *if δ_0 and T in S3) are independent of t_0;*
S5) asymptotically stable, *if it is stable and attractive;*
S6) uniformly asymptotically stable, *if it is uniformly stable and attractive.*

Definition 3.2: Let a couple of real numbers (λ, A) with $0 < \lambda < A$ be given. The system of FDE (3.1) is said to be:

PS1) practically stable w.r.t. (λ, A) *if there exists $t_0 \geq 0$ such that for any initial position $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \lambda$* implies $\|x(t; t_0, x_0)\| < A$ *for $t \geq t_0$*;
PS2) uniformly practically stable *if PS1) holds for all $t_0 \in \mathbb{R}_+$;*
PS3) practically quasi-stable w.r.t. (λ, A, T) *if there exists $t_0 \geq 0$ such that for any initial position $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \lambda$ implies $\|x(t; t_0, x_0\| < A$ for $t \geq t_0 + T$;*
PS4) uniformly practically quasi stable w.r.t. (λ, A) *if PS3) holds for all $t_0 \in \mathbb{R}_+$.*

Remark 3.1: Unlike the concept of stability the assumption on the existence of the zero solution i.e. $f(t,0) \equiv 0$ *is not essential for the concepts of boundedness and practical stability.*

The following Lemma is given for future use.

Lemma 3.1: For $0 < q < 1$ *and* $\gamma > 0$, *we have*

$$\int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-\gamma(t-\tau)^q) ds = \frac{1}{\gamma}(1 - E_q(-\gamma(t-t_0)^q)). \tag{3.2}$$

Proof 3.1: By applying change of variable in the integral (3.2) and using Preposition 2.1, we have

$$\begin{aligned} &\int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-\gamma(t-\tau)^q) ds \\ &= \int_{0}^{t-t_0} z^{q-1} E_{q,q}(-\gamma z^q) dz = -\frac{1}{\gamma}\int_{0}^{t-t_0} \frac{d}{dz} (E_q(-\gamma z^q)) dz \\ &= \frac{1}{\gamma}(1 - E_q(-\gamma(t-t_0)^q)). \ \blacksquare \end{aligned} \tag{3.3}$$

Example 3.1: To illustrate the idea in Remark 3.1, consider the following IVP for fractional relaxation equation ${}^{C}_{t_0}\mathcal{D}^q u(t) = -u(t) + \mu sint, \ \ u(t_0) = \alpha,$ *where* $\alpha > \mu > 0$. *It is obvious that above FDE has no zero solution and solution is given by*

$$u(t) = \alpha E_q(-(t-t_0)^q) + \mu \int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-(t-\tau)^q)[sin\tau] d\tau. \tag{3.4}$$

By Lemma 3.1 with $\gamma = 1$, *we obtain the following estimate for the solution*

$$|u(t)| \leq \alpha E_q(-(t-t_0)^q) + \mu(1 - E_q(-(t-t_0)^q)), \;\; t \geq t_0. \tag{3.5}$$

Then we have the following results:

- *in view of estimate (3.5) and* $0 < E_q(-(t-t_0)^q) < 1$ *the solution satisfies* $|u(t)| \leq \alpha, \; t \geq t_0$*, i.e. the solution is bounded, uniformly in* t_0*. Besides, we can find a better bound* N *with* $\alpha > N > \mu$*, it is called ultimate bound, after a transient period passed by taking into consideration of the decaying property of Mittag-Leffler function. Indeed,* $|u(t; t_0, u_0)| \leq N$ *for* $t \geq t_0 + T$ *where* $T = (-L_q(\frac{N-\mu}{\alpha-\mu}))^{\frac{1}{q}}$ *,* $L_q(z)$ *is the inverse function of the Mittag-Leffler function* $E_q(z)$ *defined as the solution of the equation* $L_q\left(E_q(z)\right) = z$ *(see [66]).*
- *consider the fractional relaxation equation with* $u(t_0) = u_0$ *and let a couple of real numbers* (λ, A) *with* $0 < \lambda < A$ *be given. Then* $|u_0| < \lambda$ *implies* $|u(t; t_0, u_0)| < A$*, i.e. fractional relaxation equation is uniformly practically stable w.r.t.* (λ, A)*. Besides, PS4) holds with* (λ, A, T) *where* $T = (-L_q(\frac{A-\mu}{\lambda-\mu}))^{\frac{1}{q}}$*.*
  Figure 3.1 shows that the exact solution for classical case $(q = 1)$ *and approximate solutions of fractional relaxation equation with* $\mu = 0.5, \; \alpha = 3$ *and different choices of* $q = 0.5, \; q = 0.8$ *and* $q = 0.9$*.*

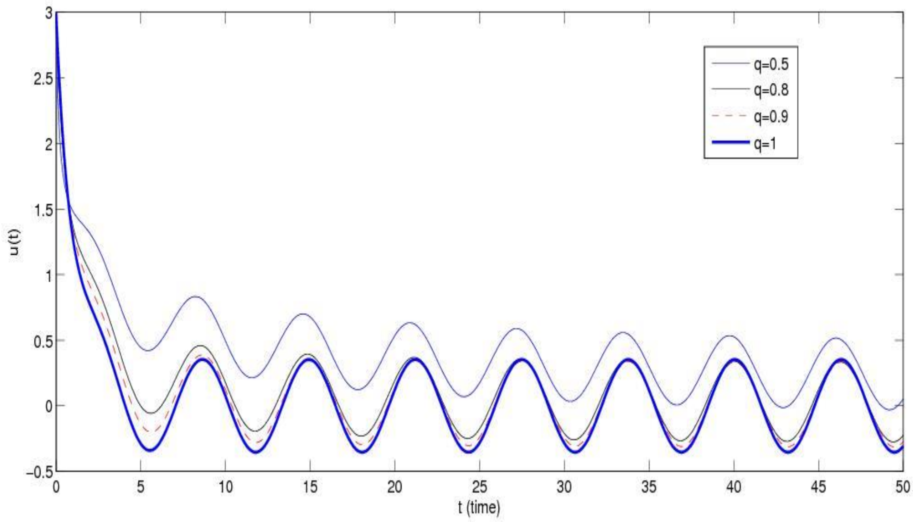

Figure 3.1: Approximate solutions with $\mu = 0.5$, $\alpha = 3$ and $q = 0.5$, $q = 0.8$, $q = 0.9$.

Corresponding to different types of stability, the concept of boundedness and Lagrange stability can be defined [3], [4], [6]-[8].

Definition 3.3: The system of FDE (3.1) is said to be:

B1) equi-bounded, *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exists a* $\beta = \beta(t_0, \alpha) > 0$ *such that any initial value* $x_0 \in \mathbb{R}^n$ *the inequality* $\|x_0\| \leq \alpha$ *implies* $\|x(t; t_0, x_0\| < \beta$ *for* $t \geq t_0$;

B2) uniformly bounded, if β *in B1) is independent of* t_0;

B3) ultimately bounded, *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exist* $N > 0$ *and* $T = T(t_0, \alpha) > 0$ *such that any initial value* $x_0 \in \mathbb{R}^n$ *the inequality* $\|x_0\| \leq \alpha$ *implies* $\|x(t; t_0, x_0)\| < N, \;\; t \geq t_0 + T$;

B4) uniformly ultimately bounded *if* T *in B3) is independent of* t_0;

A1) attractive in the large *if, for each* $\epsilon > 0$, $\alpha > 0$ *there exists a* $T = T(t_0, \epsilon, \alpha,) > 0$ *such that any initial value* $x_0 \in \mathbb{R}^n$ *the inequality* $\|x_0\| \leq \alpha$ *implies* $\|x(t; t_0, x_0)\| < \epsilon$ *for* $t \geq t_0 + T$;

A2) uniformly attractive in the large, *if* T *in A1) is independent of* t_0;

L1) Lagrange stable *if B1) and A1) hold together;*

L2) uniformly Lagrange stable *if B2) and A2) hold together.*

Proposition 3.1: In the case when (i) $f(t,0)=0$ *and* (ii) $\beta \to 0$ *as* $\alpha \to 0$ *then the definitions B1), B2) imply S1), S2).*

Proof 3.1: Suppose that the zero solution is not stable. Then there exists a $\epsilon^* > 0$ *such that* $\forall \delta > 0$ *such that* $\|x_0\| \leq \delta$ *implies* $\|x(t;t_0,x_0\| > \epsilon^*$ *for* $t \geq t_0$*. On the other hand from B1) with the choice of* $\alpha = \delta$ *we get the inequality* $\epsilon^* < \|x(t;t_0,x_0\| < \beta$ *which contradicts the property of* β*. Therefore, zero solution is stable.* ■

In order to employ the fractional order extension of Lyapunov method, following scalar FDE is used

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, u(t) = G(t,u(t)) \\ \qquad u(t_0) = u_0 \end{cases} \tag{3.6}$$

where $u, u_0 \in \mathbb{R}$, $G \in C[\mathbb{R} \times \mathbb{R}, \mathbb{R}]$. We denote the solution of the IVP for the scalar FDE (3.6) by $u(t;t_0,u_0) \in C^q([t_0,\infty),\mathbb{R})$. In case of the solution is not unique we will assume the existence of a maximal one.

Corresponding to the Definitions (3.1) - (3.3) given above, we need to recall definitions of stability, practical stability, boundedness and Lagrange stability for the scalar FDE (3.6).

Definition 3.4: The zero solution of (3.6) is called:

*S*1)* stable *if given* $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$*, there exist* $\delta = \delta(t_0,\epsilon) > 0$ *such that for any initial position* $u_0 \in \mathbb{R}$ *the inequality* $|u_0| < \delta$ *implies* $|u(t;t_0,u_0)| < \epsilon$ *for* $t \geq t_0$*, where* $u(t;t_0,u_0)$ *is a solution of (3.6).*

Definition 3.5: Let a couple of real numbers (λ, A) *with* $0 < \lambda < A$ *be given. The scalar FDE (3.6) is called:*

*PS*1)* practically stable w.r.t. (λ, A) *if there exists* $t_0 \geq 0$ *such that for any initial position* $u_0 \in \mathbb{R}$ *the inequality* $|u_0| < \lambda$ *implies* $|u(t; t_0, u_0)| < A$ *for* $t \geq t_0$*, where* $u(t; t_0, u_0)$ *is a solution of (3.6).*

Definition 3.6: The scalar FDE (3.6) is called

*B*1)* equi-bounded, *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$ *there exist* $\beta = \beta(t_0, \alpha) > 0$ *such that for any initial position* $u_0 \in \mathbb{R}$ *the inequality* $|u_0| < \alpha$ *implies* $|u(t; t_0, u_0)| < \beta$ *for* $t \geq t_0$*, where* $u(t; t_0, u_0)$ *is a solution of (3.6);*

We note that the definitions $S^*2)$ - $PS^*4)$ and $B^*2)$ - $L^*2)$ can be formulated similarly. It should also be noted that $G(t, 0) \equiv 0$ is not required in Definition (3.5) – Definition (3.6).

3.3. Lyapunov Functions and its Caputo Fractional Dini Derivative

Our aim is to establish the connection between practical stability, boundedness and Lagrange stability of the scalar FDE (3.6) and given system of FDE (3.1) by employing the fractional order extension of Lyapunov method. Hence, the class Λ of Lyapunov-like functions are given.

Definition 3.7: Let $I \subset \mathbb{R}_+$ *and* $\Delta \subset \mathbb{R}^n$*. We will say that the function* $V(t, x): I \times \Delta \to \mathbb{R}_+$ *belongs to* class $\Lambda(I, \Delta)$ *if:*

i) $V(t, x)$ *is continuous and locally Lipschitzian with respect to its second argument on* $I \times \Delta$ *and* $V(t, 0) \equiv 0$ *for* $t \in I$.

It is convenient to introduce certain class of monotone functions to characterize Lyapunov-like functions.

Definition 3.8: ([6]) A continuous function $\varphi: [0, \rho) \to \mathbb{R}_+$ *is said to belong to the* class K *if it is strictly increasing and* $\varphi(0) = 0$*. It is said to belong to* K_∞ *if* $\rho = \infty$ *and* $\varphi(r) \to \infty$ *as* $r \to \infty$.

Example 3.2: ([6]) $\varphi(r) = tan^{-1}(r)$ *is strictly increasing since* $\varphi'(r) = \frac{1}{1+r^2} > 0$. *It belongs to class* K, *but not to class* K_∞ *since* $\lim_{r\to\infty} \varphi(r) = \pi/2$.

Example 3.3: ([6]) $\varphi(r) = r^c$, *for any positive real number* c, *is strictly increasing since* $\varphi'(r) = cr^{c-1} > 0$. *Moreover* $\lim_{r\to\infty} \varphi(r) = \infty$, *thus it belongs to class* K_∞.

Definition 3.9: ([6]) $V(t,x) \in \Lambda$ *is said to be* positive definite *if there exists a function* $\varphi \in K$ *such that* $\varphi(\|x\|) \leq V(t,x)$.

Definition 3.10: ([6]) $V(t,x) \in \Lambda$ *is said to be* decrescent *if there exists a function* $\psi \in K$ *such that* $V(t,x) \leq \psi(\|x\|)$.

Definition 3.11: ([41]) $m(t)$ *is said to be* C^q continue *i.e.,* $m(t) \in C^q([t_0,T],\mathbb{R})$, *if and only if the Caputo derivative of* ${}_{t_0}^{C}\mathcal{D}^q\, m(t)$ *exists and satisfies* ${}_{t_0}^{C}\mathcal{D}^q\, m(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t} \frac{m'(s)}{(t-s)^q} ds$.

Definition 3.12: ([41]) GL fractional Dini derivative *is given by*

$$ {}_{t_0}^{GL}\mathcal{D}_+^q m(t) = \lim_{h\to 0^+} sup \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh) \tag{3.7}$$

where $\binom{q}{r}$ is the Binomial coefficients and $[\frac{t-t_0}{h}]$ means the integer part of $\frac{t-t_0}{h}$.

From the relation between the Caputo, the GL fractional derivative and (3.7), the Caputo fractional Dini derivative is defined as [39]-[40]

$$ {}_{t_0}^{C}\mathcal{D}_+^q m(t) = {}_{t_0}^{GL}\mathcal{D}_+^q [m(t) - m(t_0)] \tag{3.8}$$

i.e.

$$ {}_{t_0}^{C}\mathcal{D}_+^q m(t) = \limsup_{h\to 0^+} \frac{1}{h^q}\left[m(t) - m(t_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\binom{q}{r}\left(m(t-rh) - m(t_0)\right)\right]. \tag{3.9} $$

Remark 3.2: ([39]-[40]) If $m(t) \in C^q([t_0, T], \mathbb{R})$*, then* ${}_{t_0}^{C}\mathcal{D}_+^q m(t) = {}_{t_0}^{C}\mathcal{D}^q m(t)$.

Lyapunov second method enables us to study some stability and boundedness properties in which the appropriate definition of the derivative of Lyapunov function along the studied any type of differential equation is required. In this context, there is an approach to stability analysis of FDE without or with delay via application of continuously differentiable Lyapunov functions in literature [32]-[38]. Besides, there is another approach in which the authors [39]-[40] have investigated stability and practical stability with ITD via application of continuous Lyapunov function which could be not continuously differentiable. For this purpose Caputo fractional Dini derivative of a function $m(t)$ given by (3.9) is used and Caputo fractional Dini derivative of the function $V(t,x) \in \Lambda(I, \Delta)$ is defined along the solutions of the system of FDE (3.1) as follow

$$ {}_{(3.1)}^{C}\mathcal{D}_+^q V(t, x; t_0, x_0) = \limsup_{h\to 0^+} \frac{1}{h^q}\left[V(t,x) - V(t_0, x_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\binom{q}{r}\left(V(t-rh, x - h^q(f(t,x)) - V(t_0, x_0)\right)\right] \tag{3.10} $$

where $t, t_0 \in I$, $x, x_0 \in \Delta$.

In Example 4.5, Corollary 4.2 and Example 4.6, Caputo fractional Dini derivative relative to ITD, which is a generalization of (3.10), is introduced and applied to some Lyapunov function. In this context, the following examples from [39] are special case of it with $\eta = 0$ and we omit the details.

Example 3.4: Let the Lyapunov function does not depend on the time variable, i.e. $V(t,x) \equiv V(x)$ for $x \in \mathbb{R}$. Then, Caputo fractional Dini derivative for the considered Lyapunov function by applying formula (3.10) is

$$
{}_{(3.1)}^{\;\;C}\mathcal{D}_{+}^{q}V(t,x;t_0,x_0) = \lim_{h\to 0^+} sup \frac{V(x) - V(x - h^q(f(t,x))}{h^q} + [V(x) - V(x_0)]\frac{(t-t_0)^{-q}}{\Gamma(1-q)}. \tag{3.11}
$$

Corollary 3.1: Let $V(t,x) \equiv V(x) = x^2$ for $x \in \mathbb{R}$. According to Example 3.4 the following expression for the quadratic scalar Lyapunov function is given

$$
\begin{aligned}
{}_{(3.1)}^{\;\;C}\mathcal{D}_{+}^{q}V(t,x;t_0,x_0)
&= \lim_{h\to 0^+} sup \frac{x^2 - (x - h^q(f(t,x))^2}{h^q} + [x^2 - x_0^2]\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \\
&= 2xf(t,x) + [x^2 - x_0^2]\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{3.12}
$$

Remark 3.3: In the case when $q \to 1$ the equality (3.12) coincides with the known Dini derivative $\mathcal{D}^{+}V(t,x) = 2xf(t,x)$ [3]-[4].

Example 3.5: Let $V(t,x) = m^2(t)x^2$ for $x \in \mathbb{R}$ where $m \in C^1(\mathbb{R}_+, \mathbb{R})$. Then, Caputo fractional Dini derivative (3.10) becomes the following expression

$$
{}_{(3.1)}^{\;\;C}\mathcal{D}_{+}^{q}V(t,x;t_0,x_0) = 2xm^2(t)f(t,x) + x^2\,{}_{t_0}^{C}\mathcal{D}^{q}\,[m^2(t)] + (x^2 - x_0^2)\frac{m^2(t_0)(t-t_0)^{-q}}{\Gamma(1-q)}. \tag{3.13}
$$

It is well known that the Dini derivative of $V(t,x) = m^2(t)x^2$ for classical case $(q = 1)$ is

$$
\mathcal{D}^{+}V(t,x) = 2xm^2(t)f(t,x) + x^2\frac{d}{dt}[m^2(t)]. \tag{3.14}
$$

3.4. Fractional Order Extension of Differential Inequalities and Comparison Results

Lemma 3.2: ([39]-[40]) Let $m \in C([t_0, t_0+\theta], \mathbb{R})$, $\theta > 0$, *and there exists* $t^* \in (t_0, t_0+\theta]$ *such that* $m(t^*) = 0$ *and* $m(t) < 0$ *for* $t_0 \le t < t^*$. *Then if the Caputo fractional Dini derivative (3.10) exists for* $m(t^*)$ *then the inequality* ${}_{t_0}^{C}\mathcal{D}_{+}^{q} m(t^*) > 0$ *holds.*

Proof 3.2: From (3.7), $m(t^*) = 0$ *and the inequalities* $r - q > 0$ *for* $r = 1,2,\ldots$, *and* $0 < q < 1$ *follows*

$$ {}_{t_0}^{GL}\mathcal{D}_{+}^{q}[m(t^*)] = \lim_{h\to 0^+} sup \frac{1}{h^q} \sum_{r=0}^{\left[\frac{t^*-t_0}{h}\right]} (-1)^r \binom{q}{r} m(t^* - rh) = m(t^*) + $$

$$ \lim_{h\to 0^+} sup \frac{1}{h^q} \sum_{r=1}^{\left[\frac{t^*-t_0}{h}\right]} (-1)^r \frac{q(q-1)\ldots(q-r+1)}{r!} m(t^* - rh) \tag{3.15} $$

$$ = \lim_{h\to 0^+} sup \frac{1}{h^q} \sum_{r=1}^{\left[\frac{t^*-t_0}{h}\right]} (-1)^r \frac{q(1-q)\ldots(r-1-q)}{r!} - m(t^* - rh) > 0. $$

From the relation (3.8)

$$ {}_{t_0}^{C}\mathcal{D}_{+}^{q}[m(t^*)] = {}_{t_0}^{GL}\mathcal{D}_{+}^{q}[m(t^*)] - \frac{m(t_0)(t^*-t_0)^{-q}}{\Gamma(1-q)}. \tag{3.16} $$

From inequalities $m(t_0) < 0$, $t^* > t_0$, $\Gamma(1-q) > 0$ for $0 < q < 1$ and (3.16), the claim of Lemma 3.2 is obtained. ■

The most commonly used technique in the theory of dynamic equations is concerned with estimating a function satisfying a dynamic inequality by the extremal solutions of the related dynamic equation. The assumption with locally Hölder continuity is weakened to C^1, C^q continuity in [52], [67], respectively. For the sake of completeness we will present fractional comparison theorem in which assumption for C^1 or C^q continuity is weakened to C, i.e. continuity by employing Lemma 3.2. Here some calculations and steps in the proof are given from [39]-[40].

Lemma 3.3: Assume the following conditions are satisfied:

i) The function $G \in C[[t_0, t_0 + \theta] \times \mathbb{R}, \mathbb{R}]$ and $H > 0$ be such that for any $\epsilon \in [0, H]$ and $v_0 \in \mathbb{R}$ the scalar FDE

$$ {}_{t_0}^{C}\mathcal{D}^q u(t) = G(t, u(t)) + \epsilon, \quad u(t_0) = v_0 \tag{3.17} $$

has a solution $u(t; t_0, v_0, \epsilon) \in C^q([t_0, t_0 + \theta], \mathbb{R})$.

ii) The function $m(t) \in C([t_0, t_0 + \theta], \mathbb{R}_+)$ so that the inequality

$$ {}_{t_0}^{C}\mathcal{D}_+^q m(t) \leq G(t, m(t)), \quad t \in [t_0, t_0 + \theta] \tag{3.18} $$

holds.

Then $m(t_0) \leq u_0$ implies the validity of inequality $m(t) \leq u^*(t)$ for $t \in [t_0, t_0 + \theta]$ where $u^*(t) = u(t; t_0, u_0)$ is the maximal solution of IVP for scalar FDE (3.17) for $v_0 = u_0$ and $\epsilon = 0$.

Proof 3.3: Consider the IVP for the scalar FDE (3.17) with $v_0 = u_0 + \epsilon$ where $\epsilon \in [0, H]$ is an arbitrary fixed number. According to i) the IVP for the scalar FDE (3.17) has a solution $u_\epsilon(t) = u(t; t_0, u_0 + \epsilon, \epsilon)$. Note that $u_\epsilon(t)$ satisfy the following integral equation corresponding to IVP (3.17)

$$ u_\epsilon(t) = u_0 + \epsilon + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{[G\big(s, x(s)\big) + \epsilon]ds}{(t - s)^{1-q}} \tag{3.19} $$

for $t \in [t_0, t_0 + \theta]$. We now prove that

$$m(t) < u_\epsilon(t) \text{ for } t \in [t_0, t_0 + \theta]. \tag{3.20}$$

Note that the inequality (3.20) holds for $t = t_0$ *since* $m(t_0) \le u_0 < u_\epsilon(t_0)$*. Suppose inequality (3.20) is not true for* $t \in (t_0, t_0 + \theta]$*. Then there exists a point* t^* *such that* $m(t^*) = u_\epsilon(t^*)$*,* $m(t) < u_\epsilon(t)$ *for* $t \in [t_0, t^*)$*. Applying Lemma 3.2 to the difference* $m(t) - u_\epsilon(t)$ *yields* ${}_{t_0}^{C}\mathcal{D}_+^q[m(t^*) - u_\epsilon(t^*)] > 0$*, i.e.* ${}_{t_0}^{C}\mathcal{D}_+^q m(t^*) > G\big(t^*, u_\epsilon(t^*)\big) + \epsilon > G\big(t^*, m(t^*)\big)$*. The obtained inequality above contradicts with the ii) for* $t = t^*$*. Therefore the inequality (3.20) holds on* $[t_0, t_0 + \theta]$ *and any* $\epsilon \in (0, H]$*. We now show if* $0 < \epsilon_2 < \epsilon_1 \le H$ *then*

$$u_{\epsilon_2}(t) < u_{\epsilon_1}(t) \text{ for } t \in [t_0, t_0 + \theta]. \tag{3.21}$$

Since $u_{\epsilon_2}(t_0) = u_0 + \epsilon_2 < u_0 + \epsilon_1 = u_{\epsilon_1}(t_0)$*, the inequality (3.21) holds for* $t = t_0$*. Assume that inequality (3.21) is not true. Then there exists a* $t^* \in (t_0, t_0 + \theta]$ *such that* $u_{\epsilon_2}(t^*) < u_{\epsilon_1}(t^*)$*,* $u_{\epsilon_2}(t) < u_{\epsilon_1}(t)$ *for* $t_0 \le t < t^*$*. According to Lemma 3.2 applied to* $u_{\epsilon_2}(t) - u_{\epsilon_1}(t)$*, we obtain* ${}_{t_0}^{C}\mathcal{D}_+^q\big[u_{\epsilon_2}(t^*) - u_{\epsilon_1}(t^*)\big] > 0$*. On the other hand by using Remark 3.2 yields*

$$\begin{aligned} &{}_{t_0}^{C}\mathcal{D}_+^q\big[u_{\epsilon_2}(t^*) - u_{\epsilon_1}(t^*)\big] \\ &\qquad > G\left(t^*, u_{\epsilon_2}(t^*)\right) + \epsilon_2 - \left(G\left(t^*, u_{\epsilon_1}(t^*)\right) + \epsilon_1\right) \\ &\qquad = \epsilon_2 - \epsilon_1 < 0. \end{aligned} \tag{3.22}$$

which is a contradiction implies the validity of (3.21). Now $0 < \epsilon \le H$*, (3.21) yields that the the family of solutions* $\{u_\epsilon(t)\}$*,* $t \in [t_0, t_0 + \theta]$ *of (3.17) is uniformly bounded i.e. there exists* $K > 0$ *with* $|u_\epsilon(t)| \le K$ *for* $(t, \epsilon) \in [t_0, t_0 + \theta] \times [0, H]$*. Let* $M = sup\{|G(t, u)| : (t, u) \in [t_0, t_0 + \theta] \times [-K, K]\}$*. Take a decreasing sequence of positive numbers* $\{\epsilon_j\}_{j=0}^{\infty}$*,* $0 < \epsilon_0 \le H$ *such that* $\lim_{j\to\infty} \epsilon_j = 0$ *and consider the corresponding sequence of solutions* $u_{\epsilon_j}(t)$*. Now for* $t_1, t_2 \in [t_0, t_0 + \theta]$*,* $t_1 < t_2$*, using the inequalities* $a^q - b^q \le 2(a - b)^q$ *for* $a \ge b \ge 0$*,* $(t_1 - s)^q \le (t_2 - s)^q$

for $s \in [t_0, t_1]$ *and* $\int_{t_0}^{t_1}((t_2 - s)^{q-1} - (t_1 - s)^{q-1})\, ds = \frac{1}{q}((t_2 - t_0)^q - (t_1 - t_0)^q - (t_2 - t_1)^q) \leq \frac{(t_2-t_1)^q}{q}$, *we get an estimate for* $\left|u_{\epsilon_j}(t_2) - u_{\epsilon_j}(t_1)\right|$ *as*

$$\begin{aligned} \left|u_{\epsilon_j}(t_2) - u_{\epsilon_j}(t_1)\right| &\leq \frac{1}{q}\left|\int_{t_0}^{t_1}((t_2 - s)^{q-1} - (t_1 - s)^{q-1})\left(G\left(s, u_{\epsilon_j}(s)\right) + \epsilon_j\right) ds\right| \\ &\quad + \left|\int_{t_1}^{t_2}((t_2 - s)^{q-1})\left(G\left(s, u_{\epsilon_j}(s)\right) + \epsilon_j\right) ds\right| \\ &\leq \frac{2(M + H)}{\Gamma(1 + q)}(t_2 - t_1)^q. \end{aligned} \tag{3.23}$$

Thus, the family $\{u_{\epsilon_j}(t)\}$ *is equi-continuous on* $[t_0, t_0 + \theta]$. *The Arzela-Ascoli Theorem yields that there exists a subsequence* $\{u_{\epsilon_{j_k}}(t)\}$ *that is uniformly convergent in the interval* $[t_0, t_0 + \theta]$. *Let* $lim_{k\to\infty} u_{\epsilon_{j_k}}(t) = \bar{u}(t)$. *By taking limit in (3.19) with* $\epsilon = \epsilon_{j_k}$ *as* $k \to \infty$, *we see* $\bar{u}(t)$ *satisfies the IVP (3.18) for* $t \in [t_0, t_0 + \theta]$, *i.e.* $\bar{u}(t)$*is solution IVP for scalar FDE (3.17) for* $v_0 = u_0$ *and* $\epsilon = 0$. *Finally, we have* $m(t) \leq \bar{u}(t) \leq u^*(t)$ *for* $t \in [t_0, t_0 + \theta]$ *by taking limit in (3.20) for* $\epsilon = \epsilon_{j_k}$ *as* $k \to \infty$. ■

Remark 3.4: Note that in the case of $m(t)$ *is continuously differentiable, i.e.* $m(t)$ *is belong to class* C^q, *Lemma 3.3 generalizes and unifies the comparison results in [41], [67] in view of Remark 3.2.*

The following Lemma 3.4 is a comparison result which establish a relationship between Lyapunov functions, system of FDE (3.1) and scalar FDE (3.6). The following Lemma 3.4 and its generalization with ITD are given in [39]-[40], respectively. We will give the proof of ITD version of Lemma 3.4 in the next chapter.

