ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DYNAMICAL ANALYSIS OF SOME SYSTEMS OF NONLINEAR DIFFERENCE EQUATIONS

DEPARTMENT OF MATHEMATICS

DOCTOR OF PHILOSOPHY THESIS

İNCİ OKUMUŞ

JULY 2019

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ADVISOR: Prof. Yüksel SOYKAN

ZONGULDAK
July 2019

APPROVAL OF THE THESIS:

The thesis entitled "Dynamical Analysis of Some Systems of Nonlinear Difference Equations" and submitted by İnci OKUMUŞ has been examined and accepted by the jury as a Doctor of Philosophy thesis in Department of Mathematics, Graduate School of Natural and Applied Sciences, Zonguldak Bülent Ecevit University. 05/07/2019

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

İnci OKUMUŞ

ABSTRACT

Doctor of Philosophy Thesis

DYNAMICAL ANALYSIS OF SOME SYSTEMS OF NONLINEAR DIFFERENCE EQUATIONS

İnci OKUMUŞ

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Prof. Yüksel SOYKAN

July 2019, 143 pages

In this thesis, we present a systematic study of dynamical behavior of solutions of some specific non-linear difference equations and systems of difference equations. Especially, we investigate the stability character of equilibrium points, the exact forms, the periodicity, the oscillation and the boundedness of solutions of these equations and systems.

The organization of this thesis is as follows:

Chapter 1 is a concise overview of what this thesis is about and also is a literature summary of difference equation theory.

Chapter 2 consists of some basic important definitions and some significant theorems used throughout the thesis.

Chapter 3 includes some results about the stability, boundedness character and periodic nature of positive solutions of the system of difference equations

ABSTRACT (continued)

$x_{n+1} = A + x_{n-1} / z_n$, $y_{n+1} = A + y_{n-1} / z_n$, $z_{n+1} = A + z_{n-1} / y_n$, for $n = 0,1,...$ where the parameter A and the initial conditions $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ are arbitrary positive real numbers.

Chapter 4 presents the local asymptotic stability of the equilibrium points, boundedness character, oscillatory, and global asymptotic behavior of positive solutions of the system of rational difference equations $x_{n+1} = A + x_{n-m} / z_n$, $y_{n+1} = A + y_{n-m} / z_n$, $z_{n+1} = A + z_{n-m} / y_n$, for $n = 0,1,...$ where the parameter A and the initial values x_{-i}, y_{-i}, z_{-i}, for $i = 0,1,...,m$ are positive real numbers and m is positive integer.

Chapter 5 contains some results about the local asymptotic stability of the equilibrium points and oscillation behaviour of positive solutions of the following system of rational difference equations $x_{n+1} = A + x_{n-1}^p / z_n^p$, $y_{n+1} = A + y_{n-1}^p / z_n^p$, $z_{n+1} = A + z_{n-1}^p / y_n^p$, for $n = 0,1,...$ where the parameters $A \in (0,\infty)$, $p \in [1,\infty)$ and the initial values $x_{-i}, y_{-i}, z_{-i} \in (0,\infty)$, $i = -1, 0$.

Chapter 6 states the stability character of equilibrium points and the form of solutions and asymptotic behavior of positive solutions of the following four rational difference equations $x_{n+1} = 1/(x_n(x_{n-1} \pm 1) \pm 1)$, $x_{n+1} = -1/(x_n(x_{n-1} \pm 1) \mp 1)$, such that their solutions are associated with Tribonacci numbers.

Chapter 7 acquaints about the stability character of equilibrium points, the periodic nature of solutions and the global behavior of solutions of the following four rational difference equations $x_{n+1} = \pm 1/(x_n(x_{n-1} \pm 1) - 1)$, $x_{n+1} = \pm 1/(x_n(x_{n-1} \mp 1) + 1)$.

Chapter 8 presents some results about the stability character of equilibrium points of and the explicit form and global behavior of positive solutions of the following two systems of rational difference equations $x_{n+1} = \pm 1/(y_n(x_{n-1} \pm 1) + 1)$, $y_{n+1} = \pm 1/(x_n(y_{n-1} \pm 1) + 1)$, such that their solutions are associated with Tribonacci numbers.

ABSTRACT (continued)

Chapter 9 expresses the stability character of equilibrium points of and the explicit form and asymptotic behavior of solutions of the following nonlinear difference equation $x_{n+1} = \gamma / (x_n(x_{n-1} + \alpha) + \beta)$, such that their solutions are associated with generalized Tribonacci numbers.

Keywords: Difference equation, equilibrium point, asymptotic behaviour, global asymptotic stability, oscillation, periodicity, unbounded solutions, boundedness, recursive sequence, Tribonacci numbers.

Science Code: 403.06.01

ÖZET

Doktora Tezi

LİNEER OLMAYAN BAZI FARK DENKLEM SİSTEMLERİNİN DİNAMİK ANALİZİ

İnci OKUMUŞ

Zonguldak Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Prof. Dr. Yüksel SOYKAN
Temmuz 2019, 143 sayfa

Bu tezde, bazı özel lineer olmayan fark denklemlerinin ve fark denklem sistemlerinin çözümlerinin dinamik davranışlarının sistematik bir çalışması verilmiştir. Özellikle, bu fark denklemlerinin ve fark denklem sistemlerinin çözümlerinin sınırlılığı, salınımlılığı, periyodikliği, kesin çözüm formları ve denge noktalarının kararlılık karakteri araştırılmıştır.

Bu tezin organizasyonu aşağıdaki gibidir:

Bölüm 1, bu tezin ne ile ilgili olduğuna dair kısa bir tanıtım ve ayrıca fark denklem teorisinin literatür özetidir.

Bölüm 2, tez boyunca kullanılan bazı temel önemli tanımlardan ve teoremlerden oluşmaktadır.

ÖZET (devam ediyor)

Bölüm 3, A parametresi ve $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ başlangıç koşulları keyfi pozitif gerçel sayılar olmak üzere $n = 0,1,...$ için $x_{n+1} = A + x_{n-1} / z_n$, $y_{n+1} = A + y_{n-1} / z_n$, $z_{n+1} = A + z_{n-1} / y_n$ fark denklem sisteminin pozitif çözümlerinin kararlılığı, sınırlılık karakteri ve periyodik niteliği hakkında bazı sonuçlar içerir.

Bölüm 4, m pozitif tamsayı ve A parametresi ve $i = 0,1,...,m$ için x_{-i}, y_{-i}, z_{-i}, başlangıç koşulları pozitif gerçel sayılar olmak üzere $x_{n+1} = A + x_{n-m} / z_n$, $y_{n+1} = A + y_{n-m} / z_n$, $z_{n+1} = A + z_{n-m} / y_n$ rasyonel fark denklem sisteminin pozitif çözümlerinin sınırlılık karakterini, salınımlığını ve global asimptotik davranışını ve denge noktalarının yerel (lokal) asimptotik kararlılığını verir.

Bölüm 5, $A \in (0,\infty)$, $p \in [1,\infty)$ ve $x_{-i}, y_{-i}, z_{-i} \in (0,\infty)$, $i = -1,0$ olmak üzere $n = 0,1,...$ için $x_{n+1} = A + x_{n-1}^p / z_n^p$, $y_{n+1} = A + y_{n-1}^p / z_n^p$, $z_{n+1} = A + z_{n-1}^p / y_n^p$, rasyonel fark denklem sisteminin pozitif çözümlerinin salınım davranışı ve denge noktalarının lokal asimptotik kararlılığı hakkında bazı sonuçlar içermektedir.

Bölüm 6, çözümleri Tribonacci sayılarıyla ilişkili olan aşağıdaki $x_{n+1} = 1/(x_n(x_{n-1} \pm 1) \pm 1)$, $x_{n+1} = -1/(x_n(x_{n-1} \pm 1) \mp 1)$, dört rasyonel fark denkleminin pozitif çözümlerinin kesin çözüm formlarını ve asimptotik davranışlarını ve denge noktalarının kararlılık karakterlerini ifade eder.

Bölüm 7, aşağıdaki $x_{n+1} = \pm 1/(x_n(x_{n-1} \pm 1) - 1)$, $x_{n+1} = \pm 1/(x_n(x_{n-1} \mp 1) + 1)$ dört rasyonel fark denkleminin çözümlerinin periyodik doğası ve global davranışları ve denge noktalarının kararlılık yapıları hakkında bilgi verir.

Bölüm 8, çözümleri Tribonacci sayılarıyla ilişkili olan aşağıdaki $x_{n+1} = \pm 1/(y_n(x_{n-1} \pm 1) + 1)$, $y_{n+1} = \pm 1/(x_n(y_{n-1} \pm 1) + 1)$, iki rasyonel fark denklem sisteminin pozitif çözümlerinin kesin (açık) çözüm formları ve global davranışları ve denge noktalarının kararlılık karakterleri hakkında bazı sonuçlar sunmaktadır.

ÖZET (devam ediyor)

Bölüm 9, çözümleri genelleştirilmiş Tribonacci sayılarıyla ilişkili olan aşağıdaki $x_{n+1} = \gamma / (x_n(x_{n-1} + \alpha) + \beta),$ lineer olmayan fark denkleminin çözümlerinin asimptotik davranışlarını ve kesin çözüm formlarını ve denge noktalarının kararlılık karakterlerini ifade eder.

Anahtar Kelimeler: Fark denklem, denge noktası, asimptotik davranış, global asimptotik kararlı, salınımlılık, periyodiklik, sınırsız çözümler, sınırlılık, tekrarlı dizi, Tribonacci sayıları.

Bilim Kodu: 403.06.01

ACKNOWLEDGEMENTS

I would like to express the deepest appreciation to my advisor Prof. Yüksel Soykan who has the attitude and substance of a genius: he continually and convincingly conveyed a spirit of adventure in regard to research and scholarship, and an excitement in regard to teaching. I would like to thank to him for supporting me during my doctorate process.

Besides my advisor, I would like to thank to Prof. Yusuf Kaya and Assist. Prof. Mustafa Akıncı for their helping, suggestions and helpful career advice in general. The good advice, support and friendship of them has been invaluable on both an academic and a personal level for which I am extremely grateful.

I also appreciate to Assist. Prof. Melih Göcen, Assist. Prof. Can Murat Dikmen, Assist. Prof. Mehmet Gümüş and Assist. Prof. Erkan Taşdemir to give their support both in my studies and in my daily life adoptively.

Besides them, I thank to all my relatives, especially my aunt Elif Ümsür, for their supports during my graduate education.

Last, but by no means least, I would like to thank my parents Sadullah Okumuş and Vildane Okumuş who have a big role in my success for giving birth to me at the first place and supporting me spiritually throughout my life. Furthermore, I would like to thank to my sisters Demet Okumuş, Tuğba Okumuş, and Kübra Okumuş and my brother Metin Okumuş for their support and encouragement throughout. I owe them everything and wish I could show them just how much I love and appreciate them.

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English

FIELD OF STUDY

Major Field: Difference and Functional Equations: (Stability and Asymptotics of Difference Equations; Oscillatory and Periodic Solutions, etc.)

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEWS

Difference equation or discrete dynamical system is a diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system

$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}\right)$

determines a difference equation and vice versa.

Recently, there has been great interest in studying the difference equations and systems of difference equations. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology and so forth.

The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference eqautions of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, psychology, physics, engineering, and economics (see [1-14]). It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points. Though difference equations are very simple in their form, it is quite hard to understand throughly the global behavior of their solutions. There are many papers in which systems of difference equations have been studied, see [29-42].

Moreover, there has been a growing interest in the study of finding closed-form solutions of difference equations and systems of difference equations. Some of the forms of solutions of these equations are representable via well-known integer sequences such as Fibonacci numbers, Lucas numbers, Pell numbers and Padovan numbers, see also [74-100].

The purpose in this thesis is to investigate a systematic study of dynamical behavior of solutions of some specific nonlinear difference equations and systems of difference equations. Especially, we research the exact forms, periodicity, stability character and boundedness of solutions of difference equations and systems of nonlinear difference equations. This thesis consist of nine chapters.

The first chapter is a concise overview of what this thesis is about and also is a literature summary of difference equation theory.

The second chapter consists of some basic important definitions and some significant theorems used throughout the thesis.

The third chapter includes some results about the stability, boundedness character and periodicity of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{z_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{z_n}, \quad z_{n+1} = A + \frac{z_{n-1}}{y_n}, \quad n = 0, 1, ...,$$

where the parameter A and the initial conditions

$$x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$$

are arbitrary positive real numbers.

The fourth chapter presents the local asymptotic stability of the equilibrium points, boundedness character, oscillatory, and global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = A + \frac{x_{n-m}}{z_n}, \; y_{n+1} = A + \frac{y_{n-m}}{z_n}, \; z_{n+1} = A + \frac{z_{n-m}}{y_n}, \; n = 0, 1, ...,$$

where the parameter A and the initial values

$$x_{-i}, y_{-i}, z_{-i}, \quad \text{for } i = 0, 1, ..., m,$$

are positive real numbers and m is positive integer.

The fifth chapter contains some results about the local asymptotic stability of the equilibrium points and oscillation behaviour of positive solutions of the following system of rational difference equations

$$x_{n+1} = A + \frac{x_{n-1}^p}{z_n^p}, \; y_{n+1} = A + \frac{y_{n-1}^p}{z_n^p}, \; z_{n+1} = A + \frac{z_{n-1}^p}{y_n^p}, \; n = 0, 1, ...,$$

where the parameters

$$A \in (0, \infty), \quad p \in [1, \infty)$$

and the initial values

$$x_i, y_i, z_i \in (0, \infty), \quad i = -1, 0.$$

The sixth chapter introduces the form of solutions, stability character and asymptotic behavior of the following four rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{1}{x_n (x_{n-1} \pm 1) \pm 1}, \\ x_{n+1} &= \frac{-1}{x_n (x_{n-1} \pm 1) \mp 1}, \end{aligned}$$

such that their solutions are associated with Tribonacci numbers.

The seventh chapter acquaints about the stability character, the periodicity and the global behavior of solutions of the following four rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{\pm 1}{x_n (x_{n-1} \pm 1) - 1} \\ x_{n+1} &= \frac{\pm 1}{x_n (x_{n-1} \mp 1) + 1}. \end{aligned}$$

The eighth chapter presents some results about the explicit form, stability character and global behavior of solutions of the following two systems of rational difference equations

$$x_{n+1} = \frac{\pm 1}{y_n (x_{n-1} \pm 1) + 1}, \quad y_{n+1} = \frac{\pm 1}{x_n (y_{n-1} \pm 1) + 1}, \quad n = 0, 1, ...$$

such that their solutions are associated with Tribonacci numbers.

