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GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NONLINEAR TRANSIENT ANALYSIS OF COMPOSITE PANELS WITH VARIABLE THICKNESS

İLKE ALGÜL

A THESIS SUBMITTED FOR THE DEGREE OF
MASTER OF SCIENCE
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GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

DEĞİŞKEN KALINLIKLI KOMPOZİT PANELLERİN NONLİNEER DİNAMİK ANALİZİ

İLKE ALGÜL
YÜKSEK LİSANS TEZİ
MAKİNE MÜHENDİSLİĞİ ANABİLİM DALI

DANIŞMANI
PROF. DR. HASAN KURTARAN

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SUMMARY

In this thesis work, geometrically nonlinear dynamic analysis of composite panels with variable thickness subjected to time dependent loading using Chebyshev collocation method is studied. First-order shear deformation theory is used to consider transverse shear effect through the thickness direction. An extension of Sanders nonlinear strain-displacement relationships are used to take into account geometric nonlinearity due to large displacements. Dynamic equations for composite doubly curved panels with variable thickness are obtained using virtual work principle. The displacement fields in the governing equilibrium equations are expressed with fast converging finite double Chebyshev series. Newmark-beta average acceleration scheme is used for temporal discretization. A computer program based on Chebyshev collocation method is developed to solve the governing equilibrium equations. Several examples are solved with Chebyshev Collocation Method to emphasize the effectiveness of the proposed method. The obtained results are compared with the finite element method. Finally, parametric studies such as effects of taper ratios, materials, panel radius and boundary conditions on the response of composite panels are investigated.

Key Words: Laminated Composite Tapered Panel, Chebyshev Collocation Method, First Order Shear Deformation Theory, Nonlinear Transient Analysis.

ÖZET

Bu tez kapsamında, zamana bağlı yük altında kalınlığı değişken kompozit panellerin lineer olmayan dinamik davranışı Chebyshev kollokasyon metodu kullanılarak incelenmiştir. Kalınlık doğrultusundaki kayma etkilerini hesaba katmak için birinci mertebeden kayma teorisi kullanılmıştır. Genişletilmiş Sanders lineer olmayan birim uzama-yer değiştirme bağıntıları, büyük yer değiştirmeler sebebiyle oluşan geometrik açıdan lineer olmama etkilerini hesaba katmak için kullanılmıştır. Değişken kalınlıklı kompozit panelin dinamik denklemleri, virtüel iş prensibi kullanılarak elde edilmiştir. Hareket denklemlerindeki yer değiştirme alanları, hızlı yakınsayan sonlu çift Chebyshev serileri kullanılarak açılmıştır. Zaman ayrıklaştırması için Newmark-beta ortalama ivme metodu kullanılmıştır. Chebyshev kollokasyon metoduna dayalı bir bilgisayar programı, hareket denklemleri çözmek için geliştirilmiştir. Önerilen metodun verimliliğini vurgulamak için birçok örnek Chebyshev kollokasyon metodu ile çözülmüştür. Sonuçlar sonlu elemanlar metodu ile kıyaslanmıştır. Son olarak, kompozit panellerin cevabında kalınlık değişimi, malzeme, panel yarıçapı ve sınır şartları parametrik olarak incelenmiştir.

Anahtar Kelimeler: Değişken Kalınlıklı Kompozit Panel, Chebyshev Kollokasyon Metodu, Birinci Mertebeden Kayma Teorisi, Lineer Olmayan Dinamik Analiz.

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First, i would like to express my deepest gratitude to my supervisor Prof. Dr. Hasan Kurtaran, who not only shared his scientific knowledge with me but also taught me great lessons of life.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
$\sigma_x, \sigma_y, \tau_{xy}$	:	Stresses
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	:	Strains
E_1	:	Elasticity modulus in 1 direction
E_2	:	Elasticity modulus in 2 direction
E	:	Equivalent elasticity module
v	:	Poisson ratio
G_1	:	Shear modulus in 1 direction
G_2	:	Shear modulus in 2 direction
ρ	:	Density
FEM	:	Finite Element Method

LIST of FIGURES

1. INTRODUCTION

1.1. General

Panel structures have initially been used for the construction of mosques, domes, temple roofs, cathedrals, monuments and other historic buildings. Along with the rapid progress in science and technology, especially after World War II, the usage of panels in several industries such as aviation, naval and construction industry have considerably increased. As a consequence of this rapid development, analysis of dynamic behavior of panels under different type of loading has been attracting attention of many researchers.

Panels, which have smaller thickness comparing with its other dimensions, are structural elements. Panels contain single or double curvature and form compression, tensile and shear strength during the transfer of loading that is applied over them. In some cases, panels can be made with variable thickness due to design, geometric necessities, strength capacities and minimum weight requirements. The wing skin panels of an aircraft can be given as usage area for tapered panels. Helicopters rotor blades as shown in Figure 1.1 can also be given as another important usage area [1]. The thickness of rotor blades varies from leading edge to trailing edge along with chord line [2].

Figure 1.1: Helicopter rotor blades.

Composite materials are obtained by combining two or more materials that have different mechanical properties. Composite materials are mostly used in the field of aircraft structures, space stations, automobiles, ships, submarines due to its high strength-to-weight ratio, high resistance to corrosion etc. Thus, in this study, composite materials are chosen for panels materials.

Equations of motion including the effects of geometric nonlinearity by the reason of large displacements are derived using principle of virtual work. The displacement fields in the partial derivatives of equations are expressed by using double Chebyshev series for a rapid converging and Newmark average acceleration scheme is used in order to separate time domain. Equations of motion are solved by Chebyshev collocation method. In order to specify the effectiveness of the implemented method, cases with different parameters such as different taper ratios, materials, panel radius and boundary conditions are studied. The obtained results are compared with the commercial finite element software ANSYS.

1.2. Literature Review

Many researches have been conducted on the linear and nonlinear analysis of isotropic and laminated composite plates and panels with uniform depth under different type of loading. Thinking over the comprehensive number of studies on the topic of dynamic analysis of composite laminated flat plates, a brief summary of the literature is given in following text.

Upadhyay et al. investigated nonlinear dynamic analysis of laminated composite plates based on third-order shear deformation theory subjected to different type of pulse loading by using fast converging finite double Chebyshev series and Houbolt time marching scheme [3]. Civalek published a paper about nonlinear static and dynamic behavior of thin isotropic plates by coupling discrete singular convolution (DSC) and harmonic differential quadrature (HDQ) methods [4]. Maleki et al. carried out linear transient analysis of laminated plates for arbitrary conditions with generalized differential quadrature method (GDQ) [5]. Birman et al. studied dynamic behavior of thin laminated plates subjected to explosive loads by using Runge-Kutta procedure [6]. Tsouvalis et al. investigated the nonlinear dynamic response of composite laminated plates under lateral loads with Galerkin method [7].

Kazancı et al. implemented finite difference method to find nonlinear dynamic response of a laminated composite plate under blast load [8]. Chen et al. improved the semi-analytical finite strip method to evaluate the geometrically nonlinear response of rectangular composite laminated plates [9]. Yosibash et al. is solved nonlinear dynamic response of plates using Chebyshev collocation method [10].

Dynamic responses of laminated composite panels have also been an investigation area for researchers. Nath and Alwar used Chebyshev series to study nonlinear transient response of shallow spherical shells with and without damping [11]. Reddy et al. developed a finite element theory using a dynamic, shear deformation theory of a doubly curved shell to determine geometrically nonlinear transient response of spherical and cylindrical shells [12]. Wu et al. carried out free and forced nonlinear dynamics of composite shell structures using a 48 degrees-of-freedom shell element based on Kirchhoff-Love Theory [13]. To and Wang studied nonlinear dynamic response of laminated composite shells subjected transient excitations using the hybrid strain-based flat triangular finite element theory [14]. Kurtaran performed nonlinear transient analysis of moderately thick laminated composite shallow shells using generalized differential quadrature method (GDQM) [15]. Kundu and Sinha analyzed geometrically nonlinear transient response of the cross-ply and the specially laminated spherical, cylindrical, hyperbolic and paraboloid composite shells using a nine-noded isoparametric composite shell element [16]. Maleki et al. analyzed static-transient analysis of moderately thick laminated cylindrical shell by using GDQM in their other study. [17]. Isoldi et al. surveyed geometrically nonlinear static and dynamic behavior of laminate composite shells by using triangular finite element [18]. Kant et al. presented a C^0 continuous finite element formulation of a higher order shear deformation theory to foresee the linear and geometrically nonlinear transient responses of composite and sandwich laminated shells [19]. Türkmen et al. carried out nonlinear dynamic analysis of cylindrically curved laminated panels and solved governing equations of these panels by Runge Kutta method [20].

Extensive research has been carried out on dynamic response of plate and panel structures in the studies above. Nevertheless, no study has been conducted about the nonlinear dynamic behavior of tapered panels using Chebyshev Collocation Method. Most of the studies on the literature about tapered plate and panels have been limited to static, free vibration and buckling analysis. Ganesan and Rasul studied buckling analysis of tapered laminated shells considering uniaxial compression using Ritz method based on different first order shell theories. A comprehensive parametric study including boundary conditions, stacking sequence, taper configurations, radius and geometric parameters of the shells has been done in this research [21]. Susler et al. analyzed geometrically nonlinear transient behavior of simply supported tapered laminated composite plates to evaluate the effect of air blast loading using a closed form solution and then compared with finite element method (FEM). The effect of the taper ratio, the stacking sequence and fiber orientation angle has been taking into consideration as a parametric study to obtain displacement-time and strain-time results in this article [2]. Ashaur applied finite strip technique in conjunction with the transition matrix to examine the vibration of orthotropic tapered plates for different taper ratio, aspect ratio and different combinations of boundary conditions [22]. Turvey examined large deflection static analysis of thin tapered square plates with simply supported boundary conditions by using dynamic relaxation method [23]. Javed et al. carried out free vibration of anti-symmetric angle-ply composite plates with variable thickness using spline function approximation by taking parameter of material properties, ply orientation, number of lay ups, aspect ratio and coefficients of thickness variations [24]. Bert and Malik presented the free vibration analysis of rectangular tapered plates having simply supported conditions at two opposite edges and general boundary conditions at the other two edges by differential quadrature method which is firstly introduced by Bert as a tool for structural analysis [25]. Babu et al. investigated dynamic analysis of various configurations thickness tapered laminated composite plate by experimental study and validation of the developed finite element formulations [26]. Kobayashi et al. surveyed buckling problem of uniaxially compressed rectangular tapered plates by a power series method. The influences of thickness variation, plate aspect ratios, and boundary conditions on the buckling load have been taken as parameters in the survey [27]. Civalek solved free vibration problems of isotropic and orthotropic rectangular thickness tapered plates by coupling discrete singular convolution (DSC) [28].

1.3. Purpose and Organization of Thesis

In this dissertation, geometrically nonlinear transient analysis of composite panels with variable thickness subjected to time dependent loading is investigated using Chebyshev collocation method. The effects of taper ratios, material properties, panel radius and boundary conditions are studied. Chebyshev polynomials are used for spatial discretization of the problem, whereas Newmark beta method is used for temporal discretization. Nonlinear differential equations of motion are solved with Newton-Raphson method.

Unlike classical laminated plate theory (CPT), which ignores shear deformations and provides reasonable results for thin laminates, first order laminated shear deformation theory, known as also Kirchhoff–Love theory, takes transverse shear effect into consideration through the thickness [29] . Hence, in this study, Kirschoff-Love theory for doubly curved shells is considered. An extension of Sanders nonlinear strain-displacement relationships are taken geometric nonlinearity into account due to large displacements.

The thesis contains six chapters. Chapter one includes general review, a literature survey of some published papers about plate and panel with constant and variable thickness distribution. Chapter two contains panel theory, relationship between stress- strain and strain-displacement, derivations of equilibrium equations with virtual work principle for tapered panel. Chapter three involves the theory of Chebyshev collocation method, Newmark-beta Method, Newton Raphson method for linearization of the nonlinear differential equation and the finite element method. Chapter four contains numerical results for plate and panel with flat or variable thickness obtained by proposed method and comparisons with ANSYS. Chapter five consists of the comments about the contribution of this thesis to the literature and further studies.

2. THEORY OF COMPOSITE PANEL

In this chapter, firstly, a brief description of panel theory is given and assumptions for this theory are expressed. The assumptions to obtain displacement correlation of panel are explained and displacement equations are derived. After giving strain-displacement correlations, assumptions for mechanics of laminated composites are mentioned and constitution equations are given. Governing differential equations for tapered composite panel are obtained by using principle of virtual work.

Panel is restricted with two inclined surfaces and the distance along the thickness direction is smaller than the other dimensions. Panels contain single or double curvature and form compression, tensile and shear strength during the transfer of loading that is applied over them. Panel with variable thickness (length *a*, width *b*) is demonstrated in Figure 2.1. The points, which have equal distances to the two inclined surfaces, are known to be middle surface. *h*(x) indicates thickness function varying through the *x* direction and is linearly expressed as shown in Equation 2.1. In this equation, β is the taper ratio. *x*, *y* and *z* stated the curvilinear coordinate system.

$$h(x) = h_0.(1 + \beta \frac{x}{a}) \tag{2.1}$$

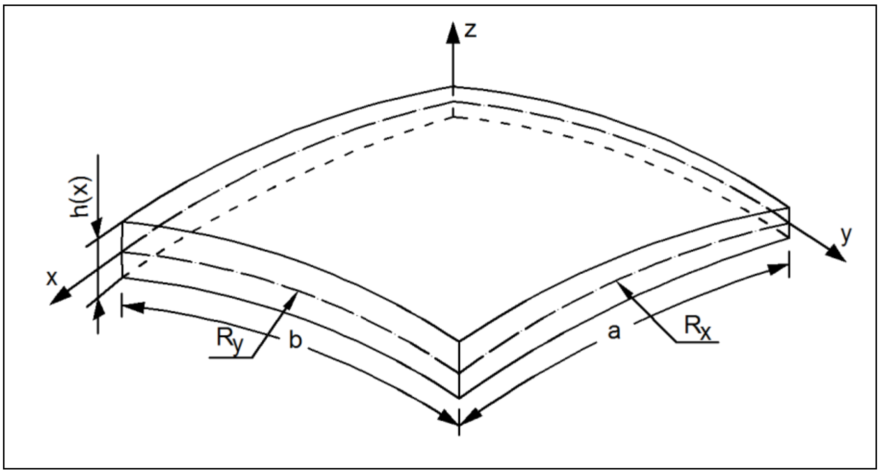

Figure 2.1: The composite panel with variable thickness.

The following assumptions are taken into account for the theory of structure. If the structure gets its former state after the removal of load applied to itself, it exhibits the behavior of linear elastic material. In this study, plate and panel structures are assumed to be linear elastic material. It is also assumed that, there is no penetration between the layers and perfectly bonded faces. Layers cannot slip on each other and displacement in the area of adhesion are continuous. Therefore, no delamination of composite layers is expected.

2.1. Displacement Field

The displacement field at general point (*x*, *y* and *z*) of the shell at *t* time based on first order shear deformation theory (FSDT) may be written as [30]:

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z.\theta_x(x,y,t) \\
v(x,y,z,t) &= v_0(x,y,t) + z.\theta_y(x,y,t) \\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned}
\tag{2.2}
$$

where u_0, v_0, w_0 are the displacement field of a point on the middle surface of the shell along the *x*, *y* and *z* axes, respectively. θ_x and θ_y are the rotations around the *y* and *x* axes, respectively and come from the rotations of the shell.

FSDT is an extension of the classical plate theory by including constant transverse shear strains and stresses through the thickness. Transverse shear stress distribution must be parabolic through the thickness direction. This conflict between the certain stress and the constant stress based on FSDT must be revised. By multiplying with shear correction factor (k_s) in FSDT in calculation of the transverse shear forces (Q_{yz}, Q_{xz}) in allows us to correct this conflict. The value of k_s is determined from the equilibrium of the shear energy due to transverse shear stress on the shell theory and the strain energy calculated by 3-D elasticity theory. Shear correction factor is given 5/6 for shell.

2.2. Strain-Displacement Relations

The strain-displacement correlations of doubly curved panels using the displacement field in Equation 2.2 and employing the case of an extension of Sanders nonlinear kinematics is written as [29]:

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \theta_x}{\partial x} + \frac{w_0}{R_x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \theta_y}{\partial y} + \frac{w_0}{R_y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2$$

$$\gamma_{xy} = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + z\frac{\partial \theta_x}{\partial y} + z\frac{\partial \theta_y}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} + c_0.\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) \tag{2.3}$$

$$\gamma_{yz} = \theta_y + \frac{\partial w_0}{\partial y} - \frac{v_0}{R_y}$$

$$\gamma_{xz} = \theta_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R_x}$$

Here R_i (i=x, y) is the radius of the curvature of the panel and c_0 specifies the $\frac{1}{2}\left(\frac{1}{R_y} - \frac{1}{R_x}\right)$ term. The strain displacement relations described in Equation 2.3, which are divided into two parts as stretching and bending, can be written in the matrix format as:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \\ {\gamma_{yz}}^0 \\ {\gamma_{xz}}^0 \end{bmatrix} + z.\begin{bmatrix} {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \\ 0 \\ 0 \end{bmatrix} \tag{2.4}$$

The terms in Equation 2.4 can be written explicitly as below:

$$
\begin{gathered}
{\varepsilon_x}^0 = \frac{\partial u_0}{\partial x} + \frac{w_0}{R_x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2, {\varepsilon_x}^1 = \frac{\partial \theta_x}{\partial x} \\
{\varepsilon_y}^0 = \frac{\partial v_0}{\partial y} + \frac{w_0}{R_y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2, {\varepsilon_y}^1 = \frac{\partial \theta_y}{\partial y} \\
{\gamma_{xy}}^0 = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} + c_0.\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) \\
{\gamma_{xy}}^1 = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \\
{\gamma_{yz}}^0 = \theta_y + \frac{\partial w_0}{\partial y} - \frac{v_0}{R_y} \\
{\gamma_{xz}}^0 = \theta_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R_x}
\end{gathered} \tag{2.5}
$$

2.3. Constitution Equations for Tapered Panel

Lamina coordinate system can be observed in Figure 2.2. 1-2 axes specify the local coordinate system of lamina. θ is the fiber angle of each laminate with the global panel axis x direction. All results should be determined in one global coordinate system of panel.

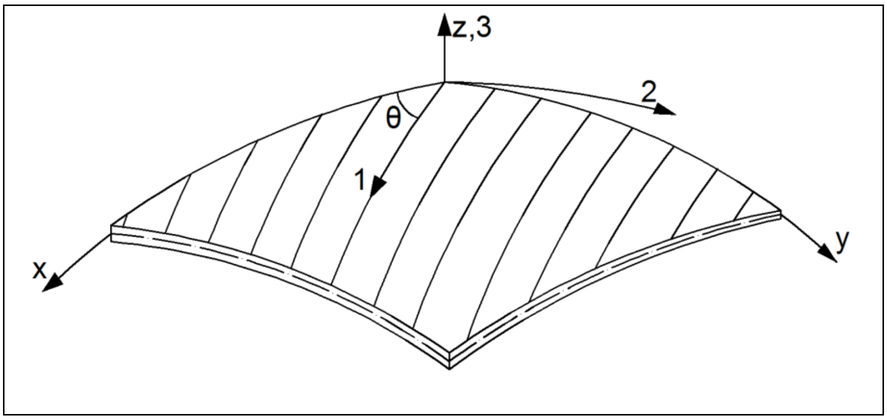

Figure 2.2: Lamina geometry.

In order to get the results in global coordinate system, stresses and strains are converted by using the transformation matrix $[T]_k$ and inverse of itself $[T]_k^{-1}$ as given in Equations 2.6-2.7.

$$[T]_k = \begin{bmatrix} cos^2\theta & sin^2\theta & 2cos\theta sin\theta \\ sin^2\theta & cos^2\theta & -2cos\theta sin\theta \\ -cos\theta sin\theta & cos\theta sin\theta & cos^2\theta - sin^2\theta \end{bmatrix} \tag{2.6}$$

$$[T]_k^{-1} = \begin{bmatrix} cos^2\theta & sin^2\theta & -2sin\theta cos\theta \\ sin^2\theta & cos^2\theta & 2sin\theta cos\theta \\ sin\theta cos\theta & -sin\theta cos\theta & cos^2\theta - sin^2\theta \end{bmatrix} \tag{2.7}$$

The stress-strain correlations for the *k*th layer of a laminated panel under plane stress using Hooke's Law are expressed in Equation 2.8 with the elements of the transformed reduced stiffness matrix, $[\bar{Q}_{ij}]_k$ [30].

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \left[\begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \\ {\gamma_{yz}}^0 \\ {\gamma_{xz}}^0 \end{bmatrix} + z \begin{bmatrix} {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \\ 0 \\ 0 \end{bmatrix} \right] \tag{2.8}$$

σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{xz} are stress components. The transformed reduced stiffness matrix can be obtained using the conversion of the $[Q_{ij}]_k$ which specifies the properties of k^{th} lamina's fiber. This conversation can be expressed as following by applying $[T]_k$ transformation matrix:

$$[\bar{Q}_{ij}]_k = [T^{-1}]_k [Q_{ij}]_k [T]_k \tag{2.9}$$

The relationship between $[\bar{Q}_{ij}]_k$ and $[Q_{ij}]_k$ is given as follows:

$$
\begin{aligned}
\bar{Q}_{11} &= Q_{11}cos^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta + Q_{22}sin^4\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})sin^2\theta cos^2\theta + Q_{12}(sin^4\theta + cos^4\theta) \\
\bar{Q}_{22} &= Q_{11}sin^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta + Q_{22}cos^4\theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})sin\theta cos^3\theta + (Q_{12} - Q_{22} \\
&\quad + 2Q_{66})sin^3\theta cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})sin^3\theta cos\theta + (Q_{12} - Q_{22} \\
&\quad + 2Q_{66})sin\theta cos^3\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})sin^2\theta cos^2\theta + Q_{66}(sin^4\theta \\
&\quad + cos^4\theta) \\
\bar{Q}_{44} &= Q_{44}cos^2\theta + Q_{55}sin^2\theta \\
\bar{Q}_{45} &= (Q_{55} - Q_{44})\, cos\theta sin\theta \\
\bar{Q}_{55} &= Q_{55}cos^2\theta + Q_{44}sin^2\theta
\end{aligned}
\tag{2.10}
$$

In Equation 2.10, the components of $[Q_{ij}]_k$ are shown as:

$$
\begin{gathered}
Q_{11} = \frac{E_1}{1 - \vartheta_{12}\vartheta_{21}};\ Q_{12} = Q_{21} = \frac{E_1\vartheta_{12}}{1 - \vartheta_{12}\vartheta_{21}} = \frac{E_2\vartheta_{21}}{1 - \vartheta_{12}\vartheta_{21}}; \\
Q_{22} = \frac{E_2}{1 - \vartheta_{12}\vartheta_{21}}; \\
Q_{66} = G_{12}; \\
Q_{44} = G_{23}; Q_{55} = G_{13}
\end{gathered}
\tag{2.11}
$$

E_1 and E_2 are the elastic modulus and v_{12} and v_{21} are the Poisson's ratios of composite layer in orthogonal directions. G_{12}, G_{23} and G_{13} are shear modulus.