Lemma 3.4: Assume the following conditions are satisfied:

i) The function $x(t) = x(t; t_0, x_0) \in C^q([t_0, t_0 + \theta], \Delta)$ *is a solution of (3.1).*
ii) The function $V \in \Lambda([t_0, t_0 + \theta], \Delta)$, $G \in C[[t_0, t_0 + \theta] \times \mathbb{R}, \mathbb{R}]$ *such that for* $t \in (t_0, t_0 + \theta]$ *the inequality*

$$ {}_{(3.1)}^{\;\;C}\mathcal{D}_+^q V(t, x(t); t_0, x_0) \leq G(t, V(t, x(t))) \tag{3.24}$$

holds.

Then $V(t_0, x_0) \leq u_0$ *implies the validity of inequality* $V(t, x(t)) \leq u^*(t)$ *for* $t \in [t_0, t_0 + \theta]$ *where* $u^*(t) = u(t; t_0, u_0)$ *is the maximal solution of IVP for scalar FDE (3.6).*

In the case when $G(t, u) = \gamma u$, $\gamma \in \mathbb{R}$ *is a constant, we deduce the following Corollary from Lemma 3.4.*

Corollary 3.2: Let the condition i), ii) of Lemma 3.4 be satisfied and the function $V \in \Lambda([t_0, t_0 + \theta], \Delta)$ *be such that the inequality* ${}_{(3.1)}^{\;\;C}\mathcal{D}_+^q V(t, x(t); t_0, x_0) \leq \gamma V(t, x)$ *holds for* $t \in [t_0, t_0 + \theta]$.
Then for $t \in [t_0, t_0 + \theta]$ *the inequality* $V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) E_q(\gamma (t - t_0)^q)$ *holds.*

Proof 3.2: The proof of Corollary 3.2 follows from the fact that corresponding IVP for scalar FDE (3.6) with $G(t, u) = \gamma u$, $u_0 = V(t_0, x_0)$, *i.e.* ${}_{t_0}^{C}\mathcal{D}^q u(t) = \gamma u(t)$ *has a unique solution* $u(t) = V(t_0, x_0) E_q(\gamma (t - t_0)^q)$ *for* $t \in [t_0, t_0 + \theta]$. ■

In the case when $G(t, u) = 0$, *the following Corollary is deduced from Lemma 3.4.*

Corollary 3.3: Let the condition i), ii) of Lemma 3.4 be satisfied and the function $V \in \Lambda([t_0, t_0 + \theta], \Delta)$ *be such that the inequality* ${}_{t_0}^{C}\mathcal{D}_+^q V(t, x(t); t_0, x_0) \leq 0$ *holds for* $t \in [t_0, t_0 + \theta]$.
Then for $t \in [t_0, t_0 + \theta]$ *the inequality* $V(t, x(t; t_0, x_0)) \leq V(t_0, x_0)$ *holds.*

Proof 3.3: The proof of Corollary 3.3 follows from the fact that corresponding IVP for scalar FDE (3.6) with $G(t,u)=0$, $u_0=V(t_0,x_0)$, *i.e.* ${}_{t_0}^{C}\mathcal{D}^q u(t)=0$ *has a unique solution* $u(t)=V(t_0,x_0)$ *for* $t\in[t_0,t_0+\theta]$. ■

The Caputo fractional Dini derivative (3.10), Lemma 3.3, 3.4, Corollary 3.2, Corollary 3.3 demand just continuity property from the candidate Lyapunov function. For the sake of completeness it is noted that if Lyapunov function $V(t,x)$ *is continuously differentiable, comparison results in terms of the Caputo fractional derivative used in the literature [32]-[38] can be given.*

Lemma 3.5: Assume the following conditions are satisfied:

i) The condition i) of Lemma 3.4 holds.

ii) The function $V:[t_0,t_0+\theta]\times\Delta\to\mathbb{R}_+$ *is continuously differentiable such that the inequality*

$$ {}_{(3.1)}^{C}\mathcal{D}^q V(t,x(t))\le G(t,V(t,x(t))) \tag{3.25}$$

holds.

Then $V(t_0,x_0)\le u_0$ *implies the validity of inequality* $V(t,x(t))\le u^*(t)$ *for* $t\in[t_0,t_0+\theta]$ *where* $u^*(t)=u(t;t_0,u_0)$ *is the maximal solution of IVP for scalar FDE (3.6).*

Proof 3.5: Let the function $m(t)\in C^1([t_0,t_0+\theta],\mathbb{R}_+)$ *be defined by* $m(t)=V(t,x(t))$. *Then, the desired result follows from Lemma 3.4 and Remark 3.2.* ■

3.5. Main Results

We will obtain sufficient conditions for practical stability, boundedness and Lagrange stability by using continuous Lyapunov-like functions from Λ class and Caputo fractional Dini derivative defined by (3.10). Such a stability result is given in [39], we give this result for the sake of completeness. Unlike the result on asymptotically stability [39], we give the same result with comparison principle, i.e. using asymptotically stability of scalar FDE.

Theorem 3.1: Assume that $f(t,0) \equiv 0, t \in \mathbb{R}_+$ *and there exists a function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$, $V(t,0) = 0$ *for* $t \in \mathbb{R}_+$ *such that*

$$b(\|x\|) \leq V(t,x) \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{3.26}$$

and the inequality

$$_{(3.1)}{}^{C}\mathcal{D}_+^q V(t,x;t_0,x_0) \leq G(t,V(t,x)) \tag{3.27}$$

holds for any $t, t_0 \in \mathbb{R}_+$, $t > t_0$ *and* $x, x_0 \in \mathbb{R}^n$, *where* $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $G(t,0) \equiv 0$ *and* $b \in K$. *Then,*

A) the stability of zero solution of scalar FDE (3.6) implies stability of zero solution of system of FDE (3.1).

B) the asymptotically stability of zero solution of scalar FDE (3.6) implies asymptotically stability of zero solution of system of FDE (3.1).

Proof 3.1: A) Let the zero solution of scalar FDE (3.6) be stable. According to Definition 3.4 for given $b(\epsilon) > 0$ *there exists* $\delta_0 = \delta_0(t_0, \epsilon)$ *such that* $|u_0| < \delta_0$ *implies*

$$|u(t;t_0,u_0)| < b(\epsilon) \ \text{ for } t \geq t_0 \tag{3.28}$$

where $u(t;t_0,u_0)$ *is a solution of scalar FDE (3.6). From the properties of the function* $V(t,x)$, *it follows that there exists a* $\delta = \delta(t_0,\epsilon) > 0$ *such that if* $x \in \mathbb{R}^n: \|x\| \leq \delta$ *then* $V(t_0,x) \leq \delta_0$. *Choose an initial position* $x_0 \in \mathbb{R}^n: \|x_0\| < \delta$ *and consider the solution* $x(t) = x(t;t_0,x_0)$ *of system of FDE (3.1) with the chosen initial data* (t_0,x_0). *Now let* $u_0^* = V(t_0,x_0)$. *From Lemma 3.4 with* $\Delta = \mathbb{R}^n$ *and* $\theta = \infty$ *it follows that*

$$V(t, x(t; t_0, x_0)) \leq u(t; t_0, u_0^*) \text{ for } t \geq t_0. \tag{3.29}$$

Then from condition (3.26), inequalities (3.28), (3.29) and $u_0^* = V(t_0, x_0) < \delta_0$ *we get to the inequalities*

$$b(\|x(t; t_0, x_0)\|) \leq V(t, x(t; t_0, x_0)) \leq u(t; t_0, u_0^*) < b(\epsilon) \tag{3.30}$$

from which it follows that $\|x(t; t_0, x_0)\| < \epsilon$ *for* $t \geq t_0$. *This proves the stability of the zero solution of system of FDE (3.1).*

B) Let the zero solution of scalar FDE (3.6) be asymptotically stable. It follows from A) that the zero solution of (3.1) is stable. We need to show that S3) holds. Since S^*3*) holds, for given* $b(\epsilon) > 0$, $t_0 \in \mathbb{R}_+$ *there exists* $\delta^* = \delta^*(t_0, \epsilon)$ *and* $T = T(t_0, \epsilon)$ *such that* $|u_0| < \delta^*$ *implies*

$$|u(t; t_0, u_0)| < b(\epsilon) \text{ for } t \geq t_0 + T. \tag{3.31}$$

From the properties of the function V, *it follows that there exists a* $\delta_0 = \delta_0(t_0, \epsilon)$ *such that if* $x \in \mathbb{R}^n$: $\|x\| \leq \delta_0$ *then* $V(t_0, x) \leq \delta^*$. *Choose an initial position* $x_0 \in \mathbb{R}^n$: $\|x_0\| < \delta_0$ *and consider the solution* $x(t) = x(t; t_0, x_0)$ *of system of FDE (3.1) with the chosen initial data* (t_0, x_0). *Now let* $u_0^* = V(t_0, x_0)$. *Then from condition (3.26), inequalities (3.29), (3.31) and* $u_0^* = V(t_0, x_0) < \delta^*$ *we get to the inequalities for* $t \geq t_0 + T$

$$b(\|x(t; t_0, x_0)\|) \leq V(t, x(t; t_0, x_0)) \leq u(t; t_0, u_0^*) < b(\epsilon) \tag{3.32}$$

from which it follows that $\|x(t; t_0, x_0)\| < \epsilon$ *for* $t \geq t_0 + T$. *This proves asymptotically stability of system of FDE (3.1).* ■

Theorem 3.2: Suppose that the assumptions of Theorem 3.1 hold except that (3.26) is replaced by

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|) \;\; for \;\; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{3.33}$$

If the zero solution of scalar FDE (3.6) is uniformly stable (uniformly asymptotically stable), then zero solution of system of FDE (3.1) is uniformly stable (uniformly asymptotically stable).

Proof 3.2: Since similar arguments are used from the proof of Theorem 3.1 except that choosing $u_0^* = a(\|x_0\|)$ and $\delta = a^{-1}(\delta_0)$, we omit it. ■

Remark 3.5: Let the conditions of Theorem 3.2 be satisfied and the inequality (3.27) is replaced by ${}_{(3.1)}^{\;\;C}\mathcal{D}_+^q V(t,x;t_0,x_0) \leq -c(\|x\|)$. Then zero solution of system of FDE (3.1) is uniformly asymptotically stable.

Proof 3.5: By choosing $V(t,x) = \|x\|$, it follows that the corresponding IVP for scalar FDE (3.6) is ${}_{t_0}^{C}\mathcal{D}^q\, u(t) =- cu(t)$. According to Corollary 3.2 zero solution of scalar FDE is asymptotically stable. Then zero solution of system of FDE (3.1) is uniformly asymptotically stable from Theorem 3.2. ■

Theorem 3.3: Assume that there exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|) \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{3.34}$$

and the inequality (3.33) holds for any $t, t_0 \in \mathbb{R}_+$, $t > t_0$ and $x, x_0 \in \mathbb{R}^n$, where $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $a, b \in K$. Then,

A) the practically stability (uniform practical stability) w.r.t. $(a(\lambda), b(A))$ of scalar FDE (3.6) implies practical stability (uniform practical stability) w.r.t. (λ, A) of system of FDE (3.1) where the positive constants λ, A: $\lambda < A$, $a(\lambda) < b(A)$ are given.

B) the practically quasi-stability (uniform practically quasi-stability) w.r.t. $(a(\lambda), b(A), T)$ *of scalar FDE (3.6) implies practically quasi-stability (uniform practically quasi-stability) w.r.t.* (λ, A, T) *of system of FDE (3.1).*

Proof 3.3: A). Let the scalar FDE (3.6) be practically stable with respect to $(a(\lambda), b(A))$ *with* $0 < \lambda < A$, $a(\lambda) < b(A)$. *According to Definition 3.5 there exists a point* $t_0 \geq 0$ *such that* $|u_0| < a(\lambda)$ *implies*

$$|u(t; t_0, u_0)| < b(A) \ \ for \ \ t \geq t_0 \tag{3.35}$$

where $u(t; t_0, u_0)$ *is a solution of scalar FDE (3.6). Choose an initial position* $x_0 \in \mathbb{R}^n$: $\|x_0\| < \lambda$ *and consider the solution* $x(t) = x(t; t_0, x_0)$ *of system of FDE (3.1) with the chosen initial data* (t_0, x_0). *Now let* $u_0^* = V(t_0, x_0)$. *From the inequality (3.34) it follows* $u_0^* = V(t_0, x_0) < a(\|x_0\|) < a(\lambda)$, *i.e.* $u_0^* < a(\lambda)$. *Therefore the maximal solution* $u^*(t) = u(t; t_0, u_0^*) \in C^q([t_0, \infty), \mathbb{R})$ *of FDE (3.6) satisfies inequality (3.35). From Lemma 3.4 with* $\Delta = \mathbb{R}^n$ *and* $\theta = \infty$ *it follows that*

$$V(t, x(t; t_0, x_0)) \leq u(t; t_0, u_0^*) \ \ for \ \ t \geq t_0. \tag{3.36}$$

Then from condition (3.34) and inequalities (3.35), (3.36) we get to the inequalities

$$b(\|x(t; t_0, x_0)\|) \leq V(t, x(t)) \leq u(t; t_0, u_0^*) < b(A) \tag{3.37}$$

from which it follows that $\|x(t; t_0, x_0)\| < A$ *for* $t \geq t_0$. *Thus, system of FDE (3.1) is practically stable. Similarly, uniform practically stability w.r.t.* (λ, A) *of system of FDE (3.1) can be proved.*

We omit the proof of claims B) since similar arguments from the proof of A) are used. ■

Corollary 3.4: Suppose that the assumptions of Theorem 3.2 and Theorem 3.3 hold except that the inequality (3.27) is replaced by ${}_{(3.1)}^{\ C}\mathcal{D}_+^q V(t, x; t_0, x_0) \leq 0$. *Then the system of FDE (3.1) is uniformly stable and uniformly practically stable w.r.t.* (λ, A).

Proof 3.4: The proof follows directly from the fact that the corresponding scalar FDE ${}_{t_0}^{C}\mathcal{D}^q u(t) = 0$ *has a constant solution which is uniformly stable and uniformly practically stable w.r.t.* $(a(\lambda), b(A))$. ■

Corollary 3.5: Suppose that the assumptions of Theorem 3.2 and Theorem 3.3 hold except that the inequality (3.33) is replaced by ${}_{(3.1)}^{C}\mathcal{D}_{+}^{q}V(t,x;t_0,x_0) \leq -\gamma V(t,x)$, $\gamma > 0$. *Then the system of FDE (3.1) is uniformly asymptotically stable and uniformly practically quasi stable w.r.t.* (λ, A, T).

Proof 3.5: The proof follows directly from the fact that the solution $u(t) = u_0 E_q(-\gamma(t-t_0)^q)$ *of the corresponding scalar FDE* ${}_{t_0}^{C}\mathcal{D}^q u(t) =- \gamma u(t)$, $u(t_0) = u_0$ *is uniformly practically quasi stable w.r.t.* (λ, A, T), *where* $T = (-\frac{1}{\gamma} L_q \left(\frac{b(A)}{a(\lambda)}\right))^{\frac{1}{q}}$. *According to Corollary 3.4 and* $u(t) \to \infty$ *as* $t \to \infty$, *zero solution of corresponding scalar FDE is uniformly asymptotically stable.* ■

Theorem 3.4: Assume that there exists a function $V \in \Lambda$ *such that*

$$b(\|x\|) \leq V(t,x) \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \qquad (3.38)$$

and the inequality (3.26) holds for any $t, t_0 \in \mathbb{R}_+$, $t > t_0$ *and* $x, x_0 \in \mathbb{R}^n$, *where* $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ *and* $b \in K$ *with* $b(r) \to \infty$ *as* $r \to \infty$. *Then,*

A) the equi-boundedness of scalar FDE (3.6) implies equi-boundedness of system of FDE (3.1).

B) the ultimately boundedness of scalar FDE (3.6) implies ultimately boundedness of system of FDE (3.1).

Proof 3.4: Let $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$. *Consider the solution* $x(t) = x(t;t_0,x_0)$ *of system of FDE (3.1) for which* $\|x_0\| \leq \alpha$. *Initially, we consider the first case A) that is the scalar FDE (3.6) is equi-bounded. From the properties of the function* V, *it follows that there exists a constant* $\gamma_1 = \gamma_1(t_0, \alpha)$ *such that if* $x \in \mathbb{R}^n$: $\|x\| \leq \alpha$ *then*

$V(t_0, x) \leq \gamma_1$. Since B^*1) holds, given $\gamma_1 = \gamma_1(t_0, \alpha) > 0$ and $t_0 \in \mathbb{R}_+$ there exists a $\beta_1 = \beta_1(t_0, \gamma_1) > 0$ such that $|u_0| < \gamma_1$ implies

$$|u(t; t_0, u_0)| < \beta_1 \ \text{for} \ t \geq t_0. \tag{3.39}$$

Choose $U_0 = V(t_0, x_0)$ and $\beta = \beta(t_0, \alpha) > 0$ so that $b(\beta) \geq \beta_1$ since for the function $b \in K$ with $b(r) \to \infty$ as $r \to \infty$. Then from condition (3.38), Lemma 3.4 and inequality (3.39) we get to the inequalities

$$b(\|x(t; t_0, x_0)\|) \leq V(t, x(t; t_0, x_0)) \leq u(t; t_0, U_0) < \beta_1 < b(\beta) \tag{3.40}$$

from which it follows that $\|x(t; t_0, x_0)\| < \beta$ for $t \geq t_0$. Thus we deduce that the system of FDE (3.1) is equi-bounded.
It should be noted that the claim B) is proved by the same arguments used in the proof of A), so we omit the details here. ■

Corollary 3.6: Suppose that the assumptions of Theorem 3.4 hold except that (3.38) is replaced by (3.26). If scalar FDE (3.7) is uniformly bounded, then the system of FDE (3.1) is uniformly bounded.

Proof 3.6: Since proofs are essentially repetitions of the arguments used in the proof of Theorem 3.4 except that choosing $u_0^* = a(\|x_0\|)$ and $\gamma_1 = a(\alpha)$, we omit it. ■

Corollary 3.7: Let $G(t, u) = 0$ in Corollary 3.6, i.e. the inequality ${}_{(3.1)}^{\ C}\mathcal{D}_+^q V(t, x; t_0, x_0) \leq 0$ holds. Then the system of FDE (3.1) is uniformly bounded.

Proof 3.7: The proof follows directly from the fact that the corresponding scalar FDE ${}_{t_0}^{C}\mathcal{D}^q\, u = 0$ has a constant solution which is uniformly bounded. ■

The following theorem present the conclusion of Corollary 3.6 with weaker assumption. Let $S^c(\rho)$ denote the set $S^c(\rho) = \{x \in \mathbb{R}^n : \|x\| \geq \rho\}$.

Theorem 3.5: Assume that there exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|) \;\; for\; (t,x) \in \mathbb{R}_+ \times S^c(\rho) \tag{3.41}$$

and the inequality (3.27) holds for any $t, t_0 \in \mathbb{R}_+$, $t > t_0$ and $x, x_0 \in \mathbb{R}^n$, where $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $a, b \in K$.

A) the uniformly boundedness of scalar FDE (3.6) implies uniformly boundedness of system of FDE (3.1).

B) the uniformly ultimately boundedness of scalar FDE (3.6) implies uniformly ultimately boundedness of system of FDE (3.1).

Proof 3.5: At first, we consider the first case A) that is the scalar FDE (3.6) is uniformly bounded . Let $\alpha > 0$ be given, it can be considered as $\alpha \geq \rho$. Consider the solution $x(t) = x(t; t_0, x_0)$ of system of FDE (3.1) for which $\|x_0\| \leq \alpha$. Since $B^*1)$ holds, given $\alpha_1 = a(\alpha) > 0$ there exist $\beta_1 = \beta_1(\alpha) > 0$ such that

$$|u_0| \leq \alpha_1 \;\; implies \;\; u(t; t_0, u_0) < \beta_1 \,,\;\; t \geq t_0. \tag{3.42}$$

We shall prove that $\|x(t; t_0, x_0)\| < \beta$ for $t \geq t_0$, where $\beta = \beta(\alpha) > 0$ verifying $b(\beta) \geq \beta_1$ since $b(r) \to \infty$ as $r \to \infty$. Assume that this is not true. Therefore, there would exist points $t_1 > t^* > t_0$ such that

$$\begin{cases} \|x(t^*, t_0, x_0)\| = \alpha, \qquad \|x(t_1, t_0, x_0)\| = \beta \; and \\ \rho < \alpha \leq \|x(t; t_0, x_0)\| \leq \beta \quad for \qquad t^* \leq t \leq t_1. \end{cases} \tag{3.43}$$

By using (3.43), condition (3.41) and Lemma 3.4 we obtain the following estimate

$$V(t, \bar{x}(t; t^*, x_0^*)) \leq u(t; t^*, u_0), \qquad t^* \leq t \leq t_1 \tag{3.44}$$

where $u_0 = V(t^*, x_0^*)$ *with* $x_0^* = x(t^*, t_0, x_0)$ *and* $\bar{x}(t, t^*, x_0^*)$ *is a solution of (3.1) through* (t^*, x_0^*)*. On the other hand, (3.44) is also valid for* $x(t; t_0, x_0)$ *because of the inclusion* $(t, x(t)) \in S^c(\rho)$ *in the interval* $t \in [t^*, t_1]$*. Finally, in view of the relations (3.42) - (3.44), condition (3.41) and* $u_0 = V(t^*, x_0^*) \le a(\|x_0^*\|) = a(\alpha) = \alpha_1$ *we obtain*

$$b(\beta) = b(\|x(t_1, t_0, x_0)\|) \le V\big(t_1, x(t_1, t_0, x_0)\big) \le u(t_1, t^*, u_0) < \beta_1 \le b(\beta). \tag{3.45}$$

The obtained contradiction shows that the claim is right, namely the system of FDE (3.1) is uniformly bounded. For $\alpha < \rho$*, we can choose* $\beta = \beta(\alpha) = \beta(\rho)$ *which also implies B2). This completes the proof of A).*

Next we consider the second case B) that is the scalar FDE (3.6) is uniformly ultimately bounded, which follows from A) that system of FDE (3.1) is uniformly bounded. Namely, there exist numbers B_0 *for* $\alpha = \rho$ *such that* $\|x_0\| \le \rho$ *implies* $\|x(t, t_0, x_0)\| < B_0$ *for* $t \ge t_0$*. In order to prove B4), let* $\alpha > \rho$ *be such that* $\rho \le \|x_0\| \le \alpha$*. Since* B^*4*) holds i.e. for* $\alpha_1 = a(\alpha) > 0$ *there exist positive numbers* N_1 *and* $T = T(\alpha)$ *such that*

$$|u_0| \le \alpha_1 \text{ implies } |u(t, t_0, u_0)| < N_1\,,\ t \ge t_0 + T. \tag{3.46}$$

Now we will prove that B6) holds with T *and* N^**, where* $N^* = max(B_0, N)$ *and* $b(N) \ge N_1$ *from* $b \in K_\infty$*. Suppose that is not true. Therefore, there exist a sequence* $\{t^{(n)}\}$*,* $t^{(n)} > t_0 + T$*,* $t^{(n)} \to \infty$ *as* $n \to \infty$ *and a solution* $x(t, t_0, x_0)$ *of (3.1) with* $\rho \le \|x_0\| \le \alpha$ *such that*

$$\big\|x\big(t^{(n)}, t_0, x_0\big)\big\| \ge N^*. \tag{3.47}$$

Here it should be note that the solutions $x(t, t_0, x_0)$ *with* $\|x_0\| \le \rho$ *satisfy* $\|x(t, t_0, x_0)\| < N^*$ *from the choice of* N^* *above. Finally, in view of the relations (3.46), (3.47), condition (3.41), the choice of* $\{t^{(n)}\}$ *and* $u_0 = a(\|x_0\|) \le a(\alpha) = \alpha_1$ *we obtain*

$$b(N^*) \leq V(t^{(n)}, x(t^{(n)}, t_0, x_0)) \leq u(t^{(n)}, t_0, u_0) < N_1 \leq b(N) \tag{3.48}$$

which contradicts with the choice of N^*. Thus B4) holds, i.e. the system of FDE (3.1) is uniformly ultimately bounded. ■

Theorem 3.6: Assume that there exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$b(\|x\|) \leq V(t, x) \text{ for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{3.49}$$

and the inequality (3.33) holds for any $t, t_0 \in \mathbb{R}_+$, $t > t_0$ and $x, x_0 \in \mathbb{R}^n$, where $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $b \in K$ with $b(r) \to \infty$ as $r \to \infty$.

If scalar FDE (3.6) is Lagrange stable, then the system of FDE (3.1) is Lagrange stable.

Proof 3.6: Let the scalar FDE (3.6) is Lagrange stable which implies that system of FDE (3.1) is equi-bounded by applying Theorem 3.4. In order to prove L1) holds, we need to show A1) holds. Let $\epsilon > 0$, $\alpha > 0$ be given. Consider the solution $x(t) = x(t; t_0, x_0)$ of system of FDE (3.1) for which $\|x_0\| \leq \alpha$. From the properties of the function V, it follows that there exists a constant $\gamma_1 = \gamma_1(t_0, \alpha)$ such that if $x \in \mathbb{R}^n$: $\|x\| \leq \alpha$ then $V(t_0, x) \leq \gamma_1$. Since $A^*1)$ holds i.e. for $b(\epsilon) > 0$ and $\gamma_1 = \gamma_1(t_0, \alpha)$ there exists a $T = T(t_0, \epsilon, \alpha)$ such that $|u_0| \leq \gamma_1$ implies

$$|u(t, t_0, u_0)| < b(\epsilon), \qquad t \geq t_0 + T. \tag{3.50}$$

Now we claim that A1) holds, i.e. $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0 + T$ provided that $\|x_0\| \leq \alpha$. In the sequel from (3.50), Lemma 3.4 and choosing $U_0 = V(t_0, x_0) \leq \gamma_1$ we obtain for $t \geq t_0 + T$

$$b(\|x(t; t_0, x_0)\|) \leq V(t, x(t; t_0, x_0)) \leq u(t; t_0, U_0) < b(\epsilon) \tag{3.51}$$

from which it follows that $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0 + T$ whenever $\|x_0\| \leq \alpha$. ■

Corollary 3.7: Suppose that the assumptions of Theorem 3.6 hold except that (3.49) is replaced by (3.33). If scalar FDE (3.6) is uniformly Lagrange stable, then the system of FDE (3.1) is uniformly Lagrange stable.

Proof 3.7: Since proofs are essentially repetitions of the arguments used in the proof of Theorem 3.6 except that choosing $U_0 = a(\|x_0\|)$ and $\gamma_1 = a(\alpha)$, we omit it. ■

3.5. Applications

We consider the following examples as an application of our main results.

Example 3.6: Consider the following FDE with order $0 < q < 1$,

$$ {}_{t_0}^{C}\mathcal{D}^q x(t) = \left(-\frac{e^{1/t}}{2\Gamma(1-q)} - sin^2(t) \right) x(t) \tag{3.52} $$

for $t > t_0$ with $x(t_0) = x_0$. Consider $V(t,x) = x^2$ and choose $a, b \in K$ such that $a(s) = 2s$, $b(s) = \frac{1}{2}s$ for the validity of the condition (3.34). From the obtained formula (3.13) in Corollary 3.1 we have

$$ \begin{aligned} &{}_{(3.52)}^{\quad C}\mathcal{D}_+^q V(t, x; t_0, x_0) \\ &\quad = 2x\left(-\frac{e^{1/t}}{2\Gamma(1-q)}x - sin^2(t)x \right) + [x^2 - x_0^2]\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \\ &\qquad \le \left(-\frac{e^{1/t}}{\Gamma(1-q)}x^2 - 2sin^2(t)x^2 \right) + \frac{e^{1/t}}{\Gamma(1-q)}x^2 \le 0. \end{aligned} \tag{3.53} $$

Then, we have following results:

- *the zero solution of FDE (3.52) is uniformly stable according to Corollary 3.4.*
- *the FDE (3.52) is uniformly practically stable with respect to (λ, A): $0 < \lambda < A$ according to Corollary 3.4.*
- *the FDE (3.52) is uniformly bounded according to Corollary 3.6.*

Figure 3.2 shows that the approximate solutions of the FDE (3.52) with $q = 0.5$, $q = 0.8$, $q = 0.9$ *and* $u(1) = 5$ *as expected from the analytical analysis already presented in Example 3.6.*

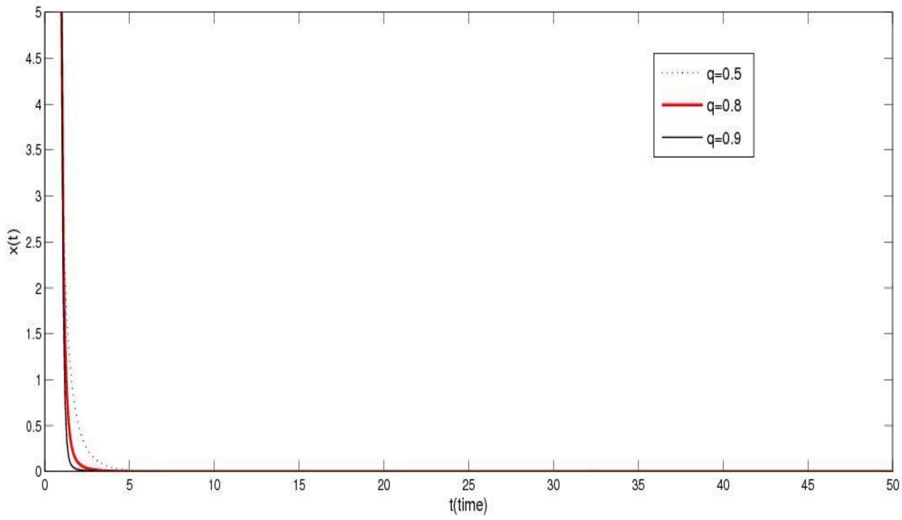

Figure 3.2: Approximate solutions with $q = 0.5, 0.8$ and $q = 0.9$.