The ninth chapter expresses the dynamical behavior of solutions of the following nonlinear difference equation

$$x_{n+1} = \frac{\gamma}{x_n (x_{n-1} + \alpha) + \beta}, \quad n = 0, 1, ...,$$

such that their solutions are associated with generalized Tribonacci numbers.

The following two sections are important summaries about difference equations and systems of difference equations which have shed light on our studies in this thesis.

1.1 LITERATURE REVIEW FOR SYSTEMS OF NONLINEAR DIFFERENCE EQUATIONS

In this section, we have divided the studies which we examine in literature into three subsections as two-dimensional, three-dimensional and multi-dimensional systems of nonlinear difference equations.

1.1.1 Review on Two-Dimensional Systems of Nonlinear Difference Equations

This subsection is concerned with the review of dynamical behavior of solutions of the systems of two-dimensional nonlinear difference equations. Then, we have classified these studies into three subsubsections as systems of rational-type difference equations, systems of exponential-type difference equations and systems of max-type difference equations.

Systems of Rational-Type Difference Equations

In [15], Papaschinopoulos and Schinas considered the system of difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, ..., \tag{1.1}$$

where $A \in (0, \infty)$, p, q are positive integers and $x_{-p}, ..., x_0$, $y_{-q}, ..., y_0$ are positive numbers. They investigated the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system (1.1). As a result, they prove that:

- Every positive nontrivial solution $\{(x_n, y_n)\}$ of system (1.1) oscillates about the positive equilibrium of system (1.1).
- If $A > 0$ and one at least of p, q is an odd number (resp. $A > 1$ and p, q are both even numbers), then any positive solution of system (1.1) is bounded away from zero and infinity.
- If $A > 1$, then the positive equilibrium (c, c) of system (1.1) is globally asymptotically stable.

In [16], Papaschinopoulos and Schinas studied the oscillatory behavior, the periodicity and the asymptotic behavior of the positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_n}, \quad n = 0, 1, ..., \tag{1.2}$$

where A is a positive constants and initial conditions are positive numbers.
They established conditions so that a positive solution (x_n, y_n) of system (1.2) oscillates about positive equilibrium of the system (1.2). Moreover, they found

- For the case $0 \leq A < 1$,
  - The unique positive equilibrium (c, c) of (1.2) is not stable.
  - The system (1.2) has unbounded solutions.
- For the case $A = 1$,
  - For every $\mu \in (1, \infty)$, there exist positive solutions (x_n, y_n) of system (1.2) which tend to the positive equilibrium $\left(\mu, \frac{\mu}{\mu-1}\right)$.
  - Every positive solution of system (1.2) tends to a period 2 solution as $n \to \infty$.
- For the case $A > 1$,
  - The unique positive equilibrium (c, c) of (1.2) is locally asymptotically stable.
  - The positive equilibrium (c, c) of system (1.2) is globally asymptotically stable.

In [17], Camouzis and Papaschinopoulos studied the boundedness, persistence, and the global asymptotic behavior of the positive solutions of the system of difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, ..., \tag{1.3}$$

where x_i, y_i, $i = -m, -m+1, ..., 0$ are positive numbers and m is a positive integer. Then the following results were exhibited in their paper:

- Every positive solution of system (1.3) is bounded and persists,
- System (1.3) has an infinite number of positive equilibrium solutions (x, y) with x, $y \in (1, \infty)$ that satisfy equation $xy = x + y$,
- Every positive solutions of system (1.3) converges to a positive equilibrium solution of system (1.3) as $n \to \infty$.

In [18], Yang studied the behavior of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{y_{n-1}}{x_{n-p}y_{n-q}}, \quad y_{n+1} = A + \frac{x_{n-1}}{x_{n-r}y_{n-s}}, \quad n = 1, 2, ..., \tag{1.4}$$

where $p \geq 2$, $q \geq 2$, $r \geq 2$, $s \geq 2$, A is a positive constant, and $x_{1-\max\{p,r\}}$, $x_{2-\max\{p,r\}}$, ...,x_0, $y_{1-\max\{q,s\}}$, $y_{2-\max\{q,s\}}$, ...,y_0 are positive real numbers.
He demonstrated that:

- The system (1.4) has the unique positive equilibrium

  $$(c,c) = \left(\frac{A+\sqrt{A^2+4}}{2}, \frac{A+\sqrt{A^2+4}}{2}\right),$$

- When $A > 1$, every positive solution of system (1.4) is bounded,
- When $A > 2/\sqrt{3}$, (c,c) is locally asymptotically stable,
- When $A > \sqrt{2}$, every positive solution of system (1.4) approaches (c,c),
- When $A > \sqrt{2}$, the positive equilibrium (c,c) of (1.4) is globally asymptotically stable for all positive solutions.

In [19], Zhang et al. considered the behavior of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{1}{y_{n-p}}, \quad y_{n+1} = A + \frac{1}{x_{n-r}y_{n-s}}, \quad n = 1,2,..., \tag{1.5}$$

where $p \geq 1$, $r \geq 1$, $s \geq 1$, $A \geq 0$, and x_{1-r}, x_{2-r}, ...,x_0, $y_{1-\max\{p,s\}}$, $y_{2-\max\{p,s\}}$, ...,y_0 are positive real numbers.
They obtained the following results:

- If $A > 0$, every positive solution of system (1.5) is bounded,
- If $A = 0$, all positive solutions of system (1.5) are periodic,
- If $A > 2/\sqrt{3}$ and $\max\{p,r,s\} \geq 2$, the positive equilibrium (c,c) of (1.5) is locally asymptotically stable where $(c,c) = \left(\frac{A+\sqrt{A^2+4}}{2}, \frac{A+\sqrt{A^2+4}}{2}\right)$,
- If $A > \sqrt{2}$, every positive solution of system (1.5) approaches (c,c),
- If $A > \sqrt{2}$ and $\max\{p,r,s\} \geq 2$, the positive equilibrium (c,c) of (1.5) is globally asymptotically stable for all positive solutions.

In [20], Zhang et al. studied the system of rational difference equations

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n}, \quad n = 0, 1, ... \tag{1.6}$$

They investigated the dynamic behavior of positive solutions of system (1.6) for the cases of $A < 1$, $A = 1$, and $A > 1$.

For the case $A < 1$, they obtained that the system (1.6) has unbounded solutions.

For the case $A = 1$, they proved that every positive solution of the system (1.6) is bounded and persists with interval $\left[L, \frac{L}{L-1}\right]$ and has prime two periodic solutions.

For the case $A > 1$, the global asymptotic stability of the unique equilibrium point of the system (1.6) is established. For this case, they proved that:

- Every positive solution of the system (1.6) is bounded and persists by interval $\left[L, \frac{L}{L-A}\right]$,
- The positive equilibrium point (c, c) of system (1.6) is locally asymptotically stable where $c = A + 1$,
- Every positive solution of system (1.6) converges to (c, c).

In [21], Zhang et al. considered the behavior of the symmetrical system of rational difference equations

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, ... \tag{1.7}$$

where $A > 0$ and x_i, $y_i \in (0, \infty)$, for $i = -k, -k+1, ..., 0$.

They investigated the dynamic behavior of positive solutions of system (1.7) for the cases of $0 < A < 1$, $A = 1$, and $A > 1$.

In the case $0 < A < 1$, they obtained similar results as in above Theorem 1 for k is odd. However, they said that they can't get some useful results for k is even.

In the case $A = 1$, the results which are obtained are similar to results in [20].

In the case $A > 1$, the following results were established:

- Every positive solution of the system (1.7) is bounded and persists by interval $\left[L, \frac{L}{L-A}\right]$,
- Every positive solution of the system (1.7) converges to the equilibrium as $n \to \infty$.

In [22], Kurbanli, Çinar and Yalcinkaya studied the behavior of the positive solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \tag{1.8}$$

where the initial conditions are arbitrary non-negative real numbers.

They found the equilibrium point and all solutions of the system (1.8). Also, they obtained the followings where $y_0 = a$, $y_{-1} = b$, $x_0 = c$ and $x_{-1} = d$ are arbitrary non-negative real numbers:

- If $b \neq 0$ and $c = 0$, $x_{2n} = 0$ and $y_{2n-1} = b$,
- If $b = 0$ and $c \neq 0$, $x_{2n} = c$ and $y_{2n-1} = 0$,
- If $d = 0$ and $a \neq 0$, $y_{2n} = a$ and $x_{2n-1} = 0$,
- If $a = 0$ and $d \neq 0$, $y_{2n} = 0$ and $x_{2n-1} = d$.

In [23], Kurbanli et al. investigated the periodicity of the solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1} + y_n}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1} + x_n}{x_n y_{n-1} - 1}, \tag{1.9}$$

where x_0, x_{-1}, y_0, $y_{-1} \in \mathbb{R}$.

They proved that the solutions of x_n and y_n are six periodic under the special conditions.

In [24], Wang, Zhang and Fu considered the system of difference equations

$$x_{n+1} = \frac{x_{n-2k+1}}{Ay_{n-k+1}x_{n-2k+1} + \alpha}, \quad y_{n+1} = \frac{y_{n-2k+1}}{Bx_{n-k+1}y_{n-2k+1} + \beta}, \quad n \geq 0, \tag{1.10}$$

where k is a positive integer, A, B, α, β and the initial conditions are positive real numbers.

Under the specific conditions, they established the convergence of the positive solutions of the system (1.10) and showed that the system (1.10) has unbounded solutions.

In [25], Zhang et al. corcerned with the dynamical behavior of positive solutions of the system of two rational difference equations

$$x_{n+1} = A + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}, \quad y_{n+1} = B + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, ..., \tag{1.11}$$

where A, B are positive constants and the initial conditions x_{-i}, $y_{-i} \in (0,\infty)$, $i = 0,1,...,k$.

They proved that under the case $A > 1/k$, $B > 1/k$ and assuming that

$\frac{k^2AB-1}{kA-1} + \frac{k^2AB-1}{kB-1} < 1$:

- Every positive solution of system (1.11) is persistent and bounded,

- The system (1.11) has a unique positive equilibrium given by

  $x = \frac{k^2AB-1}{k(kB-1)}, \ y = \frac{k^2AB-1}{k(kA-1)}$,

- Every positive solution of the system (1.11) tends to the positive equilibrium of system (1.11) as $n \to \infty$,

- The unique positive equilibrium of the system (1.11) is locally asymptotically stable,

- The unique positive equilibrium of the system (1.11) is globally asymptotically stable.

In [26], Zhang and Zhang investigated the solutions, stability character and asymptotic behavior of the system of high-order nonlinear difference equations

$$x_{n+1} = \frac{x_{n-k}}{q + \prod_{i=0}^{k} y_{n-i}}, \quad y_{n+1} = \frac{y_{n-k}}{p + \prod_{i=0}^{k} x_{n-i}}, \quad k \in \mathbb{N}^+, \ n = 0,1,..., \tag{1.12}$$

where p, $q \in (0,\infty)$, $x_{-i} \in (0,\infty)$, $y_{-i} \in (0,\infty)$ and $i = 0,1,...,k$.

First, they obtained the equilibrium points of system (1.12) as follows:

- $(0,0)$ and $\left(\sqrt[k+1]{1-p}, \sqrt[k+1]{1-q}\right)$ are equilibrium points if $p < 1$ and $q < 1$,

- Every point on the x-axis is an equilibrium point if $q = 1$,

- Every point on the y-axis is an equilibrium point if $p = 1$,

- $(0,0)$ is the unique equilibrium point if $p > 1$ and $q > 1$.

Then, they proved the following results:

- If $p > 1$ and $q > 1$, then the unique equilibrium point $(0,0)$ of system (1.12) is locally asymptotically stable,

- If $p < 1$ and $q < 1$, then the unique equilibrium point $(0,0)$ of system (1.12) is unstable,

- If $p < 1$ and $q < 1$, then the positive equilibrium point $\left(\sqrt[k+1]{1-p}, \sqrt[k+1]{1-q}\right)$ of system (1.12) is unstable,

- Every solutions of system (1.12) is bounded,

- If $p > 1$ and $q > 1$, then the unique equilibrium point $(0,0)$ of system (1.12) is globally asymptotically stable.

In [27], Zhang et al. studied the behavior of solutions of the following system

$$x_{n+1} = A + \frac{x_n}{y_{n-1}y_{n-2}}, \quad y_{n+1} = A + \frac{y_n}{x_{n-1}x_{n-2}}, \quad n = 0, 1, ..., \tag{1.13}$$

where A is positive constant and x_{-i}, $y_{-i} \in (0, \infty)$, $i = 0, 1, 2$.
They obtained the results which are listed below:

- If $A > 1$, every positive solution of system (1.13) is bounded,

- If $A > 2/\sqrt{3}$, (c, c) is locally asymptotically stable,

- If $A > \sqrt{3}$, every positive solution of system (1.13) approaches (c, c),

- If $1 < A < 2/\sqrt{3}$, (a_1, b_1) and (a_2, b_2) are locally asymptotically stable.

In [28], Stevic et al. considered the following system of difference equations

$$x_{n+1} = A + \frac{y_n^p}{x_{n-1}^q}, \quad y_{n+1} = A + \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0 \tag{1.14}$$

where parameters A, p and q are positive and investigated the boundedness character of positive solutions of system (1.14).
They proved the following results:

- If $p^2 \geq 4q > 4$, or $p \geq 1 + q$, $q \leq 1$, then system (1.14) has positive unbounded solutions where $A > 0$,

- If $p^2 < 4q$, or $2\sqrt{q} \leq p < 1+q$, $q \in (0,1)$, then all postive solutions of system (1.14) are bounded.