2.4. Equilibrium Equations for Tapered Panel

Force and moment resultants of laminated composite panel can be stated in Equation 2.12 as the summation of stress components in each layer. N_x, N_y, and N_{xy} are the in-plane force resultants, M_x, M_y, and M_{xy} are the in-plane moment resultants and Q_{yz} and Q_{xz} are the transverse shear force resultants.

$$\left(N_x\,, N_y, N_{xy}\right) = \sum_{k=1}^{n} \int_{z_{k-1}(x)}^{z_k(x)} \left(\sigma_x, \sigma_y, \tau_{xy}\right) d_z$$

$$\left(M_x\,, M_y, M_{xy}\right) = \sum_{k=1}^{n} \int_{z_{k-1}(x)}^{z_k(x)} \left(\sigma_x, \sigma_y, \tau_{xy}\right) z d_z \tag{2.12}$$

$$\left(Q_{yz}, Q_{xz}\right) = \sum_{k=1}^{n} \int_{z_{k-1}(x)}^{z_k(x)} k_s . \left(\tau_{yz}, \tau_{xz}\right) d_z$$

The force, moment and transverse shear force components can be acquired as follows by replacement of Equation 2.8.

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left(\int_{z_{k-1}(x)}^{z_k(x)} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \end{bmatrix} zdz \right.$$
$$\left. + \int_{z_{k-1}(x)}^{z_k(x)} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \end{bmatrix} z^2 dz \right) \tag{2.13}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left(\int_{z_{k-1}(x)}^{z_k(x)} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \end{bmatrix} zdz \right.$$
$$\left. + \int_{z_{k-1}(x)}^{z_k(x)} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \end{bmatrix} z^2 dz \right) \tag{2.14}$$

$$\begin{bmatrix} Q_{yz} \\ Q_{xz} \end{bmatrix} = \sum_{k=1}^{n} \left(\int_{z_{k-1}(x)}^{z_k(x)} \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} {\gamma_{yz}}^0 \\ {\gamma_{xz}}^0 \end{bmatrix} dz \right) \tag{2.15}$$

Equations 2.13, 2.14 and 2.15 can be expressed in Equations 2.16, 2.17 and 2.18, respectively when the integration is employed and the equations are rearranged.

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \end{bmatrix} \tag{2.16}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \end{bmatrix} \tag{2.17}$$

$$\begin{bmatrix} Q_{yz} \\ Q_{xz} \end{bmatrix} = \begin{bmatrix} k_s A_{44} & k_s A_{45} \\ k_s A_{45} & k_s A_{55} \end{bmatrix} \begin{bmatrix} {\gamma_{yz}}^0 \\ {\gamma_{xz}}^0 \end{bmatrix} \tag{2.18}$$

Here A_{ij}, B_{ij} and D_{ij} are the extensional, flexural-extension coupling and flexural rigidities matrices, respectively. They are obtained as:

$$\begin{gathered} \{A_{ij}, B_{ij}, D_{ij}\} = \sum_{k=1}^{N} \int_{z_{k-1}(x)}^{z_k(x)} \{1, z_k, {z_k}^2\} \left(\bar{Q}_{ij}\right)_k d_z \quad (i,j = 1,2,6) \\ \{A_{ij}\} = \sum_{k=1}^{N} k_i k_j \int_{z_{k-1}(x)}^{z_k(x)} \left(\bar{Q}_{ij}\right)_k d_z \quad (i,j = 4,5) \end{gathered} \tag{2.19}$$

Force and moment components in matrix format can also be shown as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} {\varepsilon_x}^0 \\ {\varepsilon_y}^0 \\ {\gamma_{xy}}^0 \\ {\varepsilon_x}^1 \\ {\varepsilon_y}^1 \\ {\gamma_{xy}}^1 \end{bmatrix} \tag{2.20}$$

2.5. Governing Equations of Motion for Tapered Panel

Dynamic equilibrium equations of tapered panel are obtained in curvilinear coordinate system by using principle of virtual work. This principle is based on the equilibrium and for an equilibrium of a dynamic system; the work done by internal forces and inertia forces must be equal to the work done by external forces. The dynamic version of virtual work equilibrium is expressed as:

$$\delta U + \delta T - \delta W = 0 \tag{2.21}$$

where the virtual potential work of internal forces due to internal stresses (δU), the virtual work done by inertia forces caused by accelerations (δT) and the virtual work done by distributed load (δW) are given in Equations 2.22-2.24.

$$\begin{aligned}\delta U = \int_0^a \int_0^b \int_{\frac{-h(x,y)}{2}}^{\frac{h(x,y)}{2}} \Big\{\big(\sigma_x(\delta {\varepsilon_x}^0 + z\delta {\varepsilon_x}^1) + \sigma_y(\delta {\varepsilon_y}^0 + z\delta {\varepsilon_y}^1) \\ + \tau_{xy}(\delta {\gamma_{xy}}^0 + \delta {\gamma_{xy}}^1\big) + k_s\big(\tau_{yz}\delta {\gamma_{yz}}^0 \\ + \tau_{xz}\delta {\gamma_{xz}}^0\big)\Big\} dxdydz\end{aligned} \tag{2.22}$$

$$\begin{aligned}\delta T = \int_0^a \int_0^b \int_{\frac{-h(x,y)}{2}}^{\frac{h(x,y)}{2}} \{\rho\big[(\ddot{u}_0 + z\ddot{\theta}_x)(\delta u_0 + z\delta\theta_x) \\ + (\ddot{v}_0 + z\ddot{\theta}_y)(\delta v_0 + z\delta\theta_y) + \ddot{w}_0 \delta w_0\big]\, dxdydz\}\end{aligned} \tag{2.23}$$

$$\delta W = \int_0^a \int_0^b q\, \delta w_0\, dxdy \tag{2.24}$$

where q is the distributed load, ρ is the density of panel material. Equation 2.22 can be written in terms of force and moment resultants and integrating through the thickness of laminate as below:

$$\begin{aligned}\delta U = \int_0^a \int_0^b (\, N_x\, \delta {\varepsilon_x}^0 + M_x\, \delta {\varepsilon_x}^1 + N_y\, \delta {\varepsilon_y}^0 + M_y\, \delta {\varepsilon_y}^1 + N_{xy}\, \delta {\gamma_{xy}}^0 \\ + M_{xy}\, \delta {\gamma_{xy}}^1 + Q_{yz}\delta {\gamma_{yz}}^0 + Q_{xz}\delta {\gamma_{xz}}^0) dxdy\end{aligned} \tag{2.25}$$

Integrating through the thickness of the work done by inertia forces in Equation 2.23 can be obtained as in Equation 2.26. I_0, I_1, I_2 are the mass moments of inertia. $\ddot{u}_0, \ddot{v}_0,\ \ddot{w}_0, \ddot{\theta}_x$ and $\ddot{\theta}_y$ indicate the accelerations of mid-plane.

$$\delta T = \int_0^a \int_0^b [\{(I_0\,\ddot{u}_0 + I_1\,\ddot{\theta}_x).\,\delta u_0\} + \{(I_0\,\ddot{v}_0 + I_1\,\ddot{\theta}_y).\,\delta v_0\}$$
$$+ \{(I_1\,\ddot{w}_0).\,\delta w_0\} + \{(I_1\,\ddot{u}_0 + I_2\,\ddot{\theta}_x).\,\delta\theta_x\}$$
$$+ \{(I_1\,\ddot{v}_0 + I_2\,\ddot{\theta}_y).\,\delta\theta_y\}]dxdy \tag{2.26}$$

$$(I_0\,, I_1, I_2) = \int_{-\frac{h(x,y)}{2}}^{\frac{h(x,y)}{2}} \rho.\,(1, z, z^2)\; d_z \tag{2.27}$$

The virtual strains are written in terms of the virtual displacements as shown in Equation 2.28.

$$\delta\varepsilon_x{}^0 = \frac{\partial\delta u_0}{\partial x} + \frac{\delta w_0}{R_x} + \frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial x}, \delta\varepsilon_x{}^1 = \frac{\partial\delta\theta_x}{\partial x}$$
$$\delta\varepsilon_y{}^0 = \frac{\partial\delta v_0}{\partial y} + \frac{\delta w_0}{R_y} + \frac{\partial w_0}{\partial y}\frac{\partial\delta w_0}{\partial y}, \delta\varepsilon_y{}^1 = \frac{\partial\delta\theta_y}{\partial y}$$
$$\delta\gamma_{xy}{}^0 = \frac{\partial\delta u_0}{\partial y} + \frac{\partial\delta v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial\delta w_0}{\partial x} + c_0.\,(\frac{\partial\delta v_0}{\partial x} - \frac{\partial\delta u_0}{\partial y}) \tag{2.28}$$
$$\delta\gamma_{xy}{}^1 = \frac{\partial\delta\theta_x}{\partial y} + \frac{\partial\delta\theta_y}{\partial x}$$
$$\delta\gamma_{yz}{}^0 = \delta\theta_y + \frac{\partial\delta w_0}{\partial y} - \frac{\delta v_0}{R_y}$$
$$\delta\gamma_{xz}{}^0 = \delta\theta_x + \frac{\partial\delta w_0}{\partial x} - \frac{\delta u_0}{R_x}$$

Equation 2.29 is obtained subrogating the virtual strains from Equation 2.28 into Equation 2.25.

$$\delta U = \int_0^a \int_0^b \left\{ N_x.\frac{\partial \delta u_0}{\partial x} + N_x.\frac{\delta w_0}{R_x} + N_x.\frac{\partial w_0}{\partial x}.\frac{\partial \delta w_0}{\partial x} + M_x.\frac{\partial \delta \theta_x}{\partial x} \right.$$
$$+ N_y.\frac{\partial \delta v_0}{\partial y} + N_y.\frac{\delta w_0}{R_y} + N_y.\frac{\partial w_0}{\partial y}.\frac{\partial \delta w_0}{\partial y} + M_y.\frac{\partial \delta \theta_y}{\partial y}$$
$$+ N_{xy}.\frac{\partial \delta u_0}{\partial y} + N_{xy.}\frac{\partial \delta v_0}{\partial x} + N_{xy}.\frac{\partial w_0}{\partial x}.\frac{\partial \delta w_0}{\partial y}$$
$$+ N_{xy}.\frac{\partial w_0}{\partial y}.\frac{\partial \delta w_0}{\partial x} + M_{xy}.\frac{\partial \delta \theta_x}{\partial y} + M_{xy}.\frac{\partial \delta \theta_y}{\partial x} \qquad (2.29)$$
$$+ M_{xy}.c_0.\left(\frac{\partial \delta v_0}{\partial x}\right) - M_{xy}.c_{0.}.\frac{\partial \delta u_0}{\partial y} + Q_{yz}.\delta\theta_y$$
$$+ Q_{yz}.\frac{\partial \delta w_0}{\partial y} - Q_{yz}\frac{\delta v_0}{R_y} + Q_{xz}.\delta\theta_x + Q_{xz}.\frac{\partial \delta w_0}{\partial x}$$
$$\left. - Q_{yz}\frac{\delta u_0}{R_x} \right\} d_x d_y$$

Integration of an expression can be done using Gauss Green Theorem which equates the double integral over the plane region as shown below [28]:

$$\iint N_x.\frac{\partial \delta u_0}{\partial x}.d_x.d_y = \oint N_x.\delta u_0 .n_x.d_s - \int_{\Omega} \frac{\partial N_x}{\partial x}.\delta u_{0.}d_x.d_y \qquad (2.30)$$

Equation 2.29 can be written as in Equation 2.31 by applying the mentioned Gauss Green Theorem.

$$
\begin{aligned}
\delta U = &\oint_s N_x.\delta u_0 .n_x.d_s - \int_\Omega \frac{\partial N_x}{\partial x}.\delta u_0.d_x.d_y + \int_\Omega \frac{N_x}{R_x}.\delta w_0.d_x.d_y \\
&+ \oint_s (N_x.\frac{\partial w_0}{\partial x}).\delta w_0.n_x.d_s - \int_\Omega \frac{\partial}{\partial x}(N_x.\frac{\partial w_0}{\partial x}).\delta w_0.d_x.d_y \\
&+ \oint_s M_x.\delta\theta_x .n_x.d_s - \int_\Omega \frac{\partial M_x}{\partial x}.\delta\theta_x.d_x.d_y + \oint_s N_y.\delta v_0 .n_y.d_s \\
&- \int_\Omega \frac{\partial N_y}{\partial y}.\delta v_0.d_x.d_y + \int_\Omega \frac{N_y}{R_y}.\delta w_0.d_x.d_y \\
&+ \oint_s (N_y.\frac{\partial w_0}{\partial y}).\delta w_0.n_y.d_s - \int_\Omega \frac{\partial}{\partial y}(N_y.\frac{\partial w_0}{\partial y}).\delta w_0.d_x.d_y \\
&+ \oint_s M_y.\delta\theta_y .n_y.d_s - \int_\Omega \frac{\partial M_y}{\partial y}.\delta\theta_y.d_x.d_y \\
&+ \oint_s N_{xy}.\delta u_0 .n_y.d_s - \int_\Omega \frac{\partial N_{xy}}{\partial y}.\delta u_0.d_x.d_y \\
&+ \oint_s N_{xy}.\delta v_0 .n_x.d_s - \int_\Omega \frac{\partial N_{xy}}{\partial x}.\delta v_0.d_x.d_y \\
&+ \oint_s (N_{xy}.\frac{\partial w_0}{\partial x}).\delta w_0.n_y.d_s - \int_\Omega \frac{\partial}{\partial y}(N_{xy}.\frac{\partial w_0}{\partial x}).\delta w_0.d_x.d_y \\
&+ \oint_s (N_{xy}.\frac{\partial w_0}{\partial y}).\delta w_0.n_x.d_s - \int_\Omega \frac{\partial}{\partial x}(N_{xy}.\frac{\partial w_0}{\partial y}).\delta w_0.d_x.d_y \\
&+ \oint_s M_{xy}.\delta\theta_x .n_y.d_s - \int_\Omega \frac{\partial M_{xy}}{\partial y}.\delta\theta_x.d_x.d_y \\
&+ \oint_s M_{xy}.\delta\theta_y .n_x.d_s - \int_\Omega \frac{\partial M_{xy}}{\partial x}.\delta\theta_y.d_x.d_y \\
&+ \oint_s M_{xy}.c_0.\delta v_0 .n_x.d_s - \int_\Omega \frac{\partial M_{xy}}{\partial x}.c_0.\delta v_0.d_x.d_y \\
&- \oint_s M_{xy}.c_0.\delta u_0 .n_y.d_s + \int_\Omega \frac{\partial M_{xy}}{\partial y}.c_0.\delta u_0.d_x.d_y \\
&+ \int_\Omega Q_{yz}.\delta\theta_y .d_x.d_y + \oint_s Q_{yz}.\delta w_0 .n_y.d_s \\
&- \int_\Omega \frac{\partial Q_{yz}}{\partial y}.\delta w_0.d_x.d_y - \int_\Omega \frac{Q_{yz}}{R_y}.\delta v_0.d_x.d_y \\
&+ \int_\Omega Q_{xz}.\delta\theta_x .d_x.d_y + \oint_s Q_{xz}.\delta w_0 .n_x.d_s \\
&- \int_\Omega \frac{\partial Q_{xz}}{\partial x}.\delta w_0.d_x.d_y - \int_\Omega \frac{Q_{xz}}{R_x}.\delta u_0.d_x.d_y
\end{aligned}
\tag{2.31}
$$

If the obtained statements in Equations 2.24, 2.26 and 2.31 are written in virtual work equation specified form in Equation 2.21, the following equation is obtained.

$$
\begin{aligned}
0 = \int_{\Omega} \Bigg\{ & \left[-\frac{\partial N_x}{\partial x} - \frac{\partial N_{xy}}{\partial y} - \frac{Q_{xz}}{R_x} + \frac{\partial}{\partial y}\left(M_{xy}.c_0\right) + \left(I_0\,\ddot{u}_0 + I_1\,\ddot{\theta}_x\right) \right] \delta u_0 \\
& + \left[-\frac{\partial N_y}{\partial y} - \frac{\partial N_{xy}}{\partial x} - \frac{Q_{yz}}{R_y} - \frac{\partial}{\partial x}\left(M_{xy}.c_0\right) \right. \\
& \left. + \left(I_0\,\ddot{v}_0 + I_1\,\ddot{\theta}_y\right) \right] \delta v_0 \\
& + \left[-\frac{\partial}{\partial x}\left(N_x.\frac{\partial w_0}{\partial x} + N_{xy}.\frac{\partial w_0}{\partial y}\right) \right. \\
& - \frac{\partial}{\partial y}\left(N_y.\frac{\partial w_0}{\partial y} + N_{xy}.\frac{\partial w_0}{\partial x}\right) + \frac{N_x}{R_x} + \frac{N_y}{R_y} - \frac{\partial Q_{yz}}{\partial y} \\
& \left. - \frac{\partial Q_{xz}}{\partial x} + I_1\,\ddot{w}_0 - q \right] \delta w_0 \\
& + \left[-\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_{xz} + \left(I_1\,\ddot{u}_0 + I_2\,\ddot{\theta}_x\right) \right] \delta\theta_x \\
& + \left[-\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_{yz} \right. \\
& \left. + \left(I_1\,\ddot{v}_0 + I_2\,\ddot{\theta}_y\right) \right] \delta\theta_y \Bigg\}.d_x.d_y \\
& + \oint_s \Bigg\{ \left(N_x.n_x + N_{xy}.n_y - M_{xy}.c_0.n_y\right).\delta u_0 \\
& + \left(N_y.n_y + N_{xy}.n_x + M_{xy}.c_0.n_x\right).\delta v_0 \\
& + \Bigg((N_x.\frac{\partial w_0}{\partial x}).n_x + (N_y.\frac{\partial w_0}{\partial y}).n_y + (N_{xy}.\frac{\partial w_0}{\partial x}).n_y \\
& + (N_{xy}.\frac{\partial w_0}{\partial y}).n_x + Q_{yz}.n_y + Q_{xz}.n_x \Bigg).\delta w_0 \\
& + \left(M_x.n_x + M_{xy}.n_y\right).\delta\theta_x \\
& + \left(M_y.n_y + M_{xy}.n_x\right).\delta\theta_y \Bigg\}.d_s
\end{aligned}
\tag{2.32}
$$

The governing equations are achieved as given in Equations 2.33-2.37 by setting the coefficient virtual displacements $(\delta u_0, \delta v_0, \delta w_0, \delta\theta_x, \delta\theta_y)$ in area domain (Ω) to zero. The remaining parts except space domain in Equation 2.32 gives the essential and natural boundary conditions [28].

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+\frac{Q_{xz}}{R_x}-\frac{\partial}{\partial y}\left(M_{xy}.\frac{1}{2}(\frac{1}{R_y}-\frac{1}{R_x})\right) = I_0\frac{\partial u_0}{\partial t^2}+ I_1\frac{\partial^2\theta_x}{\partial t^2}+qu \tag{2.33}$$

$$\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}+\frac{Q_{yz}}{R_y}+\frac{\partial}{\partial x}\left(M_{xy}.\frac{1}{2}(\frac{1}{R_y}-\frac{1}{R_x})\right)=I_0\frac{\partial^2 v_0}{\partial t^2}+I_1\frac{\partial^2\theta y}{\partial t^2} \tag{2.34}$$

$$\frac{\partial}{\partial x}\left(N_x.\frac{\partial w_0}{\partial x}+N_{xy}.\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_y.\frac{\partial w_0}{\partial y}+N_{xy}.\frac{\partial w_0}{\partial x}\right)-\frac{N_x}{Rx}-\frac{N_y}{Ry} +\frac{\partial Q_{yz}}{\partial y}+\frac{\partial Q_{xz}}{\partial x}=I_0\frac{\partial^2 w_0}{\partial t^2}+qw \tag{2.35}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_{xz}=I_1\frac{\partial^2 u_0}{\partial t^2}+I_2\frac{\partial^2\theta x}{\partial t^2} \tag{2.36}$$

$$\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_{yz}=I_1\frac{\partial^2 v_0}{\partial t^2}+I_2\frac{\partial^2\theta y}{\partial t^2} \tag{2.37}$$

2.6. Boundary and Initial Conditions

In this thesis work, clamped and simply support boundary conditions on all sides of panel are considered. The clamped boundary conditions on all sides are expressed as shown below:

$$\begin{aligned} &x\text{=0,a} \quad u_0=v_0=w_0=\theta_x=\theta_y=0\\ &y\text{=0,b} \quad u_0=v_0=w_0=\theta_x=\theta_y=0 \end{aligned} \tag{2.38}$$

Time derivatives of clamped boundary conditions are used in the calculation of initial acceleration and is given in Equation 2.39.

$$x=0,a \quad \ddot{u}_0 = \ddot{v}_0 = \ddot{w}_0 = \ddot{\theta}_x = \ddot{\theta}_y = 0$$
$$y=0,b \quad \ddot{u}_0 = \ddot{v}_0 = \ddot{w}_0 = \ddot{\theta}_x = \ddot{\theta}_y = 0 \tag{2.39}$$

Simply supported boundary conditions (SS3) on all sides can be expressed as below [28]:

$$x=0,a \quad u_0 = v_0 = w_0 = \theta_x = M_y = 0$$
$$y=0,b \quad u_0 = v_0 = w_0 = \theta_y = M_x = 0 \tag{2.40}$$

Time derivatives of simply supported boundary conditions are used in the calculation of initial acceleration and is given in Equation 2.39.

$$x=0,a \quad \ddot{u}_0 = \ddot{v}_0 = \ddot{w}_0 = \ddot{\theta}_x = \ddot{M}_y = 0$$
$$y=0,b \quad \ddot{u}_0 = \ddot{v}_0 = \ddot{w}_0 = \ddot{\theta}_y = \ddot{M}_x = 0 \tag{2.41}$$

where M_x and M_y are the natural boundary conditions and can be expressed in terms of displacements as below:

$$\begin{aligned} M_x = {} & B_{11}.\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) + B_{12}.\left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right) \\ & + B_{16}.\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} + c_0.\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right)\right) \\ & + D_{11}.\left(\frac{\partial \theta_x}{\partial x}\right) + D_{12}.\left(\frac{\partial \theta_y}{\partial y}\right) + D_{16}.\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) = 0 \end{aligned} \tag{2.42}$$

$$\begin{aligned} M_y = {} & B_{12}.\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R_x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) + B_{22}.\left(\frac{\partial v_0}{\partial y} + \frac{w_0}{R_y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right) \\ & + B_{26}.\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} + c_0.\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right)\right) \\ & + D_{12}.\left(\frac{\partial \theta_x}{\partial x}\right) + D_{22}.\left(\frac{\partial \theta_y}{\partial y}\right) + D_{26}.\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \end{aligned} \tag{2.43}$$

Moment type boundary conditions in simply support can be expressed in terms of accelerations in Equations 2.44-2.45 by taking the second derivative with respect to time and neglecting the nonlinear terms.

$$
\begin{aligned}
\ddot{M}_x = {} & B_{11}.\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\ddot{w}_0}{R_x}+\left(\frac{\partial \dot{w}_0}{\partial x}\right)^2+\frac{\partial w_0}{\partial x}\frac{\partial \ddot{w}_0}{\partial x}\right) \\
& + B_{12}.\left(\frac{\partial \ddot{v}_0}{\partial y}+\frac{\ddot{w}_0}{R_y}+\left(\frac{\partial \dot{w}_0}{\partial y}\right)^2+\frac{\partial w_0}{\partial y}\frac{\partial \ddot{w}_0}{\partial y}\right) \\
& + B_{16}.\left(\frac{\partial \ddot{v}_0}{\partial x}+\frac{\partial \ddot{u}_0}{\partial y}+c_0.\left(\frac{\partial \ddot{v}_0}{\partial x}-\frac{\partial \ddot{u}_0}{\partial y}\right)+\frac{\partial w_0}{\partial y}\frac{\partial \ddot{w}_0}{\partial x}\right. \\
& \left. + 2\frac{\partial \dot{w}_0}{\partial x}\frac{\partial \dot{w}_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial \ddot{w}_0}{\partial y}\right)+D_{11}.\left(\frac{\partial \ddot{\theta}_x}{\partial x}\right)+D_{12}.\left(\frac{\partial \ddot{\theta}_y}{\partial y}\right) \\
& + D_{16}.\left(\frac{\partial \ddot{\theta}_x}{\partial y}+\frac{\partial \ddot{\theta}_y}{\partial x}\right)
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
\ddot{M}_y = {} & B_{12}.\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\ddot{w}_0}{R_x}+\left(\frac{\partial \dot{w}_0}{\partial x}\right)^2+\frac{\partial w_0}{\partial x}\frac{\partial \ddot{w}_0}{\partial x}\right) \\
& + B_{22}.\left(\frac{\partial \ddot{v}_0}{\partial y}+\frac{\ddot{w}_0}{R_y}+\left(\frac{\partial \dot{w}_0}{\partial y}\right)^2+\frac{\partial w_0}{\partial y}\frac{\partial \ddot{w}_0}{\partial y}\right) \\
& + B_{26}.\left(\frac{\partial \ddot{v}_0}{\partial x}+\frac{\partial \ddot{u}_0}{\partial y}+c_0.\left(\frac{\partial \ddot{v}_0}{\partial x}-\frac{\partial \ddot{u}_0}{\partial y}\right)+\frac{\partial w_0}{\partial y}\frac{\partial \ddot{w}_0}{\partial x}\right. \\
& \left. + 2\frac{\partial \dot{w}_0}{\partial x}\frac{\partial \dot{w}_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial \ddot{w}_0}{\partial y}\right)+D_{12}.\left(\frac{\partial \ddot{\theta}_x}{\partial x}\right)+D_{22}.\left(\frac{\partial \ddot{\theta}_y}{\partial y}\right) \\
& + D_{26}.\left(\frac{\partial \ddot{\theta}_x}{\partial y}+\frac{\partial \ddot{\theta}_y}{\partial x}\right)
\end{aligned}
\tag{2.45}
$$

All initial displacement and velocity components is given zero for this thesis work. Thus, at any point of the panel (t=0) displacement and velocity values for both boundary condition types are equal to zero. This can be expressed as follows:

$$
\begin{aligned}
t=0 \quad & u_0 = v_0 = w_0 = \theta_x = \theta_y = 0 \\
t=0 \quad & \dot{u}_0 = \dot{v}_0 = \dot{w}_0 = \dot{\theta}_x = \dot{\theta}_y = 0
\end{aligned}
\tag{2.46}
$$

3. SOLUTION METHOD

In this section, obtained governing differential equations of tapered panel is solved with numerical method. Chebyshev polynomials are used for the spatial discretization of the problem, whereas Newmark-Beta method is used for the temporal discretization. The displacements and rotations in the direction of *x*, *y* and *z* of middle point and also the loadings are expressed as a summation of double Chebyshev series. Equation of motions are written in terms of displacement and nonlinear differential equations of motion are solved with Newton-Raphson method. The code to solve these differential equations were written using MATLAB programing language.