Example 3.7: Consider the following system of FDE with $0 < q < 1$

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x_1(t) = -\left(1 + \dfrac{e^{1/t}}{2\Gamma(1-q)}\right) x_1(t) - sin^2(t) x_2(t) \\ {}_{t_0}^{C}\mathcal{D}^q\, x_2(t) = -\left(1 + \dfrac{e^{1/t}}{2\Gamma(1-q)}\right) x_2(t) + sin^2(t) x_1(t) \end{cases} \tag{3.54}$$

with initial condition $x_0 = (x_{01}, x_{02}) \in \mathbb{R}^2$ *where* $x_1(t_0) = x_{01}$ *and* $x_2(t_0) = x_{02}$. *Consider* $V(t, x) = x_1^2 + x_2^2$ *for* $x = (x_1, x_2) \in \mathbb{R}^2$ *and choose* $a, b \in K$ *such that* $a(s) = 2s$, $b(s) = \frac{1}{2}s$ *for the validity of the condition (3.34). According to formula (3.13) in Corollary 3.1, we have*

$$
\begin{cases}
{}_{(3.54)}^{\;C}\mathcal{D}_{+}^{q}V(t,x;t_0,x_0) = 2x_1\left(-(1+\frac{e^{1/t}}{2\Gamma(1-q)}x_1) - sin^2(t)x_2\right) \\
\qquad +2x_2\left(-(1+\frac{e^{1/t}}{2\Gamma(1-q)}x_2) + sin^2(t)x_1\right) \\
\qquad +[x_1^2+x_2^2-x_{01}^2-x_{02}^2]\,\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \\
\le -(x_1^2+x_2^2)\left(2+\frac{e^{1/t}}{\Gamma(1-q)}\right) + \frac{e^{1/t}}{\Gamma(1-q)}(x_1^2+x_2^2) \\
\qquad = -2(x_1^2+x_2^2) = -2V(t,x)
\end{cases}
\tag{3.55}
$$

According to (3.55) the corresponding IVP for scalar FDE is

$$
\begin{cases}
{}_{t_0}^{C}\mathcal{D}^{q}\,u(t) = -2u(t) \\
\quad u(t_0) = u_0
\end{cases}
\tag{3.56}
$$

where $u_0 \in \mathbb{R}$. *The solution of IVP (3.56) is given by* $u(t) = u_0 E_q(-2(t-t_0)^q)$. *Then, we have following results*:

- *in view of Corollary 3.5 the zero solution of system of FDE (3.54) is uniformly asymptotically stable.*
- *scalar FDE (3.56) is uniformly practically stable w.r.t.* $\left(2\lambda, \frac{A}{2}\right): 0 < \lambda < \frac{A}{4}$ *and uniformly practical quasi-stable w.r.t.* $(2\lambda, \frac{A}{2}, T)$, *where* $T = (-\frac{1}{2}L_q(\frac{A}{4\lambda}))^{\frac{1}{q}}$. *According to Theorem 3.3 and Corollary 3.5, the system of FDE (3.54) is uniformly practically stable w.r.t.* (λ, A) *and uniformly practically quasi stable w.r.t* (λ, A, T), *respectively.*
- *scalar FDE (3.56) is uniformly bounded with the choice of* $\beta = \alpha$ *and uniformly attractive in the large with* $T = (-\frac{1}{2}L_q(\frac{\epsilon}{4\alpha}))^{\frac{1}{q}}$. *According to Corollary 3.7 the system of FDE (3.54) is uniformly Lagrange stable.*

4. SOME STABILITY AND BOUNDEDNESS CRITERIA FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCE

4.1. Introduction

The problem of stability of solutions is one of the major problems in the theory of differential equations. It is well known that the theory of stability in the sense of Lyapunov has been successfully investigated to understand qualitative properties of dynamic systems for many years [1]-[5]. In nonlinear systems, Lyapunov's direct method (also called the Lyapunov's second method) allows us to obtain sufficient conditions for the stability of a system without explicitly solving the differential equations. The method generalizes the idea which shows that the system is stable if there are some Lyapunov functions candidates for the system.

It is only a few decades ago, it was realized that fractional calculus provide an attractive tool for modelling the real world problems. The differentiation and integration of arbitrary order has found its applications in diverse fields of science and engineering [11]-[21]. Therefore, the qualitative theory of FDE has received much attention by many researchers.

Recently, fractional calculus was introduced to the stability analysis of FDE. Some studies on the stability of linear FDE have been done in [29]-[31]. Whereas, the stability analysis of the nonlinear FDE is much more difficult and only a few are available. In the based on Lyapunov's second method, sufficient conditions on stability for nonlinear FDE and nonlinear time-delayed FDE has been discussed in [32]-[38]. Among them in [32]-[33], the authors proposed fractional Lyapunov's second method and firstly extended the exponential stability of ODE to the Mittag-Leffler stability and generalized Mittag-Leffler stability of FDE, respectively. The authors in [35]-[36] have applied the fractional comparison principle to discussing the asymptotic stability and Mittag-Leffler stability of FDE with RL derivative, respectively. Very recently, a stability criterion for autonomous and non-autonomous nonlinear FDE with Caputo derivative is derived in [37]-[38], respectively. In these foregoing works, stability of FDE is studied with changing initial position but initial time unchanged. In practical situations, it is possible to have not only a change in initial position but also in initial time because of all kinds of disturbed factors. When we consider such a deviation in initial time, it causes measuring the difference between any two different solutions starting with different initial time [42]-[43]. In this context, we have investigated stability, practical stability, boundedness and Lagrange stability with ITD for nonlinear FDE by using fractional extension of comparison principle relative to ITD. In section 4.2, main definitions and concepts with ITD for FDE with Caputo derivative are introduced and the differences between classical notion of stability and the notion of stability with ITD are discussed, respectively. Then, in section 4.3 natural relationship between the Dini derivative of Lyapunov function with ITD for classical case ($q = 1$) and the introduced Caputo fractional Dini derivative with ITD is shown by appropriate examples. Then, comparison results relative to ITD are obtained in section 4.4. Comparison method via Lyapunov function and scalar FDE with parameter is applied to obtain several sufficient conditions on stability, practical stability, boundedness and Lagrange stability with ITD for system of FDE in section 4.5. Finally some examples are given as an application of the obtained results.

4.2. Statement of the Problem

4.2.1. Main Definitions and Concepts with ITD for FDE

Consider the following IVP for the system of FDE with order $0 < q < 1$,

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q x(t) = f(t, x(t)) \\ \qquad x(t_0) = x_0 \end{cases} \tag{4.1}$$

where $t_0 \in \mathbb{R}_+,\ x,\ x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$. Denote the solution of (4.1) by $x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$. Let $\tau_0 \in \mathbb{R}_+$, $\tau_0 \neq t_0$ be a different initial time.

Consider also the IVP (4.1) at a different initial data, i.e. $y(\tau_0) = y_0$

$$\begin{cases} {}_{\tau_0}^{C}\mathcal{D}^q\, y(t) = f(t, y(t)) \\ \qquad y(\tau_0) = y_0 \end{cases} \tag{4.2}$$

where $y,\ y_0\ \in \mathbb{R}^n$. Denote the solution of (4.2) by $y(t; \tau_0, y_0) \in C^q([\tau_0, \infty), \mathbb{R}^n)$.

We give a lemma which is necessary for future use [40].

Lemma 4.1: Let the function $x(t) \in C^q(\mathbb{R}_+, \mathbb{R}^n)$ be solution of the following IVP for FDE

$$\begin{cases} {}_{a}^{C}\mathcal{D}^q\, x(t) = f(t, x(t)) \\ \qquad x(a) = x_0 \end{cases} \tag{4.3}$$

Then the function $\tilde{x}(t) = x(t + \eta)$ satisfies the following IVP for the FDE

$$\begin{cases} {}_{b}^{C}\mathcal{D}^q\, \tilde{x}(t) = f(t + \eta, \tilde{x}(t)) \\ \qquad \tilde{x}(b) = x_0 \end{cases} \tag{4.4}$$

where $a, b \in \mathbb{R}, \eta = a - b$.

Proof 4.1: The function $x(t)$ is a solution of (4.3) if it satisfies the Volterra integral equation [63]

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_a^t \frac{f\big(s, x(s)\big)ds}{(t - s)^{1-q}} \quad , t \geq a. \tag{4.5}$$

The function $\tilde{x}(t)$ satisfies the initial condition of (4.4), i.e. $\tilde{x}(b) = x_0$. Change the variable in the integral (4.5) with $s = \xi + \eta$. Then $ds = d\xi$, $\xi = b$ for $s = a$ and $\xi = t - \eta$ for $s = t$. Therefore, from (4.5) we obtain

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_b^{t-\eta} \frac{f(\xi + \eta, x(\xi + \eta))d\xi}{(t - \eta - \xi)^{1-q}} \tag{4.6}$$

or

$$\tilde{x}(t) = x(t + \eta) = x_0 + \frac{1}{\Gamma(q)} \int_b^{t} \frac{f(\xi + \eta, \tilde{x}(\xi))d\xi}{(t - \xi)^{1-q}} \quad . \tag{4.7}$$

Therefore $\tilde{x}(t)$ satisfies (4.7), i.e. $\tilde{x}(t) = x(t + \eta)$ is a solution of IVP (4.4). ■

The relation between (4.1) and (4.2) is given by the following result.

Corollary 4.1: If $y(t) = y(t; \tau_0, y_0)$ is a solution of (4.2), then$\tilde{y}(t) = y(t + \eta)$ is a solution of IVP for FDE

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q \, \tilde{y}(t) = f(t + \eta, \tilde{y}(t)) \\ \qquad \tilde{y}(t_0) = y_0 \end{cases} \tag{4.8}$$

where $\eta = \tau_0 - t_0$.

The main goal of the present chapter is studying the stability, practical stability [40], boundedness and Lagrange stability with ITD of the system of Caputo FDE (4.1), i.e. comparing the behavior of two solution with different initial data, both initial time $\tau_0 \neq t_0$ and initial position $y_0 \neq x_0$.

We shall introduce the following definitions of stability and practical stability with ITD.

Definition 4.1: The solution $x^*(t) = x(t; t_0, x_0)$ of (4.1) is said to be:

S1) stable with ITD *if given* $\epsilon > 0$ there exist $\delta = \delta(t_0, \epsilon) > 0$ and $\sigma = \sigma(t_0, \epsilon) > 0$ *such that for any initial position $y_0 \in \mathbb{R}^n$ and any initial time $\tau_0 \in \mathbb{R}_+$ the* *inequalities* $\|y_0 - x_0\| < \delta$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t+\eta;\tau_0,y_0) - x^*(t)\| < \epsilon$ *for* $t \geq t_0$, *where* $y(t) = y(t;\tau_0,y_0)$ *is a solution of (4.2) and* $\eta = \tau_0 - t_0$;

S2) attractive with ITD *if for given* $\epsilon > 0$ *there exist* $\delta_0 = \delta_0(t_0) > 0$, $\sigma_0 = \sigma_0(t_0) > 0$ *and* $T = T(t_0,\epsilon) > 0$ *such that if* $\|y_0 - x_0\| < \delta_0$ *and* $|\tau_0 - t_0| < \sigma_0$ *imply* $\|y(t+\eta;\tau_0,y_0) - x^*(t)\| < \epsilon$ *for* $t \geq t_0 + T$;

S3) asymptotically stable with ITD, *if S1) and S2) hold simultaneously*;

PS1) practically stable with ITD *w.r.t.* (λ, A), *if there exists a number* $\sigma = \sigma(t_0,\lambda,A) > 0$ *such that for any initial position* $y_0 \in \mathbb{R}^n$ *and any initial time* $\tau_0 \in \mathbb{R}_+$ *the inequalities* $\|y_0 - x_0\| < \lambda$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t+\eta;\tau_0,y_0) - x^*(t)\| < \mathrm{A}$ *for* $t \geq t_0$, *where a couple of real numbers* (λ, A) *with* $0 < \lambda < A$ *be given;*

PS2) attractive practically stable with ITD *w.r.t.* (λ, A, T) *if there exist* $\sigma = \sigma(t_0,\lambda,A) > 0$ *and* $T = T(t_0,\lambda,A) > 0$ *such that the inequalities* $\|y_0 - x_0\| < \lambda$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t+\eta;\tau_0,y_0) - x^*(t)\| < A$ *for* $t \geq t_0 + T$;

Definition 4.2: The system of FDE of (4.1) is said to be:

US1) uniformly stable with ITD, *if given* $\epsilon > 0$ *there exist* $\delta = \delta(\epsilon) > 0$ *and* $\sigma = \sigma(\epsilon) > 0$ *such that for any initial positions* $x_0, y_0 \in \mathbb{R}^n$ *and any initial times* $\tau_0, t_0 \in \mathbb{R}_+$ *the inequalities* $\|y_0 - x_0\| < \delta$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t+\eta;\tau_0,y_0) - x(t;t_0,x_0)\| < \epsilon$ *for* $t \geq t_0$, *where* $x(t;t_0,x_0)$, $y(t;\tau_0,y_0)$ *are solutions of (4.1), (4.2), respectively;*

US2) uniformly attractive with ITD, *if for given* $\epsilon > 0$ *there exist* $\delta_0 > 0$, $\sigma_0 > 0$ *and* $T = T(\epsilon) > 0$ *such that if* $\|y_0 - x_0\| < \delta_0$ *and* $|\tau_0 - t_0| < \sigma_0$ *imply* $\|y(t+\eta;\tau_0,y_0) - x(t;t_0,x_0)\| < \epsilon$ *for* $t \geq t_0 + T$;

US3) uniformly asymptotically stable with ITD, *if it is uniformly stable and uniformly attractive;*

UPS1) uniformly practically stable with ITD *w.r.t.* (λ, A), *if there exists a number* $\sigma = \sigma(\lambda, A) > 0$ *such that for any initial positions* $x_0, y_0 \in \mathbb{R}^n$ *and any initial times* $t_0, \tau_0 \in \mathbb{R}_+$ *the inequalities* $\|y_0 - x_0\| < \lambda$ and $|\tau_0 - t_0| < \sigma$ imply $\|y(t+\eta;\tau_0,y_0) - x(t;t_0,x_0)\| < \mathrm{A}$ *for* $t \geq t_0$;

UPS2) uniformly attractive practically stable with ITD *w.r.t.* (λ, A, T) *if there exist* $\sigma = \sigma(\lambda, A) > 0$ *and* $T = T(\lambda, A) > 0$ *such that for any initial position* $x_0, y_0 \in \mathbb{R}^n$

and any initial time $t_0, \tau_0 \in \mathbb{R}_+$ *the inequalities* $\|y_0 - x_0\| < \lambda$ and $|\tau_0 - t_0| < \sigma$ imply $\|y(t+\eta; \tau_0, y_0) - x(t; t_0, x_0)\| < A$ *for* $t \geq t_0 + T$;

Remark 4.1: The concept of stability, asymptotic stability (practical stability) with ITD generalizes stability, asymptotic stability of the zero solution [39] (practical stability of (4.1)) in the literature if $x^*(t) \equiv 0$ *and* $\tau_0 = t_0$.

Corresponding to the different types of stability with ITD we can define the concepts of boundedness and Lagrange stability with ITD.

Definition 4.3: The system of FDE (4.1) is said to be:

B1) equi-bounded with ITD *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exist* $\sigma = \sigma(t_0, \alpha) > 0$ *and* $\beta = \beta(t_0, \alpha) > 0$ *such that for any initial position* $x_0, y_0 \in \mathbb{R}^n$ *and any initial times* $t_0, \tau_0 \in \mathbb{R}_+$ *the inequalities* $\|y_0 - x_0\| \leq \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t+\eta; \tau_0, y_0) - x(t; t_0, x_0)\| < \beta, t \geq t_0$*;*
B2) uniformly bounded with ITD *if B1) holds with* β *and* σ *independent of* t_0*;*
B3) ultimately bounded with ITD *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exist* $N > 0$, $\sigma_0 = \sigma_0(t_0, \alpha) > 0$ *and* $T = T(t_0, \alpha) > 0$ *such that for any initial positions* $x_0, y_0 \in \mathbb{R}^n$ *and any initial times* $t_0, \tau_0 \in \mathbb{R}_+$ *the inequalities* $\|y_0 - x_0\| \leq \alpha$ *and* $|\tau_0 - t_0| < \sigma_0$ *imply* $\|y(t+\eta; \tau_0, y_0) - x(t; t_0, x_0)\| < N, \ t \geq t_0 + T$*;*
B4) uniformly ultimately bounded with ITD, *if B3) holds with* σ_0 *and* T *independent of* t_0*;*
A1) attractive in the large *if for each* $\epsilon > 0$ *and each* $\alpha > 0$ *there exist* $\sigma = \sigma(t_0, \epsilon, \alpha,) > 0$ *and* $T = T(t_0, \epsilon, \alpha,) > 0$ *such that* $\|y_0 - x_0\| < \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *implies* $\|y(t+\eta; \tau_0, y_0) - x(t; t_0, x_0)\| < \epsilon$ *for* $t \geq t_0 + T$;
A2) uniformly attractive with ITD, *if* σ *and* T *in A1) are independent of* t_0*;*
L1) Lagrange stable *if B1) and A1) hold together;*
L2) uniformly Lagrange stable *if B2) and A2) hold simultaneously.*

We will illustrate the introduced concepts in the following example.

Example 4.1: Consider the following IVP for FDE with $0 < q < 1$,

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x(t) = -x(t) + h(t) \\ \qquad x(t_0) = x_0 \end{cases} \tag{4.9}$$

where $h(t) \in C(\mathbb{R}_+, \mathbb{R})$ *and satisfy Lipschitz condition with constant* $L > 0$.

The solution $x(t) = x(t; t_0, x_0)$ *of (4.9) satisfy* $x(t) = x_0 E_q(-(t-t_0)^q) + \int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-(t-\tau)^q) h(\tau) d\tau$. *Let* $y(t) = y(t; \tau_0, y_0)$ *be another solution of IVP (4.9) at a different data* (τ_0, y_0), $\tau_0 \neq t_0$. *According to Corollary 4.1* $\tilde{y}(t) = y(t+\eta)$ *satisfy* ${}_{t_0}^{C}\mathcal{D}^q\, \tilde{y}(t) = -\tilde{y}(t) + h(t+\eta)$, $\tilde{y}(t_0) = y_0$. *Now consider* $z(t) = \tilde{y}(t) - x(t)$ *which satisfy* ${}_{t_0}^{C}\mathcal{D}^q\, z(t) = -z(t) + h(t+\eta) - h(t)$, $z(t_0) = y_0 - x_0$. *Then difference of solutions satisfy* $\tilde{y}(t) - x(t) = (y_0 - x_0) E_q(-(t-t_0)^q) + \int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-(t-\tau)^q)[h(\tau+\eta) - h(\tau)] d\tau$. *Then we have the following estimate* $|\tilde{y}(t) - x(t)| \leq |y_0 - x_0| + L|\eta|$ *for* $t \geq t_0$. *Then,*

- *(US1) is satisfied with* $\delta = \frac{\epsilon}{2}$ *and* $\sigma = \frac{\epsilon}{2L}$.
- *(UPS1) is satisfied with* $\sigma = \frac{A-\lambda}{L}$.
- *(B2) is satisfied with* $\beta = 2\alpha$ *and* $\sigma = \frac{\alpha}{L}$.

Figure 4.1 shows that the approximate solutions $x(t)$, $\tilde{y}(t) = y(t+\eta)$ *with* $h(t) = \sin(t)$, $t_0 = 0$, $\tau_0 = 0.2$, $x_0 = 2$, $y_0 = 2.3$ *and* $L = 1$.

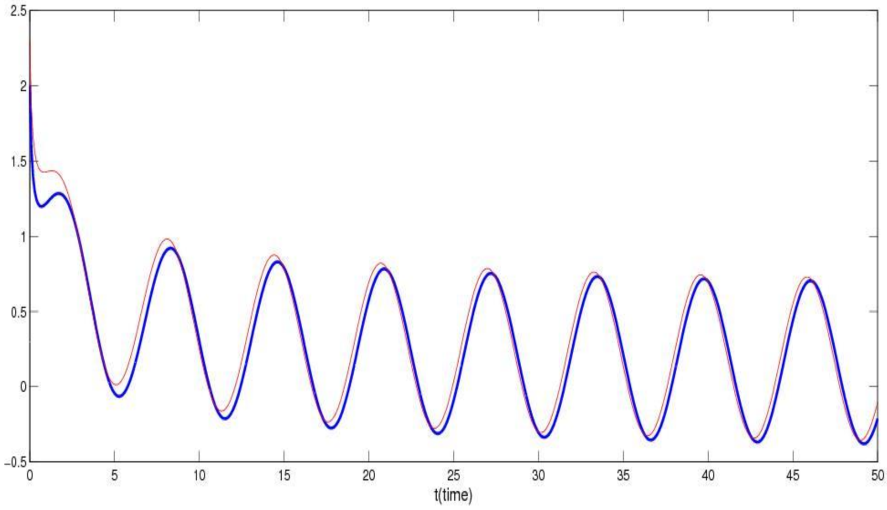

Figure 4.1: Approximate solutions with $h(t) = \sin(t)$, $t_0 = 0$, $\tau_0 = 0.2$, $x_0 = 2$, $y_0 = 2.3$ and $L = 1$.

In order to employ the fractional order extension of Lyapunov method with ITD, following scalar FDE is used

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q u(t) = g(t, u(t), \eta) \\ \qquad u(t_0) = u_0 \end{cases} \tag{4.10}$$

where $u, u_0 \in \mathbb{R}$, $g \in C[\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ and $\eta \in \mathbb{R}$ is a parameter. We denote the solution of the IVP for the scalar FDE (4.10) by $u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$.

Corresponding to the stability with ITD notions given above, we need to introduce necessary definitions of stability with respect to parameter for the scalar FDE (4.10). When (4.10) has a zero solution, i.e. $g(t, 0,0) \equiv 0$ we shall use the following stability definition.

Definition 4.4: The zero solution of scalar FDE (4.10) is said to be

*S*1)* stable with respect to parameter *if given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta = \delta(t_0, \epsilon) > 0$ and $\sigma = \sigma(t_0, \epsilon)$ such that the inequalities $|u_0| < \delta$ and $|\eta| < \sigma$ imply $|u(t; t_0, u_0, \eta)| < \epsilon$ for $t \geq t_0$, where $u(t; t_0, u_0, \eta)$ is a solution of (3.2);*
*S*2)* uniformly stable w.r.t. parameter *if $S^*1)$ holds with δ and σ independent of t_0;*

*S*3)* attractive w.r.t. parameter, *if given* $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exist* $\delta_0 = \delta_0(t_0) > 0$, $\sigma_0 = \sigma_0(t_0) > 0$ *and* $T = T(t_0, \epsilon) > 0$ *such that the inequalities* $|u_0| < \delta$ *and* $|\eta| < \sigma$ *imply* $|u(t; t_0, u_0, \eta)| < \epsilon$ *for* $t \geq t_0 + T$*;*

*S*4)* uniformly attractive w.r.t. parameter if δ_0, σ_0 *and* T *in* $S^*3)$ *is independent of* t_0*;*

*S*5)* asymptotically stable w.r.t. parameter *if it is stable and attractive;*

*S*6)* uniformly asymptotically stable w.r.t. parameter*, if it is uniformly stable and uniformly attractive;*

Definition 4.5: Let a couple of real numbers (λ, A) *with* $0 < \lambda < A$ *be given. The scalar FDE (4.10) is said to be*

*PS*1)* parametrically practically stable w.r.t. (λ, A), *if there exist* $\sigma = \sigma(t_0, \lambda, A) > 0$ *such that for any* $|\eta| < \sigma$ *the inequality* $|u_0| < \lambda$ *imply* $|u(t; t_0, u_0, \eta)| < A$ *for* $t \geq t_0$.

We note that the definitions PS*2) - PS*4) can be formulated similarly. In order to avoid repetition we omit it. To the different types of stability with ITD defined in Definition 4.4, there correspond different types of boundedness.

Definition 4.6: The scalar FDE (4.10) is said to be

*B*1)* bounded w.r.t. parameter *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$ *there exist* $\beta = \beta(t_0, \alpha) > 0$ *and* $\sigma = \sigma(t_0, \alpha)$ *such that the inequality inequalities* $|u_0| \leq \alpha$ *and* $|\eta| < \sigma$ *imply* $|u(t; t_0, u_0, \eta)| < \beta$ *for* $t \geq t_0$*;*

*B*2)* uniformly bounded w.r.t. parameter *if* $B^*1)$ *holds with* β *independent of* t_0*;*

*B*3)* ultimately bounded w.r.t. parameter *if* $B^*1)$ *holds and given* $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exist* N *and* $T = T(t_0, \alpha) > 0$ *such that the inequality* $|u_0| \leq \alpha$ *implies* $|u(t; t_0, u_0, \eta)| < N$, $t \geq t_0 + T$;

*B*4)* uniformly ultimately bounded w.r.t. parameter *if* $B^*2)$ *and* $B^*3)$ *hold with* T *in* $B^*3)$ *is independent of* t_0*;*

*A*1)* attractive in the large w.r.t. parameter *if for given* $\epsilon > 0, \alpha > 0$ *there exist* $\sigma = \sigma(t_0, \epsilon, \alpha)$ *and* $T = T(t_0, \epsilon, \alpha)$ *such that* $|u_0| \le \alpha$ *and* $|\eta| < \sigma$ *imply* $|u(t; t_0, u_0, \eta)| < \epsilon, t \ge t_0 + T$;

*A*2)* uniformly attractive in the large w.r.t. parameter if T and σ in *A*1)* is independent of t_0;

*L*1)* Lagrange stable w.r.t. parameter if *B*1)* and *A*1)* hold together;

*L*2)* uniformly Lagrange stable w.r.t. parameter if *B*2)* and *A*2)* hold simultaneously.

Example 4.2: Consider the scalar FDE (4.10) with $g(t, u(t), \eta) = -\mu u(t) + C\eta$,

$$ {}_{t_0}^{C}\mathcal{D}^q\, u(t) = -\mu u(t) + C\eta \tag{4.11} $$

where C *and* $\mu > 0$ *are constants,* η *is a parameter. The equation (4.11) with initial condition* $u(t_0) = u_0$ *has the following solution*

$$ u(t) = u_0 E_q(-\mu(t - t_0)^q) + \int_{t_0}^{t} (t - \tau)^{q-1} E_{q,q}(-\mu(t - \tau)^q) C\eta d\tau. \tag{4.12} $$

From (4.12) and Lemma 3.1 we obtain the following estimate

$$ |u(t; t_0, u_0, \eta)| \le |u_0| + \frac{1}{\mu}|C\eta|, \quad t \ge t_0. \tag{4.13} $$

Then,

- $S^*2)$ *is satisfied with* $\delta = \frac{\epsilon}{2}$ *and* $\sigma = \frac{\epsilon\mu}{2|C|}$.
- $PS^*2)$ *is satisfied with* $\sigma = \frac{\mu(A-\lambda)}{|C|}$, *i.e.* $|u_0| < \lambda$ *and* $|\eta| < \sigma$ *imply* $|u(t; t_0, u_0, \eta)| < A$ *for* $t \ge t_0$, *where* (λ, A) *is given with* $0 < \lambda < A$.
- $B^*2)$ *is satisfied with* $\beta = 2\alpha$ *and* $\sigma = \frac{\alpha\mu}{|C|}$.

Remark 4.2: For the concepts PS^*1)- B^*4) to hold, the assumption $g(t,0,0) \equiv 0$ is not necessary.

