T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

HOMOTOPY COLIMITS OF FUNCTIONS WITH G-ACTIONS BY NATURAL TRANSFORMATIONS

TÜLAY YILDIRIM

A THESIS SUBMITTED FOR THE DEGREE OF
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GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

ÜZERİNDE DOĞAL DÖNÜŞÜMLERLE G-ETKİSİ BULUNAN İZLEÇLERİN HOMOTOPİ EŞLİMİTLERİ

TÜLAY YILDIRIM

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SUMMARY

Given a finite group G, we study functors from G-categories with an action by natural transformations of G, which are firstly defined by Villaroel-Flores in 1999. We establish a one-to-one correspondence between such functors and functors from the Groethendieck construction of certain categories. Villarroel-Flores proves an equivariant version of Thomason's theorem which identifies the homotopy type of the geometric realization of the homotopy colimit of a the composition of a nerve functor with a diagram of categories functor with the geometric realization of the nerve of the Groethendieck construction of the diagram. In this thesis, we also study his proof in details and we give an alternative proof of the equivariant version of Thomason's theorem.

Key Words: Simplicial Objects, Homotopy Colimits, Equivariant Homotopy Colimits, G-Categories, Thomason's Theorem .

ÖZET

Verilen sonlu bir G grubu için, bölgeleri G-kategoriler olan ve üzerinde G grubunun doğal dönüşümlerle etkisi olan izleçleri çalıştık. Bu etkiler ilk olarak Villarroel-Flores tarafından 1999 yilinda tanımlanmıştır. Biz bu izleçlerle, ilişkili kategorilerin Groethendieck inşalarından çıkan izleçler arasında birebir eşleme kurduk. Villarroel-Flores, sinir izleclerinin kategori diagramlari ile birlesimlerinin izleçlerin homotopi eşlimitlerinin geometrik realizasyonlarının homotopy tipleri ile Groethendieck inşalarının sinirlerinin geometrik realizasyonlarını özdeşleştiren Thomasan teoreminin, ekuvaryant versiyonunu ispatlamıştır. Bu tezde, Villarroel-Flores'in ispatını detaylıca çalıştık ve bu teoremin alternatif bir ispatını verdik.

Anahtar Kelimeler: Simplicial Nesneler, Homotopi Eşlimitleri, Eşdeğişken Homotopi Eşlimitleri, G-Kategorileri, Thomason'nın Teoremi .

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
C	:	Category
$\hat{s}$**C**	:	Category of simplicial objects in **C**
$\mathcal{P}(X)$	:	Singular set of X
η	:	Action by natural transformation
N(C)	:	Nerve of category **C**
Cat	:	Category of small category
$hocolimF$	:	Homotopy colimit of F
ind_H^G	:	Induction a G-set with G-action
$mor_{\mathbf{C}}$	:	Morphisms of **C** category
Obj(**C**)	:	Objects of **C** category
Sets	:	Category of sets
Sp	:	Category of simplicial sets
$srepF$	:	Simplicial replacement of F
SSp	:	Category of simplicial spaces

1. INTRODUCTION

Let G be a finite group. One way of constructing a topological space with a G-action is to construct a functor whose values have G-action and then to glue these values by using the homotopy colimit construction. However, one mostly begins with a functor $F : \mathbf{C} \to \mathbf{D}$ where **C** is a G-category and the values of F do not necessarily have a G-action. Here, a G-category is a category on which elements of G act as functors where the identity element act as the identity functor and the composition of these functors respects the group multiplication.

In [Villarroel-Flores, 1999], Villaroel-Flores introduce a notion of an action of G on functors from G-categories by natural transformations. In this case, the action of G induces an action on the corresponding homotopy colimit and hence the above construction yields a topological space with a group action. When there is a natural transformation between two such functors respecting the actions of G by natural transformations, then the resulting homotopy colimits are weakly G-homotopy equivalent.

One of the important results about homotopy colimit construction is the Thomason's theorem [Thomason, 1979] which identifies the homotopy colimit of a diagram obtained by composing the nerve construction with a given functor F of categories with the classifying space of the Grothendieck construction of the functor, up to homotopy. The proof uses two important properties of the diagonal functor from the category of bisimplicial objects in a given category to the that of simplicial objects. First one identifies two simplicial topological spaces which are the realizations of the simplicial sets obtained by fixing one of the variables of a bisimplicial set with the diagonal. The second one states that the diagonals of pointwise weak equivalent bisimplicial sets are weak homotopy equivalent.

In [Villarroel-Flores, 1999], equivariant version of Thomason's theorem is proved when the functor in question has an action of G by natural transformations. For the proof, Villaroel-Flores first proves the equivariant versions of the above results about the diagonal functor. Then he shows that all the maps used in the proof are equivariant. In this thesis, we closely study the proof given by Villaroel-Flores. Then we give an alternative proof which uses Thomason's theorem directly and the fact that a

G-map between CW-complexes is G-homotopy equivalent if and only if its restriction to every subgroup of G is a homotopy equivalence.

Finally, we show that every functor with an action of G can be obtained as a restriction of a functor from the Grothendieck construction of a certain category, the Grothendieck construction on the domain of it.

The thesis is organized as follows:

In chapter 2, we give some background material from the category theory and the theory of simplicial objects of a given category. We also discuss some properties of bisimplicial sets.

In chapter 3, we introduce the simplicial replacement of a functor and then we define the homotopy colimits of diagrams in the category of simplicial objects. At the end of the chapter we prove the Thomason's theorem which identifies the homotopy type of the geometric realization of a nerve of a functor of categories with the geometric realization of the nerve of a certain category.

In chapter 4, we discuss the homotopy colimits of functors with an action of finite groups by natural transformations which are defined by Villaroel-Flores [Villarroel-Flores, 1999]. The main purpose of this chapter to prove the equivariant version of Thomason's theorem. For this we follow [Villarroel-Flores, 1999].

In chapter 5, we give an alternative proof to the equivariant version of the Thomason's theorem. We also establish a one-to-one correspondence between the functors with actions by natural transformations and the Grothendieck construction of certain categories.

2. PRELIMINARIES

In this chapter, we give the necessary background on the theory of simplicial objects in a given category. For the convenience of the reader, we also give the necessary definitions on the category theory. We refer reader to [Mac Lane, 1971], [Awodey, 2010], [Hatcher, 2002], and [Simmons, 2011] for more details about the category theory and to [Goerss and Jardine, 1999], [May, 1967] and [Friedman, 2011] for the theory of simplicial objects. Throught the chapter, we denote categories with boldface notation. For example, the category of sets is denoted by **Sets**, the category of small categories is denoted by **Cat**.

2.1. Preliminaries on Category

*Definition 2.1: A category **C** consists of*

- *a class Ob(**C**), whose elements are called objects*
- *a class $mor_{\mathbf{C}}$, whose elements are called morphisms. Each morphism has a unique domain and codomain which are objects of **C**. We write $f : x \to y$ if x is the domain of f and y is the codomain of f.*
- *a binary operation $\circ$, called the composition of morphisms, satisfying the associativity and the identity axioms. More precisely, given two arrows $f : x \to y$ and $g : y \to z$, the composition $g \circ f$ is a morphism from x to z such that the following axioms hold:*
  - *Associativity axiom: If $f : x \to y$, $g : y \to z$ and $h : z \to t$ then $h \circ (g \circ f) = (h \circ g) \circ f$,*
  - *Identity axiom: For every object x in **C**, there exists a morphism $1_x : x \to x$ called the identity morphism for x, such that for any morphism $f : x \to y$, we have $1_y \circ f = f = f \circ 1_x$.*

*Definition 2.2: A category **C** is called small category if both $obj(\mathbf{C})$ and $mor_{\mathbf{C}}$ are sets. Otherwise, the category **C** is said to be large.*

Now, we introduce some categories which we use throughout the thesis.

i) The class of all sets with all functions between them as morphisms forms a category. This category is denoted by **Sets**.

ii The class of all groups with all homomorphisms between them as morphisms forms a category. We denote this category by **Grp**.

iii) The class of all topological spaces with all continuous maps between them as morphisms forms a category and denoted by **Top**.

iii) The class of all small categories with all functors between them as morphisms forms a category. We denote it by **Cat**.

Example 2.1: Let X be a pre-ordered set together with a binary operation $\leq$ *which is reflexive and transtive. This can be viewed as a category, whose set of objects being the elements of X and morphisms corresponding to the ordering. Thus for* $x \leq y \in X$*, there is exactly one morphism* $x \to y$ *in the corresponding category.*

*Definition 2.3: An initial object of a category **C** is an object* ℓ *in **C** such that for every objects X in **C**, there is a unique morphism* $\ell \to X$*. The dual idea is that of a terminal object: an object T in **C** is said to be terminal if for every object X in **C** there is a unique morphism* $X \to T$*.*

Definition 2.4: If an object is both initial and terminal, then it is called a zero object.

Example 2.2: The empty set is the unique initial object in the category of sets and every one-element set is a terminal object in this category and there are no zero objects.

Example 2.3: The empty space is the unique initial object in the category of topological spaces and every one-point space is a terminal object in this category.

*Definition 2.5: Given a category **C**, the opposite category of **C**, denoted by* $\boldsymbol{C}^{op}$*, is the category with Obj(*$\boldsymbol{C}^{op}$*)=Obj(**C**) and* $mor_{\boldsymbol{C}^{op}}(A, B) = \{f | f \in mor_{\boldsymbol{C}}(B, A)\}$

*Definition 2.6: Let **C** and **D** be categories. A (covariant) functor F from **C** to **D** is a mapping that*

- *sends each object* $X \in \boldsymbol{C}$ *to an object* $F(X) \in \boldsymbol{D}$*,*
- *sends each morphism* $f : X \to Y \in \boldsymbol{C}$ *to a morphism* $F(f) : F(X) \to F(Y) \in \boldsymbol{D}$ *such that the following two conditions hold:*

- $F(1_X) = 1_{F(X)}$ *for every object* $X \in \boldsymbol{C}$
- $F(g \circ f) = F(g) \circ F(f)$ *for all morphisms* $f : X \to Y$ *and* $g : Y \to Z$.

A contravariant functor F from **C** to **D** is a covariant functor from $\mathbf{C}^{op}$ to **D**. It means that when $f : X \to Y$ in **C**, $F(f) : F(Y) \to F(X)$ in **D**.

Example 2.4: Let $F : \boldsymbol{C} \to \boldsymbol{D}$ *be a functor and* $d \in \boldsymbol{D}$*. We define the comma category* F/d *as the category with objects (c,f) with* $c \in Obj(\boldsymbol{C})$ *and* $f : F(c) \to d$ *in* ***D****. A morphism from (c, f) to* (c', f') *is a map* $g : c \to c'$ *such that* $f' \circ F(g) = f$.

Similarly, we can define the category d/F *whose objects are the pairs* (c, u) *where* $f : d \to F(c)$.

Let ***F****,* ***G*** *be functors from a category* ***C*** *to a category* ***D****. A natural transformation* $\eta : F \to G$ *is a family of maps* $\{\eta_c : F(c) \to G(c)\}_{c \in \boldsymbol{C}}$ *in* ***D*** *such that for each* $f : c \to c'$ *in* ***C*** *the following diagram commutes:*

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\eta'_c} & G(c') \end{array} \tag{2.1}$$

that is, $\eta'_c \circ F(f) = G(f) \circ \eta_c$.

Similarly, one can define a natural transformation between contravariant functors.

Definition 2.7: The functors $G : \boldsymbol{C} \to \boldsymbol{D}$ *and* $F : \boldsymbol{D} \to \boldsymbol{C}$ *are said to be adjoint if there exists an isomorphism*

$$\Phi_{Y,X} : mor_{\boldsymbol{C}}(FY, X) \xrightarrow{\cong} mor_{\boldsymbol{D}}(Y, GX) \tag{2.2}$$

which is natural in X and Y. We then say ***F*** *is left adjoint to G and G is right adjoint to F. We write* $F \dashv G$.

The above adjunction between categories **C** and **D** is called a hom-set adjunction. Equivalently, one can define adjointness between functors by using counit-unit adjuctions. A counit-unit adjuction between two categories **C** and **D** consists of two functors $F : \mathbf{D} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{D}$ and two natural transformation $\eta : 1_{\mathbf{D}} \to GF$, $\xi : FG \to 1_{\mathbf{C}}$, called the unit and the counit of the adjunction respectively, such that the compositions

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\xi} G \tag{2.3}$$

and

$$F \xrightarrow{F\eta} FGF \xrightarrow{\xi F} F \tag{2.4}$$

are the identity transformations 1_G and 1_F respectively. These equations are called counit-unit equations. These equations are satiesfied if for each $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, we have

$$1_{GX} = G(\xi_X) \circ \eta_{GX} \tag{2.5}$$

and

$$1_{FY} = \xi_{FY} \circ F(\eta_Y). \tag{2.6}$$

Theorem 2.1: Counit-unit adjuction induces a hom-set adjuction.

Proof 2.1 : Given two functors $G : \boldsymbol{C} \to \boldsymbol{D}$ and $F : \boldsymbol{D} \to \boldsymbol{C}$ and a counit-unit adjunction $(\xi, \eta) : F \dashv G$, we can construct a hom-set adjunction by defining a natural transformation

$$\Phi : mor_{\boldsymbol{C}}(F-, -) \longrightarrow mor_{\boldsymbol{D}}(-, G-) \tag{2.7}$$

as follows: For each $f : FY \to X$ in $\boldsymbol{C}$ and each $g : Y \to GX$ in $\boldsymbol{D}$, we define

$$\Phi_{Y,X}(f) = G(f) \circ \eta_Y \tag{2.8}$$

and

$$\psi_{Y,X}(g) = \xi_X \circ F(g) \tag{2.9}$$

Since ξ and η are natural transformations, Φ and ψ are also natural transformations.

Since F is a functor and ξ is a natural transformation, we have

$$\psi(\Phi(f)) = \xi_X \circ FG(f) \circ F(\eta_Y) = f \circ \xi_{FY} \circ F(\eta_Y) = f \circ 1_{FY} = f \tag{2.10}$$

*that is $\psi \circ \Phi$ is the identity transformation on **C**.*

Similary, since G is a functor and η is natural transformation, we have

$$\Phi\psi_g = G(\xi_X) \circ GF(g) \circ \eta_Y = G(\xi_X) \circ \eta_{GX} \circ g = 1_{GX} \circ g = g \tag{2.11}$$

*and therefore $\Phi \circ \psi$ is the identity transformation on **D**. Thus Φ is a natural isomorphism with inverse $\Phi^{-1} = \psi$.* ■

Theorem 2.2: Hom-set adjuction induces a counit-unit adjuction.