3.1. Chebyshev Collocation Method

Chebyshev polynomials are very usable orthogonal polynomials especially in the field of numerical analysis. The Chebyshev polynomials of the first kind denoted as T_n (x) and can be expressed [31]:

$$T_n(x) = \cos(n\theta)\,, x = \cos(\theta)\,, -1 \leq x \leq 1 \tag{3.1}$$

where n is the degree of polynomial. By combining the trigonometric relation in Equation 3.1, we can obtain

$$\cos((n+1)\theta) + \cos((n-1)\theta) = \cos(2n\theta) \tag{3.2}$$

The recurrence relations for Chebyshev polynomials can be generated as:

$$\begin{gathered} T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n \geq 1 \\ T_0(x) = 1,\ T_1(x) = x \end{gathered} \tag{3.3}$$

The first kind Chebyshev polynomials can be expanded as below by using the recurrence relation in Equation 3.3.

$$
\begin{gathered}
T_2(x) = 2x^2 - 1 \\
T_3(x) = 4x^3 - 3x \\
T_4(x) = 8x^4 - 8x^2 + 1 \\
T_5(x) = 16x^5 - 20x^3 + 5x \\
T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 \\
T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{gathered}
\tag{3.4}
$$

The displacement functions and load are written in terms of the summation of Chebyshev polynomials as following:

$$
\begin{gathered}
\mathrm{u}_0(x,y,t) = \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).T_m(x).T_n(y) \\
\mathrm{v}_0(x,y,t) = \sum_0^M \sum_0^N \delta_{mn}.v_{mn}(t).T_m(x).T_n(y) \\
\mathrm{w}_0(x,y,t) = \sum_0^M \sum_0^N \delta_{mn}.w_{mn}(t).T_m(x).T_n(y) \\
\theta_X(x,y,t) = \sum_0^M \sum_0^N \delta_{mn}.\theta x_{mn}(t).T_m(x).T_n(y) \\
\theta_y(x,y,t) = \sum_0^M \sum_0^N \delta_{mn}.\theta y_{mn}(t).T_m(x).T_n(y) \\
q = \sum_0^M \sum_0^N \delta_{mn}.q_{mn}.T_m(x).T_n(y)
\end{gathered}
\tag{3.5}
$$

M and N are the number of Chebyshev series terms. $u_{mn}(t)$, $v_{mn}(t)$, $w_{mn}(t)$, $\theta_{x_{mn}}(t)$ and $\theta_{y_{mn}}(t)$ are the unknown coefficients for displacements and rotations to be determined. The displacement fields in Equation 3.5 can be rearranged for simplicity as follows:

$$
\begin{aligned}
\mathrm{u}_0(x,y,t) &= \sum_0^M \sum_0^N \delta_{mn}.\, u_{mn}(t).\, N_{u_{mn}}(x,y) \\
\mathrm{v}_0(x,y,t) &= \sum_0^M \sum_0^N \delta_{mn}.\, v_{mn}(t).\, N_{v_{mn}}(x,y) \\
\mathrm{w}_0(x,y,t) &= \sum_0^M \sum_0^N \delta_{mn}.\, w_{mn}(t).\, N_{w_{mn}}(x,y) \\
\theta_X(x,y,t) &= \sum_0^M \sum_0^N \delta_{mn}.\, \theta_{x_{mn}}(t).\, N_{\theta_{x_{mn}}}(x,y) \\
\theta_y(x,y,t) &= \sum_0^M \sum_0^N \delta_{mn}.\, \theta_{y_{mn}}(t).\, N_{\theta_{y_{mn}}}(x,y)
\end{aligned}
\tag{3.6}
$$

$N_{u_{mn}}(x,y), N_{v_{mn}}(x,y), N_{w_{mn}}(x,y), N_{\theta_{x_{mn}}}(x,y)$ and $N_{\theta_{y_{mn}}}(x,y)$ represent the product of Chebyshev polynomials in Equation 3.6 and will be shortly denoted as $N_u, N_v, N_w, N_{\theta_x}$ and N_{θ_y}. δ_{mn} takes the following values [10]:

$$
\begin{aligned}
\delta_{ij} &= 0.25 \ \ if \ i,j = 0 \\
\delta_{ij} &= 0.5 \ \ if \ i = 0, j \neq 0 \ or \ \ i \neq 0, j = 0 \\
\delta_{ij} &= 1 \ if \ otherwise
\end{aligned}
\tag{3.7}
$$

The derivatives of the displacement function $u_0(x, y, t)$ are shown in Equation 3.8. The derivatives for other displacement functions can be also enlarged as shown in Equation 3.8.

$$
\begin{aligned}
\frac{\partial \mathrm{u}_0(x,y,t)}{\partial x} &= \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).\frac{\partial T_m(x)}{\partial x}.T_n(y) \\
&= \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).Nu,x \\
\frac{\partial \mathrm{u}_0(x,y,t)}{\partial y} &= \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).T_m(y).\frac{\partial T_n(x)}{\partial y} \\
&= \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).Nu,y \\
\frac{\partial^2 \mathrm{u}_0(x,y,t)}{\partial x \partial y} &= \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).\frac{\partial T_m(x)}{\partial x}.\frac{\partial T_n(x)}{\partial y} \\
&= \sum_0^M \sum_0^N \delta_{mn}.u_{mn}(t).Nu,xy
\end{aligned}
\tag{3.8}
$$

where $N_{u,x}, N_{u,y}$ and $N_{u,xy}$ are the derivatives of the N_u according to *x*, *y* and *xy*, respectively. Derivations are illustrated more clearly in Equation 3.9 and can be also enlarged for other derivatives of N_v, N_w, N_{θ_x} and N_{θ_y}.

$$
\begin{aligned}
N_{u,x} &= \frac{\partial N_{u_{mn}}(x,y)}{\partial x} = \sum_0^M \sum_0^N \frac{\partial T_m(x)}{\partial x}.T_n(y) \\
N_{u,y} &= \frac{\partial N_{u_{mn}}(x,y)}{\partial y} = \sum_0^M \sum_0^N \frac{\partial T_m(x)}{\partial y}.T_n(y) \\
N_{u,xy} &= \frac{\partial^2 N_{u_{mn}}(x,y)}{\partial x \partial y} = \sum_0^M \sum_0^N \frac{\partial T_m(x)}{\partial x}.\frac{\partial T_n(x)}{\partial y}
\end{aligned}
\tag{3.9}
$$

Displacements, rotations in Equation 3.6 and derivatives in Equation 3.8 are written in the governing differential equations given in Equations 2.33-2.37. The equilibrium equations are consisted of the unknown coefficients which will be calculated. Chebyshev collocation method, grid points in the structure must be determined to apply the differential equations and boundary conditions. In this study, Chebyshev-Gauss Lobatto points are selected as grid points for the solution. This is because, the points with uniform distribution can cause high oscillations at the points close to boundaries. Chebyshev-Gauss Lobatto points are distributed more extensively in internal points and also creates grid points on the boundaries. Chosen grid points can be calculated from Equations 3.10-3.11 for spatial discretization.

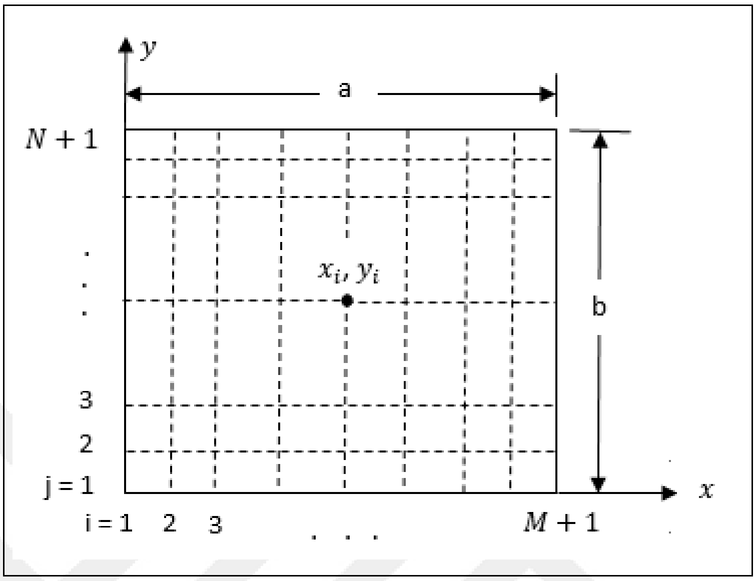

Figure 3.1: Grid points in two dimensional coordinate system.

$$x_i = \frac{a}{2}(1 - \cos(\frac{(i-1)\pi}{(M-1)}),\ \mathrm{i} = 1,2, \ldots, \mathrm{M} + 1 \tag{3.10}$$

$$y_j = \frac{b}{2}(1 - \cos\left(\frac{(j-1)\pi}{(N-1)}\right),\ \mathrm{j} = 1,2, \ldots, \mathrm{N} + 1 \tag{3.11}$$

The entire number of unknown coefficients in terms of displacements is 5(M+1)(N+1). The governing differential equations are written for internal grid points and gives us 5(M-1)(N-1) equations. The boundary conditions give 10(M+1)+10(N-1) equations. It can be seen that total number of equations is equal to the total number of unknown coefficients. Since dynamic differential equations are including nonlinear terms, Newton-Raphson method as given in section 3.3 is used for linearization and Newmark-Beta method is used for time discretization.

3.2. Newmark-Beta Method

The Newmark-beta method is a widely used numerical integration method to evaluate the dynamic response of structures. Newmark-beta method indicates that the acceleration and velocity at the nth time step can be expressed as:

$$\begin{aligned} \ddot{U}_{n+1} &= c_0(U_{n+1} - U_n) - c_1 U_n - \ddot{U}_n \\ \dot{U}_{n+1} &= \dot{U}_n + (1-\gamma)\Delta_t \ddot{U}_n + \gamma \Delta_t \ddot{U}_{n+1} \end{aligned} \tag{3.12}$$

where $c_0 = 4/\Delta t^2$, $c_1 = 4/\Delta t$ and γ=0.5 which yields the constant average acceleration method, is chosen. The governing equilibrium equations consists of the displacements and rotations in terms of unknown coefficients to be calculated after writing the displacements, rotations and their derivatives to the governing equilibrium equations. U_n in Equation 3.13 represent unknown coefficients at the nth time step.

$$U_n = \begin{bmatrix} u_{11} \\ v_{11} \\ w_{11} \\ \theta_{x_{11}} \\ \theta_{y_{11}} \\ \vdots \\ u_{1N_y} \\ v_{1N_y} \\ w_{1N_y} \\ \theta_{x_{1N_y}} \\ \theta_{y_{1N_y}} \\ \vdots \\ u_{N_xN_y} \\ v_{N_xN_y} \\ w_{N_xN_y} \\ \theta_{x_{N_xN_y}} \\ \theta_{y_{N_xN_y}} \end{bmatrix}_n \tag{3.13}$$

3.3. Newton-Raphson Method

Newton-Raphson method, which is an iterative method, can be used for solving nonlinear differential equation. It is based on the simple idea of linear approximation. Nonlinear differential equation group shown in Equations 2.33-2.37 can be written in matrix form as follows:

$$M\ddot{U} + P(U) = F \tag{3.14}$$

Equation of motion in Equation 3.14 is nonlinear in terms of unknown displacement coefficients, U_n. M is mass matrix and F, P, $\ddot{U}$ are external force, internal force and acceleration vectors, respectively.

Dynamic response of panel at (n+1)th time step can be expressed in the following equation using Newmark-beta direct integration scheme specified in section 3.2.

$$C_0 M U_{n+1} + P(U_{n+1}) = F_{n+1} + M(c_0 U_n + c_1 \dot{U}_n + \ddot{U}_n) \tag{3.15}$$

Newton-Raphson Method is applied to solve nonlinear Equation 3.15 in terms of unknown displacement coefficients. Equation 3.15 is rewritten in terms of error function or residual forces R_{n+1} as:

$$R_{n+1} = c_0 M U_{n+1} + P(U_{n+1}) - F_{n+1} - M.\left(c_0 U_n + c_1 \dot{U}_n + \ddot{U}_n\right) = 0 \tag{3.16}$$

Nonlinear terms can be found in the expression of P_{n+1}. Equation 3.17 can be linearized by taking the increment as shown in Equation 3.17.

$$\begin{aligned} \Delta R_{n+1} = c_0 M \Delta U_{n+1} + \Delta P(U_{n+1}) - \Delta F_{n+1} \\ - M.\left(c_0 \Delta U_n + c_1 \Delta \dot{U}_n + \Delta \ddot{U}_n\right) = 0 \end{aligned} \tag{3.17}$$

Linearization of ΔP_{n+1} is expressed as below.

$$\Delta P(U_{n+1}) = K_{n+1}\Delta U_{n+1} \tag{3.18}$$

Equation 3.17 takes the following form at the *i*th iteration by neglecting the zeros terms after increment [14].

$$\Delta R_{n+1}^{i} \cong \left(c_0 M + K_{n+1}^{i}\right)\Delta U_{n+1}^{i} \tag{3.19}$$

Equation 3.19, which was written in incremental form, can also be expressed as follows:

$$R_{n+1}^{i+1} \cong R_{n+1}^{i} + (C_0 M + K_{n+1}^{i})\left(U_{n+1}^{i+1} - U_{n+1}^{i}\right) \tag{3.20}$$

U_{n+1}^{i} is an approximate trial solution at the *i*th iteration. U_{n+1}^{i+1} is an improved solution at the (*i*+1)th iteration that is needed to be calculated. Equation 3.20 must be rearranged as follows since the error function at the (*i*+1)th iteration, R_{n+1}, is required to approach to zero:

$$R_{n+1}^{i+1} \approx R_{n+1}^{i} + {K_T}_{n+1}^{i}\left(U_{n+1}^{i+1} - U_{n+1}^{i}\right) = 0 \tag{3.21}$$

where ${K_T}_{n+1}^{i}$ denotes to tangent stiffness matrix and expressed in Equation 3.22.

$${K_T}_{n+1}^{i} = C_0 M + K_{n+1}^{i} \tag{3.22}$$

Equation 3.21 can be transformed into a form as below:

$${K_T}_{n+1}^{i}\Delta U_{n+1}^{i} = -R_{n+1}^{i} \tag{3.23}$$

where ΔU_{n+1}^{i}, which is the displacement increment at each iteration, can be found easily by matrix solution. Improved solution at (*i*+1)th iteration can be found easily by adding the displacement increment as in Equation 3.24.

$$U_{n+1}^{i+1} = U_{n+1}^{i} + \Delta U_{n+1}^{i} \tag{3.24}$$

To start the Newton-Raphson solution, the initial acceleration values can be found by using initial displacements and velocities values at time *t*=0. Iterative computation is expired when the convergence parameter, $conv$ is small than 0.005 ($conv \leq 0.005$).Convergence parameter is calculated as given below [32]:

$$conv = \frac{\left\| {R_{n+1}^{i}}^{2} \right\|}{1 + \left\| {F_{n+1}}^{2} \right\|} \tag{3.25}$$

Governing differential equations in incremental form mentioned in Equation 3.19 can be written as follows:

$$\frac{\partial \Delta N_x}{\partial x} + \frac{\partial \Delta N_{xy}}{\partial y} + \frac{\Delta Q_{xz}}{R_x} - \frac{\partial}{\partial y}\left(\Delta M_{xy}.\frac{1}{2}\left(\frac{1}{R_y} - \frac{1}{R_x}\right)\right) - I_0 \frac{\partial \Delta u_0}{\partial t^2} - I_1 \frac{\partial^2 \Delta \theta_x}{\partial t^2} = 0 \tag{3.26}$$

$$\frac{\partial \Delta N_y}{\partial y} + \frac{\partial \Delta N_{xy}}{\partial x} + \frac{\Delta Q_{yz}}{R_y} + \frac{\partial}{\partial x}\left(\Delta M_{xy}.\frac{1}{2}(\frac{1}{R_y} - \frac{1}{R_x})\right) - I_0 \frac{\partial^2 \Delta v_0}{\partial t^2} - I_1 \frac{\partial^2 \Delta \theta y}{\partial t^2} = 0 \tag{3.27}$$

$$\Delta N_x \frac{\partial^2 w_0}{\partial x^2} + N_x \frac{\partial^2 \Delta w_0}{\partial x^2} + \frac{\partial \Delta N_x}{\partial x}\frac{\partial w_0}{\partial x} + \frac{\partial N_x}{\partial x}\frac{\partial \Delta w_0}{\partial x} + \Delta N_{xy}\frac{\partial^2 w_0}{\partial x \partial y}$$
$$+ N_{xy}\frac{\partial^2 \Delta w_0}{\partial x \partial y} + \frac{\partial \Delta N_{xy}}{\partial x}\frac{\partial w_0}{\partial y} + \frac{\partial N_{xy}}{\partial x}\frac{\partial \Delta w_0}{\partial y}$$
$$+ \Delta N_y \frac{\partial^2 w_0}{\partial y^2} + N_y \frac{\partial^2 \Delta w_0}{\partial y^2} + \frac{\partial \Delta N_y}{\partial y}\frac{\partial w_0}{\partial y} + \frac{\partial N_y}{\partial y}\frac{\partial \Delta w_0}{\partial y} \tag{3.28}$$
$$+ \Delta N_{xy}\frac{\partial^2 w_0}{\partial x \partial y} + N_{xy}\frac{\partial^2 \Delta w_0}{\partial x \partial y} + \frac{\partial \Delta N_{xy}}{\partial y}\frac{\partial w_0}{\partial x}$$
$$+ \frac{\partial N_{xy}}{\partial y}\frac{\partial \Delta w_0}{\partial x} - \frac{\Delta N_x}{Rx} - \frac{\Delta N_y}{Ry} + \frac{\partial \Delta Q_{yz}}{\partial y} + \frac{\partial \Delta Q_{xz}}{\partial x}$$
$$- I_0 \frac{\partial^2 \Delta w_0}{\partial t^2} = 0$$

$$\frac{\partial \Delta M_x}{\partial x} + \frac{\partial \Delta M_{xy}}{\partial y} - \Delta Q_{xz} - I_1 \frac{\partial^2 \Delta u_0}{\partial t^2} - I_2 \frac{\partial^2 \Delta \theta x}{\partial t^2} = 0 \tag{3.29}$$

$$\frac{\partial \Delta M_y}{\partial y} + \frac{\partial \Delta M_{xy}}{\partial x} - \Delta Q_{yz} = I_1 \frac{\partial^2 \Delta v_0}{\partial t^2} + I_2 \frac{\partial^2 \Delta \theta y}{\partial t^2} \tag{3.30}$$

.

The Equations 3.26-3.30 are written by extracting ΔU term which is matrix as shown in Equation 3.31 in Appendix A. Thus, it is simplified by using matrix format. The details for the linearization of terms in the Equations 3.26-3.30 can be found in Appendix A.

3.4. Finite Element Method

The isotropic and composite plate and panel structures with variable thickness are modelled with FEM by using ANSYS finite element software. 40x40 shell elements (Shell 281) are used for discretization of the structure. Shell 281 is convenient for analyzing thin to moderately-thick shell structures and have geometric nonlinearity capability. The element has six degrees of freedom with three in translation and three in rotation at each node.

By modeling the composite structure in ANSYS, the faces between the laminas are assumed to be perfectly bonded and structures are assumed to be linear elastic material. The thickness function given in Equation 2.1 is used for the thickness distribution. For simply support boundary conditions, the displacements through the three direction are restricted. For clamped boundary conditions, all displacements and rotations are restricted.

Nonlinear transient analysis of tapered panel are performed in ANSYS for comparison with the obtained result. The finite element model and cross section of the structure are shown in Figures 3.2-3.3.

