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A THESIS SUBMITTED FOR THE DEGREE OF
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TUĞBA YAVUZ

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DANIŞMANI

PROF. DR. ENGİN HALİLOĞLU

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SUMMARY

In this dissertation, the coefficient problem for certain classes of analytic and normalized functions in an ellipse is discussed. First of all, the method of solution is associated with the the extreme point theory. Hence, the Faber coefficients of analytic functions in an ellipse are calculated over over the extreme points of closed convex hull of related subclass explicitly. After that, sharp upper bounds for these calculated Faber coefficients are obtained. Finally, it is shown that for each inequality there are two extremal functions, which is the number of invariant rotations of the ellipse.

Key Words: Analytic Functions, Univalent Functions, Close-to-Convex Functions, Starlike Functions, Coefficient Estimates, Faber Coefficients, Chebyshev Polynomials, Jacobi Elliptic Sine Function.

ÖZET

Bu tezde, belli bir elipste analitik ve normalize edimiş fonksiyon sınıfları için katsayı problemi ele alınmıştır. Öncelikli olarak çözüm yöntemi ekstreme nokta teorisi ile ilişkilendirilmiştir. Dolayısıyla, analitik fonksiyonun Faber katsayıları sadece ilgili alt sınıfın kapalı konvex kabuğunun ekstreme noktaları üzerinden açık olarak hesaplanmıştır. Sonrasında, hesaplanan bu Faber katsayıları için keskin üst sınırlar elde edilmiştir. Son olarak, her eşitsizlik için elipsin invaryant dönme sayısı kadar ekstremal fonksiyonun mevcut olduğu gösterilmiştir.

Anahtar Kelimeler: Analitik Fonksiyonlar, Yalınkat Fonksiyonlar, Konvekse Yakın Fonksiyonlar, Yıldızıl Fonksiyonlar, Katsayı Tahminleri, Faber Katsayıları, Chebyshev Polinomları, Jacobi Eliptik Sinüs Fonksiyonu.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		Explanations
$\mathbb{D}$	:	The open unit disc
$\mathbb{C}$	:	The complex numbers
$\mathbb{D}(z_0, r)$	:	Open disc with centre at z_0 and radius r
$\mathbb{D}_r$	:	Open disc with centre at 0 and radius r
Δ	:	Exterior of the unit disc
Δ_r	:	Exterior of the disc with centre at 0 and radius r
E_r	:	Ellipse with semi minor-axis $1 - \frac{1}{r^2}$ and semi major-axis $1 + \frac{1}{r^2}$, where $r > 1$
$\mathcal{H}(\Omega)$	:	Class of analytic functions on Ω
$\mathcal{A}(\Omega)$	:	Class of normalized analytic functions on Ω
$S(\Omega)$	:	Class of analytic, univalent and normalized functions in on Ω
$C(\Omega)$	:	Class of convex functions in on Ω
$S^*(\Omega)$	:	Class of starlike functions on Ω
$K(\Omega)$	:	Class of close-to-convex functions on Ω
$P(\Omega)$	:	Class of functions with positive real part on Ω
$T(\Omega)$	:	Class of typically real functions on Ω
$R_\alpha(\Omega)$	:	Class of analytic functions on Ω with $Re\sqrt{\frac{F(z)}{\varphi(z)}} > \frac{\alpha}{\sqrt{\varphi'(z)}}$ for $\alpha \in [0, 1)$
$S^{(2)}(\Omega)$	:	Class of odd univalent functions on Ω
$\sum_r$	:	Class of meromorphic functions with residue 1 on Δ_r
$\partial\Omega$	:	Boundary of Ω
$\overline{\Omega}$	:	Closure of Ω
$\mathcal{F}$	:	Set of functions
$\overline{co}(\mathcal{F})$	:	Closed convex hull of $\mathcal{F}$
$ext\,(\overline{co}(\mathcal{F}))$	:	Extreme points of closed convex hull of $\mathcal{F}$
$Re(z)$	:	Real part of z
$Im(z)$	:	Imaginary part of z

1. INTRODUCTION

Two important branches of complex analysis are potential theory and the geometric function theory of analytic functions. Geometric function theory is more often associated with 'geometry' and 'analysis'. This area, which deals with the geometric properties of analytic functions, was born around the turn of the 20th century, yet it remains an active field of current research. Let $\Omega \subset \mathbb{C}$ be an open and connected non-empty subset of the complex plane. A function f defined on $\Omega \subset \mathbb{C}$ is called univalent if it is one-to-one mapping of Ω onto its image.

Let $\mathcal{H}(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$ and let $\mathcal{A}$ denote the class of analytic functions defined on $\mathbb{D}$ which are normalized by the conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{1.1}$$

Each function $f \in \mathcal{A}$ has the Taylor series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

A function f is univalent in Ω if $f(z_1) = f(z_2)$ implies $z_1 = z_2$ in Ω and f is called locally univalent at $z_0 \in \Omega$ if it is univalent in some neighbourhood of z_0. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by S.

Among the first important papers which discuss topics from this area are [Koebe, 1907], [Alexander, 1915] and [Bieberbach, 1916]. Koebe initiated in 1907 the study about univalent functions, while Bieberbach presented in 1916 would soon become a famous conjecture. One of the major problems and a cornerstone for the subsequent development of this field is the Bieberbach Conjecture which asserts that the coefficients in the Taylor series expansion (1.2) of every function in the class S of normalized univalent functions in the unit disc satisfy the inequalities $|a_n| \leq n$, $n = 2, 3, 4, \cdots$. This problem stood for many years as a challenge, inspiring the development of new and elaborate research methods, such as Löwner's parametric method, the variational methods introduced by M. Schiffer and G. M. Goluzin, the extreme points method owed to L. Brickman, etc.

Although almost 70 years had passed until the Bieberbach conjecture was finally proved in the article [Branges, 1985], bounds for the Taylor coefficients were obtained in the meantime for some subclasses of univalent functions than for the full class S. After the proof of the Bieberbach conjecture the study of different classes o analytic, univalent and meromorphic functions have began to take shape, still remaining an intersting subject.

There are many books and monographs nowadays dedicated to geometric function theory or the study of univalent functions, of which we mention those of [Alfors, 1973], [Pommerenke, 1975],[Conway, 1978], [Goodman, 1983], [Duren, 1983],[Hallenbeck and MacGregor, 1984], [Miller and Mocanu, 2000].

On the other hand, it was also an interesting problem for some mathematicians to generalize the Taylor expansion for functions analytic in an arbitrary simply connected domain.

The study about generalization of Taylor expansion to simply connected domains other than unit disc has been initiated in [Faber, 1903]. For this purpose, Faber used the mapping function as follows:

Let $g(z)$ be the unique, one-to-one, analytic mapping of $\Delta = \{z : |z| > 1\}$ onto $\mathbb{C}\backslash\overline{\Omega}$ with

$$g(z) = cz + \sum_{n=0}^{\infty} \frac{c_n}{z^n}, \ \ (c > 0, z \in \Delta)\,. \tag{1.3}$$

Here $g(z)$ is called the exterior function of Ω. Without loss of generality, we can assume that Ω has capacity 1 so that $c = 1$. The Faber polynomials of Ω (or $g(z)$) are defined by the generating function relation [Duren, 1983]

$$\frac{\eta g'(\eta)}{g(\eta) - z} = \sum_{n=0}^{\infty} \Phi_n(z)\eta^{-n} \ \ (\eta \in \Delta)\,. \tag{1.4}$$

If $\partial\Omega$ is analytic and $F(z)$ is analytic in Ω, then $F(z)$ can be represented by the following series [Scober, 1975]

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z)\,,\, z \in \Omega. \tag{1.5}$$

This series is called Faber Series of $F(z)$. Here $\{A_n\}_{n=1}^{\infty}$ are called the Faber coefficients of $F(z)$ which are given by the integral

$$A_n = \frac{1}{2\pi i} \int\limits_{|z|=\rho} F(g(z))\, z^{-n-1} dz, \tag{1.6}$$

where $\rho < 1$ and close to 1.

In the article [Faber, 1907], it is proved that the series (1.5) converges uniformly to $F(z)$ on compact subsets of Ω. In [Faber, 1920], it is also stated that series in (1.5) is the best uniform approximation to $F(z)$ which is analytic in Ω. Hence, Faber expansion is important in the approximation theory.

Motivated from Faber's results we consider some subclasses of univalent functions in an elliptical domain, which is defined as

$$E_r = \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{\left(1+\frac{1}{r^2}\right)^2} + \frac{y^2}{\left(1-\frac{1}{r^2}\right)^2} < 1,\ r > 1 \right\}. \tag{1.7}$$

The main purpose of this thesis is to get sharp upper bounds for the Faber coefficients of functions which belong certain subclasses of analytic functions in E_r.

This thesis consists of five chapters. First chapter is introduction. Second chapter starts with some basic notations and a summary of the fundamental results about the class S. In the second section of this part, we define classical subclasses of S and give some important results about these classes. In the last section, we mention several conjectures other than the Bieberbach Conjecture. Also, we determine the important conjectures for the proof of de Branges [Branges, 1985].

In the third chapter, we mentioned the Faber polynomials and Faber series. Also, we give some examples of Faber polynomials for certain regions. Faber coefficient formula for analytic functions in E_r is also obtained.

In the following chapter, we define new subclasses of analytic functions in E_r. In the second section, we posed the Faber coefficient problem. Then, we associated the method of solution of this problem with extreme point theory. The third section is dedicated to give some recent results about Faber coefficient problem of univalent functions in E_r. In the following sections of this chapter, we obtain explicit expressions for the Faber coefficients of critical functions which belong to certain subclasses defined in E_r. Also, we give sharp upper bounds for these coefficients.

2. DEFINITIONS AND CLASSICAL RESULTS

This chapter introduces the class S of univalent functions and some of its subclasses defined by geometric conditions. In this part, some elementary results are given.

2.1. Basic Notations and Subclasses of S

Let $\mathbb{C}$ be the complex plane and $\mathbb{D}(z_0, r)$ be the open disc of radius $r > 0$ centered at $z_0 \in \mathbb{C}$, i. e.

$$\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}. \tag{2.1}$$

The open disc $\mathbb{D}(0, r)$ will be denoted by $\mathbb{D}_r$, and the unit disc $\mathbb{D}_1$ will be denoted by $\mathbb{D}$.

Let $\mathcal{H}(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$ and let $\mathcal{A}$ denote the class of analytic functions defined on $\mathbb{D}$ which are normalized by the conditions (1.1). Let S be the classical subclass of $\mathcal{A}$ consisting of univalent functions. Thus, functions in S have Taylor expansions of the form (1.2).

We need to give some elementary definitions to define some important subclasses of S.

Definition 2.1: A set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other $w \in E$ lies entirely in E.

Definition 2.2: The set E is said to be convex if it is starlike with respect to each point in E.

Now, we define the usual subclasses of S.

Definition 2.3: A convex function in S is one which maps the unit disc $\mathbb{D}$ onto a convex domain. The class of convex functions in S is denoted by C. It is well known that

$$C = \left\{ f \in S : \textit{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}. \tag{2.2}$$

Definition 2.4: A starlike function in S is one which maps the unit disc $\mathbb{D}$ onto a starlike domain. The class of starlike functions in S is denoted by S^*. It is also known that

$$S^* = \left\{ f \in S : \textit{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}. \tag{2.3}$$

Definition 2.5: The class of functions φ analytic and having positive real part in $\mathbb{D}$, with $\varphi(0) = 1$ is denoted by P. The class P is also called Carathedory Class.

Definition 2.6: A function $f(z)$ analytic in $\mathbb{D}$ is said to be typically real if and only if $f(z)$ is real for $-1 < z < 1$. We denote the class of such functions with conditions $f(0) = 0$ and $f'(0) = 1$ by T.

Another interesting subclass of S which contains S^* has a simple geometric description. This is the class of close-to-convex functions, introduced in the article [Kaplan, 1952].

Definition 2.7: A function f analytic in the unit disc is said to be close-to-convex if there is a convex function g such that

$$\textit{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0 \tag{2.4}$$

for all $z \in \mathbb{D}$. We shall denote by K the class of close-to-convex functions f normalized by the usual conditions (1.1).

Every convex function is obviously close-to-convex. More generally, every starlike function is close-to-convex. This implies the following inclusions

$$C \subset S^* \subset K. \tag{2.5}$$

In the following definition, we give an interesting subclass of $\mathcal{A}$ which is denoted by $R(\alpha)$.

Definition 2.8: Let $R(\alpha)$ denote the set of analytic functions f such that $f(0) = 0$, $f'(0) = 1$ and $\textit{Re}\sqrt{\frac{f(z)}{z}} > \alpha$, $z \in \mathbb{D}$, $\alpha \in [0, 1)$.