In [29], Bao investigated the local stability, oscillation and boundedness character of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{x_{n-1}^p}{y_n^p}, \quad y_{n+1} = A + \frac{y_{n-1}^p}{x_n^p}, \quad n = 0, 1, ..., \tag{1.15}$$

where $A \in (0, \infty)$, $p \in [1, \infty)$ and initial conditions x_i, $y_i \in (0, \infty)$, $i = -1, 0$.
He proved that the system (1.15) has a positive equilibrium point $(\overline{x}, \overline{y}) = (A+1, A+1)$ and the equilibrium point of system (1.15) is locally asymptotically stable if $A > 2p-1$, is unstable if $0 < A < 2p-1$ and is a sink or an attracting equilibrium if $p/(A+1) < \sqrt{2}-1$. Also, he indicated that the positive solution of system (1.15) which consists of at least two semicycles is oscillatory and the system (1.15) has unbounded solutions.
In [30], Gümüş and Soykan considered the dynamical behavior of positive solutions for a system of rational difference equations of the following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^p}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^p}, \quad n = 0, 1, ..., \tag{1.16}$$

where the parameters α, β, γ, α_1, β_1, γ_1, p and the initial values u_{-i}, v_{-i} for $i = 0, 1, 2$ are positive real numbers.
First, they reduced the system (1.16) to the following system of difference equations

$$x_{n+1} = \frac{r x_{n-1}}{1 + y_{n-2}^p}, \; y_{n+1} = \frac{s y_{n-1}}{1 + x_{n-2}^p}, \quad n = 0, 1, ..., \tag{1.17}$$

by the change of variables $u_n = (\beta_1/\gamma_1)^{1/p} x_n$ and $v_n = (\beta/\gamma)^{1/p} y_n$ with $r = \alpha/\beta$ and $s = \alpha_1/\beta_1$.
Then, they found the equilibrium points of the system (1.17) under the certain conditions and investigated their local asymptotical behavior. Also, they proved that

- If $r < 1$ and $s < 1$, the zero equilibrium point of system (1.17) is globally asymptotically stable,

- For r, $s \in (1, \infty)$, the system (1.17) has unbounded solutions,

- If $r = s = 1$, the system (1.17) possesses the prime period two solution.

In [31], Din studied the qualitative behavior of positive solutions of following second-order system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n}, \tag{1.18}$$

where the parameters α_i, β_i, a_i, b_i for $i \in \{1, 2\}$ and initial conditions are positive real numbers.

He determined the following results:

- Every positive solution of system (1.18) is bounded and persists when $\beta_1\beta_2 < a_1a_2$,
- The unique positive equilibrium point of system (1.18) is global attractor when $a_1a_2 \neq \beta_1\beta_2$,
- Under the some specific conditions the unique positive equilibrium point of system (1.18) is globally asymptotically stable.
- The system (1.18) has no prime period-two solutions when $a_1a_2 \neq \beta_1\beta_2$.

In [32], Mansour et al. got the exact form of the solutions and the periodic nature of the following systems of difference equations

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-5}y_{n-2}}, \quad y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5}x_{n-2}}, \tag{1.19}$$

where the initial conditions are real numbers.

In [33], Elsayed and El-Metwally had the periodic nature and the form of the solutions of some systems of difference equations

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1}\left(\pm 1 \pm x_n y_{n-2}\right)}, \quad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1}\left(\pm 1 \pm y_n x_{n-2}\right)}, \tag{1.20}$$

where the initial conditions are nonzero real numbers.

In [34], Elsayed obtained the form of the solutions and the periodicity of the following systems of second-order rational difference equations

$$x_{n+1} = \frac{x_n y_{n-1}}{y_n\left(\pm 1 \pm x_n y_{n-1}\right)}, \quad y_{n+1} = \frac{y_n x_{n-1}}{x_n\left(\pm 1 \pm y_n x_{n-1}\right)}, \tag{1.21}$$

with the initial conditions are nonzero real numbers.

In [35], Clark and Kulenovic investigated the asymptotic and global stability behavior of solutions of the following systems of difference equations

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}, \quad n = 0, 1, ..., \tag{1.22}$$

where the parameters are positive numbers and the initial conditions are arbitrary non-negative numbers.

Then, in [36], Clark et al. completed the investigation studied in [35] of the global behavior of system (1.22).

In [37], Kulenovic and Nurkanovic studied the system of difference equations

$$x_{n+1} = Ax_n \frac{y_n}{1 + y_n}, \quad y_{n+1} = By_n \frac{x_n}{1 + x_n}, \quad n = 0, 1, ..., \tag{1.23}$$

where the parameters A and B are in $(0, \infty)$ and the initial conditions are arbitrary nonnegative numbers. Under the special circumstances of parameters, they established the global asymptotic stability of the equilibrium points of the system (1.23).

Also, there are many similar woks, see [38, 39, 40, 41, 42].

Systems of Exponential-Type Difference Equations

In this subsubsection, we review on some papers studied related to system of difference equations of exponential form.

In [43], Papaschinopoulos, Radin and Schinas studied the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form

$$x_{n+1} = a + bx_{n-1}e^{-y_n}, \quad y_{n+1} = c + dy_{n-1}e^{-x_n} \tag{1.24}$$

where a, b, c, d are positive constants, and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real values.

They investigated the boundedness character and the existence of invariant intervals of system (1.24). Then, they found the following results. Under the conditions that $be^{-c} < 1$ and $de^{-a} < 1$, every positive solution of system (1.24) is bounded and persists. Also, they proved that the unique positive equilibrium $(\overline{x}, \overline{y})$ of system (1.24) is globally asymptotically stable under appropriate conditions.

In [44], Papaschinopoulos and Schinas considered the following systems of difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \;\; y_{n+1} = c + dx_{n-1}e^{-y_n} \tag{1.25}$$

$$x_{n+1} = a + by_{n-1}e^{-y_n}, \;\; y_{n+1} = c + dx_{n-1}e^{-x_n} \tag{1.26}$$

where the constants are positive real numbers and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real numbers.

They investigated the boundedness and the persistence of the positive solutions, the existence of a unique positive equilibrium and the global asymptotic stability of the above mentioned systems. As a result, they established that every solution of the systems (1.25) and (1.26) is positive, bounded and persists if $p = bde^{-a-c} < 1$. Also, under the specific conditions, they indicated that the systems (1.25) and (1.26) have a unique positive equilibrium and every solution of these systems tends to the unique positive equilibrium of their as $n \to \infty$, each one positive equilibrium of these systems is globally asymptotically stable and finally, these systems have unbounded solutions.

In [45], Papaschinopoulos et al. investigated the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following systems of difference equations

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \;\; y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + x_{n-1}}, \tag{1.27}$$

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \;\; y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}}, \tag{1.28}$$

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \;\; y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}}, \tag{1.29}$$

where α, β, γ, δ, ϵ, ζ are positive constant and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive constant.

They got the results are given below:

- For the system (1.27)
  - Every positive solution of the system (1.27) is bounded and persists,
  - If $\epsilon < \gamma$ and $\beta < \zeta$, the system (1.27) has a unique positive equilibrium and every solution of the system (1.27) tends to the unique positive equilibrium of the system (1.27) as $n \to \infty$,

- If $\frac{\beta\epsilon+(\beta+\epsilon)e-1}{\gamma\zeta} + \frac{(\alpha+\beta)(\delta+\epsilon)}{\gamma^2\zeta^2} < 1$, the unique positive equilibrium of the system (1.27) is globally asymptotically stable.

- For the system (1.28)
  - Every positive solution of the system (1.28) is bounded and persists,
  - If $\beta\epsilon < \gamma\zeta$, the system (1.28) has a unique positive equilibrium and every solution of the system (1.28) tends to the unique positive equilibrium of the system (1.28) as $n \to \infty$,
  - If $\frac{\alpha+\beta}{\gamma^2} + \frac{\delta+\epsilon}{\zeta^2} + \frac{\beta\epsilon}{\gamma\zeta} + \frac{(\alpha+\beta)(\delta+\epsilon)}{\gamma^2\zeta^2} < 1$, the unique positive equilibrium of the system (1.28) is globally asymptotically stable.

- For the system (1.29)
  - Every positive solution of the system (1.29) is bounded and persists,
  - If $\beta < \gamma$ and $\epsilon < \zeta$, the system (1.29) has a unique positive equilibrium and every solution of the system (1.29) tends to the unique positive equilibrium of the system (1.29) as $n \to \infty$,
  - If $\frac{\beta}{\gamma} + \frac{\epsilon}{\zeta} + \frac{\beta\epsilon}{\gamma\zeta} + \frac{(\alpha+\beta)(\delta+\epsilon)}{\gamma^2\zeta^2} < 1$, the unique positive equilibrium of the system (1.29) is globally asymptotically stable.

In [46], Elettreby and El-Metwally considered the system of difference equations, which describes an economic model,

$$x_{n+1} = (1-\alpha)\,x_n + \beta x_n\,(1-x_n)\,e^{-(x_n+y_n)}, \tag{1.30}$$

$$y_{n+1} = (1-\alpha)\,y_n + \beta y_n\,(1-y_n)\,e^{-(x_n+y_n)}, \quad n = 0, 1, ..., \tag{1.31}$$

where α and $\beta \in (0,\infty)$ with the initial conditions x_0 and $y_0 \in (0,\infty)$.
They studied the boundedness and the invariant of the solutions of system (1.30) and also investigated global convergence for the solutions of system (1.30). Then, they obtained the following main results:

- Every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1.30) is bounded. Moreover,

$$\lim_{n\to\infty} \sup x_n \leq \frac{\beta}{\alpha e}, \quad \lim_{n\to\infty} \sup y_n \leq \frac{\beta}{\alpha e}.$$

- When $\alpha \geq \beta$, the zero equilibrium $(0,0)$ is a global attractor of all positive solutions of system (1.30).

- When $\alpha + \beta e^{-2} < 1$, the unique positive equilibrium point $(\overline{x}, \overline{x})$ of system (1.30) is a global attractor of all positive solutions of system (1.30).

- When $\beta\left(\alpha e - \beta\right) \geq \alpha^2 e^3$, the unique positive equilibrium point $(\overline{x}, \overline{x})$ of system (1.30) is a global attractor of all positive solutions of system (1.30).

- When one of the following conditions hold

i) $5\beta \leq 4e^2\left(1-\alpha\right)$
ii) $\alpha + \beta < 1$
the unique positive equilibrium point $(\overline{x}, \overline{x})$ of system (1.30) is a global attractor of all positive solutions of system (1.30).
In [47], Papaschinopoulos et al. studied the asymptotic behavior of the positive solutions of the system of difference equations

$$x_{n+1} = ay_n + bx_{n-1}e^{-y_n}, \;\; y_{n+1} = cy_n + dy_{n-1}e^{-x_n}, \; n = 0, 1, ..., \tag{1.32}$$

where a, b, c, d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.
Then, they prove that under the condition that a, b, c, $d \in (0,1)$, $a+b > 1$, $c+d > 1$;

- Every positive solution of system (1.32) is bounded and persists.

- Every positive solution of system (1.32) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ of system (1.32) as $n \to \infty$, when suppose that either relations

  $c \leq a, \;\; b \leq c, \;\; d \leq c$ or $a \leq c, \;\; b \leq a, \;\; d \leq a.$

  Under the condition that $a+b \leq 1$, $c+d \leq 1$;

- Every positive solution of system (1.32) tends to the zero equilibrium $(0,0)$ of system (1.32) as $n \to \infty$.

Finally, they established that where a, b, c, d are positive constants such that either

$a+b<1,\ \ c+d<1$ or $a+b=1,\ \ c+d=1,$

the zero equilibrium $(0,0)$ of system (1.32) is globally asymptotically stable.
In [48], Khan investigated the qualitative behavior of positive solutions of the following two systems of exponential rational difference equations

$$x_{n+1}=\frac{\alpha e^{-y_n}+\beta e^{-y_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}},\quad y_{n+1}=\frac{\alpha_1 e^{-x_n}+\beta_1 e^{-x_{n-1}}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}},\quad n=0,1,..., \tag{1.33}$$

$$x_{n+1}=\frac{\alpha e^{-x_n}+\beta e^{-x_{n-1}}}{\gamma+\alpha y_n+\beta y_{n-1}},\quad y_{n+1}=\frac{\alpha_1 e^{-y_n}+\beta_1 e^{-y_{n-1}}}{\gamma_1+\alpha_1 x_n+\beta_1 x_{n-1}},\quad n=0,1,..., \tag{1.34}$$

where α, β, γ, α_1, β_1, γ_1 and the initial conditions are positive real numbers.
They obtained the results are given below:

- For the system (1.33)
  - Every positive solution of the system (1.33) is bounded and persists,
  - If $(\alpha+\beta)\,e^{-L_2}<\overline{x}\,(\gamma+(\alpha+\beta)\,L_2)$ and $(\alpha_1+\beta_1)\,e^{-L_1}<\overline{y}\,(\gamma_1+(\alpha_1+\beta_1)\,L_1)$, the unique positive equilibrium point of the system (1.33) is globally asymptotically stable.
- For the system (1.34)
  - Every positive solution of the system (1.34) is bounded and persists,
  - If $(\alpha+\beta)\,e^{-L_1}<\overline{x}\,(\gamma+(\alpha+\beta)\,L_2)$ and $(\alpha_1+\beta_1)\,e^{-L_2}<\overline{y}\,(\gamma_1+(\alpha_1+\beta_1)\,L_1)$, the unique positive equilibrium point of the system (1.34) is globally asymptotically stable.

Systems of Max-Type Difference Equations

In [49], Simsek, Demir and Cinar considered the behavior of the solutions of the following system of difference equations

$$x_{n+1}=\max\left\{\frac{A}{x_n},\frac{y_n}{x_n}\right\},\ \ y_{n+1}=\max\left\{\frac{A}{y_n},\frac{x_n}{y_n}\right\}, \tag{1.35}$$

where the constant A and the initial conditions are positive real numbers. They proved that the system (1.35) has unbounded solutions for special cases.

But then, in [50], Stevic corrected the results given in [49] and showed that the general solution to the max-type system of difference equations (1.35) for the case

$$y_0, x_0 \geq A > 0, y_0/x_0 \geq \max\{A, 1/A\},$$

is given by:

$$x_n = \left(\frac{A^{f_{k(n)-1}-\alpha(n)} x_0^{f_{k(n)}}}{y_0^{f_{k(n)}}} \right)^{(-1)^n}, \quad n \in \mathbb{N},$$

and

$$y_n = \left(\frac{y_0^{f_{k(n-1)+1}}}{A^{f_{k(n-1)}+\alpha(n)-1} x_0^{f_{k(n-1)+1}}} \right)^{(-1)^n}, \quad n \geq 2.$$

In [51], Fotiades and Papaschinopoulos studied the periodic character of the solutions of the system of the difference equations

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\}, \quad y_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\}, \tag{1.36}$$

where A, B are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

The authors established that every solution of system (1.36) is eventually periodic for the cases:

$$1 \leq A \leq B, \;\; A < 1 \leq B, \;\; A \leq B < 1, \;\; 1 \leq B \leq A, \;\; B < 1 \leq A, \;\; B \leq A < 1.$$

In [52], Stevic studied behavior of positive solutions of the max-type system of difference equations

$$x_{n+1} = \max\left\{c, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{c, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0 \tag{1.37}$$

where p,$c \in (0, \infty)$. In his work, boundedness character and global attractivity are investigated for some special cases.

