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UĞUR ODABAŞI

A THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

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GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

İKİ DÖNGÜ UZUNLUKLU HAMİLTON-WATERLOO PROBLEMİ

UĞUR ODABAŞI

DOKTORA TEZİ

MATEMATİK ANABİLİM DALI

DANIŞMANI

DOÇ. DR. SİBEL ÖZKAN

GEBZE

2016

GEBZE TEKNİK ÜNİVERSİTESİ	**DOKTORA JÜRİ ONAY FORMU**

GTÜ Fen Bilimleri Enstitüsü Yönetim Kurulu'nun 14/12/2015 tarih ve 2015/74 sayılı kararıyla oluşturulan jüri tarafından 29/12/2015 tarihinde tez savunma sınavı yapılan Uğur ODABAŞI'nın tez çalışması Matematik Anabilim Dalında DOKTORA tezi olarak kabul edilmiştir.

JÜRİ

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SUMMARY

The Hamilton-Waterloo problem with uniform cycle sizes, denoted by $(n, m) - \mathrm{URD}(v; r, s)$, asks for a resolvable cycle decomposition of the complete graph K_v (for odd v) or K_v minus a $1-$factor (for even v) where r parallel classes consist of cycles of length n and s parallel classes consist of cycles of length m with $r + s = \left\lfloor \frac{v-1}{2} \right\rfloor$. In this dissertation, firstly, the Hamilton-Waterloo problem with $4-$cycle and $m-$cycle factors for odd $m \geq 3$ is studied and all possible solutions with a few possible exceptions are determined. Then, all possible solutions for the $m-$cycle and $4m-$cycle with a few possible exceptions when m is odd are obtained.

Key Words: Cycle Decompositions, $2-$Factorizations, Resolvable Decompositions, Oberwolfach Problem, Hamilton–Waterloo Problem.

ÖZET

Çift döngülü Hamilton-Waterloo problemi, kısaca $(n,m)-\mathrm{URD}(v;r,s)$, v tek tamsayı iken tam çizge K_v'nin ya da v çift tamsayı olduğunda tam çizge eksi $1-$faktör K_v-I'nın, parallel sınıflarından r tanesi n uzunluğunda, s tanesi ise m uzunluğunda döngülerden oluşan $r+s=\left\lfloor\frac{v-1}{2}\right\rfloor$ olacak şekilde bir çözülebilir döngü parçalanışının olup olmadığını inceler. Bu tezde ilk olarak, döngü uzunluklarının birinin 4, diğerinin $m\geq 3$ olacak şekilde bir tek tamsayı olduğu durum için, bir kaç olası istisnai durum dışında, bütün mümkün sonuçlar elde edilmiştir. Daha sonra döngü uzunluklarının m ve $4m$ olduğu durum için, yani bir döngü uzunluğu diğer döngü uzunluğunun dört katı olduğunda problem, çift m değerleri için tamamen, tek m değerleri için bir kaç olası istisnai durum dışında tamamen çözülmüştür.

Anahtar Kelimeler: Döngü Parçalanışları, $2-$Faktorizasyon, Çözülebilir Parçalanışlar, Oberwolfach Problemi, Hamilton–Waterloo Problemi.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere appreciation to my advisor Dr. Sibel Özkan, for her guidance, background knowledge, insightful comments, editing skills and considerable encouragements in the completion of this thesis. It has been a great honor for me to be her first PhD. student. I would also like to thank my thesis committee Dr. Selda Küçükçifçi, Dr. Mustafa Akkurt, Dr. Didem Gözüpek and Dr. Emre Kolotoğlu for their valuable comments and suggestions.

I want to thank my former colleagues at Gebze Technical University and friends from Istanbul University for their friendship and support.

Last, but not least, I would like to thank my family: my parents, Elmas and Şükrü, for their support and encouragement, not only during the completion of the thesis but also throughout my life; my lovely wife Şafak for her constant love, all of the support she has given through all of these years; my little boy Rüzgar, for his unconditional love and his sweet smile which encourage me to finish my thesis. Their love and support provided me the energy to attain my study. This dissertation is dedicated to them.

This thesis is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) under project number 113F033.

TABLE of CONTENTS

LIST of FIGURES

1. INTRODUCTION

Combinatorial design theory and graph theory are two of the major branches of discrete mathematics.

The roots of the combinatorial design theory can be found in the recreational mathematics of the second half of the nineteen century, statistical theory of experimental design and geometry of the mid-nineteen century. Since then, the theory of combinatorial designs rapidly developed and became an active area of research of discrete mathematics that has connections with graph theory, linear and abstract algebra and number theory, and with various applications in areas such as coding theory, cryptography, and computer science.

Although graph theory has a history of more than two centuries, it has received great interest only recently. The first result on graph theory is Euler's paper on the Königsberg bridge problem [1] and now it is an essential and powerful modeling tool in mathematical research, computer science, biology, chemistry, social sciences and many more.

Most of the problems in design theory and graph theory are easy to explain, but they can be extremely difficult to solve and solutions generally involve innovative new combinatorial techniques as well as advanced tools and methods of other areas of mathematics such as algebra, geometry and number theory. The most classical problems still remain unsolved.

A great number of design theory problems can be viewed in terms of decomposition of graphs into prescribed subgraph. One of the main problem in combinatorial design theory is the $2-$factorization problem, that is, whether or not there exists a $2-$factorization of K_v, where each of the $2-$factors is of a prescribed type. In this dissertation we will focus on one particular $2-$factorization problem, so-called Hamilton-Waterloo problem, and we will give solutions to the problem for the case of two different cycle sizes.

This introduction will first give a number of very general graph and design theoretic terms and concepts that will be used throughout this dissertation, and then give a brief history of some well-known problems related to the results in this dissertation. The remainder of this dissertation will be devoted to proving the results that have obtained.

1.1. Definitions and Notation

The notation and definitions used here are mostly standard and may be found, for instance, in [2]-[5].

A *graph* G is a triple consisting of an edge set $E(G)$, a vertex set $V(G)$, and a relation that associates with each edge of G either one or two vertices called its endpoints. A graph in which each edge has distinct endpoints and no two edges have the same pair of endpoints is called a *simple graph*. A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ A *spanning subgraph* or *factor* of G is a subgraph that has the same vertex set as G.

A complete graph with v vertices is denoted by K_v. A complete *equipartite graph*, denoted by $K_{a:b}$, is a simple graph whose vertex set can be partitioned into b parts of size a each such that any two vertices in different parts connected by an edge, but no edge joining any two vertices in the same part. In particular, $K_{a:2}$ is called *complete bipartite graph* and denoted by $K_{a,a}$ as well.

A *path* is a finite sequence of distinct vertices $P = p_0p_1p_2\dots p_k$ together with the edges p_ip_{i+1}, $0 \leq i \leq m-1$. A graph is *connected* if there is a path between every pair of distinct vertices. Two endpoints of an edge are said to be *adjacent* to each other. The number of edges adjacent to a vertex v in a graph G is called the *degree of* v. A k *−regular graph* is a graph such that every vertex has degree k and a k *−factor* is a spanning k −regular subgraph. Besides regular graphs have a long history, they have nice properties such as having a lot more symmetry than arbitrary graphs.

A *cycle* is a 2 −regular connected graph. A cycle with the vertices $v_0,\ v_1,\ v_2$, $\dots, v_m$ and the edges $v_0v_1,\ v_1v_2, \dots,\ v_{m-1}v_m$ is called an m *−cycle* and denoted by $C_m = (v_0, v_1, v_2, \dots, v_m)$. The *length* of a cycle is the number of edges on the cycle (equal to the number of vertices). Also, a spanning cycle in a graph is called *Hamilton cycle*.

Given two paths $P = p_0p_1p_2\dots p_k$ and $Q = q_0q_1q_2\dots q_l$, we define the *concatenation* of paths P and Q by $PQ = p_0p_1\dots p_kq_0q_1\dots q_l$ and for $0 \leq i < j \leq k$ we write

- $Pp_j = p_0p_1p_2\dots p_j$,

- $p_iP = p_ip_{i+1}p_{i+2}\dots p_k$ and
- $p_iPp_j = p_ip_{i+1}p_{i+2}\dots p_j$.

We denote the $(k+1)$ −cycle obtained from adding an edge between p_0 and p_k in P by (P). Also P^{-1} will denote the path on the same vertex set as P but the vertices are listed in reverse order.

Let H be a finite additive group and let S be a subset of $H - \{0\}$ such that the opposite of every element of S also belongs to S. The *Cayley graph* over H with connection set S, denoted by $Cay(H, S)$, is the graph with vertex set H and edge set $E\big(Cay(H, S)\big) = \{(a, b) | a, b \in H, a - b \in S\}$. Note that the definition is not ambiguous since $S = -S$ by assumption, and that the degree of each vertex is $|S|$. As is also understood from this definition, there are some obvious necessary and sufficient conditions of Cayley graphs depending on the properties of S. For example, $Cay(H, S)$ is connected if and only if S is a generator set for H, and $Cay(H, S)$ is a complete graph if and only if $S = H - \{0\}$.

Cayley graphs provides a link between group theory and graph theory with various applications, especially in computer science. Regularity and underlying algebraic features of these graphs make them attractive to study.

Example 1.1: Figure 1.1 shows an example of a Cayley graph over the group $\mathbb{Z}_9$ *with the connection set* $\{\pm 1, \pm 2\}$.

If G_1 and G_2 are two edge disjoint graphs on the same vertex set, then $G_1 \oplus G_2$ will denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$. The *union* of graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If H is a subgraph of G, then $G - H$ denotes the graph with $V(G - H) = V(G)$ and $E(G - H) = E(G) \setminus E(H)$. Also αG will denote the vertex-disjoint union of the α copies of G.

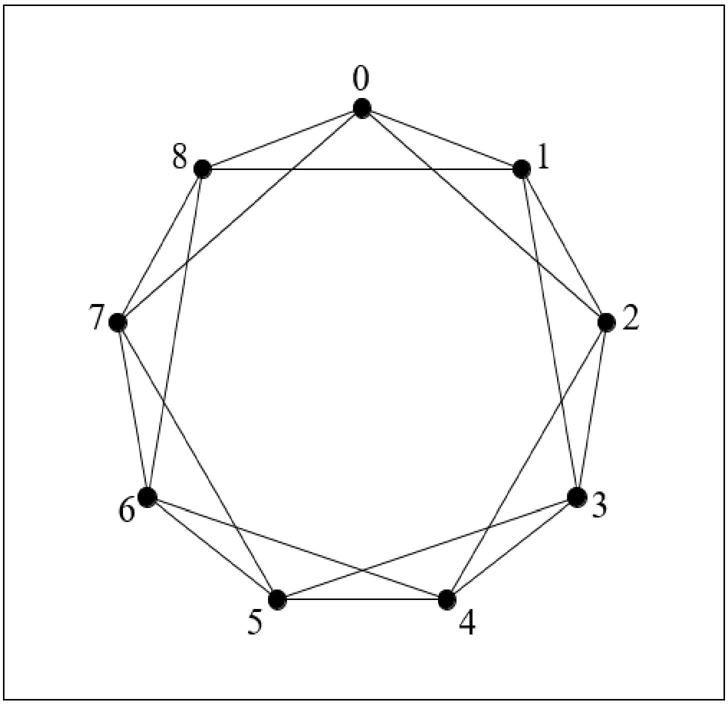

Figure 1.1: $Cay(\mathbb{Z}_9, \{\pm 1, \pm 2\})$.

2. CYCLE DECOMPOSITIONS

A *decomposition* of a graph G, is a set $\boldsymbol{\mathcal{H}} = \{H_1, H_2, \ldots, H_k\}$ of edge-disjoint subgraphs of G such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$. An $H-$*decomposition* is a decomposition of G such that $H_i \cong H$ for all $H_i \in \boldsymbol{\mathcal{H}}$. If each H_i is a cycle (or a disjoint union of cycles), then $\boldsymbol{\mathcal{H}}$ is called a *cycle decomposition*. It is easy to see that if a graph G admits a cycle decomposition, then each vertex of G must be even. The converse was shown by Veblen [6].

Theorem 2.1: [6] A graph can be decomposed into cycles if and only if every vertex has even degree.

A *parallel class* in a decomposition is a set of vertex disjoint graphs that partitions the vertex set. A cycle decomposition is called *resolvable* if it has a partition of the cycles into parallel classes. A *resolvable cycle decomposition* is also known as a $2-$*factorization* and a parallel class can be called a $2-$*factor*. If a decomposition of a graph G consists precisely of k_i parallel classes isomorphic to F_i, then we say that a $\{F_1^{k_1}, F_2^{k_2}, \ldots, F_l^{k_l}\}-$*factorization* of G exists.