Proof 2.2 : Given functors $G : \boldsymbol{C} \to \boldsymbol{D}$, $F : \boldsymbol{D} \to \boldsymbol{C}$ and a hom-set adjunction

$$\Phi : mor_{\boldsymbol{C}}(F-, -) \longrightarrow mor_{\boldsymbol{D}}(-, G-) \tag{2.12}$$

we can construct a counit-unit adjunction $(\xi, \eta) : F \dashv G$ as follows:

For each $X \in \boldsymbol{C}$, let $\xi_X = \Phi^{-1}_{GX,X}(1_{GX})$ in $mor_{\boldsymbol{C}}(FGX, X)$ where 1_{GX} in $mor_{\boldsymbol{D}}(GX, GX)$ is the identity morphism. Similarly, for each $Y \in \boldsymbol{D}$, $\eta_Y = \Phi_{Y,FY}(1_{FY})$ in $mor_{\boldsymbol{D}}(Y, GFY)$ where $1_{FY} \in mor_{\boldsymbol{C}}(FY, FY)$ is the identity morphism. Then we have,

$$\Phi_{Y,X}(f) = G(f) \circ \eta_Y \tag{2.13}$$

and

$$\Phi^{-1}_{Y,X}(g) = \xi_X \circ F(g) \tag{2.14}$$

for each $f : FY \to X$ and $g : Y \to GX$. Substituting FY for X and $\eta_Y = \Phi_{Y,FY}(1_{FY})$ for g in the second formula we obtain the first counit-unit equation

$$1_{FY} = \xi_{FY} \circ F(\eta_Y) \tag{2.15}$$

Similarly, substituting GX for Y and $\xi_X = \Phi^{-1}_{GX,x}(1_{GX})$ for f in the first formula, yields the second counit-unit equation $1_{GX} = G(\xi_X) \circ \eta_{GX}$. ■

*Example 2.5: Consider the inclusion functor G : **Ab** $\rightarrow$ **Grp** from the category of abelian groups to category of groups. It has a left adjoint which assigns to every group G the abelianization $G^{ab} = G/[G, G]$ of G.*

*Example 2.6: Suppose that F : **Set** $\rightarrow$ **Grp** is the functor assigning to each set Y the free group generated by the elements of Y, and that U : **Grp** $\rightarrow$ **Set** is the forgetful functor, which assigns to each group X its underlying set. Then F is left adjoint to U.*

*Definition 2.8: Let F : **J** $\rightarrow$ **C** be a functor in a category **C**. A cone to F is an object N of **C** together with a family $\psi_X : N \rightarrow F(X)$ of morphisms indexed by the objects of **J**, such that for every morphism $f : X \rightarrow Y$ in **J**, we have $F(f) \circ \psi_X = \psi_Y$.*

A limit of the functor F : **J** $\rightarrow$ **C** is a cone (L, φ) to F such that if (N, φ) is any other cone then there exists a unique morphism $U : N \rightarrow L$ such that $\varphi_X \circ U = \psi_X$ for all X in **J**.

*Example 2.7: Let **J** be the empty category. There there is only one diagram of type **J** which is the empty one. A cone to the empty diagram is an object of **C**. The limit of F is any object that is uniquely factored through by every other object. So, it is a terminal object.*

*Example 2.8: If **J** is a discrete category then a diagram F is a family of objects of **C**, indexed by **J**. The limit L of F is the product of these objects. The cone φ consists of a family of morphisms $\varphi_X : L \rightarrow F(X)$ called the projections of the product.*

A co-cone of a functor F : **J** $\rightarrow$ **C** is an object N of **C** together with a family of morphisms $\psi_X : F(X \rightarrow N)$ for every object of **J**, such that for every morphism $f : X \rightarrow Y$ in J, we have $\psi_Y o F(f) = \psi_X$.

*Definition 2.9: A colimit of a diagram F : **J** $\rightarrow$ **C** is a co-cone (L, ϕ) of F such that if (N, ψ) is another co-cone, then F there exists a unique morphism $U : L \rightarrow N$ such that $U \circ \phi_X = \psi_X$ for all X in **J**.*

For example, initial objects are colimits of empty diagrams. Coproducts are colimits of diagrams indexed by discrete categories.

2.2. Preliminaries on Simplicial Objects in a Category

Definition 2.10: The category Δ of finite totaly ordered sets is the category, whose objects are finite ordered sets [n]=[1, 2, . . . , n] for each natural number n, and morphisms $f : [n] \to [m]$ are order-preserving functions.

Here, we can consider [n] as a category with objects $1, 2, \ldots, n$ and a unique morphism for each $i \leq j$.

Every morphism $f : [n] \to [m]$ in Δ can be written in terms of morphisms $\delta^i : [n] \to [n+1]$ and $\sigma^i : [n] \to [n-1]$, called coface and codegeneracy maps, respectively. These are defined by

$$\delta^i(j) = \begin{cases} j, & j < i; \\ j+1, & j \geq i. \end{cases} \tag{2.16}$$

and

$$\sigma^i(j) = \begin{cases} j, & j \leq i; \\ j-1, & j > i. \end{cases} \tag{2.17}$$

These maps satisfy the following equalities:

$$\begin{cases} \delta^j \delta^i = \delta^i \delta^{j-1}, & i < j; \\ \sigma^j \sigma^i = \sigma^i \sigma^{j+1}, & i \leq j; \\ \sigma^j \delta^i = \delta^i \sigma^{j-1}, & i < j; \\ \sigma^j \delta^i = Id, & i = j+1, j; \\ \sigma^j \delta^i = \delta^{i-1} s^j, & i > j+1. \end{cases} \tag{2.18}$$

which are called the cosimplicial identities. Using the above relations one can easily show that every morphism in Δ can be written uniquely in the following form,

$$\delta^{j_i} \ldots \delta^{j_m} \sigma^{i_1} \ldots \sigma^{i_n} \tag{2.19}$$

with $i_1 < \ldots < i_n$ and $j_1 > \ldots > j_m$.

*Definition 2.11: For a given category **C**, a simplicial object in **C** is a covariant functor X:$\Delta^{op} \rightarrow$ **C** . The category of simplicial objects in **C** is defined to be the functor category $\boldsymbol{C}^{\Delta^{op}}$ and it is denoted by **sC**.*

*Definition 2.12: Let $X : \Delta^{op} \rightarrow$ **C** be a simplicial object in **C** where **C** is a category with objects being sets, we call $x_n \in X_n$ an n-simplex of X. A simplex $x \in X_n$ is called a degenerate simplex if $x = s_i y$ for some $0 \leq i \leq n-1$. A simplex that is not degenerate is said to be non-degenerate.*

When **C=Sets**, we call X a simplicial set. The category of simplicial set is denoted by **Sp**. When **C** $=$ **Top**, we call X a simplicial space. The category of simplicial space is denoted by **SSp**.

More precisely, a simplicial set X consists of a family of sets X_n $(n \geq 0)$ together with maps $d_i = (d^i)^* : X_{n+1} \rightarrow X_n$, $(0 \leq i \leq n)$ and $s_i = (s^i)^* : X_{n-1} \rightarrow X_n$, $(0 \leq i \leq n-1)$ called face maps and degeneracy maps respectively. These maps satisfy the dual of the cosimplicial identites.

Example 2.9: For every $n \in \mathbb{N}$, the standard n-simpliex Δ^n is a simplicial set

$$\Delta^n = mor_{\boldsymbol{Cat}}(., \boldsymbol{n}) \tag{2.20}$$

For example, the standard 0-simplex Δ^0 is a simplicial set with $\Delta^0_n = *$. Morever, Δ^0 is the terminal object in the category of simplicial sets.

Recall that the standard topological n-simplex is the space

$$\{|\Delta^n| = (t_0, \ldots, t_n) \subseteq \mathbb{R}^{n+1}; \sum ti = 1, 0 \leq t_i \leq 1\} \tag{2.21}$$

which has the subspace topology. There are also face maps $\delta^i_* : |\Delta^n| \rightarrow |\Delta^{n+1}|$ and degeneracy maps $\sigma^i_* : |\Delta^n| \rightarrow |\Delta^{n-1}|$ defined by

$$\delta^i_*(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \tag{2.22}$$

and

$$\sigma_*^i(t_0, \ldots, t_n) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_n). \quad (2.23)$$

Example 2.10: Let X be a topological space and $\mathcal{P}(X)_n$ be the set of continuous functions from $|\Delta^n|$ to X.

Let $\sigma : |\Delta^n| \to X$ be a continuous map representing a singular simplex. Then we define singular simplex $d_i\sigma : |\Delta^{n-1}| \to X$ is by

$$d_i\sigma(t_0, \ldots, t_{n-1}) = \sigma(t_0, \ldots t_{i-1}, 0, t_i, \ldots, t_{n-1}). \quad (2.24)$$

and the singular simplex $s_i\sigma : |\Delta^{n+1}| \to X$ by

$$s_i\sigma(t_0, \ldots, t_{n+1}) = \sigma(t_0, \ldots t_{i-1}, t_i + t_{i+1}, t_{i+2} \ldots, t_n). \quad (2.25)$$

Together with these face and degeneracy maps $\mathcal{P}(X)_n$'s constitute a simplicial set called the singular set *of X denoted by $\mathcal{P}(X)$.*

*Theorem 2.3: $\mathcal{P}(X) : Top \to$ **Sp** defined by*

$$\mathcal{P}(X)_n = mor_{\textbf{\textit{Top}}}(|\Delta_n|, X) \quad (2.26)$$

is a functor

*Proof 2.3 : Here, for $f : X \to Y$ in **Top**, $\mathcal{P}(f)$ sends an n-simplex $\varphi : |\Delta_n| \to X$ to $\varphi \circ f$. See [Dwyer and Henn, 2001] for more details.* ■

Definition 2.13: Let X be a simplicial set. The realization $|X|$ of X is the topological space

$$|X| = \coprod_{n=0}^{\infty} X_n \times |\Delta^n| / \sim \quad (2.27)$$

where $\sim$ is an equivalence relation defined by $(d_i(x), p) \sim (x, \delta_*^i p)$ for $x \in X_{n+1}$, $p \in |\Delta^n|$ and the relation $(s_i x, p) \sim (x, \sigma_*^i p)$ for $x \in X_{n-1}$, $p \in |\Delta^n|$. Here X_n has the discrete topology and $|X|$ has the quotient topology.

Theorem 2.4: The above structure makes $|-|: \textbf{Sp} \to \textbf{Top}$ *into a functor.*

Proof 2.4 : Here, for a simplicial map $f = (f_n) : X \to Y$*, we have*

$$|f|(x,t) = (f_n(x), t) \tag{2.28}$$

where $(x,t) \in X_n \times |\Delta^n|$*. See Proposition 3.8 in [Laine, 2013] for more details.* ■

Example 2.11: In each dimension, the standard 0-simplex has one simplex $[0, \cdots, 0]$*. Thus its geometric realization is* $\coprod_{i=0}^{\infty} |\Delta^i| \times [0, \cdots, 0]$*. Therefore in dimension 0 we have a single vertex v. The gluing instruction identify each* $(s_0[0], p) = ([0,0], p)$ *in* $([0,0], |\Delta^1|)$ *with* $([0], S_0(p)) = ([0], v)$*. Thus the* $|\Delta^1|$ *in dimension 1 gets collapsed to the vertex. Similarly, since each point of the 2-simplex* $([0,0,0], |\Delta^2|)$ *gets identified to a point of* $([0,0], |\Delta^1|)$*, and so on. Therefore,everything collapses down to a single vertex. Thus the geometric realization of* Δ^0 *is a point. Indeed, it is the standard topological 0-simplex* $|\Delta^0|$*. In general, the geometric realization of the standard n-simplex is the topological n-simplex.*

Theorem 2.5: If X is a simplicial set, then $|X|$ *is a CW complex with one n-cell for each nondegenerate n-simplex of X.*

Proof 2.5 : See Theorem 4.9 in [Friedman, 2011]. ■

Theorem 2.6: Geometric realization preserves colimits.

Proof 2.6 : See Proposition 2.4, Chapter II in [Goerss and Jardine, 1999]. ■

The adjuction relation: The realization functor $|-|$ turns out to be the adjoint to the singular set functor $\mathcal{P}$.

Theorem 2.7: If X is a simplicial set and Y is a topological space, then

$$mor_{\textbf{Top}}(|X|, Y) \cong mor_{\textbf{Sp}}(X, P(Y)), \tag{2.29}$$

Proof 2.7 : See Theorem 4.10 [Friedman, 2011]. ■

Definition 2.14: The product $X \times Y$ of simplicial sets X and Y is the simplicial set with

$$(X \times Y)_n = X_n \times Y_n = \{(x, y)|x \in X_n, y \in Y_n\} \tag{2.30}$$

where $d_i(x, y) = (d_i x, d_i y)$ and $s_i(x, y) = (s_i x, s_i y)$.

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projection maps given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

Theorem 2.8: If X and Y are simplicial sets, then $|X \times Y| \cong |X| \times |Y|$. In particular, if X and Y are countable or one of $|X|$ or $|Y|$ is locally finite as a CW-complex, then

$$|X \times Y| \cong |X| \times |Y| \tag{2.31}$$

as topological spaces.

Proof 2.8 : A map

$$\eta : |X \times Y| \to |X| \times |Y| \tag{2.32}$$

is defined by $\eta = |\pi_1| \times |\pi_2|$ where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. For more detail, we refer reader to Theorem 14.3 in [May, 1967] or [Milnor, 1957] for a proof in the latter situations and to Chapter III in [Gabriel and Zisman , 1967] for a proof of the general case. ■

Example 2.12: Let X be any simplicial set, and $Y = \Delta^0 = [0]$. Since Δ^0 has a unique element in each dimension, $X \times \Delta^0 \cong X$. Therefore $|X \times \Delta^0| \cong |X| \times |\Delta^0| \cong |X|$.