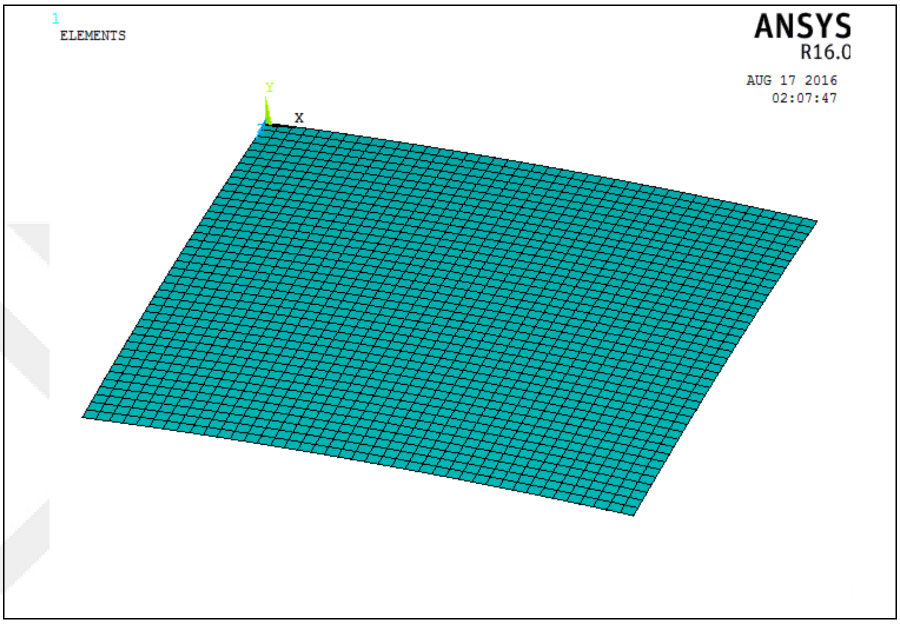

Figure 3.2: Finite Element Model of the tapered composite panel.

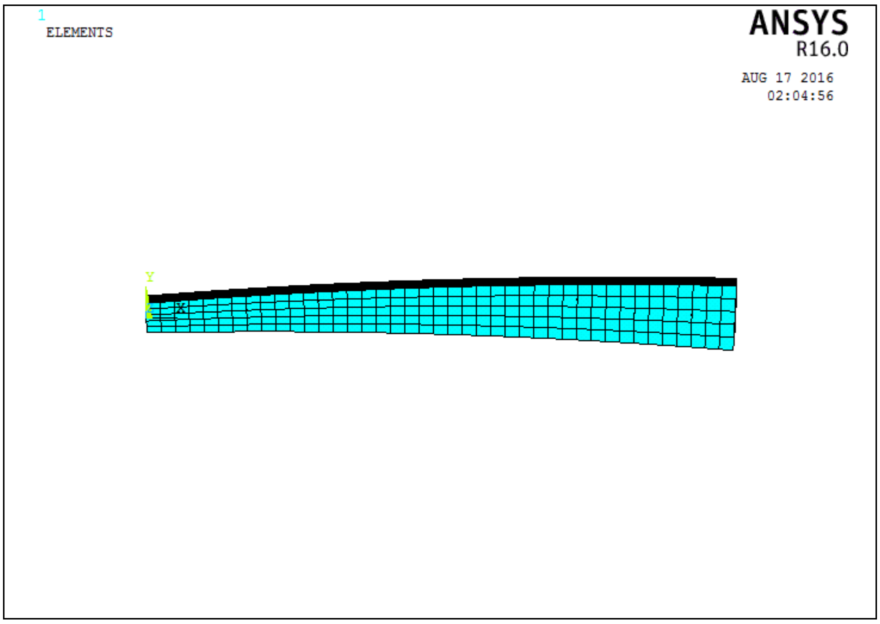

Figure 3.3: Cross section of tapered composite panel.

4. NUMERICAL EXAMPLES

Nonlinear dynamic equilibrium equations of composite doubly curved panels with variable thickness were obtained using virtual work principle. A computer program using Matlab is developed to solve these obtained equations by using Chebyshev collocation method. The structure is carried out parametrically by changing material type, boundary conditions, the taper ratios and panel radius. In all examples, the radius of panel, R_x and R_y is denoted with R. Loading is thought uniform distributed external pressure with step pulse on the structure. The external pressure is q_0 =-6x10^7 Pa in the solution of all problems. Step pulse type can be seen in Figure 4.1 in which q_0 is the peak pressure and t is the duration of loading on the structure. The final termination, t is taken 0.01 s. Time step is taken 0.1 ms. Grid points are selected as odd numbers for locating a grid point at the center of plate and panel.

Non-dimensional displacement-time histories (w/h) at the center of plate and panel are compared with the commercial finite element software ANSYS in order to confirm theoretical calculations.

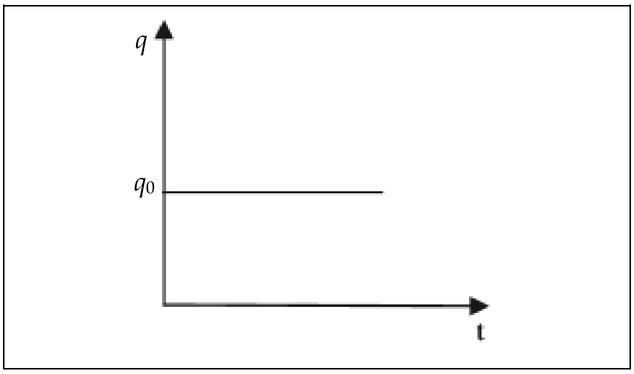

Figure 4.1: Step loading.

4.1. Validation example for isotropic plate and panel

In this section, the comparison between the Chebyshev collocation method and finite element method for the linear/nonlinear transient response of isotropic plate and panel structures for different boundary conditions are carried out parametrically. Steel material is chosen as an isotropic material. The material properties of steel are:

E=210 GPa, G=80.769 GPa, ρ=7800 kg/m3, ν=0.30. The dimensions for the plate are taken as a=b=1 m, h=0.05 m. The dimensions for the panel are taken as a=b=1 m, h=0.05 m, R_x = R_y = R =10 m. 9x9 Chebyshev collocation term was used for the converged results in all examples in this section.

The linear/nonlinear transient responses at the middle of isotropic plate and panel structures with clamped and simply supported conditions can be seen in Figures 4.2-4.13.

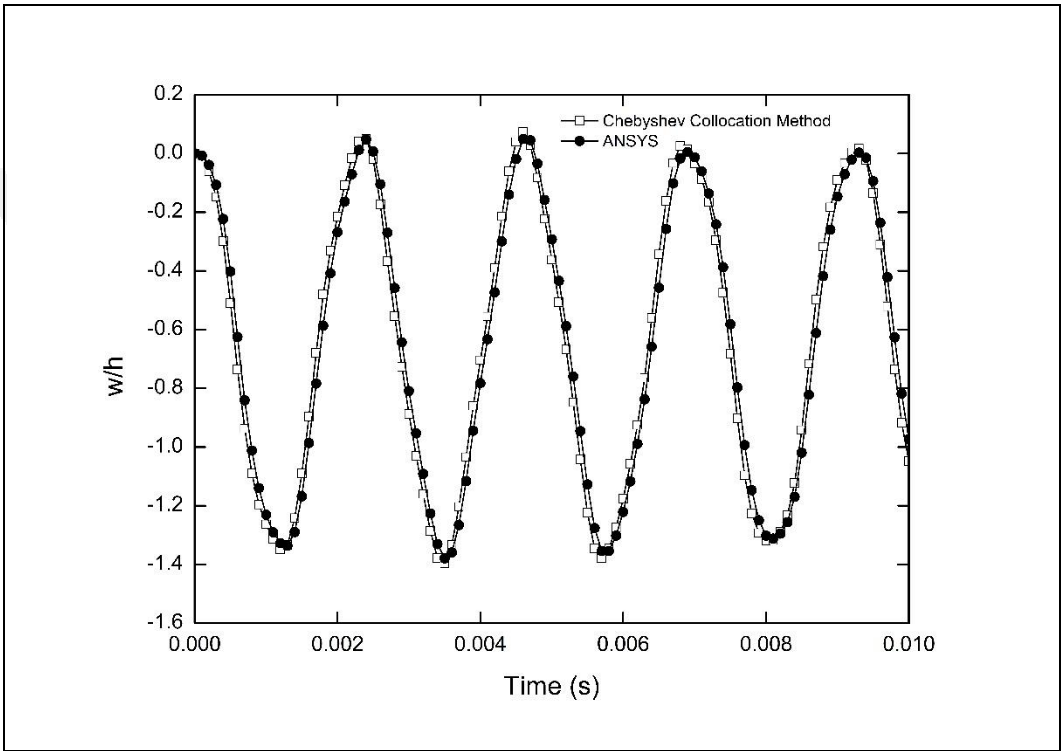

Figure 4.2: Non-dimensional displacement-time history at the center of clamped isotropic plate for linear analysis (a/h=20, R/a=∞).

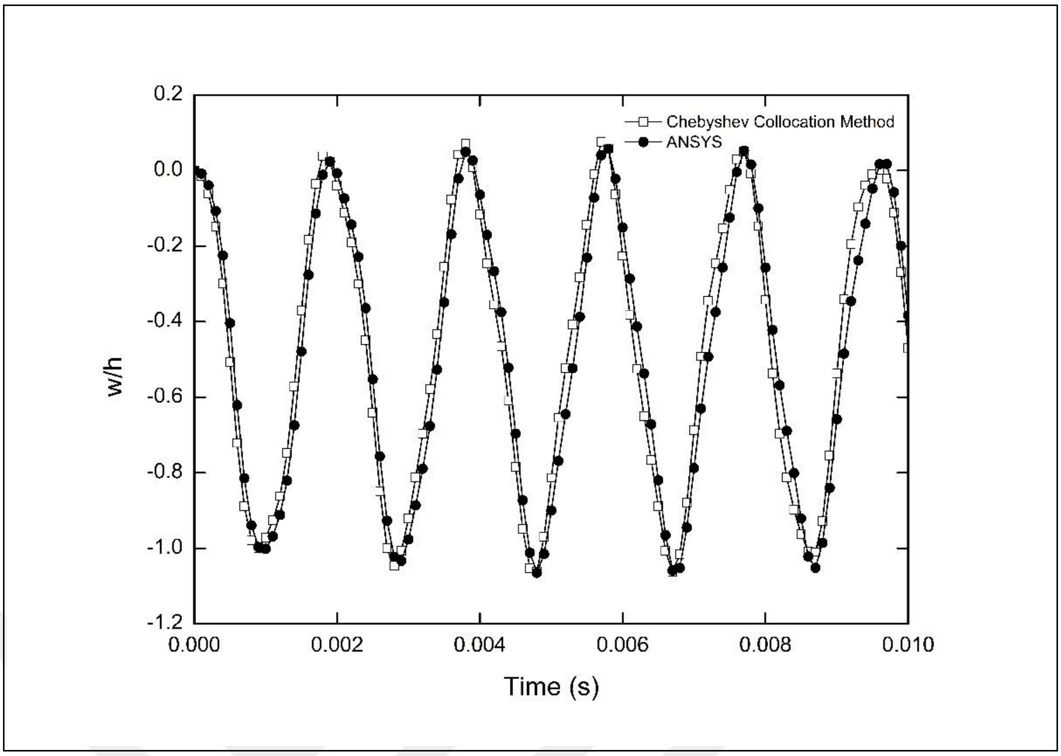

Figure 4.3: Non-dimensional displacement-time history at the center of clamped isotropic plate for nonlinear analysis (a/h=20, R/a=∞) .

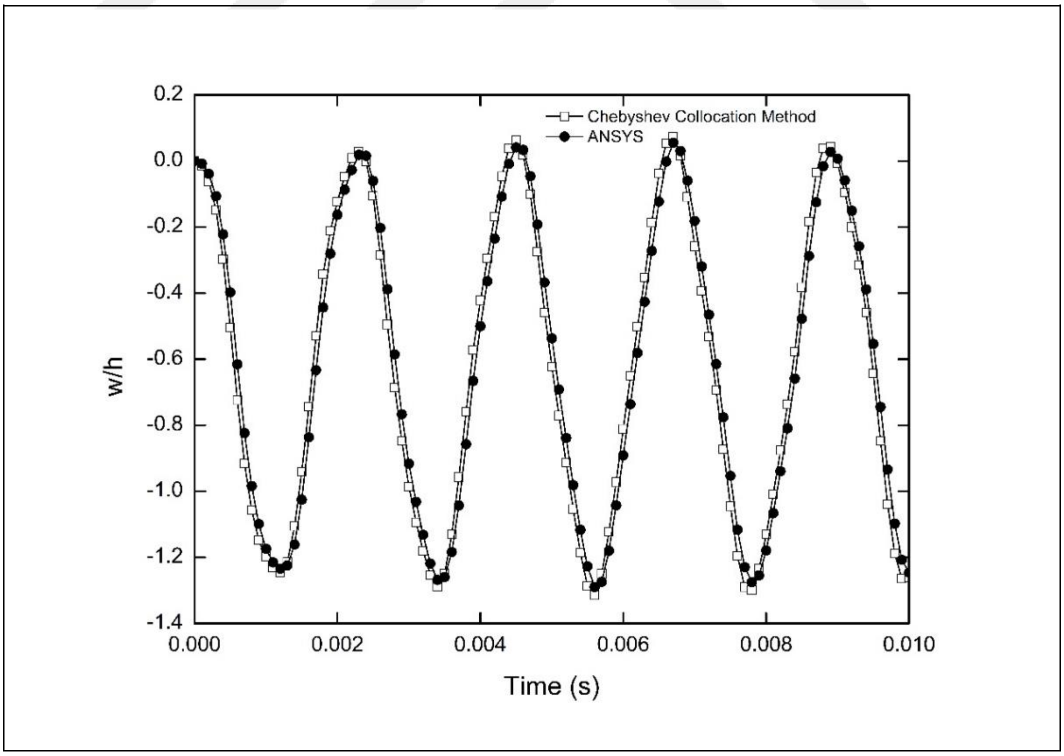

Figure 4.4: Non-dimensional displacement-time history at the center of clamped isotropic panel for linear analysis (a/h=20, R/a=10).

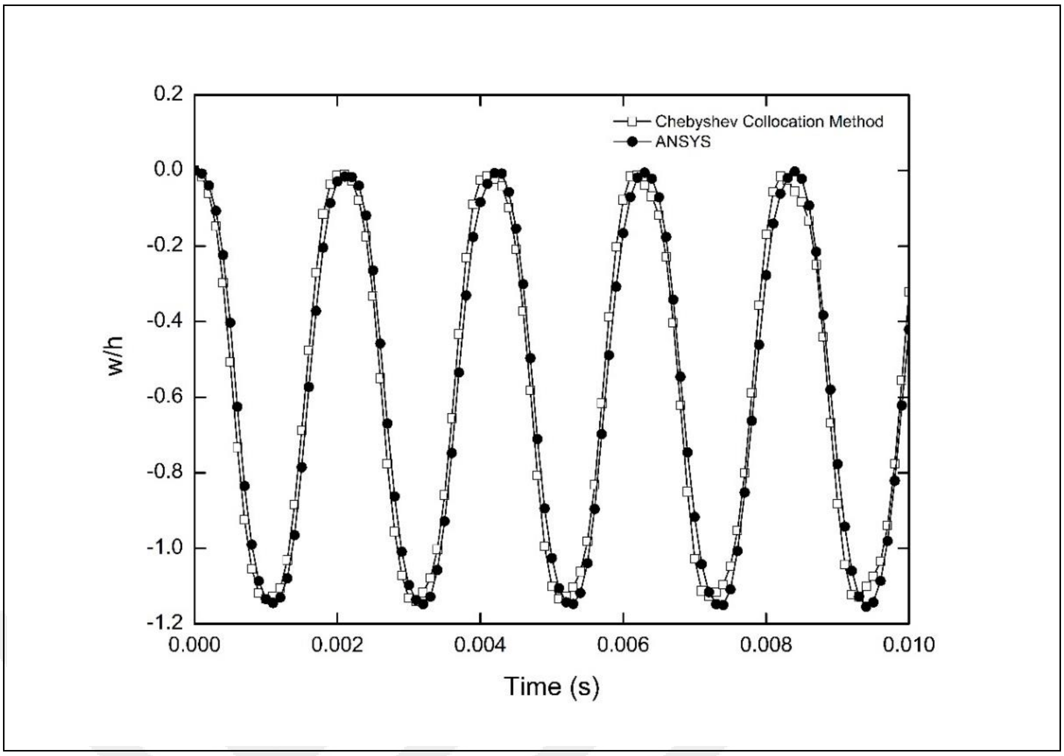

Figure 4.5: Non-dimensional displacement-time history at the center of clamped isotropic panel for nonlinear analysis (a/h=20, R/a=10).

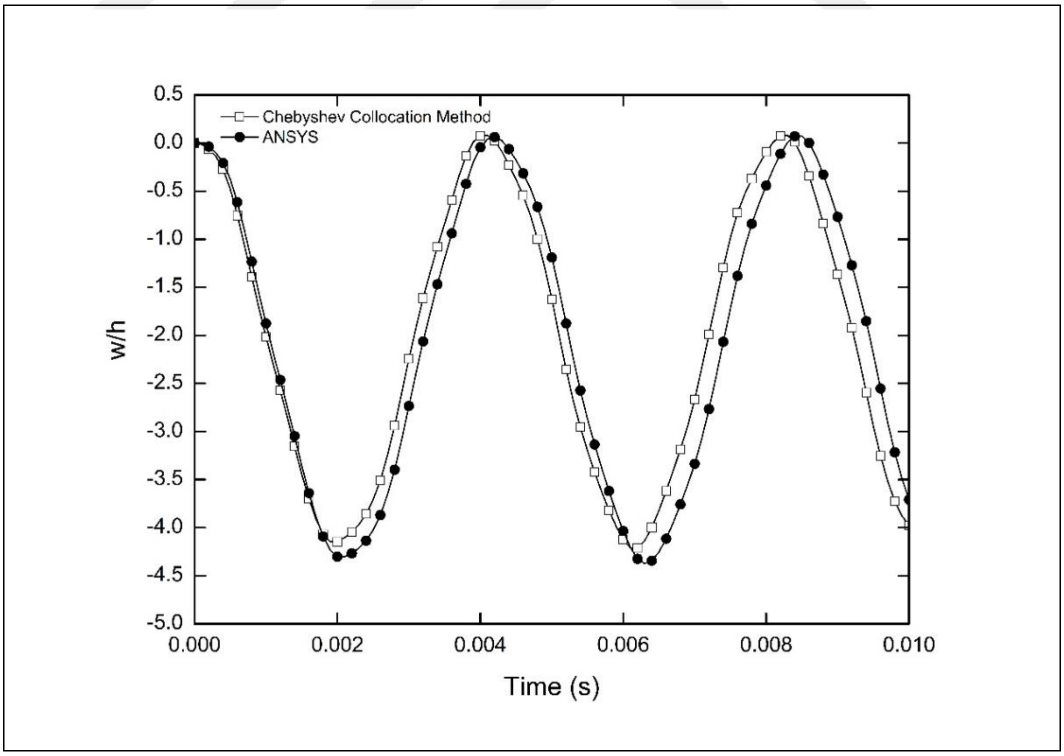

Figure 4.6: Non-dimensional displacement-time history at the center of simply supported (SS3) isotropic plate for linear analysis (a/h=20, R/a=∞).

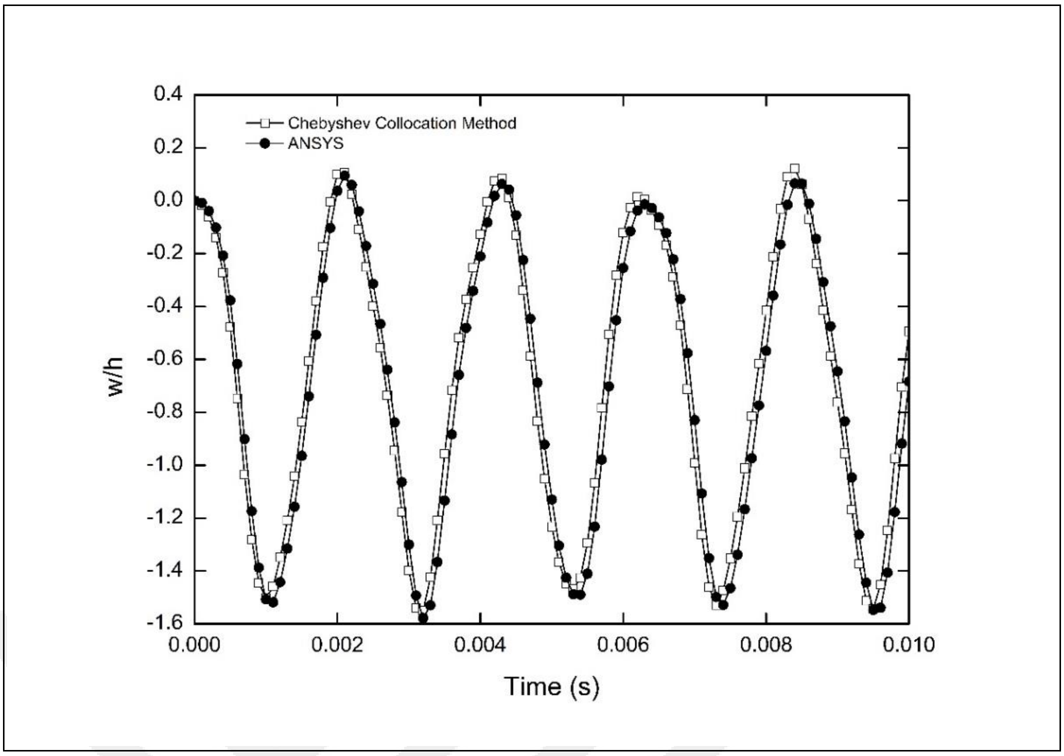

Figure 4.7: Non-dimensional displacement-time history at the center of simply supported (SS3) isotropic plate for nonlinear analysis (a/h=20, R/a=∞).

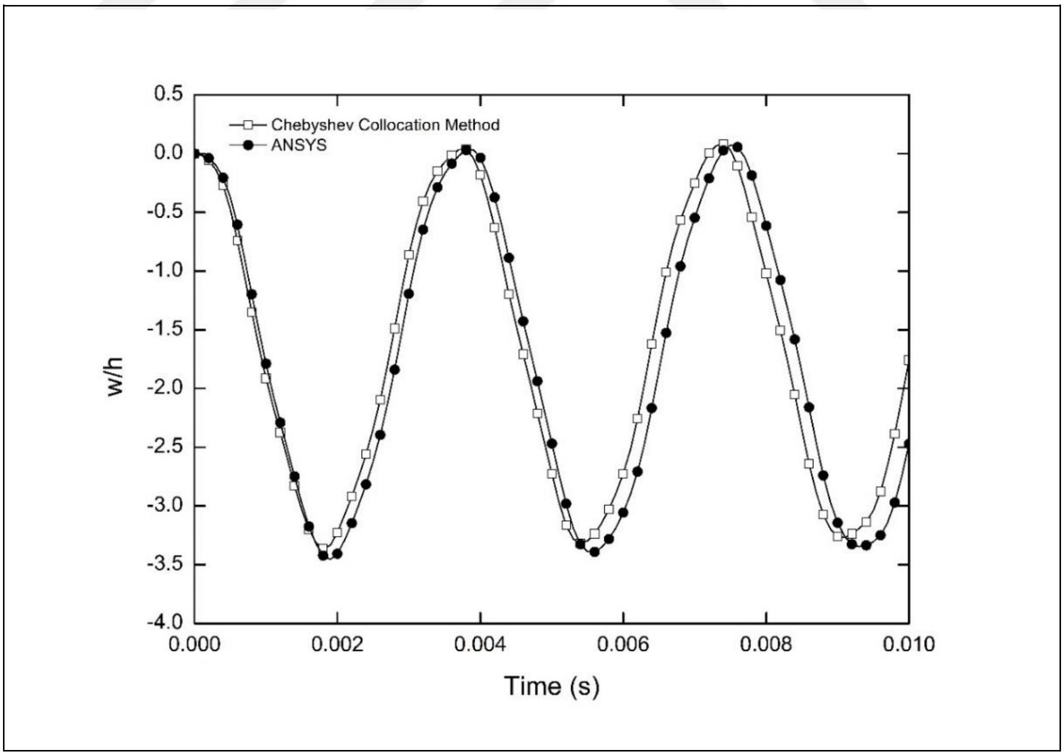

Figure 4.8: Non-dimensional displacement-time history at the center of simply supported (SS3) isotropic panel for linear analysis (a/h=20, R/a=10).