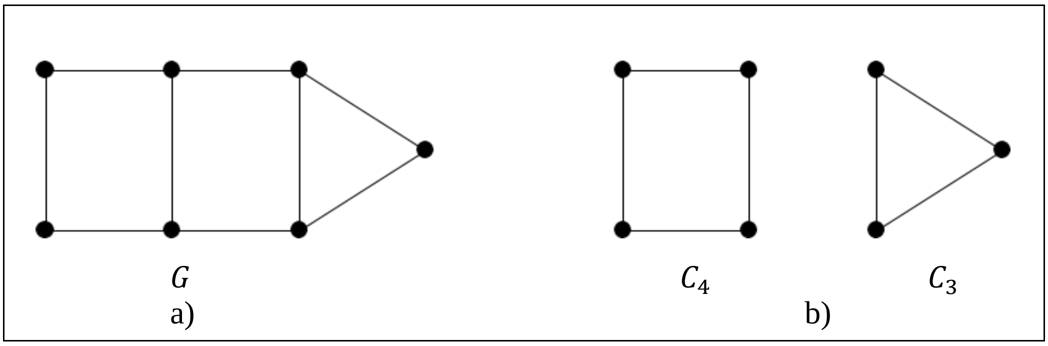

Figure 2.1: a) A given graph G, b) a $2-$factor in G.

One of the first results in graph theory that obtained by Petersen [7] in 1891 is the $2-$factor theorem about regular graphs of even degree.

Theorem 2.2: [7] Every $(2k)-$regular graph has a $2-$factor and hence can be decomposed into k edge disjoint $2-$factors.

Complete graphs are the most natural graphs to decompose and cycle decomposition is the one of the most studied family of decompositions. Since there is no cycle decomposition of K_v when v is even by Theorem 2.1, in this case it is common practice to consider cycle decompositions of $K_v - I$, the complete graph on v vertices with a $1-$factor I removed.

Example 2.1: A decomposition of K_7 into a $7-$cycle, a $6-$cycle, a $5-$cycle and a $3-$cycle is shown in Figure 1.1.

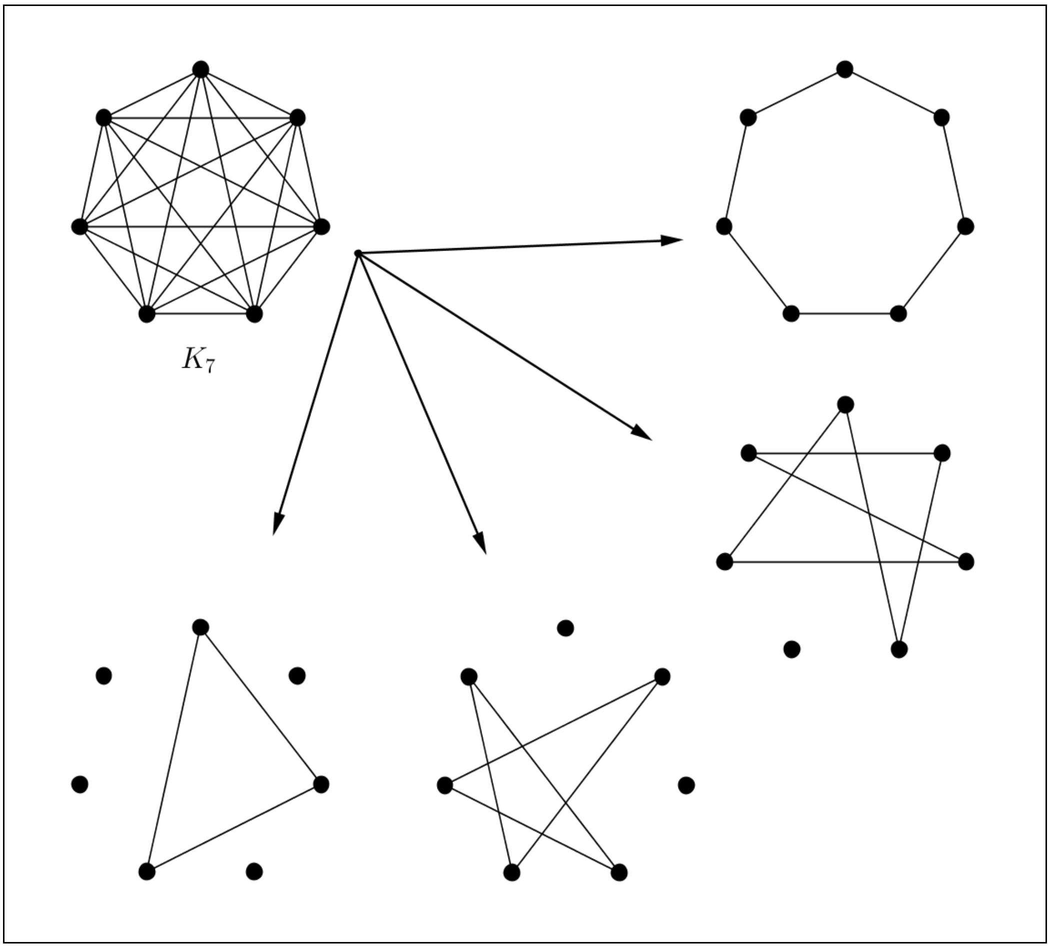

Figure 2.2: A $\{C_7, C_6, C_5, C_3\}$ $-$decomposition of K_7.

Example 2.2: A decomposition of $K_8 - I$ into two $8-$cycles and two $4-$cycles is shown in Figure 1.2.

It should be noted that decomposition of a graph into cycles is an NP-complete problem in general [8]. Dor and Tarsi [9] proved that if H contains a connected component with at least three edges, then the problem of deciding whenever a graph has an H −decomposition is NP-complete. Moreover, it has been shown in [41] that for every simple graph H and an integer v where $v \geq v_0(H)$ for some integer $v_0(H)$, an H −decomposition of complete graph K_v exists if and only if $|E(H)|$ divides $\frac{v(v-1)}{2}$ and $v-1$ is divisible by the greatest common divisor of the vertex degrees in H.

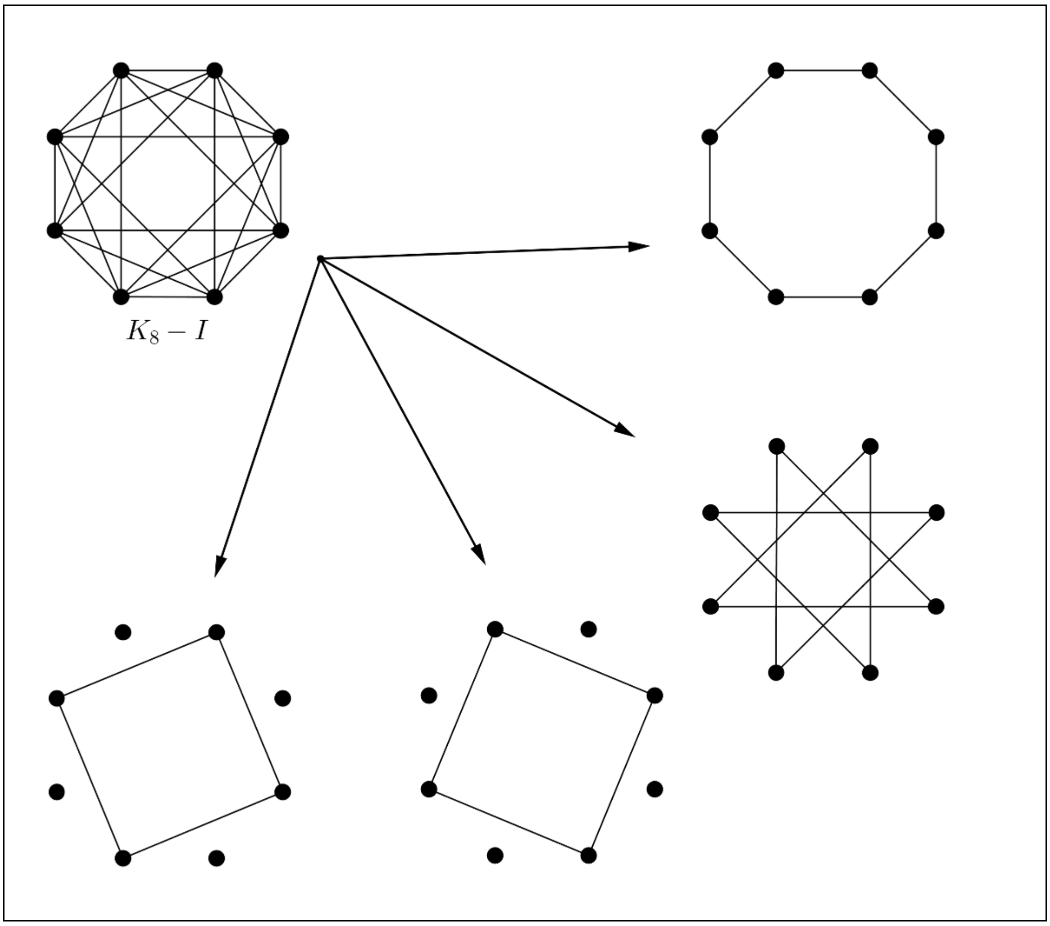

Figure 2.3: A $\{C_8, C_8, C_4, C_4\}$ −decomposition of $K_8 - I$.

2.1. History

The earliest question concerning cycle decompositions of complete graphs was posed by Kirkman [11] in 1847 which asks an H −decomposition of K_v where H is the union of $v/3$ disjoint 3 −cycles. The problem is known as *Kirkman's schoolgirl problem* and it was shown by Ray-Chadhuri and Wilson in [12] that desired decomposition exists if and only if $v \equiv 3\ (mod\ 6)$. Since then, cycle decomposition of a graph G has been attracted a lot of interests, particularly the case where G is a complete graph. In the 1890s, Walecki [13] has constructed a Hamilton cycle decomposition of K_v and $K_v - I$ In 1965, Kötzig [14] showed that $4m$ −cycle decomposition of K_v exists when $v \equiv 1\ (mod\ 8m)$. One year later, Rosa [15], [16] proved that K_v can be decomposed into C_m −cycles for $m \equiv 2\ (mod\ 4)$ and $v \equiv 1\ (mod\ 2m)$, also settled the problem for $m = 5, 7$.

It is obvious that if K_v (or $K_v - I$ for even v) has a decomposition into cycles, then length of each cycle must be greater than or equal to 3 and less than or equal to v, and the sum of the cycle lengths must equal the number of edges of K_v (or $K_v - I$ for even v). It was conjectured in 1981 by Alspach [17] that these obvious necessary conditions are also sufficient.

Conjecture 2.1: Let $v \geq 3$ *be an odd integer and* $m_1, m_2, \dots, m_t$ *be integers such that* $3 \leq m_i \leq v$ *for* $i = 1,2, \dots, t$ *with* $m_1 + m_2 + \cdots + m_t = \frac{v(v-1)}{2}$. *Then* K_v *has a* $\{C_{m_1}, C_{m_2}, \dots, C_{m_t}\}$ *−decomposition.*

Conjecture 2.2: Let $v \geq 4$ *be an even integer,* I *is a* 1 *−factor in* K_v *and* $m_1, m_2, \dots, m_t$ *be integers such that* $3 \leq m_i \leq v$ *for* $i = 1,2, \dots, t$ *with* $m_1 + m_2 + \cdots + m_t = \frac{v(v-2)}{2}$. *Then* $K_v - I$ *has a* $\{C_{m_1}, C_{m_2}, \dots, C_{m_t}\}$ *−decomposition.*

In a number of special cases of this problem there are additional obvious necessary conditions. One of the most studied case is the uniform cycle decomposition, that is, all the cycle lengths are the same, and in [18], [19] it has been shown that Alspach's conjecture is true when the cycle lengths are all the same.

Also, a great deal of work has been done for the case where the lengths of the cycles vary. Finally, Alspach’s Conjecture was verified in 2012 by Bryant et al. [20].

One variation of the cycle decomposition problem is the Oberwolfach problem was first formulated by Ringel in 1967 at a graph theory conference in Oberwolfach, Germany and first mentioned in [21]. The problem is related to the possible seating arrangements at the conference and was inspired by a question of whether v mathematicians could be seated in such a way that each mathematician sits next to each other mathematician exactly once over $\frac{v-1}{2}$ days, where there are k_i round tables with m_i seats for $1 \leq i \leq t$ satisfying $\sum_{i=1}^{t} k_i m_i = v$. In graph theory language, the problem asks for a decomposition of the complete graph K_v into $2-$factors each of which is isomorphic to a given $2-$factor H. If H consists of k_i m_i-cycles, $1 \leq i \leq t$, then the corresponding *Oberwolfach problem* is denoted by OP($m_1^{k_1}$, $m_2^{k_2}$, … , $m_t^{k_t}$). As previously mentioned, a decomposition into $2-$factors clearly requires that the degree of each vertex be even and so v must be odd, in the case when v is even it is natural to consider a decomposition of the graph $K_v - I$ into $2-$factors instead, where I is a $1-$factor of K_v. The corresponding problem when v is even is called the *spouse-avoiding* version of the Oberwolfach problem.