We also note that the class $S \cap R(\frac{1}{2})$ was introduced in the article [Dvorak, 1934]. Some fundamental properties of this class has been discused in [Duren and Schober, 1971].

Definition 2.9: The class $S^{(2)}$ consist of all odd univalent functions in $\mathbb{D}$ with normalization conditions (1.1) .

Every function f in $S^{(2)}$ is the square-root transform of a function $g(z) \in S$, i.e

$$f(z) = \sqrt{g(z^2)}. \tag{2.6}$$

2.2. Coefficient Inequalities For Univalent Functions in Unit Disc

Coefficient problem, which means finding the maximum value of $|a_n|$, is one of the most popular problems in univalent function theory. According to following theorem, this problem is well posed.

Theorem 2.1: The class S of univalent functions is a compact normal family. [Duren, 1983]

According to Theorem 2.1, there exist a M_n for fixed n such that

$$\max_{f \in S} |a_n| = M_n, \tag{2.7}$$

since the functional $\mathcal{L}(f) = a_n$ is linear and continuous.

The function $k(z)$ defined by

$$k(z) = \frac{z}{(1-z)^2} \tag{2.8}$$

is called Koebe function. The rotations of $k(z)$ is denoted by

$$k_\theta(z) = e^{-i\theta} k(e^{i\theta} z) = \frac{z}{(1-e^{i\theta}z)^2}, \tag{2.9}$$

which plays an extremal role in class S.

Now, we mention the coefficient estimates for functions which belong to certain subclasses of analytic functions in the unit disc. The following result for functions in the class P was given in [Carathedory, 1907].

Lemma 2.1: If $\varphi \in P$ and

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.10}$$

then $|c_n| \leq 2$, $n = 1, 2, \cdots$. Equality holds only for the functions

$$p_\theta(z) = \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}. \tag{2.11}$$

The class P is closely related to the classes C and S^*. This was first observed in [Alexander, 1915].

Theorem 2.2: Let f be analytic in $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in C$ if and only if $zf'(z) \in S^*$ [Alexander, 1915].

Bieberbach Conjecture for the class S^* of starlike functions proved in the article [Nevalinna, 1921].

Theorem 2.3: The coefficients of each function $f \in S^*$ satisfy $|a_n| \leq n$ for $n = 2, 3, \cdots$. Strict inequality holds for all n unless f is a rotation of Koebe function given by (2.9) [Nevalinna, 1921].

The following result for functions in the class C follows from Theorem 2.2. and Theorem 2.3, which can be found in [Loewner, 1917].

Corollary 2.1: If $f \in C$, then $|a_n| \leq 1$ for $n = 2, 3, \cdots$ .Strict inequality holds for all n unless f is in the form

$$c_\theta(z) = \frac{z}{1 - e^{i\theta} z}. \tag{2.12}$$

The following coefficient inequality for functions in T is given in [Rogosinki, 1932].

Theorem 2.4: If $f(z) \in T$ is given by (1.2), then

$$|a_n| \leq n, n = 2, 3, 4, \cdots . \tag{2.13}$$

Strict inequality occurs for all even n unless f is the Koebe function or its real rotation $-k(z)$. Strict inequality occurs for all odd n unless f is a convex combination of these two functions.

2.3. Bieberbach's Conjecture by De Branges

Before de Branges' proof of the Bieberbach conjecture in [Branges, 1985] via the Milin conjecture, the following seven conjectures in their fully generality were open problems. Details of the logical non-trivial relationship between these seven conjectures can be found in the book [Gong, 1999].

- Bieberbach Conjecture. For any $f \in S$ the inequality $|a_n| \leq n$ holds for all $n \geq 2$. The equality occurs if and only if $f(z)$ is in the form (2.3) [Bieberbach, 1916].
- Littlewood Conjecture. If $f \in S$ and $f(z) \neq w$ for any $z \in \mathbb{D}$, then $|a_n| \leq 4|w|\, n$ holds for all $n \geq 2$ [Littlewood, 1925].
- Robertson Conjecture. For any odd function $h(z) = z + c_3 z^3 + c_5 z^5 + \ldots$ in S, the inequality

$$1 + |c_3|^2 + \ldots |c_{2n-1}|^2 \leq n, \tag{2.14}$$

is true for all $n \geq 2$ [Robertson, 1936].

- Rogosinski (Generalized Bieberbach) Conjecture. Let $g(z)$ be an analytic function in $\mathbb{D}$ with

$$g(z) = b_1 z + \ldots + b_n z^n + \ldots \tag{2.15}$$

If $g(\mathbb{D}) \subset f(\mathbb{D})$ and $f \in S$, then the inequality $|b_n| \leq n$ holds for all $n \geq 2$ [Rogosinski, 1943].

- Asymptotic Bieberbach Conjecture, Connected With Hayman's Regularity Theorem. If

$$A_n = \max_{f \in S} |a_n|, \tag{2.16}$$

then

$$\lim_{n \to \infty} \frac{A_n}{n} = 1 \tag{2.17}$$

[Hayman, 1958].

- Milin Conjecture. For any $f \in S$, let γ_n be defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n. \tag{2.18}$$

Then the inequality

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k |\gamma_k|^2 - \frac{1}{k} \right) \leq 0 \tag{2.19}$$

holds for all $n \geq 1$ [Milin, 1967].

- Sheil-Small Conjecture. For any $f \in S$ and any polynomial $P(z) = b_0 + b_1 z + \ldots + b_n z^n$ the convolution (Hadamard product) defined by

$$(P * f)(z) = b_1 z + a_2 b_2 z^2 + \ldots + a_n b_n z^n \tag{2.20}$$

satisfies the inequality

$$\max_{|z| \leq 1} |(P * f)(z)| \leq n \max_{|z| \leq 1} |P(z)|, \tag{2.21}$$

for all $n \geq 2$ [Sheil-Small, 1973].

There is also some implication between three of these conjectures which is given in he following theorem.

Theorem 2.5: Milin conjecture implies the Robertson conjecture. Robertson conjecture implies the Bieberbach conjecture.

De Branges only considered the implications in Theorem 2.5 when he proved the Bieberbach Conjecture. The easiest step is given in the following theorem.

Theorem 2.6: Robertson conjecture implies the Bieberbach conjecture.

Proof 2.6: For $f \in S$ let $\widetilde{h}(z) = \sqrt{\frac{f(z^2)}{z^2}}$. Then the odd function

$$h(z) = z\widetilde{h}(z) = z + \sum_{n=2}^{\infty} c_{2n-1} z^{2n-1} \tag{2.22}$$

belongs to the family S as well. If we take into account that $\widetilde{h}$ is an even function we see that $s(z) = \widetilde{h}(\sqrt{z})$ is holomorphic in $\mathbb{D}$ and that $f(z) = z\,(s(z))^2$. This identity implies that, for $n \geq 2$,

$$a_n = \sum_{k=0}^{n-1} c_{2k+1} c_{2(n-k)-1}, \tag{2.23}$$

where $c_1 = 1$. According to the Cauchy inequality, this yields

$$|a_n| \leq \sum_{k=0}^{n-1} |c_{2k+1}|^2 . \tag{2.24}$$

Hence, the proof is completed. ■

Second implication is obtained by the following result which is called Second Lebedev-Milin Inequality.

Lemma 2.2: Let $\Phi(z)$ is an analytic function at $z = 0$, $\Phi(0) = 0$ and has a Taylor expansion

$$\Phi(z) = \sum_{k=1}^{\infty} \alpha_k z^k \tag{2.25}$$

and

$$\Psi(z) = e^{\Phi(z)} = \sum_{k=0}^{\infty} \beta_k z^k. \tag{2.26}$$

Then, for $n = 1, 2, \ldots,$

$$\sum_{k=0}^{n} |\beta_k|^2 \leq (n+1) \exp\left\{ \frac{1}{n+1} \sum_{m=1}^{n} \sum_{k=1}^{m} \left(k\,|\alpha_k|^2 - \frac{1}{k} \right) \right\}. \tag{2.27}$$

Equality occurs for a given n *if and only if* $\alpha_k = \frac{\gamma^k}{k}$, $k = 1, 2, \ldots, n$ *for some constant* γ *with* $|\gamma| = 1$ (*[Milin, 1967], [Lebedev and Milin, 1965]*) .

Now, we can prove the second implication.

Theorem 2.7: Milin conjecture implies the Robertson conjecture.

Proof 2.7: Assume that Milin conjecture holds. Let $g(z) \in S^{(2)}$. *Then, there exist a function* $f \in S$ *such that* $g(z) = \sqrt{f(z^2)}$, *where* $f(z)$ *and* $g(z)$ *have Taylor expansions,*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2.28}$$

respectively. From square-root transformation, we get

$$\log\left(\frac{g(z)}{z}\right) = \frac{1}{2}\log\left(\frac{f(z^2)}{z^2}\right). \tag{2.29}$$

Let

$$\log\left(\frac{f(z)}{z}\right) = \sum_{k=1}^{\infty} \alpha_k z^k \tag{2.30}$$

and

$$\log\left(\frac{g(z)}{z}\right) = \sum_{k=1}^{\infty} \beta_k z^k. \tag{2.31}$$

Since

$$\log\left(\frac{g(z)}{z}\right) = \sum_{k=1}^{\infty} \beta_k z^k = \frac{1}{2}\log\left(\frac{f(z^2)}{z^2}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \alpha_k z^{2k} \tag{2.32}$$

so that

$$\beta_k = \begin{cases} 0, \ k = 2n - 1 \\ \frac{1}{2}\alpha_k, \ k = 2n \end{cases}. \tag{2.33}$$

Hence,

$$\log\left(\frac{g(\sqrt{z})}{\sqrt{z}}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \alpha_k z^k. \tag{2.34}$$

On the other hand,

$$\frac{g\left(\sqrt{z}\right)}{\sqrt{z}} = \sum_{k=0}^{\infty} b_{2k+1} z^k$$

$$= \exp\left(\log\left(\frac{g\left(\sqrt{z}\right)}{\sqrt{z}}\right)\right) \tag{2.35}$$

$$= \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} \alpha_k z^k\right)$$

According to Lemma 2.2, the following inequality

$$\sum_{k=0}^{n} \left|b_{2k+1}\right|^2 \leq (n+1)\exp\left\{\frac{1}{n+1}\sum_{k=1}^{n}\sum_{l=1}^{k}\left(l\frac{\left|\alpha_l\right|^2}{4} - \frac{1}{l}\right)\right\} \tag{2.36}$$

is obtained. From Milin conjecture we have

$$\exp\left\{\frac{1}{n+1}\sum_{k=1}^{n}\sum_{l=1}^{k}\left(l\frac{\left|\alpha_l\right|^2}{4} - \frac{1}{l}\right)\right\} \leq 1, \tag{2.37}$$

which implies the desired result. ■

3. FABER POLYNOMIALS AND FABER SERIES

By Runge's Theorem [Markushevich, 1983] every function $f(z)$ that is analytic in a simply connected domain Ω not containing the point at infinity can be expanded in a series of polynomials converging uniformly within Ω. In connection with this theorem, Faber formulated in 1903 the problem of constructing a generalized Taylor series for an arbitrary simply connected domain in [Faber, 1903]. He showed that an analytic function f in a simple connected domain can be represented by the following series

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_n(z) \tag{3.1}$$

where the coefficients $\{a_n\}$ are determined by Ω and $f(z)$. Here, the polynomials $\Phi_n(z)$ are called Faber Polynomials of Ω which are investigated in Section 3.1.

Since then his work has found applications in many areas of mathematics and a large number of papers on Faber polynomials have been published. Examples of applications and further properties can be found in [Curtiss, 1971], [Smirnov and Lebedev, 1968], [Suetin, 1998]. Suetin's recent book [Suetin, 1998] additionaly contains a comprehensive bibliography of the literature on Faber polynomials.

Recently, Faber polynomials for particular regions in the complex plane heve been the subject of many researches. For example, Faber polynomials of cicular arc and cicular lunes were studied in [He, 1994], [He, 1995]. Circular sectors were considered in [Coleman and Smith, 1987], [Gaterman et al., 1992]. On the other hand, annular sectors were studied in [Coleman and Myers, 1995]. There are also many studies about Faber polynomials of hypocycloidal domains such as [Eiermann and Varga, 1993], [He, 1996] and [He and Saff, 1994].

3.1. Definition of Faber Polynomials

We need to define the following class of functions, before we mention Faber polynomials.