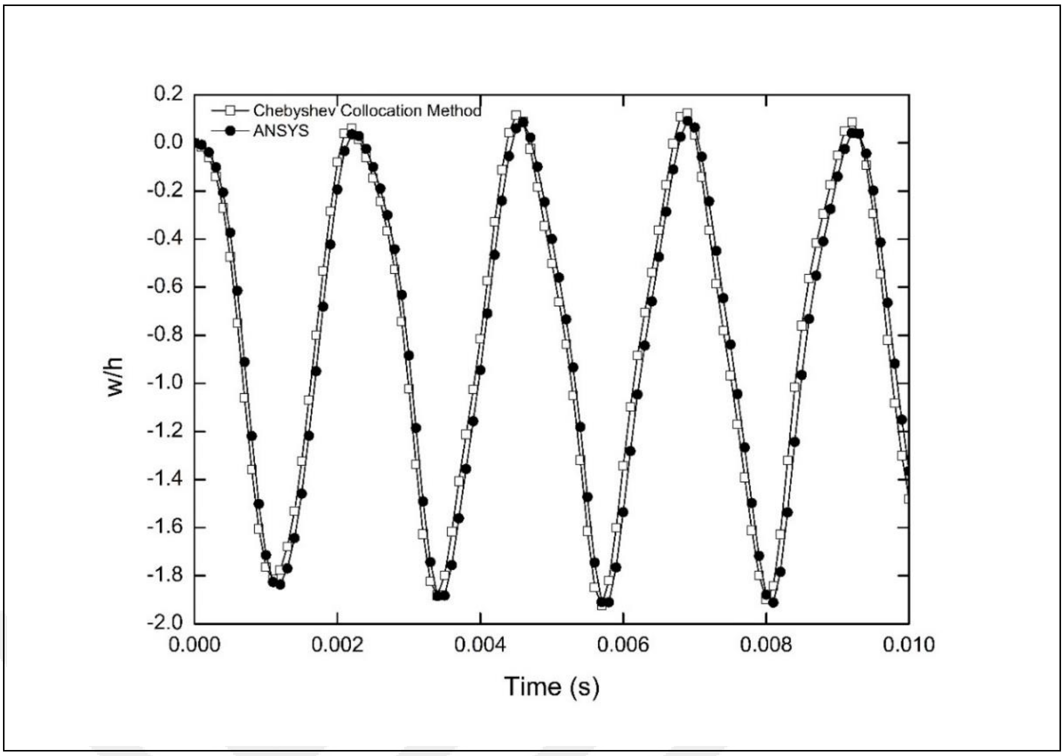

Figure 4.9: Non-dimensional displacement-time history at the center of simply supported (SS3) isotropic panel for nonlinear analysis (a/h=20, R/a=10).

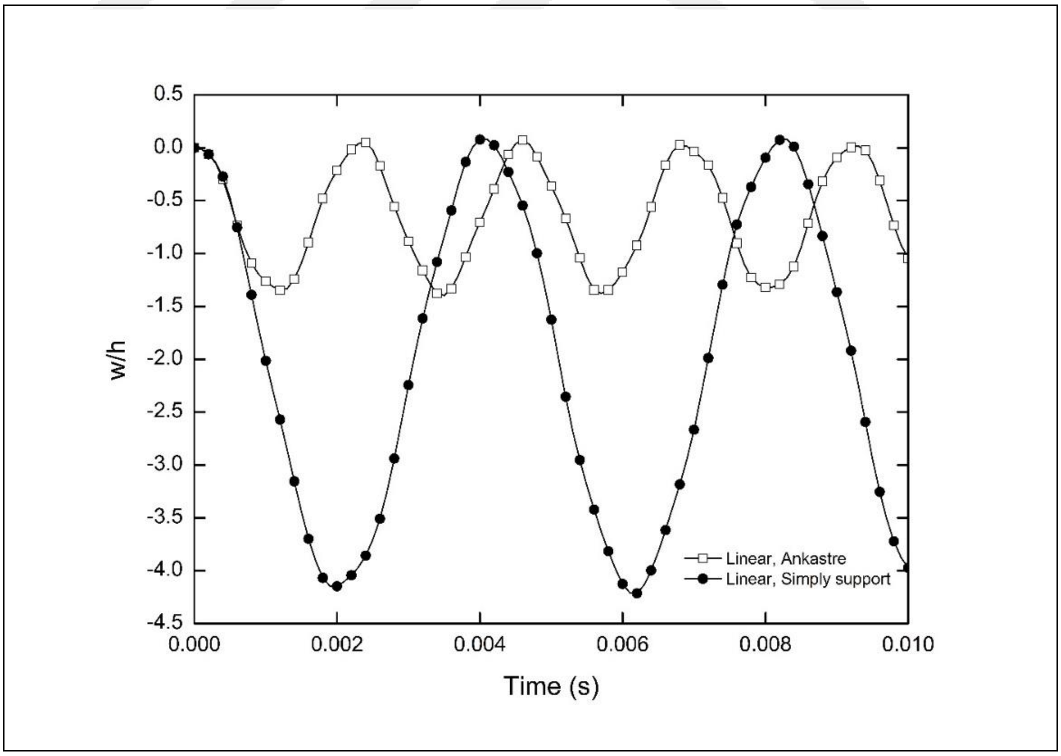

Figure 4.10: The comparison of simply support and clamped isotropic plate for linear analysis (a/h=20, R/a=∞).

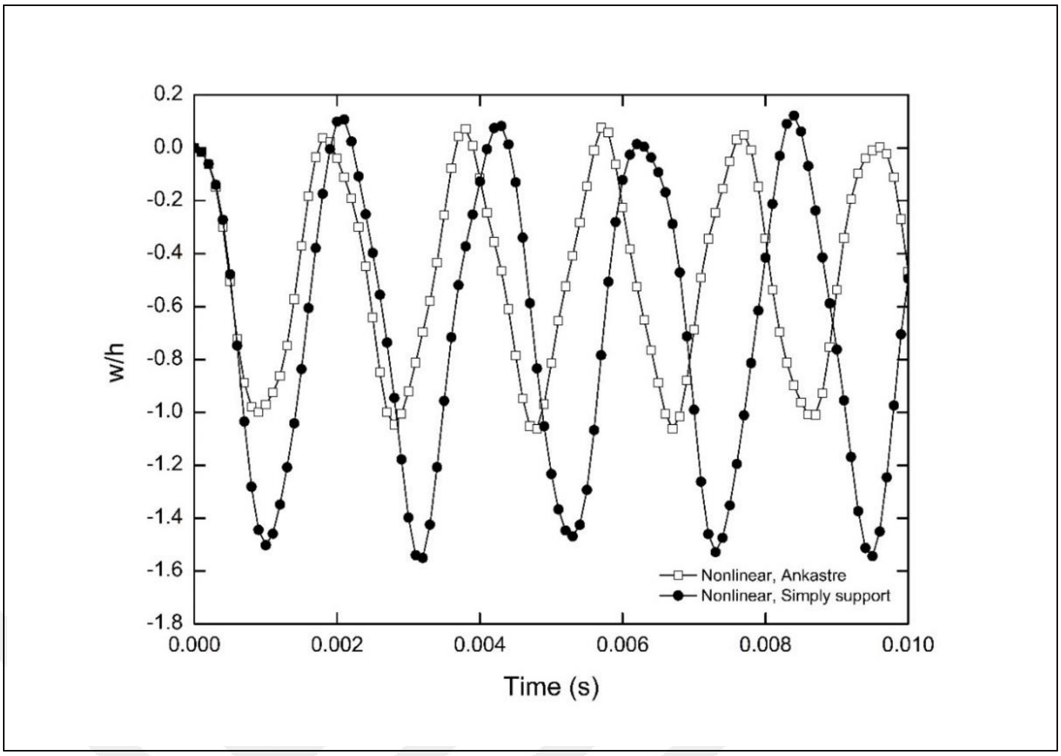

Figure 4.11: The comparison of simply support and clamped isotropic plate for nonlinear analysis (a/h=20, R/a=∞).

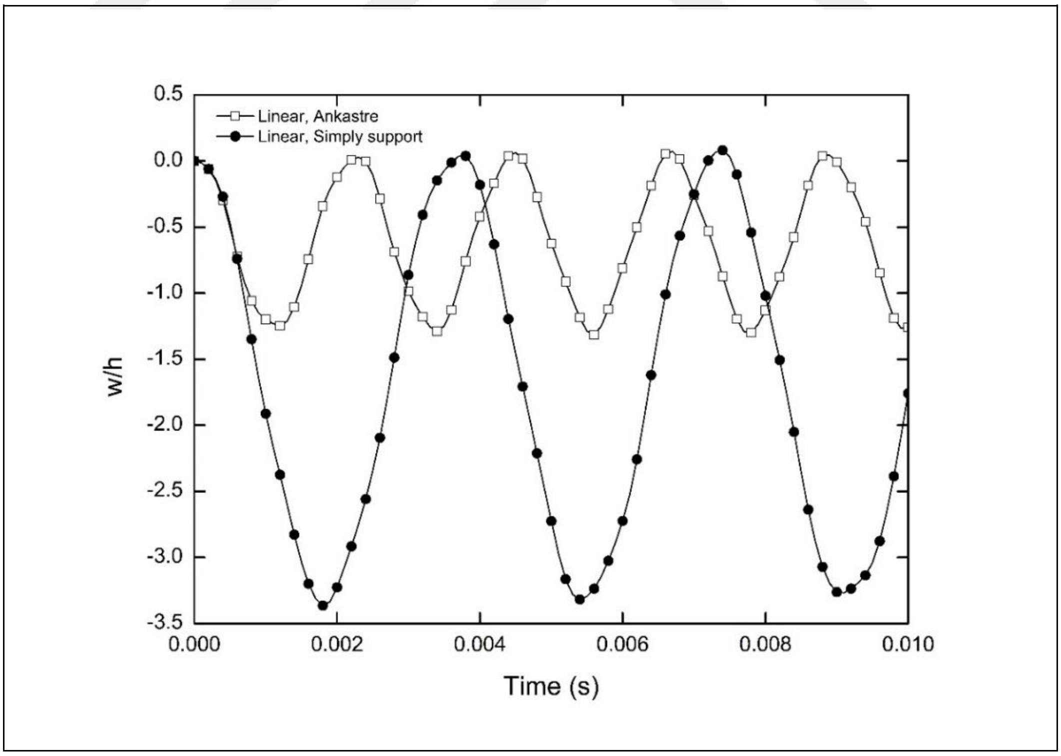

Figure 4.12: The comparison of simply support and clamped isotropic panel for linear analysis (a/h=20, R/a=10).

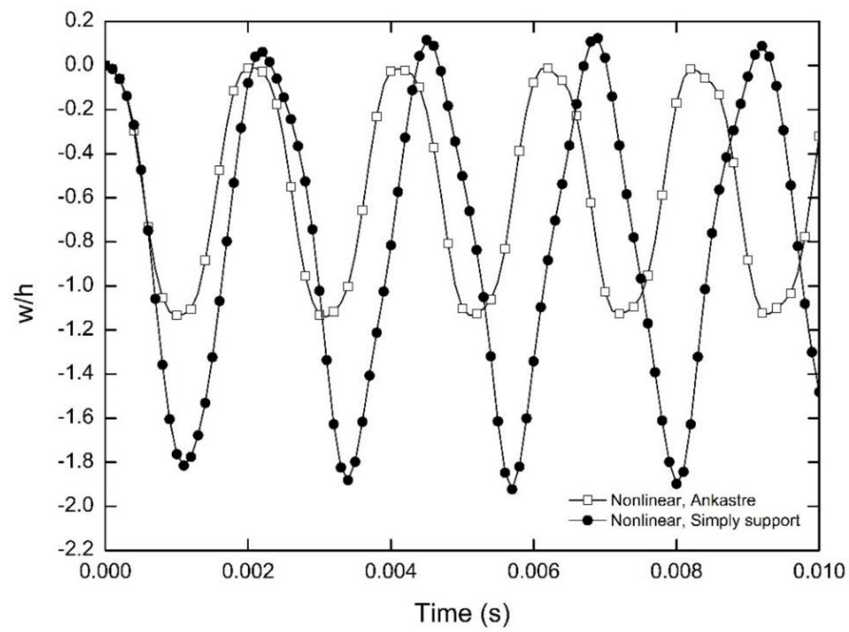

Figure 4.13: The comparison of simply support and clamped isotropic panel for nonlinear analysis (a/h=20, R/a=10).

4.2. Validation example for composite plate and panel

In this section, the comparison between the Chebyshev collocation method and finite element method for the linear/nonlinear transient response of composite plate and panel structures for different taper ratios and boundary conditions are carried out parametrically. Boron-Epoxy composite material is used for the all examples in this section. Stacking sequence of symmetric composite layers are [0°/90°/30°/90°/0°] and 5 layers have equal thicknesses. The material properties are: E_1=204 GPa, E_2=18.5 GPa, G_{12}=5.59 GPa, ρ=2100 kg/m^3, ν_{12}=0.23. The dimensions for the plate and panel structures are taken as the same with the previous section. Taper ratios β of the plate and panel structures in the samples are taken as 0, 0.7 and 1.2 to understand the effect of taper ratio. Thickness changes of the plate and panel edge through x direction are given in Figures 4.14-4.15, respectively.

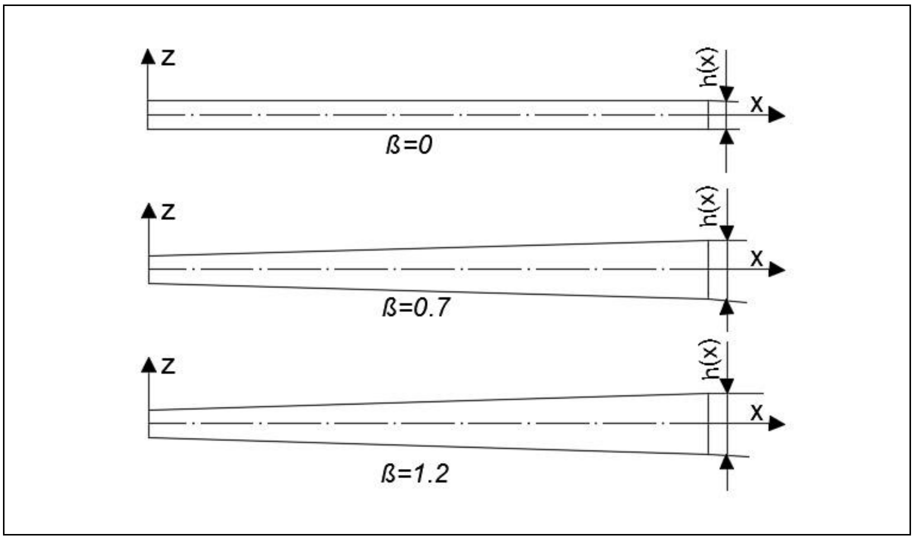

Figure 4.14: Thickness changes of the plate edge through the x direction for β =0,0.7,1.2 (R/a=10).

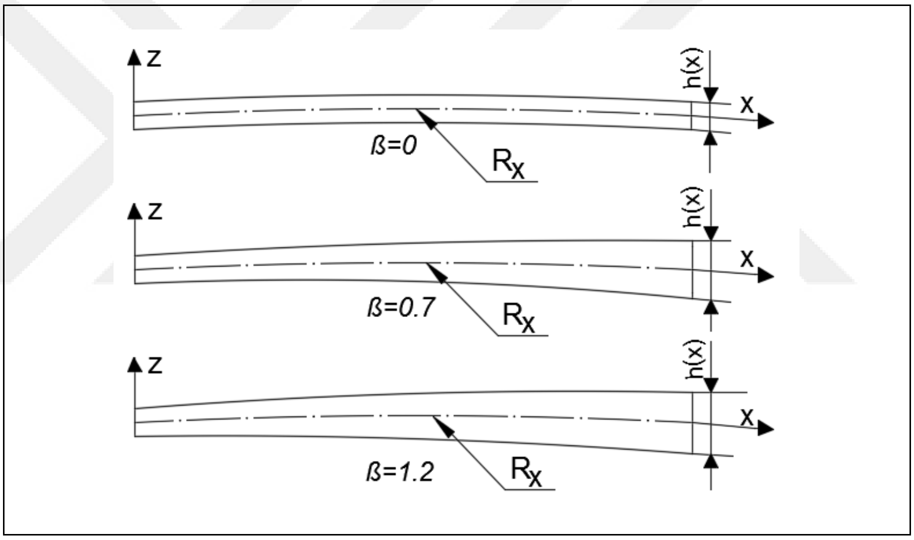

Figure 4.15: Thickness changes of the panel edge through the x direction for β =0,0.7,1.2 (R/a=10).

Some of the examples in this section were converged with 11x11 Chebyshev terms and specified in related figures with these examples. The rest of the examples converged with 9x9 Chebyshev terms. The linear/nonlinear transient responses at the middle of composite plate and panel structures with clamped and simply supported conditions can be seen in Figures 4.16-4.21.

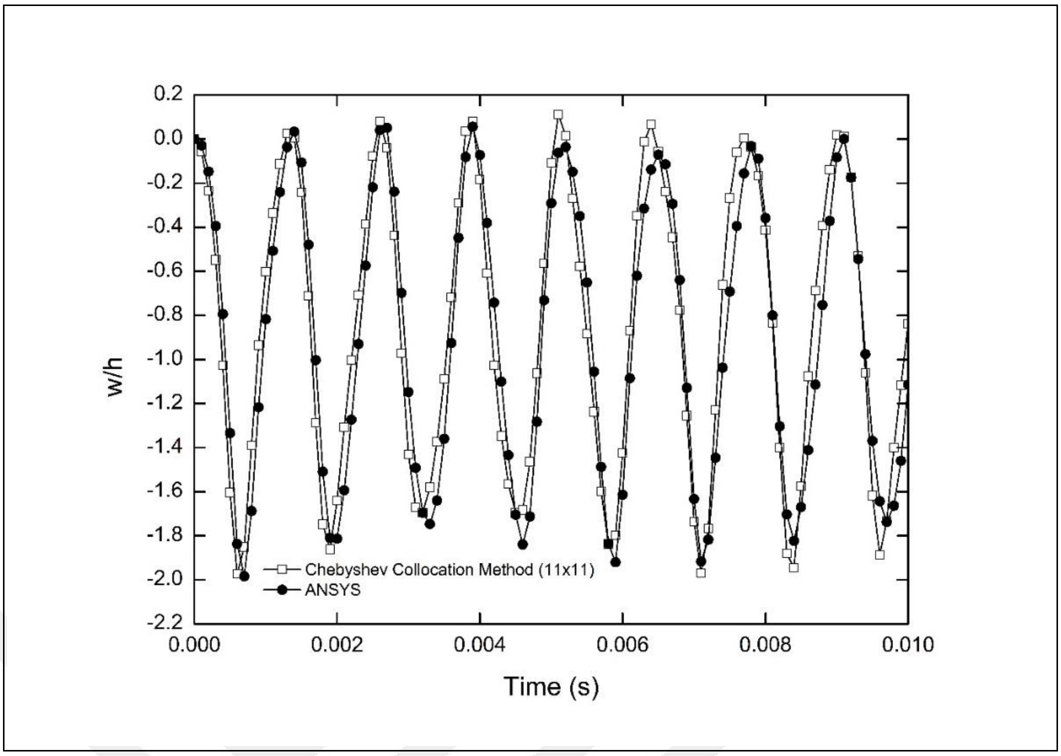

Figure 4.16: Non-dimensional displacement-time history at the center of clamped composite plate for nonlinear analysis (a/h=20, R/a=∞, $\beta = 0$).

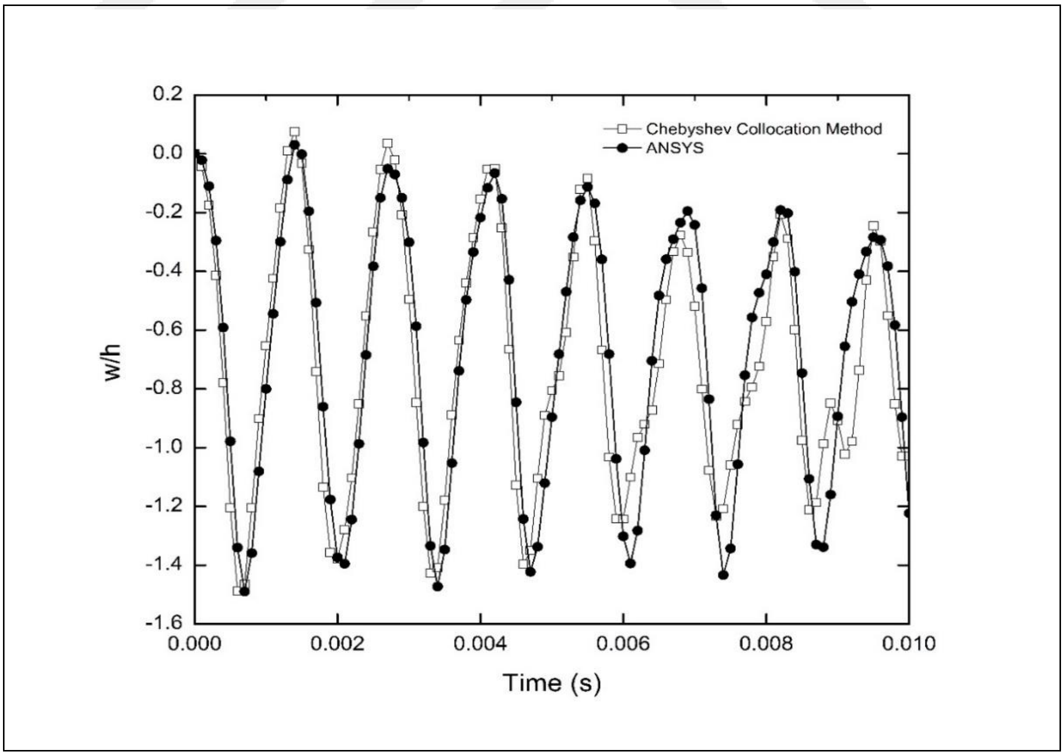

Figure 4.17: Non-dimensional displacement-time history at the center of clamped composite plate for nonlinear analysis (a/h=20, R/a=∞, $\beta = 0.7$).