A generalization of the Oberwolfach Problem is the *Hamilton-Waterloo Problem* where the conference takes places in two venues; Hamilton and Waterloo, the first of which has k round tables, each seating n_i people for $i = 1,2,\dots,k$, the second of which has l round tables each seating m_i people for $i = 1,2,\dots,l$ (necessarily $\sum_{i=1}^{k} n_i = \sum_{i=1}^{l} m_i = v$).

If we let $n = n_1 = n_2 = \cdots = n_k$ and $m = m_1 = m_2 = \cdots = m_k$, then each $2-$factor is composed of either $n-$cycles or $m-$cycles. This corresponds to the *uniformly resolvable cycle decomposition* of K_v (or $K_v - I$ for even v) into $n-$cycles and $m-$cycles. This version of the Hamilton-Waterloo problem, with uniform cycle sizes, has attracted most of the attention and we use the notation to denote the problem with r factors of $n-$cycles and s factors of $m-$cycles by $(n, m)-$URD$(v; r, s)$.

Example 2.3: Figure 2.4 shows a $\{C_3^2, C_4^3\}$ *−factorization of* $K_{12} - I$.

The obvious necessary conditions for the existence of a solution to $(n, m)-$URD$(v; r, s)$ are given by Adams et al. in [22]:

Lemma 2.1: [22] Let v, n, m, r *and* s *be non-negative integers with* $m,n \geq 3$*. If there exists a solution to* $(n, m) -$ URD$(v; r, s)$*, then*

i) $s > 0$ *for* $v \equiv 0 \ (mod\ m)$

ii) $r + s = \left\lfloor \frac{v-1}{2} \right\rfloor$.

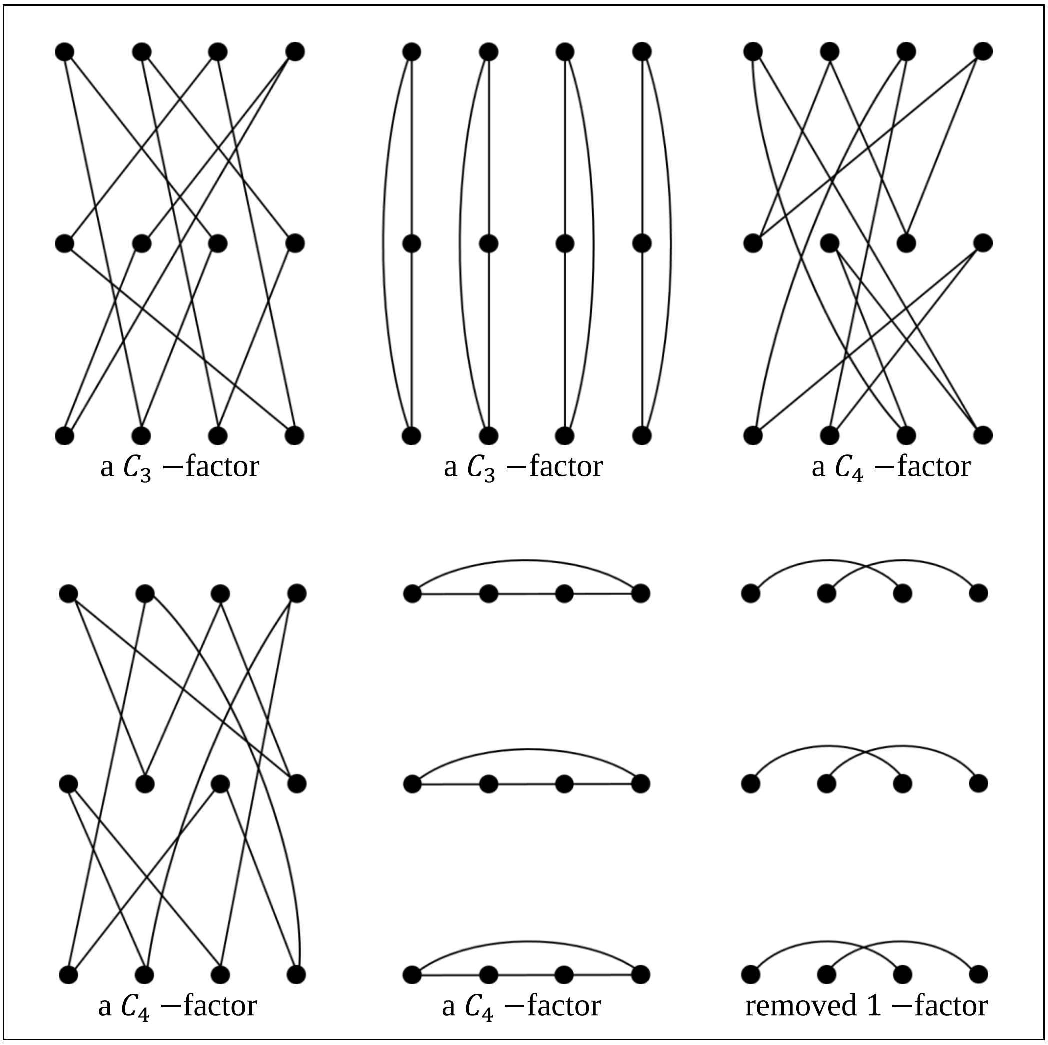

Figure 2.4: A solution to $(3,4) - \mathrm{URD}(12; 2,3)$.

Solving the problem completely for n −cycles and m −cycles means finding a solution to the problem for all possible r and s satisfying the obvious necessary conditions.

2.2. Preliminary Results

Let G be a graph and $\{G_0, G_1, G_2, \ldots, G_{k-1}\}$ be vertex disjoint copies of G with $v_i \in V(G_i)$ for each $v \in V(G)$. Then the graph $G[k]$ is a graph with vertex set $V(G[k]) = V(G_0) \cup V(G_1) \ldots \cup V(G_{k-1})$ and edge set $E(G[k]) = \{u_i v_j \ : \ uv \in$

$E(G)$ *and* $0 \leq i, j \leq k-1\}$. For example $K_m[2] \cong K_{2m} - I$ and $K_2[m] \cong K_{m,m}$ where I is a $1-$factor of K_{2m}.

It is easy to see that if a graph G has an $H-$decomposition, then there exists an $H[k]-$decomposition of $G[k]$. Moreover if a graph G has an $H-$factorization, then there exists an $H[k]-$factorization of $G[k]$.

In fact, this graph operation is a generalization of Häggkvist's *doubling construction* and it coincides with a special case of a graph product called lexicographic product. Häggkvist [23] constructed $2-$factorizations containing even cycles using $G[2]$.

Lemma 2.2: [23] Let G *be a path or a cycle with* m *edges and let* H *be a* $2-$*regular graph on* $2m$ *vertices where each component of* H *is a cycle of even length. Then* $G[2]$ *has an* $H-$*decomposition.*

Example 2.4: Let G *be a* $6-$*cycle. Then* $G[2]$ *can be decomposed two Hamilton cycle.*

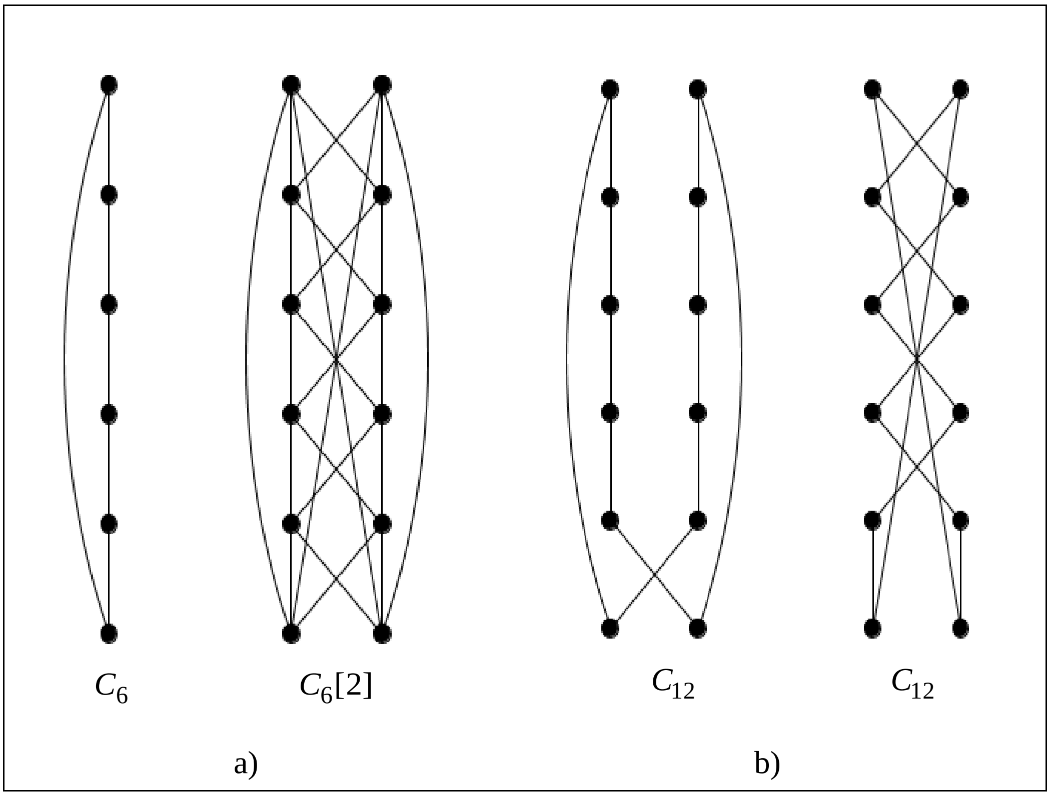

Figure 2.5: a) C_6 and $C_6[2]$, b) $C_{12}-$decomposition of $C_6[2]$.

Baranyai and Szasz [24] have shown that if a graph G can be decomposed into x Hamilton cycles and if H is a graph with y vertices and can be decomposed into z Hamilton cycles then their lexicographic product is decomposable into $xy+z$ Hamilton cycles. So, $C_m[n]$ has a C_{mn} −factorization. Also Alspach et al. [25] have shown that for an odd integer m and a prime p with $3 \leq m \leq p$, $C_m[p]$ has a C_p −factorization.

It is known that the solutions to the cases $\mathrm{OP}(3^2)$, $\mathrm{OP}(3^4)$, $\mathrm{OP}(4,5)$, and $\mathrm{OP}(3^2,5)$ do not exist [25]-[27]. The Oberwolfach Problem for a single cycle size $\mathrm{OP}(m^k)$, for all $m \geq 3$ has been solved in two separate cases: odd cycles in 1989 by Alspach et al. [25] and the even cycle case in 1991 by Hoffman and Schellenberg [28]. This results will be used in the main construction.

Theorem 2.3: [25],[28] A resolvable m −cycle decomposition of K_v (or $K_v - I$ for even v) exists if and only if $m|v$ and $m \neq 3$ when $v = 6, 12$.

The following theorem summarizes the known results on the Oberwolfach problem for non-uniform length cycles given in the survey [4] and more recent results in [29]-[32].

Theorem 2.4: [23],[29]-[34] The following Oberwolfach problems all have solutions:

- $\mathrm{OP}(m_1^{k_1}, m_2^{k_2}, \dots, m_t^{k_t})$ *for* $m_1k_1 + m_2k_2 + \dots + m_tk_t \leq 40$;
- $\mathrm{OP}(3^k, 4)$ *for all odd* $k \geq 1$;
- $\mathrm{OP}(3^k, 5)$ *for all even* $k \geq 4$;
- $\mathrm{OP}(m^k, v - mk)$ *for* $v \geq 6km - 1$, $k \geq 1$, $m \geq 3$;
- $\mathrm{OP}(m, v - m)$ *for* $m = 3,4,5,6,7,8,9$ *and* $v \geq m + 3$;
- $\mathrm{OP}(m^2, v - 2m)$ *for* $m = 3,4$ *and* $v \geq 2m + 3$;
- $\mathrm{OP}(2m_1, 2m_t, \dots, 2m_t)$ *for all* $m_i \geq 2$ *and* $m_1 + m_2 + \dots + m_t$ *odd*;
- $\mathrm{OP}(m, m+1)$ *and* $\mathrm{OP}(m, m+2)$ *for* $m \geq 3$;
- $\mathrm{OP}((2s+1)^2, 2s+2)$ *for* $s \geq 1$;
- $\mathrm{OP}(3, (4s)^2)$ *for* $s \geq 1$;

- $\mathrm{OP}(4^k, 2s+1)$ $OP(4^k, 2s+1)$ *for* $s > 1$, $k \geq 0$;
- $\mathrm{OP}((4s)^2, 2s+1)$ *for* $s > 1$, $k \geq 0$;
- $\mathrm{OP}(m_1, m_2)$ *for all* $m_1, m_2 \geq 3$ with $m_1 \neq m_2$ except $(m_1, m_2) = (4,5)$.

Also in [35], the Oberwolfach problem is completely solved for an infinite set of prime orders.