For the case $p \in (0, 4)$ and $c > 0$, boundedness of all positive solutions of system (1.37) is determined. Also, for $p \in (0, 4)$ and $c \geq 1$, it is given that every positive solution $(x_n, y_n)_{n\geq 1}$ of system (1.37) is eventually equal to (c, c). Besides, the system (1.37) has positive unbounded solutions when $p \geq 4$ and $c > 0$. Finally, every positive solution of system (1.37) converges to $(1, 1)$ when $p \in (0, 1)$ and $c \in (0, 1)$.

In [53], Stevic et al. studied the boundedness character of positive solutions of system of difference equations

$$x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\}, \quad y_{n+1} = \max\left\{A, \frac{x_n^p}{y_{n-1}^q}\right\}, \quad n \in \mathbb{N}_0 \tag{1.38}$$

with $\min\{A, p, q\} > 0$.

Consequently, the following statements are obtained:

- All positive solutions of system (1.38) are bounded when $A > 0$, $2\sqrt{q} \leq p < 1 + q$ and $q \in (0, 1)$.
- All positive solutions of system (1.38) are bounded when $A > 0$, $p > 0$ and $p^2 < 4q$.
- All positive solutions of system (1.38) are bounded when $A > 0$, $p = 1 + q$ and $q \in (0, 1)$.
- The system (1.38) has positive unbounded solutions if $A > 0$, $p^2 \geq 4q \geq 4$, or $p > 1 + q$ and $q \in (0, 1)$.

1.1.2 Review on Three-Dimensional Systems of Nonlinear Difference Equations

This subsection is concerned with review of dynamical behavior of solutions of the systems of three-dimensional nonlinear difference equations.

Systems of Rational-Type Difference Equations

In [54], Kulenovic and Nurkanovic studied the global behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{c + y_n}{d + z_n}, \quad z_{n+1} = \frac{e + z_n}{f + x_n}, \quad n = 0, 1, ..., \tag{1.39}$$

where the parameters a, b, c, d, e and f are in $(0, \infty)$ and the initial conditions are arbitrary non-negative numbers.

They indicated that the equilibrium of system (1.39) is locally asymptotically stable if $b \geq 1$, $d \geq 1$, $f \geq 1$, and obtained the global asymptotic stability of the unique positive equilibrium for several cases depending of some special values of the parameters.

In [55] Kurbanli, in [56] Kurbanli and in [57] Kurbanli et al. investigated the behavior of the solutions of the difference equations systems

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1} \tag{1.40}$$

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{1}{y_n z_n}, \tag{1.41}$$

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}, \tag{1.42}$$

where the initial conditions are arbitrary real numbers, respectively.

They found all exact solutions of systems (1.40), (1.41), and (1.42) under special conditions and showed that the systems have unbounded solutions.

In [58], Özkan and Kurbanli studied the periodical solutions of the systems of difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2} x_{n-1} y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}, \quad z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}, \quad n \in \mathbb{N}_0,$$

where the initial conditions are arbitrary real numbers. They obtained all six-period solutions of given systems under special conditions.

In [59], Stevic showed that the following system of difference equations

$$x_{n+1} = \frac{a_1 x_{n-2}}{b_1 y_n z_{n-1} x_{n-2} + c_1}, \quad y_{n+1} = \frac{a_2 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2} + c_2}, \quad z_{n+1} = \frac{a_3 z_{n-2}}{b_3 x_n y_{n-1} z_{n-2} + c_3}, \quad n \in \mathbb{N}_0,$$

where the parameters and the initial conditions are real numbers, can be solved.

1.1.3 Review on Multi-Dimensional Systems of Nonlinear Difference Equations

This subsection is concerned with review of dynamical behavior of solutions of the systems of multi-dimensional nonlinear difference equations.

Systems of Max-Type Difference Equations

In [60], Stevic studied the system of max-type difference equations

$$\begin{aligned} x_n^{(1)} &= \max_{1\leq i\leq m_1}\left\{ f_{1i}\left(x^{(1)}_{n-k^{(1)}_{i,1}}, x^{(2)}_{n-k^{(1)}_{i,2}}, ..., x^{(l)}_{n-k^{(1)}_{i,l}}, n\right), x^{(1)}_{n-s}\right\}, \\ x_n^{(2)} &= \max_{1\leq i\leq m_2}\left\{ f_{2i}\left(x^{(1)}_{n-k^{(2)}_{i,1}}, x^{(2)}_{n-k^{(2)}_{i,2}}, ..., x^{(l)}_{n-k^{(2)}_{i,l}}, n\right), x^{(2)}_{n-s}\right\}, \\ &\vdots \\ x_n^{(l)} &= \max_{1\leq i\leq m_l}\left\{ f_{li}\left(x^{(1)}_{n-k^{(l)}_{i,1}}, x^{(2)}_{n-k^{(l)}_{i,2}}, ..., x^{(l)}_{n-k^{(l)}_{i,l}}, n\right), x^{(l)}_{n-s}\right\}, \end{aligned} \tag{1.43}$$

$n\in\mathbb{N}_0$, where $s,\ l,\ m_j,\ k^{(j)}_{i,t}\in\mathbb{N},\ j,\ t\in\{1,...,l\}$ and for a fixed j, $i\in\{1,...,m_j\}$, and where the functions $f_{ij}:(0,\infty)^l\times\mathbb{N}_0\to(0,\infty)$, $j\in\{1,...,l\}$, $i\in\{1,...,m_j\}$.

He proved that every positive solution to system (1.43) is eventually periodic with period s under some conditions. Also, he proved some related results for the corresponding system of min-type difference equations.

In [61], Stevic and Iricanin investigated the long-term behavior of positive solutions of the cyclic system of difference equations

$$x^{(i)}_{n+1} = \max\left\{ \alpha, \frac{\left(x^{(i+1)}_n\right)^p}{\left(x^{(i+2)}_{n-1}\right)^q}\right\},\ \ i=1,...,k,\ n\in\mathbb{N}_0, \tag{1.44}$$

where $k\in\mathbb{N}$, $\min\{\alpha,p,q\}>0$.

They showed that the system (1.44) has bounded and unbounded solutions depending on the status of the parameters and gave some sufficient conditions which guaranty the global attractivity of all positive solutions of system (1.44).

1.2 LITERATURE REVIEWS FOR DIFFERENCE EQUATIONS AND DISCRETE SYSTEMS VIA INTEGER SEQUENCES

In this section, we study the recent investigations on the forms of solutions of systems difference equations and difference equations in terms of well-known integer sequences such as Fibonacci numbers, Horadam numbers, Padovan numbers. We focus on the papers given some interesting relationships both between the exact solutions of difference equations and the integer sequences and between the equilibrium points of difference equations and golden ratio, plastic number.

In [74], Tollu et al. considered the following difference equations

$$x_{n+1} = \frac{1}{1+x_n}, \quad y_{n+1} = \frac{1}{-1+y_n}, \quad n = 0, 1, ..., \tag{1.45}$$

such that their solutions are associated with Fibonacci numbers, where initial conditions are $x_0 \in \mathbb{R} - \left\{-\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}$ and $y_0 \in \mathbb{R} - \left\{-\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}$ and F_m is the mth Fibonacci number.

They investigated the some relationships both between Fibonacci numbers and solutions of equations (1.45) and between the golden ratio and equilibrium points of equations (1.45). Then, they proved that: the solutions of equations (1.45) are given by

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}, \; y_n = \frac{F_{-n} + F_{-(n-1)}y_0}{F_{-(n+1)} + F_{-n}y_0},$$

where F_n is the nth Fibonacci number, and the nontrival solutions of equations (1.45) converge to $-\beta$ and β, so that β is conjugate to the golden ratio.

Next, Rabago [75] presented a theoretical explanation in deriving the closed-form solution of Eq.(1.45) which Tollu et al. studied in [74] and provided another approach in proving Sroysang's conjecture (2013).

Then, in [76], Yazlik et al. studied the following rational difference equation systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}}, \quad n = 0, 1, ..., \tag{1.46}$$

such that their solutions associated with Padovan numbers. In their study, they obtained that the forms of solutions of system (1.46) are as follows

$$x_n = \begin{cases} \mp\frac{P_n x_{-1}y_0 \mp P_{n+1}x_{-1} + P_{n-1}}{P_{n-1}x_{-1}y_0 \mp P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \mp\frac{P_n y_{-1}x_0 \mp P_{n+1}y_{-1} + P_{n-1}}{P_{n-1}y_{-1}x_0 \mp P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

$$y_n = \begin{cases} \mp\frac{P_n y_{-1}x_0 \mp P_{n+1}y_{-1} + P_{n-1}}{P_{n-1}y_{-1}x_0 \mp P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \mp\frac{P_n x_{-1}y_0 \mp P_{n+1}x_{-1} + P_{n-1}}{P_{n-1}x_{-1}y_0 \mp P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

where P_n is the nth Padovan number. Also, they demonstrated that every solutions of the systems (1.46) converge to point (p, p) and $(-p, -p)$, where p is the plastic number.

Tollu et al. [77] considered the following four Riccati difference equations

$$x_{n+1} = \frac{1+x_n}{x_n}, \quad y_{n+1} = \frac{1-y_n}{y_n}, \quad u_{n+1} = \frac{1}{u_n+1}, \quad v_{n+1} = \frac{1}{v_n-1}, \tag{1.47}$$

in which the initial conditions are real numbers. They derived the formulae for the solutions of equations (1.47) are given by

$$\begin{aligned} x_n &= \frac{F_{n+1}x_0 + F_n}{F_n x_0 + F_{n-1}}, \\ y_n &= \frac{F_{-(n+1)}y_0 + F_{-n}}{F_{-n}y_0 + F_{-(n-1)}}, \\ u_n &= \frac{F_n + F_{n-1}u_0}{F_{n+1} + F_n u_0}, \\ v_n &= \frac{F_{-n} + F_{-(n-1)}v_0}{F_{-(n+1)} + F_{-n}v_0}, \end{aligned}$$

where F_n is nth Fibonacci number, F_{-n} is nth negative Fibonacci number. In addition to, they stated the asymptotic behaviors of the solutions of these equations and introduced that every solutions of these equations converge to their positive or negative equilibrium points.

Also, they in [78] studied the systems of difference equations

$$x_{n+1} = \frac{1 + p_n}{q_n}, \quad y_{n+1} = \frac{1 + r_n}{s_n}, \quad n \in \mathbb{N}_0,$$

where each of the sequences p_n, q_n, r_n and s_n is some of the sequences x_n or y_n by their own. They solved fourteen systems out of sixteen possible systems. In particularly, the representation formulae of solutions of twelve systems were stated via Fibonacci numbers. Also, for ten systems, they expressed that the solutions of these systems tend to the unique point (α, α) where α is the golden ratio.

In [79], Halim concerned with the following systems of rational difference equations

$$x_{n+1} = \frac{1}{1 + y_n}, \quad y_{n+1} = \frac{1}{1 + x_n}, \quad n = 0, 1, ..., \tag{1.48}$$

and

$$x_{n+1} = \frac{1}{1 - y_n}, \quad y_{n+1} = \frac{1}{1 - x_n}, \quad n = 0, 1, ..., \tag{1.49}$$

initial conditions are arbitrary nonzero real numbers. He determined the form of solutions of system (1.48) as given below

$$\begin{aligned} x_{2n-1} &= \tfrac{F_{2n-1} + F_{2n-2}y_0}{F_{2n} + F_{2n-1}y_0}, \quad x_{2n} = \tfrac{F_{2n} + F_{2n-1}x_0}{F_{2n+1} + F_{2n}x_0}, \\ y_{2n-1} &= \tfrac{F_{2n-1} + F_{2n-2}x_0}{F_{2n} + F_{2n-1}x_0}, \quad y_{2n} = \tfrac{F_{2n} + F_{2n-1}y_0}{F_{2n+1} + F_{2n}y_0}, \end{aligned}$$

and proved that the equilibrium point E of system (1.48) is globally asymptotically stable, where $E = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) = \left(\frac{1}{\alpha}, \frac{1}{\alpha}\right)$, where α is the golden ratio. Furthermore, he established the solutions of system (1.49) are periodic with period six and are unstable.

In [80], Bacani and Rabago studied the behavior of solutions of the following nonlinear difference equations

$$x_{n+1} = \frac{q}{p + x_n^v} \quad \text{and} \quad y_{n+1} = \frac{q}{-p + y_n^v}, \tag{1.50}$$

where p, $q \in \mathbb{R}^+$ and $v \in \mathbb{N}$. They proved that the solutions of equations (1.50) are as follows

$$\begin{aligned} x_n &= \frac{qW_n + x_0 qW_{n-1}}{W_{n+1} + x_0 W_n}, \\ y_n &= \frac{qW_{-n} + y_0 qW_{-(n-1)}}{W_{-(n+1)} + y_0 W_{-n}}, \end{aligned}$$

where W_n is the nth Horadam number.

In [81], Halim and Bayram investigated the solutions, stability character, and asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha}{\beta + \gamma x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.51}$$

where the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonzero real numbers, such that its solutions are associated to Horadam numbers, which are generalized Fibonacci numbers. Firstly, they had the difference equation

$$x_{n+1} = \frac{q}{p + x_{n-k}}, \tag{1.52}$$

by putting $q = \frac{\alpha}{\gamma}$ and $p = \frac{\beta}{\gamma}$. Then, they proved that the forms of the solutions of difference equation (1.52) are as follows

$$x_{(k+1)n+i} = \frac{W_{n+1} + W_n x_{i-(k+1)}}{W_{n+2} + W_{n+1} x_{i-(k+1)}} q, \quad i = 1, 2, ..., k+1,$$

where W_n is the nth Horadam number. Also, they obtained that the equilibrium point E of difference equation (1.52) is globally asymptotically stable, where $E = \frac{-p+\sqrt{p^2+4q}}{2}$.