*Definition 2.15: For a given category **C**, the nerve $N(\boldsymbol{C})$ of **C** is defined to be the simplicial set*

$$N(\boldsymbol{C}) = mor_{\boldsymbol{Cat}}(-, \boldsymbol{C}) \tag{2.33}$$

Note that an n-simplex $X \in N(\mathbf{C}_n)$ is a sequence

$$\sigma : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n \tag{2.34}$$

of composible morphisms in **C**. Here, σ corresponds to a functor with $\sigma(i) = X_i$ and $\sigma((i-1) \to i) = f_i$. The face and degeneracy maps are defined as follows:

$$d_i(\sigma) = \begin{cases} \sigma(1) \to \ldots \to \sigma(n), & i = 0 \\ \sigma(0) \to \ldots \to \sigma(i-1) \xrightarrow{f_{i+1}f_i} \sigma(i+1)\ldots \to \sigma(n), & 0 < i < n \\ \sigma(0) \to \ldots \to \sigma(n-1), & i = n \end{cases} \tag{2.35}$$

and

$$s_i(\sigma) = \sigma(0) \to \ldots \to \sigma(i) \xrightarrow{1_{\sigma(i)}} \sigma(i) \to \sigma(i+1) \to \ldots \to \sigma(n) \tag{2.36}$$

A functor $F : C_1 \to C_2$ between small categories induces a map N(F) from $N(\mathbf{C}_1)$ to $N(\mathbf{C}_2)$ by sending

$$(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_q} X_q) \longrightarrow (FX_0 \xrightarrow{Ff_1} FX_1 \xrightarrow{Ff_2} \ldots \xrightarrow{Ff_q} FX_q) \tag{2.37}$$

Proposition 2.1: $N : \boldsymbol{Cat} \to \boldsymbol{Sp}$ *is a functor*

Proof 2.1 : Clearly, the map N(F) defined above is a simplicial map. See Proposition 4.2 in [Laine, 2013] for more details. ■

Proposition 2.2: $N : \boldsymbol{Cat} \to \boldsymbol{Sp}$ *respects products.*

Proof 2.2 : When the product $\boldsymbol{C} \times \boldsymbol{D}$ *of categories* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *are considered, we have*

$$\begin{aligned} N(\boldsymbol{C} \times \boldsymbol{D})_n &= mor_{\boldsymbol{Sp}}(\boldsymbol{n}, \boldsymbol{C} \times \boldsymbol{D}) \\ &\cong mor_{\boldsymbol{Sp}}(\boldsymbol{n}, \boldsymbol{C}) \times mor_{\boldsymbol{Sp}}(\boldsymbol{n}, \boldsymbol{D}) \\ &= N(\boldsymbol{C})_n \times N(\boldsymbol{D})_n \end{aligned} \tag{2.38}$$

This isomorphism clearly commutes with d_i, s_i. So we have

$$N(\boldsymbol{C} \times \boldsymbol{D}) \cong N(\boldsymbol{C}) \times N(\boldsymbol{D}). \tag{2.39}$$

as desired. ■

Example 2.13: Let $\boldsymbol{C} = [2]$ with morphism $0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2$. Then nerve of $\boldsymbol{C}$ is the simplicial set whose n-simplices are given as follows:

- $(N\boldsymbol{C})_0 = \{0, 1, 2\}$,
- $(Nd)(N\boldsymbol{C})_1 = \{0 \xrightarrow{1_0} 0, 1 \xrightarrow{1_1} 1, 2 \xrightarrow{1_2} 2, 0 \xrightarrow{f_1} 1, 0 \xrightarrow{f_2 f_1} 2, 1 \xrightarrow{f_2} 2\}$,
- $(Nd)(N\boldsymbol{C})_2 = \{0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2\}$

where (Nd) denotes the set of nondegenerate simplicies. All the higher degree simplices are degenerate. So $|N(\boldsymbol{C})| = |\Delta^2|$

Example 2.14: Let $\boldsymbol{C} = [3]$ with morphism $0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2 \xrightarrow{f_3} 3$, the nerve of $\boldsymbol{C}$ can be written as follows:

- $(N\boldsymbol{C})_0 = \{0, 1, 2, 3\}$,
- $(Nd)(N\boldsymbol{C})_1 = \{0 \xrightarrow{f_1} 1, 0 \xrightarrow{f_2 f_1} 2, 0 \xrightarrow{f_3 f_2 f_1} 3, 1 \xrightarrow{f_2} 2, 1 \xrightarrow{f_3 f_2} 3, 2 \xrightarrow{f_3} 3\}$,
- $(Nd)(N\boldsymbol{C})_2 = \{0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2, 0 \xrightarrow{f_1} 1 \xrightarrow{f_3 f_2} 3, 0 \xrightarrow{f_2 f_1} 2 \xrightarrow{f_3} 3, 1 \xrightarrow{f_2} 2 \xrightarrow{f_3} 3\}$
- $(Nd)(N\boldsymbol{C})_3 = \{0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2 \xrightarrow{f_3} 3\}$

all the higher simplices are degenerate. So $|[3]| = |\Delta^3|$.

Definition 2.16: A bisimplicial object $X : \Delta^{op} \times \Delta^{op} \to \boldsymbol{C}$ in a category $\boldsymbol{C}$ is the simplicial object $\Delta^{op} \to \boldsymbol{C}^{\Delta^{op}}$ in the category of simplicial objects in $\boldsymbol{C}$.

We denote the external face and degeneracy maps with d_i^h, s_i^h, where the h stands for the horizontal and the inner face and degeneracy maps with d_i^v, s_i^v, where v stands for the vertical ones.

There is a diagonal functor

$$diag : \Delta^{op} \to \Delta^{op} \times \Delta^{op} \tag{2.40}$$

which induces a functor

$$\mathbf{C}^{\Delta^{op}\times\Delta^{op}} \to \mathbf{C}^{\Delta^{op}} \tag{2.41}$$

which also denote by diag. Therefore, $diag(X)_n = X_{n,n}$ and $d_i = d_i^h \circ d_i^v$,$s_i = s_i^h \circ s_i^v$ for a bisimplicial object X in **C**.

Theorem 2.9: Let X be a bisimplicial set. Construct a simplicial topological space X^1 by sending [p] to the realization of the simplicial set $[q] \to X_{pq}$. Similarly, construct another simplicial topological space X^2 by sending [q] to the realization of the simplicial set $[p] \to X_{pq}$. Then we have a homeomorphism of topological spaces

$$|diag(X)| \cong |X^1| \cong |X^2| \tag{2.42}$$

Proof 2.9 : We refer reader to see p. 19 in [Gelfand and Manin, 1996]. ■

3. HOMOTOPY COLIMITS

In this chapter, we introduce homotopy colimits of a diagram $F : \mathbf{C} \to \mathbf{sD}$. We refer reader to [Bousfield and Kan, 1972] and [Goerss and Jardine, 1999] for more details. The main purpose of this chapter is to give the proof of the famous Thomason's theorem. For this, we follow [Thomason, 1979].

3.1. Homotopy Colimits

In this section we introduce the homotopy colimit of a diagram $F : \mathbf{C} \to \mathbf{sD}$ where $\mathbf{C}$ and $\mathbf{D}$ are small categories. For this, we need the definition of a simplicial replacement of a functor.

Definition 3.1: Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. The simplicial replacement of F is defined to be the simplicial object srepF in $\boldsymbol{D}$ with

$$(srepF)_n = \coprod_{\sigma \in N(\boldsymbol{C})_n} F(\sigma(0)) \tag{3.1}$$

where s_i sends $F(\sigma(0))$ indexed by σ to $F(s_i(\sigma(0)))$ indexed by $s_i \circ \sigma$ by the identity map and d_i sends $F(\sigma(0))$ indexed by σ to $F(d_i(\sigma(0)))$ indexed by $d_i \circ \sigma$ by the identity map when $i > 0$ and by the map $F(\alpha_1)$ when i=0. Here, $d_i\sigma(0) = \sigma(0)$ if $i > 0$ and $d_0\sigma(0) = \sigma(1)$.

We denote the component $F(\sigma(0))$ indexed by $\sigma \in N(\mathbf{C})_n$ with $F(\sigma(0))^\sigma$. We denote the elements of $F(\sigma(0))^\sigma_k$ by x^σ_k, y^σ_k and so on when $F(\sigma(0))$ is a set.

Theorem 3.1: If $\boldsymbol{C}$ is a small category, then

$$srep : \boldsymbol{D}^{\boldsymbol{C}} \to \boldsymbol{sD} \tag{3.2}$$

is a functor.

Proof 3.1:Let $F, G : \boldsymbol{C} \to \boldsymbol{D}$ be functors and let $\tau : F \to G$ be a natural transformation. For each n, define

$$(srep\tau)_n : \coprod_{\sigma \in N(C)_n} F(\sigma(0)) \to \coprod_{\beta \in N(C)_n} G(\beta(0)) \tag{3.3}$$

to be the map which sends $F(\sigma(0))^\sigma$ *to* $G(\sigma(0))^\sigma$ *by* $\tau_{\sigma(0)}$.

We need to prove that $srep\tau : srepF \to srepG$ *is a map of simplicial spaces, i.e. the following diagrams commute:*

$$\begin{array}{ccc} (srepF)_n & \xrightarrow{d_i} & (srepF)_{n-1} \\ {\scriptstyle (srep\tau)_n}\downarrow & & \downarrow{\scriptstyle (srep\tau)_{n-1}} \\ (srepG)_n & \xrightarrow{d_i} & (srepG)_{n-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} (srepF)_n & \xrightarrow{s_i} & (srepF)_{n+1} \\ {\scriptstyle (srep\tau)_n}\downarrow & & \downarrow{\scriptstyle (srep\tau)_{n+1}} \\ (srepG)_n & \xrightarrow{s_i} & (srepG)_{n+1} \end{array} \tag{3.4}$$

for $0 \leq i \leq n$.

We first show the commutativity of the first diagram. When $i > 0$, d_i *sends* $F(\sigma(0))^\sigma$ *to* $F(\sigma(0))^{d_i\sigma}$ *by the identity map and hence* $(srep\tau)_{n-1} \circ d_i$ *sends* $F(\sigma(0))^\sigma$ *to* $G(\sigma(0))^{d_i\sigma}$ *by* $\tau_{\sigma(0)}$. *On the other hand,* $(srep\tau)_n$ *sends* $F(\sigma(0))^\sigma$ *to* $G(\sigma(0))^\sigma$ *by* $\tau_{\sigma(0)}$ *and* d_i *sends* $G(\sigma(0))^\tau$ *to* $G(\sigma(0))^{d_i\tau}$ *by the identity. So we have,*

$$(srep\tau)_{n-1} \circ d_i = d_i \circ (srep\tau)_n \tag{3.5}$$

When i=0, d_0 *sends* $F(\sigma(0))^\sigma$ *to* $F(\sigma(1))^{d_0(\sigma)}$ *by* $F(\sigma)$ *and hence* $(srep\tau)_{n-1} \circ d_0$ *sends* $F(\sigma(0))^\sigma$ *to* $G(\sigma(1))^{d_0(\sigma)}$ *by* $\tau_{\sigma(1)} \circ F(\alpha_1)$. *On the other hand,* $d_0 \circ (srep\tau)_n$ *sends* $F(\sigma(0))^\sigma$ *to* $G(\sigma(1))^{d_0(\sigma)}$ *by* $G(\alpha_1) \circ \tau_{\sigma(0)}$.

Since τ *is a natural transformation, the following diagram commutes:*

$$\begin{array}{ccc} F(\sigma(0)) & \xrightarrow{F(\alpha_1)} & F(\sigma(1)) \\ {\scriptstyle \tau_{\sigma(0)}}\downarrow & & \downarrow{\scriptstyle \tau_{\sigma(1)}} \\ G(\sigma(0)) & \xrightarrow{G(\alpha_1)} & G(\sigma(1)) \end{array} \tag{3.6}$$

i. e. $G(\alpha(1))\tau_{\sigma(0)} = \tau_{\sigma(1)} \circ F(\alpha_1)$. *Therefore we have,*

$$(srep\tau)_{n-1} \circ d_0 = d_0 \circ (srep\tau)_n \tag{3.7}$$

For the second diagram, note that $(srep\tau)_{n+1} \circ s_i$ *sends* $F(\sigma(0))^\sigma$ *to* $G(\sigma(0))^\sigma$ *by* $\tau_{\sigma(0)}$. *On the other hand,* $(srep\tau)_n$ *sends* $F(\sigma(0))^\sigma$ *to* $G(\sigma(0))^\sigma$ *by* $\tau_{\sigma(0)}$ *and hence*

$s_i \circ (srep\tau)_n$ sends $F(\sigma(0))^\sigma$ to $G(\sigma(0))^\sigma$ by $\tau_{\sigma(0)}$. This proves that $srep\tau$ is a map of simplicial spaces.

*Finally, let $F \xrightarrow{\tau} G \xrightarrow{\theta} H$ be a sequence of natural transformation. Then for every $d \in$ **D**,*

$$\begin{array}{ccc} F(d) & \xrightarrow{\tau_d} & G(d) \\ & \searrow^{(\theta\tau)_d} & \downarrow \theta_d \\ & & H(d) \end{array} \tag{3.8}$$

commutes and so by the definition of $srep\tau$,

$$srep(\theta\tau) = (srep\theta) \circ (srep\tau) \tag{3.9}$$

as desired. ■

*Definition 3.2: If $F : \mathbf{C} \to \mathbf{sD}$ is a diagram in **sD**, then we define the homotopy colimit of F to be the simplicial object*

$$hocolim(F) = diag(srepF) \tag{3.10}$$

*in **D**.*

When $\mathbf{D} = Set$, the hocolim(F) is a simplicial set with n-simplices:

(hocolimF)$_n = \{(\bar{X}, y) \mid \bar{X} = (X_0 \xrightarrow{\Phi_1} X_1 \to \ldots \xrightarrow{\Phi_n} X_n) \in N(\mathbf{C})_n, y \in F(X_0)_n\}$
where the face and degeneracy maps are given by:

$$d_i(\bar{X}, y) = \begin{cases} (d_0^{N(\mathbf{C})}\bar{X}, F(\Phi_1)(d_0^{F(X_0)}y)), & i = 0 \\ (d_i^{N(\mathbf{C})}\bar{X}, d_i^{F(X_0)}y), & i > 0. \end{cases} \tag{3.11}$$

and

$$s_i(\bar{X}, y) = (s_i^{N(\mathbf{C})}\bar{X}, s_i^{F(X_0)}y) \tag{3.12}$$

*Example 3.1: Let **C** be the category **1** with two objects 0, 1 and a unique morphism* $0 \leq 1$. *Let* $F : \boldsymbol{C} \to \boldsymbol{Sp}$ *be the functor defined by* $F(0) = F(1) = \Delta[0]$ *in **Sp**, then the homotopy colimit is* $\Delta[1]$.