Piotrowski [36] solved the bipartite analogue of the Oberwolfach problem.

Theorem 2.5: [36] There is a $2-$*factorization of* $K_{a,a}$ *in which each* $2-$*factor consists of vertex disjoint cycles* $C_{n_1}, C_{n_2}, \ldots, C_{n_l}$ *if and only if* a *is even,* $n_i \geq 4$ *is even for* $1 \leq i \leq l$ *and* $n_1 + n_2 + \ldots + n_l = 2a$*, except that* $a = 6$ *and* $n_i = 6$ *for* $1 \leq i \leq l$.

Moreover, Liu [37] gave a complete solution to the Oberwolfach Problem for complete equipartite graphs where all cycles have the same length and we will use this result in our main construction.

Theorem 2.6: [37] The complete equipartite graph $K_{a:b}$ *has a* C_l-*factorization for* $l \geq 3$ *and* $a \geq 2$ *if and only if* $l|ab$, $a(b-1)$ *is even,* l *is even if* $b = 2$ *and* $(a,b,l) \neq (2,3,3), (6,3,3), (2,6,3), (6,2,6)$.

The first results on the Hamilton-Waterloo Problem [22], settled the problem for all $v \leq 17$ and in addition solved the cases $(n,m) \in \{(4,6), (4,8), (4,16), (8,16), (3,5), (3,15), (5,15)\}$ except that a solution to $(3,5)-URD(15;6,1)$ does not exist and the case $(3,5)-\mathrm{URD}(v; \frac{v-3}{2}, 1)$ is unresolved for $v \equiv 0 \ (mod\ 15)$ with $v > 15$. With a few possible exceptions when $m = 24$ and 48, Danziger et al. [38] solved the problem for the case $(n,m) = (3,4)$.

Theorem 2.7: [38] For all positive integers r *and* s*, a solution to* $(3,4)-\mathrm{URD}(v; r, s)$ *exists if and only if* $12|v$, $r + s = \frac{v-2}{2}$ *and* $(v,r) \neq (12,5)$ *except possibly when* $v = 24$ *and* $r = 5,7,9$ *or* $v = 48$ *and* $r = 5,7,9,13,15,17$.

The case $n = 3$ and $m = v$, that is triangle-factors and Hamilton cycles, has attracted much attention and remarkable progress has been made by Horak et al. [23], Dinitz and Ling [40], [41], Lei and Shen [42]. The following theorem summarizes the results for this case.

Theorem 2.8: [39]-[42] Let r *and* s *be non-negative integers with* $r + s = \left\lfloor \frac{v-1}{2} \right\rfloor$. *Then, a solution to*$(3, v) - \mathrm{URD}(v; r, s)$ *exists except when*

- $r = 5$ *and* $v = 6,12$;
- $r = 3$ *and* $v = 9$;

except possibly when

- $v = 18$ *and* $s = 1$, $v = 36$ *and* $s \in \{2,4\}$;
- $v \equiv 6\ (mod\ 36)$ *and* $s = 1$;
- $v \equiv 12\ (mod\ 18)$ *and* $s \in \left\{1,2, \ldots, \frac{v-6}{6}\right\}$;
- $v \in \{93,111,123,129,141,153,159,177,183,201,207,213,249\}$ *and* $s = 1$;
- $v \equiv 15\ (mod\ 18)$ *and* $s \in \left\{2,3, \ldots, \frac{v-3}{6}, \frac{v+9}{6}\right\}$.

In [43], Bryant et al. have settled the Hamilton-Waterloo Problem for bipartite $2-$factors, and in [44] Buratti and Rinaldi studied regular $2-$factorizations leading to some cyclic solutions to Oberwolfach and Hamilton-Waterloo Problems, and also in [45], an infinite class of cyclic solutions to the Hamilton-Waterloo Problem is given.

Fu and Huang [46] solved the case of $4-$cycles and $m-$cycles for even m, and also settled all cases where $m = 2n$ and n is even in 2008.

Theorem 2.9: [46] Let r *and* s *be non-negative integers with* $r + s = \frac{v-2}{2}$. *Then*

- *for all even* $m \geq 6$, *a solution to* $(4, m) - \mathrm{URD}(v; r, s)$ *exists if and only if* $4|v$ *and* $m|v$;

- *for all even* $m \geq 4$*, a solution to* $(m, 2m) - \mathrm{URD}(v; r, s)$ *exists if and only if* $2m|v$.

Two years later Keranen and Özkan [47] solved the case of $4-$cycles and a single factor of $m-$cycles for odd m.

Theorem 2.10: [47] For all odd m*, a solution to* $(4, m) - \mathrm{URD}(v; r, 1)$ *exists if and only if* $4m|v$ *and* $r = \frac{v-4}{2}$.

In a recent paper [48], a complete solution to the problem for $3-$cycles and $7-$cycles is given.

Uniformly resolvable decompositions of K_v or $K_v - I$ into graphs other than cycles have also been considered in [49]-[54].

Most of the results about uniformly resolvable decomposition of K_v (or $K_v - I$ for even v) involve the cases of even cycles or some graphs other than cycle. Solving the Hamilton-Waterloo Problem for cycles with different parity is a more difficult problem and is not studied much.

In this dissertation, firstly $4-$cycle and odd cycle factors is considered, and the remaining cases in [47] are completed. This result also complements the results of Fu and Huang [46] and shows that the necessary conditions are sufficient also for odd m with a few exceptions. Then, the problem for the case of $m-$cycles and $4m-$cycles is studied, and a complete solution is given for even m as well as all possible solutions with a few possible exceptions are determined for odd m.

When we give solutions to $(4, m) - \mathrm{URD}(v; r, s)$ and $(m, 4m) - \mathrm{URD}(v; r, s)$, first we give some cycle decompositions of $C_m[4]$ and $C_m[4] \oplus mK_4 - I$ in Section 3, then we show how to decompose $K_v - I$ into subgraphs including $C_m[4]$'s and one $C_m[4] \oplus mK_4 - I$ in Sections 4 and 5.

3. 2-FACTORIZATIONS OF $C_m[4]$

It is obvious that a $2-$factorization of $C_m[4]$ has exactly four factors. The following results will be shown:

- $C_m[4]$ has a C_4-factorization (Lemma 3.1),
- $C_m[4]$ has a C_m-factorization (Lemma 3.2),
- $C_m[4]$ has a $\{C_4^2, C_m^2\}-$factorization (Lemma 3.3),
- $C_m[4]$ has no $\{C_4^1, C_m^3\}-$factorization (Lemma 3.4),
- $C_m[4]$ has a $\{C_m^{2a}, C_{4m}^{2b}\}-$factorization for $a, b \in \{0,1,2\}$ with $a + b = 2$ (Lemma 3.5).

Lemma 3.1: For every integer $m \geq 3$, $C_m[4]$ has a C_4-factorization.

Proof 3.1: Note that $C_m[4] \cong C_m[2][2]$. By Lemma 2.2, $C_m[2]$ can be decomposed into $C_{2m}-$factors, and each C_{2m} can be decomposed into two $1-$factors. So $C_m[2]$ has a $1-$factorization. If F is a $1-$factor in $C_m[2]$, $F[2]$ is a C_4-factor in $C_m[4]$ since $K_2[2] \cong C_4$. Hence $C_m[4]$ has a C_4-factorization.

Lemma 3.2: For every integer $m \geq 3$, $C_m[4]$ has a C_m-factorization.

Proof 3.2: We can represent $C_m[4]$ as the Cayley graph over $V_4 \times \mathbb{Z}_m$ with connection set $V_4 \times \{1, -1\}$ where V_4 is the additive group of $\mathbb{F}_4 = \{0,1,x,x^2\}$, the finite field of order 4. Let $C = (v_0, v_1, v_2, \ldots, v_{m-1})$ be an $m-$cycle of $C_m[4]$ where $v_i = (x^i, i)$ for $0 \leq i \leq m-1$. In the case of $m \equiv 1 \ (mod\ 3)$ replace v_{m-1} with $(x, m-1)$. Then

$$F = C \cup (x,1) \cdot C \cup (x^2,1) \cdot C \cup (0,1) \cdot C \quad (3.1)$$

is a $2-$factor of $C_m[4]$. And also

$$\mathcal{F} = \{F, F + (1,0), F + (x,0), F + (x^2,0)\} \quad (3.2)$$

is a $2-$*factorization of* $C_m[4]$.

It is evident that the addition by $(1,0)$ and multiplication by $(x,1)$ are automorphisms of the above factorization $\mathcal{F}$. These automorphisms clearly generate $AGL(1,4)$ (the $1-$ dimensional affine general linear group over $\mathbb{F}_4$).

Example 3.1: The Figure 3.1 shows a decomposition of $C_5[4]$ *into four* C_5-*factors.*

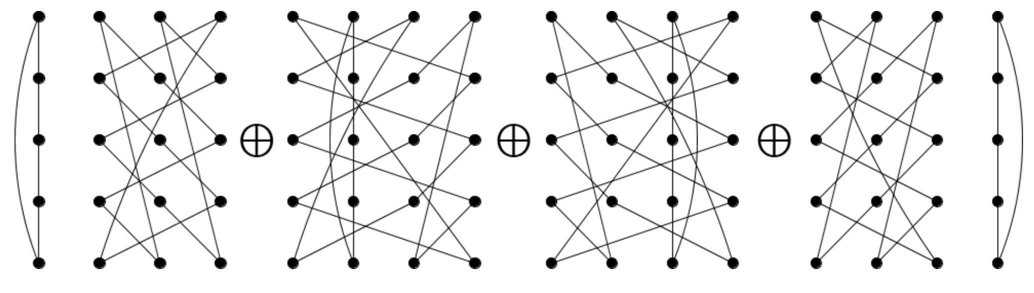

Figure 3.1: A C_5-factorization of $C_5[4]$.

Lemma 3.3: For every integer $m \geq 3$, $C_m[4]$ *has a* $\{C_4^2, C_m^2\}-$*factorization.*

Proof 3.3: We can represent $C_m[4]$ *as the Cayley graph* Γ *over* $\mathbb{Z}_4 \times \mathbb{Z}_m$ *with connection set* $\mathbb{Z}_4 \times \{1,-1\}$.

When m *is even, let* $C = (v_0, v_1, v_2, \dots, v_{m-1})$ *and* $C' = (v'_0, v'_1, v'_2, \dots, v'_{m-1})$ *be the* $m-$*cycles of* Γ *where* $v_i = (2i, i)$ *and* $v'_i = (0, i)$ *for* $0 \leq i \leq m-1$. *Then*

$$F_1 = C \cup (C+(1,0)) \cup (C+(2,0)) \cup (C+(3,0)) \text{ and}$$
$$F'_1 = C' \cup (C'+(1,0)) \cup (C'+(2,0)) \cup (C'+(3,0)) \quad (3.3)$$

are two edge-disjoint $m-$*cycle factors of* Γ.

Also let $C_* = ((0,1),(1,0),(2,1),(3,0))$ *be a* $4-$*cycle of* Γ. *Then*

$$F_2 = \bigcup_{i=0}^{m-1} (C_* + (0,i)) \quad (3.4)$$

is a $4-$*cycle factor of* Γ. *Moreover*

$$\mathcal{F} = \{F_1, F_1', F_2, F_2 + (1,0)\} \tag{3.5}$$

is a $2-$*factorization of* Γ.

When m *is odd, let* C, C' *and* C_* *be defined as above with* $v_{m-1} = (1, m-1)$. *Also let* $C_*' = ((0,0), (2, m-1), (1, m-2), (3, m-1))$ *be a* $4-$*cycle of* Γ. *Then*

$$F_1 = C \cup (C + (1,0)) \cup (C + (2,0)) \cup (C + (3,0)),$$
$$F_1' = C' \cup (C' + (1,0)) \cup (C' + (2,0)) \cup (C' + (3,0)) \text{ and} \tag{3.6}$$
$$F_2 = \bigcup_{i=0}^{m-3} ((C_* + (0,i)) \cup C_*' \cup (C_*' + (2,0)))$$

are $2-$*factors of* Γ. *Moreover*

$$\mathcal{F} = \{F_1, F_1', F_2, F_2 + (1,0)\} \tag{3.7}$$

is a $2-$*factorization of* Γ.

Example 3.2: The Figure 3.2 shows a $\{C_4^2, C_5^2\}-$*factorization of* $C_5[4]$.