Then, in [82] Halim considered the system of difference equations

$$x_{n+1} = \frac{1}{1 + y_{n-2}}, \quad y_{n+1} = \frac{1}{1 + x_{n-2}}, \quad n = 0, 1, ..., \tag{1.53}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, and y_0 are real numbers. He presented the relationship between Fibonacci numbers and the solutions of system (1.53), i.e., the form of the solutions of system (1.53) are given by

$$\begin{aligned} x_{6n+i} &= \tfrac{F_{2n+1}+F_{2n}y_{i-3}}{F_{2n+2}+F_{2n+1}y_{i-3}}, \quad i = 1, 2, 3, \\ y_{6n+i} &= \tfrac{F_{2n+1}+F_{2n}x_{i-3}}{F_{2n+2}+F_{2n+1}x_{i-3}}, \quad i = 1, 2, 3, \\ x_{6n+i} &= \tfrac{F_{2n+2}+F_{2n+1}x_{i-6}}{F_{2n+3}+F_{2n+2}x_{i-6}}, \quad i = 4, 5, 6, \\ y_{6n+i} &= \tfrac{F_{2n+2}+F_{2n+1}y_{i-6}}{F_{2n+3}+F_{2n+2}y_{i-6}}, \quad i = 4, 5, 6, \end{aligned}$$

where F_n is the nth Fibonacci number. Furthermore, he shown that the equilibrium point E of system (1.53) is globally asymptotically stable, where $E = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$.
El-Dessoky in [83] dealt with the following difference equation

$$x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \quad n = 0, 1, ..., \tag{1.54}$$

where the parameters α, β, γ and a and the initial conditions x_{-t}, x_{-t+1}, $,x_{-1}$ and x_0 where $t = \max\{l, k\}$ are positive real numbers. He introduced the explicit formula of solutions of some special cases of Eq.(1.54) via Fibonacci numbers and also, discussed the global behavior of solutions of Eq.(1.54).
In [84], Halim and Rabago studied the systems of difference equaions

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}}, \quad y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}}, \quad n,\, k \in \mathbb{N}_0, \tag{1.55}$$

where the initial conditions x_{-k}, x_{-k+1}, ..., x_0, y_{-k}, y_{-k+1}, ..., y_0 are nonzero real numbers. Initially, they examined the form and behavior of solutions of system of difference equations

$$x_{n+1} = \frac{1}{1 + y_{n-k}}, \quad y_{n+1} = \frac{1}{1 + x_{n-k}}. \tag{1.56}$$

Therefore, they determined that the exact solutions of system (1.56) are as follows

$$\begin{aligned}
x_{2(k+1)n+i} &= \tfrac{F_{2n+1}+F_{2n}y_{i-(k+1)}}{F_{2n+2}+F_{2n+1}y_{i-(k+1)}}, \quad i = 1, 2, ..., k+1,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+1}+F_{2n}x_{i-(k+1)}}{F_{2n+2}+F_{2n+1}x_{i-(k+1)}}, \quad i = 1, 2, ..., k+1,\\
x_{2(k+1)n+i} &= \tfrac{F_{2n+2}+F_{2n+1}x_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}x_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2,\\
y_{2(k+1)n+i} &= \tfrac{F_{2n+2}+F_{2n+1}y_{i-(2k+2)}}{F_{2n+3}+F_{2n+2}y_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2,
\end{aligned}$$

and the equilibrium point of system (1.56) is globally asymptotically stable. In addition, the authors given some results for other systems.
Then, in [85], the authors studied the rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.57}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, α, β, $\gamma \in \mathbb{R}^+$ and the initial conditions nonzero real numbers and also investigated the two-dimensional case of the this equation given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, \quad y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1.58}$$

Firstly, they reduced the difference equation (1.57) to the difference equation

$$x_{n+1}=\frac{px_{n-1}+q}{x_nx_{n-1}} \tag{1.59}$$

by using changes variables $p=\frac{\alpha}{\gamma}$ and $q=\frac{\beta}{\gamma}$. Then, they presented that the closed-form solution of difference equation (1.59) is given by

$$x_n=\frac{S_{n+1}x_{-1}+S_nx_0x_{-1}+qS_{n-1}}{S_nx_{-1}+S_{n-1}x_0x_{-1}+qS_{n-2}},$$

where S_n is the nth generalized Padovan number and the equilibrium point of Eq.(1.59) is globally asymptotically stable.

Later, they reduced the system of difference equation (1.58) to the system

$$x_{n+1}=\frac{px_{n-1}+q}{y_nx_{n-1}},\ \ y_{n+1}=\frac{py_{n-1}+q}{x_ny_{n-1}} \tag{1.60}$$

by using changes variables $p=\frac{\alpha}{\gamma}$ and $q=\frac{\beta}{\gamma}$. Then, they presented that the closed-form solutions of system (1.60) are given by

$$x_n = \begin{cases} \frac{S_{n+1}y_{-1}+S_nx_0y_{-1}+qS_{n-1}}{S_ny_{-1}+S_{n-1}x_0y_{-1}+qS_{n-2}}, & \text{if } n \text{ is even,} \\ \frac{S_{n+1}x_{-1}+S_ny_0x_{-1}+qS_{n-1}}{S_nx_{-1}+S_{n-1}y_0x_{-1}+qS_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}$$

$$y_n = \begin{cases} \frac{S_{n+1}x_{-1}+S_ny_0x_{-1}+qS_{n-1}}{S_nx_{-1}+S_{n-1}y_0x_{-1}+qS_{n-2}}, & \text{if } n \text{ is even,} \\ \frac{S_{n+1}y_{-1}+S_nx_0y_{-1}+qS_{n-1}}{S_ny_{-1}+S_{n-1}x_0y_{-1}+qS_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}$$

and the equilibrium point of the system (1.60) is global attractor.

Then, in [86], Stevic et al. the following nonlinear second-order difference equation

$$x_{n+1}=a+\frac{b}{x_n}+\frac{c}{x_nx_{n-1}},\ \ n\in\mathbb{N}_0, \tag{1.61}$$

in which parameters a, b, c and the initial values x_{-1} and x_0 are complex numbers such that $c\neq 0$. Next, they used the following change of variables

$$x_n=\frac{y_n}{y_{n-1}},$$

and obtained the following third-order linear difference equation with constant coefficients

$$y_{n+1}=ay_n+by_{n-1}+cy_{n-2}.$$

After, they introduced that the representation formula of every solution of Eq.(1.61) is

$$x_n=\frac{\left(s_{n+1}-as_n\right)x_{-1}+s_nx_0x_{-1}+cs_{n-1}}{\left(s_n-as_{n-1}\right)x_{-1}+s_{n-1}x_0x_{-1}+cs_{n-2}},$$

where s_n is the nth generalized Padovan number. Note that, Eq.(1.57) is a special case of Eq.(1.61) such that $a = 0$.

Alotaibi et al. in [87] considered the following systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \ y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}}, \ n = 0, 1, ..., \tag{1.62}$$

where the initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 are arbitrary positive real numbers. They analyzed the solutions of the systems (1.62) such that their solutions are associated with Fibonacci numbers.

In [88], El-Dessoky et al. examined the following difference equation

$$y_{n+1} = \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \ n = 0, 1, ..., \tag{1.63}$$

where α, β, A, and B are real numbers and the initial values y_{-4}, y_{-3}, y_{-2}, y_{-1} and y_0 are positive real numbers. They presented the solutions of Eq.(1.63) in terms of Fibonacci numbers according to some special cases of the parameters α, β, A, and B.

Then, in [89], Matsunaga and Suzuki studied the following system of rational difference equations

$$x_{n+1} = \frac{a y_n + b}{c y_n + d}, \ y_{n+1} = \frac{a x_{n-1} + b}{c x_n + d}, \ n = 0, 1, ..., \tag{1.64}$$

where the parameters a, b, c, d and the initial values x_0, y_0 are real numbers. They obtained that the explicit solutions of system (1.64) are as follows

$$\begin{aligned} x_{2n-1} &= \frac{(a y_0 + b) G_{2n-1} + (bc - ad) y_0 G_{2n-2}}{G_{2n} + (c y_0 - a) G_{2n-1}}, \ x_{2n} = \frac{(a x_0 + b) G_{2n} + (bc - ad) x_0 G_{2n-1}}{G_{2n+1} + (c x_0 - a) G_{2n}}, \\ y_{2n-1} &= \frac{(a x_0 + b) G_{2n-1} + (bc - ad) x_0 G_{2n-2}}{G_{2n} + (c x_0 - a) G_{2n-1}}, \ y_{2n} = \frac{(a y_0 + b) G_{2n} + (bc - ad) y_0 G_{2n-1}}{G_{2n+1} + (c y_0 - a) G_{2n}}, \end{aligned}$$

where G_n is a generalized Fibonacci sequence defined by

$$G_{n+2} = (a + d) G_{n+1} + (bc - ad) G_n,$$

with $G_0 = 0$ and $G_1 = 1$. Moreover, they presented that every solution of system (1.64) converges to its equilibrium points.

In [90], Öcalan and Duman considered the following nonlinear recursive difference equation

$$x_{n+1} = \frac{x_{n-1}}{x_n}, \ n = 0, 1, ..., \tag{1.65}$$

with any nonzero initial values x_{-1} and x_0. Then, they extended their all results to solutions of the following nonlinear recursive equations

$$x_{n+1} = \left(\frac{x_{n-1}}{x_n}\right)^p, \ \ p > 0 \text{ and } n = 0, 1, ..., \tag{1.66}$$

with any nonzero initial values x_{-1} and x_0. Later, they obtained that the exact solution of Eq.(1.65) is

$$x_n = \begin{cases} \frac{x_{-1}^{f_{n-1}}}{x_0^{f_n}} & \text{if } n = 1, 3, 5, ..., \\ \frac{x_0^{f_n}}{x_{-1}^{f_{n-1}}} & \text{if } n = 2, 4, 6, ..., \end{cases}$$

where f_n is the nth Fibonacci number. Under the special case of initial values, they determined that there exist non-oscillatory positive solutions of Eq.(1.65), which converge monotonically to the equilibrium point 1.

Furthermore, they given that the exact solution of Eq.(1.66) is

$$x_n = \begin{cases} \frac{x_{-1}^{g_{n-1}(p)}}{x_0^{f_n(p)}} & \text{if } n = 1, 3, 5, ..., \\ \frac{x_0^{f_n(p)}}{x_{-1}^{f_{n-1}(p)}} & \text{if } n = 2, 4, 6, ..., \end{cases}$$

where $f_n(p)$ and $g_n(p)$ are the nth Fibonacci-type number. And also, under the special case of initial values, they demonstrated that there exist non-oscillatory positive solutions of Eq.(1.66), which converge monotonically to the equilibrium point 1 and the Eq.(1.66) has unbounded solutions.

Next, Akrour et al. [91] studied the following system of difference equations

$$x_{n+1} = \frac{a y_n x_{n-1} + b x_{n-1} + c}{y_n x_{n-1}}, \ y_{n+1} = \frac{a x_n y_{n-1} + b y_{n-1} + c}{x_n y_{n-1}}, \ n = 0, 1, ...,$$

where the parameters a, b, c are arbitrary real numbers with $c \neq 0$ and the initial values x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers. They examined that the explicit solutions of system (1.56) are given by

$$\begin{aligned} x_{2n+1} &= \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\, x_{-1} + J_{2n+2} x_{-1} y_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})\, x_{-1} + J_{2n+1} x_{-1} y_0}, \\ x_{2n+2} &= \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})\, y_{-1} + J_{2n+3} x_0 y_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\, y_{-1} + J_{2n+2} x_0 y_{-1}}, \\ y_{2n+1} &= \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\, y_{-1} + J_{2n+2} x_0 y_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})\, y_{-1} + J_{2n+1} x_0 y_{-1}}, \\ y_{2n+2} &= \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1} + J_{2n+3} x_{-1} y_0}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\, x_{-1} + J_{2n+2} x_{-1} y_0}, \end{aligned}$$

where J_n is defined by the recurrent relation

$$J_{n+3} = aJ_{n+2} + bJ_{n+1} + cJ_n, \ \ n \in \mathbb{N},$$

such that $J_0 = 0, \ J_1 = 1, \ J_2 = a.$

For related studies on solving difference equations and systems of difference equations and investigating the asymptotic behavior of their solutions, see [92, 100].

CHAPTER 2

THE GENERAL DEFINITIONS AND THEOREMS

In this chapter, we state some definitions and theorems used in this thesis. For details, see [1-14, 105-120].

2.1 DIFFERENCE EQUATIONS

In this section, we give some important definitions and theorems about difference equations and systems of difference equations (discrete dynamical systems).
Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{2.1}$$

A solution of Eq.(2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(2.1) for all $n \geq -k$.

Definition 2.1 *A solution of Eq.(2.1) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(2.1). If*

$x_n = \overline{x}$, *for all* $n \geq -k$

is an equilibrium solution of Eq.(2.1), then $\overline{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(2.1).

Definition 2.2 (Stability) *Let $\overline{x}$ an equilibrium point of Eq.(2.1).*

(a) *An equilibrium point $\overline{x}$ of Eq.(2.1) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) *An equilibrium point $\overline{x}$ of Eq.(2.1) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) with*

$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$

then we have

$\lim_{n\to\infty} x_n = \overline{x}.$

(c) *An equilibrium point $\overline{x}$ of Eq.(2.1) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1), we have*

$\lim_{n\to\infty} x_n = \overline{x}.$

(d) *An equilibrium point $\overline{x}$ of Eq.(2.1) is called globally asymptotically stable if it is locally stable, and a global attractor.*

(e) *An equilibrium point $\overline{x}$ of Eq.(2.1) is called unstable if it is not locally stable.*

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\overline{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}), \quad \text{for } i = 0, 1, ..., k$$

denote the partial derivative of $f(u_0, u_1, ..., u_k)$ with respect to u_i evaluated at the equilibrium point $\overline{x}$ of Eq.(2.1).

Definition 2.3 *The equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_k y_{n-k}, \; n = 0, 1, ... \tag{2.2}$$

is called the linearized equation of Eq.(2.1) about the equilibrium point $\overline{x}$, and the equation

$$\lambda^{k+1} - q_0\lambda^k - ... - q_{k-1}\lambda - q_k = 0 \tag{2.3}$$

is called the characteristic equation of Eq.(2.2) about $\overline{x}$.