Example 3.2: Let X be a simplicial set and $F : \boldsymbol{C} \to \boldsymbol{Sp}$ *a constant functor, such that* $F(c) = X$ *for every* $c \in \boldsymbol{C}$ *and* $F(c \xrightarrow{f} c') = 1_X$ *for every morphism f. We show that*

$$hocolim(F) \cong X \times N(\boldsymbol{C}) \tag{3.13}$$

Indeed, since $F(c) = X$ *for every* $c \in \boldsymbol{C}$, *we have*

$$hocolim(F)_n = diag(srepF)_n = \coprod_{\sigma \in N(\boldsymbol{C})} X_n^\sigma \tag{3.14}$$

where $X_n^\sigma = X_n$ *for every* $\sigma \in N(\boldsymbol{C})_n$. *For* $i > 0$, *we have* $d_i(x_n^{(\sigma)}) = (d_i x_n)^{(d_i\sigma)}$. *For i=0, F(f) is the identity map for any* $f : \sigma(0) \to \sigma(1)$ *and hence* $d_0(x_n^{(\sigma)}) = (d_0 x_n)^{(d_0\sigma)}$. *Morever,* $s_i(x_n^{(\sigma)}) = (s_i x_n)^{(s_i\sigma)}$.

Now, let $\varphi_n : hocolim(F)_n \to X_n \times N(\boldsymbol{C})_n$ *be defined by* $\varphi_n(x_n^\sigma) = (x_n, \sigma)$. *It is obviously a bijection. So it remains to show that* φ *is a simplicial map, that is, the following diagrams commute:*

$$\begin{array}{ccc} hocolim(F)_n & \xrightarrow{\varphi_n} & X_n \times N(\boldsymbol{C})_n \\ d_i\downarrow & & \downarrow d_i \\ hocolim(F)_{n-1} & \xrightarrow{\varphi_{n-1}} & X_{n-1} \times N(\boldsymbol{C})_{n-1} \end{array} \tag{3.15}$$

and

$$\begin{array}{ccc} hocolim(F)_n & \xrightarrow{\varphi_n} & X_n \times N(\boldsymbol{C})_n \\ s_i\downarrow & & \downarrow s_i \\ hocolim(F)_{n+1} & \xrightarrow{\varphi_{n+1}} & X_{n-1} \times N(\boldsymbol{C})_{n+1} \end{array} \tag{3.16}$$

For the face map, we have

$$\varphi d_i(x_n^{(\sigma)}) = \varphi((d_i x_n)^{(d_i\sigma)}) = (d_i x_n, d_i\sigma) = d_i(x_n, \sigma) = d_i\varphi(x_n^{(\sigma)}) \tag{3.17}$$

and also for the degeneracy map, we have

$$\varphi s_i(x_n^{(\sigma)}) = \varphi((s_i x_n)^{(s_i\sigma)}) = (s_i x_n, s_i\sigma) = s_i(x_n, \sigma) = s_i\varphi(x_n^{(\sigma)}) \tag{3.18}$$

Therefore φ is a simplicial map.

*Theorem 3.2: $hocolim$: **sD**$^{\boldsymbol{C}}$ $\rightarrow$ **sD** is a functor.*

Proof 3.2 :As a composition of two functors srep and diag

$$\boldsymbol{sD}^{\boldsymbol{C}} \xrightarrow{srep} (\Delta^{op})^{\boldsymbol{sD}} \xrightarrow{diag} \boldsymbol{sD}. \tag{3.19}$$

hocolim is a functor. ■

3.2. Thomason's Theorem

Recall that maps $f, g : X \rightarrow Y$ between topological spaces are said to be homotopic if there is a continuous map $H : X \times [0,1] \rightarrow Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If the maps f and g are homotopic, then we write $f \simeq g$. Topological spaces X and Y are said to be homotopy equivalent if there is continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Definition 3.3: Let X and Y be simplicial sets. A simplicial map $f : X \rightarrow Y$ is said to be a weakly homotopy equivalence if $|f| : |X| \rightarrow |Y|$ is a homotopy equivalence. In this case, we say that X is weakly homotopy equivalent to Y and we write $X \simeq Y$.

Morever, two functors between categories $F_1, F_2 : \mathbf{C} \rightarrow \mathbf{D}$ are weakly homotopic if there exists a homotopy from $|\mathbf{C}|$ to $|\mathbf{D}|$ such that

$$H(x,0) = |F_1|(x) \quad \text{and} \quad H(x,1) = |F_2|(x) \tag{3.20}$$

Also, we say that two categories **C** and **D** are weakly homotopy equivalent if $|\mathbf{C}|$ and $|\mathbf{D}|$ are homotopy equivalent.

Definition 3.4: Let $f, g : X \to Y$ be simplicial maps, we say that f is homotopic to g if there is a simplicial map $H : X \times I \to Y$ such that $H|_{X\times 0} = g$ and $H|_{X\times 1} = f$.

Definition 3.5: Let X and Y be two simplicial sets and $\phi, \psi : X \to Y$ are maps, we say that ϕ is strongly homotopic to ψ if there exists a simplicial map

$$H : X \times \Delta^1 \to Y \tag{3.21}$$

such that H restricted to $X \times 0$ can be identified with ϕ and H restricted to $X \times 1$ can be identified with ψ.

Analogously, we say that two functors between categories $F_1, F_2 : \mathbf{C} \to \mathbf{D}$ are strongly homotopic if $N(F_1)$ and $N(F_2)$ are strongly homotopic.

Theorem 3.3: Let $\varphi : X \to Y$ be a map of the simplicial sets. Suppose that for all p, the simplicial map $\varphi_p : X_p \to Y_p$ is a weak homotopy equivalence. Then $diag(\varphi)$ is a weak homotopy equivalence.

Proof 3.3 : We refer reader to see Chapter IV, Proposition 1.9 in [Goerss and Jardine, 1999]. ■

Lemma 3.1: A natural transformation $\eta : F \to F'$ between the functors induces a strong homotopy between F and F'.

*Proof 3.1 : Let $F, F' : \boldsymbol{C} \to \boldsymbol{D}$ be functors between categories **C** and **D**. Define $\sigma : \boldsymbol{C} \times [1] \to \boldsymbol{D}$ on objects by:*

$$\sigma(c, 0) = F(c), \sigma(c, 1) = F'(c) \tag{3.22}$$

and on morphisms by :

- $(c, 0) \xrightarrow{(f,1_0)} (d, 0) \rightsquigarrow F(f) : F(c) \to F(d)$,
- $(c, 0) \xrightarrow{(f,(0\leq 1))} (d, 1) \rightsquigarrow F(c) \xrightarrow{F'(f)\circ\eta_c} F'(d)$,
- $(c, 1) \xrightarrow{(f,1_1)} (d, 1) \rightsquigarrow F(f) : F'(c) \to F'(d)$

Since $N(\boldsymbol{C} \times (0 \leq 1)) \cong_G N(\boldsymbol{C} \times \Delta^1)$, we have that $N(\sigma)$ is a homotopy equivalence.
■

Theorem 3.4: If $F : \boldsymbol{C} \to \boldsymbol{D}$ is left adjoint to the functor $F' : \boldsymbol{D} \to \boldsymbol{C}$, then F and F' are strong homotopy equivalences.

Proof 3.4 : Apply the above lemma to the unit-counit adjuction $\eta : 1_{\boldsymbol{C}} \to F' \circ F$ and $\varepsilon : FF' \to 1_{\boldsymbol{D}}$. ■

*Corollary 3.1: If **C** is a category with an initial object X, then **C** is strongly contractible.*

*Proof 3.1 : Let **X** be the category with an object X and a morphism and $S : \boldsymbol{C} \to \boldsymbol{X}$ be a functor which sends every object to X, and all maps to the identity. Let $T : \boldsymbol{X} \to \boldsymbol{C}$ contains the object X in **C**. So, ST is the identity in **X**. Since for any object C in **C** there exists a unique map $\boldsymbol{C} \to \boldsymbol{X}$ as defining the component of a natural transformation η_C between TS and the identity, TS is homotopic to the identity. For any object C in **C**, define $\eta_C : TS(c) \to 1_C$ to be the unique map $X \to C$. Then the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\eta_C} & C \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\eta_{C'}} & C' \end{array} \tag{3.23}$$

by the uniquness of $\eta_{C'} : X \to C'$. ■

*Definition 3.6: Let $F : \boldsymbol{C} \to$ **Cat** be a functor. The Grothendieck construction on F, $\int_{\mathcal{G}} F$, is the category whose objects are the pairs (c,x) where c an object of **C** and x an object of $F(\boldsymbol{C})$, and whose morphisms are pairs $(\alpha, f) : (c, x) \to (d, y)$ given by a morphism $\alpha : c \to d$ in **C** and an $f : F(\alpha)(x) \to y$ in F(d). Composition is defined by*

$$(\alpha, f) \circ (\beta, g) = (\alpha\beta, f \circ F(\alpha)g) \tag{3.24}$$

Note that a natural transformation $\eta : F \to F'$ where $F, F' :$ **C** $\to$ **Cat** induces a functor $\int_{\mathcal{G}} h$ from $\int_{\mathcal{G}} F$ to $\int_{\mathcal{G}} F'$ by

$$(\int_{\mathcal{G}} \eta)(c, x) = (c, h(c)(x)) \tag{3.25}$$

and

$$(\int_{\mathcal{G}} \eta)(\alpha, f) = (\alpha, h(d)(f)) \tag{3.26}$$

for $(\alpha, f) : (c, x) \to (d, y)$. Thus $\int_{\mathcal{G}}$ is a functor from $\mathbf{Cat}^{\mathbf{C}}$ to **Cat**.

Theorem 3.5: [Thomason, 1979] Let $F : \mathbf{C} \to \mathbf{Cat}$ be a functor. There is a weak homotopy equivalence

$$\psi : hocolim N(F) \to N(\int_{\mathcal{C}} F) \tag{3.27}$$

between the homotopy colimit of $N \circ F$ and the nerve of the Groethendieck construction.

As in [Thomason, 1979], to prove the above theorem, we first define a natural transformation then we construct a functor $\tilde{F} : \mathbf{C} \to \mathbf{Cat}$ and produce weak homotopy equivalences

$$hocolim N(F) \xleftarrow{\lambda_1} hocolim N(\tilde{F}) \xrightarrow{\lambda_2} N(\int_{\mathcal{C}} F) \tag{3.28}$$

Finally, we construct a simpicial homotopy $H : \psi.\lambda_1 \simeq \lambda_2$. Since λ_1, λ_2 are weak homotopy equivalences, so is ψ.

Lemma 3.2: There is a simplicial map $\psi : hocolim N(F) \to N(\int_{\mathcal{C}} F)$.

Proof 3.2 : The map

$$\psi : hocolim N(F) \to N(\int_{\mathcal{C}} F) \tag{3.29}$$

is defined on n-simplicies to be the map which sends

$$(X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_p} X_p,\ a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_p} a_p) \tag{3.30}$$

to

$$\begin{aligned}(X_0, a_0) &\xrightarrow{(\varphi_1, F(\varphi_1)(\alpha_1))} (X_1, F(\varphi_1)(a_1)) \xrightarrow{(\varphi_2, F(\varphi_2)(\alpha_2))} \ldots \\ &\xrightarrow{(\varphi_p, F(\varphi_p)(\alpha_p))} (X_p, F(\varphi_p)(a_p))\end{aligned} \tag{3.31}$$

where

$$(X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_p} X_p) \tag{3.32}$$

*is a string of p-composible morphisms of **C** and*

$$(a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_p} a_p) \tag{3.33}$$

is a string of p-composible morphism of $F(X_0)$. Clearly, the above map defines a simplicial map ψ. ■

Now, we define a functor $\tilde{F} : \mathbf{C} \to \mathbf{Cat}$ for $X \in Obj(\mathbf{C})$ as follows. Let $\tilde{F}(X)$ be the category whose objects are the pairs (c,x), where $c : Y \to X$ is a map in **C** and $x \in ObjF(Y)$ and whose morphisms are pairs $(\alpha, f) : (c, x) \to (c', x')$ given by a map $\alpha : Y \to Y'$ in **C** such that $c = c'\alpha$ and $f : F(\alpha)(x) \to x'$ in $F(Y')$. Composition in $\tilde{F}(X)$ is given by

$$(\alpha_1, f_1)(\alpha_2, f_2) = (\alpha_1\alpha_2, f_1 \circ F(\alpha_2)(f_2)) \tag{3.34}$$

A map $g : X \to Y$ in **C** gives a functor $\tilde{F}(g)$ defined on objects by $\tilde{F}(g)(c, x) = (gc, x)$ and on maps by $\tilde{F}(g)(\alpha, f) = (\alpha, f)$.

Lemma 3.3: There is a natural equivalence $\lambda_1 : hocolimN\tilde{F} \to hocolimNF$.

*Proof 3.3: For every $x \in Obj($**C**$)$, there is a functor $K(x) : \tilde{F}(x) \to F(x)$ defined on objects by $K(x)(c) = F(c)(x)$ and on morphisms by $K(x)(\alpha, f) = F(c'(f))$ where $(\alpha, f) : (c, x) \to (c', x')$. This functor has a right adjoint $L(x) : F(x) \to \tilde{F}$ defined by $L(x)(c) = (1_x, c)$. Then by Theorem 3.4, $N(K(x)) : N\tilde{F}(x) \to NF(x)$ is a strong homotopy equivalence. Morever, $K(x) : \tilde{F} \to F(x)$ gives a natural transformation $K : \tilde{F} \to F$ of functors. So $N(K) : N\tilde{F} \to NF$ is also a natural transformation. Consider the induced map*

$$\Phi_p : (srepN\tilde{F})_p \to (srepNF)_p \tag{3.35}$$

which is defined on $N\tilde{F}(\sigma(0))$ indexed by $\sigma \in N_p(\boldsymbol{C})$ as $K(\sigma(0))$ and hence send it to $NF(\sigma(0))$ indexed by σ. It is clearly a weak homotopy equivalence and hence $\lambda_1 = diag(\Phi)$ is a weak homotopy equivalence by Theorem 3.4. ■

Lemma 3.4: There is a natural equivalence $\lambda_2 : hocolimN\tilde{F} \to N(\int_{\mathcal{C}} F)$.

Proof 3.4 :A p-simplex of $N\tilde{F}(X)$

$$(c_0, x_0) \xrightarrow{(\alpha_1, f_1)} (c_1, x_1) \xrightarrow{(\alpha_2, f_2)} \ldots \xrightarrow{(\alpha_p, f_p)} (c_p, x_p) \tag{3.36}$$

corresponds to a p-simplex in $N(\int_{\mathcal{C}} F)$

$$(Y_0, x_0) \xrightarrow{(\alpha_1, f_1)} (Y_1, x_1) \xrightarrow{(\alpha_2, f_2)} \ldots \xrightarrow{(\alpha_p, f_p)} (Y_p, x_p) \tag{3.37}$$

together with the map $c_p : Y_p \to X$. Thus $srepN(\tilde{F})$ has as (p,q)-simplices

$$X_0 \to \ldots \to X_p, \quad Y_q \to X_0, \quad (Y_0, x_0) \to \ldots \to (Y_q, x_q). \tag{3.38}$$

We define λ_2 to be the map which sends such a (q,q)-simplex to the q-simplex $(Y_0, x_0) \to \ldots \to (Y_q, x_q)$ in $N(\int_{\mathcal{C}} F)$. Clearly, it is a simplicial map.