Example 4.3: To motivate the idea, consider the following IVP for scalar FDE

$$ {}_{t_0}^{C}\mathcal{D}^q u(t) = -u(t) + \eta sint + cost, \qquad u(t_0) = u_0 \tag{4.14} $$

where $g(t,0,0) \neq 0$ and solution of (4.14) is given by

$$ u(t) = u_0 E_q(-(t-t_0)^q) + \int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-(t-\tau)^q)[\eta sin\,\tau + cos\tau]d\tau. \tag{4.15} $$

The solution has following estimate

$$ |u(t)| \leq |u_0| E_q(-(t-t_0)^q) + (|\eta| + 1)\left[1 - E_q(-(t-t_0)^q)\right]. \tag{4.16} $$

From (4.16) we have $|u(t;t_0,u_0,\eta)| \leq |u_0| + |\eta| + 1$ for $t \geq t_0$. Then B^*2) is satisfied with $\beta = \frac{3\alpha}{2} + 1$ and $\sigma = \frac{\alpha}{2}$. PS^*2) is also satisfied with $\sigma = A - \lambda - 1$ with $A - \lambda > 1$.

4.2.2. Stability versus Stability with ITD

In the real situations, it is often not possible to keep measurements with the expected initial time. So, when we study the influence of parameters, sometimes we need to consider two solutions which have not only different initial points, but also different initial time. The stability with ITD gives us an opportunity to compare solutions of FDE which both initial time and position are different. As a connected with Remark 4.1, we will give a brief overview between the introduced stability with ITD and the known stability of a nonzero solution in the sense of Lyapunov [1]-[6].

• Stability of a solution in the sense of Lyapunov

Let $x(t) = x(t, t_0, x_0)$ be the solution of (4.1). To study the stability of $x(t)$ we consider another solution $Y(t) = Y(t, t_0, y_0)$ of (4.1). The difference between both solutions $z(t) = Y(t) - x(t)$ is a solution of the following IVP

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q z = F(t, z) \\ z(t_0) = y_0 - x_0 \end{cases} \tag{4.17}$$

with $F(t, z) = f\big(t, z + x(t)\big) - f\big(t, x(t)\big)$, which has a zero solution and study of stability properties of $x(t)$ of (4.1) are reduced to the stability of the zero solution of transformed system (4.17).

• Stability with initial time difference

Study the stability with ITD of $x(t)$. Consider the solution of (4.1) with different initial data as $y(t) = y(t, \tau_0, y_0)$. Then the difference $z(t) = y(t + \eta) - x(t)$ is a solution of

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q z = F(t, z; \eta) \\ z(t_0) = y_0 - x_0 \end{cases} \tag{4.18}$$

where $F(t, z; \eta) = f\big(t + \eta, z + x(t)\big) - f\big(t, x(t)\big)$. In the non-autonomous case, i.e. $f(t, x) \neq f(x)$, the IVP (4.18) has no zero solution since $F(t, 0; \eta) = f\big(t + \eta, x(t)\big) - f\big(t, x(t)\big) \neq 0$. Therefore, study of stability with ITD of $x(t)$ could not be reduced to the study of stability of the zero solution in this case.

Example 4.4: To illustrate the idea presented above, consider IVP (4.9) with $f(t, x(t)) = -x(t) + h(t)$ *in Example 4.1. Difference of solutions* $z(t) = y(t + \eta) - x(t)$ *satisfy* ${}_{t_0}^{C}\mathcal{D}^q z(t) = -z(t) + h(t + \eta) - h(t)$, $z(t_0) = y_0 - x_0$ *which has no zero solution.*

4.3. Lyapunov functions and its Caputo fractional Dini derivative with ITD

Our aim is to establish the connection between the stability, practical stability, boundedness and Lagrange stability of the scalar FDE (4.10) and given system of FDE (4.1) via fractional order extension of Lyapunov method relative to ITD. The concept of stability with ITD requires a new definition of derivative of Lyapunov-like functions and a new type of comparison results. In this context, we define Caputo fractional Dini derivative of the function $V(t,x) \in \Lambda(I,\Delta)$ along solutions of the system of FDE (4.1) relative to ITD as follow [40]:

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t,x,y,\eta,x_0,y_0) = \lim_{h\to 0^+} sup \frac{1}{h^q}[V(t,y-x) - V(t_0,y_0-x_0) \\ \quad - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\binom{q}{r}(V(t-rh, y-x-h^q(f(t+\eta,y) \ - f(t,x)) \\ \qquad -V(t_0,y_0-x_0))] \end{cases} \tag{4.19}$$

where $t, t_0 \in I$, $y-x,\ y_0-x_0 \in \Delta$.

Now we will consider the introduced Caputo fractional Dini derivative (4.19), which is generalization of (3.10) with respect to ITD, for some particular Lyapunov functions. The following examples are obtained in [40].

Example 4.5: Let the Lyapunov function does not depend on the time variable, i.e. $V(t,x) \equiv V(x)$ *for* $x \in \mathbb{R}$. *Then, applying formula (4.19)*

$$
\begin{cases}
{}^{C}_{t_0}\mathcal{D}^{q}_{(4.1)}V(t,x,y,\eta,x_0,x_0) = \\
\lim_{h\to 0^+} sup \frac{1}{h^q}[V(y-x)-V(y_0-x_0)- \\
\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r+1}\binom{q}{r}(V(t-rh,x-h^q(f(t+\eta,y)-f(t,x))-V(t_0,x_0))] \\
= \lim_{h\to 0^+} sup \frac{V(y-x)-V(y-x-h^q(f(t+\eta,y)-f(t,x))}{h^q} \\
-V(y_0-x_0)\lim_{h\to 0^+} sup \frac{1}{h^q}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\binom{q}{r}+ \\
\lim_{h\to 0^+} sup[V(y-x-h^q(f(t+\eta,y)-f(t,x))]\frac{1}{h^q}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\binom{q}{r}
\end{cases}
\tag{4.20}
$$

By using $lim_{N\to\infty}\sum_{r=0}^{N}(-1)^r\binom{q}{r}=0$, *where* N *is a natural number,* $\lim_{h\to 0^+} sup \frac{1}{h^q}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\binom{q}{r} = {}^{GL}_{t_0}\mathcal{D}^q\,[1] = {}_{t_0}\mathcal{D}^q\,[1] = \frac{(t-t_0)^{-q}}{\Gamma(1-q)}$ *and*

$\lim_{h\to 0^+} sup \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\binom{q}{r} = -1$, *then it follows that the following formula*

$$
{}^{C}_{t_0}\mathcal{D}^{q}_{(4.1)}V(t,x,y,\eta,x_0,y_0) = \lim_{h\to 0^+} sup \frac{V(y-x)-V(x-h^q(f(t+\eta,y)-f(t,x))}{h^q} + [\,V(y-x)-V(y_0-x_0)]\frac{(t-t_0)^{-q}}{\Gamma(1-q)}. \tag{4.21}
$$

Here the expression with limit in the right hand side of (4.21) is coincides with the fractional derivative of $V(t,x)\equiv V(x)$ *used in [55].*

Corollary 4.2: Let $V(t,x)\equiv V(x)=x^2$ *for* $x\in\mathbb{R}$. *Then according to Example 4.5 we obtain the following expression for the quadratic scalar Lyapunov function as*

$$
\begin{aligned}
&{}_{t_0}^{C}\mathcal{D}^{q}_{(4.1)}V(t,x,y,\eta,x_0,y_0) \\
&\quad = 2(y-x)\big(f(t+\eta,y)-f(t,x)\big) + [\,(y-x)^2-(y_0-x_0)^2]\,\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{4.22}
$$

Remark 4.3: In the case when $q \to 1$ the formula (4.22) is coincide with the known Dini derivative with ITD of $V(t,x) = x^2$ i.e. $\mathcal{D}^{+}V(t,y-x) = 2(y-x)\big(f(t+\eta,y)-f(t,x)\big)$ [52]-[53], [60].

We will compare the introduced Caputo fractional Dini derivative given by (4.19) and classical derivative ($q = 1$) with ITD of Lyapunov function.

Example 4.6: Let $V(t,x) = m^2(t)x^2$ for $x \in \mathbb{R}$ where $m \in C^1(\mathbb{R}_+,\mathbb{R})$. Then, Caputo fractional Dini derivative of the function $V(t,x)$:

$$
\begin{aligned}
&{}_{t_0}^{C}\mathcal{D}^{q}_{(4.1)}V(t,x,y,\eta,x_0,y_0) \\
&\qquad = 2(y-x)m^2(t)\big(f(t+\eta,y)-f(t,x)\big) + ((y-x)^2 \times \\
&\qquad\qquad {}_{t_0}^{C}\mathcal{D}^{q}\,[m^2(t)]) + [\,(y-x)^2-(y_0-x_0)^2]\,\frac{m^2(t_0)(t-t_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{4.23}
$$

On the other hand, it is well known that the Dini derivative of Lyapunov function with ITD for classical case ($q = 1$) is

$$
\begin{aligned}
\mathcal{D}^{+}V(t,y-x) &= 2(y-x)m^2(t)\big(f(t+\eta,y)-f(t,x)\big) \\
&\qquad + (y-x)^2\,\frac{d}{dt}[m^2(t)].
\end{aligned} \tag{4.24}
$$

It is noteworthy that the derivative of $m(t)$ in (4.24) is replaced by the fractional derivative in (4.23), which shows that formula (4.19) is a natural generalization of the classical case ($q = 1$) to fractional case ($0 < q < 1$).

4.4. Comparison Results with ITD for Scalar FDE

Now we will give a comparison theorem which gives us a relationship between Lyapunov functions, system of FDE (4.1) and scalar FDE (4.10).

Lemma 4.2: Assume the following conditions are satisfied:

i) The function $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, t_0 + \theta], \mathbb{R}^n)$ *and* $y(t) = y(t; \tau_0, y_0) \in C^q([\tau_0, \tau_0 + \theta], \mathbb{R}^n)$ *are solutions of system of FDE (4.1), (4.2) respectively,* $y(t + \eta^*) - x(t) \in \Delta$ where $\eta^* = \tau_0 - t_0$, $\Delta \in \mathbb{R}^n$ and θ is a given number.

ii) The function $V \in \Lambda([t_0, t_0 + \theta], \Delta)$, $g \in C[[t_0, t_0 + \theta] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ *such that for* $t \in [t_0, t_0 + \theta]$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y(t + \eta^*), \eta^*, x_0, y_0) \leq g(t, V\big(t, y(t + \eta^*) - x^*(t)\big), \eta^*) \tag{4.25} $$

holds.

Then $V(t_0, y_0 - x_0) \leq u_0$ *implies* $V(t, y(t + \eta^*) - x^*(t)) \leq u^*(t)$ *for* $t \in [t_0, t_0 + \theta]$ *where* $u^*(t) = u(t; t_0, u_0, \eta^*)$ *is the maximal solution of IVP for scalar FDE (4.10) with* $\eta = \eta^*$.

Proof 4.2: Let the function $m(t) \in C([t_0, t_0 + \theta], \mathbb{R}_+)$ *be defined by* $m(t) = V(t, y(t + \eta^*) - x^*(t))$. *Then from Remark 3.2 we obtain for* $t \in (t_0, t_0 + \theta]$ *the equality* $\limsup_{h \to 0^+} \frac{1}{h^q} (y(t + \eta^*) - x^*(t) - (y_0 - x_0) - S(y(t + \eta^*), x^*(t), h)) = f(t + \eta^*, y(t + \eta^*)) - f(t, x^*(t))$,

where $S(y(t + \eta^*), x^*(t), h) = \sum_{r=1}^{\left[\frac{t - t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} [y(t + \eta^* - rh) - x^*(t - rh) - (y_0 - x_0)]$. *Therefore* $S(y(t + \eta^*), x^*(t), h) = y(t + \eta^*) - x^*(t) - (y_0 - x_0) - h^q(f(t + \eta^*, y(t + \eta^*)) - f(t, x^*(t))) - \epsilon(h^q)$ *or* $y(t + \eta^*) - x^*(t) - h^q(f(t + \eta^*, y(t + \eta^*)) - f(t, x^*(t))) = S(y(t + \eta^*), x^*(t), h) + (y_0 - x_0) + \epsilon(h^q)$ *with* $\frac{\epsilon(h^q)}{h^q}$ *as* $h \to 0^+$. *Then for any* $t \in (t_0, t_0 + \theta]$ *using (3.9) we obtain*

$$
\begin{cases}
m(t) - m(t_0) - \left[\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \left(m(t-rh) - m(t_0)\right)\right] \\
= V\left(t, z(t)\right) - V(t_0, z_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \times \\
[V(t-rh, z(t) - z_0 - h^q \left(f\left(t+\eta^*, y(t+\eta^*)\right) - f\left(t, x^*(t)\right)\right) \\
-V(t_0, z_0)] + \left[\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \times \right. \\
\left. \left(V\left(t-rh, S(y(t+\eta^*), x^*(t), h) + z_0 + \epsilon(h^q)\right)\right)\right] \\
-\left[\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \left(V(t-rh, z(t-rh))\right)\right]
\end{cases} \tag{4.26}
$$

where $z(t) = y(t+\eta^*) - x^*(t)$, $z_0 = y_0 - x_0$ *and* η^* *is defined in i). After arrangement in the expression (4.25) via* V *is locally Lipschitzian in its second argument with a Lipschitz constant* $L > 0$ *we obtain*

$$\begin{cases} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} [V(t-rh, S(y(t+\eta^*), x^*(t), h) + z_0 + \epsilon(h^q)) \\ \qquad -V(t-rh, z(t-rh))] \\ \leq L \left\| \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} \binom{q}{r} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{j+1} \binom{q}{j} (z(t-jh) - z_0) \right. \\ \left. - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} (z(t-rh) - z_0) \right\| + L\epsilon(h^q) \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} \binom{q}{r} \\ = L \left\| \sum_{j=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \sum_{j=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{j+1} \binom{q}{j} (z(t-jh) - z_0) \right\| \\ \qquad +L\epsilon(h^q) \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} \binom{q}{r}. \end{cases} \tag{4.27}$$

Substitute (4.26) in (4.25), divide both sides by h^q, take a limit as $h \to 0^+$, use (4.21), (3.11), ii) and $\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]} \binom{q}{r} z^r = (1+z)^q$ if $|z| \leq 1$ we obtain for any $t \in (t_0, t_0 + \theta]$ the inequality

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}_{+}^{q} m(t) \leq {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y(t+\eta^*), \eta^*, x_0, y_0) \\ +L \lim_{h\to 0^+} sup \left\| \sum_{j=1}^{\left[\frac{t-t_0}{h}\right]} \frac{1}{h^q} (-1)^{j+1} \binom{q}{j} (z(t-jh) - z_0) \sum_{r=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \right\| \\ +L \lim_{h\to 0^+} sup \frac{\epsilon(h^q)}{h^q} \lim_{h\to 0} sup \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} \binom{q}{r} \\ = {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y(t+\eta^*), \eta^*, x_0, y_0) \\ \leq g(t, V(t, y(t+\eta^*) - x^*(t), \eta^*) = g(t, m(t), \eta^*). \end{cases} \tag{4.28}$$

Finally, we get $m(t) = V(t, y(t+\eta^*) - x^*(t)) \leq u(t; t_0, u_0, \eta^*)$ *for* $t \in [t_0, t_0 + \theta]$ *by applying Lemma 3.3.* ■

Corollary 4.3: Let the condition i) of Lemma 4.2 be satisfied and the function $V \in \Lambda([t_0, t_0+\theta], \Delta)$ *be such that the inequality*
${}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y(t+\eta^*), \eta^*, x_0, y_0) \leq 0$ *holds for* $t \in [t_0, t_0+\theta]$.
Then for $t \in [t_0, t_0+\theta]$ *the inequality* $V(t, y(t+\eta^*) - x^*(t)) \leq V(t_0, y_0 - x_0)$ *holds.*

Proof 4.3: The proof of Corollary 4.3 follows directly from the fact that corresponding IVP for scalar FDE ${}_{t_0}^{C}\mathcal{D}^{q}\, u = 0$ *with* $u_0 = V(t_0, y_0 - x_0)$ *has a unique solution* $u(t) = V(t_0, y_0 - x_0)$ *for* $t \in [t_0, t_0+\theta]$. ■

Corollary 4.4: Let the condition i) of Lemma 4.2 be satisfied and the function $V \in \Lambda([t_0, t_0+\theta], \Delta)$ *be such that the inequality*
${}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y(t+\eta^*), \eta^*, x_0, y_0) \leq -\gamma V\big(t, y(t+\eta^*) - x^*(t)\big) + C\eta^*$ *holds for* $t \in [t_0, t_0+\theta]$, *where* $\gamma > 0$ *and* $C \in R$ *are constants.*
Then for $t \in [t_0, t_0+\theta]$ *the inequality* $V\big(t, y(t+\eta^*) - x^*(t)\big) \leq [V(t_0, y_0 - x_0) - \frac{1}{\mu} C\eta^*] E_q(-\gamma (t-t_0)^q) + \frac{1}{\mu} C\eta^*$ *holds.*

Proof 4.4: The proof of Corollary 4.4 follows directly from the fact that corresponding IVP for scalar FDE with $g(t, u, \eta^*) = -\mu u + C\eta^*$, $u_0 = V(t_0, y_0 - x_0)$, *i.e.* ${}_{t_0}^{C}\mathcal{D}^{q}\, u =- \mu u + C\eta^*$ *has a unique solution* $u(t; t_0, u_0, \eta^*) = [V(t_0, y_0 - x_0) - \frac{1}{\mu} C\eta^*] E_q(-\gamma (t-t_0)^q) + \frac{1}{\mu} C\eta^*$ *for* $t \in [t_0, t_0+\theta]$. ■

The result of Lemma 4.2 is also true on the half line.

Lemma 4.3: Let the conditions of Lemma 4.2 are satisfied for $\theta = \infty$, *i.e. for* $t \geq t_0$ *and* $t \geq \tau_0$ *respectively. Then* $V(t_0, y_0 - x_0) \leq u_0$ *implies* $V(t, y(t+\eta^*) - x^*(t)) \leq u^*(t)$ *for* $t \geq t_0$.

In the case when the Lyapunov function $V(t,x)$ is continuously differentiable we also give comparison results in terms of the Caputo fractional derivative used in the literature [32]-[38].

Lemma 4.4: Assume the following conditions are satisfied:

i) The condition i) of Lemma 4.2 holds.

ii) The function $V \in \Lambda([t_0, t_0+\theta], \Delta)$ is continuously differentiable such that the inequality

$$ {}_{t_0}^{C}\mathcal{D}^q V(t, y(t+\eta^*) - x^*(t)) \leq g(t, V(t, y(t+\eta^*) - x^*(t))) \quad (4.29) $$

holds.

Then $V(t_0, y_0 - x_0) \leq u_0$ implies the validity of inequality $V(t, y(t+\eta^*) - x^*(t)) \leq u^*(t)$ for $t \in [t_0, t_0+\theta]$ $u^*(t) = u(t; t_0, u_0, \eta^*)$ is the maximal solution of IVP for scalar FDE (4.10) with $\eta = \eta^*$.

Proof 4.4: Let the function $m(t) \in C^1([t_0, t_0+\theta], \mathbb{R}_+)$ be defined by $m(t) = V(t, y(t+\eta^*) - x^*(t))$. Then, the desired result follows from Lemma 3.5 and Remark 3.2. ■

4.5. Main Results

In this part, we will obtain sufficient conditions on stability, practical stability, boundedness and Lagrange stability with ITD. We will use Lyapunov-like functions from class Λ. The proof is based on the fractional order extension of Lyapunov method combined with comparison result with ITD for scalar FDE with parameter.

4.5.1. Stability and Practical Stability Criteria with ITD

Firstly, we will give sufficient conditions on stability, asymptotically stability with ITD.

Theorem 4.1: Let the following conditions be satisfied:

A1) The function $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$ *is a solution of system of FDE (4.1), where* $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ *are given points.*

A2) The function $g \in C[[t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $g(t, 0,0) \equiv 0$ *and for any* η, $u_0 \in \mathbb{R}$, *the IVP for scalar FDE (4.10) has a solution* $u(t) = u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$.

A3) There exists a function $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ *such that*

i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *where* $a, b \in K$.

ii) for any y, $y_0 \in \mathbb{R}^n$, $t > t_0$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y, \eta, x_0, y_0) \leq g(t, V(t, y - x^*(t)), \eta) \tag{4.30} $$

holds.

A4) The zero solution of scalar FDE (4.10) is stable (attractive) w.r.t. parameter.

Then the solution $x^*(t) = x(t; t_0, x_0)$ *of the system of FDE (4.1) is stable (attractive) with ITD.*

Proof 4.1: Initially, we consider the first case of A4) that is the zero solution of the scalar FDE (4.10) is stable w.r.t. parameter. Since $S^*1)$ *holds, given* $b(\epsilon) > 0$, *there exist* $\delta_1 = \delta_1(t_0, \epsilon) > 0$ *and* $\sigma = \sigma(t_0, \epsilon)$ *such that for* $|u_0| < \delta_1$ *and* $|\eta| < \sigma$ *we have*

$$ |u(t; t_0, u_0, \eta)| < b(\epsilon) \quad , \qquad t \geq t_0 \tag{4.31} $$

where $u(t; t_0, u_0, \eta)$ *is a solution of (4.10). Choose arbitrary points* $y_0 \in \mathbb{R}^n$ *and* $\tau_0 \in \mathbb{R}_+$ *such that* $\|y_0 - x_0\| < \delta$ *and* $|\eta^*| < \sigma$ *where* $\delta < a^{-1}(\delta_1)$ *and* $\eta^* = \tau_0 - t_0$. *Consider a solution* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.2) with the chosen initial data* (τ_0, y_0). *Let* $U_0 = V(t_0, y_0 - x_0)$. *From Condition A3) i) and choice of* δ *it follows* $U_0 = V(t_0, y_0 - x_0) < a(\|y_0 - x_0\|) < a(\delta) < \delta_1$. *Therefore, the maximal solution* $u^*(t) = u(t; t_0, U_0, \eta^*) \in C^q([t_0, \infty), \mathbb{R})$ *of FDE (4.10) satisfies* *inequality (4.31). The conditions of Lemma 4.2 are satisfied for $\Delta = \mathbb{R}^n$, $\theta = \infty$. According to Lemma 4.2 the inequality*

$$V(t, y(t+\eta^*, \tau_0, y_0) - x^*(t)) \leq u(t; t_0, U_0, \eta^*), \qquad t \geq t_0. \tag{4.32}$$

is valid. Consequently, in view of the relations (4.30), (4.31), condition A3) and the choice of U_0 we obtain

$$b(y(t+\eta^*) - x^*(t)) \leq V(t, y(t+\eta) - x^*(t)) \leq u^*(t) < b(\epsilon) \tag{4.33}$$

which implies that $\|y(t+\eta^*, \tau_0, y_0) - x^*(t)\| < \epsilon$ for $t \geq t_0$. Therefore, according to Definition 4.1 the solution $x^*(t)$ is stable with ITD.

Secondly, consider the other case of A4) that is zero solution of (4.10) is attractive w.r.t. parameter. Since $S^*3)$ holds, given $b(\epsilon) > 0$, there exists a $\delta_0 = \delta_0(t_0) > 0$, $\sigma_0 = \sigma_0(t_0)$ and $T = T(t_0, \epsilon)$ such that for $|u_0| < \delta_0$ and $|\eta| < \sigma_0$ we have

$$|u(t; t_0, u_0, \eta)| < b(\epsilon) \quad , \qquad t \geq t_0 + T. \tag{4.34}$$

Choose arbitrary points $y_0 \in \mathbb{R}^n$ and $\tau_0 \in \mathbb{R}_+$ such that $\|y_0 - x_0\| < \delta_1$ and $|\eta^*| < \sigma_0$ where $\delta_1 < a^{-1}(\delta_0)$. Consider a solution $y(t) = y(t; \tau_0, y_0)$ of system of FDE (4.2) with the chosen initial data (τ_0, y_0). Let $U_0 = V(t_0, y_0 - x_0)$. After applying similar step in previous proof, we have

$$b(y(t+\eta^*) - x^*(t)) \leq V(t, y(t+\eta) - x^*(t)) \leq u^*(t) < b(\epsilon) \tag{4.35}$$

for $t \geq t_0 + T$. It follows that $\|y(t+\eta^*, \tau_0, y_0) - x^*(t)\| < \epsilon$ for $t \geq t_0 + T$. Therefore, according to Definition 4.1 the solution $x^*(t)$ is attractive with ITD. ■

Corollary 4.5: Suppose that the A1) - A3) of Theorem 4.1 hold.

If the zero solution of scalar FDE (4.10) is asymptotically stable w.r.t. parameter, then the solution $x^*(t) = x(t; t_0, x_0)$ of the system of FDE (4.1) asymptotically stable with ITD.

Theorem 4.2: Let the following conditions be satisfied:

A1) The condition A2) of Theorem 4.1 is satisfied.

A2) There exists a function $V \in \Lambda([t_0,\infty), S(\rho))$ *such that*

(i) $b(\|x\|) \leq V(t,x) \leq a(\|x\|)$ *for* $(t,x) \in \mathbb{R}_+ \times S(\rho)$ *where* $a, b \in K$.

(ii) for any $t > t_0 \in \mathbb{R}_+, x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in S(\rho)$, $y_0 - x_0 \in S(\rho)$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}^{q}_{(4.1)} V(t,x,y,\eta,x_0,y_0) \leq g(t, V(t, y-x), \eta) \tag{4.35} $$

holds.

A3) The zero solution of scalar FDE (4.10) is uniformly stable (uniformly asymptotically stable) w.r.t. parameter.

Then the system of FDE (4.1) is uniformly stable (uniformly asymptotically stable) with ITD.

Proof 4.2: First consider the first case of A3) that is the zero solution of the scalar FDE (4.10) is uniformly stable w.r.t. parameter. Let $\epsilon > 0$ *be a number,* $\epsilon \leq \rho$. *Since* $S^*2)$ *holds, given* $b(\epsilon) > 0$, *there exists a* $\delta_1 = \delta_1(\epsilon) > 0$ *and* $\sigma = \sigma(\epsilon)$ *such that for* $|u_0| < \delta_1$ *and* $|\eta| < \sigma$ *we have*

$$ |u(t; t_0, u_0, \eta)| < b(\epsilon) \quad , \qquad t \geq t_0. \tag{4.36} $$

Now let points $x_0, y_0 \in \mathbb{R}^n$ *and* $\tau_0, t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \delta$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$ *and* $\delta < a^{-1}(\delta_1)$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.1) and (4.2) correspondingly with the chosen initial data* (τ_0, y_0) *and* (t_0, x_0) *respectively. Let* $U_0 = a(\|y_0 - x_0\|)$. *From Condition A2) i) and choice of* δ *it follows* $U_0 = a(\|y_0 - x_0\|) < a(\delta) < \delta_1$. *Therefore, the maximal solution* $u^*(t) = u(t; t_0, U_0, \eta^*) \in C^q([t_0,\infty), \mathbb{R})$ *of FDE (4.10) satisfies inequality (4.36). Then we claim that*

$$\|y(t+\eta^*)-x(t)\|<\epsilon \quad for\ t\geq t_0. \tag{4.37}$$

If the inequality (4.37) is not true there would exist a $t^*>t_0$ *such that* $\|y(t^*+\eta^*)-x(t)\|=\epsilon$ *and* $\|y(t+\eta^*)-x(t)\|<\epsilon$ *for* $t_0\leq t<t^*$*. Therefore the inclusion* $(t,y(t+\eta^*)-x(t))\in S(\rho)$ *is valid for* $t\in[t_0,t^*]$*. By using (4.35), and applying Lemma 4.2 for* $\Delta=\mathbb{R}^n$, $\theta=t^*-t_0$ *we obtain the following estimate*

$$V(t,y(t+\eta^*)-x(t))\leq u^*(t), \qquad t_0\leq t<t^*. \tag{4.38}$$

In the sequel from the relations (4.36), (4.38), condition A2) and the choice of t^**, we obtain*

$$\begin{aligned} b(\epsilon)\leq b(\|y(t^*+\eta^*)-x(t^*)\|)& \\ \leq V\big(t^*,y(t^*+\eta^*)-x(t^*)\big)\leq u^*(t) &< b(\epsilon). \end{aligned} \tag{4.39}$$

Obtained contradiction proves the validity of (4.37), i.e. the system of FDE (4.1) is uniformly stable with ITD. Secondly, consider the other case of A3) that is scalar FDE (4.10) is asymptotically stable w.r.t. parameter. From the first part of the proof it follows that of system of FDE (4.1) is uniformly stable with ITD. Consequently, from the definition US1) there exist $\delta_0=\delta_0(\rho)$ *and* $\sigma_0=\sigma_0(\rho)$ *for* $\epsilon=\rho$ *such that*

$$\begin{cases} \|y_0-x_0\|<\delta_0\ and\ \ |\tau_0-t_0|<\sigma_0 \quad imply \\ \|y(t+\eta,\tau_0,y_0)-x(t,t_0,x_0)\|<\ \rho\ , \qquad t\geq t_0 \end{cases} \tag{4.40}$$

In order to prove asymptotically stability with ITD, let $\epsilon>0$ *be a number,* $\epsilon<\rho$*. Since* $S^*3)$ *holds, given* $b(\epsilon)>0$*, there exists a* $\delta_1>0$*,* $\sigma_1>0$ *and* $T=T(\epsilon)$ *such that for* $|u_0|<\delta_1$ *and* $|\eta|<\sigma_1$ *we have*

$$|u(t;t_0,u_0,\eta)|<b(\epsilon)\ \ , \qquad t\geq t_0+T. \tag{4.41}$$

Now let points $x_0,y_0\in\mathbb{R}^n$ *and* $\tau_0,t_0\in\mathbb{R}_+$ *be such that* $\|y_0-x_0\|<\delta$ *and* $|\eta^*|<\sigma$ *where* $\eta^*=\tau_0-t_0$, $\delta=min(\delta_0,a^{-1}(\delta_1))$ *and* $\sigma=min(\sigma_0,\sigma_1)$*. Consider any* *solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.1) and (4.2) correspondingly with the chosen initial data* (τ_0, y_0) *and* (t_0, x_0) *respectively. On the other hand the estimate (4.38) is valid for all* $t \geq t_0$ *by using (4.40), condition A2) and Lemma 4.2 with* $\Delta = S(\rho)$, $\theta = \infty$. *We will prove that if* $\|y_0 - x_0\| < \delta$ *and* $|\tau_0 - t_0| < \sigma$ *are satisfied then*

$$\|y(t + \eta^*) - x(t)\| < \epsilon \quad \text{for } t \geq t_0 + T. \tag{4.42}$$

Assume the opposite, i.e. there exist a sequence $\{t^{(n)}\}, t^{(n)} \geq t_0 + T,\ t^{(n)} \to \infty$ *as* $n \to \infty$ *such that*

$$\|y(t^{(n)} + \eta^*) - x(t^{(n)})\| \geq \epsilon \tag{4.43}$$

with $\|y_0 - x_0\| < \delta$ *and* $|\eta^*| < \sigma$. *Finally, in view of the relations (4.41), (4.42), (4.43), condition A2), the choice of* $t^{(n)}$ *and* $u_0 = a(\|y_0 - x_0\|) < a(\delta) < \delta_1$ *we obtain*

$$\begin{aligned} b(\epsilon) > u^*(t^{(n)}) \geq V\left(t^{(n)}, y(t^{(n)} + \eta^*) - x(t^{(n)})\right) \\ \geq b\left(\|y(t^{(n)} + \eta^*) - x(t^{(n)})\|\right) \geq b(\epsilon). \end{aligned} \tag{4.44}$$

The obtained contradiction proves validity of inequality (4.42) which implies US2) holds. Consequently, system of FDE (4.1) is uniformly stable and uniformly attractive with ITD i.e. uniformly asymptotically stable with ITD. ■

In this part we will give sufficient conditions on practical stability, attractive practical stability with ITD [40].