Theorem 2.1 (The Linearized Stability Theorem) *Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:*

(a) *When all the roots of characteristic equation (2.3) have absolute value less than one, then the equilibrium point* $\overline{x}$ *of Eq.(2.1) is locally asymptotically stable.*

(b) *If at least one root of characteristic equation (2.3) has absolute value greater than one, then the equilibrium point* $\overline{x}$ *of Eq.(2.1) is unstable.*

(c) *The equilibrium point* $\overline{x}$ *of Eq.(2.1) is called hyperbolic if no root of characteristic equation (2.3) has absolute value equal to one. If there exists a root of characteristic equation (2.3) with absolute value equal to one, then the equilibrium* $\overline{x}$ *is called nonhyperbolic.*

(d) *An equilibrium point* $\overline{x}$ *of Eq.(2.1) is called a repeller if all roots of characteristic equation (2.3) have absolute value greater than one.*

(e) *An equilibrium point* $\overline{x}$ *of Eq.(2.1) is called a saddle if one of the roots of characteristic equation (2.3) is greater and another is less than one in absolute value.*

The following two theorems state necessary and sufficient conditions for all the roots of a real polynomial of degree two or three, respectively, to have modulus less than one.

Theorem 2.2 ***([1], p.6)*** *Assume that* a_1 *and* a_0 *are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$$\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_1| < 1 + a_0 < 2.$$

Theorem 2.3 ***([1], p.6)*** *Assume that* a_2, a_1, *and* a_0 *are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1 \quad \text{and} \quad a_0^2 + a_1 - a_0a_2 < 1.$$

Theorem 2.4 (Clark Theorem) *([1], p.6) Assume that $q_0, q_1, ..., q_k$ are real numbers such that*

$$|q_0| + |q_1| + ... + |q_k| < 1$$

Then all roots of Eq.(2.3) lie inside the unit disk.

Let us introduce the discrete dynamical system:

$$\begin{aligned} x_{n+1} &= f_1\left(x_n, x_{n-1}, ..., x_{n-k}, y_n, y_{n-1}, ..., y_{n-k}, z_n, z_{n-1}, ..., z_{n-k}\right), \\ y_{n+1} &= f_2\left(x_n, x_{n-1}, ..., x_{n-k}, y_n, y_{n-1}, ..., y_{n-k}, z_n, z_{n-1}, ..., z_{n-k}\right), \\ z_{n+1} &= f_3\left(x_n, x_{n-1}, ..., x_{n-k}, y_n, y_{n-1}, ..., y_{n-k}, z_n, z_{n-1}, ..., z_{n-k}\right), \end{aligned} \tag{2.4}$$

$n \in \mathbb{N}$, where $f_1 : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_1$, $f_2 : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_2$ and $f_3 : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_3$ are continuously differentiable functions and I_1, I_2, I_3 are some intervals of real numbers. Also, a solution $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ of system (2.4) is uniquely determined by initial values $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$ for $i \in \{0, 1, ..., k\}$.

Definition 2.4 *An equilibrium point of system (2.4) is a point $(\overline{x}, \overline{y}, \overline{z})$ that satisfies*

$$\begin{aligned} \overline{x} &= f_1\left(\overline{x}, \overline{x}, ..., \overline{x}, \overline{y}, \overline{y}, ..., \overline{y}, \overline{z}, \overline{z}, ..., \overline{z}\right), \\ \overline{y} &= f_2\left(\overline{x}, \overline{x}, ..., \overline{x}, \overline{y}, \overline{y}, ..., \overline{y}, \overline{z}, \overline{z}, ..., \overline{z}\right), \\ \overline{z} &= f_3\left(\overline{x}, \overline{x}, ..., \overline{x}, \overline{y}, \overline{y}, ..., \overline{y}, \overline{z}, \overline{z}, ..., \overline{z}\right). \end{aligned}$$

Together with system (2.4), if we consider the associated vector map

$$F = \left(f_1, x_n, x_{n-1}, ..., x_{n-k}, f_2, y_n, y_{n-1}, ..., y_{n-k}, f_3, z_{n-1}, ..., z_{n-k}\right),$$

then the point $(\overline{x}, \overline{y}, \overline{z})$ is also called a fixed point of the vector map F.

Definition 2.5 *Let $(\overline{x}, \overline{y}, \overline{z})$ be an equilibrium point of system (2.4).*

(a) *An equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ is called stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that for every initial value $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$, with*

$$\sum\nolimits_{i=-k}^{0} |x_i - \overline{x}| < \delta, \ \sum\nolimits_{i=-k}^{0} |y_i - \overline{y}| < \delta, \ \sum\nolimits_{i=-k}^{0} |z_i - \overline{z}| < \delta$$

implying $|x_n - \overline{x}| < \varepsilon$, $|y_n - \overline{y}| < \varepsilon$, $|z_n - \overline{z}| < \varepsilon$for $n \in \mathbb{N}$.

(b) *If an equilibrium point* $(\overline{x}, \overline{y}, \overline{z})$ *of system (2.4) is called unstable if it is not stable.*

(c) *An equilibrium point* $(\overline{x}, \overline{y}, \overline{z})$ *of system (2.4) is called locally asymptotically stable if, it is stable, and if in addition there exists* $\gamma > 0$ *such that*

$$\sum_{i=-k}^{0} |x_i - \overline{x}| < \gamma, \ \sum_{i=-k}^{0} |y_i - \overline{y}| < \gamma, \ \sum_{i=-k}^{0} |z_i - \overline{z}| < \gamma$$

and $(x_n, y_n, z_n) \to (\overline{x}, \overline{y}, \overline{z})$ *as* $n \to \infty$.

(d) *An equilibrium point* $(\overline{x}, \overline{y}, \overline{z})$ *of system (2.4) is called a global attractor if,* $(x_n, y_n, z_n) \to (\overline{x}, \overline{y}, \overline{z})$ *as* $n \to \infty$.

(e) *An equilibrium point* $(\overline{x}, \overline{y}, \overline{z})$ *of system (2.4) is called globally asymptotically stable if it is stable, and a global attractor.*

Definition 2.6 *Let* $(\overline{x}, \overline{y}, \overline{z})$ *be an equilibrium point of the map* F *where* f_1, f_2 *and* f_3 *are continuously differentiable functions at* $(\overline{x}, \overline{y}, \overline{z})$. *The linearized system of system (2.4) about the equilibrium point* $(\overline{x}, \overline{y}, \overline{z})$ *is*

$$X_{n+1} = F(X_n) = BX_n,$$

where

$$X_n = (x_n, ..., x_{n-k}, y_n, ..., y_{n-k}, z_n, ..., z_{n-k})^T$$

and B *is a Jacobian matrix of system (2.4) about the equilibrium point* $(\overline{x}, \overline{y}, \overline{z})$.

Theorem 2.5 (The Linearized Stability Theorem) *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, ...,$$

be a system of difference equations such that $\overline{X}$ *is a fixed point of* F.

(a) *If all eigenvalues of the Jacobian matrix* B *about* $\overline{X}$ *lie inside the open unit disk* $|\lambda| < 1$, *that is, if all of them have absolute value less than one, then* $\overline{X}$ *is locally asymptotically stable.*

(b) *If at least one of them has a modulus greater than one, then* $\overline{X}$ *is unstable.*

Corollary 2.1 *Assume that*

$$X_{n+1} = F\left(X_n\right), n = 0, 1, ...,$$

be a system of difference equations such that $\overline{X}$ *is a fixed point of* F*. If no eigenvalues of the Jacobian matrix* B *about* $\overline{X}$ *have absolute value equal to one, then* $\overline{X}$ *is called hyperbolic. If there exists an eigenvalue of the Jacobian matrix* B *about* $\overline{X}$ *with absolute value equal to one, then* $\overline{X}$ *is called nonhyperbolic.*

The so-called Schur-Cohn criterion provides necessary and sufficient conditions for all roots of the equation

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{2.5}$$

with real coefficients to lie in the open disk $|\lambda| < 1$.
Before we can explain the Schur-Cohn criterion, we need the so-called Routh-Hurwitz criterion.

Theorem 2.6 *(Routh-Hurwitz criterion) Assume that*

$$X_{n+1} = F(X_n),\ n = 0, 1, ...,$$

is a system of difference equations and $\overline{X}$ *is a fixed point of* F*, the characteristic polynomial of this system about the equilibrium point* $\overline{X}$ *is given by (2.5) with real coefficients and* $a_0 > 0$*. Then all roots of the polynomial* $P(\lambda)$ *lie inside the open unit disk* $|\lambda| < 1$ *if and only if*

$$\Delta_k > 0 \text{ for } k = 1, 2, \ldots, n$$

where Δ_k *is the principal minor of order* k *of the* $n \times n$ *matrix*

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Theorem 2.7 *(Schur-Cohn criterion) The equation*

$$P(\lambda) = \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

has all its roots in the open unit disk $|\lambda| < 1$ *if and only if the equation*

$$P\left(\frac{z+1}{z-1}\right) = 0$$

has all its roots in the left-half plane

$$\mathrm{Re}(z) < 0.$$

Definition 2.7 *A solution* $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ *of system (2.4) is bounded and persists if there exist positive constants* M, N *such that*

$$M \leq x_n, y_n, z_n \leq N, \quad n = -m, -m+1, \ldots.$$

Definition 2.8 *A solution* $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ *of system (2.4) is periodic with period* p *if*

$$x_{n+p} = x_n,\ y_{n+p} = y_n,\ z_{n+p} = z_n, \text{ for all } n \geq -1.$$

Definition 2.9 *Let* $(\overline{x}, \overline{y}, \overline{z})$ *be an equilibrium point of system (2.4), and assume that* $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ *is a solution of the system (2.4).*

A "string" of consecutive terms $\{x_s..., x_m\}$ *(resp.* $\{y_s..., y_m\}$, $\{z_s..., z_m\}$*),* $s \geq -1$, $m \leq \infty$ *is said to be a positive semicycle if* $x_i \geq \overline{x}$ *(resp.* $y_i \geq \overline{y}$, $z_i \geq \overline{z}$*),* $i \in \{s, ..., m\}$, $x_{s-1} < \overline{x}$ *(resp.* $y_{s-1} < \overline{y}$, $z_{s-1} < \overline{z}$*), and* $x_{m+1} < \overline{x}$ *(resp.* $y_{m+1} < \overline{y}$, $z_{m+1} < \overline{z}$*).*

A "string" of consecutive terms $\{x_s..., x_m\}$ *(resp.* $\{y_s..., y_m\}$, $\{z_s..., z_m\}$*),* $s \geq -1$, $m \leq \infty$ *is said to be a negative semicycle if* $x_i < \overline{x}$ *(resp.* $y_i < \overline{y}$, $z_i < \overline{z}$*),* $i \in \{s, ..., m\}$, $x_{s-1} \geq \overline{x}$ *(resp.* $y_{s-1} \geq \overline{y}$, $z_{s-1} \geq \overline{z}$*), and* $x_{m+1} \geq \overline{x}$ *(resp.* $y_{m+1} \geq \overline{y}$, $z_{m+1} \geq \overline{z}$*).*

A "string" of consecutive terms $\{(x_s, y_s, z_s), ..., (x_m, y_m, z_m)\}$ *is said to be a positive (resp. negative) semicycle if* $\{x_s..., x_m\}$, $\{y_s..., y_m\}$, $\{z_s..., z_m\}$ *are positive (resp. negative) semicycles.*

Definition 2.10 *A solution* $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ *of system (2.4) is called nonoscillatory about* $(\overline{x}, \overline{y}, \overline{z})$*, or simply nonoscillatory, if there exists* $N \geq -k$ *such that either*

$$x_n \geq \overline{x},\ y_n \geq \overline{y},\ z_n \geq \overline{z}, \text{ for all } n \geq N$$

or

$x_n < \overline{x}$, $y_n < \overline{y}$, $z_n < \overline{z}$, *for all* $n \geq N$.

Otherwise, the solution $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ *is called oscillatory about* $(\overline{x}, \overline{y}, \overline{z})$, *or simply oscillatory.*

Theorem 2.8 (Rouche's Theorem) *([105],* ***p.365****) Let* C *be a simple closed contour lying entirely with a domain* D. *Suppose* f *and* g *are analytic in* D. *If the strict inequality* $|f(z) - g(z)| < |f(z)|$ *holds for all* z *on* C, *then* f *and* g *have the same number of zeros (counted according to their order or multiplicities) inside* C.

Now, we give some notifications about centre manifold theorem see [106-113].
Centre manifold theory [106] may be utilized to refer to the stability of non-hyberbolic fixed points. A centre manifold is a set M_c in a lower dimensional space where the dynamics of the original systems can be derived by examining the dynamics on M_c. Regard the $m-$parameter map $F(m, u)$, $F : \mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^k$, where $m \in \mathbb{R}^s$ is a parameter and $u \in \mathbb{R}^k$. Let $F(m, u^*(m)) = u^*(m)$ be a fixed point of F. It is notice that the stability of the hyperbolic fixed points of F is established from the stability of the fixed points under the linear map $J = D_u F(m, u^*(m))$.
Centre manifold theory make use of when one of the eigenvalues lies on the unit circle and the other eigenvalues are inside the unit circle.
Assuming, without loss of generality that $u^* = 0_k = (0, 0, ..., 0)$ the $k-$dimensional zero vector, the map F can be written in the form

$$\begin{aligned} x &\longmapsto Ax + f(x, y) \\ y &\longmapsto Bx + g(x, y) \end{aligned} \tag{2.6}$$

where J on (2.6) has the form

$$J = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Notice that all of the eigenvalues of A lie on the unit circle and all of the eigenvalues of B are off the unit circle. Hereby, A is a $t \times t$ matrix and B is an $s \times s$ matrix, with $t + s = k$. The following theorem alleges the existence of a (non-unique) centre manifold (a curve $y = h(x)$) on which the dynamics of system (2.6) is provided with the map on the centre manifold.

Theorem 2.9 *There is a C^r centre manifold for system (2.6) that can be represented locally as*

$$M_c = \left\{(x, y) \in \mathbb{R}^t \times \mathbb{R}^s : y = h(x),\ \|x\| < \delta,\ h(0) = 0,\ Dh(0) = 0\right\}.$$

Furthermore, the dynamics restricted to M_c are given locally by the map

$$x \longmapsto Ax + f(x, h(x)),\ x \in \mathbb{R}^t.$$

The following theorem indicates that the dynamics on the centre manifold M_c determines the dynamics on (2.6).