To show that λ_2 is a weak homotopy equivalence, let $N(\int_{\mathcal{C}} F)$ be a bisimplicial set which is constant in the p-direction i.e. a bisimplicial set whose (p,q)-simplices are

$$N(\int_{\mathcal{C}} F)_{pq} = N(\int_{\mathcal{C}} F)_q \tag{3.39}$$

So $N(\int_{\mathcal{C}} F)_* = diagN(\int_{\mathcal{C}} F)_{**}$.

Let $\wedge : srepN(\tilde{F}) \to N(\int_{\boldsymbol{C}} F)_{**}$ be the simplicial map which sendes

$$X_0 \to \ldots \to X_p, \quad Y_q \to X_0, \quad (Y_0, x_0) \to \ldots \to (Y_q, x_q) \tag{3.40}$$

to

$$(Y_0, x_0) \to \ldots \to (Y_q, x_q) \tag{3.41}$$

Then, $\lambda_2 = diag(\wedge)$. *Therefore, by Theorem 3.3 it suffices to show that* $\wedge$ *is a weak homotopy equivalence. Here* $\wedge$ *can be expessed as the coproduct of simplicial maps* $N(Y_q \setminus \boldsymbol{C}) \to \Delta^0$. *More precisely*

$$\wedge : \bigcup_{(Y_0, x_0 \to \ldots \to (Y_q, x_q)} N(Y_q \setminus \boldsymbol{C}) \longrightarrow \bigcup_{(Y_0, x_0) \to \ldots \to (Y_q, x_q)} \Delta^0 \tag{3.42}$$

On the other hand, since $I_{1_q} : Y_q \to Y_q$ *is a an initial object in* Y_q*, by Theorem 3.4, the map* $|N(Y_q \setminus \boldsymbol{C})| \to |\Delta^2|$ *is a homotopy equivalence. Since the geometric realization commutes with coproducts,* $|\wedge|$ *is a homotopy equivalence and hence* $\wedge$ *is a weak homotopy equivalence.* ■

Lemma 3.5: There is a simplicial homotopy

$$H : (hocolim N\tilde{F}) \times \triangle^1 \to N(\int_{\mathcal{C}} F) \tag{3.43}$$

from $\psi.\lambda_1$ *to* λ_2

Proof 3.5 : Define

$$H : (hocolim N\tilde{F}) \times \triangle^1 \to N(\int_{\mathcal{C}} F) \tag{3.44}$$

to be the map which sends

$$\begin{aligned}(X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_p} X_p,\ (c_0, x_0) \xrightarrow{(\alpha_1, f_1)} (c_1, x_1) \xrightarrow{(\alpha_2, f_2)} \\ \ldots \xrightarrow{(\alpha_p, f_p)} (c_p, x_p), v_i : p \to [1])\end{aligned} \tag{3.45}$$

where $v(0) = \ldots = v(i) = 0$ *and* $v(i+1) = \ldots = v(p) = 1$*, to*

$$(Y_0, x_0) \xrightarrow{(\alpha_1, f_1)} (Y_1, x_1) \to \cdots \xrightarrow{(\alpha_{i-1}, f_{i-1})} (Y_{i-1}, x_{i-1})$$

$$\xrightarrow{(\Phi_i c_{i-1}, F(\Phi_i c_i)(f_i))} (X_i, F(\Phi_i c_i)(x_i)) \to (X_{i+1}, F(\Phi_{i+1})(c_{i+1})(x_{i+1})) \tag{3.46}$$

$$\xrightarrow{(\Phi_{i+1} c_i, F(\Phi_{i+1} c_{i+1})(f_{i+1}))} \cdots \xrightarrow{(\Phi_p c_{p-1}, F(\Phi_p c_p)(f_p))} (X_p, F(\Phi_p c_p)(x_p))$$

where $\Phi_i = \varphi_i \circ \varphi_{i-1} \circ \cdots \circ \varphi_1$.

Clearly, $H|_0 = \psi \circ \lambda_1$ *and* $H|_1 = \lambda_2$*. It remains to show that H is a simplicial map.*

For the commutativity of H with the face maps. Note that sends

$$X_0 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_p} X_p, \ (c_0, x_0) \xrightarrow{(\alpha_1, f_1)} \ldots \xrightarrow{(\alpha_p, f_p)} (c_p, x_p), v_i \tag{3.47}$$

to

$$X_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_p} X_p, \ (\varphi_1 c_1, x_1) \xrightarrow{(\alpha_2, f_2)} \ldots \xrightarrow{(\alpha_p, f_p)} (\varphi_1 c_p, x_p), v_i \circ d_0 \tag{3.48}$$

Therefore, Hd_0 *sends it to*

$$(Y_1, x_1) \to \cdots \to (Y_{i-1}, x_{i-1}) \xrightarrow{(\Phi_i c_{i-1}, F(\Phi_i c_i)(f_i))} (X_i, F(\Phi_i c_i)(x_i)) \tag{3.49}$$

$$\to \cdots \xrightarrow{(\Phi_p c_p, F(\Phi_p c_p)(f_p))} (X_p, F(\Phi_p c_p)(x_p) \tag{3.50}$$

Since $(\varphi_i \cdots \circ \varphi_2) \circ (\varphi_1 c_k) = \Phi_i c_k$.

On the other hand, d_0 *sends*

$$(Y_0, x_0) \to \cdots \to (Y_{i-1}, x_{i-1}) \xrightarrow{((\varphi_i c_{i-1}, F(\varphi_i c_i)(f_i)))} (X_i, F(\varphi_i c_i)(x_i)) \tag{3.51}$$

$$\to (X_{i+1}, F(\varphi_{i+1})(c_{i+1} x_{i+1})) \to \ldots \to (X_p, F(\varphi_p c_p)(x_p)) \tag{3.52}$$

to

$$((Y_1, x_1) \to \quad \cdots \quad \to (Y_{i-1}, x_{i-1}) \xrightarrow{(\varphi_i c_{i-1}, F(\varphi_i c_i)(f_i))} (X_i, F(\varphi_i c_i)(x_i)) \tag{3.53}$$

$$\to \cdots \xrightarrow{(\varphi_p c_p, F(\varphi_p c_p)(f_p))} (X_p, F(\varphi_p c_p)(x_p)) \tag{3.54}$$

So, we can identify $Hd_0 = d_0H$. The other identifications hold, similarly. ■

4. EQUIVARIANT HOMOTOPY COLIMIT

In this section, we discuss theorems about homeomorphisms and homotopy equivalences of simplicial sets in the equivariant setting and we introduce the equivariant homotopy colimit. We also give the equivariant version of the Thomason's theorem. For this we follow the [Villarroel-Flores, 1999].

4.1. Basic Definitions and Theorems

For a finite group G, let **G** be the category with one object, whose morphisms are the elements of the group G where the composition is given by the group multiplication. The category of **G**-objects in a category **C** is the category $\mathbf{C}^{\mathbf{G}}$ of functors from **G** to **C**.

Theorem 4.1: Given a bisimplicial G-set X, let X^1 be the simplicial G space which sends [p] to the realization of the simplicial set $[q] \to X_{pq}$ and let X^2 be the simplicial G space which sends [q] to the realization of the simplicial set $[p] \to X_{pq}$. Then we have a homomorphism of G-topological spaces

$$\mid diag(X) \mid \cong_G \mid X^1 \mid \cong_G \mid X^2 \mid \tag{4.1}$$

Proof 4.1 : The non-equivariant version of this theorem is proved in [Gelfand and Manin, 1996]. It can be easily show that all the maps defined are G-maps when X is a bisimplicial G-set. ■

For example, **G**-category or a G-object in the **Cat** is a functor $\gamma : \mathbf{G} \to \mathbf{Cat}$. It consists of a category $\gamma(*) = \mathbf{D}$, and actions of G on the set of objects of **D** and on the set of morphisms $\underset{A,B \in Obj\mathbf{D}}{\cup} mor_{\mathbf{D}}(A, B)$ which satisfy the following axioms:

i) $g1_A = 1_{gA}$ for all $g \in G$ and $A \in Obj\mathbf{D}$,

ii) $g\phi \in mor_{\mathbf{D}}(gA, gB)$ for all $g \in G$ and $\phi \in mor_{\mathbf{D}}(A, B)$,

iii) $g(\phi \circ \psi) = (g\phi) \circ (g\psi)$ for all $g \in G$, and ϕ, ψ morphisms in **D**.

Then, we say **D** is a G-category.

Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be G-categories, given by functors $\gamma_1, \gamma_2 : \mathbf{G} \to \mathbf{Cat}$ respectively.

An equivariant G-functor $\mathbf{D}_1 \to \mathbf{D}_2$ is defined to be a natural transformation from γ_1 to γ_2. So it is determined by a functor $\eta : \mathbf{D}_1 \to \mathbf{D}_2$ such that $\eta(gA) = g\eta(A)$ for all objects A in $\mathbf{D}$ and $g \in G$, and $\eta(g\phi) = g\eta(\phi)$ for all morphisms ϕ in $\mathbf{D}$ and $g \in G$.

Definition 4.1: Given a finite group G, let ε_G be the category whose objects are pairs $(G/H, aH)$ where $H \leq G$ and $a \in G$ and morphisms from $(G/H, aH)$ to $(G/K, bK)$ are G-maps $f : G/H \to G/K$ such that $F(aH) = bK$.

Note that there is a morphism f from $(G/H, aH)$ to $(G/K, bK)$ if and only if $H^{a^{-1}b} \leq K$. In this case, the morphism is unique and we denote it by $f_{a^{-1}b}$. Here ε_G is a G-category with a G-action given by

- $g(G/H, aH) = (G/H, gaH)$
- $gf : (G/H, gaH) \to (G/K, gbK)$ is the map $gf : G/H \to G/K$ defined by $(gf)(xH) = gf(xH)$.

Example 4.1: Let G be a cyclic group of order two with a generator t. Then objects of ε_G are (G/1, 1.1), (G/1, t.1) and (G/G, 1.G) and the category ε_G is

$$\begin{array}{ccc} (G/1, 1.1) & \longleftrightarrow & (G/1, t.1) \\ & \searrow & \downarrow \\ & & (G/G, 1.G) \end{array} \tag{4.2}$$

Example 4.2: Let G be a symmetric group of order three that is

$$G = S_3 = \{r, s | r^2 = s^3 = 1, rsr = s^2\} = \{1, r, s, rs, rs^2, s^2\} \tag{4.3}$$

The subgroups of G are

$$
\begin{aligned}
&G \\
&H_1 =< r > \\
&H_2 =< rs > \\
&H_3 =< rs^2 > \\
&K =< s > \\
&E =< 1 >
\end{aligned}
\tag{4.4}
$$

Then, objects of ε_G *are* $(G/G, 1G)$ $(G/H_1, H_1)$, $(G/H_1, sH_1)$, $(G/H_1, s^2H_1)$, $(G/H_2, H_2)$, $(G/H_2, sH_2)$, $(G/H_2, s^2H_2)$, $(G/H_3, H_3)$, $(G/H_3, sH_3)$, $(G/H_3, s^2H_3)$, $(G/K, K)$, $(G/K, rK)$, $(G/E, E)$, $(G/E, rE)$, $(G/E, sE)$, $(G/E, rsE)$, $(G/E, rs^2E)$, $(G/E, s^2E)$. *There is a unique morphism from* $(G/E, gE)$ *to every object of* ε_G *and there is a unique morphism from every object to* $(G/G, 1.G)$*. The other morphisms are given as follows:*

- $mor((G/K, K), (G/K, rK)) = \{f_r\}$,
- $mor((G/K, rK), (G/K, K)) = \{f_r\}$,
- $mor((G/H_2, s^2H_2), (G/H_,s^2H_3)) = \{f_1\}$,
- $mor((G/H_,s^2H_3), (G/H_2, s^2H_2)) = \{f_1\}$,
- $mor((G/H_1, H_1), (G/H_3, sH_3)) = \{f_s\}$
- $mor((G/H_2, H_2), (G/H_1, sH_1)) = \{f_s\}$
- $mor((G/H_1, sH_1), (G/H_3, s^2H_3)) = \{f_s\}$
- $mor((G/H_2, s^2H_2), (G/H_1, H_1)) = \{f_s\}$
- $mor((G/H_1, s^2H_1), (G/H_3, H_3)) = \{f_s\}$
- $mor((G/H_3, s^2H_3), (G/H_2, H_2)) = \{f_s\}$
- $mor((G/H_3, H_3), (G/H_2, sH_2)) = \{f_s\}$
- $mor((H/H_2, sH_2), (G/H_1, s^2H_1)) = \{f_s\}$
- $mor((G/H_1, sH_1), (G/H_2, H_2)) = \{f_{s^2}\}$
- $mor((G/H_1, H_1), (G/H_2, s^2H_2)) = \{f_{s^2}\}$
- $mor((G/H_2, H_2), (G/H_3, s^2H_3)) = \{f_{s^2}\}$

- $mor((G/H_2, sH_2), (G/H_3, H_3)) = \{f_{s^2}\}$
- $mor((G/H_1, s^2H_1), (G/H_2, sH_2)) = \{f_{s^2}\}$
- $mor((G/H_3, sH_3), (G/H_1, H_1)) = \{f_{s^2}\}$
- $mor((G/H_3, s^2H_3), (G/H_1, sH_1)) = \{f_{s^2}\}$
- $mor((H/H_3, H_3), (G/H_1, s^2H_1)) = \{f_{s^2}\}$

Now, we recall the following definitions from equivariant homotopy theory.

Definition 4.2: Let $f, g : X \to Y$ be G-maps between G-spaces X and Y. A G-homotopy from f to g is a homotopy $H : X \times [0,1] \to Y$ such that $H(gx,t) = gH(x,t)$ for all $g \in G$, $x \in X$ and $t \in [0,1]$. In this case, the G-maps f and g are called G-homotopic and we write $f \simeq_G g$.

Definition 4.3: Let X and Y be two G-spaces. We say that X and Y are G-homotopy equivalent if there exists G-maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_G 1_Y$ and $g \circ f \simeq_G 1_X$.

For simplicial maps, we have the analoguous definitions.