Definition 3.1: A function f belongs to $\sum_r$ if it is an analytic and univalent in the *domain* $\Delta_r = \{z : |z| > r\}$, *except for a simple pole at infinity with residue* 1.

Each function $g \in \sum_r$ maps Δ_r onto the complement of a compact connected set Ω. Then, each $g \in \sum_r$ has an expansion

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}, \ \ |z| > r. \tag{3.2}$$

Consider the function $g(z) = \frac{g(z)-w}{z}$, where $w \in \mathbb{C}$. This is an analytic function in a neighbourhood of infinity and equal to 1 at infinity. Hence, it does not vanish at infinity. Consequently, it has an analytic logarithm which is equal to 0 at the point $z = 1$. Then, we get the following expansion

$$\log\left(\frac{g(z)-w}{z}\right) = \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right) P_k(w) z^{-k}, \tag{3.3}$$

where $\left(-\frac{1}{k}\right)$ is for convenience. If we take the derivative of both sides of (3.3) with respect to z, we obtain

$$-\frac{1}{z} + \frac{g'(z)}{g(z)-w} = \sum_{k=1}^{\infty} P_k(w) z^{-k-1} \tag{3.4}$$

or

$$\frac{zg'(z)}{g(z)-w} = \sum_{k=0}^{\infty} P_k(w) z^{-k}, \ P_0(z) = 1. \tag{3.5}$$

We need to determine the form of polynomials $P_k(w)$ from (3.5) . If we substitute (3.2) into (3.5) , we get

$$z - \sum_{n=1}^{\infty} n b_n z^{-n} = \left[z + \sum_{n=0}^{\infty} b_n z^{-n} - w\right]\left[\sum_{n=0}^{\infty} P_n(w) z^{-n}\right]. \tag{3.6}$$

Performing Cauchy's product and comparing coefficients yield the recursion formula.

$$P_0(w) \ = \ 1, \tag{3.7}$$

$$P_1(w) \ = \ w - b_0, \tag{3.8}$$

$$P_{n+1}(w) = (w - b_0)\, P_n(w) - \sum_{k=1}^{n-1} b_k P_{n-k}(w) - (n+1)\, b_n,\ n = 1, 2, \cdots . \quad (3.9)$$

Definition 3.2: The polynomials $\{P_n(w)\}_{n=0}^{\infty}$ *are called the Faber polynomials of* $g(z)$ *(or* Ω*).*

From equlity (3.9) it is obvious that $P_n(w)$ is a monic polynomial of degree n.

The next theorem can be considered another definition of the Faber polynomials.

Theorem 3.1: Let $\{P_n(w)\}$ *be the Faber polynomials of* $g\,(z) \in \sum_r$*. Then*

$$\left(g^{-1}\,(w)\right)^n = P_n(w) + O\left(\frac{1}{w}\right) \quad (3.10)$$

at a neigborhood of infinity. It means that $P_n(w)$ *actuaclly is a monic polynomial of degree* n *and consists of the principal part of the expansion of* $\left(g^{-1}\,(w)\right)^n$ *near* ∞*.*

Proof 3.1: Let $r' > r$*. Let* $\gamma_{r'} = \{g(z)\,||z| = r'\}$ *and let* $\Omega_{r'}$ *denote the interior of* $\gamma_{r'}$*. Choose a* ρ *such that* $r' < \rho < \infty$*. By using Cauchy Integral formula, we obtain*

$$P_n(z) = \frac{1}{2\pi i} \int\limits_{|\tau|=\rho} \frac{\tau^n g'(\tau)}{g\,(\tau) - z} d\tau = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{\left(g^{-1}(w)\right)^n}{w - z} dw, \textit{ for } z \in \overline{\Omega}_r. \quad (3.11)$$

Since $g\,(\infty) = \infty$ *and* $g'\,(\infty) = 1$*, we get the following*

$$g^{-1}(z) = z + \sum_{n=0}^{\infty} d_n z^{-n}, \quad (3.12)$$

for z *sufficiently large. Hence, we see that* $\left(g^{-1}(z)\right)^n$ *has a pole of order* n *at infinity. We can rewrite (3.11) as*

$$P_n(z) = \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{\left(g^{-1}(w)\right)^n}{w - z} dw, \quad (3.13)$$

for sufficiently large R*. Let*

$$\left(g^{-1}(w)\right)^n = w^n + D_1^{(n)}w^{n-1} + \ldots + D_n^{(n)} + \frac{D_{-1}^{(n)}}{w} + \frac{D_{-2}^{(n)}}{w^2} + \ldots \tag{3.14}$$

and

$$\frac{1}{w-z} = \frac{1}{w} + \frac{z}{w^2} + \ldots + \frac{z^n}{w^{n+1}} + \ldots. \tag{3.15}$$

By taking the Cauchy product and using residue theorem, $P_n(z)$ is obtained from (3.13) as

$$P_n(z) = z^n + D_1^{(n)}z^{n-1} + \ldots + D_n^{(n)}. \tag{3.16}$$

If we substitute (3.16) in (3.14), we get the desired result (3.10). ■

Now, we give examples of Faber polynomials of some certain regions.

Example 3.1: The function $g(z) = z + a$ maps the exterior of $|z| = r$ onto the exterior of the cicle with center a and radius r. So, the Faber polynomials of $g(z)$ are $P_n(z) = (z-a)^n$, $n = 0, 1, 2, \cdots$.

Note that the Faber polynomials of both the exterior and the interior of circles centred at the origin are just the powers of z.

Example 3.2: The function

$$f(z) = \frac{z}{1 - \frac{z}{2p}} \tag{3.17}$$

maps $|z| < 2|p|$ onto the half-plane containing the origin whose nearest boundary point is p. So, its Faber polynomials are

$$P_n(z) = \left[z - \frac{1}{2p}\right]^n, \; n = 0, 1, 2, \cdots . \tag{3.18}$$

Example 3.3: $g(z) = z + \frac{1}{r^2 z}$, $r > 1$, maps the exterior of the unit disc $\mathbb{D}$ onto the exterior of E_r which is given by (1.7). By the generating function relation (3.5) for function $g : \Delta \to \mathbb{C}\backslash\overline{E_r}$, we have

$$\frac{zg'(z)}{g(z)-w} = 1 + \frac{r^2wz-2}{r^2z^2-r^2wz+1}. \tag{3.19}$$

It is also equivalent to

$$\frac{zg'(z)}{g(z)-w} = 1 + \frac{1}{r^2}\left[\frac{r^2wz-2}{(z-z_1)(z-z_2)}\right], \tag{3.20}$$

where

$$z_{1,2} = \frac{rw \pm \sqrt{r^2w^2-4}}{2r}. \tag{3.21}$$

Then, (3.20) can be written as

$$\frac{zg'(z)}{g(z)-w} = 1 + \frac{1}{r^2}\left\{\frac{\frac{r}{2}\sqrt{r^2w^2-4}+\dfrac{r^2w}{2}}{z-z_1} + \frac{-\frac{r}{2}\sqrt{r^2w^2-4}+\frac{r^2w}{2}}{z-z_2}\right\}$$

$$= 1 + \frac{1}{2r}\left\{\frac{rw+\sqrt{r^2w^2-4}}{z-\dfrac{rw+\sqrt{r^2w^2-4}}{2r}} + \frac{rw-\sqrt{r^2w^2-4}}{z-\frac{rw-\sqrt{r^2w^2-4}}{2r}}\right\} \tag{3.22}$$

$$= 1 + \frac{1}{rz}\left\{\frac{\frac{rw}{2}+\sqrt{\left(\frac{rw}{2}\right)^2-1}}{1-\dfrac{\dfrac{rw}{2}+\dfrac{\sqrt{\left(\frac{rw}{2}\right)^2-1}}{r}}{z}} + \frac{\frac{rw}{2}-\sqrt{\left(\frac{rw}{2}\right)^2-1}}{1-\dfrac{\dfrac{rw}{2}-\dfrac{\sqrt{\left(\frac{rw}{2}\right)^2-1}}{r}}{z}}\right\}$$

and

$$\frac{zg'(z)}{g(z)-w} = 1 + \sum_{n=1}^{\infty} r^{-n}\left\{\left[\frac{rw}{2}+\sqrt{\left(\frac{rw}{2}\right)^2-1}\right]^n + \left[\frac{rw}{2}-\sqrt{\left(\frac{rw}{2}\right)^2-1}\right]^n\right\} z^{-n}. \tag{3.23}$$

Hence, we obtain the Faber polynomials of E_r as

$$P_n(z) = 2^n r^{-n} \Phi_n(\frac{rz}{2}),\ n = 0, 1, 2, \cdots, \tag{3.24}$$

where $\Phi_n(z)$ *are the monic Chebyschev polynomials which are defined by*

$$\Phi_0(z) = 1, \tag{3.25}$$

$$\Phi_n(z) = 2^{-n} \left\{ \left[z + \sqrt{z^2 - 1} \right]^n + \left[z - \sqrt{z^2 - 1} \right]^n \right\},\ n = 1, 2, \cdots. \tag{3.26}$$

3.2. Properties of Faber Series

It is well known that a function $f(z)$ which is analytic in $\mathbb{D}(z_0, r)$ can be represented by Taylor series as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,\ z \in \mathbb{D}(z_0, r). \tag{3.27}$$

Before we mention the properties of Faber series, remember the following properties of Taylor series.

- The series in (3.27) converges uniformly to $f(z)$ on compact subsets of $\mathbb{D}(z_0, r)$.
- Radius of convergence ρ of this series in (3.27), is obtained by the formula

$$\frac{1}{\rho} = \lim_{n \to \infty} \sup \left| \frac{f^{(n)}(z_0)}{n!} \right|^{\frac{1}{n}},\ \rho \geq r. \tag{3.28}$$

- The coefficients $\frac{f^{(n)}(z_0)}{n!}$ are unique. It means that, if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,\ z \in \mathbb{D}(z_0, r), \tag{3.29}$$

then

$$a_n = \frac{f^{(n)}(z_0)}{n!},\ n = 0, 1, 2, \cdots. \tag{3.30}$$

Let $g(z)$ be an analytic function which maps the exterior of the circle $|z| = r$ to exterior of the closed curve γ. If f is analytic in $Int\,(\gamma)$, can be represented by a series in terms of Faber polynomials of $g(z)$. Moreover, this series has similar properties to the properties of Taylor series listed above. These properties may be found in the book [Scober, 1975].

Theorem 3.2: Let $g(z) \in \sum_r$ *and let* $P_n(z)$, $n = 0, 1, 2, \cdots$ *be its Faber polynomials. Suppose* f *is analytic in the interior of*

$$\gamma_R = \{w = g(z)\,||z| = R\} \tag{3.31}$$

for $R > r$*. Then the following result holds :*

i) For $w \in Int\,(\gamma_R)$*,*

$$f(w) = \sum_{n=0}^{\infty} c_n P_n(w), \tag{3.32}$$

where the coefficients c_n*,* $n = 0, 1, 2, \cdots$ *are obtained by the formula*

$$c_n = \frac{1}{2\pi i} \oint\limits_{|\tau|=\rho} \frac{f\,(g(\tau))}{\tau^{n+1}} d\tau,\ n = 0, 1, 2, \cdots \tag{3.33}$$

where $r < \rho < R$*.*

ii) Coefficients of the series (3.32) satisfy

$$\lim_{n\to\infty} \sup |c_n|^{\frac{1}{n}} \leq \frac{1}{R}. \tag{3.34}$$

iii) The series in (3.32) converges uniformly to $f(z)$ *on compact subset of* $Int\,(\gamma_R)$ *.*

Proof 3.2: *i) Let* $w \in Int\,(\gamma_R)$ *. Choose* ρ *such that s* $r < \rho < R$ *and let*

$$\gamma_\rho = \{w = g(z)\,||z| = \rho\}\,. \tag{3.35}$$

From Cauchy Integral Formula, we get

$$f(w) = \frac{1}{2\pi i}\oint\limits_{\gamma_\rho} \frac{f(z)}{z-w}dz$$

$$= \frac{1}{2\pi i}\oint\limits_{|\tau|=\rho} \frac{f(g(\tau))\,g'(\tau)}{g(\tau)-w}d\tau \tag{3.36}$$

$$= \frac{1}{2\pi i}\oint\limits_{|\tau|=\rho} f(g(\tau))\sum_{n=0}^{\infty}\frac{P_n(w)}{\tau^{n+1}}d\tau.$$