Theorem 2.10 *Suppose that the t-dimensional zero vector $0_t = (0, ..., 0)$ is a fixed point for the map $x \longmapsto Ax + f(x, h(x))$, $x \in \mathbb{R}^t$. If 0_t is stable, asymptotically stable, or unstable, then the fixed point 0_k of system (2.6) is stable, asymptotically stable, or unstable, respectively.*

2.2 INTEGER SEQUENCES

Now, we give information about integer sequences that establish a large part of our study. Some properties of the above sequences were studied in various papers, see [114-120].

2.2.1 Fibonacci Numbers

The Fibonacci sequence is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{2.7}$$

with initial conditions $F_0 = 0$, $F_1 = 1$. Also, it is obtained to extend the Fibonacci sequence backward as

$$F_{-n} = (-1)^{n+1} F_n.$$

The characteristic equation of (2.7) is $x^2 - x - 1 = 0$ such that the roots

$\alpha = \dfrac{1+\sqrt{5}}{2}$ (golden ratio) and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Also, there exists the following limit

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha,$$

where F_n is nth Fibonacci number.

2.2.2 Padovan Numbers

The Padovan sequence is defined by

$$P_n = P_{n-2} + P_{n-3}, \quad n \in \mathbb{N} \tag{2.8}$$

with initial conditions $P_{-2} = 0$, $P_{-1} = 0$, $P_0 = 1$.
The characteristic equation of (2.8) is $x^3 - x - 1 = 0$ such that the roots

$$\begin{aligned}
p &= \frac{r^2+12}{6r} \\
q &= -\frac{r^2+12}{6r} - i\frac{\sqrt{3}}{2}\left(\frac{r}{6} - \frac{2}{3r}\right) \\
t &= -\frac{r^2+12}{6r} + i\frac{\sqrt{3}}{2}\left(\frac{r}{6} - \frac{2}{3r}\right)
\end{aligned}$$

where $r = \sqrt[3]{108 + 12\sqrt{69}}$ and the unique real root is p named as plastic number. Also, there exists the following limit

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = p,$$

where P_n is nth Padovan number.

2.2.3 Horadam Numbers

Horadam sequence, a generalization of Fibonacci sequence, $(W_n(a, b; p, q))_{n\geq 0}$ or simply $(W_n)_{n\geq 0}$ is defined by

$$W_n = pW_{n-1} + qW_{n-2}, \quad W_0 = a, W_1 = b, \qquad n \geq 2, \tag{2.9}$$

where a, b, p and q are arbitrary real numbers.
The characteristic equation of (2.9) is $x^2 - px - q = 0$ such that the roots

$$\lambda = \frac{p + \sqrt{p^2+4q}}{2} \text{ and } \mu = \frac{p - \sqrt{p^2+4q}}{2}.$$

Also, there exists the following limit

$$\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = \lambda,$$

where W_n is nth Horadam number.

2.2.4 Generalized Padovan Numbers

The generalized Padovan sequence, an extension of the padovan sequence, is defined by

$$S_n = pS_{n-2} + qS_{n-3}, \quad n \in \mathbb{N} \tag{2.10}$$

with initial conditions $S_{-2} = 0$, $S_{-1} = 0$, $S_0 = 1$, where p and q are arbitrary real numbers.

The characteristic equation of (2.10) is $x^3 - px - q = 0$ such that the roots

$$\begin{aligned}
\phi &= \frac{R^2 + 12p}{6R} \\
\varphi &= -\frac{R^2 + 12p}{12R} + i\frac{\sqrt{3}}{2}\left(\frac{R}{6} - \frac{2p}{R}\right) \\
\psi &= -\frac{R^2 + 12p}{12R} - i\frac{\sqrt{3}}{2}\left(\frac{R}{6} - \frac{2p}{R}\right)
\end{aligned}$$

where $R = \sqrt[3]{108q + 12\sqrt{-12p^3 + 81q^2}}$. Also, there exists the following limit

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \phi,$$

where S_n is nth generalized Padovan number.

Also, the other integer sequences are as follows:

- Lucas sequence is defined by

  $$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ L_1 = 1,$$

- Pell sequence is defined by

  $$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \ P_1 = 1,$$

- Pell-Lucas sequence is defined by

  $$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 2, \ P_1 = 2,$$

- Jacobsthal sequence is defined by

  $$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \ J_1 = 1,$$

- Jacobsthal-Lucas sequence is defined by

  $$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2,\ j_1 = 1,$$

- Perrin sequence is defined by

  $$Q_n = Q_{n-2} + Q_{n-3}, \quad Q_0 = 3, Q_1 = 0, Q_2 = 2.$$

Now, we give information about Tribonacci numbers that we afterwards need in the paper.

2.2.5 Tribonacci Numbers

The Tribonacci sequence $\{T_n\}_{n=0}^{\infty}$ is defined by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \tag{2.11}$$

with initial conditions $T_0 = 0$, $T_1 = 1$, $T_2 = 1$. Also, it can be extended the Tribonacci sequence backward (negative subscripts) as

$$T_{-n} = T_{-n+3} - T_{-n+2} - T_{-n+1}. \tag{2.12}$$

It can be clearly obtained that the characteristic equation of (2.11) has the form

$$x^3 - x^2 - x - 1 = 0 \tag{2.13}$$

such that the roots

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3} \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3} \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}
\end{aligned}$$

where α is called Tribonacci constant and

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

is a primitive cube root of unity. Therefore, Tribonacci sequence can be expressed using Binet formula

$$T_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)}.$$

Furthermore, there exist the following limit

$$\lim_{n\to\infty} \frac{T_{n+r}}{T_n} = \alpha^r, \tag{2.14}$$

where $r \in \mathbb{Z}$ and T_n is the nth Tribonacci number.

2.2.6 Generalized Tribonacci Numbers

The generalized Tribonacci sequence in [121] $\{V_n\}_{n\geq 0}$ is defined as follows:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, \quad V_0 = a, V_1 = b, V_2 = c, \quad n \geq 3 \tag{2.15}$$

where a, b, c are arbitrary integers and r, s, t are real numbers.

The sequence $\{V_n\}_{n\geq 0}$ can be expanded to negative subscripts by defining

$$V_{-n} = -\frac{s}{t}V_{-(n-1)} - \frac{r}{t}V_{-(n-2)} + \frac{1}{t}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Thus, recurrence (2.15) holds for all integer n.

If we set $r = s = t = 1$ and $V_0 = 0, V_1 = 1, V_2 = 1$ then $\{V_n\}_{n\geq 0}$ is the well-known Tribonacci sequence and if we set $r = s = t = 1$ and $V_0 = 3, V_1 = 1, V_2 = 3$ then $\{V_n\}_{n\geq 0}$ is the well-known Tribonacci-Lucas sequence.

Actually, the generalized Tribonacci sequence is the generalization of the renowned sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas. In literature, for instance, the following names and notations (see Table 1) are used for the special cases of r, s, t and initial values.

Table 2.1 A few values of generalized Tribonacci sequences.

Sequences (Numbers)	Notation
Tribonacci	$\{T_n\} = \{V_n(0,1,1;1,1,1)\}$
Tribonacci-Lucas	$\{K_n\} = \{V_n(3,1,3;1,1,1)\}$
Padovan (Cordonnier)	$\{P_n\} = \{V_n(1,1,1;0,1,1)\}$
Pell-Padovan	$\{R_n\} = \{V_n(1,1,1;0,2,1)\}$
Jacobsthal-Padovan	$\{JP_n\} = \{V_n(1,1,1;0,1,2)\}$
Perrin	$\{Q_n\} = \{V_n(3,0,2;0,1,1)\}$
Pell-Perrin	$\{pQ_n\} = \{V_n(3,0,2;0,2,1)\}$
Jacobsthal-Perrin	$\{JQ_n\} = \{V_n(3,0,2;0,1,2)\}$
Padovan-Perrin	$\{S_n\} = \{V_n(0,0,1;0,1,1)\}$
Narayana	$\{N_n\} = \{V_n(0,1,1;1,0,1)\}$
third order Jacobsthal	$\{J_n\} = \{V_n(0,1,1;1,1,2)\}$
third order Jacobsthal-Lucas	$\{j_n\} = \{V_n(2,1,5;1,1,2)\}$

As $\{V_n\}_{n\geq 0}$ is a third order recurrence sequence (difference equation), it's characteristic equation is $x^3 - rx^2 - sx - t = 0$, whose roots are

$$\begin{aligned}
\alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B \\
\beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\
\gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\
\Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \\
\omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}$$

Note that we obtain the following identities

$$\begin{aligned}
\alpha + \beta + \gamma &= r, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\
\alpha\beta\gamma &= t.
\end{aligned}$$

From now on, we assume that $\Delta(r,s,t) > 0$, so that the Eq.(2.15) has one real (α) and two non-real solutions with the latter being conjugate complex. Therefore, in this case, it is known that generalized Tribonacci numbers can be declared, for all integers n, using Binet's formula

$$V_n = \frac{P\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{Q\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{R\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{2.16}$$

where

$$P = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0, \ Q = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0, \ R = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.$$

Notice that the Binet form of a sequence satisfying (2.15) for non-negative integers is valid for all integers n, for a proof of this result see [122]. This result of Howard and Saidak [122] is even true in the case of higher-order recurrence relations.

We can present Binet's formula of the generalized Tribonacci numbers for the negative subscripts: for $n = 1, 2, 3, ...$ we get

$$\begin{aligned}
V_{-n} &= \frac{\alpha^2 - r\alpha - s}{t}\frac{P\alpha^{1-n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^2 - r\beta - s}{t}\frac{Q\beta^{1-n}}{(\beta-\alpha)(\beta-\gamma)} \\
&\quad + \frac{\gamma^2 - r\gamma - s}{t}\frac{R\gamma^{1-n}}{(\gamma-\alpha)(\gamma-\beta)}.
\end{aligned}$$

CHAPTER 3

GLOBAL BEHAVIOR OF SOLUTIONS OF A SYSTEM OF THREE-DIMENSIONAL NONLINEAR DIFFERENCE EQUATIONS

Firstly, we state that the results of this chapter are cited from [65] which has been published by us.

In this chapter, we investigate the stability, boundedness character and periodicity of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{z_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{z_n}, \quad z_{n+1} = A + \frac{z_{n-1}}{y_n}, \quad n = 0, 1, ..., \tag{3.1}$$

where the parameter A and the initial values x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0 are positive real numbers.

3.1 MAIN RESULTS

In this section, we prove our main results.

Theorem 3.1 *The following statements are true:*

(i) *If $(\overline{x}, \overline{y}, \overline{z})$ is a positive equilibrium point of system (3.1), then*

$$(\overline{x}, \overline{y}, \overline{z}) = \begin{cases} (A+1, A+1, A+1), & \text{if } A \neq 1, \\ \left(\mu, \mu, \frac{\mu}{\mu-1}\right), \ \mu \in (1, \infty) & \text{if } A = 1. \end{cases}$$

(ii) *If $A > 1$, then the equilibrium point of system (3.1) is locally asymptotically stable.*

(iii) *If $0 < A < 1$, then the equilibrium point of system (3.1) is locally unstable.*

(iv) *If $A = 1$, then for every $\mu \in (1, \infty)$ there exist positive solutions $\{(x_n, y_n, z_n)\}$ of system (3.1) which tend to the positive equilibrium point $\left(\mu, \mu, \frac{\mu}{\mu-1}\right)$.*

Proof. **(i)** It is easily seen from the definition of equilibrium point that the equilibrium points of system (3.1) are the nonnegative solution of the equations

$$\overline{x} = A + \frac{\overline{x}}{\overline{z}},\ \overline{y} = A + \frac{\overline{y}}{\overline{z}},\ \overline{z} = A + \frac{\overline{z}}{\overline{y}}.$$

From this, we get

$$\begin{aligned}
\overline{x}.\overline{z} &= A\overline{z} + \overline{x},\ \ \overline{y}.\overline{z} = A\overline{z} + \overline{y},\ \ \overline{z}.\overline{y} = A\overline{y} + \overline{z} \\
&\Rightarrow\ \overline{x}.\overline{z} - \overline{x} = \overline{y}.\overline{z} - \overline{y},\ \ A\overline{z} + \overline{y} = A\overline{y} + \overline{z} \\
&\Rightarrow\ \overline{x}\left(\overline{z} - 1\right) = \overline{y}\left(\overline{z} - 1\right),\ \ \overline{z}\left(A - 1\right) = \overline{y}\left(A - 1\right).
\end{aligned}$$

From which it follows that if $A \neq 1$,

$$\overline{x} = \overline{y} = \overline{z} = A + 1 \Rightarrow \left(\overline{x}, \overline{y}, \overline{z}\right) = \left(A + 1, A + 1, A + 1\right).$$

Also, we have

$$\begin{aligned}
\frac{\overline{x}.\overline{z} - \overline{x}}{\overline{z}} &= A,\ \ \frac{\overline{y}.\overline{z} - \overline{y}}{\overline{z}} = A,\ \ \frac{\overline{z}.\overline{y} - \overline{z}}{\overline{y}} = A \\
&\Rightarrow\ \frac{\overline{x}.\overline{z} - \overline{x}}{\overline{z}} = \frac{\overline{y}.\overline{z} - \overline{y}}{\overline{z}},\ \ \frac{\overline{y}.\overline{z} - \overline{y}}{\overline{z}} = \frac{\overline{z}.\overline{y} - \overline{z}}{\overline{y}} \\
&\Rightarrow\ \overline{x}.\overline{z} - \overline{x} = \overline{y}.\overline{z} - \overline{y},\ \ \overline{y}^2\overline{z} - \overline{y}^2 = \overline{z}^2\overline{y} - \overline{z}^2 \\
&\Rightarrow\ \overline{x}\left(\overline{z} - 1\right) = \overline{y}\left(\overline{z} - 1\right),\ \ \overline{y}.\overline{z}\left(\overline{y} - \overline{z}\right) = \left(\overline{y} - \overline{z}\right)\left(\overline{y} + \overline{z}\right).
\end{aligned}$$

From which it follows that if $A = 1$,

$$\overline{x} = \overline{y} \text{ and } \overline{y}.\overline{z} = \overline{y} + \overline{z} \Rightarrow \left(\overline{x}, \overline{y}, \overline{z}\right) = \left(\mu, \mu, \frac{\mu}{\mu - 1}\right),\ \mu \in \left(1, \infty\right).$$