Theorem 4.3: Let the conditions A1), A2) of Theorem 4.1 be satisfied and A3), A4) are replaced by as follow:

A3) There exists a function $V \in \Lambda([t_0,\infty),\mathbb{R}^n)$ *such that*

i) $b(\|x\|) \le V(t,x) \le a(\|x\|)$ *for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *where* $a, b \in K$.

ii) for any $y, y_0 \in \mathbb{R}^n$ *and* $\eta \in B_H$, $t > t_0$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x^*(t), y, \eta^*, x_0, y_0) \le g(t, V(t, y - x^*(t)), \eta) \qquad (4.45) $$

holds, where $g(t,0,0) \equiv 0$ *is not required.*

A4) The scalar FDE (4.10) is parametrically practically stable with respect to $(a(\lambda), b(A))$, *where the constant* λ, A *are given such that* $\lambda \in (0, A)$ *and* $a(\lambda) < b(A)$.

Then the solution $x^*(t) = x(t; t_0, x_0)$ *of the system of FDE (4.1) is practically stable with ITD with respect to* (λ, A).

Proof 4.3: From condition A3) according to Definition 4.5 there exists a positive number $\sigma = \sigma(t_0, \lambda, A) < H$ *such that for* $u_0 \in \mathbb{R}$: $|u_0| < a(\lambda)$ *and* η: $|\eta| < \sigma$ *we have*

$$ |u(t; t_0, u_0, \eta)| < b(A) \quad , \qquad t \ge t_0 \qquad (4.46) $$

where $u(t; t_0, u_0, \eta)$ *is a solution of (4.10). Now let points* $x_0, y_0 \in \mathbb{R}^n$ *and* $\tau_0, t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \lambda$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.1) and (4.2) correspondingly with the chosen initial data* (τ_0, y_0) *and* (t_0, x_0) *respectively. Let* $\tilde{u}_0 = V(t_0, y_0 - x_0)$. *From Condition A3) i) it follows* $\tilde{u}_0 = V(t_0, y_0 - x_0) < a(\|y_0 - x_0\|) < a(\lambda)$. *Therefore, the maximal solution* $u^*(t) = u(t; t_0, \tilde{u}_0, \eta^*) \in C^q([t_0,\infty),\mathbb{R})$ *of FDE (4.10) satisfies inequality (4.46). The conditions of Lemma 4.2 are satisfied for* $\Delta = \mathbb{R}^n$, $\theta = \infty$. *According to Lemma 4.2 the inequality*

$$ V(t, y(t + \eta^*, \tau_0, y_0) - x^*(t)) \le u(t; t_0, \tilde{u}_0, \eta^*), \qquad t \ge t_0 \qquad (4.47) $$

is valid. Consequently, in view of the relations (4.46), (4.47), condition A3) and the choice of $\tilde{u}_0$ *we obtain*

$$b(A) > u(t; t_0, \tilde{u}_0, \eta^*) \geq V(t, y(t+\eta^*) - x^*(t)) \geq b(y(t+\eta^*) - x^*(t)) \quad (4.48)$$

which implies that $\|y(t+\eta^*, \tau_0, y_0) - x^*(t)\| < A$ *for* $t \geq t_0$. *Therefore, according to Definition 4.1 the solution* $x^*(t)$ *is practically stable with ITD.* ■

Theorem 4.4: Let the following conditions be satisfied:

A1) The conditions A1), A2) of Theorem 4.1 be satisfied

A2) The scalar FDE (4.10) is attractive parametrically practically stable with respect to $(a(\lambda), b(A))$, *where the constant* λ, A *are given such that* $\lambda \in (0, A)$ *and* $a(\lambda) < b(A)$.

A3) The condition A3) of Theorem 4.3 is satisfied where inequality (4.45) holds for $t \geq t_0 + T$ *with* $T > 0$ *from A2).*

Then the solution $x^*(t) = x(t; t_0, x_0)$ *of the system of FDE (4.1) is attractive practically stable with ITD with respect to* (λ, A).

Proof 4.4: The proof of Theorem 4.4 is similar the one of Theorem 4.3 and we omit it. ■

Corollary 4.6: The function $g(t, u, \eta) = 0$ *is admissible in Theorem 4.4.*

Theorem 4.5: Let the conditions of Theorem 4.3 be satisfied and A3), A4) are replaced by as follow:

A3) There exists a function $V \in \Lambda(\mathbb{R}_+, S(A))$ *such that*

i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $(t, x) \in \mathbb{R}_+ \times S(A)$ *where* $a, b \in K$.

ii) for any $t > t_0 \in \mathbb{R}_+, x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in S(A)$, $y_0 - x_0 \in S(A)$ *and* $\eta \in B_H$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x, y, \eta, x_0, y_0) \leq g(t, V(t, y - x), \eta) \quad (4.49)$$

holds.

A4) The scalar FDE (4.10) is uniformly parametrically practically stable $(a(\lambda), b(A))$, *where the constant* λ, A *are given such that* $\lambda \in (0, A)$ *and* $a(\lambda) < b(A)$.

Then the system of FDE (4.1) is uniformly practically stable with ITD with respect to (λ, A).

Proof 4.5: From condition A4) according to Definition 4.5 there exists a positive number $\sigma = \sigma(\lambda, A) < H$ *such that for* $u_0 \in \mathbb{R}$: $|u_0| < a(\lambda)$ *and* η: $|\eta| < \sigma$ *we have*

$$|u(t; t_0, u_0, \eta)| < b(A) , \qquad t \geq t_0 \tag{4.50}$$

where $u(t; t_0, u_0, \eta)$ *is a solution of (4.10). Choose arbitrary points* $y_0 \in \mathbb{R}^n$ *and* $\tau_0 \in \mathbb{R}_+$ *such that* $\|y_0 - x_0\| < \lambda$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Consider a solution* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.2) with the chosen initial data* (τ_0, y_0). *Let* $\tilde{u}_0 = V(t_0, y_0 - x_0)$. *From Condition A3) i) it follows* $\tilde{u}_0 = V(t_0, y_0 - x_0) < a(\|y_0 - x_0\|) < a(\lambda)$.
Therefore, the maximal solution $u^*(t) = u(t; t_0, \tilde{u}_0, \eta^*) \in C^q([t_0, \infty), \mathbb{R})$ *of FDE (4.10) satisfies inequality (4.50). Then we claim that*

$$\|y(t + \eta^*) - x(t)\| < A \text{ for } t \geq t_0. \tag{4.51}$$

Assume the opposite, i.e. there exists a point $t_1 > t_0$ *such that* $\|y(t_1 + \eta^*) - x(t)\| = A$ *and* $\|y(t + \eta^*) - x(t)\| < A$ *for* $t_0 \leq t < t_1$. *Therefore the inclusion* $(t, y(t + \eta^*) - x(t)) \in S(A)$ *is valid for* $t \in [t_0, t_1]$. *By using (4.49) and applying Lemma 4.2 for* $\Delta = S(A)$, $\theta = t_1 - t_0$ *we obtain the following estimate*

$$V(t, y(t + \eta^*) - x(t)) \leq u^*(t), \qquad t_0 \leq t < t_1. \tag{4.52}$$

From the choice of t_1, *condition A3) and inequalities (4.50), (4.51), (4.52) we obtain*

$$\begin{aligned} b(A) &= b(\|y(t_1 + \eta^*) - x(t_1)\|) \\ &\leq V(t_1, y(t^* + \eta^*) - x(t_1)) \leq u^*(t_1) < b(A). \end{aligned} \tag{4.53}$$

The obtained contradiction proves the validity of inequality (4.51). Therefore, according to Definition 4.1 the system of FDE (4.1) is uniformly practically stable with ITD w.r.t. (λ, A). ■

Corollary 4.7: If the inequality (4.35), (4.49) are satisfied with $g(t, u, \eta) = -\mu u + C\eta$ *in Theorem 4.2 and Theorem 4.5, then the system of FDE (4.1) is uniformly stable and uniformly practically stable with ITD.*

Proof 4.7: In this case the corresponding scalar FDE (4.10) reduces to ${}_{t_0}^{C}\mathcal{D}^q u = -\mu u + C\eta$, $u(t_0) = u_0$. *For any parameter* η, $u_0 \in \mathbb{R}$ *the above scalar FDE has a solution* $u(t) = u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$ *which is uniformly stable and uniformly parametrically stable w.r.t parameter from Example 4.2. Hence, system of FDE (4.1) is uniformly stable and practically stable with ITD by using Theorem 4.2 and Theorem 4.5.* ■

4.5.2. Boundedness and Lagrange Stability Criteria with ITD

In this part we will give sufficient conditions on boundedness and Lagrange stability with ITD.

Theorem 4.6: Let the conditions A1), A2) of Theorem 4.1 be satisfied and A3), A4) are replaced by as follow:

A3) There exists a function $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ *such that*

i) $b(\|x\|) \le V(t, x) \le a(\|x\|)$ *for* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *where* $a, b \in K_\infty$.

ii) for any $t > t_0 \in \mathbb{R}_+$, *and* $x, y, x_0, y_0 \in \mathbb{R}^n$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}_{(4.1)}^{q} V(t, x, y, \eta, x_0, y_0) \le g(t, V(t, y - x), \eta) \tag{4.54}$$

holds.

(A4) The scalar FDE (4.10) is equi-bounded (ultimately bounded) w.r.t. parameter.

Then the system of FDE (4.1) is equi-bounded (quasi-ultimately bounded) with ITD.

Proof 4.6: Let $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$. *Consider the solution* $x(t) = x(t; t_0, x_0)$ *of system of FDE (4.1). Initially, we consider the first case of A4) that is the scalar FDE (4.10) is equi-bounded w.r.t. parameter. Since* $B^*1)$ *holds, given* $\gamma_1 = a(\alpha) > 0$ *and* $t_0 \in \mathbb{R}_+$ *there exist* $\beta_1 = \beta_1(t_0, \gamma_1) > 0$ *and* $\sigma = \sigma(t_0, \gamma_1)$ *such that* $|u_0| < \gamma_1$ *and* $|\eta| < \sigma$ *imply*

$$|u(t; t_0, u_0, \eta)| < \beta_1 \ \text{for } t \geq t_0. \tag{4.55}$$

Now let point $y_0 \in \mathbb{R}^n$ *and* $t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \alpha$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Consider a solution* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.2) with the chosen initial data* (τ_0, y_0). *Choose* $\tilde{u}_0 = V(t_0, y_0 - x_0)$ *and* $\beta = \beta(t_0, \alpha) > 0$ *such that* $\tilde{u}_0 = V(t_0, y_0 - x_0) < a(\|y_0 - x_0\|) < a(\alpha) = \gamma_1$ *and* $b(\beta) \geq \beta_1$ *since for the function* $b \in K$ *we have* $b(r) \to \infty$ *as* $r \to \infty$, *respectively. Then from condition (A3), Lemma 4.2 and (4.54), (4.55) we get to the inequalities*

$$b\big(y(t + \eta^*) - x(t)\big) \leq V\big(t, y(t + \eta) - x(t)\big) \leq u^*(t) < \beta_1 < b(\beta) \tag{4.56}$$

which implies that $\|y(t + \eta^*, \tau_0, y_0) - x(t)\| < \beta$ *for* $t \geq t_0$.
Next we consider the second case of A4) that is the scalar FDE (4.10) is equi-ultimately bounded w.r.t. parameter. In order to prove B3), let $\alpha > 0$ *and* $t_0 \in \mathbb{R}_+$. *Consider the solution* $x(t) = x(t; t_0, x_0)$ *of system of FDE (4.1). Since* $B^*3)$ *holds i.e. for* $\alpha_1 = a(\alpha) > 0$ *there exist positive numbers* N, $\sigma = \sigma(t_0, \alpha) > 0$ *and* $T = T(t_0, \alpha)$ *such that* $|u_0| < \alpha_1$ *and* $|\eta| < \sigma$ *imply* $|u(t; t_0, u_0, \eta)| < N$, $t \geq t_0 + T$. *Now let point* $y_0 \in \mathbb{R}^n$ *and* $\tau_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \alpha$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Consider a solution* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.2) with the chosen initial data* (τ_0, y_0). *We claim that B3) holds with* T, σ *and* N^*, *where* $b(N^*) \geq N$. *Suppose that is not true. Therefore, there exist a sequence* $\{t^{(n)}\}$, $t^{(n)} > t_0 + T$, $t^{(n)} \to \infty$ *as* $n \to \infty$ *such that*

$$\|y(t^{(n)} + \eta^*) - x(t^{(n)})\| \geq N^* \tag{4.57}$$

with $\|y_0 - x_0\| < \alpha$ and $|\eta^*| < \sigma$. Finally, in view of the relation (4.57), condition A3), the choice of $t^{(n)}$ and $\tilde{u}_0 = a(\|y_0 - x_0\|) < a(\alpha) < \alpha_1$ we obtain

$$b(N^*) \geq N > u^*\big(t^{(n)}\big) \geq V\left(t^{(n)}, y\big(t^{(n)} + \eta^*\big) - x\big(t^{(n)}\big)\right) \geq b\big(\big\|y\big(t^{(n)} + \eta^*\big) - x\big(t^{(n)}\big)\big\|\big) \geq b(N^*). \tag{4.58}$$

The obtained contradiction proves that B3) holds. Thus, the system of FDE (4.1) is ultimately bounded with ITD. ■

Corollary 4.8: Let the conditions of Theorem 4.6 hold except A4). If the scalar FDE (4.10) is uniformly bounded (uniformly quasi-ultimately bounded), then the system of FDEs (4.1) is uniformly bounded (uniformly quasi-ultimately bounded) with ITD.

Corollary 4.9: If the inequality (4.54) is satisfied with $g(t, u, \eta) = -\mu u + C\eta$ in Theorem 4.6, then the system of FDE (4.1) is uniformly bounded with ITD.

Proof 4.9: In this case the corresponding scalar FDE (4.10) reduces to ${}_{t_0}^{C}\mathcal{D}^q u = -\mu u + C\eta$, $u(t_0) = u_0$. For any parameter η, $u_0 \in \mathbb{R}$, the above scalar FDE has a solution $u(t) = u(t; t_0, u_0, \eta) \in C^q([t_0, \infty), \mathbb{R})$ which is uniformly bounded w.r.t parameter from Example 4.2. Hence, system of FDE (4.1) is uniformly bounded with ITD by using Theorem 4.6. ■

Theorem 4.7: Let the conditions of Theorem 4.6 hold except A4). If the scalar FDE (4.10) is Lagrange stable (uniformly Lagrange stable) w.r.t. parameter, then the system of FDE (4.1) is Lagrange stable (uniformly Lagrange stable) with ITD.

Proof 4.7: Let the scalar FDE (4.10) is Lagrange stable w.r.t. parameter which implies that system of FDE (4.1) is equi-bounded with ITD by applying Theorem 4.6. In order to prove L1) holds, we need to show A1) holds. Let $\epsilon > 0$, $\alpha > 0$ be given. Since A^*1) holds i.e. for $b(\epsilon) > 0$ and $\alpha_1 = a(\alpha) > 0$ there exists a $T = T(t_0, \epsilon, \alpha)$ and $\sigma = \sigma(t_0, \alpha, \epsilon) > 0$ such that $|u_0| < \alpha_1$ and $|\eta| < \sigma$ imply $|u(t; t_0, u_0, \eta)| < b(\epsilon)$, $t \geq t_0 + T$. Now let point $y_0 \in \mathbb{R}^n$ and $\tau_0 \in \mathbb{R}_+$ be such that $\|y_0 - x_0\| < \alpha$ and $|\eta^*| < \sigma$ where $\eta^* = \tau_0 - t_0$. Consider a solution $y(t) = y(t; \tau_0, y_0)$ of system *of FDE (4.2) with the chosen initial data* (τ_0, y_0). *We claim that A1) holds with* T, σ. *Suppose that is not true. Therefore, there exist a sequence* $\{t^{(n)}\}$, $t^{(n)} > t_0 + T$, $t^{(n)} \to \infty$ *as* $n \to \infty$ *such that*

$$\|y(t^{(n)} + \eta^*) - x(t^{(n)})\| \geq \epsilon \tag{4.59}$$

with $\|y_0 - x_0\| < \alpha$ *and* $|\eta^*| < \sigma$. *Finally, in view of the relation (4.59), condition A3), the choice of* $t^{(n)}$ *and* $\tilde{u}_0 = a(\|y_0 - x_0\|) < a(\alpha) < \alpha_1$ *we obtain*

$$\begin{aligned} b(\epsilon) > u^*\big(t^{(n)}\big) \geq V\Big(t^{(n)}, y\big(t^{(n)} + \eta^*\big) - x\big(t^{(n)}\big)\Big) \\ \geq b\big(\big\|y\big(t^{(n)} + \eta^*\big) - x\big(t^{(n)}\big)\big\|\big) \geq b(\epsilon). \end{aligned} \tag{4.60}$$

which implies that A1) holds. Since Theorem 4.6 implies that is equi-bounded with ITD, then system of FDE (4.1) is Lagrange stable with ITD. ■

Theorem 4.1 - Theorem 4.7 require using Caputo fractional Dini derivative with ITD for Lyapunov functions, i.e. they require only continuity property for Lyapunov function. In the case of differentiable Lyapunov function $V(t, x)$ we generalize the stability results of [32]-[38] to stability, practical stability, boundedness and Lagrange stability with ITD by using Caputo fractional derivative of Lyapunov functions. The proofs of the next theorems are similar to those in the proof of Theorem 4.1 – Theorem 4.7 where Lemma 4.4 can be used instead of Lemma 4.2.

Theorem 4.8: Let the conditions A1), A2) of Theorem 4.1 be satisfied and A3), A4) are replaced by as follow:

A3) There exists a function $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ *such that*

i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *where* $a, b \in K$.

ii) for any solution $y(t; \tau_0, y_0)$ *of system of FDE (4.2) such that* $\eta = \tau_0 - t_0 \in B_H$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}^q V\big(t, y(t+\eta) - x^*(t)\big) \le g(t, V\big(t, y(t+\eta) - x^*(t)\big), \eta) \qquad (4.61) $$

holds for $t \ge t_0$+*T.*

A4) The scalar FDE (4.10) is parametrically practically stable, attractive practically stable.

Then if $T = 0$ *the solution* $x^*(t) = x(t; t_0, x_0)$ *of the system of FDE (4.1) is practically stable with ITD otherwise it is attractive practically stable with ITD.*

Proof 4.8: The proof is similar to those in the proof of Theorem 4.1 – Theorem 4.7 in which Lemma 4.2 should be used with replacing Caputo fractional derivative in Lemma 4.2. ■

Theorem 4.9: Let the conditions A1) of Theorem 4.1 be satisfied and A2), A3) are replaced by as follow:

A2) There exists a function ($V \in \Lambda(\mathbb{R}_+, S(A))$) *such that*

i) $b(\|x\|) \le V(t,x) \le a(\|x\|)$ *for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *where* $a, b \in K$.

ii) for any solutions $x(t; t_0, x_0)$ *and* $y(t; \tau_0, y_0)$ *of systems of FDE(4.1) and (4.2) such that* $y(t+\eta) - x(t) \in S(A)$ *the inequality*

$$ {}_{t_0}^{C}\mathcal{D}^q V\big(t, y(t+\eta) - x(t)\big) \le g(t, V\big(t, y(t+\eta) - x(t)\big), \eta) \qquad (4.62) $$

holds for $t \ge t_0$.

A3) The scalar FDE (4.10) is uniformly stable w.r.t parameter (uniformly parametrically practically stable).

Then the solution $x^*(t) = x(t; t_0, x_0)$ *of the system of FDE (4.1) is uniformly stable, (uniformly practically stable) with ITD.*

Remark 4.4: It should be noted that similar results for other introduced concepts in this chapter can be stated.

4.6. Applications

Now we will illustrate the application of the defined concepts with ITD and the obtained above sufficient conditions on examples [40].

Example 4.7: Consider the following IVP for the system of FDE with $0 < q < 1$,

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x_1(t) = -x_1(t) - x_2(t) + h_1(t) \\ {}_{t_0}^{C}\mathcal{D}^q\, x_2(t) = -x_2(t) + x_1(t) + h_2(t) \end{cases} \tag{4.63}$$

for $t > t_0$ *with* $x_1(t_0) = x_0^1$ *and* $x_2(t_0) = x_0^2$ *where the functions* $h_1, h_2 \in C(\mathbb{R}_+, \mathbb{R})$ *satisfy Lipschitz condition with* $L_1, L_2 > 0$, *respectively. Consider* $V(t,x) = x^T x = x_1^2 + x_2^2$ *for* $x = (x_1, x_2) \in \mathbb{R}^2$ *and choose* $a, b \in K$ *such that* $a(s) = 2s$, $b(s) = \frac{1}{2}s$ *for the validity of the condition A3) i) of Theorem 4.1.*

Let a couple of real numbers (λ, A) *with* $0 < \lambda < \frac{A}{4}$ *be given. Now, let* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *be solutions of system of FDE (4.1) and (4.2) respectively such that* $y(t+\eta) - x(t) \in S(A)$ *where* $x(t) = (x_1(t), x_2(t))$, $y(t) = \big(y_1(t), y_2(t)\big)$, $x_0 = (x_0^1, x_0^2)$ *and* $y_0 = (y_0^1, y_0^2)$. *By using Corollary 4.1, Remark 1 [38], linearity of Caputo derivative, Lipschitz property of* $h_1(t)$, $h_2(t)$ *we get the following inequality for the Caputo fractional derivative of candidate Lyapunov function* $V(t,x) = x^T x$ *as follow*

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, V\big(t, y(t+\eta) - x(t)\big) \\ \le 2\big(y(t+\eta) - x(t)\big)^T\, {}_{t_0}^{C}\mathcal{D}^q\, y(t+\eta) - x(t) = -2\big(y_1(t+\eta) - x_1(t)\big)^2 \\ -2\big(y_2(t+\eta) - x_2(t)\big)^2 + 2\big(y_1(t+\eta) - x_1(t)\big)\big(h_1(t+\eta) - h_1(t)\big) \\ +2\big(y_2(t+\eta) - x_2(t)\big)\big(h_2(t+\eta) - h_2(t)\big) \\ \le -2\, V\big(t, y(t+\eta) - x(t)\big) + 2AL\eta \end{cases} \tag{4.64}$$

where $L = max\{L_1, L_2\}$. *According to (4.64), the corresponding IVP for scalar FDE*

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, u(t) = -2u(t) + C\eta \\ \qquad u(t_0) = u_0 \end{cases} \tag{4.65}$$

where $u, u_0 \in \mathbb{R}$ *and* $C = 2AL$. *According to Example 4.2 with* $\mu = 2$ *scalar FDE (4.65) is uniformly parametrically stable w.r.t.* $\left(2\lambda, \frac{A}{2}\right)$ *and uniformly stable w.r.t. parameter. Then the system of FDE (4.63) is uniformly practically stable with ITD, uniformly stable with ITD by using Theorem 4.9. Figure 4.2 shows that the approximate solutions* $x(t)$, $y(t+\eta)$ *with* $h_1(t) = \sin(t)$, $h_2(t) = \cos(t)$, $t_0 = 0$, $\tau_0 = 0.2$, $x_0 = (2,3)$, $y_0 = (2.3, 3.2)$ *and* $L = 1$.

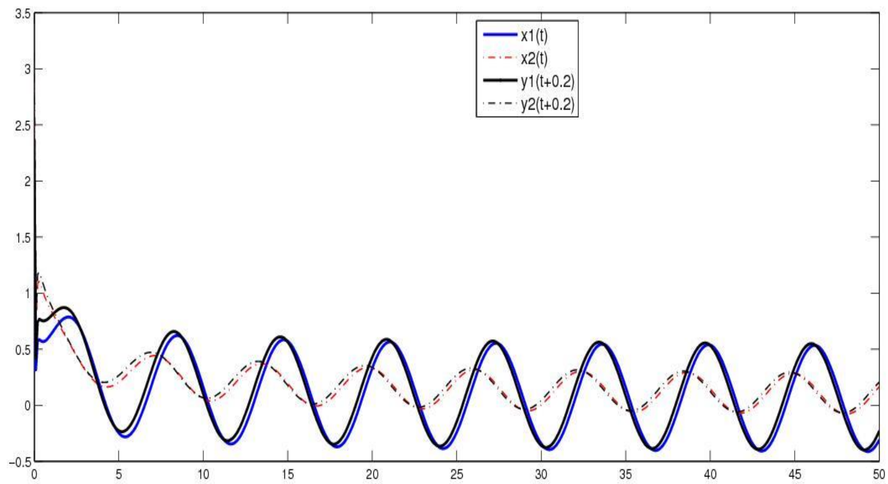

Figure 4.2: Approximate solutions with $h_1(t) = sin(t)$, $h_2(t) = \cos(t)$, $t_0 = 0$, $\tau_0 = 0.2$ and $L = 1$.