Since, the series $\sum\limits_{n=0}^{\infty}\frac{P_n(w)}{\tau^{n+1}}$ *is uniformly convergent inside* $|\tau| = \rho$, *we can interchange integration and summation. Hence, we get the desired coefficient formula (3.33).*

ii) Since $f(g(\tau))$ *is an analytic function in a compact subset of* $|\tau| = \rho$, *it is also bounded. Hence, there exists a* M_ρ, *such that*

$$M_\rho = \max_{|\tau|=\rho}|f(g(\tau))| < \infty. \tag{3.37}$$

Then, we have the following inequality

$$|c_n| \le \left|\frac{1}{2\pi i}\oint\limits_{|\tau|=\rho}\frac{f(g(\tau))}{\tau^{n+1}}d\tau\right| \le \frac{M_\rho}{\rho^n}. \tag{3.38}$$

If we take the limit of both sides, we obtain

$$\lim_{n\to\infty}\sup|c_n|^{\frac{1}{n}} \le \lim_{n\to\infty}\frac{(M_\rho)^{\frac{1}{n}}}{\rho} \le \frac{1}{\rho}. \tag{3.39}$$

iii) Let $K \subset Int\,(\gamma_R)$ *be a compact set. Choose* ρ *such that* $K \subset Int\,(\gamma_\rho)$. *For* $w \in K$, *we obtain*

$$\frac{g'(z)}{g(z)-w} = \sum_{k=0}^{\infty} \frac{P_k(w)}{z^{k+1}}, \ \ |z| \geq R. \tag{3.40}$$

After multiplying both sides by z^n and integrating, we get

$$\frac{1}{2\pi i} \oint\limits_{|z|=\rho} \frac{z^n g'(\tau)}{g(z)-w} dz = \sum_{k=0}^{\infty} P_k(w) \left(\frac{1}{2\pi i} \oint\limits_{|z|=\rho} \frac{dz}{z^{k+1-n}} \right). \tag{3.41}$$

Since

$$\frac{1}{2\pi i} \oint\limits_{|z|=\rho} \frac{dz}{z^{k+1-n}} = \begin{cases} 1, \ k=n \\ 0, \ k \neq n \end{cases}, \tag{3.42}$$

From (3.41), we obtain

$$P_n(w) = \frac{1}{2\pi i} \oint\limits_{|z|=\rho} \frac{z g'(z)}{g(z)-w} dz. \tag{3.43}$$

For $w \in K$, it is true that

$$P_n(w) \leq Q_\rho \rho^{n+1}, \tag{3.44}$$

since the number

$$Q_\rho = \max_{|z|=\rho, w\in K} \left| \frac{g'(z)}{g(z)-w} \right| < \infty \tag{3.45}$$

exists. Now choose a ρ' such that $\rho < \rho' < R$.Then as in the proof of (ii), there exists a c such that

$$|c_n| \leq \frac{c}{(\rho')^n}. \tag{3.46}$$

Finally, the series, which is given by (3.32) is uniformly convergent in K by the Weierstrass M-test since the majorant series $\sum\limits_{n=0}^{\infty} \left(\frac{\rho}{\rho'}\right)^n$ is convergent. The proof is completed by letting $\rho \to R^-$. ■

4. COEFFICIENT PROBLEM FOR ANALYTIC FUNCTIONS IN AN ELLIPSE

In this chapter, we state our main problem. Then, we solve this problem for certain subclasses of normalized analytic functions in E_r.

4.1. Determination of Our Problem

We know that the function

$$g(z) = z + \frac{1}{r^2 z},\ r > 1 \tag{4.1}$$

is one-to-one, analytic mapping of Δ_1 onto $\mathbb{C}\backslash\overline{E_r}$, where E_r is given by (1.7). In Example 3.3, we have shown that the Faber polynomials of E_r can be represented in terms of the monic Chebyshev polynomials in (3.24).

Let sn$(z; q)$ be the Jacobi elliptic sine function which has nome q and modulus k_0. The modulus k_0 can be expressed in terms of q as

$$k_0 = 4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4. \tag{4.2}$$

Let

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k_0^2t^2}}, \tag{4.3}$$

which is known as complete elliptic integral of first kind [Lawden, 1989]. Then the function

$$\varphi(z) = \sqrt{k_0}\,\mathrm{sn}\left(\frac{2K}{\pi}\sin^{-1}\frac{rz}{2};\frac{1}{r^4}\right) \tag{4.4}$$

is the one-to-one, analytic mapping of E_r onto $\mathbb{D}$ with $\varphi(0) = 0$ and $\varphi'(0) = \frac{r\sqrt{k_0}K}{\pi}$ [Nehari, 1952].

Now, we define certain subclasses of analytic functions in E_r.

Let $\mathcal{H}(E_r)$ be the class of all analytic functions on E_r and let $\mathcal{A}(E_r)$ denote the class of analytic functions defined on E_r with the following conditions

$$F(0) = 0 \text{ and } F'(0) = 1. \tag{4.5}$$

The subclass of $\mathcal{A}(E_r)$ consisting of univalent functions is denoted by $\mathcal{S}(E_r)$.

Also, we define some subclasses of $\mathcal{S}(E_r)$, analogous to the subclasses of S in $\mathbb{D}$ as follows:

Definition 4.1: A function $F(z) \in \mathcal{S}(E_r)$ is said to be convex if it maps the E_r onto a convex domain. The set of convex functions in E_r is denoted by $\mathcal{C}(E_r)$.

Definition 4.2: A function $F(z) \in \mathcal{S}(E_r)$ is said to be starlike if it maps the E_r onto a starlike domain with respect to the orijin. The set of starlike functions in E_r is denoted by $\mathcal{S}^*(E_r)$.

Definition 4.3: Define $K(E_r)$ to be the subset of $\mathcal{S}(E_r)$ consisting of close-to-convex functions in E_r.

We know that

$$C(E_r) \subset \mathcal{S}^*(E_r) \subset K(E_r). \tag{4.6}$$

Definition 4.4: Let $P(E_r)$ denote the class of functions $P(z)$ analytic in E_r with $P(0) = \frac{1}{\varphi'(0)} = \frac{\pi}{r\sqrt{k_0}K}$ and Re $\{P(z)\} > 0$. (The condition $P(0) = \frac{1}{\varphi'(0)}$ is imposed for the convenience.)

Definition 4.5: $T(E_r)$ denote the class of functions $F(z)$ analytic in E_r, satisfying the conditions in (4.1) and having real values for $-1 - \frac{1}{r^2} < z < 1 + \frac{1}{r^2}$ and nonreal values elsewhere.

Definition 4.6: Let $R_\alpha(E_r)$ denote the class of analytic functions in E_r with

$$F(0) = 0,\ F'(0) = 1 \textit{ and } \mathit{Re}\sqrt{\frac{F(z)}{\varphi(z)}} > \frac{\alpha}{\sqrt{\varphi'(0)}},\ \alpha \in [0, 1). \tag{4.7}$$

Definition 4.7: Let $\mathcal{S}^{(2)}(E_r)$ denote the class of odd functions in $\mathcal{S}(E_r)$.

According to orthogonality property of Chebyshev polynomials, we can calculate the Faber coefficients as follows:

Lemma 4.1: The Faber coefficients of F(z) analytic in E_r are given by

$$A_n = \frac{r^n}{\pi}\int_0^{\pi} F\left(\frac{2\cos\theta}{r}\right)\cos n\theta d\theta,\ n = 0, 1, 2, \cdots \tag{4.8}$$

[Haliloglu and Johnston, 2005].

Proof 4.1: Let $F(z) \in \mathcal{A}(E_r)$, then we know that $F(z)$ can be represented by the series

$$F(z) = \sum_{n=0}^{\infty} A_n(f) 2^n r^{-n} \Phi\left(\frac{rz}{2}\right), \tag{4.9}$$

where $\Phi_n(z)$, $n = 0, 1, 2, \cdots$ are monic Chebyshev polynomials of degree n. Substituting $z = \frac{2\cos\theta}{r}$ and using $\Phi_n(\cos\theta) = 2^{1-n}\cos n\theta$, we get

$$F\left(\frac{2\cos\theta}{r}\right) = \sum_{n=0}^{\infty} A_n(f) 2r^{-n}\cos n\theta. \tag{4.10}$$

After multiplying by $\cos m\theta$ and integrating both sides from 0 to π, we obtain the desired result. ■

It is clear that $F(z)$ belongs to the one of the classes defined above if and only if

$$F(z) = \frac{f(\varphi(z))}{\varphi'(0)} \tag{4.11}$$

for some $f(z)$ in the associated class. So, we can rewrite Faber series of $F(z)$ which belongs to one of the classes defined above as

$$F(z) = \sum_{n=0}^{\infty} A_n(f)\Phi_n(z), \tag{4.12}$$

where f is the associated analytic function in $\mathbb{D}$ determined by (4.11).

Hence, we obtain the following corollary.

Corollary 4.1: If $F(z)$ belongs to $\mathcal{A}(E_r)$ then the coefficient formula (4.8) is given by

$$A_n(f) = \frac{r^{n-1}}{K\sqrt{k_0}} \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) \cos n\theta d\theta,\ n = 0, 1, 2, \cdots \tag{4.13}$$

[Haliloglu and Johnston, 2005].

Another representation formula for the Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, is given in the following corollary.

Corollary 4.2: The Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, of functions belonging to $\mathcal{A}(E_r)$ are given by

$$A_n(f) = \frac{2^n n! r^{n-1}}{K\sqrt{k_0}(2n)!} \int_0^{\pi} \left(f(\varphi(x))\right)^{(n)} \Big|_{x=\frac{2\cos\theta}{r}} \sin^{2n}\theta d\theta \tag{4.14}$$

for $n = 0, 1, 2, \cdots$ [Haliloglu and Johnston, 2005].

The following result is about the Faber coefficients of functions in $S^{(2)}(E_r)$.

Corollary 4.3: If $F(z) \in S^{(2)}(E_r)$, then $A_{2n}(f) = 0$, $n = 0, 1, 2, \cdots$ [Haliloglu and Johnston, 2005].

Let $\mathcal{F}$ denote one of the classes S^*, K and $R(\alpha)$, respectively. Since $\mathcal{F}$ is a compact set, the closed convex hull of $\mathcal{F}$, $\overline{\text{co}}(\mathcal{F})$ is also compact. The number

$$M = \max_{f \in \overline{\text{co}}(\mathcal{F})} |A_n(f)| \tag{4.15}$$

exists, because $A_n(f)$ is a continuous linear functional. According to the Krein Milman Theorem, we have

$$\max_{f \in \mathcal{F}} |A_n(f)| = \max_{\text{ext}(\overline{\text{co}}(\mathcal{F}))} |A_n(f)|, \tag{4.16}$$

where ext $(\overline{\text{co}}(\mathcal{F}))$ denotes the extreme points of $\overline{\text{co}}(\mathcal{F})$.

The upper bounds for Faber coefficients of functions belonging to the classes $C(E_r)$, $T(E_r)$ and $P(E_r)$ were obtained in [Haliloglu, 1997]. According to equality (4.16), the extreme points of closed convex hulls of these subclasses are important for obtaining maximum of $|A_n(f)|$.

The extreme points of $\overline{\text{co}}(C)$ and $\overline{\text{co}}(T)$ are determined in [Brickman et al., 1971] as follows

$$\text{ext}\,(\overline{\text{co}}\,(C)) = \{f : f(z) = c_\theta(z),\ 0 \le \theta < 2\pi\} \tag{4.17}$$

and

$$\text{ext}\,(\overline{\text{co}}\,(T)) = \{f : f(z) = t_\theta(z),\ 0 \le \theta \le \pi\}, \tag{4.18}$$

where $c_\theta(z)$ and $t_\theta(z)$ are given by

$$c_\theta(z) = \frac{z}{1 - e^{i\theta}z} \tag{4.19}$$

and

$$t_\theta(z) = \frac{z}{1 - 2z\cos\theta + z^2}, \tag{4.20}$$

respectively. The extreme points of $\overline{\text{co}}(P)$ are given in the article [Brannan et al., 1973] by

$$\text{ext}\,(\overline{\text{co}}\,(P)) = \{f : f(z) = p_\theta(z),\ 0 \le \theta < 2\pi\}, \tag{4.21}$$

where

$$p_\theta(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z}. \tag{4.22}$$

In this dissertation, we get sharp upper bounds for the Faber coefficients of functions in $S^*(E_r)$, $K(E_r)$ and $R_\alpha(E_r)$, respectively. Therefore, we utilize the extreme points of $\overline{\text{co}}\,(S^*)$, $\overline{\text{co}}\,(K)$ and $\overline{\text{co}}\,(R_\alpha)$, respectively. The extreme points of $\overline{\text{co}}\,(S^*)$ are given in the article [Brickman et al., 1971] as

$$\text{ext}\,(\overline{\text{co}}\,(S^*)) = \{f : f(z) = k_\theta(z),\ 0 \le \theta < 2\pi\}\,, \tag{4.23}$$

where

$$k_\theta(z) = \frac{z}{(1 - e^{i\theta} z)^2}. \tag{4.24}$$

The extreme points of $\overline{\text{co}}\,(K)$ can be found in [Grasmann et al., 1976] as

$$\text{ext}\,(\overline{\text{co}}\,(K)) = \{f : f(z) = K(z;\theta,\alpha)\ \alpha,\ \theta \in [0, 2\pi)\,, \theta \neq \alpha\}\,, \tag{4.25}$$

where

$$K(z;\theta,\alpha) = \frac{z - \frac{1}{2}\left(e^{i\theta} + e^{i\alpha}\right) z^2}{(1 - e^{i\theta} z)^2}. \tag{4.26}$$

Finally, the extreme points of $\overline{\text{co}}\,(R_\alpha)$ are determined in [Hallenbeck and MacGregor, 1984] as

$$\text{ext}\,(\overline{\text{co}}\,(R_\alpha)) = \{f : f(z) = g(z;\theta,\alpha)\ \theta \in [0, 2\pi)\,, \alpha \in [0, 1)\}\,, \tag{4.27}$$

where

$$g(z;\theta,\alpha) = z\left(\frac{1 + (1 - 2\alpha)\, e^{i\theta} z}{1 - e^{i\theta} z}\right)^2. \tag{4.28}$$

Note that, $\text{ext}\left(\overline{\text{co}}\left(R\left(\frac{1}{2}\right)\right) \cap S\right) = \text{ext}\,(\overline{\text{co}}\,(S^*))$.