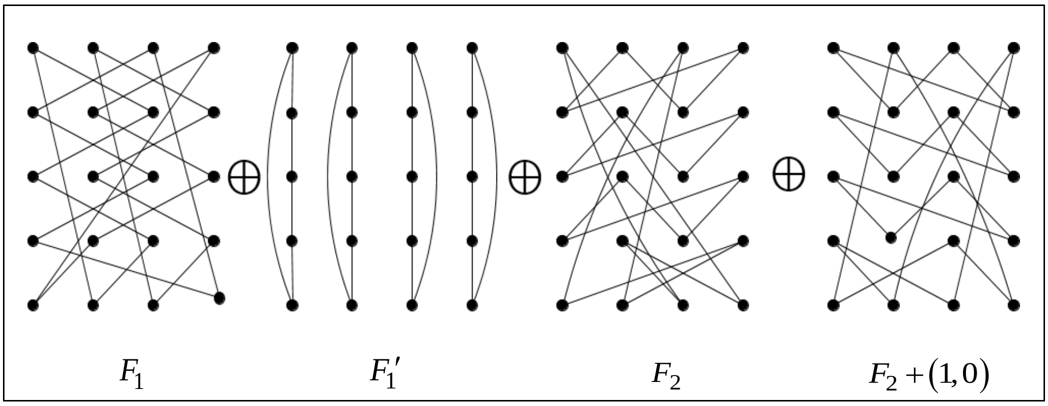

Figure 3.2: A $\{C_4^2, C_5^2\}-$factorization of $C_5[4]$.

Lemma 3.4: For every odd integer $m \geq 3$, $C_m[4]$ *has no* $\{C_4^1, C_m^3\}-$*factorization; that is,* $C_m[4] \ncong mC_4 \oplus 4C_m \oplus 4C_m \oplus 4C_m$.

Proof 3.4: Consider $C_m[4]$ as the Cayley graph Γ over $\mathbb{Z}_4 \times \mathbb{Z}_m$ with connection set $\mathbb{Z}_4 \times \{1, -1\}$ as before.

We prove the Lemma by contradiction. So assume that Γ can be decomposed into three C_m −factors and a single C_4 −factor.

Since m is odd, each m −cycle in Γ contains one and only one vertex (a, i) of Γ for each $i \in \mathbb{Z}_m$. When we remove the three C_m −factors, we are left with a 2 −regular graph where each vertex (a, i) is adjacent to only one vertex $(b, i-1)$ and only one vertex $(c, i+1)$ for some $b, c \in \mathbb{Z}_4$. So, this 2 −regular graph can not contain any 4 −cycles.

Hence, Γ has no $\{C_4^1, C_m^3\}$ −factorization.

Let $C_m[4]$ be the Cayley graph over $\mathbb{Z}_4 \times \mathbb{Z}_m$ with connection set $\mathbb{Z}_4 \times \{1, -1\}$ and in the K_4 −factor of K_{4m}, each K_4 consists of vertices $(0, i), (1, i), (2, i)$ and $(3, i)$ for $0 \leq i \leq m-1$.

Now, define paths in $C_m[4]$;

$$\begin{aligned} P_0 &= p_0 p_1 \dots p_{m-1} \\ Q_0 &= q_0 q_1 \dots q_{m-1} \\ R_0 &= r_0 r_1 \dots r_{m-1} \\ S_0 &= s_0 s_1 \dots s_{m-1} \end{aligned} \tag{3.8}$$

where $p_i = (0, i)$, $q_i = (2i, i)$, $r_i = \begin{cases} (0, i) & if\ i\ is\ even \\ (1, i) & if\ i\ is\ odd \end{cases}$, $s_i = (3,1) \cdot r_i$ for $0 \leq i \leq m-2$, and $p_{m-1} = (1, m-1)$, $q_{m-1} = (2m-2, m-1)$, $r_{m-1} = (0, m-1)$ and $s_{m-1} = (3, m-1)$. Also for $j = 1,2,3$, $P_j = P_0 + (j, 0)$, $Q_j = Q_0 + (j, 0)$, $R_j = R_0 + (j, 0)$ and $S_j = S_0 + (j, 0)$.

For the sake of brevity, we use the following notations to denote the following four m −cycle factors and three $4m$ −cycle factors of $C_m[4]$:

$$F_P = \big((P_0) \cup (P_1) \cup (P_2) \cup (P_3)\big)\ and\ F_P' = (P_0 P_1 P_2 P_3), \tag{3.9}$$

$$F_R = \big((R_0) \cup (R_1) \cup (R_2) \cup (R_3)\big)\ and\ F_R' = (R_0 R_1 R_2 R_3), \tag{3.10}$$

$$F_S = \big((S_0) \cup (S_1) \cup (S_2) \cup (S_3)\big) \text{ and } F'_S = (S_0 S_1 S_2 S_3), \tag{3.11}$$

and

$$F_Q = \big((Q_0) \cup (Q_1) \cup (Q_2) \cup (Q_3)\big). \tag{3.12}$$

Lemma 3.5: Let $a, b \in \{0,1,2\}$ *with* $a + b = 2$. *Then for every integer* $m \geq 3$, $C_m[4]$ *has a* $\{C_m^{2a}, C_{4m}^{2b}\}$ *−factorization.*

Proof 3.5: Note that $C_m[4] \cong C_{2m}[2][2]$. *Since* $C_{2m}[2]$ *can be decomposed into two* C_{2m} *−factors by Lemma 2.2,* $C_m[4]$ *has a* $C_{2m}[2]$ *−factorization, that is* $C_m[4] \cong C_{2m}[2] \oplus C_{2m}[2]$.

For even m, *each* $C_{2m}[2]$ *−factor of* $C_m[4]$ *can be decomposed into two* C_l *−factors for* $l \in \{m, 4m\}$ *by Lemma 2.2. Thus we have the desired* 2 *−factorizations for even* m.

Similarly for odd m, $C_m[4]$ *has a* C_{4m} *−factorization since each* $C_{2m}[2]$ *can be decomposed into two* C_{4m} *−factors. Also by Lemma 3.2,* $C_m[4]$ *can be decomposed into four* C_m *−factors for all integers with* $m \geq 3$. *In addition to these* $\{F_P, F_Q, F'_R, F'_S\}$ *is a* $\{C_m^2, C_{4m}^2\}$ *−factorization of* $C_m[4]$ *for odd* m.

4. SOLUTION TO $(4, m) - \mathrm{URD}(v; r, s)$

In this section, first we decompose $K_v - I$ into subgraphs including $C_m[4]$'s, and then we give solutions to the problem for $4-$cycles and $m-$cycles using appropriate factorizations of $C_m[4]$.

By [25] and [28], solutions to $\mathrm{OP}(4^{v/4})$ and $\mathrm{OP}(m^{v/m})$ exist except $m = 3$ and $v = 6$ or $v = 12$. That is a solution to $(4, m) - \mathrm{URD}(v; r, s)$ exists for $r = 0$ or $s = 0$ with exceptions $(v, m, r) = (6,3,0)$ and $(v, m, r) = (12,3,0)$. So, we can assume that $r \neq 0$ and $s \neq 0$.

In our case $4|v$, $m|v$ and m is odd. Then there exists a $t \in \mathbb{Z}^+$ such that $v = 4mt$.

Note that;

$$K_{4mt} \cong K_{mt}[4] \oplus mtK_4 \tag{4.1}$$

or equivalently

$$K_{4mt} - I \cong K_{mt}[4] \oplus mtC_4 \tag{4.2}$$

where $V(K_{4mt}) = V(K_{mt}[4])$ and I is a $1-$factor in K_{4mt}. Since K_{mt} has a C_m-factorization for odd t [25] and by the equivalence (4.1), K_{4mt} has $\{(C_m[4])^{(mt-1)/2}, K_4\}-$factorization for odd t. In short, for odd t we have

$$K_{4mt} \cong \overbrace{tC_m[4] \oplus tC_m[4] \ldots \oplus tC_m[4]}^{(mt-1)/2} \oplus mtK_4 \tag{4.3}$$

Similarly, since K_{mt} has $\left\{C_m^{(mt-2)/2}, K_2\right\}-$factorization for even t [16] and by the equivalence (4.1), K_{4mt} has a $\{(C_m[4])^{(mt-2)/2}, K_{4,4}, K_4\}-$factorization for even t. In short, for even t, we have

$$K_{4mt} \cong \overbrace{tC_m[4] \oplus tC_m[4] \dots \oplus tC_m[4]}^{(mt-2)/2} \oplus \frac{mt}{2} K_{4,4} \oplus mtK_4 \tag{4.4}$$

with exceptions $m = 3$ and $t = 2$ or $t = 4$.

In our proofs, we will use these decompositions with appropriate factorizations of $C_m[4]$'s.

4.1. When r is Odd

Now, we can prove that for odd $m \geq 3$, a solution to $(4, m) - \text{URD}(v; r, s)$ exists for all odd r (or even s) satisfying the necessary conditions.

Theorem 4.1: For all positive odd integers r and $m \geq 3$, a solution to $(4, m) - \text{URD}(v; r, s)$ exists if and only if $4|v$, $m|v$ and $r + s = \frac{v-2}{2}$ except possibly $v = 24,48$ when $m = 3$.

Proof 4.1: *If a solution to $(4, m) - \text{URD}(v; r, s)$ exists, then by Lemma 2.1, $m|v$, $4|v$* and $r + s = \frac{v-2}{2}$ since v is even.

For the sufficiency part, assume $m \geq 3$ is odd, $m|v$ and $4|v$. Then, since $gcd(4, m) = 1$, $4m|v$. Thus, there exists a positive integer t such that $v = 4mt$.

We will prove the theorem in two cases; t is odd or even.

- Case 1: *Assume t is odd.*

By (4.3), $K_{4mt} - I$ has a $\{(C_m[4])^{(mt-1)/2}, C_4\}$ −factorization. Now, let r_1, s_1 and x be non-negative integers with $r_1 + s_1 + x = \frac{mt-1}{2}$. Placing a C_4 −factorization on r_1 of the $C_m[4]$ −factors by Lemma 3.1, a C_m −factorization on s_1 of the $C_m[4]$ −factors by Lemma 3.2 and a $\{C_4^2, C_m^2\}$ −factorization on the remaining x $C_m[4]$ −factors by Lemma 3.3 gives us a $\{C_4^{4r_1+2x+1}, C_m^{4s_1+2x}\}$ −factorization of the $K_{4mt} - I$. That is, a solution to $(4, m) - \text{URD}(v; r, s)$ exists for $r = 4r_1 + 2x + 1$ (any positive odd integer can be written in this form for non-negative r_1 and x) and

$s = 4s_1 + 2x$. *It is not difficult to see that* $r \geq 1$ *is odd and* $s \geq 0$ *is even with* $r + s = 4r_1 + 2x + 1 + 4s_1 + 2x = 2mt - 1 = \frac{v-2}{2}$.

Therefore, a solution to $(4, m) - \mathrm{URD}(4mt; r, s)$ *exists for all odd integers* r *and* t *with* $r + s = 2mt - 1$.

• Case 2: *Now assume* t *is even, except* $t \neq 2,4$ *when* $m = 3$.

By (4.4), $K_{4mt} - I$ *has a* $\left\{(C_m[4])^{(mt-2)/2}, C_4^3\right\}$ *−factorization. Now, let* r_1, s_1 *and* x *be non-negative integers with* $r_1 + s_1 + x = \frac{mt-2}{2}$. *Similarly, placing a* C_4-*factorization on* r_1 *of the* $C_m[4]$ *−factors by Lemma 3.1, a* C_m *−factorization on* s_1 *of the* $C_m[4]$ *−factors by Lemma 3.2 and a* $\{C_4^2, C_m^2\}$ *−factorization on the remaining* x $C_m[4]$ *−factors by Lemma 3.3 gives us a* $\left\{C_4^{4r_1+2x+3}, C_m^{4s_1+2x}\right\}$ *−factorization of the* $K_{4mt} - I$.

Since any odd integer $r \geq 3$ *can be written as* $r = 4r_1 + 2x + 3$ *for non-negative integers* r_1 *and* x, *we obtain that for even* t, *a solution to* $(4, m) - \mathrm{URD}(4mt; r, s)$ *exists for all odd integers* $r \geq 3$ *(or even* $s \geq 0$*) with and* $m \geq 3$, *except* $t \neq 2,4$ *when* $m = 3$.

For $r = 1$, *since* $K_{mt}[4] \cong K_{4:mt}$ *and by the equivalence (4.2), we can write* $K_{4mt} - I \cong K_{4:mt} \oplus mtC_4$. *From [18],* $K_{4:mt}$ *has a* C_m *−factorization. So, placing a* C_m *−factorization on the* $K_{4:mt}$ *−factor yields a factorization of* $K_{4mt} - I$ *with* $s = 2mt - 2$.

4.2. When r is Even

Since $C_m[4]$ has no $\{C_4^1, C_m^3\}$ −factorization, we cannot obtain a solution to $(4, m) - \mathrm{URD}(4mt; r, s)$ for even r (or equivalently odd s) using the construction in the proof of Theorem 4.1. However, we will use a similar construction via switching the edges of a 1 −factor from K_4's with some edges of $C_m[4]$ in (4.3) and (4.4), then we will get a $\{C_4^2, C_m^3\}$ −factorization of the new graph. In short, if we let $C_m^*[4] \cong C_m[4] \oplus K_4$ and I is a 1 −factor of $C_m[4]$, then we will show that

$$C_m^*[4] - I \cong mC_4 \oplus mC_4 \oplus 4C_m \oplus 4C_m \oplus 4C_m \tag{4.5}$$

that is, $C_m^*[4] - I$ has a $\{C_4^2, C_m^3\}$ −factorization.