Example 4.8: Consider the following FDE with $0 < q < 1$,

$${}_{t_0}^{C}\mathcal{D}^q\, x(t) = x(t)(1 - 0.1t) \tag{4.66}$$

for $t > t_0$ *with* $x(t_0) = x_0$. *The IVP (4.66) has a zero solution* $x^*(t) \equiv 0$ *with* $x(0) = 0$. *Consider the quadratic function* $V(t,x) = x^2$. *The condition A3) i) of Theorem 4.1 is satisfied for* $a, b \in K$ *such that* $a(s) = 2s$, $b(s) = \frac{1}{2}s$. *Let a couple of real numbers* (λ, A) *with* $0 < \lambda < \frac{A}{4}$ *be given. Now, let* $y(t) = y(t; \tau_0, y_0)$ *be a* *solution of FDE* ${}_{\tau_0}^{C}\mathcal{D}^{q}\, y(t) = y(t)(1-0.1t),\ \ t>\tau_0$ *with* $y(\tau_0)=y_0$ *where* $\eta=\tau_0\in B_H,\ H>0$ *is a given constant. According to Corollary 4.1 the function* $y(t+\eta)$ *is a solution of* ${}_{0}^{C}\mathcal{D}^{q}\, y = y(1-0.1(t+\eta)),\ t>0$ *with* $y(0)=y_0$*. Then apply Remark 1 [38] and get the following inequality for the Caputo fractional derivative of the Lyapunov function* $V(t,x)=x^2$

$$
\begin{aligned}
&{}_{0}^{C}\mathcal{D}^{q}\, V\big(t, y(t+\eta)-x^*(t)\big) \\
&\qquad \le 2y(t+\eta)\,{}_{0}^{C}\mathcal{D}^{q}\,\big(y(t+\eta)\big) = g(t,\eta)\big(y(t+\eta)\big)^2
\end{aligned}
\tag{4.67}
$$

where the function $g(t,\eta)=2(1-0.1(t+\eta))$*. Then* $g(t,\eta)\le 0$ *for* $t>11$ *with the choice of* $H=1$*. According to Theorem 4.8 the zero solution* $x^*(t)\equiv 0$ *of FDE (4.66) is attractive practically stable and the inequality* $\|y(t+\eta)-x^*(t)\|<A$ *for* $t>11$ *holds whenever* $\|y_0-x_0\|<\lambda$*. Now use the Lyapunov function* $V(t,x)=m^2(t)x^2$*. From (4.20), Example 4.6 we get*

$$
\begin{cases}
{}_{0}^{C}\mathcal{D}^{q}_{(4.66)} V(t,0,y,\eta,0,y_0) \\
= 2(y-0)^2 m^2(t)\big(f(t+\eta,y)-f(t,0)\big) \\
+(y-0)^2\,{}_{0}^{C}\mathcal{D}^{q}\,[m^2(t)] + ((y-0)^2-(y_0-0)^2)\dfrac{m^2(0)t^{-q}}{\Gamma(1-q)} \\
\le y^2(2m^2(t)\left(\big(1-0.1(t+\eta)\big)\right) + {}_{0}^{C}\mathcal{D}^{q}\,[m^2(t)] + \dfrac{m^2(0)t^{-q}}{\Gamma(1-q)}
\end{cases}
\tag{4.68}
$$

Choose the function $m^2(t)$ *such that it satisfy the scalar fractional linear inequality* ${}_{0}^{C}\mathcal{D}^{q}\, u(t) \le -2u(t)\big(1-0.1(t+\eta)\big) - \frac{t^{-q}}{\Gamma(1-q)}$ *for* $t\ge T,\ T>0$*. Let, for example,* $m(t)=\frac{1}{1+t}$*. Then for* $q=0.5$*, we have* ${}_{0}^{C}\mathcal{D}^{q}\,[\frac{1}{(1+t)^2}] = \frac{-4t^{0.5}{}_2F_1(1,3;1.5;-t)}{\Gamma(0.5)}$ *where* ${}_2F_1(1,3;1.5;-t)$ *is the hypergeometric function. Then the inequality* $M(t,\eta) = 2m^2(t)\big(1-0.1(t+\eta)\big) + {}_{t_0}^{C}\mathcal{D}^{q}\,[m^2(t)] + \frac{t^{-0.5}}{\Gamma(0.5)} \le 0$ *is satisfied for* $t\ge T$ *where* $T>0$ *depends on* η*, see Figure 4.3. From Figure 4.4 it can be seen* $T=5.5$ *for* $\eta=\tau_0\le 1$*. Therefore,* ${}_{0}^{C}\mathcal{D}^{q}_{(4.66)} V(t,x^*(t),y,\eta,0,y_0)\le 0$ *for* $t>5.5$ *and according to Corollary 4.6 the (4.66) is attractive practically stable with ITD, where the* *interval* $t > 11$ *is substituted by* $t > 5.5$*. Therefore, in this case the application of the introduced formula (4.20) gives us better result than the application of Caputo fractional derivative of Lyapunov function.*

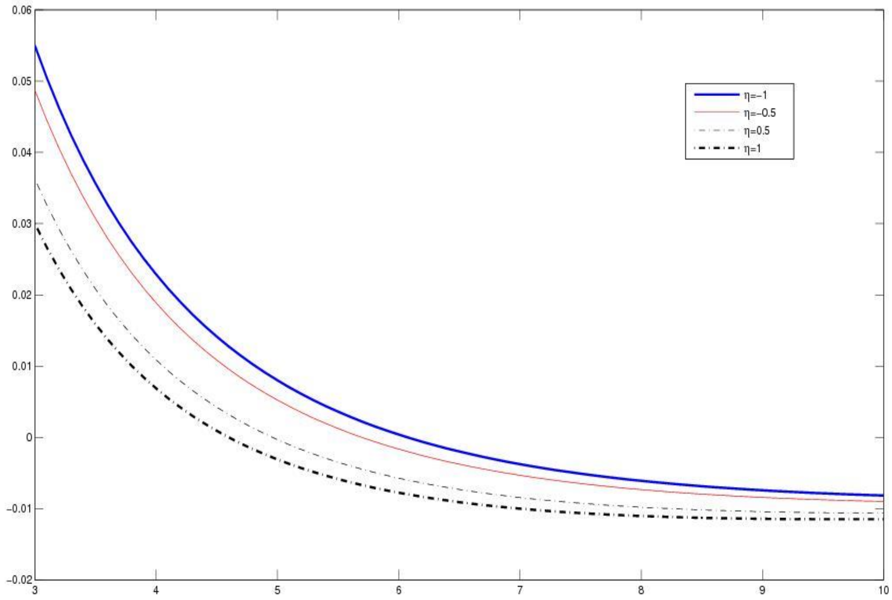

Figure 4.3: Graph of $M(t,\eta)$ for various η.

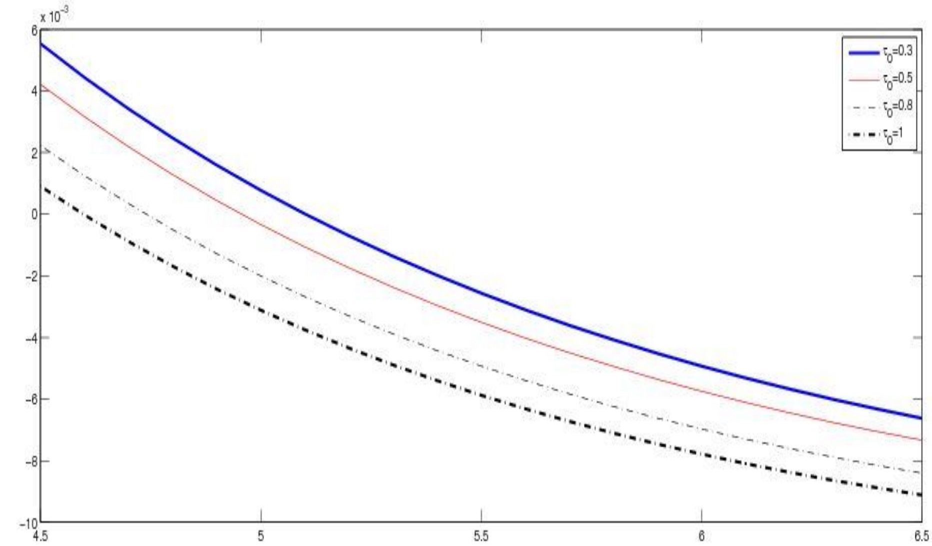

Figure 4.4: Graph of $M(t,\tau_0)$ for various τ_0.

5. STABILITY AND BOUNDEDNESS OF PERTURBED FRACTIONAL DIFFERENTIAL EQUATION WITH INITIAL TIME DIFFERENCE

5.1. Introduction

The concept of a Lyapunov function has been employed with great success in a wide variety of investigations to understand qualitative and quantitative properties of dynamic systems for many years. Lyapunov' s second method is a standard technique used in the study of the qualitative behavior of differential systems along with a comparison result that allows the prediction of behavior of a differential system when the behavior of the null solution of a comparison system is known. The application of Lyapunov' s second method in stability and boundedness theory [3]-[4] has the advantage of not requiring knowledge of solutions.

An important problem in stability theory is to determine which stability properties of a particular differential system are preserved under sufficiently small perturbations. This problem was investigated in several ways in [1-6]. The author in [50] investigated the problem of determining the behavior of the solutions of a perturbed differential equation with respect to the solutions of the original unperturbed differential equation. The principal mathematical technique employed is a modification of Lyapunov's direct method which is applied to the difference of the solutions of perturbed and unperturbed system where the initial positions are sufficiently close. In [51], the authors applied variational Lyapunov method (VLM), combines the method of variation of parameters and the method of Lyapunov, to connect the solutions of perturbed and unperturbed system with initial time unchanged.

However, the possibility of making error in initial time as well as in initial position when we deal with real world problems needs to be considered. We call this type of stability analysis, initial time difference stability analysis [42]-[49], [52]-[56]. A significant difference between ITD stability of perturbed system and the classical notions of stability is that the classical notions of stability are with respect to the null solution, but ITD stability of perturbed system is with respect to the unperturbed differential system. So far, several studies have been made on this problem for ODE to explore the ITD stability, boundedness, etc. criteria by using generalized variation of parameters and comparison method via Lyapunov functions in [52], [53] and references therein. However, there are a few results for FDE. In [52], VLM is applied to connect between the solutions of system of perturbed and unperturbed FDE that have the same initial time. On the other hand sufficient conditions on stability with ITD are obtained in [54]. In chapter 5, we have investigated stability, practical stability, boundedness and Lagrange stability for system of nonlinear perturbed FDE with ITD by using fractional comparison method via Lyapunov function and scalar FDE with parameter. We begin with section 5.2 which includes the necessary some stability and boundedness definitions of system of perturbed FDE relative to unperturbed FDE with ITD and Caputo fractional Dini derivative of Lyapunov function with respect to the system of perturbed FDE and unperturbed FDE in relation with definition in [55]-[56]. In section 5.3, firstly we present a comparison result which uses Lyapunov function to connect the solutions of the perturbed and the unperturbed systems in terms of solution of a scalar FDE. We have obtained some sufficient conditions for ITD stability, boundedness and Lagrange stability of nonlinear system of perturbed FDE.

5.2. Statement of the problem

Consider the following IVP for the system of FDE for $0 < q < 1$

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x(t) = f(t, x(t)) \\ \qquad x(t_0) = x_0 \end{cases} \tag{5.1}$$

where $t_0 \in \mathbb{R}_+,\ f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$. Denote the solution of (5.1) by $x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$. Let $\tau_0 \in \mathbb{R}_+$, $\tau_0 \neq t_0$ be a different initial time. In addition to (5.1), we also consider the associated system of perturbed FDE with different initial data

$$\begin{cases} {}_{\tau_0}^{C}\mathcal{D}^q\, y(t) = F(t, y(t) \\ \qquad y(\tau_0) = y_0 \end{cases} \tag{5.2}$$

where $F(t, y(t) = f(t, y(t)) + R(t, y(t))$ and $R \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is called the perturbation term.

We shall introduce the following definitions of stability, practical stability of system of perturbed FDE (5.2) relative to unperturbed system (5.1) with ITD.

Definition 5.1: The solution $y(t, \tau_0, y_0)$ of system of perturbed FDE (5.2) is said to be

S1) equi-stable with ITD relative to (5.1) *if given* $\epsilon > 0$, *there exist* $\delta = \delta(\tau_0, \epsilon) > 0$ *and* $\sigma = \sigma(\tau_0, \epsilon) > 0$ *such that the inequalities* $\|y_0 - x_0\| < \delta$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \epsilon$ *for* $t \geq \tau_0$, *where* $x(t, t_0, x_0)$ *is a solution of (5.1) and* $\eta = \tau_0 - t_0$;
S2) uniformly stable with ITD, *if* δ *and* σ *in S1) is independent of* τ_0*;*
S3) attractive with ITD *if for given* $\epsilon > 0$ *and there exist* $\delta_0 = \delta_0(\tau_0) > 0$, $\sigma_0 = \sigma_0(\tau_0) > 0$ *and a* $T = T(\tau_0, \epsilon) > 0$ *such that the inequalities* $\|y_0 - x_0\| < \delta_0$ *and* $|\tau_0 - t_0| < \sigma_0$ *imply* $\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \epsilon$ *for* $t \geq \tau_0 + T$*;*
S4) uniformly attractive *if* δ_0, σ_0 *and* T *in S3) is independent of* τ_0*;*
S5) asymptotically stable with ITD *if S1) and S3) hold simultaneously*;
S6) uniformly asymptotically stable with ITD, *if S2) and S4) hold simultaneously;*
PS1) practically stable with ITD *w.r.t.* (λ, A), *if there exists a number* $\sigma = \sigma(\tau_0, \lambda, A) > 0$ *such that the inequalities* $\|y_0 - x_0\| < \lambda$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \epsilon$ *for* $t \geq \tau_0$, *where a couple of real numbers* (λ, A) *with* $0 < \lambda < A$ *be given;*
PS2) uniformly practical stable with ITD *w.r.t.* (λ, A) *if* $\sigma = \sigma(\lambda, A) > 0$ *in PS1);*
PS3) attractive practical stable with ITD *w.r.t.* (λ, A, T) *if there exist* $\sigma = \sigma(\tau_0, \lambda, A) > 0$ *and* $T = T(\tau_0, \lambda, A) > 0$ *such that the inequalities* $\|y_0 - x_0\| < \lambda$ and $|\tau_0 - t_0| < \sigma$ imply $\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < A$ *for* $t \geq t_0 + T$*;*
PS4) uniformly attractive practically stable with ITD *w.r.t.* (λ, A) *if* $\sigma(\lambda, A) > 0$ *and* $T = T(\lambda, A) > 0$ *in PS3);*

Remark 5.1: It should be note that the definitions S1) - PS4) are equivalent to the statement that a solution of system of perturbed FDE (5.2) which start sufficiently close to the initial data of the unperturbed solution respectively remain close to it or eventually approach it.

Remark 5.2: All of the above definitions are independent of the behavior of the solutions of the system of unperturbed FDE (5.1).

In connection with Remark 5.2, we specifically indicate that the solution of the system of unperturbed FDE (5.1) may be stable, asymptotically stable or even unstable. To motivate the idea, we give the following example.

Example 5.1: Consider the following fractional differential equation

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x(t) = -ax(t), \quad a > 0, \\ \qquad\qquad x(t_0) = x_0 \end{cases} \tag{5.3}$$

It can be seen from Example 3.7 that solution of (5.3) is given by $x^*(t) = x(t, t_0, x_0) = x_0 E_q(-a(t - t_0)^q)$*. We also know that solution* $x^*(t)$ *is uniformly asymptotically stable and uniformly practically stable in the sense of Lyapunov studied in Chapter 3. On the other hand, consider the associated perturbed FDE*

$$\begin{cases} {}_{\tau_0}^{C}\mathcal{D}^q\, y(t) = -(a+b)y(t), \\ \qquad\qquad y(\tau_0) = y_0 \end{cases} \tag{5.4}$$

whose solution is $y(t, \tau_0, y_0) = y_0 E_q(-(a+b)(t - \tau_0)^q)$*. Under the idea of Definition 5.1 we are interested in the difference of solutions as follow*

$$y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0) = \\ y_0 E_q(-(a+b)(t - \tau_0)^q) - x_0 E_q(-a(t - \tau_0)^q)). \tag{5.5}$$

Now we consider the cases that is possible here.

- *Case-1*

Let $a + b > 0$*. Then the difference of solutions (5.5) approaches* 0 *as* $t \to \infty$*. Thus the perturbed FDE (5.4) is asymptotically stable relative to (5.3) with ITD.*

• *Case-2*

Let $a + b < 0$. *Then the perturbed FDE (4.4) is unstable relative to (4.1) with ITD.*

Corresponding to the different types of stability defined above, we can define the concepts of boundedness and related Lagrange stability of system of perturbed FDE (5.2) relative to unperturbed system (5.1) with ITD.

Definition 5.2: The system of perturbed FDE (5.2) is said to be

B1) equi-bounded with ITD *if given* $\alpha > 0$, *there exist* $\sigma = \sigma(\tau_0, \alpha) > 0$ *and* $\beta = \beta(\tau_0, \alpha) > 0$ *such that* $\|y_0 - x_0\| \leq \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \beta,\ t \geq \tau_0$;

B2) uniformly bounded with ITD, *if B1) holds with* β *and* σ *independent of* τ_0;

B3) ultimately bounded with ITD *if for each* $\alpha > 0$ *there exist* $N > 0$, $\sigma = \sigma(\tau_0, \alpha) > 0$ *and* $T = T(\tau_0, \alpha) > 0$ *such that inequalities* $\|y_0 - x_0\| \leq \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)\| < N,\ \ t \geq \tau_0 + T$;

B4) uniformly ultimately bounded with ITD, *if* σ *and* T *in B3) is independent of* τ_0;

B5) ultimately bounded with ITD, *if B1) and B3) hold simultaneously;*

B6) uniformly ultimately bounded with ITD, *if B2) and B4) hold simultaneously;*

A1) attractive in the large with ITD *if for each* $\epsilon > 0$, $\alpha > 0$ *there exist* $\sigma = \sigma(\tau_0, \epsilon, \alpha) > 0$ *and* $T = T(\tau_0, \epsilon, \alpha) > 0$ *such that* $\|y_0 - x_0\| < \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $\|y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)\| < \epsilon$ *for* $t \geq \tau_0 + T$;

A2) uniformly attractive with ITD, *if* σ *and* T *in A1) are independent of* τ_0;

L1) Lagrange stable *if B1) and A1) hold together;*

L2) uniformly Lagrange stable *if B2) and A2) hold together.*

In our further investigations we will use scalar FDE (4.10) and related Definition (4.4) – Definition (4.6) as in Chapter 4. We will study the connection between stability, practical stability, boundedness and Lagrange stability of the scalar FDE (4.10) and corresponding stability, practical stability, boundedness and Lagrange stability of system of perturbed FDE (5.2) relative to unperturbed system (5.1) with ITD. The principal mathematical technique employed is a fractional order extension of Lyapunov's method which is applied to the difference of the solutions studied. In this context, we define Caputo fractional Dini derivative of the function $V(t,x) \in \Lambda(I,\Delta)$ with ITD along solutions of the system (5.1) and (5.2) ITD as follow

$$\begin{cases} {}_{\tau_0}^{C}\mathcal{D}_+^q V(t,x,y,\eta,x_0,y_0) = \lim_{h\to 0^+} sup \frac{1}{h^q}[V(t,y-x) - V(\tau_0, y_0 - x_0) \\ - \sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1}\binom{q}{r}(V(t-rh, y-x-h^q(F(t,y) \ - f(t-\eta,x)) \\ \qquad -V(\tau_0, y_0 - x_0))] \end{cases} \tag{5.6}$$

where $F(t,y) = f(t,y) + R(t,y)$, $t, \tau_0 \in I$, $y - x,\ y_0 - x_0 \in \Delta$.

Now we will apply the introduced Caputo fractional Dini derivative (5.6) for some Lyapunov functions with generalization relative to perturbed system and ITD. In order to avoid repetitions of arguments used in Example 4.5, Corollary 4.2 and Example 4.6 we state the formulas directly.

Example 5.2: Let the Lyapunov function does not depend on the time variable, i.e. $V(t,x) \equiv V(x)\ for\ x \in \mathbb{R}$. *Then, applying formula (5.6) we obtain*

$$\begin{aligned} &{}_{\tau_0}^{C}\mathcal{D}_+^q V(t,x,y,\eta,x_0,y_0) \\ &\quad = \lim_{h\to 0^+} sup \frac{V(y-x) - V(y-x-h^q(F(t,y) - f(t-\eta,x))}{h^q} \\ &\qquad\qquad + [\,V(y-x) - V(y_0 - x_0)]\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)}. \end{aligned} \tag{5.7}$$

Corollary 5.1: Let $V(t,x) \equiv V(x) = x^2$ *for* $x \in \mathbb{R}$. *We deduce the following expression from Example 5.2*

$$\begin{aligned} {}_{\tau_0}^{C}\mathcal{D}_+^q V(t,x,y,\eta,x_0,y_0) &= 2(y-x)\big((F(t,y) - f(t-\eta,x))\big) \\ &\quad + [\,(y-x)^2 - (y_0 - x_0)^2]\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)}. \end{aligned} \tag{5.8}$$

Remark 5.3: In the case when $q \to 1$ the equality (5.8) is coincide with the known Dini derivative $\mathcal{D}^{+}V(t, y-x) = 2(y-x)(F(t,y) - f(t-\eta, x))$.

Example 5.3: Let $V(t,x) = m^2(t)x^2$ for $x \in \mathbb{R}$ where $m \in C^1(\mathbb{R}_+, \mathbb{R})$. Firstly, using the definition (5.6) we obtain Caputo fractional Dini derivative of the function $V(t,x)$:

$$
\begin{aligned}
&{}_{\tau_0}^{C}\mathcal{D}_{+}^{q}V(t,x,y,\eta,x_0,y_0) \\
&\quad = 2(y-x)m^2(t)\big(F(t,y) - f(t-\eta,x)\big) + ((y-x)^2 \times \\
&\qquad {}_{\tau_0}^{C}\mathcal{D}^{q}\,[m^2(t)]) + [\,(y-x)^2 - (y_0 - x_0)^2]\,\frac{m^2(\tau_0)(t-\tau_0)^{-q}}{\Gamma(1-q)}.
\end{aligned}
\tag{5.9}
$$

In the case $m(t) = 1$, the formula (5.9) is reduced to (5.8). On the other hand the Dini derivative of Lyapunov function with ITD for classical case $(q = 1)$ is

$$
\begin{aligned}
\mathcal{D}^{+}V(t, y-x) = 2(y-x)m^2(t)\big(F(t,y) - f(t-\eta,x)\big) \\
+(y-x)^2\frac{d}{dt}[m^2(t)].
\end{aligned}
\tag{5.10}
$$

Notice that first derivative of $m(t)$ in (5.10) is replaced by the fractional derivative in (5.9).

5.3. Main Results

In this part, we give a comparison theorem which establish a relation between the solutions of (5.1), (5.2) and scalar FDE with parameter (4.10). Then, we will obtain sufficient conditions for some stability and boundedness of system of perturbed FDE (5.2) relative to unperturbed system (5.1) with ITD. The proof is based on the second method of Lyapunov which is applied to the difference of the solutions of (5.1) and (5.2) where not only initial position but also initial time are different.

5.3.1. Comparison Results with ITD for Scalar FDE

Now we give a comparison theorem which establishes a relation between the solutions of (5.1), (5.2) and scalar FDE with parameter (4.10).

Lemma 5.1: Assume the following conditions are satisfied:

i) The function $x(t) = x(t; t_0, x_0) \in C^q([t_0, \tau_0 + \theta], \mathbb{R}^n)$ *and* $y(t) = y(t; \tau_0, y_0) \in C^q([\tau_0, \tau_0 + \theta], \mathbb{R}^n)$ *are solutions of system of FDE (5.1), (5.2) respectively,* $y(t) - x(t - \eta^*) \in \Delta$ *where* $\eta^* = \tau_0 - t_0$, $\Delta \in \mathbb{R}^n$ *and* θ *is a given number.*

ii) The function $V \in \Lambda([\tau_0, \tau_0 + \theta], \Delta)$, $g \in C[[\tau_0, \tau_0 + \theta] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$ *such that for* $t \in (\tau_0, \tau_0 + \theta]$ *the inequality*

$$ {}_{\tau_0}^{C}\mathcal{D}_{+}^{q} V(t, x(t - \eta^*), y(t), \eta^*, x_0, y_0) \leq g(t, V\big(t, y(t) - x(t - \eta^*)\big), \eta^*) \qquad (5.11)$$

holds.

Then $V(\tau_0, y_0 - x_0) \leq u_0$ *implies* $V(t, y(t) - x(t - \eta^*)) \leq u^*(t)$ *for* $t \in [\tau_0, \tau_0 + \theta]$ *where* $u^*(t) = u(t; \tau_0, u_0, \eta^*)$ *is the maximal solution of IVP for scalar FDE (4.10) with* $\eta = \eta^*$.

Proof 5.1: Let the function $m(t) \in C([\tau_0, \tau_0 + \theta], \mathbb{R}_+)$ *be defined by* $m(t) = V(t, y(t) - x(t - \eta^*))$. *Then from Remark 3.2 we obtain for* $t \in (\tau_0, \tau_0 + \theta]$ *the equality* $\lim_{h \to 0} sup \frac{1}{h^q} (y(t) - x(t - \eta^*) - (y_0 - x_0) - S(y(t), x(t - \eta^*), h)) = F(t, y(t)) - f(t - \eta^*, x(t - \eta^*))$,

where $S(y(t), x(t - \eta^*), h) = \sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1} \binom{q}{r} [y(t - rh) - x(t - \eta^* - rh) - (y_0 - x_0)]$. *Therefore* $S(y(t), x(t - \eta^*), h) = y(t) - x(t - \eta^*) - (y_0 - x_0) - h^q(F(t, y(t)) - f(t - \eta^*, x(t - \eta^*))) - \epsilon(h^q)$

or $y(t) - x(t - \eta^*) - h^q(F(t, y(t)) - f(t - \eta^*, x(t - \eta^*))) = S(y(t), x(t - \eta^*), h) + (y_0 - x_0) + \epsilon(h^q)$ *with* $\frac{\epsilon(h^q)}{h^q}$ *as* $h \to 0^+$. *Then for any* $t \in (\tau_0, \tau_0 + \theta]$ *using (3.9) and (3.10) we obtain*

$$
\begin{cases}
m(t) - m(\tau_0) - \left[\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \left(m(t-rh) - m(\tau_0)\right)\right] \\
= V\big(t, z(t)\big) - V(\tau_0, z_0) - \sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \times \\
[V(t-rh, z(t) - z_0 - h^q \left(F\big(t, y(t)\big) - f\big(t-\eta^*, x(t-\eta^*)\big)\right) \\
-V(\tau_0, z_0)] + \left[\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \left(V\big(t-rh, S(z(t), h) + z_0 + \epsilon(h^q)\big)\right)\right] \\
-\left[\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1} \binom{q}{r} \big(V(t-rh, z(t-rh))\big)\right]
\end{cases}
\tag{5.12}
$$

where $z(t) = y(t) - x(t-\eta^*)$, $z_0 = y_0 - x_0$ *and* η^* *is defined in i). After arrangement in the expression (5.12) via* V *is locally Lipschitzian in its second argument with a Lipschitz constant* $L > 0$ *we obtain*

$$
\begin{cases}
\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]} (-1)^{r+1}\binom{q}{r}\left[V(t-rh, S(y(t), x(t-\eta^*), h) + z_0 + \epsilon(h^q))\right. \\
\qquad \left. -V\big(t-rh, z(t-rh)\big)\right] \\
\qquad \le L\left\|\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]}\binom{q}{r}\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]}(-1)^{j+1}\binom{q}{j}(z(t-jh)-z_0)\right. \\
\left. -\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]}(-1)^{r+1}\binom{q}{r}(z(t-rh)-z_0)\right\| + L\epsilon(h^q)\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]}\binom{q}{r} \\
= L\left\|\sum_{j=0}^{\left[\frac{t-\tau_0}{h}\right]}(-1)^{r+1}\binom{q}{r}\sum_{j=1}^{\left[\frac{t-\tau_0}{h}\right]}(-1)^{j+1}\binom{q}{j}(z(t-jh)-z_0)\right\| \\
\qquad +L\epsilon(h^q)\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]}\binom{q}{r}
\end{cases}
\tag{5.13}
$$

Substitute (5.13) in (5.12), divide both sides by h^q, take a limit as $h \to 0^+$, use (5.6), ii) and $\sum_{r=0}^{\left[\frac{t-\tau_0}{h}\right]}\binom{q}{r} z^r = (1+z)^q$ if $|z| \le 1$ we obtain for any $t \in (\tau_0, \tau_0+\theta]$ the inequality

$$
\begin{cases}
{}_{\tau_0}^{C}\mathcal{D}_+^q m(t) \le {}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x(t-\eta^*), y(t), \eta^*, x_0, y_0) \\
+L\lim_{h\to 0^+} sup\left\|\sum_{j=1}^{\left[\frac{t-\tau_0}{h}\right]}\frac{1}{h^q}(-1)^{j+1}\binom{q}{j}(z(t-jh)-z_0)\sum_{r=0}^{\left[\frac{t-\tau_0}{h}\right]}(-1)^{r+1}\binom{q}{r}\right\| \\
\qquad +L\lim_{h\to 0^+} sup\frac{\epsilon(h^q)}{h^q}\lim_{h\to 0} sup\sum_{r=1}^{\left[\frac{t-\tau_0}{h}\right]}\binom{q}{r} \\
\qquad = {}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x(t-\eta^*), y(t), \eta^*, x_0, y_0) \\
\qquad \le g(t, V(t, y(t)-x(t-\eta^*), \eta^*) = g(t, m(t), \eta^*).
\end{cases}
\tag{5.14}
$$

Finally, we get $m(t) = V(t, y(t) - x(t - \eta^*) \leq u^*(t)$ *for* $t \in [\tau_0, \tau_0 + \theta]$ *by applying Lemma 3.3.* ■

The result of Lemma 5.1 is also true on the half line.