According to the equality (4.16), the problem of finding maximum value of $|A_n\,(f)|$ over the classes S^*, K and R_α means finding maximum of $|A_n\,(k_\theta)|$, $|A_n\,(K(\theta,\alpha))|$ and $|A_n\,(g(\theta,\alpha))|$ over $\theta \in [0, 2\pi)$, respectively. Firstly, we evaluate these coefficients explicitly. Then, we determine the maximum values of them. We show that equality occurs only in two cases which is the number of invariant rotations of E_r.

4.2. Auxiliary Results

We mention some earlier results, before our main results. In [Haliloglu, 1997], coefficient bounds for analytic functions belonging to certain subclasses of $\mathcal{A}(E_r)$ are obtained.

Faber coefficients of the extreme points of convex functions are given in the next four theorems.

Theorem 4.1: If $c_\theta(z)$ *is given by (4.19), then for* $0 \leq \theta \leq \frac{\pi}{2}$ *and* $n = 0, 1, 2, \cdots$

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta} \left\{ e^{in\alpha(\theta)} - r^{-2n} e^{-in\alpha(\theta)} \right\}}{2rK^2\sqrt{k_0}\left(1 - r^{-4n}\right)\left[(1-k_0)^2 + 4k_0 \sin^2\theta\right]^{\frac{1}{2}}}, \tag{4.29}$$

where $0 \leq \alpha(\theta) \leq \frac{\pi}{2}$ *is given by*

$$\varphi\left[\frac{2}{r}\cos\left(\alpha(\theta) + \frac{\pi\tau}{4}\right)\right] = e^{-i\theta},\ 0 \leq \theta \leq \frac{\pi}{2} \text{ with } \tau = \frac{4i\ln r}{\pi} \tag{4.30}$$

[Haliloglu, 1997].

Theorem 4.2: If $c_\theta(z)$ *is given by (4.19), then for* $\frac{\pi}{2} \leq \theta \leq \pi$ *and* $n = 0, 1, 2, \cdots$

$$A_n(c_\theta) = \frac{(-1)^n \pi^2 e^{-i\theta} \left\{ e^{-in\alpha(\pi-\theta)} - r^{-2n} e^{in\alpha(\pi-\theta)} \right\}}{2rK^2\sqrt{k_0}\left(1 - r^{-4n}\right)\left[(1-k_0)^2 + 4k_0 \sin^2\theta\right]^{\frac{1}{2}}}, \tag{4.31}$$

where $\alpha(\theta)$ *is as in Theorem 4.1. [Haliloglu, 1997].*

Theorem 4.3: If $c_\theta(z)$ *is given by (4.19), then for* $\pi \leq \theta \leq \frac{3\pi}{2}$ *and* $n = 0, 1, 2, \cdots$

$$A_n(c_\theta) = \frac{(-1)^n \pi^2 e^{-i\theta} \left\{ e^{in\alpha(\theta-\pi)} - r^{-2n} e^{-in\alpha(\theta-\pi)} \right\}}{2rK^2\sqrt{k_0}\left(1 - r^{-4n}\right)\left[(1-k_0)^2 + 4k_0 \sin^2\theta\right]^{\frac{1}{2}}}, \tag{4.32}$$

where $\alpha(\theta)$ *is as in Theorem 4.1. [Haliloglu, 1997].*

Theorem 4.4: If $c_\theta(z)$ *is given by (4.19), then for* $\frac{3\pi}{2} \leq \theta \leq 2\pi$ *and* $n = 0, 1, 2, \cdots$

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta} \left\{ e^{-in\alpha(2\pi-\theta)} - r^{-2n} e^{in\alpha(2\pi-\theta)} \right\}}{2rK^2\sqrt{k_0}\left(1 - r^{-4n}\right)\left[(1-k_0)^2 + 4k_0 \sin^2\theta\right]^{\frac{1}{2}}}, \tag{4.33}$$

where $\alpha(\theta)$ is as in Theorem 4.1. [Haliloglu, 1997].

The following theorem gives sharp upper bounds for Faber coefficients of convex functions in E_r.

Theorem 4.5: If $f \in C$ and

$$c(z) = c_0(z) = \frac{z}{1-z}, \tag{4.34}$$

then for each $r > 1$

$$|A_n(f)| \leq A_n(c) = \frac{\pi^2}{2rK^2\sqrt{k_0}\,(1-k_0)\,(1+r^{-2n})}, \tag{4.35}$$

$n = 0, 1, 2, \cdots$. Equality occurs only for the functions $f(z) = c(z)$ and $f(z) = -c(-z)$ [Haliloglu, 1997].

Sharp upper bounds for the Faber coefficients of typically real functions in E_r is given in the following theorem.

Theorem 4.6: If $f \in T$ and $k(z)$ is the Koebe function given by (2.8), then for each $r > 1$,

$$|A_n(f)| \leq A_n(k) = \frac{\pi^3 n}{4rK^3\sqrt{k_0}\,(1-k_0)^2\,(1-r^{-2n})}, \tag{4.36}$$

$n = 1, 2, 3 \cdots$. Equality occurs only for the functions $f(z) = k(z)$ and $f(z) = -k(-z)$ [Haliloglu, 1997].

Finally, the next theorem gives sharp upper bounds for the Faber coefficients of functions in the class $P(E_r)$.

Theorem 4.7: If $f \in P$ and $c(z)$ as in Theorem 4.5, then for each $r > 1$,

$$|A_n(f)| \leq 2A_n(c), \tag{4.37}$$

$n = 0, 1, 2, \cdots$. Equality occurs only for the functions $f(z) = p(z)$ and $f(z) = p(-z)$ where

$$p(z) = p_0(z) = \frac{1+z}{1-z} \tag{4.38}$$

[Haliloglu, 1997].

4.3. Coefficient Estimates For Starlike Functions in an Ellipse

In this part, the Faber coefficients of extreme points of starlike functions in E_r are evaluated explicitly. In order to evaluate $A_n(k_\theta)$ we use different contours for different quadrants of θ. So, we write a theorem for each quadrant of θ. After that, sharp upper bounds for the Faber coefficients of starlike functions in E_r are given.

Theorem 4.8: If $k_\theta(z)$ is given by (4.24), then for $0 \le \theta \le \frac{\pi}{2}$ and $n = 0, 1, 2, \cdots$, we have

$$A_n(k_\theta) = \frac{i\pi^2 e^{-i\theta}\sqrt{k_0}\sin 2\theta \left\{e^{in\alpha(\theta)} - r^{-2n}e^{-in\alpha(\theta)}\right\}}{rK^2(1-r^{-4n})\left[(1-k_0)^2 + 4k_0\sin^2\theta\right]^{\frac{3}{2}}} + \frac{n\pi^3 e^{-i\theta}\left\{e^{in\alpha(\theta)} + r^{-2n}e^{-in\alpha(\theta)}\right\}}{4rK^3\sqrt{k_0}(1-r^{-4n})\left[(1-k_0)^2 + 4k_0\sin^2\theta\right]}, \tag{4.39}$$

where $0 \le \alpha(\theta) \le \frac{\pi}{2}$ is given by

$$\varphi\left[\frac{2}{r}\cos\left(\alpha(\theta) + \frac{\pi\tau}{4}\right)\right] = e^{-i\theta},\ 0 \le \theta \le \frac{\pi}{2} \text{ with } \tau = \frac{4i\ln r}{\pi}. \tag{4.40}$$

Proof 4.8: We know that $\varphi\left(\frac{2\cos z}{r}\right)$ maps the rectangle R which has the vertices at the points $-\frac{\pi\tau}{4}, \pi - \frac{\pi\tau}{4}, \pi + \frac{\pi\tau}{4}$ and $\frac{\pi\tau}{4}$ onto $\mathbb{D}$ with

$$\varphi\left(\frac{2}{r}\cos\left(\alpha(\theta) + \frac{\pi\tau}{4}\right)\right) = e^{-i\theta},\ 0 \le \theta \le \frac{\pi}{2}, \tag{4.41}$$

where $\alpha(\theta)$ is an increasing function on $\left[0, \frac{\pi}{2}\right]$, with $\alpha(0) = 0$ and $\alpha\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.

We consider the integral of $h(z) = k_\theta\left(\varphi\left(\frac{2\cos z}{r}\right)\right)e^{inz}$ over the parallelogram $ABCD$ which has the vertices at the points $-\pi, \pi, \pi\tau$ and $\pi\tau - 2\pi$, respectively. From the equality (4.41), $h(z)$ has a pole at $z = \alpha(\theta) + \frac{\pi\tau}{4}$ inside $ABCD$.

We use snz *as abbreviation for* $sn\left(z;\frac{1}{r^4}\right)$. *Since* snz *has double periods* $2iK'$ *and* $4K$, *we have*

$$\varphi\left(\frac{2}{r}\cos\left(\pi\tau-z\right)\right)=\sqrt{k_0}sn\left(\frac{2K}{\pi}\left(\frac{\pi}{2}-\pi\tau+z\right)\right) \tag{4.42}$$

and

$$\varphi\left(\frac{2}{r}\cos\left(\pi\tau-z\right)\right)=\sqrt{k_0}sn\left(\frac{2K}{\pi}\left(\frac{\pi}{2}+z\right)\right). \tag{4.43}$$

Therefore, we have the relations

$$\varphi\left(\frac{2\cos z}{r}\right)=\varphi\left(\frac{2\cos\left(-z\right)}{r}\right)=\varphi\left(\frac{2\cos\left(\pi\tau-z\right)}{r}\right). \tag{4.44}$$

From (4.44), it is obtained that $z=-\alpha\left(\theta\right)+\frac{3\pi\tau}{4}$ *is the other pole of* $h(z)$ *inside* $ABCD$. *By the residue theorem, we have*

$$\oint\limits_{ABCD} h\left(z\right)dz=2\pi i\left\{\textbf{\textit{Res}}\left(h(z);\alpha\left(\theta\right)+\frac{\pi\tau}{4}\right)\right.$$

$$\left.+\textbf{\textit{Res}}\left(h(z);-\alpha\left(\theta\right)+\frac{3\pi\tau}{4}\right)\right\}, \tag{4.45}$$

where ***Res*** $\left(h(z);z_0\right)$ *denotes the residue of the function* $h(z)$ *at* $z=z_0$. *Since,* $h(z)$ *has a period* 2π, *we can write*

$$\int\limits_{BC} h\left(z\right)dz+\int\limits_{DA} h\left(z\right)dz=0. \tag{4.46}$$

On the other hand, the integrals over other sides of $ABCD$ *can be calculated as*

$$\int\limits_{AB} h\left(z\right)dz=\int\limits_{-\pi}^{\pi} h\left(x\right)dx=2\int\limits_{0}^{\pi} k_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)\cos nxdx \tag{4.47}$$

and

$$\int_{CD} h(z)\,dz = \int_{2\pi}^{0} h(x+\pi\tau-2\pi)\,dx = -\int_{0}^{2\pi} h(x+\pi\tau)\,dx \tag{4.48}$$