Definition 4.4: If X and Y are G-simplicial sets and $\phi, \psi : X \to Y$ are G-maps, we say that ϕ is weakly G-homotopic to ψ if there is a G-homotopy from $|X|$ to $|Y|$ such that $H(x,0) = |\phi|(x)$ and $H(x,1) = |\psi|(x)$.

Similarly, we say that two G-functors between G-categories $F_1, F_2 : \mathbf{D} \to \mathbf{C}$ are weakly G-homotopic if there is a G-homotopy from $|\mathbf{D}|$ to $|\mathbf{C}|$. Hence two G-categories **D** and **C** are weakly G-homotopy equivalent if $|\mathbf{D}|$ and $|\mathbf{C}|$ are G-homotopy equivalent.

Definition 4.5: Let X and Y be G-simplicial sets and $\phi, \psi : X \to Y$ be simplicial G-maps. The simplicial map ϕ is called strongly G-homotopic to ψ if there exists a G-simplicial map

$$H : X \times \Delta^1 \to Y \tag{4.5}$$

such that H restricted to $X \times 0$ is ϕ and H restricted to $X \times 1$ is ψ.

Similarly, two G-functors between G-categories $F_1, F_2 : \mathbf{C} \to \mathbf{D}$ are called strongly G-homotopic if $N(F_1)$ and $N(F_2)$ are strongly G-homotopic.

Theorem 4.2: If X and Y are G-simplicial sets, we have a G-homeomorphism

$$|X \times Y| \cong_G |X| \times |Y| \tag{4.6}$$

if the topology on the right side is taken to be compactly generated.

Proof 4.2 : It can be easily checked that the maps, in the proof of Theorem 2.8 are G-maps when X and Y are G-simplicial sets. ■

Corollary 4.1: Let X and Y be G-simplicial sets and $\phi, \psi : X \to Y$ be strogly G-homotopic G-maps. Then ϕ and ψ are also weakly G-homotopic maps.

Proof 4.1 : Let ϕ and ψ be two strongly G-homotopic maps, i.e. there is a G-map $H : X \times \Delta^1 \to Y$ such that $H|_{X\times 0} = \phi$ and $H|_{X\times 1} = \psi$. Then that $|H||_{X\times 0} = |\phi|$ and $|H||_{X\times 1} = |\psi|$. Therefore, $|H| : |X| \times |\Delta^1| \to |Y|$ gives the desired weak G-homotopy from ϕ to ψ. ■

Lemma 4.1: A natural transformation $\eta : F \to F'$ between the G-functors induces a strong G-homotopy between F and F'.

Proof 4.1 : Clearly, the maps defined in Lemma 3.1 are G-maps. ■

*Corollary 4.2: If the G-functor F : **C** $\to$ **D** is left adjoint to the G-functor F' : **D** $\to$ **C**, then F and F' are strongly G-homotopy equivalences.*

*Definition 4.6: Let **C** be a G-category. If **C** is a (strongly) weakly G-homotopy equivalent to a point, then **C** is called (strongly) weakly G-contractible.*

*Corollary 4.3: If **C** is a G-category with an initial object X fixed by G, then **C** is strongly G-contractible.*

*Proof 4.3 : Since * and X are fixed objects, the functors T and S defined in Corollary 3.1 are G-functors.*■

Theorem 4.3: Let X and Y be G-CW-complexes and $\phi : X \to Y$ be a G-equivariant cellular map. For each subgroup H of G, $\phi^H : X^H \to Y^H$ is a homotopy equivalence if and only if ϕ is a G-homotopy equivalence.

Proof 4.3 : See Section II in [Bredon, 1967] ■

If F is a G-functor, then both F/d and d/F have an action of the stabilizer G_d induced by the action of G on **C** and **D** such that $g(c, v) = (gc, gv)$. In [Villarroel-Flores, 1999], the following generalization of Quillen's Theorem [Quillen, 1978] and (1.4) from [Thevenaz and Webb, 1991] is proved.

Theorem 4.4: Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a G-functor. F is a weak G-homotopy equivalence if for every object $d \in \boldsymbol{D}$ the category d/F is weakly G_d-contractible.

*Proof 4.4 :Let H be a subgroup of G and $\boldsymbol{C}^H$ be the subcategory of **C** whose objects are the objects of **C** which are fixed by H and morphisms are those of **C** which are fixed by H. Since the geometric realization preserves small limits, we have $|\boldsymbol{C}^H| = |\boldsymbol{C}|^H$. By the Theorem 4.3, it suffices to show that $F^H : \boldsymbol{C}^H \to \boldsymbol{D}^H$ is a homotopy equivalence where F^H is the restriction of F. Let $d \in Obj(\boldsymbol{D}^H)$ i.e. hd=d for all $h \in H$. Then we have $H \leq G_d$. Now, consider d/F^H. It is the category whose objects are the pairs $(c, f : d \to F^H(c))$ where $c \in \boldsymbol{C}^H$, $f \in mor^H_{\boldsymbol{C}}(d, F(c))$ and a morphism in $mor_{d/F^H}((c, f), (c', f'))$ is a morphism $g : c \to c'$ in $\boldsymbol{C}^H$ such that $f' = F^H(g)f$. As a subgroup of G_d, it acts on d/F by*

$$h(c, f) = (hc, hf) \tag{4.7}$$

and

$$h.g = hg : (hc, hf) \to (hc', hf') \tag{4.8}$$

Therefore,

$$\begin{aligned} Obj((d/F)^H) &= \{(c, f : d \to F(c)) \mid hc = c, hf = f, \forall h \in H\} \\ &= Obj(d/F^H) \end{aligned} \tag{4.9}$$

and

$$mor_{(d/F)^H}((c,f),(c',f')) = \{g:(c,f)\to(c',f') \mid F(g)\circ f = f, hg = g, \forall h\in H\}$$

$$= \{g : c \to c' \; in \; \boldsymbol{c}^H \mid f' = F^H(g)(f)\} \tag{4.10}$$

$$= mor_{d/F^H}((c,f),(c',f')$$

So, we can identify the categories d/F^H and $(d/F)^H$ when $d \in Obj(\boldsymbol{D}^H)$. Since d/F is weakly G_d-contractible, $(d/F)^H = d/F^H$ is weakly contractible. So, F^H is a weak homotopy equivalent by the nonequivariant version. ■

4.2. Actions of G by Natural Transformations on Functors from G-Categories

*Definition 4.7: Let **C** be a G-category and $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. We say that η acts by natural transformations on F if for each $g \in G$, $X \in Obj(\boldsymbol{C})$, there is a map $\eta_{g,X} : F(X) \to F(gX)$ such that*

- *i) $\eta_{1,X} = 1_{F(X)}$ for all $X \in Obj\boldsymbol{C}$,*
- *ii) the following diagram commutes for $X \in Obj(\boldsymbol{C})$, $g_1, g_2 \in G$:*

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_{g_2,X}} & F(g_1X) \\ & \searrow^{\eta_{g_1g_2,X}} & \downarrow{\scriptstyle \eta_{g_1,g_2X}} \\ & & F(g_1g_2X) \end{array} \tag{4.11}$$

- *iii) the following diagram is commutative for $g \in G$ and $f : X \to Y$ a map in **C**:*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_{g,X}}\downarrow & & \downarrow{\scriptstyle \eta_{g,Y}} \\ F(gX) & \xrightarrow{F(gf)} & F(gY) \end{array} \tag{4.12}$$

Remark 4.1: Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor with an action of η by natural transformations. Then for any functor $T : \boldsymbol{D} \to \boldsymbol{E}$, we can define an action of $T\eta$ by natural transformations on $TF : \boldsymbol{C} \to \boldsymbol{E}$ by $(T\eta)_{g,X} = T \circ \eta_{g,X}$. Clearly,

- *i) $(T\eta)_{1,X} : TF(X) \to TF(X)$, so $(T\eta)_{1,X} = 1_{TF(X)}$ for all $X \in Obj\boldsymbol{C}$,* *ii) the following diagram is commutative for $X \in Obj\mathbf{C}$, $g_1, g_2 \in G$:*

$$\begin{array}{ccc} TF(X) & \xrightarrow{(T\eta)_{g_2,X}} & TF(g_2X) \\ & \searrow^{(T\eta)_{g_1g_2,X}} & \downarrow{\scriptstyle (T\eta)_{g_1,g_2X}} \\ & & TF(g_1g_2X) \end{array} \tag{4.13}$$

*iii) the following diagram is commutative for $g \in G$ and $f : X \to Y$ a map in **C**:*

$$\begin{array}{ccc} TF(X) & \xrightarrow{TF(f)} & TF(Y) \\ {\scriptstyle T\eta_{g,X}}\downarrow & & \downarrow{\scriptstyle T\eta_{g,Y}} \\ TF(gX) & \xrightarrow{TF(gf)} & TF(gY) \end{array} \tag{4.14}$$

*Each object FX obtains an action of the subgroup G_X by $\eta_{g,X} : FX \to FX$ in such a way that if $\phi : X \to Y$ is a map in **C**, then $F\phi : FX \to FY$ is $G_X \cap G_Y$-equivaiant.*

Recall that a G-simplicial set is a functor $G \to$ **Sp**. Let H be a subgroup of G, **C** a G-category and F : **C** $\to$ **D** be a functor with an action of η by natural transformation. A category of H-fixed points of **C** is the category whose objects are $c \in Obj(\mathbf{C})$ with hc=c for all $h \in H$ and morphisms $f : c \to c'$ are $f \in mor_{\mathbf{C}}(c, c')$ with hf=f for all $h \in H$.

A functor F : **C** $\to$ **D** restricts to a functor $F^H : \mathbf{C}^H \to \mathbf{D}$ by $F^H(X) = F(X)$ and $F^H(f) = F(f)$. There is an induced action on F^H given by

$$\eta'_{gH,X} = \eta_{g,X} : F(X) \to F((gH), X) \tag{4.15}$$

where $(gH).X =: gX$.

*Proposition 4.1: Given a G-category **C** and an action of η by natural transformations on F : **C** $\to$ **D** there is a natural structure of G-simplicial object on srepF.*

Proof 4.1 : For $g \in G$, let $g_n : (srepF)_n \to (srepF)_n$ be the map which sends $F(\sigma(0))$ indexed by $\sigma \in N(\mathbf{C})_n$ by η_{g,σ_0} to $F((g\sigma)(0))$ indexed by $g\sigma$. Then the simplicial map $g = (g_n) : srepF \to srepF$ defines a simplicial map. ■

*Corollary 4.4: Let **C** be a G-category, $F : \mathbf{C} \to \mathbf{sD}$ a functor and η an equivariant automorphism of F. Then hocolimF is a G-simplicial object in **D**, with action on the p-simplices given by*

$$g(X_0 \xrightarrow{\phi_1} X_1 \to \ldots \xrightarrow{\phi_p} X_p, y) = (gX_0 \xrightarrow{g\phi_1} gX_1 \to \ldots \xrightarrow{g\phi_p} gX_p, \eta_{g,X_0}(y)) \quad (4.16)$$

for every $y \in (FX_0)_p$.

Proof 4.4 : Since the G-action on diag(srepF) induced from the G-action on srepF, the corollary holds. ■

Lemma 4.2: Let X be a G-simplicial set. Then $|X|^G$ is naturally homeomorpic to $|X^G|$.

Proof 4.2 : Since $|\cdot|$ preserves finite limits, we have $| X |^G \cong | X^G |$. ■

Theorem 4.5: Let $\phi : X \to Y$ be a map of bisimplicial G-sets. Then $diag(\phi)$ is a weak G-homotopy equivalence if $\phi_p : X_p \to Y_p$ is a weak G-homotopy equivalence for all p.

Proof 4.5 : If we show that $| diag(\phi) |^H$ is an ordinary homotopy equivalence, then it follows by Theorem 4.3 that $|diag(\phi)|$ is a G-homotopy equivalence and hence $diag\phi$ is a weak G-homotopy equivalence. By the Lemma 4.2, it suffices to show that the simplicial map

$$diag(\phi^H) : diag(X^H) \to diag(Y^H) \quad (4.17)$$

is a weak homotopy equivalence because $diag(Y)^H$ can be identified with $diag(Y^H)$. Since $|(X^H)_p| = |X_p|^H$ and the equivariant homotopy equivalence $|\phi_p| : |X_p| \to |Y_p|$ restricts to fixed points, the map

$$|(\phi^H)_p| : |(X^H)_p| \to |(Y^H)_p| \quad (4.18)$$

is a homotopy equivalence. Then the result follows by the non-equivariant version of the Theorem 4.5. ■

Now, we give the equivariant version of the Thomason's theorem.