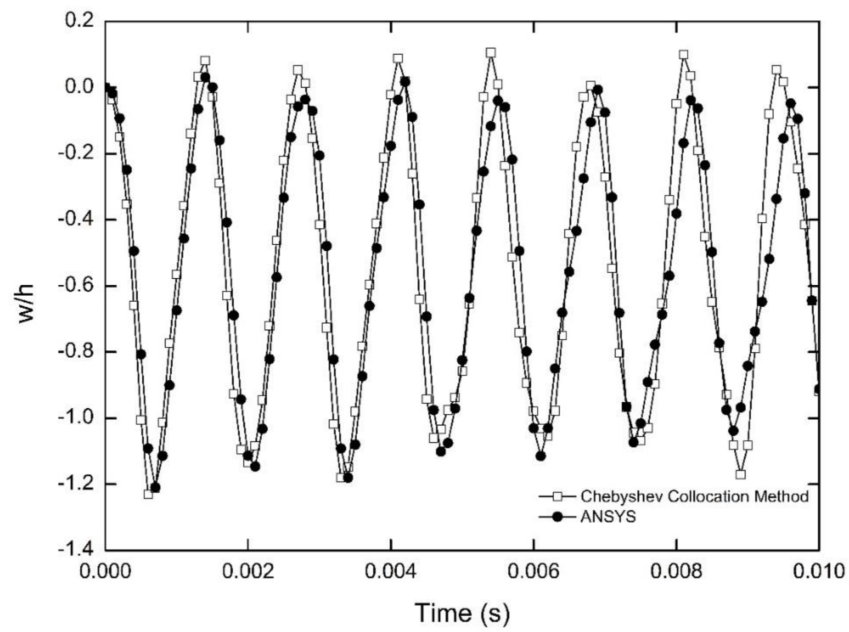

Figure 4.18: Non-dimensional displacement-time history at the center of clamped composite plate for nonlinear analysis (a/h=20, R/a=∞, $\beta = 1.2$).

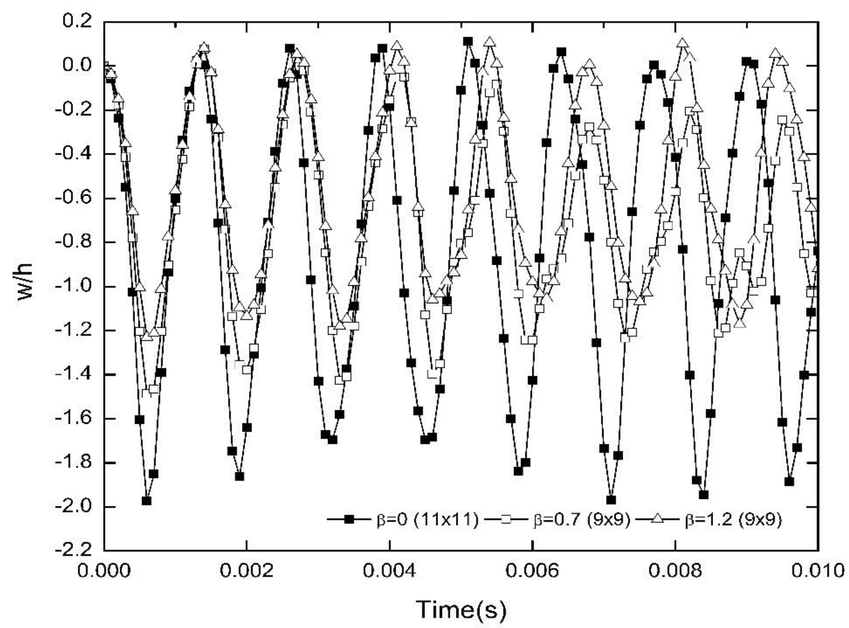

Figure 4.19: The comparison of taper ratios (β) of clamped composite plate for nonlinear analysis (a/h=20, R/a=∞).

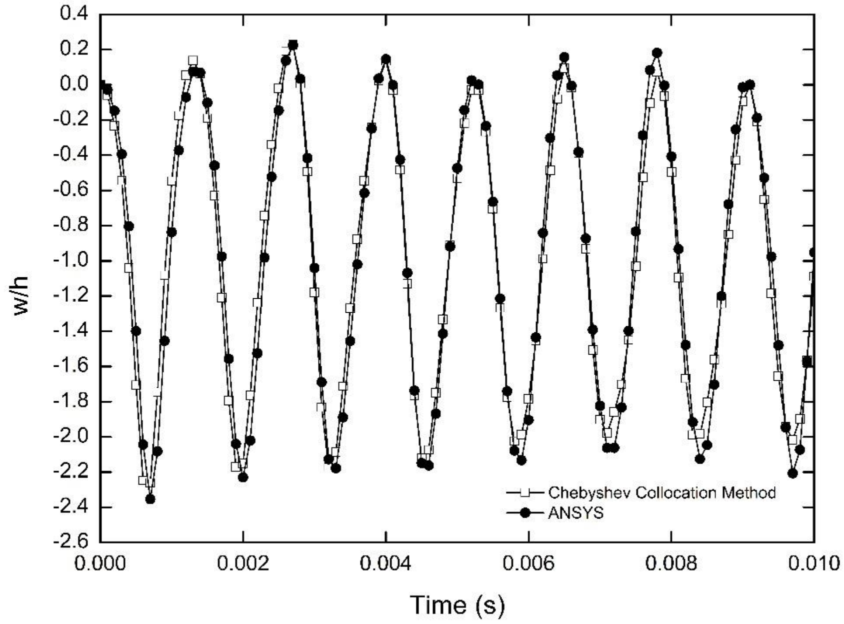

Figure 4.20: Non-dimensional displacement-time history at the center of simply supported composite plate for nonlinear analysis (a/h=20, R/a=∞, $\beta = 0$).

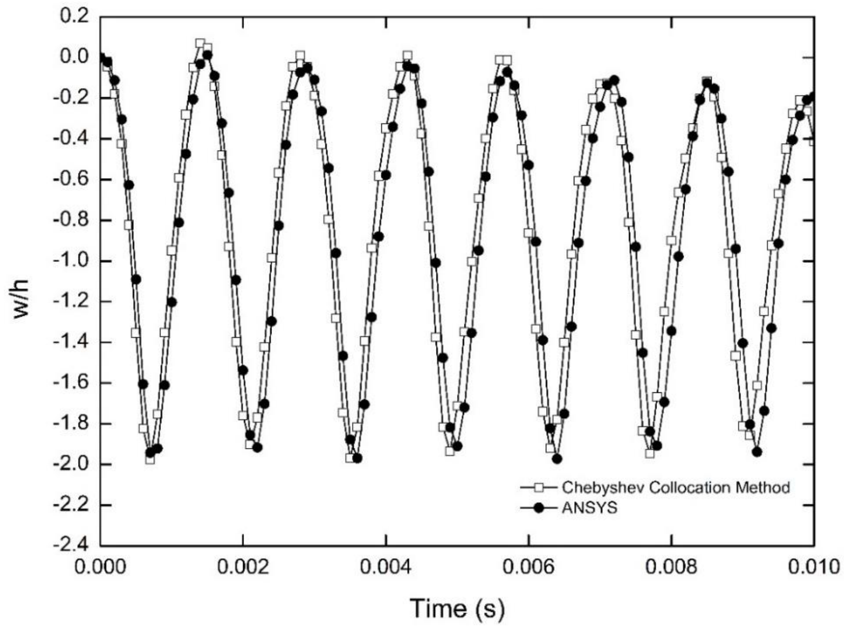

Figure 4.21: Non-dimensional displacement-time history at the center of simply supported composite plate for nonlinear analysis (a/h=20, R/a=∞, $\beta = 0.7$).

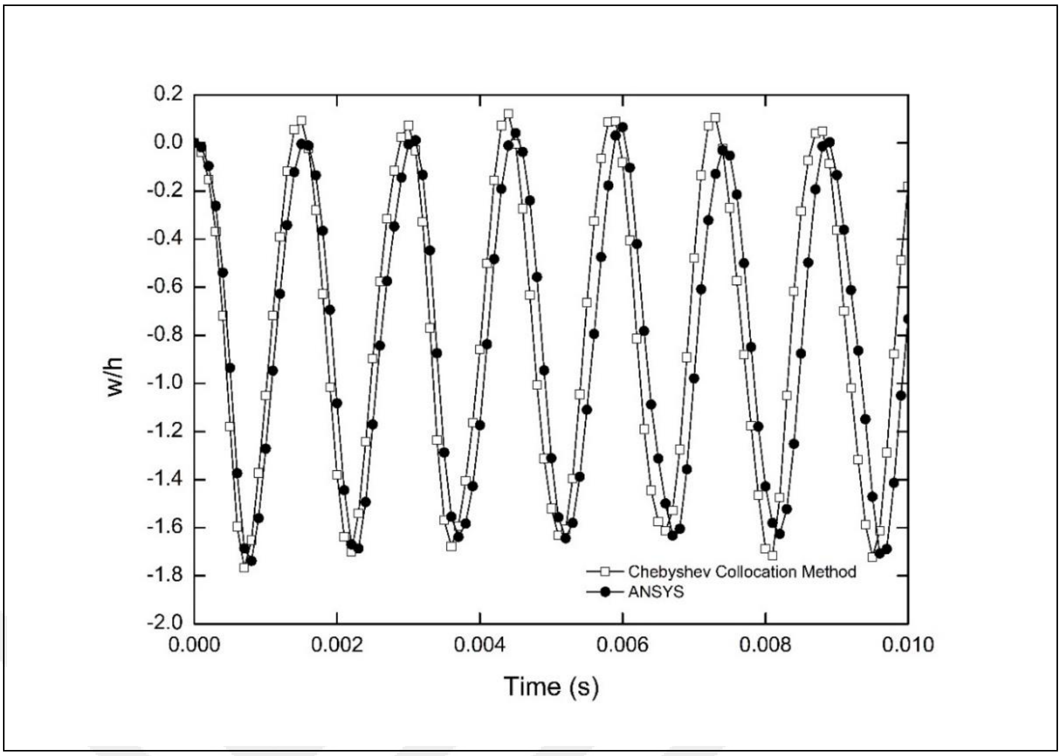

Figure 4.22: Non-dimensional displacement-time history at the center of simply supported composite plate for nonlinear analysis (a/h=20, R/a=∞, $\beta = 1.2$).

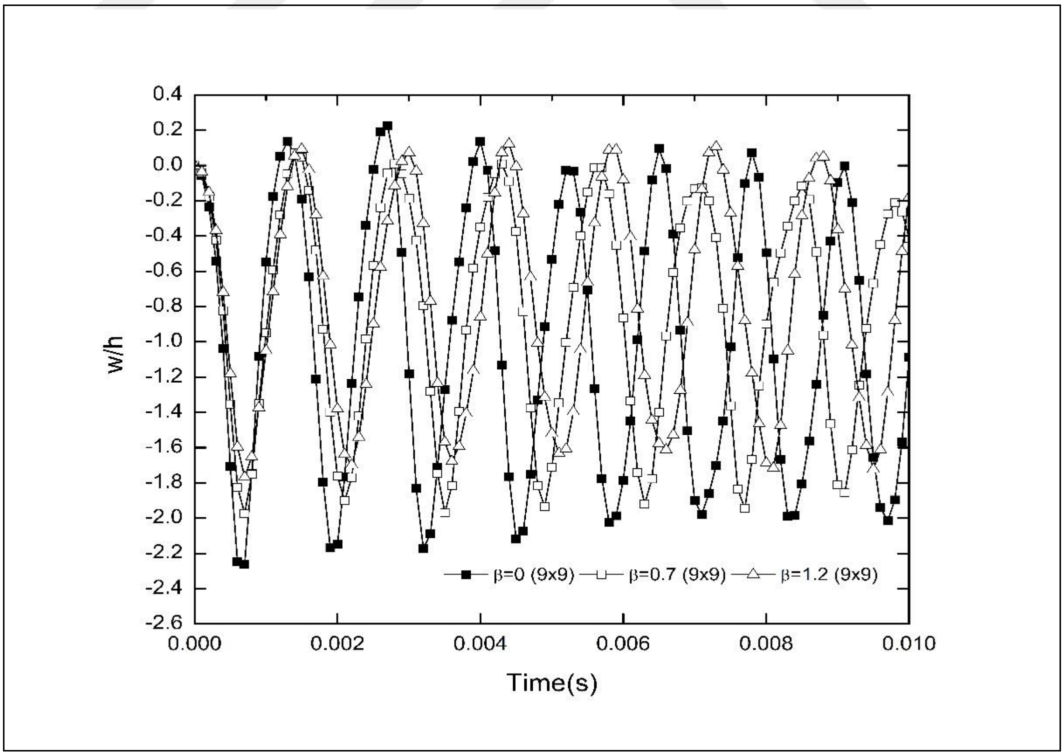

Figure 4.23: The comparison of taper ratios (β) of simply supported composite plate for nonlinear analysis (a/h=20, R/a=∞).

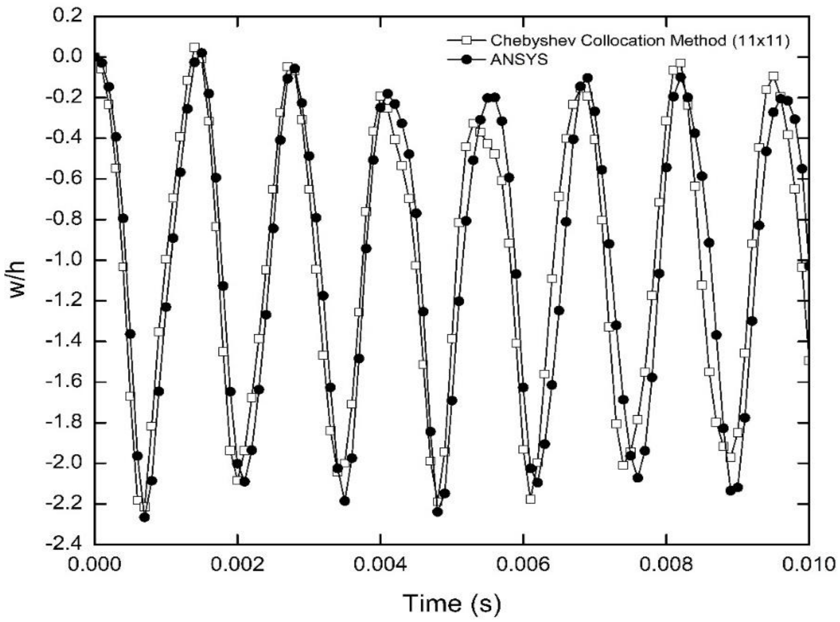

Figure 4.24: Non-dimensional displacement-time history at the center of clamped composite panel for nonlinear analysis (a/h=20, R/a=10, $\beta = 0$).

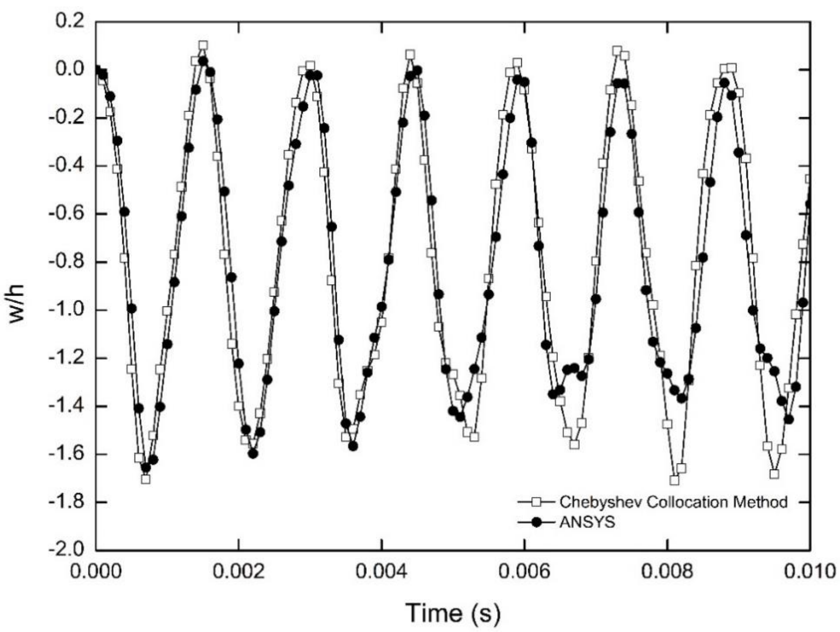

Figure 4.25: Non-dimensional displacement-time history at the center of clamped composite panel for nonlinear analysis (a/h=20, R/a=10, $\beta = 0.7$).

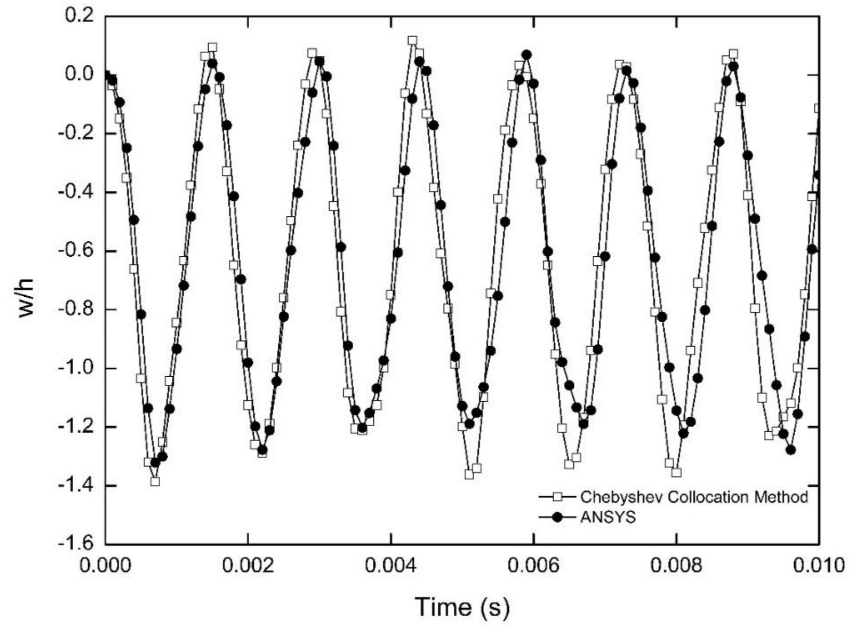

Figure 4.26: Non-dimensional displacement-time history at the center of clamped composite panel for nonlinear analysis (a/h=20, R/a=10, $\beta = 1.2$).

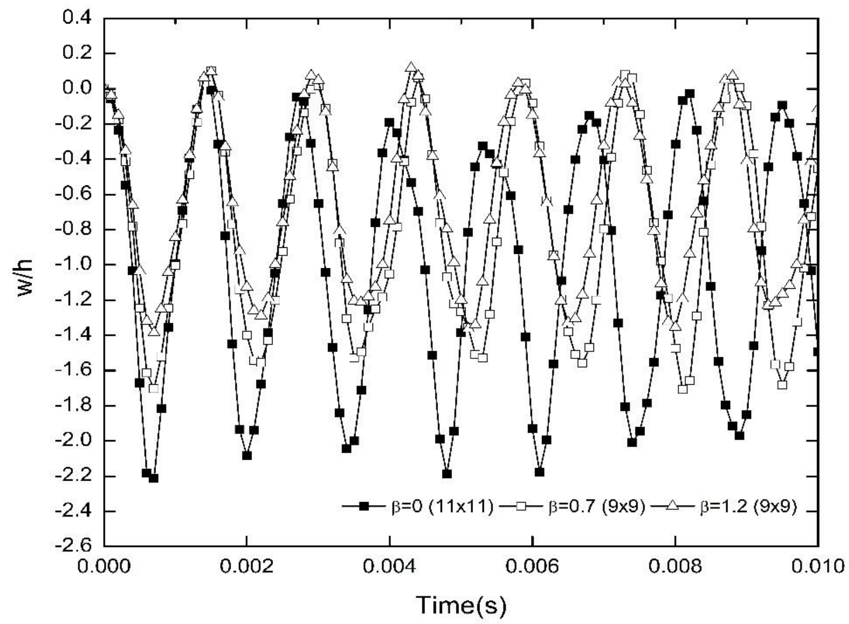

Figure 4.27: The comparison of taper ratios (β) of clamped composite panel for nonlinear analysis (a/h=20, R/a=10).

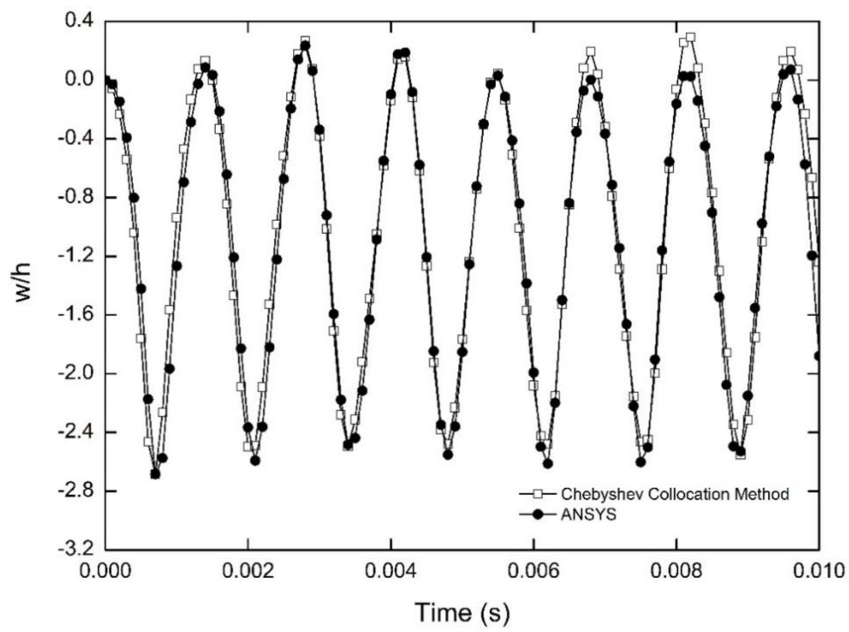

Figure 4.28: Non-dimensional displacement-time history at the center of simply supported composite panel for nonlinear analysis (a/h=20, R/a=10, $\beta = 0$).

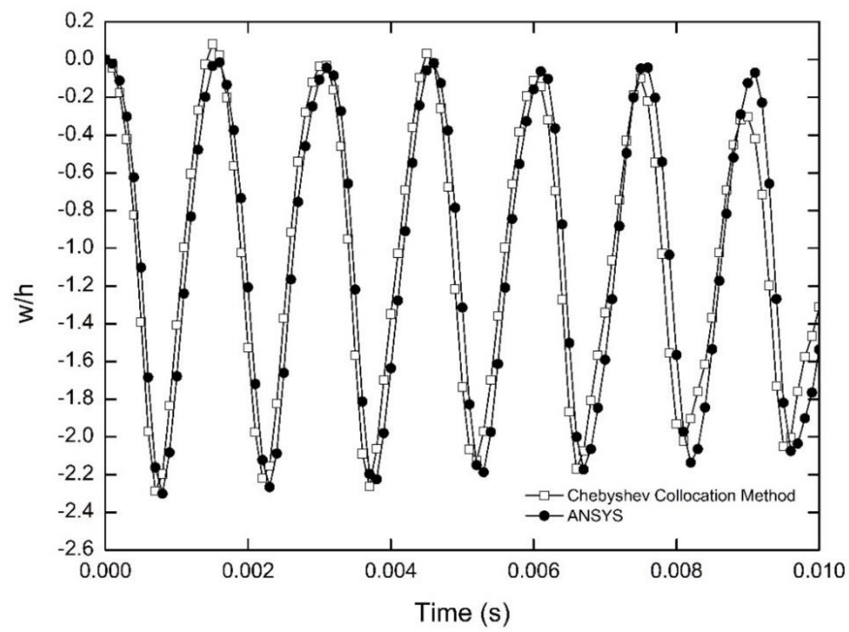

Figure 4.29: Non-dimensional displacement-time history at the center of simply supported composite panel for nonlinear analysis (a/h=20, R/a=10, $\beta = 0.7$).