By using the equality (4.44), we obtain

$$\int_{CD} h(z)\,dz = -2r^{-4n}\int_{0}^{\pi} k_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)\cos nx dx. \tag{4.49}$$

Then, adding (4.47) and (4.49) yields

$$\oint_{ABCD} h(z)dz = 2\left(1-r^{-4n}\right)\int_{0}^{\pi} k_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)\cos nx dx. \tag{4.50}$$

To evaluate $Res\left(h(z);\alpha(\theta)+\frac{\pi\tau}{4}\right)$, *it is necessary to expand the function* $k_\theta\left(\sqrt{k_0}sn\,(u+u_0)\right)$ *about* $u=0$, *where* $u_0=\frac{2K}{\pi}\left(\frac{\pi}{2}-\alpha(\theta)-\frac{\pi\tau}{4}\right)$. *By using the addition formula for* snu, *we have*

$$\sqrt{k_0}sn\,(u+u_0) = \frac{\sqrt{k_0}snucnu_0dnu_0+\sqrt{k_0}snu_0cnudnu}{1-k_0^2sn^2u_0sn^2u}, \tag{4.51}$$

where cnz *and* dnz *are used for the abbreviations for* $cn\left(z;\frac{1}{r^4}\right)$ *and* $dn\left(z;\frac{1}{r^4}\right)$, *respectively. From (4.41), it is clear that*

$$\sqrt{k_0}snu_0 = e^{-i\theta}\ ,\ 0\le\theta\le\frac{\pi}{2}. \tag{4.52}$$

To find the values of cnu_0 *and* dnu_0 *we use the following identities*

$$sn^2z+cn^2z=1 \tag{4.53}$$

and

$$k_0^2sn^2z+dn^2z=1. \tag{4.54}$$

The sign of cnu_0 *and* dnu_0 *is determined by checking the signs of Re* $\left\{cn\left(x-\frac{iK'}{2}\right)\right\}$ *and Re* $\left\{dn\left(x-\frac{iK'}{2}\right)\right\}$, *respectively. The sum formulas for* cnu *and* dnu *give that*

$$cn\left(x-\frac{iK'}{2}\right)=\sqrt{\frac{1+k_0}{k_0}}\,\frac{cnx+isn^2xdnx}{1+k_0^2sn^2x} \tag{4.55}$$

$$dn\left(x-\frac{iK'}{2}\right)=\sqrt{1+k_0}\,\frac{dnx+ik_0sn^2xcnx}{1+k_0^2sn^2x}. \tag{4.56}$$

Since, cnx *decreases from 1 to 0 and* dnx *decreases from* 1 *to* $\sqrt{1-k_0^2}$ *for* $x\in[0,K]$, *it follows from (4.55) and (4.56) that Re* $\left\{cn\left(x-\frac{iK'}{2}\right)\right\}\geq 0$ *and Re* $\left\{dn\left(x-\frac{iK'}{2}\right)\right\}\geq 0$. *By using the relations in (4.53) and (4.54), one can obtain that*

$$cnu_0=\sqrt{1-\frac{e^{-2i\theta}}{k_0}} \tag{4.57}$$

and

$$dnu_0=\sqrt{1-k_0e^{-2i\theta}}. \tag{4.58}$$

If we choose the principal branch as $-\pi<\arg z\leq\pi$, *we get*

$$0\leq\arg\left(cnu_0\right)\leq\frac{\pi}{2} \tag{4.59}$$

and

$$0\leq\arg\left(dnu_0\right)\leq\frac{\pi}{2}. \tag{4.60}$$

Therefore

$$0\leq\arg\left(cnu_0dnu_0\right)\leq\pi \tag{4.61}$$

which implies that

$$\sqrt{k_0}cnu_0dnu_0=ie^{-i\theta}\left(1+k_0^2-2k_0\cos 2\theta\right)^{\frac{1}{2}}. \tag{4.62}$$

Using the expansions

$$sn u = u - \frac{1}{3!}\left(1+k_0^2\right)u^3 + \cdots \tag{4.63}$$

$$cn u = 1 - \frac{1}{2!}u^2 + \cdots \tag{4.64}$$

$$dn u = 1 - \frac{1}{2!}k_0^2 u^2 + \cdots \tag{4.65}$$

and (4.62) in (4.51) and then doing necessary calculations, we have

$$\sqrt{k_0}sn\left(u+u_0\right) = e^{-i\theta}\left\{1 + i\sqrt{1+k_0^2-2k_0\cos 2\theta}\, u \right.$$

$$\left. + \left(k_0 e^{-2i\theta} - \frac{1+k_0^2}{2}\right)u^2 + \cdots\right\}. \tag{4.66}$$

Thus

$$k_\theta\left(\sqrt{k_0}sn\left(u+u_0\right)\right) = \frac{-e^{-i\theta}}{\left(1+k_0^2-2k_0\cos 2\theta\right)u^2}$$

$$\times\left\{1 + \frac{2k_0\sin 2\theta}{\sqrt{1+k_0^2-2k_0\cos 2\theta}}u + \cdots\right\} \tag{4.67}$$

or

$$k_\theta\left(\sqrt{k_0}sn\left(\frac{2K}{\pi}\left(\frac{\pi}{2}-z\right)\right)\right) = \frac{-\pi^2 e^{-i\theta}}{4K^2\left(1+k_0^2-2k_0\cos 2\theta\right)\left(z-\alpha\left(\theta\right)-\frac{\pi\tau}{4}\right)^2}$$

$$\times\left\{1 - \frac{4Kk_0\sin 2\theta}{\pi\sqrt{1+k_0^2-2k_0\cos 2\theta}}\left(z-\alpha\left(\theta\right)-\frac{\pi\tau}{4}\right) + \cdots\right\}. \tag{4.68}$$

Hence, the residue at $z = \alpha(\theta) + \frac{\pi\tau}{4}$ is obtained as

$$Res\left(h(z); \alpha(\theta) + \frac{\pi\tau}{4}\right) = \frac{-\pi^2 e^{-i\theta} e^{in\alpha(\theta)} r^{-n}}{4K^2\left(1 + k_0^2 - 2k_0 \cos 2\theta\right)} \times \left\{ in - \frac{4Kk_0 \sin 2\theta}{\pi\sqrt{1 + k_0^2 - 2k_0 \cos 2\theta}} \right\}. \quad (4.69)$$

In similar way, we obtain that

$$Res\left(h(z); -\alpha(\theta) + \frac{3\pi\tau}{4}\right) = \frac{-\pi^2 e^{-i\theta} e^{-in\alpha(\theta)} r^{-3n}}{4K^2\left(1 + k_0^2 - 2k_0 \cos 2\theta\right)} \times \left\{ in + \frac{4Kk_0 \sin 2\theta}{\pi\sqrt{1 + k_0^2 - 2k_0 \cos 2\theta}} \right\}. \quad (4.70)$$

Substituting (4.69) and (4.70) into (4.45) yields that

$$\oint_{ABCD} h(z)dz = \frac{2i\pi^2 e^{-i\theta} r^{-n} k_0 \sin 2\theta}{K\left(1 + k_0^2 - 2k_0 \cos 2\theta\right)^{\frac{3}{2}}} \left\{ r^{-2n} e^{-in\alpha(\theta)} - e^{in\alpha(\theta)} \right\} + \frac{n\pi^3 e^{-i\theta} r^{-n}}{2K^2\left(1 + k_0^2 - 2k_0 \cos 2\theta\right)} \left\{ e^{in\alpha(\theta)} + r^{-2n} e^{-in\alpha(\theta)} \right\} \quad (4.71)$$

Comparing (4.50) and (4.71), we obtain the desired result. ■

The next three theorems give the values of $A_n(k_\theta)$ for the values of θ in the second, third and fourth quadrants, respectively. The proofs are outlined since they are similar to the proof of Theorem 4.8.

Theorem 4.9: If $k_\theta(z)$ is given by (4.24), then for $\frac{\pi}{2} \leq \theta \leq \pi$ and $n = 0, 1, 2, \cdots$

$$A_n(k_\theta) = \frac{i\pi^2(-1)^n e^{-i\theta}\sqrt{k_0} \sin 2\theta \left\{ e^{-in\alpha(\pi-\theta)} - r^{-2n} e^{in\alpha(\pi-\theta)} \right\}}{rK^2\left(1 - r^{-4n}\right)\left((1 - k_0)^2 + 4k_0 \sin^2\theta\right)^{\frac{3}{2}}} + \frac{n\pi^3(-1)^n e^{-i\theta}\left\{ e^{-in\alpha(\pi-\theta)} + r^{-2n} e^{in\alpha(\pi-\theta)} \right\}}{4rK^3\sqrt{k_0}\left(1 - r^{-4n}\right)\left((1 - k_0)^2 + 4k_0 \sin^2\theta\right)}, \quad (4.72)$$

where $\alpha(\theta)$ is as in Theorem 4.8.

Proof 4.9: Consider the integral of function $h(z) = k_\theta\left(\varphi\left(\frac{2\cos z}{r}\right)\right)e^{inz}$ *over the parallelogram* $ABCD$ *with vertices at the points* $0,\ 2\pi, 3\pi+\pi\tau,$ *and* $\pi+\pi\tau,$ *respectively. It is obvious that* $h(z)$ *has two poles at the points* $\pi-\alpha(\pi-\theta)+\frac{\pi\tau}{4}$ *and* $\pi+\alpha(\pi-\theta)+\frac{3\pi\tau}{4}$ *inside this parallelogram.* ■

Theorem 4.10: If $k_\theta(z)$ *is given by (4.24), then for* $\pi\le\theta\le\frac{3\pi}{2}$ *and* $n=0,1,2,\cdots$

$$A_n(k_\theta) = \frac{i\pi^2(-1)^n e^{-i\theta}\sqrt{k_0}\sin 2\theta\left\{e^{in\alpha(\theta-\pi)} - r^{-2n}e^{-in\alpha(\theta-\pi)}\right\}}{rK^2(1-r^{-4n})\left((1-k_0)^2+4k_0\sin^2\theta\right)^{\frac{3}{2}}}$$

$$-\frac{n\pi^3(-1)^n e^{-i\theta}\left\{e^{in\alpha(\theta-\pi)} + r^{-2n}e^{-in\alpha(\theta-\pi)}\right\}}{4rK^3\sqrt{k_0}(1-r^{-4n})\left((1-k_0)^2+4k_0\sin^2\theta\right)}, \quad (4.73)$$

where $\alpha(\theta)$ *is as in Theorem 4.8.*

Proof 4.10: Consider the integral of function $h(z) = k_\theta\left(\varphi\left(\frac{2\cos z}{r}\right)\right)e^{inz}$ *over the parallelogram* $ABCD$ *with vertices at the points* $0,\ 2\pi, 3\pi-\pi\tau,$ *and* $\pi-\pi\tau,$ *respectively. Inside* $ABCD$ *there are two poles of* $h(z)$*, at the points* $\pi-\alpha(\pi-\theta)-\frac{\pi\tau}{4}$ *and* $\pi+\alpha(\pi-\theta)-\frac{3\pi\tau}{4}$. ■

Theorem 4.11: If $k_\theta(z)$ *is given by (4.24), then for* $\frac{3\pi}{2}\le\theta\le 2\pi$ *and* $n=0,1,2,\cdots$

$$A_n(k_\theta) = \frac{i\pi^2 e^{-i\theta}\sqrt{k_0}\sin 2\theta\left\{e^{-in\alpha(2\pi-\theta)} - r^{-2n}e^{in\alpha(2\pi-\theta)}\right\}}{rK^2(1-r^{-4n})\left((1-k_0)^2+4k_0\sin^2\theta\right)^{\frac{3}{2}}}$$

$$-\frac{n\pi^3 e^{-i\theta}\left\{e^{-in\alpha(2\pi-\theta)} + r^{-2n}e^{in\alpha(2\pi-\theta)}\right\}}{4rK^3\sqrt{k_0}(1-r^{-4n})\left((1-k_0)^2+4k_0\sin^2\theta\right)}, \quad (4.74)$$

where $\alpha(\theta)$ *is as in Theorem 4.8.*

Proof 4.11: We need to calculate the integral of function $h(z) = k_\theta\left(\varphi\left(\frac{2\cos z}{r}\right)\right)e^{inz}$ *over the parallelogram* $ABCD$ *with vertices at the points* $-\pi,\ \pi,\ -\pi\tau$ *and* $-2\pi-\pi\tau,$ *respectively. We obtain* $h(z)$ *has two poles at the points* $\alpha(2\pi-\theta)-\frac{\pi\tau}{4}$ *and* $-\alpha(2\pi-\theta)-\frac{3\pi\tau}{4}$ *inside* $ABCD$. ■

The next theorem gives sharp upper bounds for Faber coefficients of functions starlike in E_r.