Lemma 4.1: For any odd integer $m \geq 3$, $(C_m[4] - I) \oplus mK_4$ *has a* $\{C_4^2,\ C_m^3\}$ *−factorization for some* 1 *−factor in* $C_m[4]$ *where each* K_4 *consists of four copies of the vertex* v_i *for any* $v_i \in C_m$.

Proof 4.1: Consider $C_m[4]$ *as the Cayley graph over* $\mathbb{Z}_4 \times \mathbb{Z}_m$ *with connection set* $\mathbb{Z}_4 \times \{1, -1\}$, *so each* K_4 *consists of the vertices* $(0, i), (1, i), (2, i)$ and $(3, i)$ *for* $i \in \mathbb{Z}_m$. *And let* $C_{(1)} = (u_0, u_1, u_2, \ldots, u_{m-1})$, $C_{(2)} = (v_0, v_1, v_2, \ldots, v_{m-1})$ *and* $C_{(3)} = (y_0, y_1, y_2, \ldots, y_{m-1})$ *be* m *−cycles of* Γ *defined by the vertices* $u_i = (0,1)$, $v_i = (i^2, i)$, *and* $y_i = (-i^2, i)$ *for* $u_{m-1} = (3, m-1)$, $v_{m-1} = (1, m-1)$, $y_{m-1} = (0, m-1)$. *Then*

$$F_1 = C_{(1)} \cup (C_{(1)} + (1,0)) \cup (C_{(1)} + (2,0)) \cup (C_{(1)} + (3,0))$$

$$F_2 = C_{(2)} \cup (C_{(2)} + (1,0)) \cup (C_{(2)} + (2,0)) \cup (C_{(2)} + (3,0)) \qquad (4.6)$$

$$F_3 = C_{(3)} \cup (C_{(3)} + (1,0)) \cup (C_{(3)} + (2,0)) \cup (C_{(3)} + (3,0))$$

are m *−cycle factors of* Γ.

Let $C_{(4)} = ((1,0), (2,0), (0,1), (3,1))$ *and* $C_{(5)} = ((0,0), (1,0), (3,0), (2,0))$ *be* 4 *−cycles of* Γ. *Then*

$$F_4 = \bigcup_{i=0}^{m-1} ((C_{(4)} + (0, i))$$

$$(4.7)$$

$$F_5 = \bigcup_{i=0}^{m-1} ((C_{(5)} + (0, i))$$

are 4 *−cycle factors of* $(\Gamma - I) \oplus mK_4$. *Then*

$$\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\} \qquad (4.8)$$

is a $2-$*factorization of* $(\Gamma - I) \oplus mK_4$ *where* I *is a* $1-$*factor of* Γ *with the edges* $(0,i)(2,i+1)$ *and* $(3,i)(1,i+1)$ *for each* $i \in \mathbb{Z}_m$.

Now, we give solutions to the Hamilton-Waterloo Problem for some small cases and improve the results given in [38].

Th*eorem 4.2: For all positive integer* r, *a solution to* $(4,3) - \mathrm{URD}(24; r, s)$ *exists if and only if* $r + s = 11$ *except possibly when* $r = 2$ *and* $r = 6$.

Proof 4.2: All the cases are covered by [22] with possible exceptions when $r = 2,4,6$. *Let the vertex set* K_{24} *be* $\mathbb{Z}_{24}$. *Then, let*

- $F_1 = (0,1,10,9) \cup (2,3,17,16) \cup (4,5,19,18) \cup (6,7,8,15) \cup (11,12,21,20) \cup (13,14,23,22)$,
- $F_2 = (0,2,4,6) \cup (1,3,5,7) \cup (8,10,12,14) \cup (16,18,20,22) \cup (17,19,21,23) \cup (9,11,13,15)$,
- $F_3 = (0,3,4,7) \cup (1,2,5,6) \cup (10,11,14,15) \cup (16,19,20,23) \cup (17,18,21,22) \cup (9,12,13,8)$,
- $F_4 = (0,4,1,5) \cup (2,6,3,7) \cup (11,15,12,8) \cup (16,20,17,21) \cup (18,22,19,23) \cup (9,13,10,14)$,
- $F_5 = (0,8,16) \cup (1,9,17) \cup (2,10,18) \cup (3,11,19) \cup (4,12,20) \cup (5,13,21) \cup (6,14,22) \cup (7,15,23)$,
- $F_6 = (0,13,19) \cup (1,14,20) \cup (2,15,21) \cup (3,8,22) \cup (4,9,23) \cup (5,10,16) \cup (6,11,17) \cup (7,12,18)$,
- $F_7 = (0,14,18) \cup (1,15,19) \cup (2,8,10) \cup (3,9,21) \cup (4,10,22) \cup (5,11,23) \cup (6,12,16) \cup (7,13,17)$,
- $F_8 = (0,15,20) \cup (1,8,21) \cup (2,9,22) \cup (3,10,23) \cup (4,11,16) \cup (5,12,17) \cup (6,13,18) \cup (7,14,19)$,
- $F_9 = (0,12,23) \cup (1,13,16) \cup (2,14,17) \cup (3,15,18) \cup (4,8,19) \cup (5,9,20) \cup (6,10,21) \cup (7,11,22)$,
- $F_{10} = (0,11,21) \cup (1,12,22) \cup (2,13,23) \cup (3,14,16) \cup (4,15,17) \cup (5,8,18) \cup (6,9,19) \cup (7,10,20)$,

- $F_{11} = (0,10,17) \cup (1,11,18) \cup (2,12,19) \cup (3,13,20) \cup (4,14,21) \cup (5,15,22) \cup (6,8,23) \cup (7,9,16)$.

Then,

$$\mathcal{F} = \{F_1, F_2, \dots, F_{11}\} \tag{4.9}$$

is a $2-$*factorization of* $K_{24} - I$ *with four* C_4-*factors where* $I = \{(0,22), (1,23), (3,12), (4,13), (5,14), (6,20), (7,21), (8,17), (9,18), (10,19), (15,16)\}$. *This completes the case* $r = 4$.

Here we give a sketch of the construction of this result. We assumed that the vertex set of K_{24} *is* $A \cup B \cup C$ *where* $A = \{a_i \ : i = 0,1,2, \dots, 7\}$*,* $B = \{b_i \ : i = 0,1,2, \dots, 7\}$ *and* $C = \{c_i \ : i = 0,1,2, \dots, 7\}$. *Also we let the sets of edges*

- $(AB)_d = \{(a_i, b_{i+d}) \ : \ i = 0,1,2, \dots, 7\}$,
- $(BC)_d = \{(b_i, c_{i+d}) \ : \ i = 0,1,2, \dots, 7\}$ *and*
- $(CA)_d = \{(c_i, a_{i+d}) \ : \ i = 0,1,2, \dots, 7\}$

for $0 \le d \le 27$ *and* $E(K_{24}) = E(A) \cup E(B) \cup E(C) \cup \bigcup_{i=0}^{7} \{(AB)_d \cup (BC)_d \cup (CD)_d\}$ *where* $E(A)$, $E(B)$ *and* $E(C)$ *are the edges of the complete graphs induced by the set of vertices* A, B *and* C *respectively. Then,*

- $F_5 = (AB)_0 \cup (BC)_0 \cup (CA)_0$,
- $F_6 = (AB)_5 \cup (BC)_6 \cup (CA)_5$,
- $F_7 = (AB)_6 \cup (BC)_4 \cup (CA)_6$,
- $F_8 = (AB)_7 \cup (BC)_5 \cup (CA)_4$,
- $F_9 = (AB)_4 \cup (BC)_3 \cup (CA)_1$,
- $F_{10} = (AB)_3 \cup (BC)_2 \cup (CA)_3$ *and*
- $F_{11} = (AB)_2 \cup (BC)_7 \cup (CA)_7$

are edge-disjoint $3-$*factors of* K_{24}.

Finally, using the remaining edges $(AB)_1, (BC)_1, (CA)_2, E(A), E(B)$ *and* $E(C)$ *we construct four* C_4 *−factors* F_1, F_2, F_3, F_4 *and* 1 *−factor* I. *And at the end we rename the vertices as* $x_i = i$, $y_i = i + 8$ *and* $z_i = i + 16$.

Theorem 4.3: For all positive integer r *a solution to* $(4,3) - \mathrm{URD}(48; r, s)$ *exists if and only if* $r + s = 23$ *wih a possible exception when* $r = 6$.

Proof 4.3: It is known that $(4,3) - \mathrm{URD}(48; r, s)$ *has a solution with the possible exceptions when* $r = 6,8,10,14,16,18$ *[22]. By (4.1),* $K_{48} \cong K_{12}[4] \oplus 12K_4$ *and a solution to* $(4,3) - \mathrm{URD}(12; 1, 4)$ *has given in the Appendix of [21]. Hence a* $\{(C_3[4])^4, C_4[4], K_{4,4}, K_4\}$ *−factorization of* K_{48} *exists. Also by Lemma 3.1,* $C_4[4]$ *can be decomposed into four* C_4 *−factors, by Lemma 4.2,* $(C_3[4] - I) \oplus 3K_4$ *has a* $\{C_4^2, C_3^3\}$ *−factorization, and it is easy to see that* $K_{4,4}$ *can be decomposed into two* C_4 *−factors. So we now have 8* C_4 *−factors and 3* C_3 *−factors already. For the remaining three* $C_3[4]$*'s, decompose* r_1 *of them into* C_4 *−factors,* s_1 *of them into* C_3 *−factors and* x *of them into two* C_4 *−factors and two* C_3 *−factors where* $r_1 + s_1 + x = 3$ *by Lemmas 3.1, 3.2 and 3.3 respectively. Hence, we get* $r = 4r_1 + 2x + 8$ *and* $s = 4s_1 + 2x + 3$ *and this gives us a* $\{C_4^r, C_3^s\}$ *−factorization of* $K_{48} - I$ *for* $r = 8,10,12,14,16,18,20$.

We would like to note that, in the preparation of this dissertation, we have discovered that in their preprint [55] Bonvicini and Buratti have given an independent solution to all of the nine remaining cases from [38]. We include our independent solutions for six of these cases.

Theorem 4.4: For all positive even r *and odd* $m \geq 3$, *a solution to* $(4, m) - \mathrm{URD}(v; r, s)$ *exists if and only if* $4|v$, $m|v$ *and* $r + s = \frac{v-2}{2}$ *except possibly* $v = 8m$ *when* $r = 2$, *and* $v = 24, 48$ *when* $m = 3$.

Proof 4.4: We will consider two cases depending on the parity of t.

- Case 1: *Assume* t *is odd*.

  By (4.3), K_{4mt} *has a* $\{(C_m[4])^{(mt-1)/2}, K_4\}$ *−factorization.*

Let I be a $1-$*factor of* K_{4m} *as defined in Lemma 4.2 and,* r_1*,* s_1 *and* x *be non-negative integers with* $r_1+s_1+x=\frac{mt-3}{2}$*. Placing a* C_4-*factorization on* r_1 of the $C_m[4]$*'s by Lemma 3.1, a* C_m-*factorization on* s_1 of the $C_m[4]$*'s by Lemma 3.2, a* $\{C_4^2, C_m^2\}-$*factorization on* x of the $C_m[4]$*'s by Lemma 3.3 and a* $\{C_4^2, C_m^3\}-$*factorization on the remaining* $(C_m[4]-I)\oplus mK_4-$*factor by Lemma 4.2 gives us a* $\{C_4^{4r_1+2x+2}, C_m^{4s_1+2x+3}\}-$*factorization of the* $K_{4mt}-tI$ *where* tI *gives a* $1-$*factor in* K_{4mt}.

Then, since any integer $r\geq 2$ *can be written as* $r=4r_1+2x+2$ *for non-negative integers* r_1 *and* x*, a solution to* $(4,m)-\mathrm{URD}(4mt;r,s)$ *exists*
For any even $r\geq 2$ *and odd* t *satisfying* $r+s=2mt-1=\frac{v-2}{2}$.

- Case 2: *Let* t *be even*.

By (4.4), K_{4mt} *has a* $\{(C_m[4])^{(mt-1)/2}, K_{4,4}, K_4\}-$*factorization*.

For $r_1+s_1+x=\frac{mt-2}{2}$*, placing a* C_4-*factorization on* r_1 of the $C_m[4]$*'s, a* C_m-*factorization on* s_1 of the $C_m[4]$*'s, a* $\{C_4^2, C_m^2\}-$*factorization on* x of the $C_m[4]$*'s and two* C_4-*factor on the* $K_{4,4}-$*factor and a* $\{C_4^2, C_m^3\}-$*factorization on the remaining* $(C_m[4]-I)\oplus mK_4-$*factor yields a solution to* $(4,m)-\mathrm{URD}(4mt;r,s)$ *for all even* $r\geq 4$ *except* $t=2$ *or* $t=4$ *when* $m=3$.