*Theorem 4.6: [Theorem 3.23 in [Villarroel-Flores, 1999]]. Let **C** be a G-category and* $F : \textbf{\textit{C}} \rightarrow \textbf{\textit{Cat}}$ *be a functor with an action of* η *by natural transformations. Then there is a weak G-homotopy equivalence*

$$\psi : hocolimN(F) \rightarrow N(\int_{\textbf{\textit{C}}} F) \tag{4.19}$$

between the homotopy colimit of N(F) and the nerve of the Groethendieck construction. Here, the G-action on hocolimN(F) is given by $N(\eta)$ *and the G-action on* $N(\int_{\textbf{\textit{C}}} F)$ *is given by* η*.*

Proof 4.6 : In the [Thomason, 1979], it is proved that ψ *is a homotopy equivalence. Now, we first show that* ψ *is equivariant, then we descibe an action by natural transformation* $\tilde{\eta}$ *on* $\tilde{F}$ *which is constructed in Theorem 3.5 and then produce weak G-homotopy equivalences*

$$\lambda_1 : hocolimN(\tilde{F}) \rightarrow hocolimN(F) \tag{4.20}$$

and

$$\lambda_2 : hocolimN(\tilde{F}) \rightarrow N(\int_{\textbf{\textit{C}}} F) \tag{4.21}$$

Finally, we prove that $\psi\lambda_1$ *is strongly G-homotpic to* λ_2*. Since* λ_1*,* λ_2 *are weak G-homotopy equivalence, it follows that* ψ *is also weak G-homotopy equivalence.*■

Lemma 4.3: There is an equivariant map $\psi : hocolimN(F) \rightarrow N(\int_{\textbf{\textit{C}}} F)$

Proof 4.3: We prove in Lemma 3.2 that ψ *is a simplicial map. We now have*

$$\psi(g(X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_p} X_p, a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_p} a_p))$$

$$= \psi(gX_0 \xrightarrow{g\varphi_1} \dots \xrightarrow{g\varphi_p} gX_p, \eta_{g,X_0}(a_0) \xrightarrow{\eta_{g,X_0}(\alpha_1)} \dots \xrightarrow{\eta_{g,X_0}(\alpha_p)} \eta_{g,X_0}(a_p))$$

$$= (gX_0, \eta_{g,X_0}(a_0)) \xrightarrow{(g\varphi_1, F(g\varphi_1)(\eta_{g,X_0}(\alpha_1)))} (gX_1, F(g\varphi_1)\eta_{g,X_0}(a_1)) \cdots \tag{4.22}$$

$$(gX_{p-1}, F(g\varphi_{p-1})\eta_{g,X_0}(a_{p-1})) \xrightarrow{(g\varphi_p, F(g\varphi_p)(\eta_{g,X_0}(\alpha_p)))} (gX_p, F(g\varphi_p)\eta_{g,X_0}(a_p))$$

$$= (gX_0, \eta_{g,X_0}(a_0)) \xrightarrow{(g\varphi_1, \eta_{g,X_1}(F(\varphi_1)(\alpha_1))} (gX_1, \eta_{g,X_1}(F(\varphi_1)(a_1)) \cdots$$

$$(gX_{p-1}, \eta_{g,X_{p-1}}(F(\varphi_{p-1})(a_{p-1}))) \xrightarrow{(g\varphi_p, \eta_{g,x_{p-1}}F(\varphi_p)(\alpha_p))} (gX_p, \eta_{g,X_{p-1}}F(\varphi_p)(a_p))$$

$$= g\psi((X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_p} X_p, a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_p} a_p))$$

and hence ψ is equivariant. ■

In the Theorem 3.5, we define a functor $\tilde{F} : \mathbf{C} \to \mathbf{Cat}$ associated to the functor F. Now, from the action by natural transformation η on F, we define an action by natural transformation $\tilde{\eta}$ on $\tilde{F}$ by

$$\tilde{\eta}_{g,X}(c, x) = (gc, \eta_{g,X}(x)) \tag{4.23}$$

and

$$\tilde{\eta}_{g,X}(\alpha, f) = (g\alpha, \eta_{g,X}(f)) \tag{4.24}$$

Lemma 4.4: There is a weak G-homotopy equivalence

$$\lambda_1 : hocolimN\tilde{F} \to hocolimNF \tag{4.25}$$

Proof 4.4 : Here λ_1 is defined as in the proof of Lemma 3.3. The result follows from Corollary 4.2 and Theorem 4.5 which are the equivariant versions of Theorem 3.4 and Theorem 3.5 respectively. ■

Remark 4.2: Let G be a group, H be a subgroup of G and X be a H-set. The product $G \times X$ carries an H-action $(h, (g, x)) \mapsto (gh^{-1}, hx)$. Define an equivalence relation *on $G \times X$ by $(g, x) \sim (g', x')$ if there exists $h \in H$ such that $g = g'h$ and $x' = hx$. The orbit space is denoted by*

$$ind_H^G X = \{[g, x] \mid g \in G, x \in X\} \tag{4.26}$$

Here, $ind_H^G X$ is a G-set with a G-action given by

$$g'[g, x] = [g'g, x] \tag{4.27}$$

for all $g' \in G$ and $[g, x] \in ind_H^G X$.

For any H-map $f : X \to Y$, we have an induced G-map

$$ind_H^G f : G \times_H X \to G \times_H Y \tag{4.28}$$

defined by $f([g, x]) = [g, f(x)]$. Clearly, ind_H^G is a functor from the category of H-sets to the category of G-sets.

Similarly, we can define ind_H^G for H-spaces, H-simplicial sets and so on.

Proposition 4.2: Let $H \leq G$. If $f : X \to Y$ is a H-homotopy equivalence between H-spaces, then $ind_H^G f$ is a G-homotopy equivalence.

Proof 4.2 : Let $f : X \to Y$ be a H-homotopy equivalent with a homotopy inverse $g : Y \to X$. Let $H_1 : X \times I \to X$ be H-homotopy from $g \circ f$ to 1_X and $H_2 : Y \times I \to Y$ be H-homotopy from $f \circ g$ to 1_X. Then $ind_H^G H_2$ and $ind_H^G H_1$ give the G-homotopies between $(ind_H^G f) \circ (ind_H^G g) \simeq 1_X$ and $(ind_H^G g) \circ (ind_H^G f) \simeq 1_Y$, respectively . ■

Lemma 4.5: There is a weak equivariant homotopy equivalence

$$\lambda_2 : hocolim N\tilde{F} \to N(\int_{\boldsymbol{C}} F) \tag{4.29}$$

Proof 4.5 : As we have shown in Lemma 3.4, λ_2 is a natural homotopy equivalence between $hocolim N\tilde{F}$ and the nerve of the Groethendieck construction. It is a G-map, since

$$\lambda_2(g(X_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_p} X_p, (c_0, x_0) \xrightarrow{(\alpha_1,f_1)} (c_1, x_0) \xrightarrow{(\alpha_2,f_2)} \cdots \xrightarrow{(\alpha_p,f_p)} (c_p, x_p)))$$

$$= \lambda_2(gX_0 \xrightarrow{g\varphi_1} \cdots \xrightarrow{g\varphi_p} gX_p, \tilde{\eta}_{g,X_0}(c_0, x_0) \xrightarrow{\tilde{\eta}_{g,X_0}(\alpha_1,f_1)} \cdots \xrightarrow{\tilde{\eta}_{g,X_0}(\alpha_p,f_p)} \tilde{\eta}_{g,X_0}(c_p, x_p))$$

$$= (gX_0, \tilde{\eta}_{g,X_0}(c_0, x_0)) \xrightarrow{(g\varphi_1, F(g\varphi_1)(\tilde{\eta}_{g,X_0}(\alpha_1,f_1)))} (gX_1, F(g\varphi_1)\tilde{\eta}_{g,X_0}(c_1, x_1))$$

$$\cdots \xrightarrow{(g\varphi_{p-1}, F(g\varphi_{p-1})(\tilde{\eta}_{g,X_0}(\alpha_{p-1},f_{p-1})))} (gX_{p-1}, F(g\varphi_{p-1})\tilde{\eta}_{g,X_0}(c_{p-1}, x_{p-1}))$$

$$\xrightarrow{(g\varphi_p, F(g\varphi_p)(\tilde{\eta}_{g,X_0}(\alpha_p,f_p)))} (gX_p, F(g\varphi_p)\tilde{\eta}_{g,X_0}(c_p, x_p))$$

$$= (gX_0, \tilde{\eta}_{g,X_0}(c_0, x_0)) \xrightarrow{(g\varphi_1, \tilde{\eta}_{g,X_1}(F(\varphi_1)(\alpha_1,f_1))} (gX_1, \tilde{\eta}_{g,X_1}(F(\varphi_1)(c_1, x_1)) \tag{4.30}$$

$$\cdots \xrightarrow{(g\varphi_{p-1}, \tilde{\eta}_{g,x_{p-2}}F(\varphi_{p-1})(\alpha_{p-1},f_{p-1}))} (gX_{p-1}, \tilde{\eta}_{g,X_{p-1}}(F(\varphi_{p-1})(c_{p-1}, x_{p-1})))$$

$$\xrightarrow{(g\varphi_p, \tilde{\eta}_{g,x_{p-1}}F(\varphi_p)(\alpha_p,f_p))} (gX_p, \tilde{\eta}_{g,X_{p-1}}F(\varphi_p)(c_p, x_p))$$

$$= g\lambda_2((X_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_p} X_p, \ (c_0, x_0) \xrightarrow{(\alpha_1,f_1)} (c_1, x_1) \xrightarrow{(\alpha_2,f_2)} \cdots \xrightarrow{(\alpha_p,f_p)} (c_p, x_p))$$

As in the non-equivariant case, $\lambda_2 = diag(\wedge)$ *where* $\wedge : srep(N\tilde{F}) \to N(\int_{\boldsymbol{C}} F)$. *By Theorem 4.1 , we have*

$$|diagN(\int_{\boldsymbol{C}} F)_{**}| \cong_G |N(\int_{\boldsymbol{C}} F)| \tag{4.31}$$

Therefore, it suffices to show that

$$\wedge_q : srep(N\tilde{F})_q \to N(\int_{\boldsymbol{C}} F)_q \tag{4.32}$$

is a weak G-homotopy equivalence by Theorem 4.5.

As in the non-equivariant case, $\wedge_q$ *can be expressed as the coproduct of simplicial maps*

$$N(Y_q/\boldsymbol{C}) \to \Delta(0) \tag{4.33}$$

taken over the points of $N(\int_{\boldsymbol{C}} F)_q$. *Since the geometric realization commutes with the coproducts, it suffices to show that the map*

$$\coprod_{(Y_0,x_0)\to\ldots\to(Y_q,x_q)} |N(Y_q/\boldsymbol{C})| \longrightarrow \coprod_{(Y_0,x_0)\to\ldots\to(Y_q,x_q)} |\Delta^0| \tag{4.34}$$

is a G-homotopy equivalence. Let I_q be the set of representatives of the G-orbits of the action of G on $N(\int F)_q$. Let $i = (Y_0, x_0) \to \ldots \to (Y_q, x_q) \in I$. Since 1_{Y_q} is an initial object in $Y_q/\boldsymbol{C}$ which is fixed by the stabilizer G_i of i, the map

$$ind_{G_i}^G |N(Y_q/\boldsymbol{C})| \to ind_{G_i}^G |\Delta^0| \tag{4.35}$$

is a G-homotopy equivalence. Therefore the induced map

$$\coprod_{i\in I_q} ind_{G_i}^G |N(Y_q/\boldsymbol{C})| \to \coprod_{i\in I_q} ind_{G_i}^G |\Delta^0| \tag{4.36}$$

which equals to

$$\coprod_{(Y_0,x_0)\to\ldots\to(Y_q,x_q)} |N(Y_q/\boldsymbol{C})| \longrightarrow \coprod_{(Y_0,x_0)\to\ldots\to(Y_q,x_q)} |\Delta^0| \tag{4.37}$$

is a G-homotopy equivalence as desired.

Lemma 4.6: There is a G-homotopy

$$H : (hocolim N\tilde{F}) \times \triangle^1 \to N(\int_{\mathcal{C}} F) \tag{4.38}$$

from $\psi\lambda_1$ to λ_2.

Proof 4.6 : In the Lemma 3.5, we proof that $H : \psi\lambda_1 \simeq \lambda_2$ is a simplicial homotopy. It remains to show that, H is equivariant. Here, g sends

$$\begin{aligned}(X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_p} X_p,\ \ (c_0, x_0) \xrightarrow{(\alpha_1,f_1)} (c_1, x_1) \xrightarrow{(\alpha_2,f_2)} \\ \ldots \xrightarrow{(\alpha_p,f_p)} (c_p, x_p),\ \ v_i)\end{aligned} \tag{4.39}$$

to

$$(gX_0 \xrightarrow{g\varphi_1} \cdots \xrightarrow{g\varphi_p} gX_p,\ \ gY_q \to gX_0,\ \ (gc_0,\eta_{g,Y_0}(x_0)) \xrightarrow{(g\alpha_1,\eta_{g,Y_0}(f_1))}$$
$$\cdots \to (gc_p, \eta_{g,Y_p}(x_p))\ \ , v_i) \tag{4.40}$$

and H sends it to

$$(gY_0,\eta_{g,Y_0}(x_0)) \to \cdots (gY_{i-1}, \eta_{g,Y_{i-1}}(x_{i-1})) \xrightarrow{g\Phi_i c_{i-1}, F(g\Phi_i c_i)(\eta_{g,Y_0}(f_i))}$$
$$(gX_i, F(g\Phi_i c_i)(\eta_{g,Y_i}(x_i))) \to \cdots \to (gX_p, F(g\Phi_p c_p)(\eta_{g,Y_p}(x_p))) \tag{4.41}$$

since, $(g\Phi_k c_k) = (g\varphi_k) \circ \cdots \circ (g\varphi_1) \circ (g\varphi_k)$.

On the other hand, H sends

$$(X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_p} X_p,\ \ (c_0, x_0) \xrightarrow{(\alpha_1,} f_1)\, (c_1, x_1) \xrightarrow{(\alpha_2,f_2)}$$
$$\ldots \xrightarrow{(\alpha_p,f_p)} (c_p, x_p),\ \ v_i) \tag{4.42}$$

to

$$(Y_0, x_0) \to (Y_1, x_1) \to \ldots \to (Y_{i-1}, x_{i-1}) \xrightarrow{((\Phi_i c_{i-1}, F(\Phi_i c_i)(f_i)))} (X_i, F(\Phi_i c_i)(x_i))$$
$$\to (X_{i+1}, F(\Phi_{i+1})(c_{i+1}x_{i+1})) \to \ldots \to (X_p, F(\Phi_p c_p)(x_p)) \tag{4.43}$$

which is send by g to

$$(gY_0, \eta_{g,Y_0}(x_0)) \to \cdots \to (gY_{i-1}, \eta_{g,Y_{i-1}}(x_{i-1}) \xrightarrow{(g\Phi_g c_{i-1}, gF(\Phi_i c_i)(f_i))}$$
$$(gX_i, gF(\Phi_i c_i)(x_i)) \to \cdots \to (gX_p, gF(\Phi_p c_p)(x_p))) \tag{4.44}$$

Since $gF(\Phi_k c_k)(f_k) = F(g\Phi_k c_k)(\eta_{g,Y_k}(f_k))$, *H is G-equivariant.*

This completes the proof of the theorem. ■

5. AN ALTERNATIVE PROOF OF THOMASON'S THEOREM

In this chapter, we give an alternative proof of the equivariant version of the Thomason's theorem which identifies the G-homotopy type of the geometric realization of the equivariant homotopy colimit of a nerve of a functor with an action by natural transformation to the geometric realization of the nerve of its Groethendieck construction. We also establish one to one correspondence with such functors and functors from the Groethendieck contruction of related categories.