Lemma 5.2: Let the conditions of Lemma 5.1 are satisfied for $\theta = \infty$, *i.e. for* $t \geq t_0$ *and* $t \geq \tau_0$ *respectively. Then* $V(\tau_0, y_0 - x_0) \leq u_0$ *implies* $V(t, y(t) - x(t - \eta^*)) \leq u^*(t)$ *for* $t \geq \tau_0$.

Corollary 5.2: Let the conditions of Lemma 5.2 be satisfied and the inequality ${}_{\tau_0}^{C}\mathcal{D}_{+}^{q} V(t, x(t - \eta^*), y(t), \eta^*, x_0, y_0) \leq 0$ *holds for* $t > \tau_0$.
Then the estimate $V(t, y(t) - x(t - \eta^*)) \leq V(\tau_0, y_0 - x_0)$ *holds for* $t \geq \tau_0$.

Proof 5.2: The proof follows directly from the fact that corresponding IVP for scalar FDE ${}_{\tau_0}^{C}\mathcal{D}^{q}\, u(t) = 0,\ u(\tau_0) = V(\tau_0, y_0 - x_0)$ *has a unique solution* $u(t) = V(\tau_0, y_0 - x_0)$ *for* $t \geq \tau_0$. ■

Corollary 5.3: Let the condition of Lemma 5.2 be satisfied and the inequality ${}_{\tau_0}^{C}\mathcal{D}_{+}^{q} V(t, x(t - \eta^*), y(t), \eta^*, x_0, y_0) \leq -\gamma V\big(t, y(t) - x(t - \eta^*)\big) + C\eta^*$ *holds for* $t > \tau_0$, *where* $\gamma > 0$ *and* $C \in \mathbb{R}$ *are constants.*
Then the inequality $V\big(t, y(t) - x(t - \eta^*)\big) \leq [V(\tau_0, y_0 - x_0) - \frac{1}{\mu} C\eta^*] E_q(-\gamma(t - \tau_0)^q) + \frac{1}{\mu} C\eta^*$ *holds for* $t \geq \tau_0$.

Proof 5.3: The proof of Corollary 5.3 follows directly from the fact that corresponding IVP for scalar FDE ${}_{\tau_0}^{C}\mathcal{D}^{q}\, u = -\mu u + C\eta^*$, $u_0 = V(\tau_0, y_0 - x_0)$ *has a unique solution* $u(t) = [V(\tau_0, y_0 - x_0) - \frac{1}{\mu} C\eta^*] E_q(-\gamma(t - \tau_0)^q) + \frac{1}{\mu} C\eta^*$ *for* $t \geq \tau_0$. ■

5.3.2. Some Stability and Boundedness Criteria

Theorem 5.1: Let the following conditions be satisfied:

A1) The function $g \in C[\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $g(t, 0,0) \equiv 0$ *and for any* η, $u_0 \in \mathbb{R}$, *the IVP for scalar FDE (4.10) has a solution* $u(t) = u(t; \tau_0, u_0, \eta) \in C^q([\tau_0, \infty), \mathbb{R})$.

A2) There exists a function $V \in \Lambda([\mathbb{R}_+, S(\rho))$ *such that*

i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $(t, x) \in \mathbb{R}_+ \times S(\rho)$ *where* $a, b \in K$.

ii) for any $t > \tau_0 \in \mathbb{R}_+$, $x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in S(\rho)$, $y_0 - x_0 \in S(\rho)$ *the inequality*

$${}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x, y, \eta, x_0, y_0) \leq g(t, V(t, y - x), \eta) \tag{5.15}$$

holds.

Then the stability properties of zero solution of the scalar FDE (4.10) imply the corresponding stability properties of system of perturbed FDE (5.2) relative to unperturbed system (5.1) with ITD.

Proof 5.1: Initially, we assume that the zero solution of scalar FDE (4.10) is uniformly stable w.r.t parameter. Let $\epsilon > 0$ *be a number,* $\epsilon < \rho$. *Then by definition* $S^*1)$ *for given* $b(\epsilon) > 0$ *there exist* $\delta_1 = \delta_1(\epsilon) > 0$ *and* $\sigma = \sigma(\epsilon) > 0$ *such that* $|u_0| < \delta_1$ *and* $|\eta| < \sigma$ *imply*

$$|u(t; \tau_0, u_0, \eta)| < b(\epsilon) \text{ for } t \geq \tau_0 \tag{5.16}$$

where $u(t, \tau_0, u_0, \eta)$ *is any solution of (4.10). From the condition i) of A2) the inequalities*

$$\|y_0 - x_0\| \leq \delta, \qquad a(\|y_0 - x_0\|) \leq \delta_1 \tag{5.17}$$

holds together with $\delta = a^{-1}(\delta_1)$*. Now let points* $x_0, y_0 \in \mathbb{R}^n$ *and* $\tau_0, t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \delta$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$*. Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (5.1) and (5.2) with the chosen initial data* (τ_0, y_0) *and* (t_0, x_0)*, respectively. Let* $u_0^* = V(\tau_0, y_0 - x_0)$*. From condition A2) i) and choice of* δ *it follows* $u_0^* = V(\tau_0, y_0 - x_0) < a(\|y_0 - x_0\|) < a(\delta) < \delta_1$*. Therefore, the maximal solution* $u^*(t) = u(t; \tau_0, u_0^*, \eta^*) \in C^q([\tau_0, \infty), \mathbb{R})$ *of FDE (4.10) satisfies inequality (5.16). We claim that*

$$\|y(t; \tau_0, y_0) - x(t - \eta^*, t_0, x_0)\| < \epsilon \text{ for } t \geq \tau_0. \tag{5.18}$$

Suppose inequality (5.18) is not true. Therefore, there would exist a point $t_1 > \tau_0$ *and a solution* $y(t, \tau_0, y_0)$ *of (5.2) such that*

$$\begin{cases} \|y(t_1, \tau_0, y_0) - x(t_1 - \eta^*, t_0, x_0)\| = \epsilon \text{ and} \\ \|y(t, \tau_0, y_0) - x(t - \eta^*, t_0, x_0)\| < \epsilon \text{ for } \tau_0 \leq t < t_1. \end{cases} \tag{5.19}$$

After now in order not to repeat, we use $y(t)$*,* $x(t - \eta^*)$ *instead of* $y(t, \tau_0, y_0)$*,* $x(t - \eta^*, t_0, x_0)$ *respectively. In view of (5.20), the inclusion* $(t, y(t) - x(t - \eta^*)) \in S(\rho)$ *is valid for* $t \in [\tau_0, t_1]$*. By using (5.15) and applying Lemma 5.1 for* $\Delta = S(\rho)$*,* $\theta = t_1 - \tau_0$ *we have*

$$V(t, y(t) - x(t - \eta^*)) \leq u(t; t_0, u_0^*, \eta^*), \ t \in [\tau_0, t_1]. \tag{5.20}$$

Consequently, in view of the relations (5.16), (5.18), (5.20), condition A2), the choice of t_1 *and* u_0^* *we obtain*

$$\begin{aligned} b(\epsilon) &\leq b(\|y(t_1) - x(t_1 - \eta^*)\|) \\ &\leq V\big(t_1, y(t_1) - x(t_1 - \eta^*)\big) \leq u^*(t_1) < b(\epsilon). \end{aligned} \tag{5.21}$$

The obtained contradiction proves the validity of inequality (5.18) which proves that system of perturbed FDE (5.2) is uniformly stable with ITD relative to unperturbed system (5.1).

Secondly, we assume that the scalar FDE (4.10) is uniformly asymptotically stable w.r.t parameter. From the first part of the proof it follows that system of perturbed FDE (5.2) is stable with ITD relative to unperturbed system (5.1). Therefore, from the definition S1) there exist $\delta_0 = \delta_0(\rho)$ *and* $\sigma_0 = \sigma_0(\rho)$ *for* $\epsilon = \rho$ *such that*

$$\begin{cases} \|y_0 - x_0\| < \delta_0 \text{ and } \ |\tau_0 - t_0| < \sigma_0 \ \text{ imply} \\ \qquad \|y(t) - x(t-\eta)\| < \rho \,, \qquad t \geq \tau_0 \end{cases} \tag{5.22}$$

In order to prove S3), we let $0 < \epsilon < \rho$. *Since* $S^*3)$ *holds under assumption, given* $b(\epsilon) > 0$ *there exist* $\delta_1^* > 0$, $\sigma_1^* > 0$ *and* $T = T(\epsilon) > 0$ *such that* $|u_0| < \delta_1^*$ *and* $|\eta| < \sigma_1^*$ *imply*

$$|u(t; t_0, u_0, \eta)| < b(\epsilon) \ , \qquad t \geq \tau_0 + T. \tag{5.23}$$

Now let points $x_0, y_0 \in \mathbb{R}^n$ *and* $\tau_0, t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \delta$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$, $\delta = min(\delta_0, a^{-1}(\delta_1^*))$ *and* $\sigma = min(\sigma_0, \sigma_1^*)$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (4.1) and (4.2) with the chosen initial data* (τ_0, y_0) *and* (t_0, x_0) *respectively.*
On the other hand the estimate (5.20) is valid for all $t \geq \tau_0$ *by using (5.22), condition A2) and Lemma 5.1 with* $\Delta = S(\rho)$, $\theta = \infty$. *We will prove that*

$$\|y(t) - x(t - \eta^*)\| < \epsilon \quad \text{for } t \geq \tau_0 + T. \tag{5.24}$$

Assume the opposite, i.e. there exist a sequence $\{t^{(n)}\}, t^{(n)} \geq \tau_0 + T,\ t^{(n)} \to \infty$ *as* $n \to \infty$ *such that*

$$\left\|y\left(t^{(n)}\right) - x\left(t^{(n)} - \eta^*\right)\right\| \geq \epsilon. \tag{5.25}$$

Finally, in view of the relations (5.23) - (5.25), condition A2), the choice of $t^{(n)}$ *and* $u_0 = a(\|y_0 - x_0\|) < a(\delta) < \delta_1^*$ *we obtain*

$$b(\epsilon) > u^*\left(t^{(n)}\right) \geq V\left(t^{(n)}, y\left(t^{(n)}\right) - x\left(t^{(n)} - \eta^*\right)\right) \\ \geq b\left(\left\|y\left(t^{(n)}\right) - x\left(t^{(n)} - \eta^*\right)\right\|\right) \geq b(\epsilon). \tag{5.26}$$

The obtained contradiction proves validity of inequality (5.24) which implies S4) holds, i.e. system of perturbed FDE (5.2) is asymptotically stable with ITD relative to unperturbed system (5.1). ■

Theorem 5.2: Let the conditions A1) of Theorem 5.1 be satisfied and A2) is replaced by as follow:

A2) There exists a function $V \in \Lambda([\mathbb{R}_+, S(A))$ *such that*

i) $b(\|x\|) \leq V(t,x) \leq a(\|x\|)$ *for* $(t,x) \in \mathbb{R}_+ \times S(A)$ *where* $a, b \in K$.

ii) for any $t > \tau_0 \in \mathbb{R}_+, x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in S(A)$, $y_0 - x_0 \in S(A)$ *the inequality*

$$ {}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x, y, \eta, x_0, y_0) \leq g(t, V(t, y - x), \eta) \tag{5.27}$$

holds.

Then the parametrically practically stability properties of the scalar FDE (4.10) with respect to $(a(\lambda), b(A))$ *imply the corresponding practical stability properties w.r.t.* (λ, A) *of system of perturbed FDE with ITD relative to unperturbed system (5.1).*

Proof 5.2: From condition A2) according to Definition 4.5 there exists a positive number $\sigma = \sigma(\lambda, A) > 0$ *such that for* $u_0 \in \mathbb{R}$: $|u_0| < a(\lambda)$ *and* η: $|\eta| < \sigma$ *we have*

$$|u(t; t_0, u_0, \eta)| < b(A), \qquad t \geq \tau_0. \tag{5.28}$$

Now let points $x_0, y_0 \in \mathbb{R}^n$ *and* $\tau_0, t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < a(\lambda)$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and*

$y(t) = y(t; \tau_0, y_0)$ of system of FDE (5.1) and (5.2) correspondingly with the chosen initial data (τ_0, y_0) and (t_0, x_0) respectively. Let $\tilde{u}_0 = V(\tau_0, y_0 - x_0)$. From condition A2) i) it follows $\tilde{u}_0 = V(\tau_0, y_0 - x_0) < a(\|y_0 - x_0\|) < a(\lambda)$. Therefore, the maximal solution $u^*(t) = u(t; \tau_0, \tilde{u}_0, \eta^*) \in C^q([\tau_0, \infty), \mathbb{R})$ of FDE (4.10) satisfies inequality (5.28). Then we claim that $\|y(t) - x(t - \eta^*)\| < A$ for $t \geq \tau_0$. Assume the opposite, i.e. there exists a point $t_1 > \tau_0$ such that $\|y(t_1) - x(t_1 - \eta^*)\| = A$ and $\|y(t) - x(t - \eta^*)\| < A$ for $\tau_0 \leq t < t_1$. Therefore the inclusion $(t, y(t) - x(t - \eta^*)) \in S(A)$ is valid for $t \in [\tau_0, t_1]$. By using (5.27) and applying Lemma 5.1 for $\Delta = S(A)$, $\theta = t_1 - \tau_0$ we obtain (5.20). From the choice of t_1, condition A2) and (5.18), (5.20) we obtain

$$\begin{aligned} b(A) &= b(\|y(t_1) - x(t_1 - \eta^*)\|) \\ &\leq V\big(t_1, y(t_1) - x(t_1 - \eta^*)\big) \leq u^*(t_1) < b(A). \end{aligned} \tag{5.29}$$

The obtained contradiction proves the system of perturbed FDE (5.2) is uniformly practically stable with ITD relative to (5.1) w.r.t. (λ, A). ■

Now we give boundedness and Lagrange stability criteria for system of perturbed FDE (5.2) with ITD relative to unperturbed system (5.1).

Theorem 5.3: Let the conditions of conditions A1) of Theorem 5.1 be satisfied and A2) is replaced by as follow:

A2) There exists a function $V \in \Lambda([\mathbb{R}_+, \mathbb{R}^n)$ such that

i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ where $a, b \in K_\infty$.

ii) for any $t > \tau_0 \in \mathbb{R}_+$, $x, y, x_0, y_0 \in \mathbb{R}^n$ the inequality

$$ {}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x, y, \eta, x_0, y_0) \leq g(t, V(t, y - x), \eta) \tag{5.30}$$

holds.

If scalar FDE (4.10) is uniformly equi-bounded (uniformly Lagrange stable) w.r.t. parameter, then the system of perturbed FDE (5.2) is uniformly equi-bounded (uniformly Lagrange stable) with ITD relative to unperturbed system (5.1).

Proof 5.3: Initially, we consider the first case of our assumption that is the FDE (4.10) is uniformly equi-bounded w.r.t. parameter. Let $\alpha > 0$ *be given. Since* $B^*1)$ *holds, given* $\alpha_1 = a(\alpha) > 0$ *there exist* $\beta_1 = \beta_1(\alpha_1) > 0$ *and* $\sigma = \sigma(\alpha_1) > 0$ *such that* $|u_0| < \alpha_1$ *and* $|\eta| < \sigma$ *imply*

$$|u(t; t_0, u_0, \eta)| < \beta_1 \text{ for } t \geq \tau_0. \tag{5.31}$$

Choose $u_0 = a(\|y_0 - x_0\|)$ *and* $\beta = \beta(\alpha) > 0$, *where* $b(\beta) \geq \beta_1$ *since* $b \in K_\infty$. *Consider solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (5.1) and (5.2) such that* $\|y_0 - x_0\| \leq \alpha$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *With* $\beta = \beta(\alpha) > 0$ *we claim that B2) holds i.e.* $\|y(t) - x(t - \eta^*)\| < \beta,\ t \geq \tau_0$ *holds. Assume the claim is not true. Therefore, there exist a point* $t^* > \tau_0$ *such that* $\|y(t^*) - x(t^* - \eta)\| = \beta$. *Since* $u_0 = a(\|y_0 - x_0\|) \leq a(\alpha) = \alpha_1$, *we get by (5.30), (5.31), condition A2) and Lemma 5.1*

$$b(\beta) \leq V(t^*, y(t^*) - x(t^* - \eta)) \leq u^*(t^*) < \beta_1 \leq b(\beta). \tag{5.32}$$

This contradiction shows that the perturbed system (5.2) is uniformly equi-bounded with ITD relative to unperturbed system (5.1).

Next we consider the second case of the assumption of Theorem 5.3 that is the FDE (4.10) is uniformly Lagrange stable w.r.t. parameter, which implies that perturbed system (5.2) is uniformly equi-bounded with ITD relative to unperturbed system (5.1) from the first part. In order to prove L2), we need to show A2) holds. Let $\epsilon > 0$, $\alpha > 0$ *be given. Since* $A^*1)$ *holds, given* $\alpha_1 = a(\alpha) > 0$ *and* $b(\epsilon) > 0$ *there exist* $\sigma = \sigma(\epsilon, \alpha)$ *and* $T = T(\epsilon, \alpha)$ *such that* $|u_0| < \alpha_1$ *and* $|\eta| < \sigma$ *imply* $|u(t; \tau_0, u_0, \eta)| < b(\epsilon),\ t \geq \tau_0 + T$. *Now let points* $x_0, y_0 \in \mathbb{R}^n$ *and* $\tau_0, t_0 \in \mathbb{R}_+$ *be such that* $\|y_0 - x_0\| < \alpha$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (5.1) and (5.2)* *correspondingly with the chosen initial data* (τ_0, y_0) *and* (t_0, x_0) *respectively. We claim that A2) holds with* T, σ. *Suppose that is not true. Therefore, there exist a sequence* $\{t^{(n)}\}$, $t^{(n)} > \tau_0 + T$, $t^{(n)} \to \infty$ *as* $n \to \infty$ *such that*

$$\|y(t^{(n)}) - x(t^{(n)} - \eta^*)\| \geq \epsilon^*. \tag{5.33}$$

Finally, in view of the relation (5.33), condition A2), the choice of $t^{(n)}$ *and* $u_0 = a(\|y_0 - x_0\|) < a(\alpha) < \alpha_1$ *we obtain*

$$\begin{aligned} b(\epsilon) > u^*(t^{(n)}) \geq V\left(t^{(n)}, y(t^{(n)}) - x(t^{(n)} - \eta^*)\right) \\ \geq b(\|y(t^{(n)}) - x(t^{(n)} - \eta^*)\|) \geq b(\epsilon) \end{aligned} \tag{5.34}$$

which implies that A2) holds. Then system of perturbed FDE (5.2) is Lagrange stable with ITD relative to (5.1) since B2) and A2) holds together. ■

6. SOME STABILITY AND BOUNDEDNESS OF SYSTEM OF PERTURBED FRACTIONAL DIFFERENTIAL EQUATION WITH INITIAL TIME DIFFERENCE IN TERMS OF TWO MEASURES

6.1. Introduction

There are many stability concepts presented in the literature such as the partial stability, eventual stability, conditional stability, Lipschitz stability, relative stability and so on. In 1960, [57] introduced the concept of stability in terms of two measures which unified the foregoing stability concepts. Then, the theories of the stability in terms of two measures have been successfully developed in [58] and some stability and boundedness results are obtained by means of various types of Lyapunov functions for several kinds of differential equations in [59]-[61] and references therein. We have investigated some stability and boundedness in terms of two measures for system of perturbed FDE with ITD relative to system of unperturbed FDE in this chapter. We begin with section 6.2 which includes the necessary definitions of stability, practical stability, boundedness and Lagrange stability in terms of two measures with ITD. Then, we have generalized the main results obtained in previous chapter 5 by using the notion of two measures.

6.2. Main Definitions and Concepts with Two Measures

Consider the following IVP for the system of FDE for $0 < q < 1$,

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}^q\, x(t) = f(t, x(t)) \\ \qquad x(t_0) = x_0 \end{cases} \tag{6.1}$$

where $t_0 \in \mathbb{R}_+,\ f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$. Denote the solution of (6.1) by $x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$. Let $\tau_0 \in \mathbb{R}_+$, $\tau_0 \neq t_0$ be a different initial time. In addition to (6.1), we also consider the associated system of perturbed FDE with different initial data

$$\begin{cases} {}_{\tau_0}^{C}\mathcal{D}^q\, y(t) = F(t, y(t)) \\ \qquad y(\tau_0) = y_0 \end{cases} \tag{6.2}$$

where $F(t, y(t)) = f(t, y(t)) + R(t, y(t))$, $R \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is called the perturbation term.

We give the following set of measures for future use.

Definition 6.1: ([58]) A function $h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *is said to belong to the class* Γ *if* $\inf_{x\in\mathbb{R}^n} h(t, x) = 0$ *for all* $t \in \mathbb{R}_+$.

Before giving our main definitions on stability, practical stability, boundedness and Lagrange stability in terms of two measures with ITD, we recall the definition of stability in terms of two measures for (6.1) which can be found in literature [16], [58].

Definition 6.2: Let $h, h_0 \in \Gamma$. Then system (6.1) is said to be

S1) (h_0, h)-equistable *if for each* $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exists* $\delta = \delta(t_0, \epsilon) > 0$ *such that* $h_0(t_0, x_0) < \delta$ *implies* $h(t, x(t, t_0, x_0)) < \epsilon$, $t \geq t_0$;
S2) (h_0, h)-uniformly stable, *if S1) holds with* δ *independent of* $t_0 \in \mathbb{R}_+$.

Remark 6.1: ([58]) Stability in terms of two measures enable us to unify a variety of stability notions found in the literature. When we endow h_0, h *with explicit form, the* (h_0, h)*-stability reduces to the other stability such as:*

1) set $h_0(t, x) = h(t, x) = \|x\|$, *then* (h_0, h)*-stability means the corresponding Lyapunov stability of the zero solution;*
2) set $h_0(t, x) = h(t, x) = \|x - x^*\|$, *then* (h_0, h)*-stability means the corresponding Lyapunov stability of solution* x^*;
3) set $h_0(t, x) = \|x\|$, $h(t, x) = |x|_s$, $1 \leq s < n$, *then* (h_0, h)*-stability means the corresponding partial stability of the trivial solution;*
4) set $h_0(t, x) = h(t, x) = d(x, A)$, *where* $A \subset \mathbb{R}^n$, *then* (h_0, h)*-stability means the corresponding stability of an invariant set* A;

5) set $h_0(t,x) = d(x,A)$, $h(t,x) = d(x,B)$, *where* $A \subset B \subset \mathbb{R}^n$, *then* (h_0, h)-*stability means the corresponding stability of a conditionally invariant set* B *with respect to* A.

We shall introduce the following definitions of stability, practical stability in terms of two measures (h_0, h), in short (h_0, h)-stability, of system of perturbed FDE (6.2) with ITD relative to unperturbed system (6.1).

Definition 6.3: The solution $y(t, \tau_0, y_0)$ *of system of perturbed FDE (6.2) is said to be:*

S1) (h_0, h)-equistable with ITD relative to (6.1) *if given* $\epsilon > 0$ *and* $\tau_0 \in \mathbb{R}_+$, *there exist* $\delta = \delta(\tau_0, \epsilon) > 0$ *and* $\sigma = \sigma(\tau_0, \epsilon) > 0$ *such that* $h_0(\tau_0, y_0 - x_0) < \delta$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $h(y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) < \epsilon$ *for* $t \geq \tau_0$, *where* $x(t, t_0, x_0)$ *is a solution of (6.1) and* $\eta = \tau_0 - t_0$;

S2) (h_0, h)-uniformly stable with ITD, *if S1) holds with* δ *and* σ *independent of* $\tau_0 \in \mathbb{R}_+$;

S3) (h_0, h)-asymptotically stable with ITD, *if S1) holds and given* $\epsilon > 0$ *and* $\tau_0 \in \mathbb{R}_+$, *there exist* $\delta_0 = \delta_0(\tau_0) > 0$, $\sigma_0 = \sigma_0(\tau_0) > 0$ *and* $T = T(\tau_0, \epsilon) > 0$ *such that* $h_0(\tau_0, y_0 - x_0) < \delta_0$ *and* $|\tau_0 - t_0| < \sigma_0$ *imply* $h(y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) < \epsilon$ *for* $t \geq \tau_0 + T$;

S4) (h_0, h)-uniformly asymptotically stable with ITD, *S2) and S3) hold with* δ_0, σ_0 *and* T *in S3) are independent of* τ_0;

PS1) (h_0, h)-practically stable with ITD *w.r.t.* (λ, A), *if there exists a* $\sigma = \sigma(\tau_0, \lambda, A) > 0$ *such that the inequalities* $h_0(\tau_0, y_0 - x_0) < \lambda$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $h(y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) < A$ *for* $t \geq \tau_0$, *where a couple of real numbers* (λ, A) *with* $0 < \lambda < A$ *be given;*

PS2) (h_0, h)-uniformly practically stable with ITD *if* $\sigma = \sigma(\lambda, A) > 0$ *in PS1) is independent of* τ_0;

PS3) (h_0, h)-attractive practically stable with ITD w.r.t. (λ, A, T) *if there exist* $\sigma = \sigma(\tau_0, \lambda, A) > 0$ *and* $T = T(\tau_0, \lambda, A) > 0$ *such that the inequalities* $h_0(\tau_0, y_0 - x_0) < \lambda$ and $|\tau_0 - t_0| < \sigma$ *imply* $h(y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) < A$ *for* $t \geq \tau_0 + T$;

PS4) (h_0, h)-uniformly attractive practically stable with ITD *if* $\sigma(\lambda, A) > 0$ *and* $T = T(\lambda, A) > 0$ *in PS3) is independent of* τ_0.

Remark 6.2: We note that in the case when $h_0(t, x) = \|x\|$ *the* (h_0, h)*-stability and practically stability properties with ITD reduces to stability and practically stability properties with ITD studied in previous chapter 5.*

Corresponding to the different types of (h_0, h)-stability with ITD defined above, we can define the concepts of (h_0, h)-boundedness and related (h_0, h)-Lagrange stability with ITD.

Definition 6.4: The system of perturbed FDE (6.2) is said to be:

B1) (h_0, h)-equi-bounded with ITD *if given* $\alpha > 0$ *and* $\tau_0 \in \mathbb{R}_+$ *there exist* $\sigma = \sigma(\tau_0, \alpha) > 0$ *and* $\beta = \beta(\tau_0, \alpha) > 0$ *such that* $h_0(\tau_0, y_0 - x_0) \leq \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $h(y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) < \beta$ *for* $t \geq \tau_0$*;*
B2) (h_0, h)-uniformly bounded with ITD, *if (B1) holds with* β *and* σ *independent of* $\tau_0 \in \mathbb{R}_+$*;*
A1) (h_0, h)-attractive in the large with ITD, *if for each* $\epsilon > 0$, $\alpha > 0$ *there exist* $\sigma = \sigma(\tau_0, \epsilon, \alpha) > 0$ *and* $T = T(\tau_0, \epsilon, \alpha) > 0$ *such that* $h_0(\tau_0, y_0 - x_0) < \alpha$ *and* $|\tau_0 - t_0| < \sigma$ *imply* $h(y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) < \epsilon$ *for* $t \geq \tau_0 + T$*;*
A2) (h_0, h)-uniformly attractive with ITD, *if* σ *and* T *in A1) are independent of* $\tau_0 \in \mathbb{R}_+$*;*
L1) (h_0, h)- Lagrange stable *if B1) and A1) hold;*
L2) (h_0, h)- uniformly Lagrange stable *if B2) and A2) hold simultaneously.*

Remark 6.3: We note that in the case when $h_0(t, x) = \|x\|$ *the* (h_0, h)*-boundedness and Lagrange stability with ITD reduces to boundedness and Lagrange stability with ITD studied in previous chapter.*

In our further investigations we will use scalar FDE (4.10) and related Definition 4.4 – Definition 4.6 as in Chapter 5. We will study the connection between stability, practical stability, boundedness and Lagrange stability of the scalar

FDE (4.10) and corresponding (h_0, h)-stability, (h_0, h)-practical stability, (h_0, h)-boundedness and (h_0, h)-Lagrange stability of system of perturbed FDE (6.2) relative to unperturbed system (6.1) with ITD.

It is convenient to give following definition from [58] to characterize Lyapunov functions.

Definition 6.5: Let $h \in \Gamma$. The function $V(t,x) \in \Lambda$ is said to be

i) h-positive definite *if there exists a* $\rho > 0$ *and a function* $b \in K$ *such that* $h(t,x) < \rho$ *implies* $b(h(t,x)) \leq V(t,x)$;

ii) h-descrescent *if there exists a* $\rho > 0$ *and a function* $a \in K$ *such that* $h(t,x) < \rho$ *implies* $V(t,x) \leq a(h(t,x))$;

We will use a property of the functions from class Γ in the following definition from [58].

Definition 6.6: Let $h_0, h \in \Gamma$. Then h_0 is called uniformly finer than *h if there exists a* $\rho > 0$ *and a function $\varphi \in K$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \varphi(h_0(t,x))$;*

6.3. Main Results

In this part, we will obtain sufficient conditions for (h_0, h)-stability, practical stability, boundedness and Lagrange stability of system of perturbed FDE (6.2) with ITD relative to unperturbed system (6.1). The obtained results generalize the main results in chapter 5.