Now, we consider the case $r=2$ *and* t *is even. Partitioning the vertices of* K_{4mt} *into* t *sets of size* $4m$ *gives the equivalence:* $K_{4mt}-I\cong t(K_{4m}-I')\oplus K_{4m:t}$ *where* I' *is a* $1-$*factor of* K_{4m}*. By case* 1*,* $K_{4m}-I'$ *has a* $\{C_4^2, C_m^{2m-3}\}-$*factorization and also from Theorem 2.8,* $K_{4m:t}$ *has a* C_m-*factorization for* $t\neq 2$*. Thus,* $K_{4mt}-I$ *has a* $\{C_4^2, C_m^{2mt-3}\}-$*factorization.*

4.3. First Main Result

Combining the results of the previous section it is now possible to obtain the proof of the following Theorem.

Theorem 4.5: For all positive integers r, s *and odd* $m \geq 3$, *a solution to* $(4, m) - \mathrm{URD}(v; r, s)$ *exists if and only if* $4|v$, $m|v$ *and* $r + s = \frac{v-2}{2}$ *except possily when* $r = 2$ *and* $v = 8m$ *or* $v = 24{,}48$ *when* $m = 3$ *and* $r = 6$.

Proof 4.5: Odd r follows from Theorem 4.1 and even r *follows from Theorem 4.5 with possible exceptions when* $r = 2$ *and* $v = 8m$, *and* $v = 24$ *or* $v = 48$ *when* $m = 3$. *Theorem 4.3 and Theorem 4.4 cover some of these exceptions for* $m = 3$ *and the remaining cases are* $r = 2$ *when* $v = 8m$, *and* $v = 24{,}48$ *and* $r = 6$ *when* $m = 3$.

Although our solution is for odd m, our results in Lemmas are valid for even m as well and will be used in the forthcoming section. Our result also complements the result of Fu and Huang [46]; altogether, existence of a solution to $(4, m) - \mathrm{URD}(v; r, s)$ is shown for all integers $m \geq 3$ with a few possible exceptions. Regarding the results of Bonvicini and Buratti, only exception would be $r = 2$ when $v = 8m$, for odd $m \geq 5$.

We can then combine these results as follows.

Theorem 4.6: For all positive integers r, s *and* $m \geq 3$, *a solution to* $(4, m) - \mathrm{URD}(v; r, s)$ *exists if and only if* $4|v$, $m|v$ *and* $r + s = \frac{v-2}{2}$ *except possibly when* $r = 2$ *and* $v = 8m$ *for* $m \geq 5$ *odd.*

5. SOLUTION TO $(m,4m)-\mathrm{URD}(v;r,s)$

In this section, firstly we decompose the $C_m[4]\oplus mK_4-I$ into C_m and C_{4m} −cycles, then we show how to decompose K_v-I into subgraphs including $C_m[4]$'s and one $C_m[4]\oplus mK_4-I$. Secondly, using these decompositions we will be able to obtain solutions to the problem for even and odd m.

Lemma 5.1: For all positive integer $m\geq 3$, $C_m[4]\oplus mK_4-I$ has a $\{C_m^a,\ C_{4m}^b\}$ −factorization for some 1 −factor I in $C_m[4]\oplus mK_4$, $a=0,1,2,3$ and also $a=4$ when m is even, satisfying $a+b=5$ where each K_4consists of four copies of the vertex v_i for any $v_i\in C_m$.

Proof 5.1: Let $C_m[4]$ be the Cayley graph over $\mathbb{Z}_4\times\mathbb{Z}_m$ with connection set $\mathbb{Z}_4\times\{1,-1\}$ and in the K_4 −factor of K_{4m}, each K_4 consists of vertices $(0,i),(1,i),(2,i)$ and $(3,i)$ for $0\leq i\leq m-1$. Now define paths in $C_m[4]\oplus mK_4$;

$$\begin{aligned}X&=x_0x_1x_2\ldots x_{2m-1},\\ Y&=y_0y_1y_2\ldots y_{2m-1},\\ Z&=z_0z_1z_2\ldots z_{2m-1},\\ W&=Y+(1,0)\end{aligned}\tag{5.1}$$

where $x_{4i}=(3,2i)$, $x_{4i+1}=(0,2i)$, $x_{4i+2}=(2,2i+1)$, $x_{4i+3}=(1,2i+1)$, $y_{2i}=(0,i)$, $y_{2i+1}=(2,i)$, $z_{4i}=(2,2i)$, $z_{4i+1}=(1,2i)$, $z_{4i+2}=(3,2i+1)$ and $z_{4i+3}=(0,2i+1)$. Then,

- *$F_P'\oplus F_R'\oplus F_S'\oplus(YW^{-1})\oplus(ZX^{-1})$ is a factorization with $(a,b)=(0,5)$,*
- *$F_P\oplus F_R'\oplus F_S'\oplus(YW^{-1})\oplus(ZX^{-1})$ is a factorization with $(a,b)=(1,4)$,*
- *$F_P\oplus F_R\oplus F_S'\oplus(YW^{-1})\oplus(ZX^{-1})$ is a factorization with $(a,b)=(2,3)$ and*
- *$F_P\oplus F_R\oplus F_S\oplus(YW^{-1})\oplus(ZX^{-1})$ is a factorization with $(a,b)=(3,2)$*

where F_P, F_P', F_R, F_R', F_S and F_S' are 2 −factors defined as before.

Now for even m define paths in $C_m[4] \oplus mK_4$;

$Y_0 = Yy_{\frac{m}{2}-1}$, $Y_1 = y_{\frac{m}{2}}Yy_{m-1}$, $Y_2 = y_mYy_{\frac{3m}{2}-1}$, $Y_3 = y_{\frac{3m}{2}}Yy_{2m-1}$ *and* $W_j = Y_j + (1,0)$ *for* $j = 0,1,2,3$. *Then,*

- $F_P \oplus F_R \oplus F_S \oplus \left((Y_0W_0^{-1}) \cup (Y_1W_1^{-1}) \cup (Y_2W_2^{-1}) \cup (Y_3W_3^{-1})\right) \oplus (ZX^{-1})$

is a factorization with $(a,b) = (4,1)$ *for even* m.

Example 5.1: Figure 5.1 shows a $\{C_5^1, C_{20}^4\}$ *−factorization of* $C_5[4] \oplus 5K_4 - I$.

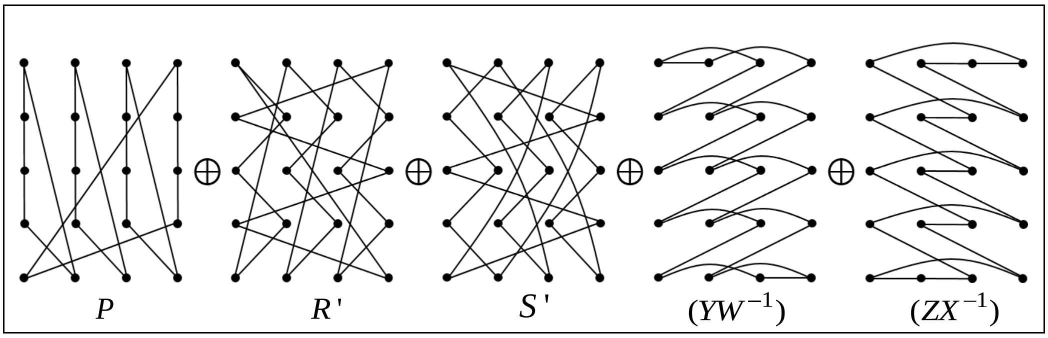

Figure 5.1: A $\{C_5^1, C_{20}^4\}$ −factorization of $C_5[4] \oplus 5K_4 - I$.

5.1. When m is Even

First we give a well-known result of Walecki [13] for Hamilton cycle decompositions of complete graph minus a 1 −factor, then using the result we will obtain solutions to our problems when $v = 4m$ which will be generalized at the end of this section.

Lemma 5.2: [10] *For all even* $m \geq 4$, $K_m - I^*$ *has a Hamilton decomposition with prescribed cycles* $\left\{C^*, \rho(C^*), \rho^2(C^*), \ldots, \rho^{\frac{m-4}{2}}(C^*)\right\}$ *for some permutation* ρ *of* $\{v_0, v_1, \ldots, v_{m-1}\}$ *where* $C^* = (v_0, v_1, \ldots, v_{m-1})$ *and* $E(I^*) = \left\{v_0v_{m/2}, v_iv_{m-i} : 1 \leq i \leq \frac{m}{2} - 1\right\}$.

By (4.1),

$$K_{4m} \cong K_m[4] \oplus mK_4 \tag{5.2}$$

where $V(K_{4m}) = V(K_m[4])$. Then, by (5.2) and Lemma 5.2, we have a $\{(C_m[4])^{(m-6)/2}, (C^* \oplus I^*)[4], C_m[4] \oplus mK_4\}$ −factorization of K_{4m} for $m \geq 6$.

In short, for even $m \geq 6$ we have

$$K_{4m} \cong \overbrace{C_m[4] \oplus C_m[4] \dots \oplus C_m[4]}^{(m-4)/2} \oplus (C^* \oplus I^*)[4] \oplus mK_4 \tag{5.3}$$

or equivalently for some $1-$factor I in $C_m[4] \oplus K_4$,

$$K_{4m} - I \cong \overbrace{C_m[4] \oplus C_m[4] \dots \oplus C_m[4]}^{\frac{m-6}{2}} \oplus (C^* \oplus I^*)[4] \oplus (C_m[4] \oplus mK_4 - I) \tag{5.4}$$

Now we give some $2-$factorizations of $(C^* \oplus I^*)[4]$ using the following lemma.

Lemma 5.3: Let $m \geq 4$ be an even integer and $G_m^* = C^* \oplus I^*$ where $C^* = (v_0, v_1, \dots, v_{m-1})$ be an $m-$cycle and I^* is a $1-$factor of K_m with $E(I^*) = \left\{v_0 v_{m/2}, v_i v_{m-i} \;:\; 1 \leq i \leq \frac{m}{2} - 1\right\}$. Then $G_m^*[2]$ has a C_{2m} −factorization.

Proof 5.3: Let the vertex set of G_m^* be $\mathbb{Z}_2 \times \mathbb{Z}_m$ and define a cycle and two paths in G_m^* as follow:

$$\begin{aligned} C &= (u_0, u_1, \dots, u_{2m-1}) \\ A &= (a_0 a_1 \dots a_{m-1}) \\ B &= (b_0 b_1 \dots b_{m-1}) \end{aligned} \tag{5.5}$$

where $u_i = \begin{cases} (i,i) & for \quad 0 \leq i \leq m-1 \\ (i+1,i) & for \quad 0 \leq i \leq 2m-1 \end{cases}$, $a_i = (0,1)$ *and* $b_0 = (1,0)$, *and for* $1 \leq i \leq m-1$ *if* $m \equiv 0 \ (mod\ 4)$, *then*

$$b_i = \begin{cases} \left(1, \frac{m}{2} + \left\lfloor \frac{i}{2} \right\rfloor\right) & for \quad i \equiv 0{,}3 \ (mod\ 4) \\ \left(1, \frac{m}{2} - \left\lfloor \frac{i}{2} \right\rfloor\right) & for \quad i \equiv 1{,}2 \ (mod\ 4) \end{cases}$$ *, and if* $m \equiv 2\ (mod\ 4)$*, then*

$$b_i = \begin{cases} \left(1, \frac{m}{2} + \left\lfloor \frac{i}{2} \right\rfloor\right) & for \quad i \equiv 1{,}2 \ (mod\ 4) \\ \left(1, \frac{m}{2} - \left\lfloor \frac{i}{2} \right\rfloor\right) & for \quad i \equiv 0{,}3 \ (mod\ 4) \end{cases}$$ *. Then* C *and* (AB) *are two edge-disjoint* $2m$ *−cycles in* $G_m^*[2]$*. Also it can be checked that* $C' = G_m^* - (C \oplus (AB))$ *is an* $2m$ *−cycle. Thus* $\{C, C', (AB)\}$ *is a* C_{2m} *−factorization of* $G_m^*[2]$.

Corollary 5.1: Let $m \geq 4$ *be an even integer and* $G_m^* = C^* \oplus I^*$ *where* $C^* = (v_0, v_1, \dots, v_{m-1})$ *be an* m *−cycle and* I^* *is a* 1 *−factor of* K_m *with* $E(I^*) = \left\{v_0 v_{m/2}, v_i v_{m-i} \ : \ 1 \leq i \leq \frac{m}{2} - 1\right\}$*. Then* $G_m^*[4]$ *has a* $\{C_m^{2a}, C_{4m}^{2b}\}$ *−factorization for all non-negative integers* a *and* b *with* $a + b = 3$.