5.1. Another Proof of Equivariant Version of Thomason's Theorem

We discussed the nonequivariant version of the Thomason's theorem [Thomason, 1979] in the Chapter 3 and we provided a proof of the equivariant version of the Thomason's theorem in the Chapter 4 given by [Villarroel-Flores, 1999]. For each step of the proof, we have closely followed [Villarroel-Flores, 1999]. In this section, we give an altenative proof to the equivariant version of the Thomason's theorem by using fixed point categories. For this, we need the following observations:

Lemma 5.1: Let $\boldsymbol{C}$ *be a G-category. Then* $N(\boldsymbol{C})^H$ *and* $N(\boldsymbol{C}^H)$ *are identical as simplicial sets.*

Proof 5.1 : An n-simplex in $N(\boldsymbol{C}_n)$ *is a sequence*

$$\sigma : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n \tag{5.1}$$

of composible morphism in $\boldsymbol{C}$*. The set* $N(\boldsymbol{C}_n)^H$ *is the subset of* $N(\boldsymbol{C}_n)$ *consisting of* σ*'s which are fixed by H that is* $h\sigma = \sigma$ *for ever* $h \in H$ *where*

$$h\sigma : hX_0 \xrightarrow{hf_1} hX_1 \xrightarrow{hf_2} \dots \xrightarrow{hf_n} hX_n \tag{5.2}$$

So, the elements of $N(\boldsymbol{C}_n)^H$ *are* $\sigma : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n$ *with* $hX_i = X_i$ *and* $hf_i = f_i$ *for all* $h \in H$*.*

On the other hand, $\boldsymbol{C}^H$ is the subcategory of $\boldsymbol{C}$ whose objects are $X \in Obj(\boldsymbol{C})$ with hx=x for all $h \in H$ and morphisms are $f : X \to X'$ with hf=f for all $h \in H$. So the set of $\boldsymbol{C}_n^H$ consists of sequences:

$$\sigma : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n \tag{5.3}$$

where $hX_i = X_i$ and $hf_i = f_i$. Since the coface and codegeneracy maps are also the same, we can identify $N(\boldsymbol{C})^H$ with $N(\boldsymbol{C}^H)$. ■

Let $F : \mathbf{C} \to \mathbf{Cat}$ be a functor with an action of G by natural transformations. Then for every object c in **C**, we consider F(c) as a G_c-category. Here, the G_c-action on objects and morphisms are given by $\eta_{g,c}$ that is $g.x = \eta_{g,c}(x)$ and $gf = \eta_{g,c}(f)$ for all $g \in G_c$.

If $c \in Obj(\mathbf{C}^H)$, then $H \leq G_c$. Since F(c) is a G_c-category, we can take the H fixed points of F(c). Then we can define a functor $F^H : \mathbf{C}^H \to \mathbf{Cat}$ by $F^H(c) = F(c)^H$ and $F^H(f) = F(f)$. This is well-defined since

i) Let $x \in Obj(F(c)^H)$, that is $x \in Obj(F(c))$ with $hx = \eta_{h,c}(x) = x$ for all $h \in H$. So, we have

$$\begin{aligned} hF(f)(x) &= \eta_{h,\mathbf{C}'}(F(f)(x)) \\ &= F(hf) \circ \eta_{h,\mathbf{C}}(x) \\ &= F(f)(x) \end{aligned} \tag{5.4}$$

that is $F(f)(x)$ in $Obj(F(\mathbf{C})^H)$

ii) For a morphism $\alpha : X \to Y$ in $F(c)^H$ that is $h\alpha = \eta_{h,c}\alpha = \alpha$ for all $h \in H$, we have

$$\begin{aligned} hF(f)(\alpha) &= \eta_{h,\mathbf{C}'}(F(f)(\alpha)) \\ &= F(hf) \circ \eta_{h,\mathbf{C}}(\alpha) \\ &= F(f)(\alpha) \end{aligned} \tag{5.5}$$

So, $F(f)(\alpha)$ in $Obj(F(\mathbf{C})^H)$ is a morphism in $F(\mathbf{C}')^H$ as desired.

Lemma 5.2: $(\int_{\mathbf{C}} F)^H = \int_{\mathbf{C}^H} F^H$.

Proof 5.2 : Here $(\int_{\mathbf{C}} F)^H$ *is the category whose objects are the pairs (c,x) where* $c \in Obj(\mathbf{C})$ *and* $x \in Obj(F(c))$ *such that* $h(c,x) = (hc, \eta_{h,\mathbf{C}}(x)) = (c,x)$ *that is hc=c,* $\eta_{h,\mathbf{C}}(x) = x$. *So the objects of* $(\int_{\mathbf{C}} F)^H$ *are the pairs (c,x) where* $c \in Obj\mathbf{C}^H$ *and* $x \in F(\mathbf{C})^H$. *Also, the mophisms of* $(\int_{\mathbf{C}} F)^H$ *are* $(\alpha, f) : (c,x) \to (d,y)$ *such that* $(h\alpha, \eta_{h,\mathbf{C}} f)$ *that is* $h\alpha = \alpha$ *and* $\eta_{h,\mathbf{C}} f = f$. *Since the coface and the codegeneracy maps are also the same, we can identify* $(\int_{\mathbf{C}} F)^H$ *with* $(\int_{\mathbf{C}^H} F^H)$. ■

Lemma 5.3: $(srepNF)^H = srepN(F^H)$

Proof 5.3 : A (p,q)-simplices of a bisimplicial G-set srepNF are the expression of the form

$$(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_p} X_p,\ a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_q} a_q) \tag{5.6}$$

where the second sequence is a q-simplex in $N(F(X_0))$. *Here the G-action is given by*

$$\begin{aligned} g(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_p} X_p,\ a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_q} a_q) = (gx_0 \xrightarrow{gf_1} \dots \xrightarrow{gf_p} X_p, \\ \eta_{g,X_0}(a_0) \xrightarrow{\eta_{g,X_1}(\alpha_1)} \dots \xrightarrow{\eta_{g,X_q}(\alpha_q)} \eta_{g,X_0}(a_q)) \end{aligned} \tag{5.7}$$

So, the element of the set $(srepNF)^H_{p,q}$ *are pairs*

$$(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_p},\ a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_q} a_q) \tag{5.8}$$

with $hX_i = X_i$, $hf_i = f_i$, $\eta_{h,X_0}(a_i) = a_i$ *and* $\eta_{h,X_i}(\alpha_i) = \alpha_i$ *for all* $h \in H$. *These are exactly the element of* $srepN(F^H)_{p,q}$. *Since the coface and codegeneracy maps are the same, we identify* $srepN(F^H)$ *with* $srepN(F^H)$.

Since diag also respects H-fixed points, we can identify

$$(hocolimNF)^H = hocolimN(F^H) \tag{5.9}$$

by the above Lemma 5.3.

By the Lemma 5.1 and Lemma 5.2 we have

$$(N\int_{\mathbf{C}} F)^H = N(\int_{\mathbf{C}} F)^H = N(\int_{\mathbf{C}^H} F^H) \tag{5.10}$$

Therefore, for all $H \leq G$ there exists a weak homotopy equivalence

$$(hocolimNF)^H \simeq (N\int_{\mathbf{C}} F)^H \tag{5.11}$$

This implies that there is a weak G-homotopy equivalence

$$hocolimNF \simeq N(\int_{\mathbf{C}} F) \tag{5.12}$$

between the homotopy colimit of $N \circ F$ and the nerve of the Groethendieck construction of F. ■

5.2. Lifting of Functors F : C $\rightarrow$ D with η Actions to Functors from $\int_{\mathcal{G}}$ C to D

*Theorem 5.1: Let F : **C** $\rightarrow$ **D** be a functor where **C** is a G-category determined by the functor $\gamma_c : \mathcal{G} \rightarrow$ **Cat**. Then there is an action on F by natural transformation if and only if F factors through*

$$\tilde{F} : \int_{\mathcal{G}} \gamma_C \longrightarrow \mathbf{D} \tag{5.13}$$

i. e. the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{i} & \int_{\mathcal{G}} \gamma_C \\ & \searrow^{F} & \downarrow{\tilde{F}} \\ & & \mathbf{D} \end{array} \tag{5.14}$$

where the functor i defined by $i(c) = (*, c)$ and $i(f) = (1_*, f)$.

Proof 5.1 : We firstly prove the necessity part of the Theorem 5.1. Now given a functor $F : \boldsymbol{C} \to \boldsymbol{D}$ with an action by natural transformations. Define the functor $\tilde{F} : \int_{\mathcal{G}} \gamma_C \to$ $\boldsymbol{D}$ on objects by

$$\tilde{F}(*, c) = F(c) \tag{5.15}$$

and on morphisms by

$$\tilde{F}((g, f)) = F(f) \circ \eta_{g,c} \tag{5.16}$$

where $(g, f) : (*, c_1) \to (*, c_2)$.

A map $\int_{\mathcal{G}} \gamma_C \longrightarrow \boldsymbol{D}$ gives a functor $\tilde{F}$ since

$$\tilde{F}(1, 1_c) = F(1_c) \circ \eta_{1,c} = 1_{F(c)} \circ 1_{F(c)} = 1_{F(c)} \tag{5.17}$$

and

$$\begin{aligned} \tilde{F}((g_2, f_2)) \circ (g_1 \circ f_1)) &= \tilde{F}(g_2 g_1, f_2(g_2 f_1)) \\ &= F(f_2(g_2 f_1)) \circ \eta_{g_2 g_1, c_1} \\ &= F(f_2(g_2 f_1)) \circ \eta_{g_2, g_1 c_1} \circ \eta_{g_1 c_1} \\ &= F(f_2) \circ F(g_2 f_1) \circ \eta_{g_2, g_1 c_1} \circ \eta_{g_1, c_1} \\ &= F(f_2) \circ \eta_{g_2, c_2} \circ F(f_1) \circ \eta_{g_1, c_1} \\ &= \tilde{F}((g_2, f_2)) \circ \tilde{F}((g_1, f_1)) \end{aligned} \tag{5.18}$$

Also, F factors through $\tilde{F}$ since

- $\tilde{F}(*, c) = F(c)$
- $\tilde{F}(1, f) = F(f) \circ \eta_{1,c} = F(f) \circ 1_{F(c)} = F(f)$

*Now, we prove the sufficiency of the Theorem 5.1, let $F : \boldsymbol{C} \to$ **Cat** be a functor that factors through $\tilde{F}$. Define $\eta_{g,c} : F(c) \to F(gc)$ by $\eta_{g,c} := \tilde{F}(g., 1_c)$. Then we have* *i) For $c \in Obj(\boldsymbol{C})$, we have $\eta_{1,c} = \tilde{F}(1, 1_c) = 1_{F(c)}$*

ii) For $c \in Obj(\boldsymbol{C})$, $g_1, g_2 \in G$, we have

$$\begin{aligned}\eta_{g_2,g_1c} \circ \eta_{g_1,c} &= \tilde{F}(g_2, 1_{g_2g_1c}) \circ \tilde{F}(g_1, 1_{g,c}) = \tilde{F}((g_2, 1_{g_2g_1,c}) \circ (g_1, 1_{g_1c})) \\ &= \tilde{F}(g_2g_1, 1_{g_2g_1,c} \circ g_2(1_{g,c})) = \tilde{F}(g_2g_1, 1_{g_2g_1,c}) = \eta_{g_2g_1,c}\end{aligned} \tag{5.19}$$

*iii) For $g \in G$ and $f : c_1 \to c$ in **C**, we have*

$$\begin{aligned}\eta_{g_1,c_1} \circ F(f) &= \tilde{F}(g_1, 1_{g_1,c}) \circ \tilde{F}(1, f) = \tilde{F}((g_1, 1_{g_1,c}) \circ (1, f)) \\ &= \tilde{F}(g_1, 1_{g_1,c} \circ g_1(1, f)) = \tilde{F}(g_1, 1_{g_1,c \circ gf}) \\ &= \tilde{F}(g_1, gf) = \tilde{F}((1, gf) \circ (g_1, 1_{gc_1})) \\ &= \tilde{F}(1, gf) \circ \tilde{F}(g_1, 1_{gc_1}) = F(gf) \circ \eta_{g_1,c_1}\end{aligned} \tag{5.20}$$

This proves the theorem. ■

Here, the map i induces a map

$$i_* : hocolimNF \to hocolimN\tilde{F} \tag{5.21}$$

which sends

$$(X_0 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_n} X_n, a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} a_n) \tag{5.22}$$

to $(hocolimNF)_n$ such that

$$((*, X_0) \xrightarrow{(1,\varphi_1)} \dots \xrightarrow{(1,\varphi_n)} (*, X_n), a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} a_n) \tag{5.23}$$

in $(hocolimN\tilde{F})_n$. Clearly, this is a simplicial map.

It also induces a map

$$i_* : N(\int_{\mathbf{C}} F) \to N(\int_{\int_{\mathcal{G}} \gamma_c} \tilde{F}) \tag{5.24}$$

by sending an n-simplex

$$(c_0, a_0) \xrightarrow{(\varphi_1,\alpha_1)} \dots \xrightarrow{(\varphi_n,\alpha_n)} (c_n, a_n) \tag{5.25}$$

where c_i is an object in **C** and a_i is an object in $F(c_i)$ to an n-simplex

$$((*, c_0), a_0) \xrightarrow{((1_*,\varphi_1),\alpha_1)} \dots \xrightarrow{((1_*,\varphi_n),\alpha_n)} ((*, c_n), a_n) \tag{5.26}$$

Proposition 5.1: The following diagram commutes:

$$\begin{array}{ccc} hocolim NF & \xrightarrow{i_*} & hocolim N\tilde{F} \\ \psi_F \downarrow & & \downarrow \psi_{\tilde{F}} \\ N(\int_{\boldsymbol{C}} F) & \xrightarrow{i_*} & N(\int_{\int_{\mathcal{G}} \gamma_c} \tilde{F}) \end{array} \tag{5.27}$$

where ψ_F and $\psi_{\tilde{F}}$ are the maps given in Thomason's theorem.

Proof 5.1 : For the commutativity of the diagram

$$\begin{aligned} &\psi_{\tilde{F}} \circ i_*(X_0 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_p} X_p, a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} a_p) \\ &= \psi_{\tilde{F}}((*, X_0) \xrightarrow{(1,\varphi_1)} \dots \xrightarrow{(1,f_n)} (*, X_p), a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} a_p) \\ &= ((*, X_0), a_0) \xrightarrow{(\varphi_1, F(\varphi_1)(\alpha_1))} \dots \xrightarrow{(\varphi_p, F(\varphi_p)(\alpha_p))} ((*, X_p), F(\varphi_p)(a_p)) \end{aligned} \tag{5.28}$$

On the other hand, we have

$$\begin{aligned} &i_* \circ \psi_F(X_0 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_p} X_p, a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_p} a_p) \\ &= i_*((X_0, a_0) \xrightarrow{\varphi_1, F(\varphi_1)(\alpha_1)} \dots \xrightarrow{\varphi_p, F(\varphi_p)(\alpha_p)} (X_p, F(\varphi_p)(a_p)) \\ &= ((*, X_0), a_0) \xrightarrow{(\varphi_1, F(\varphi_1)(\alpha_1))} \dots \xrightarrow{(\varphi_p, F(\varphi_p)(\alpha_p))} ((*, X_p), F(\varphi_p)(a_p)) \end{aligned} \tag{5.29}$$

and hence $\psi_{\tilde{F}} \circ i_* = i_* \circ \psi_F$. ■

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