Theorem 6.1: Let the following conditions be satisfied:

A1) The function $g \in C[\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $g(t,0,0) \equiv 0$ and for any η, $u_0 \in \mathbb{R}$, the IVP for scalar FDE (4.10) has a solution $u(t) = u(t;\tau_0, u_0, \eta) \in C^q([\tau_0, \infty), \mathbb{R})$.

A2) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h.

A3) There exists a function $V \in \Lambda([\mathbb{R}_+, S(h,\rho))$ such that

i) V is h-positive definite, h_0-decrescent.

ii) for any $t > \tau_0 \in \mathbb{R}_+, x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in S(h, \rho)$, $y_0 - x_0 \in S(h, \rho)$ *the inequality*

$$ {}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x, y, \eta, x_0, y_0) \le g(t, V(t, y - x), \eta) \tag{6.3}$$

holds, where $S(h, \rho) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t, x) < \rho\}$.

Then the stability w.r.t parameter properties of the zero solution of scalar FDE (4.10) imply the corresponding (h_0, h)*-stability properties of system of perturbed FDE (6.2) with ITD relative to (6.1).*

Proof 6.1: Since V is h-positive definite from condition i) of A3), there exist a $\lambda \in (0, \rho]$ *and* $b \in K$ *such that*

$$b\big(h(t, x)\big) \le V(t, x), \qquad (t, x) \in S(h, \lambda). \tag{6.4}$$

Let $\epsilon > 0$ *be a number,* $\epsilon \le \lambda$*. Initially, we assume that zero solution of the scalar FDE (4.10) is uniformly stable w.r.t. parameter. Then, given* $b(\epsilon) > 0$ *and* $\tau_0 \in \mathbb{R}_+$*, there exist a* $\delta_1 = \delta_1(\epsilon) > 0$ *and* $\sigma = \sigma(\epsilon) > 0$ *such that* $|u_0| < \delta_1$ *and* $|\eta| < \sigma$ *imply that*

$$|u(t, \tau_0, u_0, \eta)| < b(\epsilon) \ \text{ for } \ t \ge \tau_0. \tag{6.5}$$

On the other hand, there exist a $\lambda_0 = \varphi^{-1}(\lambda) > 0$ *and a function* $a \in K$ *such that*

$$h(\tau_0, y_0 - x_0) < \lambda \ \text{ and } \ V(\tau_0, y_0 - x_0) \le a(h_0(\tau_0, y_0 - x_0)) \tag{6.6}$$

for $(\tau_0, y_0 - x_0) \in S(h_0, \lambda_0)$ *by using V is* h_0*-decrescent and* h_0 *is uniformly finer than* h *in view of condition A3). Then we have the following inequality from (6.4)-(6.6)*

$$b(h(\tau_0, y_0 - x_0)) \le V(\tau_0, y_0 - x_0) \le a(h_0(\tau_0, y_0 - x_0)) \quad (6.7)$$

for $(\tau_0, y_0 - x_0) \in S(h_0, \lambda_0)$. *Now, choose* $\delta = \delta(\epsilon)$ *satisfying* $\delta \in (0, \lambda_0]$ *and* $a(\delta) < \delta_1$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (6.1) and (6.2) such that* $h_0(\tau_0, y_0 - x_0) < \delta$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Then from the inequality (6.5), (6.7) and the choice of* δ *follows that*

$$b(h(\tau_0, y_0 - x_0)) \le a(h(\tau_0, y_0 - x_0)) < a(\delta) < \delta_1 < b(\epsilon) \quad (6.8)$$

i.e. we get $h(\tau_0, y_0 - x_0) < \epsilon$. *We claim that* $h(t, y(t) - x(t - \eta^*)) < \epsilon$ *holds for* $t \ge \tau_0$. *Assume that the claim is not true, then there exists a point* $t_1 > \tau_0$ *such that*

$$h(t_1, y(t_1) - x(t_1 - \eta^*)) = \epsilon \ \text{ and } \ h(t, y(t) - x(t - \eta^*)) < \epsilon \quad (6.9)$$

for $\tau_0 \le t < t_1$ *in view of the fact that* $h(\tau_0, y_0 - x_0) < \epsilon$ *whenever* $h_0(\tau_0, y_0 - x_0) < \delta$. *Therefore the inclusion* $(t, y(t) - x(t - \eta^*)) \in S(h, \lambda)$ *is valid for* $t \in [\tau_0, t_1]$. *Hence, by using (6.3), A1) and Lemma 5.1, we have*

$$V(t, y(t, \tau_0, y_0) - x(t - \eta^*, t_0, x_0)) \le u(t, \tau_0, u_0, \eta^*), \ \ \tau_0 \le t \le t_1 \quad (6.10)$$

where $u_0 = V(\tau_0, y_0 - x_0)$. *Consequently, in view of the relations (6.4), (6.9), (6.10), condition A4), the choice of* t_1 *and* $u_0 = V(\tau_0, y_0 - x_0) \le a(h(\tau_0, y_0 - x_0)) < a(\delta) < \delta_1$ *we obtain*

$$\begin{aligned} b(\epsilon) &= b\left(h\big(t_1, y(t_1) - x(t_1 - \eta^*)\big)\right) \\ &\le V(t_1, y(t_1) - x(t_1 - \eta^*)) \le u^*(t_1) < b(\epsilon). \end{aligned} \quad (6.11)$$

The obtained contradiction proves the validity of the claim which implies that system of perturbed FDE (6.2) is (h_0, h)*-uniformly stable with ITD relative to unperturbed system (6.1).*

Secondly, we assume that the zero solution of scalar FDE (4.10) is uniformly asymptotically stable w.r.t. parameter. From the first part of the proof it follows that system of perturbed FDE (6.2) is (h_0, h)*-uniformly stable relative to unperturbed system (6.1) with ITD. Therefore, from the definition S1) there exist* $\delta_0 = \delta_0(\lambda)$ *and* $\sigma_0 = \sigma_0(\lambda)$ *for* $\epsilon = \lambda$ *such that* $h_0(\tau_0, y_0 - x_0) < \delta_0$ *and* $|\tau_0 - t_0| < \sigma_0$ *imply* $h(t, y(t) - x(t-\eta)) < \lambda$, $t \geq \tau_0$. *In order to prove S4), we let* $0 < \epsilon < \lambda$. *Since* $S^*4)$ *holds under assumption, given* $b(\epsilon) > 0$ *there exist* $\delta_1^* = \delta_1^*(\epsilon) > 0$, $\sigma_1^* = \sigma_1^*(\epsilon) > 0$ *and* $T = T(\epsilon) > 0$ *such that* $|u_0| < \delta_1^*$ *and* $|\eta| < \sigma_1^*$

$$|u(t, \tau_0, u_0, \eta)| < b(\epsilon) \quad for \;\; t \geq \tau_0 + T. \tag{6.12}$$

On the other hand the estimate (6.10) holds for $t \geq \tau_0$ *because of the inclusion* $(t, y(t) - x(t-\eta)) \in S(h, \lambda)$ *is valid for all* $t \geq \tau_0$. *Now choose* $\delta = min(\delta_0, \delta_1^*)$ *with* $a(\delta_0^*) < \delta_1^*$ *and* $\sigma = min(\sigma_0, \sigma_1^*)$. *Consider any solutions* $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (6.1) and (6.2) such that* $h_0(\tau_0, y_0 - x_0) < \delta$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *We claim that* $h(t, y(t) - x(t-\eta^*)) < \epsilon$ *for* $t \geq \tau_0 + T$. *Assume the claim is not true. Therefore, there exist a sequence* $\{t^{(n)}\}, t^{(n)} \geq \tau_0 + T$, $t^{(n)} \to \infty$ *as* $n \to \infty$ *such that*

$$h(y(t^{(n)}) - x(t^{(n)} - \eta^*)) \geq \epsilon. \tag{6.13}$$

Finally from the choice of $t^{(n)}$, *condition A2), inequalities (6.12), (6.13) and* $u_0 = V(\tau_0, y_0 - x_0) \leq a(h(\tau_0, y_0 - x_0)) < a(\delta) < \delta_1^*$ *we obtain*

$$\begin{aligned} b(\epsilon) &\leq b(h(y(t^{(n)}) - x(t^{(n)} - \eta^*)) \\ &\leq V(t^{(n)}, y(t^{(n)}) - x(t^{(n)} - \eta^*)) \leq u^*(t^{(n)}, \tau_0, u_0, \eta^*) < b(\epsilon) \end{aligned} \tag{6.14}$$

which proves that the claim is right, namely system of perturbed FDE (6.2) is (h_0, h)*-uniformly asymptotically stable with ITD relative to unperturbed system (6.1).* ■

Theorem 6.2: Let the conditions A1) of Theorem 6.1 be satisfied and A2), A3) are replaced by as follow:

A2) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than* h, *i.e.* $h(t,x) \leq \varphi(h_0(t,x))$, $\varphi \in K$, *whenever* $h_0(t,x) < \lambda$.

A3) There exists a function $V \in \Lambda([\mathbb{R}_+, S(h,A))$ *such that*

i) V is h-positive definite, h_0*-decrescent.*

ii) for any $t > \tau_0 \in \mathbb{R}_+$, $x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in S(h,A)$, $y_0 - x_0 \in S(h,A)$ *the inequality*

$$ {}_{\tau_0}^{C}\mathcal{D}_+^{q} V(t,x,y,\eta,x_0,y_0) \leq g(t, V(t,y-x),\eta) \tag{6.15}$$

holds, where $S(h,A) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t,x) < A\}$.

Then the parametrically practically stability properties of scalar FDE (4.10) with respect to $(a(\lambda), b(A))$ *imply the corresponding* (h_0,h)*-practically stability properties w.r.t.* (λ, A) *of system of perturbed FDE (6.2) with ITD relative to (6.1), where the constants* (λ, A) *are given such that* $\varphi(\lambda) < A$.

Proof 6.2: We assume that the scalar FDE (4.10) is uniformly parametrically practically stable w.r.t. $(a(\lambda), b(A))$. *Then, there exists a* $\sigma = \sigma(\lambda, A) > 0$ *such that for* $u_0 \in \mathbb{R}$: $|u_0| < a(\lambda)$ *and* η: $|\eta| < \sigma$ *we have*

$$ |u(t,\tau_0,u_0,\eta)| < b(A), \qquad t \geq \tau_0. \tag{6.16}$$

Consider any solutions $x(t) = x(t;t_0,x_0)$ *and* $y(t) = y(t;\tau_0,y_0)$ *of system of FDE (6.1) and (6.2) such that* $h_0(\tau_0, y_0 - x_0) < \lambda$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *Let* $\tilde{u}_0 = V(\tau_0, y_0 - x_0)$. *From condition i) of A3) i.e.* h_0*-decrescent it follows* $\tilde{u}_0 = V(\tau_0, y_0 - x_0) < a(h_0(\tau_0, y_0 - x_0)) < a(\lambda)$. *We claim that* $h(t, y(t) - x(t - \eta^*)) < A$ *holds for* $t \geq \tau_0$. *We have* $h(\tau_0, y_0 - x_0) < \varphi\big(h_0(\tau_0, y_0 - x_0)\big) < \varphi(\lambda) < A$ *i.e. desired inequality above holds at* $t = \tau_0$. *Assume that the claim is not true, then there exists a point* $t_1 > \tau_0$ *such that*

$$h(t_1, y(t_1) - x(t_1 - \eta^*)) = A \ \ and \ \ h(t, y(t) - x(t - \eta^*)) < A \tag{6.17}$$

for $\tau_0 \leq t < t_1$ *in view of the fact that* $h(\tau_0, y_0 - x_0) < A$ *whenever* $h_0(\tau_0, y_0 - x_0) < \lambda$. *Therefore the inclusion* $(t, y(t) - x(t - \eta^*)) \in S(h, A)$ *is valid for* $t \in [\tau_0, t_1]$. *Hence, by using (6.15), (6.17), A1) and Lemma 5.1, we have (6.10). Consequently, in view of the relations (6.10), (6.15)-(6.17), condition A3), the choice of* t_1 *and* $\tilde{u}_0$ *we obtain*

$$\begin{aligned} b(A) &= b\left(h\big(t_1, y(t_1) - x(t_1 - \eta^*)\big)\right) \\ &\leq V(t_1, y(t_1) - x(t_1 - \eta^*)) \leq u^*(t_1\ ) < b(A). \end{aligned} \tag{6.18}$$

The obtained contradiction proves the validity of the claim which implies that system of perturbed FDE (6.2) is (h_0, h)*-uniformly practically stable with ITD relative to unperturbed system (6.1). The proof of attractive practical stability of system (6.2) can be done with arguments used in the proof of Theorem 6.2.* ■

Theorem 6.3: Let the conditions A1), A2) of Theorem 6.2 be satisfied and A3) is replaced by as follow:

A3) There exists a function $V \in \Lambda([\mathbb{R}_+, \mathbb{R}^n)$ *such that*

i) V is h-positive definite and h_0*-decrescent with* $a, b \in K_\infty$*, respectively.*

ii) for any $t > \tau_0 \in \mathbb{R}_+$, $x, y, x_0, y_0 \in \mathbb{R}^n$ *the inequality*

$${}_{\tau_0}^{C}\mathcal{D}_+^q V(t, x, y, \eta, x_0, y_0) \leq g(t, V(t, y - x), \eta) \tag{6.19}$$

holds.

Then Lagrange stability w.r.t parameter of the scalar FDE (4.10) imply the corresponding (h_0, h)*- Lagrange stability with ITD of system of perturbed FDE (6.2) relative to (6.1).*

Proof 6.3: Assume that the scalar FDE (4.10) is Lagrange stable w.r.t. parameter i.e. $B^*1)$ *and* $A^*1)$ *are satisfied. Let* $\alpha > 0$ *be given. Since* $B^*1)$ *holds, given* $\alpha_1 = a(\alpha) > 0$ *there exist* $\beta_1 = \beta_1(\alpha_1) > 0$ *and* $\sigma = \sigma(\alpha_1) > 0$ *such that* $|u_0| < \alpha_1$ *and* $|\eta| < \sigma$ *imply*

$$|u(t, \tau_0, u_0, \eta)| < \beta_1 \text{ for } t \geq \tau_0. \tag{6.20}$$

Consider any solutions $x(t) = x(t; t_0, x_0)$ *and* $y(t) = y(t; \tau_0, y_0)$ *of system of FDE (6.1) and (6.2) such that* $h_0(\tau_0, y_0 - x_0) < \alpha$ *and* $|\eta^*| < \sigma$ *where* $\eta^* = \tau_0 - t_0$. *From the condition i) of A3) and* $h_0(\tau_0, y_0 - x_0) \leq \alpha$ *it follows that* $b\big(h(\tau_0, y_0 - x_0)\big) \leq a(\alpha) = \alpha_1 < \beta_1$. *Then we get* $h(\tau_0, y_0 - x_0) < \beta$ *by choosing* $\beta = b^{-1}(\beta_1)$. *We will prove that* $h(t, y(t) - x(t - \eta^*)) < \beta$ *for* $t \geq \tau_0$. *Assume the opposite, i.e. there exists a point* $t_1 > \tau_0$ *such that* $h(t_1, y(t_1) - x(t_1 - \eta^*)) = \beta$. *We also have (6.10) for* $t \geq \tau_0$ *with* $u_0 = V(\tau_0, y_0 - x_0)$. *Consequently, in view of the relations (6.10), (6.20), condition A3), the choice of* t_1 *and* $u_0 = V(\tau_0, y_0 - x_0) \leq a(h(\tau_0, y_0 - x_0)) < a(\alpha) = \alpha_1$ *we obtain*

$$\begin{aligned} b(\beta) = b\left(h\big(t_1, y(t_1) - x(t_1 - \eta^*)\big)\right) &\leq V\big(t_1, y(t_1) - x(t_1 - \eta^*)\big) \\ &\leq u^*(t_1, \tau_0, V(\tau_0, y_0 - x_0), \eta^*) < \beta_1 = b(\beta). \end{aligned} \tag{6.21}$$

The obtained contradiction proves the validity of the claim which implies that system of perturbed FDE (6.2) is (h_0, h)*-equi-bounded with ITD relative to (6.1), i.e. B1) holds. On the other hand, the proof of A1) can be done with the arguments used above, we omit it. Hence, L1) holds, i.e. system of perturbed FDE (6.2) is* (h_0, h)*-Lagrange stable with ITD relative to (6.1).* ■

REFERENCES

[1] Hahn W., (1967), “Stability of Motion”, 1st Edition, Springer-Verlag.

[2] Hahn W., (1963), “Theory and Application of Liapunov’s Direct Method”, 2nd Edition, Prentice-Hall.

[3] Lakshmikantham V., Leela S., (1969), “Differential and Integral Inequalities Volume-1”, 1st Edition, Academic Press.

[4] Lakshmikantham V., Leela S., Martynyuk A. A., (1989), “Stability Analysis of Nonlinear Systems”, 1st Edition, Marcel Dekker.

[5] Brauer F., Nohel J. A., (1989), “The Qualitative Theory of Ordinary Differential Equations: An Introduction”, 2nd Edition, Dover Publications.

[6] Khalil H. K., (2002), “Nonlinear Systems”, 3rd Edition, Prentice Hall.

[7] Yoshizawa T., (1958), “Lyapunov functions and boundedness of solutions”, Funkcialaj Ekvacioj, 2, 71-103.

[8] Stamov G. T., (2012), “Almost Periodic Solutions of Impulsive Differential Equations”, Springer, Heidelberg.

[9] LaSalle J. P., Lefschetz S., (1961), “Stability by Liapunov’s Direct Method with Applications”, Academic Press, New York.

[10] Lakshmikantham V., Leela S., Martynyuk A. A., (1990), “Practical Stability of Nonlinear Systems”, World Scientific, Singapore.

[11] Oldham K. B., Spanier J., (1974), “The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary Order”, Academic Press, London.

[12] Samko G., Kilbas A. A., Marichev O. I., (1993), “Fractional Integrals and Derivatives: Theory and Applications”, Gordon and Breach, Yverdon.

[13] Miller K. S., Ross B., (1993), “An introduction to the Fractional Calculus and Fractional Differential Equations”, John Wiley and Sons, New York.

[14] Podlubny I., (1999), “Fractional Differential Equations”, Academic Press, San Diego.

[15] Kilbas A. A., Srivastava H. M., Trujillo J. J., (2006), “Theory and Applications of Fractional Differential Equations”, Elsevier B. V.

[16] Lakshmikantham V., Leela S., Devi V. J., (2009), “Theory of Fractional Dynamical Systems”, Scientific Publishers, Cambridge.

[17] Metzler R., Klafter J., (2000), “The random walk’s guide to anomalous diffusion: a fractional dynamics approach”, Physics Reports, 339, 1-77.

[18] Hilfer R., (2000), “Applications of Fractional Calculus in Physics”, World Scientific, Singapore.

[19] Sabatier J., Agrawal O. P., Machado J. A. T., (2007), “Advances in Fractional Calculus-Theoretical Developments and Applications in Physics and Engineering”, Springer, Netherlands.

[20] Baleanu D., Diethelm K., Scalas E., Trujillo J. J., (2012), “Fractional Calculus Models and Numerical Methods”, Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston.

[21] Kilbas A. A., Trujillo J. J., (2001), “Differential equations of fractional order: methods results and problem I”, Applicable Analysis, 78 (1-2), 153-192.

[22] Lakshmikantham V., Vatsala A. S., (2007), “Theory of fractional differential inequalities and applications”, Communications in Applied Analysis, 11, 395-402.

[23] Lakshmikantham V., Vatsala A. S., (2008), “Basic theory of fractional differential equations”, Nonlinear Analysis, 69, 3337-3343.

[24] Lakshmikantham V., Vatsala A. S., (2008), “General uniqueness and monotone iterative technique for fractional differential equations”, Applied Mathematic Letters, 21, 828-834.

[25] Ahmad B., Nieto J. J., (2009), “Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions”, Computers and Mathematics with Applications, 58 (9), 1838-1843.

[26] McRae F. A., (2009), “Monotone iterative technique and existence results for fractional differential equations”, Nonlinear Analysis, 71, 6093-6096.

[27] Deithelm K, Ford N. J., (2002), “Analysis of fractional differential equations”, Journal of Mathematical Analysis and Applications, 265, 229-248.

[28] Denton Z., Vatsala A. S., (2010), “Fractional integral inequalities and applications”, Computers and Mathematics with Applications, 59, 1087-1094.

[29] Matignon D., (1996), “Stability results for fractional differential equations with applications to control processing”, Computational Engineering in Systems Applications, 963-968, Lille, France.

[30] Deng W. H., Li C. P., Lü J. H., (2007), "Stability analysis of linear fractional differential system with multiple time delays", Nonlinear Dynamics, 48 (4), 409-416.

[31] Qian D., Li C. P., Agarwal R. P., Wong J. Y., (2010), "Stability analysis of fractional differential system with Riemann Liouville derivative", Mathematical and Computer Modelling, 52, 862-874.

[32] Li Y., Chen Y., Podlubny I., (2009), "Mittag--Leffler stability of fractional order nonlinear dynamic systems", Automatica, 45, 1965-1969.

[33] Li Y., Chen Y., Podlubny I., (2010), "Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability", Computers and Mathematics with Applications, 59 (5), 1810-1821.

[34] Sadati J. R, Baleanu D., Ghaderi R., Ranjbar N. A., Abdeljawad T., Jarad F., (2010), "Mittag-Leffler stability theorem for fractional nonlinear systems with delay", Abstract and Applied. Analysis, Article ID 108651.

[35] Zhang F., Li C., Chen Y., (2011), "Asymptotical Stability of Nonlinear Fractional Differential System with Caputo Derivative", International journal of Differential equations, Article ID 635165.

[36] Choi S. K., Koo N., (2011), "The monotonic property and stability of solution of fractional differential equations", Nonlinear Analysis, 74, 6530-6536.

[37] Zhou X. F., Hu L. G., Liu S., Jiang W., (2014), "Stability criterion for a class of nonlinear fractional differential systems", Applied Mathematics Letters, 28, 25-29.

[38] Aguila-Camacho N., Duarte-Mermoud M. A., Gallegos J. A., (2014), "Lyapunov functions for fractional order systems", Communications in Nonlinear Science and Numerical Simulation, 19, 2951-2957.

[39] Agarwal R. P., O'Regan D., Hristova S., (2015), "Stability of Caputo fractional differential equations by Lyapunov's functions", Applications of Mathematics, 60 (6), 653-676.

[40] Agarwal R. P., O'Regan D., Hristova S., Cicek M., (2015), "Practical stability with respect to initial time difference for Caputo fractional differential equations", preprint.

[41] Devi J. V., McRae F. A., Drici Z., (2012), "Variational Lyapunov Method for Fractional Differential Equations", Computers and Mathematics with Applications, 64, 2982-2989.

[42] Lakshmikantham V., Vatsala A. S., (1999), "Differential inequalities with initial time difference and applications", Journal of Inequalities and Applications, 3, 233-244.

[43] Lakshmikantham V., Leela S., Devi J. V., (1999), “Stability criteria for solutions of differential equations relative to initial time difference”, International Journal of Nonlinear Differential Equations Theory-Methods and Application, 5 (1-2), 109-114.

[44] Shaw M. D., Yakar C., (1999), “Generalized variation of parameters with initial time difference and a comparison result in terms of Lyapunov-like functions”, International Journal of Nonlinear Differential Equations Theory-Methods and Applications, 5 (1-2), 86-108.

[45] Yakar C., Deo S. G., (2006), “Variation of parameters formulae with initial time difference for linear integrodifferential equations”, Applicable. Analysis, 85 (4), 333-343.

[46] McRae F. A., (2001), “Perturbing Lyapunov functions and stability criteria for initial time difference”, Applied Mathematics and Computation, 117, 313-320.

[47] Li A., Feng E., Li S., (2009), “Stability and boundedness criteria for nonlinear differential systems relative to initial time difference and applications”, Nonlinear Analysis: Real World Applications, 10, 1073-1080.

[48] Li A., Lv W., Ye J., (2011), “Exponential and global stability of nonlinear dynamical systems relative to initial time difference”, Applied Mathematics and Computation, 217, 5923-5929.

[49] Song X., Li A., Li S., Feng E., (2008), “Practical stability of nonlinear differential equation with initial time difference”, Applied Mathematics and Computation, 203 157 162.

[50] Gordon S. P. (1979), “A stability theory for perturbed differential equations”, International Journal of Mathematic & Mathematical Science, 2, 283-297.

[51] Lakshmikantham V., Liu X., Leela S., (1998), “Variational Lyapunov method and stability theory”, Mathematical problems in engineering 3, 555-571.

[52] Shaw, M.D. and Yakar, C., (2000), “Stability criteria and slowly growing motions with initial time difference”, Problems of Nonlinear Analysis in Engineering Systems, 1, 50-66.

[53] Yakar C., Shaw M.D., (2005), “A comparison result and Lyapunov stability criteria with initial time difference”, Dynamics of Continuous, Discrete and Impulsive Systems. A: Mathematical Analysis, 12 (6), 731-741.

[54] Yakar C., (2010), “Fractional differential equations in terms of comparison results and Lyapunov stability with initial time difference”, Abstract and Applied Analysis, Article ID 762857.

[55] Yakar C., Cicek M., Gücen M. B., (2011), "Boundedness and Lagrange stability of fractional order perturbed system related to unperturbed systems with initial time difference in Caputo's sense", Advances in Difference Equations, 54.

[56] Yakar C., Cicek M., Gücen M. B., (2014), "Practical stability in terms of two measures for fractional order dynamic systems in Caputo's sense with initial time difference", Journal of Franklin Institute, 351, 732-742.

[57] Movchan A. A., (1960), "Stability of process with respect to two measures", Prikladnoi Matematiki i Mekhaniki, 24, 988-1001.

[58] Lakshmikantham V., Liu X. Z., (1993), "Stability Analysis in Terms of Two Measures", World Scientific, Singapore.

[59] Wang P. G., Lian H. R., (2006), "On the stability in terms of two measures for perturbed impulsive integro-differential equations", Journal of Mathematical Analysis and Applications, 313, 642-653.

[60] Yakar C., Shaw M. D., (2009), "Practical stability in terms of two measures with initial time difference", Nonlinear Analysis: Theory Methods & Applications, 10, 781-785.

[61] Yakar C., Cicek M., (2011), "Theory, methods and applications of initial time difference boundedness and Lagrange stability in terms of two measures for nonlinear systems", Hacettepe Journal of Mathematics and Statistics, 40 (2), 305-330.

[62] Podlubny I., (2002), "Geometric and physical interpretations of fractional integration and fractional differentiation", Fractional Calculus and Applied. Analysis, 5 (4), 367-386.

[63] Li C., Deng W., (2007), "Remarks on fractional derivatives", Applied Mathematics and Computation, 187, 777-784.

[64] Diethelm K., (2004), "The Analysis of Fractional Differential Equations", Springer- Verlag, Berlin Heidelberg.

[65] Baleanu D., Mustafa O. G., (2010), "On the global existence of solutions to a class of fractional differential equations", Computers and Mathematics with Applications, 59, 1835-1841.

[66] Hilfer R., Seybold H. J., (2006), "Computation of the generalized Mittag-Leffler function and its inverse in the complex plane", Integral Transforms and Special Functions, 17 (9), 637-652.

[67] Ramirez J. D., Vatsala A. S., (2012), "Generalized Monotone Iterative Technique for Caputo Fractional Differential Equation with Periodic Boundary Condition via Initial Value Problem", International Journal of Differential Equations, Article ID 842813.

BIOGRAPHY

Muhammed ÇİÇEK was born in 1986. He completed primary and secondary education in Kocaeli. He graduated from Uludag University, Department of Mathematics, in 2008. He also received M.Sc. degree in Mathematics from Gebze Institute of Technology in 2010. He has been working as a research assistant at Department of Mathematics, Gebze Technical University since 2010.

APPENDICES

Appendix A: Publications Based on the Thesis

Yakar C., Cicek M., Gücen M. B., (2011), "Boundedness and Lagrange stability of fractional order perturbed system related to unperturbed systems with initial time difference in Caputo's sense", Advances in Difference Equations, 54.

Agarwal R. P., O'Regan D., Hristova S., Cicek M., (2015), "Practical stability with respect to initial time difference for Caputo fractional differential equations", preprint.

Cicek M., Yakar C., Gücen M. B., (2014), "Practical stability in terms of two measures for fractional order dynamic systems in Caputo's sense with initial time difference", Journal of Franklin Institute, 351, 732-742.

Appendix B: Conference Presentations Based on the Thesis

Cicek M., (2012), "Initial Time Difference Criteria in Terms of Two Measures for Lagrange Stability for Fractional Order Dynamic Systems in Caputo's sense", Jubilee National Scientific conference with international participation: Traditions, Directions and Challenges, Smolyan, Bulgaria, 19-21 October.

Yakar C., Cicek M., Gücen M. B., (2011), "Practical Stability In Terms Of Two Measures For Fractional Order Dynamic Systems in Caputo's Sense with Initial Time Difference", International Conference on Applied Analysis and Algebra, Istanbul, Turkey, 29 June- 2 July.