Proof 5.1: By Lemma 5.3, $G_m^*[2]$ *can be decomposed into three* C_{2m} *−factors, that is,* $G_m^*[2] \cong C_{2m} \oplus C_{2m} \oplus C_{2m}$*. So we have* $G_m^*[4] \cong C_{2m}[2] \oplus C_{2m}[2] \oplus C_{2m}[2]$*, since* $G_m^*[4] \cong G_m^*[2][2]$*. Also by Lemma* 2.4*, each* $C_{2m}[2]$ *− factor of* $G_m^*[4]$ *can be decomposed into two* C_l *−factors for* $l \in \{m, 4m\}$*. Hence* $G_m^*[4]$ *has desired* 2 *−factorization.*

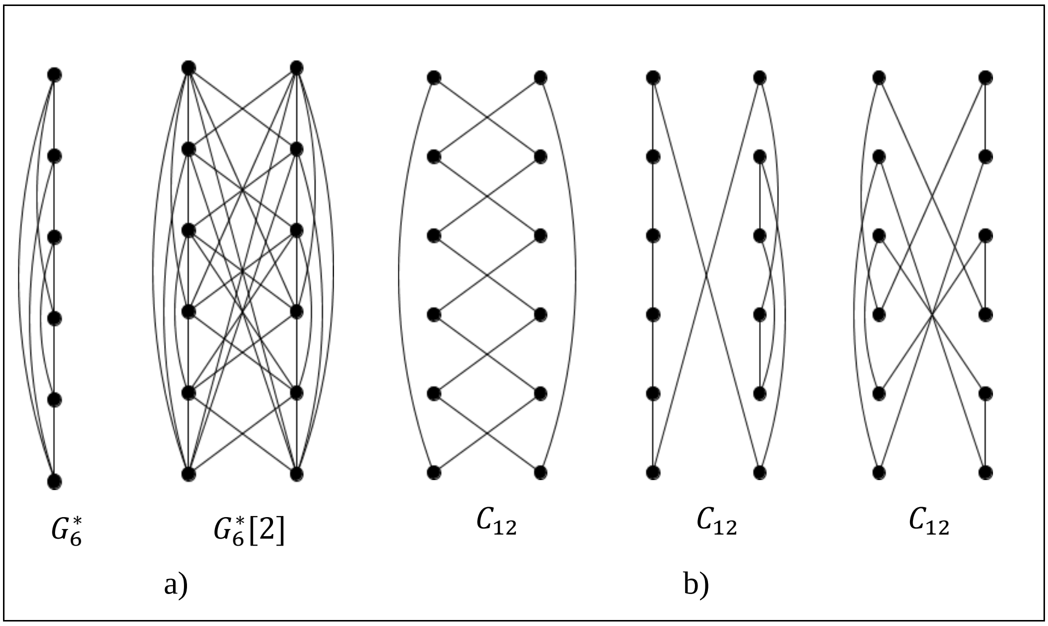

Figure 5.2: a) G_6^* and $G_6^*[2]$, b) C_{12} −factorization of $G_6^*[2]$.

In our proofs, we will use these decompositions with appropriate factorizations of $C_m[4] \oplus mK_4 - I$, $(C^* \oplus I^*)[4]$ and $C_m[4]$'s.

Lemma 5.5: For all positive integers r, s and even $m \geq 4$, a solution to $(m, 4m) -$ URD$(4m; r, s)$ exists if and only if $r + s = 2m - 1$.

Proof 5.5: Since the problem has a solution for $m = 4$ in [30], we may assume that $m > 4$. Also by (10), a $\{(C_m[4])^{(m-6)/2}, (C^* \oplus I^*)[4], C_m[4] \oplus mK_4 - I\}-$ factorization of $K_{4m} - I$ exists. Let r_i and s_i be integers for $i = 1,2,3$ with $r_1 + s_1 = m - 6$, $r_2 + s_2 = 3$, $r_3 + s_3 = 5$ and $0 \leq r_1 \leq m - 6$, $0 \leq r_2 \leq 3$, $0 \leq r_3 \leq 4$. Now, factor $\frac{m-6}{2}$ many $C_m[4]-$factors of K_{4m} into a $\{C_m^{2r_1}, C_{4m}^{2s_1}\}-$factor by Lemma 3.5, and $(C^* \oplus I^*)[4]$ into a $\{C_m^{2r_2}, C_{4m}^{2s_2}\}-$factor by Corollary 5.4. Then, factoring $C_m[4] \oplus mK_4 - I$ into a $\{C_m^{r_3}, C_{4m}^{s_3}\}-$factor by Lemma 5.1 gives us a $\{C_m^r, C_{4m}^s\}-$factorization of $K_{4m} - I$ where $r = r_3 + 2(r_2 + r_1)$ and $s = s_3 + 2(s_2 + s_1)$ It can be checked that $r + s = 2m - 1$ with $1 \leq r, s \leq 2m - 2$.

Now, we give some general solutions to our problem for even *m*.

Theorem 5.1: For all non-negative integers r, s and even $m \geq 4$, a solution to $(m, 4m) -$ URD$(v; r, s)$ exists if and only if $r + s = \frac{v-2}{2}$ and $4m|v$.

Proof 5.1: The cases where one of the r and s is zero and other is non-zero have been

solved in [28]. So, we can assume that $r \neq 0$ and $s \neq 0$.

In our case, $m|v$ and $4m|v$. Then there exists a $t \in \mathbb{Z}^+$ such that $v = 4mt$. Since $K_{2t} - I$ has a $1-$factorization, $K_{4mt} - I$ has a $\{t(K_{4m} - I'), (tK_{2m,2m})^{2t-2}\}-$factorization where I' is a $1-$factor in K_{4m} and $I = tI'$. That is,

$$K_{4mt} - I \cong t(K_{4m} - I') \oplus \overbrace{tK_{2m,2m} \oplus tK_{2m,2m} \dots \oplus tK_{2m,2m}}^{2t-2} \tag{5.6}$$

Then by Theorem 2.4, factor r_1 *of the* $K_{2m,2m}$ *into* C_m *−factors and* s_1 *of the* $K_{2m,2m}$ *into* C_{4m} *−factors with* $0 \leq r_1, s_1 \leq 2t-2$ *and* $r_1 + s_1 = 2t - 2$*. Also by Lemma 5.5,* $K_{4m} - I'$ *has a* $\{C_m^{r_2}, C_{4m}^{s_2}\}$ *−factorization for all* r_2 *and* s_2 *satisfying* $r_2 + s_2 = 2m - 1$ *with* $0 \leq r_2, s_2 \leq 2m - 1$.

Thus, $\{C_m^r, C_{4m}^s\}$ *−factorization of* $K_{4mt} - I$ *exists where* $r = r_2 + mr_1$ *and* $s = s_2 + ms_1$*. It can be checked that* $r + s = 2mt - 1$ *with* $0 \leq r, s \leq 2mt - 1$.

5.2. When m is Odd

For a given odd integer t, we have

$$K_{4mt} - I \cong \overbrace{tC_m[4] \oplus tC_m[4] \ldots \oplus tC_m[4]}^{(mt-3)/2} \oplus tC_m[4] \oplus mK_4 - I. \tag{5.7}$$

And for even t, we have

$$K_{4mt} - I \cong \overbrace{tC_m[4] \oplus tC_m[4] \ldots \oplus tC_m[4]}^{\frac{mt-4}{2}} \oplus \frac{mt}{2} K_{4,4} \oplus tC_m[4] \oplus mK_4 - I. \tag{5.8}$$

with exceptions $m = 3$ and $t = 2, 4$.

Theorem 5.2: For all positive integers r*,* s *and odd* $m \geq 3$, a solution to $(m, 4m)-$URD$(v; r, s)$ *exists if and only if* $4m|v$ and $r + s = \frac{v-2}{2}$ *except possibly when* $v \equiv 0 \ (mod\ 8)$ *or* $v \equiv 4 \ (mod\ 8)$ *and* $s = 1$.

Proof 5.2: Since $4m|v$*, there exists a* $t \in \mathbb{Z}^+$ *such that* $v = 4mt$*. Assume that* t *be an odd integer, that is* $v \equiv 4 \ (mod\ 8)$*. Also* r_i and s_i *be integers for* $i = 1,2$ *with* $r_1 + s_1 = mt - 3$, $r_2 + s_2 = 5$ *and* $0 \leq r_1 \leq mt - 3$, $0 \leq r_2 \leq 3$*. By (5.7),* $K_{4mt} - I$ *has* $\{(tC_m[4])^{(mt-3)/2}, tC_m[4] \oplus mtK_4 - I\}$ *−factorization. Now, factor* $(mt-3)/2$ *many* $C_m[4]$ *−factors of* K_{4mt} *into a* $\{C_m^{2r_1}, C_{4m}^{2s_1}\}$ *−factor by Lemma 3.5 and* $tC_m[4] \oplus mtK_4 - I$ *into a* $\{C_m^{r_2}, C_{4m}^{s_2}\}$ *−factor with* $s_2 \neq 1$ *by Lemma 5.1. Then, we* *have* $\{C_m^r, C_{4m}^s\}$ *−factorization of* $K_{4mt} - I$ *where* $r = r_2 + 2r_1$ *and* $s = s_2 + 2s_1$. *It can be checked that* $r + s = 2mt - 1$ *with* $1 \leq r < 2mt - 2$ *and* $1 < r \leq 2mt - 2$.

5.3. Second Main Result

Combining the results of the previous section it is now possible to obtain the following result.

Theorem 5.3: For all non-negative integers $r,\ s$ *and* $m \geq 3$, *a solution to* $(m, 4m) -$ URD$(v; r, s)$ *exists if and only if* $r + s = \frac{v-2}{2}$ *and* $4m|v$ *except possibly when* m *is odd and*

- $v \equiv 0\ (mod\ 8)$, *or*
- $v \equiv 4\ (mod\ 8)$ *and* $s = 1$.

6. CONCLUSION

In this dissertation, we gave solutions to the Hamilton-Waterloo problem for two different cycle sizes.

In section 3, we gave some $2-$factorizations of $C_m[4]$ which are also generalizations of cycle decompositions of Cayley graphs given in [23]-[25]. Then using these results, we obtained the following result in section 4.

Theorem 4.5: For all positive integers $r,\ s$ and odd $m \geq 3$, a solution to $(4, m) -$ URD$(v; r, s)$ exists if and only if $4|v$, $m|v$ and $r + s = \frac{v-2}{2}$ except possily when $r = 2$ and $v = 8m$ or $v = 24, 48$ when $m = 3$ and $r = 6$.

In section 5, we decomposed the graph $C_m[4] \oplus mK_4 - I$ *into* C_m and C_{4m} $-$factors. Then using these results and results of Walecki [13], we gave a complete solution to the problem for m and $4m$ $-$cycle factors when m is even. At the end of the section 5, we determined all possible solutions to the problem for the odd case with a few possible exceptions. The following theorem summarizes these results:

Theorem 5.3: For all non-negative integers $r,\ s$ and $m \geq 3$, a solution to $(m, 4m) -$ URD$(v; r, s)$ exists if and only if $r + s = \frac{v-2}{2}$ and $4m|v$ except possibly when m is odd and

- $v \equiv 0\ (mod\ 8)$, *or*
- $v \equiv 4\ (mod\ 8)$ *and* $s = 1$.

The lemmas proved in section 3, 4 and 5 provide some powerful methods which we used in our main construction and can be used in different constructions such as generalization of the Hamilton-Waterloo problem for three or more different cycle sizes.

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BIOGRAPHY

Uğur ODABAŞI was born on January 5, 1985 in Ordu, Turkey. After finishing high-school in 2002, he attended Istanbul University and received a bachelor of science degree in mathematics. After a short teaching career, he enrolled in Gebze Institute of Technology and received his Master of Science degree in mathematics in 2012. Between 2012 and 2014, he worked as a research assistant at Gebze Institute of Technology. Since 2012, he has been a PhD student in Mathematics Department in Sciences Institute of Gebze Technical University. He currently works as a research assistant at the Department of Engineering Sciences at Istanbul University.

APPENDICES

Appendix A: Publications Based on the Thesis

Odabaşı U., Özkan S., (2016), "The Hamilton-Waterloo Problem with C_4 and C_m Factors", Discrete Mathematics, 339, (1), 263–269.

Odabaşı U., Özkan S., "Uniformly Resolvable Cycle Decompositions with Four Different Factors", submitted.

Odabaşı U., Özkan S., (2015), "The Hamilton-Waterloo Problem with C_4 and C_m Factors", 13th Cologne-Twente Workshop on Graphs & Combinatorial Optimization, Istanbul, Turkey, 26-28 May.

Odabaşı U., Özkan S., (2015), "Uniformly Resolvable Cycle Decomposition with Four Cycles", Antalya Algebra Days XVII, Izmir, Turkey, 20-24 May.

Odabaşı, U., (2014), "On The Hamilton-Waterloo Problem", Antalya Algebra Days XVI, Antalya, Turkey, 9-13 May (Poster Presentation).