Theorem 4.12: If $f \in S^*$ and $k(z) = \frac{z}{(1-z)^2}$, then

$$|A_n(f)| \leq A_n(k) = \frac{n\pi^3}{4rK^3\sqrt{k_0}\left(1-r^{-2n}\right)\left(1-k_0\right)^2},\ n = 0, 1, 2, \cdots. \tag{4.75}$$

Equality holds only for the functions $f(z) = k(z)$ and $f(z) = -k(-z)$.

Proof 4.12: By using (4.16) and (4.23), we need to show that $|A_n(k_\theta)|$ attains its maximum only for $\theta = 0$ and $\theta = \pi$ where $\theta \in [0, 2\pi)$. We will give the proof only for $\theta \in \left[0, \frac{\pi}{2}\right]$. The proof is similar for θ in other quadrants. Since $k(z) = \frac{z}{(1-z)^2}$ is a starlike function, the case $n = 0$ follows from Theorem 1 in [Haliloglu and Johnston, 2005]. According to Theorem 4.1, we have

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta}\left\{e^{in\alpha(\theta)} - r^{-2n}e^{-in\alpha(\theta)}\right\}}{2rK^2\sqrt{k_0}\left(1-r^{-4n}\right)\sqrt{\left(1-k_0\right)^2 + 4k_0\sin^2\theta}} \tag{4.76}$$

where $c_\theta(z)$ is given by (4.19). It follows from Theorem 4.8 that

$$A_n(k_\theta) = \frac{A_n(c_\theta)}{\sqrt{\left(1-k_0\right)^2 + 4k_0\sin^2\theta}}\left\{\frac{2ik_0\sin 2\theta}{\sqrt{\left(1-k_0\right)^2 + 4k_0\sin^2\theta}}\right.$$

$$\left. + \frac{n\pi}{2K}\,\frac{e^{in\alpha(\theta)} + r^{-2n}e^{-in\alpha(\theta)}}{e^{in\alpha(\theta)} - r^{-2n}e^{-in\alpha(\theta)}}\right\}. \tag{4.77}$$

After some elementary calculations, one obtains that

$$A_n(k_\theta) = \frac{A_n(c_\theta)}{\sqrt{\left(1-k_0\right)^2 + 4k_0\sin^2\theta}}\left\{g(\theta) + f(\theta)\right\}, \tag{4.78}$$

where

$$g(\theta) = \frac{n\pi}{2K}\,\frac{1-r^{-4n}}{1+r^{-4n}-2r^{-2n}\cos\left(2n\alpha(\theta)\right)}, \tag{4.79}$$

$$f(\theta) = 2i\left\{\frac{k_0\sin 2\theta}{\sqrt{\left(1-k_0\right)^2 + 4k_0\sin^2\theta}} - \frac{n\pi}{2K}\,\frac{r^{-2n}\sin\left(2n\alpha(\theta)\right)}{1+r^{-4n}-2r^{-2n}\cos\left(2n\alpha(\theta)\right)}\right\}. \tag{4.80}$$

Applying the triangle inequality, we get

$$|A_n(k_\theta)| \le \frac{|A_n(c_\theta)|}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} \{|g(\theta)| + |f(\theta)|\}. \tag{4.81}$$

In the inequality

$$|g(\theta) + f(\theta)| \le |g(\theta)| + |f(\theta)| \tag{4.82}$$

equality occurs if and only if

$$Re\left(\overline{g(\theta)} f(\theta)\right) = |g(\theta)| \, |f(\theta)|. \tag{4.83}$$

Considering the fact that $g(\theta) > 0$ *for* $\theta \in [0, 2\pi)$, *we obtain that equality holds if and only if*

$$Re(f(\theta)) = |f(\theta)|. \tag{4.84}$$

Since $f(\theta)$ *is pure imaginary it follows that* $f(\theta) = 0$. *Hence,*

$$|A_n(k_\theta)| \le \frac{|A_n(c_\theta)|}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} |g(\theta)|. \tag{4.85}$$

From (4.35) and $|g(\theta)| \le g(0) = g(\pi)$ *we obtain the inequality*

$$|A_n(k_\theta)| \le \frac{n\pi(1 + r^{-2n})}{2K(1-k_0)(1 - r^{-2n})} A_n(c) = A_n(k) \tag{4.86}$$

which completes the proof. ■

4.4. Coefficient Estimates For Close-to-Convex Functions In an Ellipse

In this section, we address the coefficient problem for close-to-convex functions in E_r. We calculate the functional $A_n(f)$ over the extreme points of close-to-convex functions in terms of $A_n(c_\theta)$ and $A_n(k_\theta)$. Then, we give sharp upper bounds for these coefficients.

Theorem 4.13: If $K(z;\theta,\alpha)$ is given by (4.26), then for $0 \leq \theta \leq 2\pi$, $0 \leq \alpha \leq 2\pi$ and $\theta \neq \alpha$

$$A_n(K(\theta,\alpha)) = \frac{\left(1 - e^{i(\alpha-\theta)}\right)}{2} A_n(k_\theta) + \frac{\left(1 + e^{i(\alpha-\theta)}\right)}{2} A_n(c_\theta). \tag{4.87}$$

Proof 4.13: Since the functional

$$A_n(K(\theta,\alpha)) = \frac{r^{n-1}}{K\sqrt{k_0}} \int_0^\pi K\left(\frac{2\varphi(\cos x)}{r};\theta,\alpha\right) \cos nx dx. \tag{4.88}$$

is linear and the function $K(z;\theta,\alpha)$ is expressed by

$$K(z;\theta,\alpha) = \frac{\left(1 - e^{i(\alpha-\theta)}\right)}{2} k_\theta(z) + \frac{\left(1 + e^{i(\alpha-\theta)}\right)}{2} c_\theta(z), \tag{4.89}$$

we obtain the desired result. ■

In the following theorem, we obtain sharp upper bounds for Faber coefficients of close-to-convex functions in E_r.

Theorem 4.14: If $f \in K$ and $k(z) = \frac{z}{(1-z)^2}$, then

$$|A_n(f)| \leq A_n(k) = \frac{n\pi^3}{4rK^3\sqrt{k_0}\left(1 - r^{-2n}\right)\left(1 - k_0\right)^2},\ n = 0, 1, 2, \cdots. \tag{4.90}$$

Equality holds only for the functions $f(z) = k(z)$ and $f(z) = -k(-z)$.

Proof 4.14: According to (4.16) and (4.25), it suffices to show that $|A_n(K(\theta,\alpha))|$ attains its maximum value only for $\theta = 0$ and $\theta = \pi$. The case $n = 0$ is obvious from Theorem 1 in [Haliloglu and Johnston, 2005], since $k(z) = \frac{z}{(1-z)^2}$ is also a close-to-convex function. From Theorem 4.13 and the triangle inequality, we have

$$|A_n(K(\theta,\alpha))| \leq \left|\frac{1 - e^{i(\alpha-\theta)}}{2}\right| |A_n(k_\theta)| + \left|\frac{1 + e^{i(\alpha-\theta)}}{2}\right| |A_n(c_\theta)|. \tag{4.91}$$

Equality holds in (4.91) if and only if

$$|\sin(\alpha-\theta)|\,|A_n(c_\theta)|\,|A_n(k_\theta)| = \sin(\alpha-\theta)\, \mathit{Im}\left(\overline{A_n(c_\theta)}A_n(k_\theta)\right). \tag{4.92}$$

It is possible for either $\sin(\alpha-\theta)=0$ *or* $\overline{A_n(c_\theta)}A_n(k_\theta)$ *is pure imaginary with*

$$\sin(\alpha-\theta)\, \mathit{Im}\left(\overline{A_n(c_\theta)}A_n(k_\theta)\right) \geq 0. \tag{4.93}$$

From (4.78), we obtain that

$$\overline{A_n(c_\theta)}A_n(k_\theta) = \frac{|A_n(c_\theta)|^2}{\sqrt{(1-k_0)^2+4k_0\sin^2\theta}}\{g(\theta)+f(\theta)\} \tag{4.94}$$

and $g(\theta) > 0$, *for* $n = 1, 2, \cdots$ *and* $\theta \in [0, 2\pi)$, *imply that* $\overline{A_n(c_\theta)}A_n(k_\theta)$ *can not be pure imaginary. Therefore, we conclude that* $\sin(\alpha-\theta) = 0$ *which implies that* $\alpha = \theta+\pi$ *or* $\alpha = \theta-\pi$. *So, we obtain the following inequality*

$$|A_n(K(\theta,\alpha))| \leq |A_n(K(\theta,\theta+\pi))| = |A_n(K(\theta,\theta-\pi))|. \tag{4.95}$$

From (4.75), we get the following inequality

$$|A_n(K(\theta,\alpha))| \leq |A_n(k_\theta)| \leq A_n(k). \tag{4.96}$$

This completes the proof. ■

4.5. Coefficient Estimates for a Certain Subclass of Analytic Functions in an Ellipse

In this section, we consider the coefficient problem for functions in the class $R_\alpha(E_r)$. First of all, we calculate $A_n(f)$ over the extreme points of $R(\alpha)$ related with the coefficients $A_n(k_\theta)$ and $A_n(c_\theta)$. After that, sharp upper bounds for $|A_n(g(\theta,\alpha))|$ is also determined.

Before we give our coefficient estimates, we need the following lemma.

Lemma 4.2: The Faber coefficients of $F(z) = \frac{\varphi(z)}{\varphi'(0)}$ are given by

$$A_n(z) = -\frac{2\pi}{r^{n+1}Kk_0^{\frac{3}{2}}\left(1 - r^{-4n}\right)}\sin\left(\frac{n\pi}{2}\right),\ n = 0, 1, 2, \cdots. \tag{4.97}$$

Proof 4.2: According to formula in (4.13), we need to calculate the integral

$$A_n(z) = \frac{r^{n-1}}{K\sqrt{k_0}}\int_0^{\pi}\varphi\left(\frac{2\cos x}{r}\right)\cos nx dx,\ n = 0, 1, 2, \cdots. \tag{4.98}$$

We consider the integral of

$$h(z) = \varphi\left(\frac{2\cos z}{r}\right)e^{inz} \tag{4.99}$$

over the rectangle $ABCD$ with vertices at the points $-\pi$, π, $\pi + \pi\tau$ and $-\pi + \pi\tau$, respectively. Note that $\tau = \frac{4i\ln r}{\pi}$. It is known that snz has poles at the points

$$z = 2mK + i\left(2n + 1\right)K',\ m, n \in \mathbb{Z} \tag{4.100}$$

Therefore, the pole of

$$h(z) = \varphi\left(\frac{2\cos z}{r}\right)e^{inz} = \sqrt{k_0}sn\left(K - \frac{2Kz}{\pi}\right)e^{inz} \tag{4.101}$$

inside the rectangle $ABCD$ is $z = \frac{\pi}{2} + \frac{\pi\tau}{2}$. Then, we have the following relations

$$\varphi\left(\frac{2\cos z}{r}\right) = \varphi\left(\frac{2\cos\left(-z\right)}{r}\right) = \varphi\left(\frac{2\cos\left(\pi\tau - z\right)}{r}\right). \tag{4.102}$$

since snz has double periods $2iK'$ and $4K$.