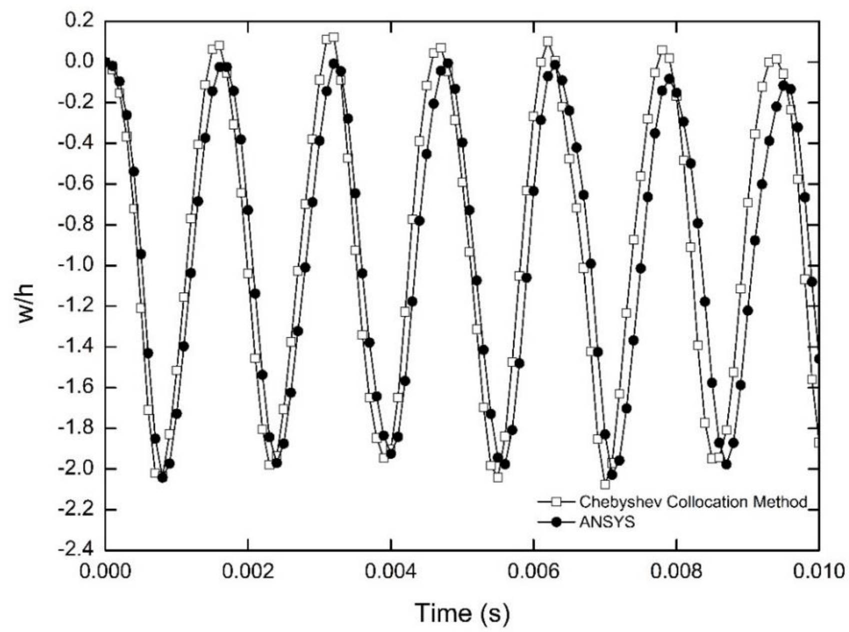

Figure 4.30: Non-dimensional displacement-time history at the center of simply supported composite panel for nonlinear analysis (a/h=20, R/a=10, $\beta = 1.2$).

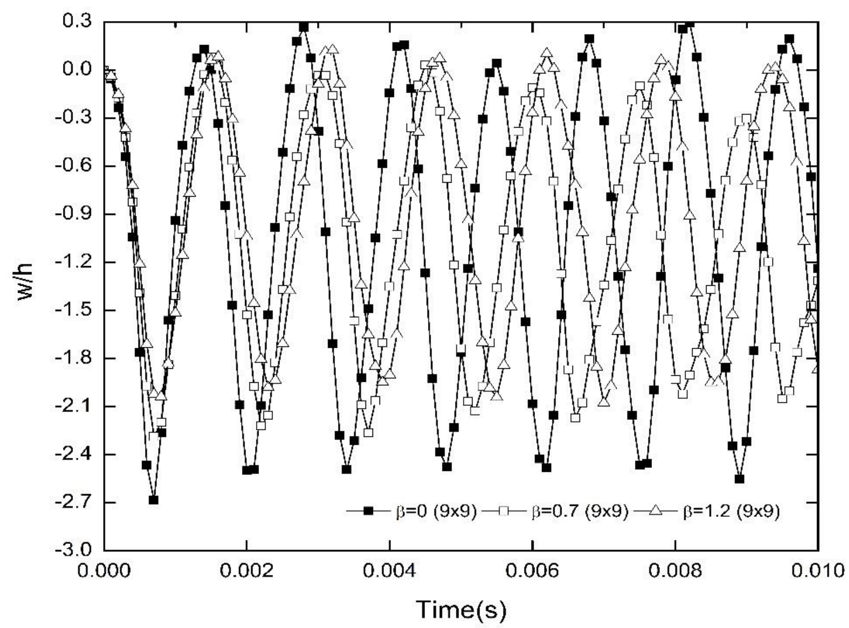

Figure 4.31: The comparison of taper ratios (β) of simply supported composite panel for nonlinear analysis (a/h=20, R/a=10).

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4.3 The effect of radius on dynamic behavior of composite panel

In this section, the effect of different radius values (R/a=5, 10, 20, 50) for the nonlinear transient responses of composite panels with clamped and simply supported boundary conditions and also different taper ratios is considered parametrically. The Boron-Epoxy composite material properties can be taken from the Sec.4.2. The angle orientations of symmetric composite layers and the dimensions for panel structures are the same with the previous section. Taper ratios, β of the panel structures in the samples are taken as 0, 0.7 and 1.2.

Some of the examples in this section were converged with 11x11 Chebyshev terms and specified in related figures with these examples. The rest of the examples converged with 9x9 Chebyshev terms. The comparison of different radius values for the nonlinear transient responses of composite panels with different taper ratios is shown in Figures 4.32-34 for clamped boundary condition and in Figures 4.35-4.37 for simply supported boundary conditions. Peak displacement values in the middle of the clamped and simply supported panel with respect to taper ratio and panel radius is shown in Figures 4.38-4.39, respectively.

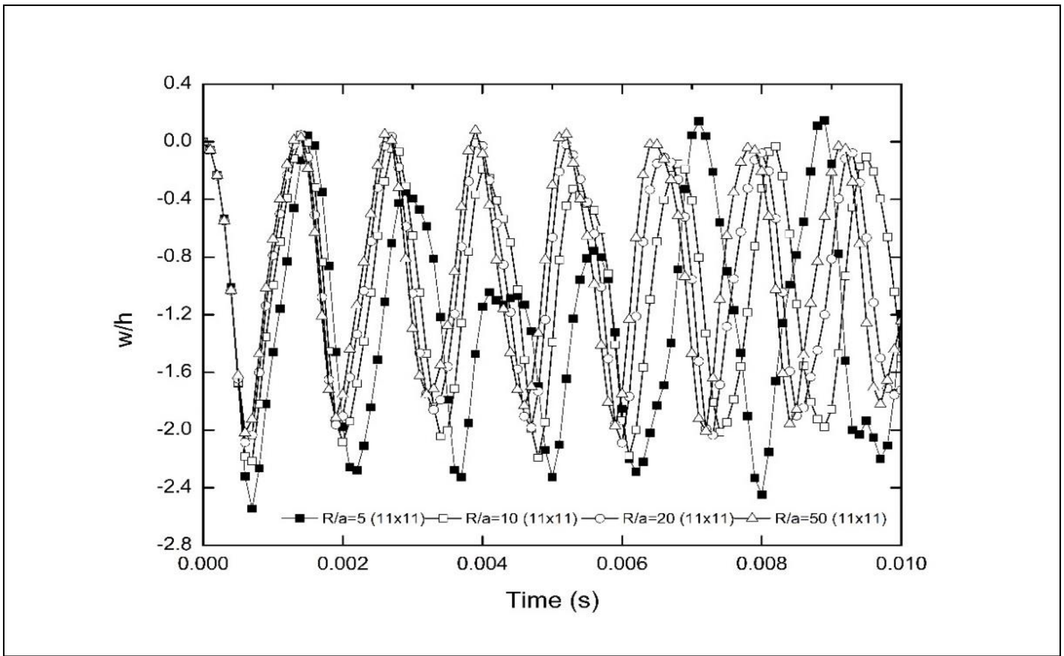

Figure 4.32: The comparison of different radius values of clamped composite panel for nonlinear analysis ($\mathrm{a/h} = 20, \beta = 0$).

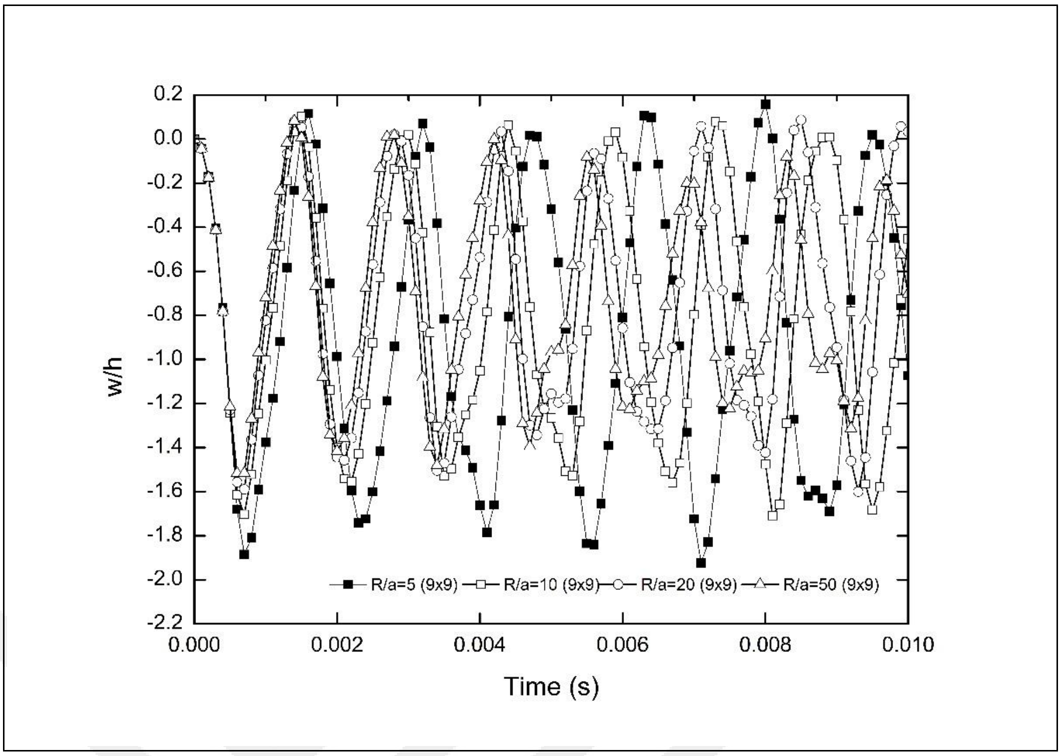

Figure 4.33: The comparison of different radius values of clamped composite panel for nonlinear analysis $(a/h = 20, \beta = 0.7)$.

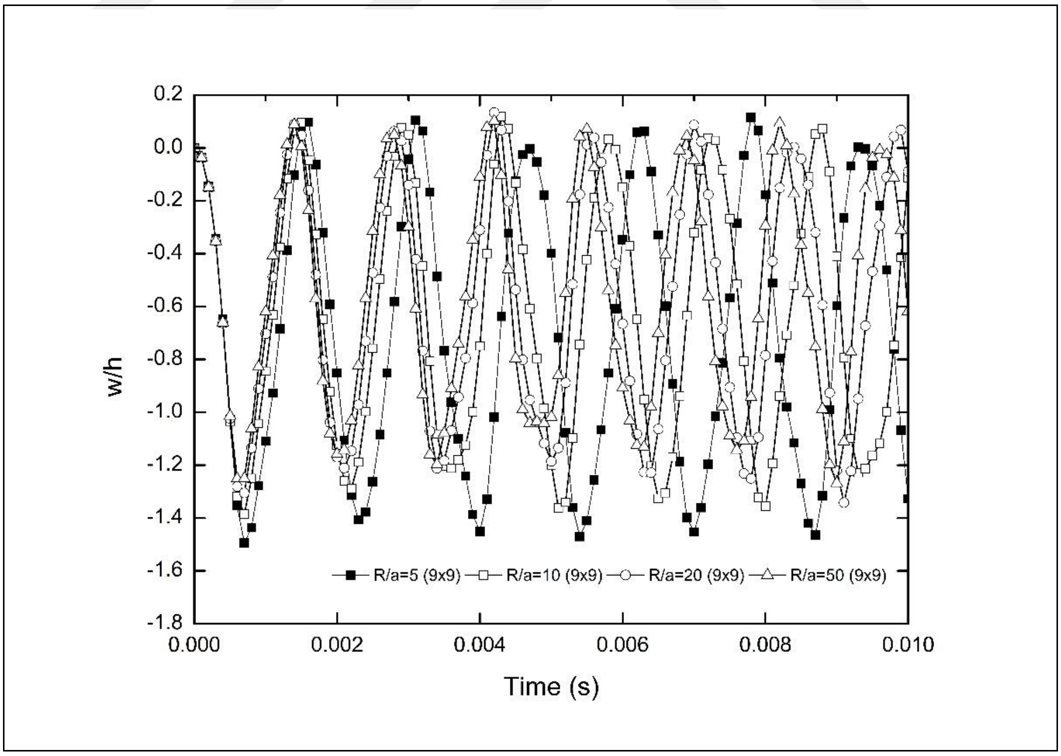

Figure 4.34: The comparison of different radius values of clamped composite panel for nonlinear analysis $(a/h = 20, \beta = 1.2)$.

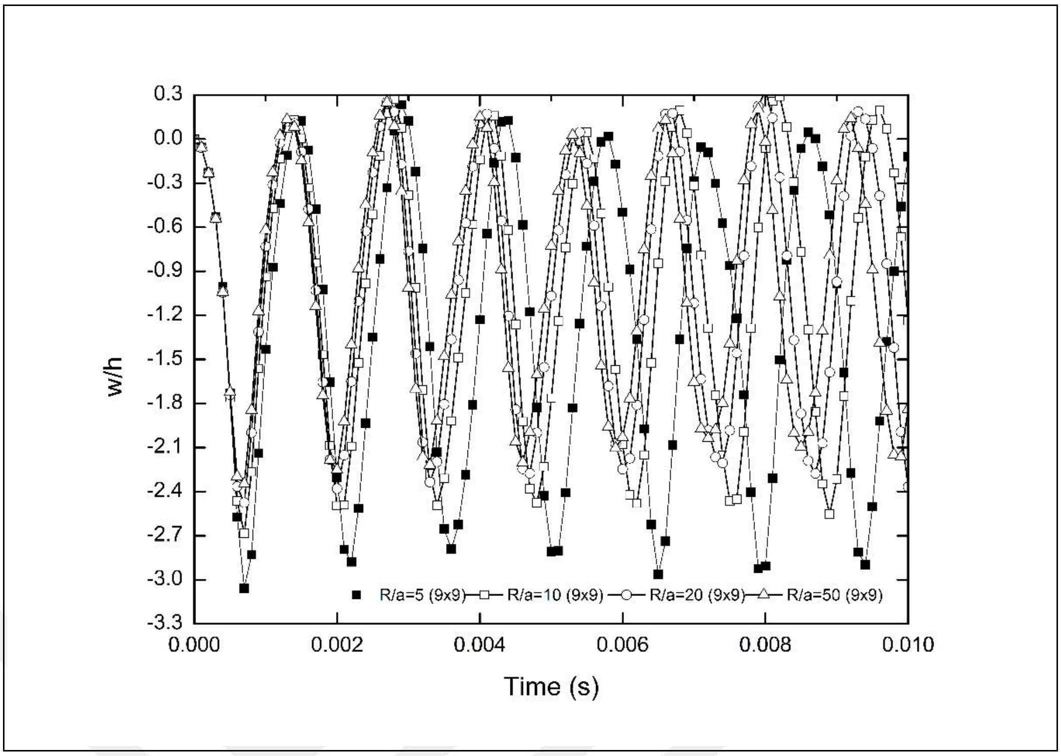

Figure 4.35: The comparison of different radius values of simply supported composite panel for nonlinear analysis ($a/h = 20, \beta = 0$).

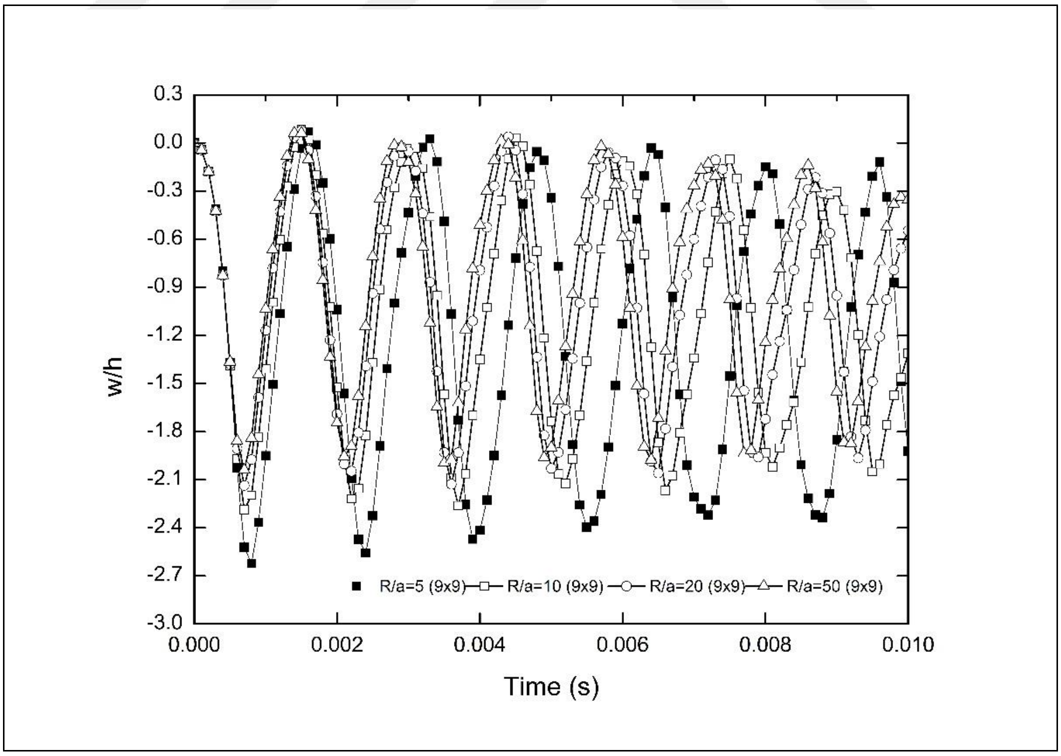

Figure 4.36: The comparison of different radius values of simply supported composite panel for nonlinear analysis ($a/h = 20, \beta = 0.7$).

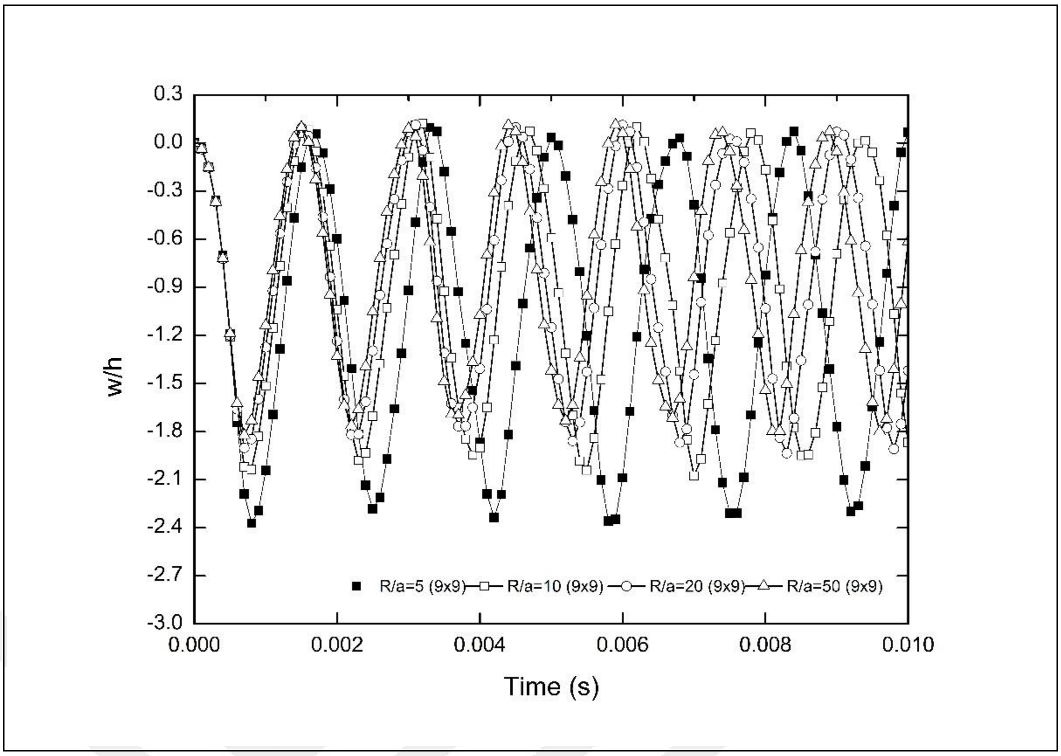

Figure 4.37: The comparison of different radius values of simply supported composite panel for nonlinear analysis $(a/h = 20, \beta = 1.2)$.

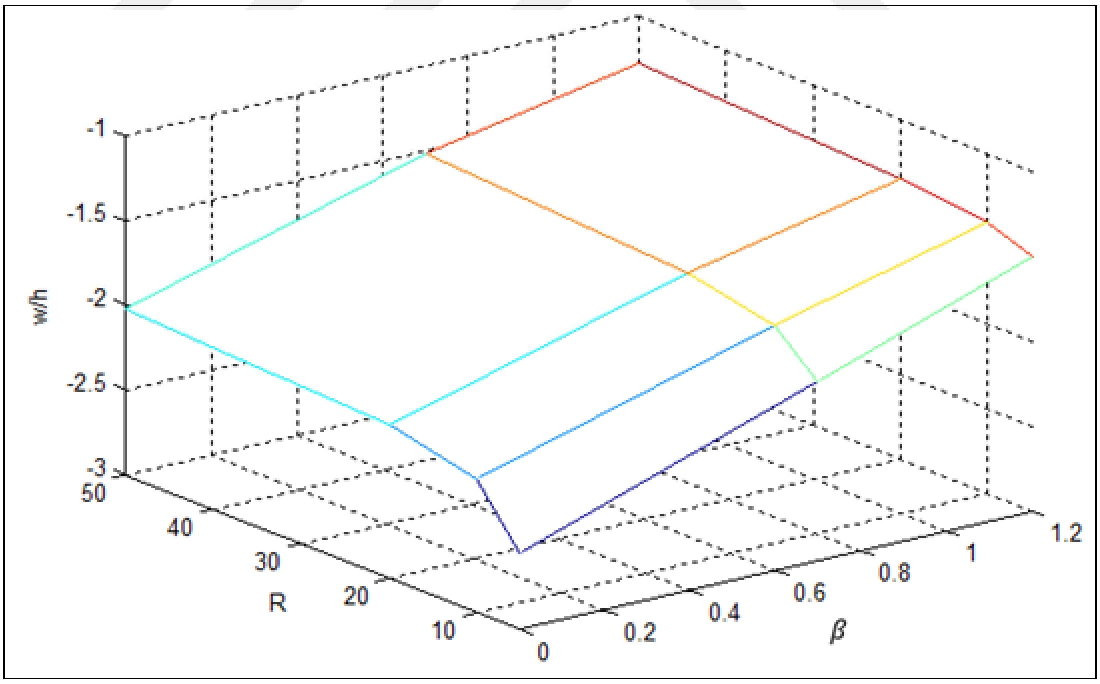

Figure 4.38: Peak non-dimensional displacement values in the middle of the clamped panel with respect to taper ratio and panel radius $(a/h = 20)$.