From (4.102), we see that $z = -\frac{\pi}{2} + \frac{\pi\tau}{2}$ is the other pole of $h(z)$ inside $ABCD$. So by the residue theorem,

$$\oint_{ABCD} h\left(z\right)dz = 2\pi i\left\{\textbf{\textit{Res}}\left(h(z); \frac{\pi}{2} + \frac{\pi\tau}{2}\right) + \textbf{\textit{Res}}\left(h(z); -\frac{\pi}{2} + \frac{\pi\tau}{2}\right)\right\}, \tag{4.103}$$

where Res $(h(z); z_0)$ *denotes the residue of the function* $h(z)$ *at* $z = z_0$. *Since* $h(z)$ *has a period* 2π, *we may write*

$$\int_{BC} h(z)\, dz + \int_{DA} h(z)\, dz = 0. \tag{4.104}$$

Integrals over other sides of $ABCD$ *can be calculated as*

$$\int_{AB} h(z)\, dz = \int_{-\pi}^{\pi} h(x)\, dx = 2\int_{0}^{\pi} \varphi\left(\frac{2\cos x}{r}\right) \cos nx dx \tag{4.105}$$

and

$$\int_{CD} h(z)\, dz = \int_{\pi}^{-\pi} h(x + \pi\tau - 2\pi)\, dx = -r^{-4n} \int_{-\pi}^{\pi} \varphi\left(\frac{2\cos(x + \pi\tau)}{r}\right) e^{inx} dx. \tag{4.106}$$

From (4.102), we can write

$$\int_{CD} h(z)\, dz = -2r^{-4n} \int_{0}^{\pi} \varphi\left(\frac{2\cos x}{r}\right) \cos nx dx. \tag{4.107}$$

After adding (4.105) and (4.107), we obtain

$$\oint_{ABCD} h(z)dz = 2\left(1 - r^{-4n}\right) \int_{0}^{\pi} \varphi\left(\frac{2\cos x}{r}\right) \cos nx dx. \tag{4.108}$$

In order to evaluate Res $\left(h(z), \frac{\pi}{2} + \frac{\pi\tau}{2}\right)$, *we need to expand the function* $\sqrt{k_0} sn\,(u + u_0)$ *about* $u = 0$, *where*

$$u_0 = \frac{2K}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi\tau}{2}\right) = -iK'. \tag{4.109}$$

We know that

$$sn\,(u - iK') = \frac{1}{k_0 snu} = \frac{1}{k_0 u} + \frac{1}{6k_0}\left(1 + k_0^2\right) u + \cdots. \tag{4.110}$$

Hence, one can obtain that

$$\textit{Res}\left(h(z), \frac{\pi}{2} + \frac{\pi\tau}{2}\right) = \frac{1}{k_0} e^{in\left(\frac{\pi}{2} + \frac{\pi\tau}{2}\right)}. \qquad (4.111)$$

In similar way, we can calculate other residue as

$$\textit{Res}\left(h(z), -\frac{\pi}{2} + \frac{\pi\tau}{2}\right) = -\frac{1}{k_0} e^{in\left(-\frac{\pi}{2} + \frac{\pi\tau}{2}\right)}. \qquad (4.112)$$

Finally, we get from (4.103) and

$$\oint_{ABCD} h(z)dz = -\frac{4\pi r^{-2n}}{k_0} \sin\left(\frac{n\pi}{2}\right). \qquad (4.113)$$

From (4.108) and after doing elementary calculations, we obtain the desired result. ■

In order to prove our coefficient inequalities, we need one more theorem given below.

Theorem 4.15: If $g(z;\theta,\alpha)$ is given by (4.28), then

$$A_n(g_\theta) = -\frac{2\pi(1-2\alpha)^2}{r^{n+1} K k_0^{\frac{3}{2}} (1-r^{-4n})} \sin\left(\frac{n\pi}{2}\right) + 4(1-\alpha)^2 A_n(k_\theta) - 4(1-\alpha)(1-2\alpha) A_n(c_\theta) \qquad (4.114)$$

where $\theta \in (0, 2\pi]$ and $\alpha \in [0,1)$.

Proof 4.15: Since

$$g(z;\theta,\alpha) = (1-2\alpha)^2 z + 4(1-\alpha)^2 k_\theta(z) - 4(1-\alpha)(1-2\alpha) c_\theta(z), \qquad (4.115)$$

we obtain from the linearity of $A_n(f)$ and Lemma 4.2 that

$$A_n(g(\theta,\alpha)) = -\frac{2\pi(1-2\alpha)^2}{r^{n+1} K k_0^{\frac{3}{2}} (1-r^{-4n})} \sin\left(\frac{n\pi}{2}\right) + 4(1-\alpha)^2 A_n(k_\theta) - 4(1-\alpha)(1-2\alpha) A_n(c_\theta). \qquad (4.116)$$

This completes the proof of Theorem 4.15. ■

Now, we can mention our coefficient estimates for functions which belong to the class $R_\alpha(E_r)$.

Theorem 4.16: If $f \in R(\alpha)$ *and* $g(z) = z\left(\frac{1+(1-2\alpha)z}{1-z}\right)^2$, *then*

$$|A_0(f)| \leq A_0(g). \tag{4.117}$$

Equality holds only for the functions $f(z) = g(z)$ *and* $f(z) = -g(-z)$.

Proof 4.16: According the formula in (4.13), we have

$$A_0(f) = \frac{1}{rK\sqrt{k_0}} \int_0^\pi f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) d\theta. \tag{4.118}$$

We may write (4.118) as

$$A_0(f) = \frac{1}{rK\sqrt{k_0}} \int_0^{\frac{\pi}{2}} \left[f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) + f\left(-\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right] d\theta, \tag{4.119}$$

since our mapping function $\varphi(z)$ *is an odd function. By substituting (1.2) into (4.119), we obtain*

$$A_0(f) = \frac{2}{rK\sqrt{k_0}} \int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} a_{2n}\varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right) d\theta. \tag{4.120}$$

Thus

$$|A_0(f)| \leq \frac{2}{rK\sqrt{k_0}} \int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} |a_{2n}|\,\varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right) d\theta \tag{4.121}$$

since $\varphi(x) \geq 0$ *for* $x \in \left[0, \frac{2}{r}\right]$. *We know that*

$$|a_n| \leq 4(1-\alpha)\left[n(1-\alpha) - (1-2\alpha)\right] \tag{4.122}$$

for $f \in R(\alpha)$. Therefore, we obtain the following inequality

$$|A_0(f)| \leq$$

$$\frac{2}{rK\sqrt{k_0}} \int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} 4(1-\alpha)\left[2n(1-\alpha)-(1-2\alpha)\right]\varphi^{2n}\left(\frac{2\cos\theta}{r}\right)\right) d\theta \quad (4.123)$$

Hence, we have the desired result

$$|A_0(f)| \leq A_0(g(z)) \quad (4.124)$$

This completes the proof. ■

In the following theorem, we obtain sharp coefficient estimates for functions belonging $R_\alpha(E_r)$.

Theorem 4.17: If $f \in R(\alpha)$ and $g(z) = z\left(\frac{1+(1-2\alpha)z}{1-z}\right)^2$, then

$$|A_n(f)| \leq A_n(g) \ , n = 1, 2, \cdots . \quad (4.125)$$

Equality holds only for the functions $f(z) = g(z)$ and $f(z) = -g(-z)$.

Proof 4.17: From (4.15) and (4.25), we need to show that $|A_n(g(\theta,\alpha))|$ attains its maximum only for $\theta = 0$ and $\theta = \pi$. From the result in Theorem 4.15 and triangle inequality, we have

$$|A_n(g(\theta,\alpha))| \leq \frac{2\pi(1-2\alpha)^2}{r^{n+1}Kk_0^{\frac{3}{2}}(1-r^{-4n})}\left|\sin\left(\frac{n\pi}{2}\right)\right|$$

$$+\left|4(1-\alpha)^2 A_n(k_\theta) - 4(1-\alpha)(1-2\alpha)A_n(c_\theta)\right| . \quad (4.126)$$

From Theorem 4.8, we know that

$$A_n(k_\theta) = \frac{A_n(c_\theta)}{\sqrt{(1-k_0)^2 + 4k_0\sin^2\theta}}\{g(\theta) + if(\theta)\} , \quad (4.127)$$

where the functions $g(\theta)$ and $f(\theta)$ are given in (4.79) and (4.80), respectively. Also, $A_n(c_\theta)$ is calculated in Theorem 4.1. Let define

$$C_n(\theta) = 4(1-\alpha)^2 A_n(k_\theta) - 4(1-\alpha)(1-2\alpha) A_n(c_\theta) \tag{4.128}$$

or

$$C_n(\theta) = 4(1-\alpha)^2 \left[A_n(k_\theta) - A_n(c_\theta)\right] + 4\alpha(1-\alpha) A_n(c_\theta). \tag{4.129}$$

Then, we obtain from (4.127)

$$C_n(\theta) = 4(1-\alpha)^2 A_n(c_\theta) \left\{ \frac{g(\theta)}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} - 1 \right.$$

$$\left. + i \frac{f(\theta)}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} \right\} + 4\alpha(1-\alpha) A_n(c_\theta), \tag{4.130}$$

$\alpha \in [0,1)$. *By using the triangle inequality, we have*

$$|C_n(\theta)| \leq 4(1-\alpha)^2 |A_n(c_\theta)| \left| \frac{g(\theta)}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} - 1 \right.$$

$$\left. + i \frac{f(\theta)}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} \right| + 4\alpha(1-\alpha) |A_n(c_\theta)|. \tag{4.131}$$

In (4.131), the equality case occurs if and only if

$$f(\theta) = 0 \text{ and } \frac{g(\theta)}{\sqrt{(1-k_0)^2 + 4k_0 \sin^2\theta}} - 1 > 0. \tag{4.132}$$

Hence, it follows from (4.35) and $g(\theta) \leq g(0) = g(\pi)$ that

$$|C_n(\theta)| \leq 4(1-\alpha)^2 A_n(c) \left(\frac{g(0)}{1-k_0} - 1 \right) + 4\alpha(1-\alpha) A_n(c). \tag{4.133}$$

From (4.127), we obtain the following

$$|C_n(\theta)| \leq 4(1-\alpha)^2 A_n(k) - 4(1-\alpha)(1-2\alpha) A_n(c). \tag{4.134}$$

We consider the following three cases according to values of n :

- *If* n *is even, the result is obtained from* $A_n(g(\theta,\alpha)) = C_n(\theta)$ *and (4.134).*
- *If* n *is odd and* $n = 4k+3$, $k = 0, 1, 2, \cdots$ *. Then* $A_{4k+3}(z) > 0$ *and*

$$A_{4k+3}(g(\theta,\alpha)) = (1-2\alpha)^2 A_{4k+3}(z) + C_{4k+3}(\theta). \tag{4.135}$$

From (4.134), it is clear that

$$|A_{4k+3}(g(\theta,\alpha))| \leq A_{4k+3}(g). \tag{4.136}$$

- *If* n *is odd and* $n = 4k+1$, $k = 0, 1, 2, \cdots$ *. Then* $A_{4k+1}(z) < 0$ *and*

$$A_{4k+1}(g(\theta,\alpha)) = (1-2\alpha)^2 A_{4k+1}(z) + C_{4k+1}(\theta). \tag{4.137}$$

Consider the function

$$C_{4k+1}(\theta) = A_{4k+1}(g(\theta,\alpha)) - (1-2\alpha)^2 A_{4k+1}(z), \tag{4.138}$$

Applying the triangle inequality, we have

$$|C_{4k+1}(\theta)| \leq \left| A_{4k+1}(g(\theta,\alpha)| - (1-2\alpha)^2 A_{4k+1}(z) \right|. \tag{4.139}$$

In (4.139), the equality is possible if and only if $A_{4k+1}(g(\theta,\alpha) > 0$*. It means that*

$$\mathit{Im}(A_{4k+1}(g(\theta,\alpha))) = \mathit{Im}(C_{4k+1}(\theta)) = 0. \tag{4.140}$$

Since $C_{4k+1}(0) > 0$, *we obtain the following*

$$|C_{4k+1}(\theta)| \le |A_{4k+1}(g(\theta,\alpha))| - (1-2\alpha)^2 A_{4k+1}(z) \le C_{4k+1}(0). \quad (4.141)$$

Hence,

$$|A_{4k+1}(g(\theta,\alpha))| \le A_{4k+1}(g). \quad (4.142)$$

This result is sharp. ■

5. CONCLUSIONS

In this desertion, coefficient problem for analytic functions in $\mathbb{D}$ is generalized to coefficient problem for functions in some certain subclasses of analytic and normalized functions in E_r. In the last chapter of this thesis, we consider the Faber coefficients of analytic functions in E_r. We obtained sharp upper bounds for the Faber coefficients of functions which belong to these subclasses. We see that the equality case occurs only for $\theta = 0$ and $\theta = \pi$. It means that there are only two cases which is the number of the invariant rotations of E_r.

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BIOGRAPHY

Tuğba YAVUZ was born in Istanbul, Turkey, in 1984. In 2002, she graduated from Çorlu Mehmet Akif Ersoy Anatolian High School. After that, she got Bsc degree in Mathematics with the second highest GPA from Gebze Technical University, in 2007. She also received M.Sc degree in Mathematics from Gebze Technical University in 2009. She worked as a research assistant at Gebze Technical University between 2007-2015. She has been a doctorate student in Mathematics, department of Graduate School of Natural and Applied Sciences since 2009. Her research interest is about geometric function theory, especially univalent function theory.

APPENDICES

Appendix A: Publications Based on the Thesis

Haliloglu E., Yavuz T., (2015), “Coefficient bounds for starlike and close-to-convex functions in an ellipse ”, Complex Variables and Elliptic Equations, 60 (7) 926-937.

Yavuz T., (2015), “Coefficient bounds for analytic functions in an ellipse ”, Integral Transforms and Special Functions, 26 (11) 900-909.