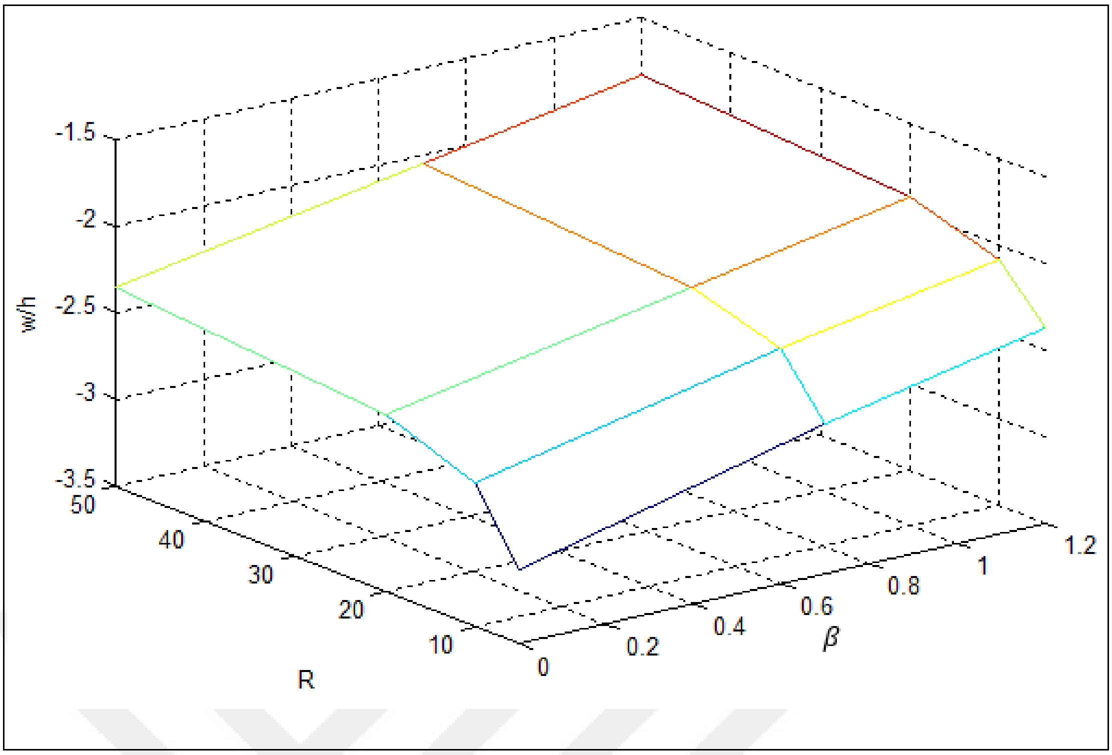

Figure 4.39: Peak non-dimensional displacement values in the middle of the simply supported panel with respect to taper ratio and panel radius ($a/h = 20$).

5. CONCLUSIONS AND FUTURE WORK

In this thesis, several plate and panel examples were solved with Chebyshev Collocation Method. The effects of taper ratios, material properties, panel radius and boundary conditions were studied. The results of Chebyshev collocation method were compared with those of finite element method and very similar results were observed.

The brief conclusion obtained from the results in the previous chapter is summarized as follows:

- The deflection of tapered plate and panel structures decreases for increasing taper ratios.
- The displacement decreases for increasing curvature ratios (R/a) from 5-50. The effects of curvature ratios are comparatively less than after the R/a ratio exceeds 20.
- The displacement in dynamic analysis is much higher for simply supported panels than those for clamped panels.
- Generally 9x9 terms can yield sufficiently accurate results with Chebyshev Collocation Method.
- The computational cost of Chebyshev Collocation Method is lower than FEM in transient analyzes.

As a conclusion, Chebyshev Collocation Method can be applied to the efficient solution of other engineering problems and can serve as a bench work for forthcoming surveys.

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BIOGRAPHY

İlke Algül was born in Tokat in 1989. She graduated from Dokuz Eylül University with a bachelor’s degree in the field of Mechanical Engineering in 2013. She worked in a crane factory as a project engineer for a time in İzmir. In 2014, she started her master education in the field of Mechanical Engineering in Graduate School of Natural and Applied Sciences in Gebze Technical University. She has been working in Gebze Technical University, in Department of Mechanical Engineering as a Research Assistant since 2014.

APPENDICES

Appendix A: Linearization of the Terms in Governing Differential Equations

The each terms in governing differential equations in incremental form can be written as shown below.

$$\Delta U = \begin{bmatrix} \Delta u_{11} \\ \Delta v_{11} \\ \Delta w_{11} \\ \Delta \theta_{x_{11}} \\ \Delta \theta_{y_{11}} \\ \vdots \\ \Delta u_{1N_y} \\ \Delta v_{1N_y} \\ \Delta w_{1N_y} \\ \Delta \theta_{x_{1N_y}} \\ \Delta \theta_{y_{1N_y}} \\ \vdots \\ \Delta u_{N_xN_y} \\ \Delta v_{N_xN_y} \\ \Delta w_{N_xN_y} \\ \Delta \theta_{x_{N_xN_y}} \\ \Delta \theta_{y_{N_xN_y}} \end{bmatrix} \tag{A1.1}$$

$$\begin{bmatrix} \Delta N_x \\ \Delta N_y \\ \Delta N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \Delta {\varepsilon_x}^0 \\ \Delta {\varepsilon_y}^0 \\ \Delta {\gamma_{xy}}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \Delta {\varepsilon_x}^1 \\ \Delta {\varepsilon_y}^1 \\ \Delta {\gamma_{xy}}^1 \end{bmatrix} \tag{A1.2}$$

$$\begin{bmatrix} \Delta M_x \\ \Delta M_y \\ \Delta M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \Delta {\varepsilon_x}^0 \\ \Delta {\varepsilon_y}^0 \\ \Delta {\gamma_{xy}}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \Delta {\varepsilon_x}^1 \\ \Delta {\varepsilon_y}^1 \\ \Delta {\gamma_{xy}}^1 \end{bmatrix} \tag{A1.3}$$

$$\begin{bmatrix} \Delta Q_{yz} \\ \Delta Q_{xz} \end{bmatrix} = \begin{bmatrix} k_s A_{44} & k_s A_{45} \\ k_s A_{45} & k_s A_{55} \end{bmatrix} \begin{bmatrix} \Delta {\gamma_{yz}}^0 \\ \Delta {\gamma_{xz}}^0 \end{bmatrix} \tag{A1.4}$$

$$\begin{bmatrix} \frac{\partial \Delta N_x}{\partial x} \\ \frac{\partial \Delta N_y}{\partial x} \\ \frac{\partial \Delta N_{xy}}{\partial x} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^0}{\partial x} \\ \frac{\partial \Delta {\varepsilon_y}^0}{\partial x} \\ \frac{\partial \Delta {\gamma_{xy}}^0}{\partial x} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^1}{\partial x} \\ \frac{\partial \Delta {\varepsilon_y}^1}{\partial x} \\ \frac{\partial \Delta {\gamma_{xy}}^1}{\partial x} \end{bmatrix} \tag{A1.5}$$

$$\begin{gathered} \begin{bmatrix} \frac{\partial \Delta M_x}{\partial x} \\ \frac{\partial \Delta M_y}{\partial x} \\ \frac{\partial \Delta M_{xy}}{\partial x} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^0}{\partial x} \\ \frac{\partial \Delta {\varepsilon_y}^0}{\partial x} \\ \frac{\partial \Delta {\gamma_{xy}}^0}{\partial x} \end{bmatrix} \\ + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^1}{\partial x} \\ \frac{\partial \Delta {\varepsilon_y}^1}{\partial x} \\ \frac{\partial \Delta {\gamma_{xy}}^1}{\partial x} \end{bmatrix} \end{gathered} \tag{A1.6}$$

$$\begin{bmatrix} \frac{\partial \Delta Q_{yz}}{\partial x} \\ \frac{\partial \Delta Q_{xz}}{\partial x} \end{bmatrix} = \begin{bmatrix} k_s A_{44} & k_s A_{45} \\ k_s A_{45} & k_s A_{55} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\gamma_{yz}}^0}{\partial x} \\ \frac{\partial \Delta {\gamma_{xz}}^0}{\partial x} \end{bmatrix} \tag{A1.7}$$

$$\begin{bmatrix} \frac{\partial \Delta N_x}{\partial y} \\ \frac{\partial \Delta N_y}{\partial y} \\ \frac{\partial \Delta N_{xy}}{\partial y} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^0}{\partial y} \\ \frac{\partial \Delta {\varepsilon_y}^0}{\partial y} \\ \frac{\partial \Delta {\gamma_{xy}}^0}{\partial y} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^1}{\partial y} \\ \frac{\partial \Delta {\varepsilon_y}^1}{\partial y} \\ \frac{\partial \Delta {\gamma_{xy}}^1}{\partial y} \end{bmatrix} \tag{A1.8}$$

$$\begin{bmatrix} \frac{\partial \Delta M_x}{\partial y} \\ \frac{\partial \Delta M_y}{\partial y} \\ \frac{\partial \Delta M_{xy}}{\partial y} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^0}{\partial y} \\ \frac{\partial \Delta {\varepsilon_y}^0}{\partial y} \\ \frac{\partial \Delta {\gamma_{xy}}^0}{\partial y} \end{bmatrix}$$

$$+ \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\varepsilon_x}^1}{\partial y} \\ \frac{\partial \Delta {\varepsilon_y}^1}{\partial y} \\ \frac{\partial \Delta {\gamma_{xy}}^1}{\partial y} \end{bmatrix} \tag{A1.9}$$

$$\begin{bmatrix} \frac{\partial \Delta Q_{yz}}{\partial y} \\ \frac{\partial \Delta Q_{xz}}{\partial y} \end{bmatrix} = \begin{bmatrix} k_s A_{44} & k_s A_{45} \\ k_s A_{45} & k_s A_{55} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta {\gamma_{yz}}^0}{\partial y} \\ \frac{\partial \Delta {\gamma_{xz}}^0}{\partial y} \end{bmatrix} \tag{A1.10}$$

$$\Delta {\varepsilon_x}^0 = \frac{\partial \Delta u_0}{\partial x} + \frac{\Delta w_0}{R_x} + \frac{\partial w_0}{\partial x}\frac{\partial \Delta w_0}{\partial x} = \begin{bmatrix} 1 & \frac{1}{R_X} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} \frac{\partial \Delta u_0}{\partial x} \\ \Delta w_0 \\ \frac{\partial \Delta w_0}{\partial x} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{R_X} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} Nu,x \\ Nw \\ Nwx \end{bmatrix} . \Delta U \tag{A1.11}$$

$$\Delta {\varepsilon_x}^1 = \frac{\partial \Delta \theta_x}{\partial x} = N\theta_x, x. \Delta U \tag{A1.12}$$

$$\Delta {\varepsilon_y}^0 = \frac{\partial \Delta v_0}{\partial y} + \frac{\Delta w_0}{R_y} + \frac{\partial w_0}{\partial y}\frac{\partial \Delta w_0}{\partial y} = \begin{bmatrix} 1 & \frac{1}{R_y} & \frac{\partial w_0}{\partial y} \end{bmatrix} . \begin{bmatrix} \frac{\partial \Delta v_0}{\partial y} \\ \Delta w_0 \\ \frac{\partial \Delta w_0}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{R_y} & \frac{\partial w_0}{\partial y} \end{bmatrix} . \begin{bmatrix} Nvy \\ Nw \\ Nwy \end{bmatrix} . \Delta U \tag{A1.13}$$

$$\Delta\varepsilon_y{}^1 = \frac{\partial\Delta\theta_y}{\partial y} = N\theta_y, y.\Delta U \qquad \text{(A1.14)}$$

$$\Delta\gamma_{xy}{}^0 = \frac{\partial\Delta u_0}{\partial y} + \frac{\partial\Delta v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial\Delta w_0}{\partial y} + \frac{\partial\Delta w_0}{\partial x}\frac{\partial w_0}{\partial y}$$

$$= \begin{bmatrix} 1 & 1 & \frac{\partial w_0}{\partial x} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} \frac{\partial\Delta u_0}{\partial y} \\ \frac{\partial\Delta v_0}{\partial x} \\ \frac{\partial\Delta w_0}{\partial y} \\ \frac{\partial\Delta w_0}{\partial x} \end{bmatrix} \qquad \text{(A1.15)}$$

$$= \begin{bmatrix} 1 & 1 & \frac{\partial w_0}{\partial x} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} Nu, y \\ Nv, x \\ Nwy \\ Nwx \end{bmatrix} .\Delta U$$

$$\Delta\gamma_{xy}{}^1 = \frac{\partial\Delta\theta_x}{\partial y} + \frac{\partial\Delta\theta_y}{\partial x} + c_0\left(\frac{\partial\Delta v_0}{\partial x} - \frac{\partial\Delta u_0}{\partial y}\right)$$

$$= [1 \quad 1 \quad c_0 \quad -c_0] . \begin{bmatrix} \frac{\partial\Delta\theta_x}{\partial y} \\ \frac{\partial\Delta\theta_y}{\partial x} \\ \frac{\partial\Delta v_0}{\partial x} \\ \frac{\partial\Delta u_0}{\partial y} \end{bmatrix} \qquad \text{(A1.16)}$$

$$= [1 \quad 1 \quad c_0 \quad -c_0] . \begin{bmatrix} N\theta_x, y \\ N\theta_y, x \\ Nv, x \\ Nu, y \end{bmatrix} .\Delta U$$

$$\Delta\gamma_{yz}{}^0 = \Delta\theta_y + \frac{\partial\Delta w_0}{\partial y} - \frac{\Delta v_0}{R_y} = \begin{bmatrix} 1 & 1 & -\frac{1}{R_y} \end{bmatrix} . \begin{bmatrix} \Delta\theta_y \\ \frac{\partial\Delta w_0}{\partial y} \\ \Delta v_0 \end{bmatrix} = \qquad \text{(A1.17)}$$

$$\begin{bmatrix} 1 & 1 & -\frac{1}{R_y} \end{bmatrix} . \begin{bmatrix} N\theta_y \\ Nw, y \\ Nv \end{bmatrix} .\Delta U$$

$$\Delta\gamma_{xz}{}^{0} = \Delta\theta_x + \frac{\partial\Delta w_0}{\partial x} - \frac{\Delta u_0}{R_x} = \begin{bmatrix} 1 & 1 & -\frac{1}{R_x} \end{bmatrix} . \begin{bmatrix} \Delta\theta_x \\ \frac{\partial\Delta w_0}{\partial x} \\ \Delta u_0 \end{bmatrix} \tag{A1.18}$$

$$= \begin{bmatrix} 1 & 1 & -\frac{1}{R_x} \end{bmatrix} . \begin{bmatrix} N\theta_x \\ Nw,x \\ Nu \end{bmatrix} . \Delta U$$

$$\frac{\partial\Delta\varepsilon_x{}^{0}}{\partial x} = \frac{\partial^2\Delta u_0}{\partial x^2} + \frac{\partial\Delta w_0}{\partial x} . \frac{1}{R_x} + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial\Delta w_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial^2\Delta w_0}{\partial x^2}$$

$$= \begin{bmatrix} 1 & (\frac{1}{R_x} + \frac{\partial^2 w_0}{\partial x^2}) & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} \frac{\partial^2\Delta u_0}{\partial x^2} \\ \frac{\partial\Delta w_0}{\partial x} \\ \frac{\partial^2\Delta w_0}{\partial x^2} \end{bmatrix} \tag{A1.19}$$

$$= \begin{bmatrix} 1 & (\frac{1}{R_X} + \frac{\partial^2 w_0}{\partial x^2}) & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} Nu,xx \\ Nw,x \\ Nw,xx \end{bmatrix} . \Delta U$$

$$\frac{\partial\Delta\varepsilon_x{}^{0}}{\partial y} = \frac{\partial^2\Delta u_0}{\partial x\partial y} + \frac{\partial\Delta w_0}{\partial y} . \frac{1}{R_x} + \frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial\Delta w_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial^2\Delta w_0}{\partial x\partial y}$$

$$= \begin{bmatrix} 1 & \frac{1}{R_x} & \frac{\partial^2 w_0}{\partial x\partial y} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} \frac{\partial^2\Delta u_0}{\partial x\partial y} \\ \frac{\partial\Delta w_0}{\partial y} \\ \frac{\partial\Delta w_0}{\partial x} \\ \frac{\partial^2\Delta w_0}{\partial x\partial y} \end{bmatrix} \tag{A1.20}$$

$$= \begin{bmatrix} 1 & \frac{1}{R_X} & \frac{\partial^2 w_0}{\partial x\partial y} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} Nu,xy \\ Nw,y \\ Nw,x \\ Nw,xy \end{bmatrix} . \Delta U$$

$$\frac{\partial \Delta \varepsilon_y{}^0}{\partial x} = \frac{\partial^2 \Delta v_0}{\partial x \partial y} + \frac{\partial \Delta w_0}{\partial x}.\frac{1}{R_y} + \frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial \Delta w_0}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial^2 \Delta w_0}{\partial x \partial y}$$

$$= \begin{bmatrix} 1 & \frac{1}{R_y} & \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial w_0}{\partial y} \end{bmatrix} . \begin{bmatrix} \frac{\partial^2 \Delta v_0}{\partial x \partial y} \\ \frac{\partial \Delta w_0}{\partial x} \\ \frac{\partial \Delta w_0}{\partial y} \\ \frac{\partial^2 \Delta w_0}{\partial x \partial y} \end{bmatrix} \qquad \text{(A1.21)}$$

$$= \begin{bmatrix} 1 & \frac{1}{R_y} & \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial w_0}{\partial y} \end{bmatrix} . \begin{bmatrix} Nv,xy \\ Nw,x \\ Nw,y \\ Nw,xy \end{bmatrix} . \Delta U$$

$$\frac{\partial \Delta \varepsilon_y{}^0}{\partial y} = \frac{\partial^2 \Delta v_0}{\partial y^2} + \frac{\partial \Delta w_0}{\partial y}.\frac{1}{R_y} + \frac{\partial w_0}{\partial y}\frac{\partial^2 \Delta w_0}{\partial y^2} + \frac{\partial \Delta w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}$$

$$= \begin{bmatrix} 1 & \left(\frac{1}{R_y} + \frac{\partial^2 w_0}{\partial y^2}\right) & \frac{\partial w_0}{\partial y} \end{bmatrix} . \begin{bmatrix} \frac{\partial^2 \Delta v_0}{\partial y^2} \\ \frac{\partial \Delta w_0}{\partial y} \\ \frac{\partial^2 \Delta w_0}{\partial y^2} \end{bmatrix} \qquad \text{(A1.22)}$$

$$= \begin{bmatrix} 1 & \left(\frac{1}{R_y} + \frac{\partial^2 w_0}{\partial y^2}\right) & \frac{\partial w_0}{\partial y} \end{bmatrix} . \begin{bmatrix} Nv,yy \\ Nw,y \\ Nw,yy \end{bmatrix} . \Delta U$$

$$\frac{\partial \Delta {\gamma_{xy}}^0}{\partial x} = \frac{\partial^2 \Delta u_0}{\partial x \partial y} + \frac{\partial^2 \Delta v_0}{\partial x^2} + \frac{\partial^2 \Delta w_0}{\partial x^2}\frac{\partial w_0}{\partial y} + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial \Delta w_0}{\partial y} + \frac{\partial \Delta w_0}{\partial x}\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x}\frac{\partial^2 \Delta w_0}{\partial x \partial y}$$

$$= \begin{bmatrix} 1 & 1 & \frac{\partial w_0}{\partial y} & \frac{\partial^2 w_0}{\partial x^2} & \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} \frac{\partial^2 \Delta u_0}{\partial x \partial y} \\ \frac{\partial^2 \Delta v_0}{\partial x^2} \\ \frac{\partial^2 \Delta w_0}{\partial x^2} \\ \frac{\partial \Delta w_0}{\partial y} \\ \frac{\partial \Delta w_0}{\partial x} \\ \frac{\partial^2 \Delta w_0}{\partial x \partial y} \end{bmatrix} \tag{A1.23}$$

$$= \begin{bmatrix} 1 & 1 & \frac{\partial w_0}{\partial y} & \frac{\partial^2 w_0}{\partial x^2} & \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} Nu,xy \\ Nv,xx \\ Nw,xx \\ Nw,y \\ Nw,x \\ Nw,xy \end{bmatrix} . \Delta U$$

$$\frac{\partial \Delta {\gamma_{xy}}^0}{\partial y} = \frac{\partial^2 \Delta u_0}{\partial y^2} + \frac{\partial^2 \Delta v_0}{\partial x \partial y} + \frac{\partial^2 \Delta w_0}{\partial x \partial y}\frac{\partial w_0}{\partial y} + \frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial \Delta w_0}{\partial y} + \frac{\partial \Delta w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial w_0}{\partial x}\frac{\partial^2 \Delta w_0}{\partial y^2}$$

$$= \begin{bmatrix} 1 & 1 & \frac{\partial w_0}{\partial y} & \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial^2 w_0}{\partial y^2} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} \frac{\partial^2 \Delta u_0}{\partial y^2} \\ \frac{\partial^2 \Delta v_0}{\partial x \partial y} \\ \frac{\partial^2 \Delta w_0}{\partial x \partial y} \\ \frac{\partial \Delta w_0}{\partial y} \\ \frac{\partial \Delta w_0}{\partial x} \\ \frac{\partial^2 \Delta w_0}{\partial y^2} \end{bmatrix} \tag{A1.24}$$

$$= \begin{bmatrix} 1 & 1 & \frac{\partial w_0}{\partial y} & \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial^2 w_0}{\partial y^2} & \frac{\partial w_0}{\partial x} \end{bmatrix} . \begin{bmatrix} Nu,yy \\ Nv,xy \\ Nw,xy \\ Nw,y \\ Nw,x \\ Nw,yy \end{bmatrix} . \Delta U$$

$$\frac{\partial \Delta\gamma_{xy}{}^{1}}{\partial x} = \frac{\partial^2 \Delta\theta_x}{\partial x \partial y} + \frac{\partial^2 \Delta\theta_y}{\partial x^2} + c_0\left(\frac{\partial^2 \Delta v_0}{\partial x^2} - \frac{\partial^2 \Delta u_0}{\partial x \partial y}\right) \tag{A1.25}$$

$$= \left(N\theta_x, xy + N\theta_y, xx + c_0 . Nv, xx - c_0 Nu, xy\right). \Delta U$$

$$\frac{\partial \Delta\gamma_{xy}{}^{1}}{\partial y} = \frac{\partial^2 \Delta\theta_x}{\partial y^2} + \frac{\partial^2 \Delta\theta_y}{\partial x \partial y} + c_0\left(\frac{\partial^2 \Delta v_0}{\partial x \partial y} - \frac{\partial^2 \Delta u_0}{\partial y^2}\right) \tag{A1.26}$$

$$= \left(N\theta_x, yy + N\theta_y, xy + c_0 . Nv, xy - c_0 Nu, yy\right). \Delta U$$

$$\frac{\partial \Delta\gamma_{yz}{}^{0}}{\partial x} = \frac{\partial \Delta\theta_y}{\partial x} + \frac{\partial^2 \Delta w_0}{\partial x \partial y} - \frac{1}{R_y}\frac{\partial \Delta v_0}{\partial x} \tag{A1.27}$$

$$\frac{\partial \Delta\gamma_{yz}{}^{0}}{\partial y} = \frac{\partial \Delta\theta_y}{\partial y} + \frac{\partial^2 \Delta w_0}{\partial y^2} - \frac{1}{R_y}\frac{\partial \Delta v_0}{\partial y} \tag{A1.28}$$

$$\frac{\partial \Delta\gamma_{xz}{}^{0}}{\partial x} = \frac{\partial \Delta\theta_x}{\partial x} + \frac{\partial^2 \Delta w_0}{\partial x^2} - \frac{1}{R_x}\frac{\partial \Delta u_0}{\partial x} \tag{A1.29}$$

$$\frac{\partial \Delta\gamma_{xz}{}^{0}}{\partial y} = \frac{\partial \Delta\theta_x}{\partial y} + \frac{\partial^2 \Delta w_0}{\partial x \partial y} - \frac{1}{R_x}\frac{\partial \Delta u_0}{\partial y} \tag{A1.30}$$