In that case, we have a continuous of positive equilibriums which lie on the hyperboloid

$$\overline{y}.\overline{z} = \overline{y} + \overline{z}. \tag{3.2}$$

(ii) We consider the following transformation to build the corresponding linearized form of system (3.1):

$$\left(x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}\right) \rightarrow \left(f, f_1, g, g_1, h, h_1\right)$$

where

$$\begin{aligned} f &= A + \frac{x_{n-1}}{z_n} \\ f_1 &= x_n \\ g &= A + \frac{y_{n-1}}{z_n} \\ g_1 &= y_n \\ h &= A + \frac{z_{n-1}}{y_n} \\ h_1 &= z_n. \end{aligned}$$

The Jacobian matrix about the equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ under the above transformation is given by

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} 0 & \frac{1}{\overline{z}} & 0 & 0 & -\frac{\overline{x}}{\overline{z}^2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\overline{z}} & -\frac{\overline{y}}{\overline{z}^2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\overline{z}}{\overline{y}^2} & 0 & 0 & \frac{1}{\overline{y}} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.3}$$

Hence, the linearized system of system (3.1) about the equilibrium point

$(\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1)$

is

$X_{n+1} = B(\overline{x}, \overline{y}, \overline{z}) X_n,$

where

$X_n = ((x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}))^T$

and

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} 0 & \frac{1}{A+1} & 0 & 0 & -\frac{1}{A+1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{A+1} & -\frac{1}{A+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{A+1} & 0 & 0 & \frac{1}{A+1} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic equation of $B(\overline{x}, \overline{y}, \overline{z})$ about

$$(\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1)$$

is

$$\lambda^6 - \frac{(3A+4)}{(A+1)^2}\lambda^4 + \frac{(3A+4)}{(A+1)^3}\lambda^2 - \frac{1}{(A+1)^3} = 0. \tag{3.4}$$

From this, the roots of characteristic equation (3.4) are

$$\begin{aligned}
\lambda_1 &= \frac{1}{\sqrt{A+1}}, \\
\lambda_2 &= -\frac{1}{\sqrt{A+1}}, \\
\lambda_3 &= \frac{1}{2}\frac{\sqrt{4A+5}-1}{A+1}, \\
\lambda_4 &= -\frac{1}{2}\frac{\sqrt{4A+5}+1}{A+1}, \\
\lambda_5 &= \frac{1}{2}\frac{\sqrt{4A+5}+1}{A+1}, \\
\lambda_6 &= -\frac{1}{2}\frac{\sqrt{4A+5}-1}{A+1}.
\end{aligned}$$

From the Linearized Stability Theorem, since $A > 1$, all roots of the characterictic equation lie inside the open unit disk $|\lambda| < 1$. Therefore, the positive equilibrium point of system (3.1) is locally asymptotically stable.

(iii) From the proof of (ii), it is true.

(iv) From (3.3), the linearized system of system (3.1) about the equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\mu, \mu, \frac{\mu}{\mu-1}\right)$$

is

$$X_{n+1} = B(\overline{x}, \overline{y}, \overline{z})\, X_n,$$

where

$$X_n = ((x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}))^T$$

and

$$B\left(\overline{x},\overline{y},\overline{z}\right)=\begin{pmatrix} 0 & \frac{\mu-1}{\mu} & 0 & 0 & -\frac{(\mu-1)^2}{\mu} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu-1}{\mu} & -\frac{(\mu-1)^2}{\mu} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\mu(\mu-1)} & 0 & 0 & \frac{1}{\mu} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, the characteristic equation of the matrix B is

$$\lambda^6-\left(\frac{2\mu^2-1}{\mu^2}\right)\lambda^4+\left(\frac{\mu^3+\mu^2-3\mu+1}{\mu}\right)\lambda^2-\frac{(\mu-1)^2}{\mu^3}=0. \qquad (3.5)$$

Therefore, the roots of the equation (3.5) are:

$$\begin{aligned} \lambda_1 &= -1, \\ \lambda_2 &= 1, \\ \lambda_3 &= \frac{\sqrt{\mu-1}}{\mu}, \\ \lambda_4 &= -\frac{\sqrt{\mu-1}}{\mu}, \\ \lambda_5 &= \sqrt{\frac{\mu-1}{\mu}}, \\ \lambda_6 &= -\sqrt{\frac{\mu-1}{\mu}}. \end{aligned}$$

Then, the modulus of four of the roots of (3.5) are less than 1. So, there exist positive solutions of system (3.1) which tend to the positive equilibrium point $\left(\mu,\mu,\frac{\mu}{\mu-1}\right)$ of system (3.1) (this follows from the following proposition). This completes the proof.

In the following proposition we find positive solutions of system (3.1) which tend to $(\overline{x},\overline{y},\overline{z})$ as $n\to\infty$.

Proposition 3.2 *Let $\{(x_n,y_n,z_n)\}$ be a positive solution of system (3.1). Then, if there exists an $s\in\{-1,0,...\}$ such that for $n\geq s$, $x_n\geq\overline{x}$, $y_n\geq\overline{y}$, $z_n\geq\overline{z}$ (resp., $x_n<\overline{x}$, $y_n<\overline{y}$, $z_n<\overline{z}$), the solution $\{(x_n,y_n,z_n)\}$ tends to the positive equilibrium $(\overline{x},\overline{y},\overline{z})$ of system (3.1) as $n\to\infty$.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a positive solution of system (3.1) such that

$$x_n \geq \overline{x}, \;\; y_n \geq \overline{y}, \;\; z_n \geq \overline{z}, \;\; n \geq s, \tag{3.6}$$

where $s \in \{-1, 0, ...\}$. Then from (3.1) and (3.6) we have

$$x_{n+1} = A + \frac{x_{n-1}}{z_n} \leq A + \frac{x_{n-1}}{\overline{z}}, \quad n \geq 1. \tag{3.7}$$

Set

$$u_{n+1} = A + \frac{u_{n-1}}{\overline{z}}, n \geq 1 \tag{3.8}$$

such that

$$u_s = x_s, \; u_{s+1} = x_{s+1}, \; s \in \{-1, 0, 1, ...\}, \quad n \geq s. \tag{3.9}$$

Then, the solution u_n of the difference equation (3.8) is as follows:

$$u_n = c_1 \left(\frac{1}{\sqrt{\overline{z}}}\right)^n + c_1 \left(-\frac{1}{\sqrt{\overline{z}}}\right)^n + \frac{A\overline{z}}{A\overline{z} - 1} = c_1 \left(\frac{1}{\sqrt{\overline{z}}}\right)^n + c_1 \left(-\frac{1}{\sqrt{\overline{z}}}\right)^n + \overline{x}, \tag{3.10}$$

where c_1, c_2 depend on x_s, x_{s+1}. In addition, the relations (3.7) and (3.8) imply that

$$x_{n+1} - u_{n+1} \leq \frac{x_{n-1} - u_{n-1}}{\overline{z}}, \quad n > s. \tag{3.11}$$

Then, by using (3.9) and (3.11) and induction, we have

$$x_n \leq u_n, \quad n \geq s. \tag{3.12}$$

Therefore, from (3.6), (3.10), and (3.12), it is clear that

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{3.13}$$

Similarly, we can prove that

$$\lim_{n \to \infty} y_n = \overline{y} \;\text{ and }\; \lim_{n \to \infty} z_n = \overline{z}. \tag{3.14}$$

Thus, from (3.13) and (3.14), the solution $\{(x_n, y_n, z_n)\}$ tends to $(\overline{x}, \overline{y}, \overline{z})$ as $n \to \infty$. Arguing as above we can show that if $x_n < \overline{x}$, $y_n < \overline{y}$, $z_n < \overline{z}$ for $n \geq s$, then $\{(x_n, y_n, z_n)\}$ tends to $(\overline{x}, \overline{y}, \overline{z})$ as $n \to \infty$. The proof of the proposition is completed.

Theorem 3.3 *Assume that* $0 < A < 1$ *and* $\{(x_n, y_n, z_n)\}$ *is an arbitrary positive solution of system (3.1). Then, the following statements are true.*

(i) *If*

$$x_{-1} < 1,\ y_{-1} < 1,\ \ z_{-1} < 1,\ \ x_0 > \frac{1}{1-A},\ \ y_0 > \frac{1}{1-A},\ z_0 > \frac{1}{1-A}, \tag{3.15}$$

then

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= A,\ \lim_{n\to\infty} y_{2n+1} = A,\ \lim_{n\to\infty} z_{2n+1} = A, \\ \lim_{n\to\infty} x_{2n} &= \infty,\ \lim_{n\to\infty} y_{2n} = \infty,\ \lim_{n\to\infty} z_{2n} = \infty. \end{aligned}$$

(ii) *If*

$$x_0 < 1,\ y_0 < 1,\ z_0 < 1,\ \ x_{-1} > \frac{1}{1-A},\ y_{-1} > \frac{1}{1-A},\ z_{-1} > \frac{1}{1-A}, \tag{3.16}$$

then

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= \infty,\ \lim_{n\to\infty} y_{2n+1} = \infty,\ \lim_{n\to\infty} z_{2n+1} = \infty, \\ \lim_{n\to\infty} x_{2n} &= A,\ \lim_{n\to\infty} y_{2n} = A,\ \ \lim_{n\to\infty} z_{2n} = A. \end{aligned}$$

Proof. (i) From (3.1) and (3.15), we get

$$\begin{aligned} x_1 &= A + \frac{x_{-1}}{z_0} < A + \frac{1}{z_0} < A + (1-A) = 1, \\ y_1 &= A + \frac{y_{-1}}{z_0} < A + \frac{1}{z_0} < A + (1-A) = 1, \\ z_1 &= A + \frac{z_{-1}}{y_0} < A + \frac{1}{y_0} < A + (1-A) = 1, \\ x_2 &= A + \frac{x_0}{z_1} > x_0 > \frac{1}{1-A}, \\ y_2 &= A + \frac{y_0}{z_1} > y_0 > \frac{1}{1-A}, \\ z_2 &= A + \frac{z_0}{y_1} > z_0 > \frac{1}{1-A}. \end{aligned}$$

By induction for $n = 0, 1, 2, ...$, we obtain

$$\begin{aligned} x_{2n-1} &< 1,\ \ y_{2n-1} < 1,\ z_{2n-1} < 1, \\ x_{2n} &> \frac{1}{1-A},\ \ y_{2n} > \frac{1}{1-A},\ z_{2n} > \frac{1}{1-A}. \end{aligned} \tag{3.17}$$

Thus, relations (3.1) and (3.17) imply that

$$\begin{aligned} x_{2n} &= A + \frac{x_{2n-2}}{z_{2n-1}} > A + x_{2n-2} > 2A + \frac{x_{2n-4}}{z_{2n-3}} > 2A + x_{2n-4}, \\ y_{2n} &= A + \frac{y_{2n-2}}{z_{2n-1}} > A + y_{2n-2} > 2A + \frac{y_{2n-4}}{z_{2n-3}} > 2A + y_{2n-4}, \\ z_{2n} &= A + \frac{z_{2n-2}}{y_{2n-1}} > A + z_{2n-2} > 2A + \frac{z_{2n-4}}{y_{2n-3}} > 2A + z_{2n-4}. \end{aligned}$$

From which we get

$$\lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} z_{2n} = \infty.$$

Noting that (3.17) and taking limits on both sides of three equations

$$x_{2n+1} = A + \frac{x_{2n-1}}{z_{2n}}, \; y_{2n+1} = A + \frac{y_{2n-1}}{z_{2n}}, \quad z_{2n+1} = A + \frac{z_{2n-1}}{y_{2n}},$$

we have

$$\lim_{n\to\infty} x_{2n+1} = A, \quad \lim_{n\to\infty} y_{2n+1} = A, \quad \lim_{n\to\infty} z_{2n+1} = A.$$

(ii) The proof is similar to the proof of (i), so we leave it to readers.

Theorem 3.4 *Assume that $A = 1$. Then every positive solution of system (3.1) is bounded and persists.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a positive solution of the system (3.1).
Obviously, $x_n > 1$, $y_n > 1$, $z_n > 1$, for $n \geq 1$. So, we have

$$x_i, y_i, z_i \in \left[K, \frac{K}{K-1}\right], \quad i = 1, 2, ..., m+1,$$

where

$$K = \min\left\{\alpha, \frac{\beta}{\beta - 1}\right\} > 1, \quad \alpha = \min_{1\leq i\leq m+1} \{x_i, y_i, z_i\}, \quad \beta = \max_{1\leq i\leq m+1} \{x_i, y_i, z_i\}.$$

Then, we obtain

$$\begin{aligned}
K &= 1 + \frac{K}{K/(K-1)} \leq x_{m+2} = 1 + \frac{x_1}{z_{m+1}} \leq 1 + \frac{K/(K-1)}{K} = \frac{K}{K-1}, \\
K &= 1 + \frac{K}{K/(K-1)} \leq y_{m+2} = 1 + \frac{y_1}{z_{m+1}} \leq 1 + \frac{K/(K-1)}{K} = \frac{K}{K-1}, \\
K &= 1 + \frac{K}{K/(K-1)} \leq z_{m+2} = 1 + \frac{z_1}{y_{m+1}} \leq 1 + \frac{K/(K-1)}{K} = \frac{K}{K-1}.
\end{aligned}$$

By induction, we get

$$x_i, y_i, z_i \in \left[K, \frac{K}{K-1}\right], \quad i = 1, 2, ...$$

Theorem 3.5 *Assume $A = 1$. Then, every positive solution of system (3.1) is periodic of period 2.*

Proof. From system (3.1), we have

$$
\begin{array}{lll}
x_{n+1}=1+\frac{x_{n-1}}{z_n}, & y_{n+1}=1+\frac{y_{n-1}}{z_n}, & z_{n+1}=1+\frac{z_{n-1}}{y_n} \\
x_{n+2}=1+\frac{x_n}{z_{n+1}}, & y_{n+2}=1+\frac{y_n}{z_{n+1}}, & z_{n+1}=1+\frac{z_{n-1}}{y_n} \\
=1+\frac{x_n}{1+\frac{z_{n-1}}{y_n}} & =1+\frac{y_n}{1+\frac{z_{n-1}}{y_n}} & =1+\frac{z_n}{1+\frac{y_{n-1}}{z_n}} \\
=1+\frac{x_n y_n}{y_n+z_{n-1}} & =1+\frac{y_n^2}{y_n+z_{n-1}} & =1+\frac{z_n^2}{z_n+y_{n-1}} \\
\overset{(\text{from } 3.2)}{=} \frac{z_{n-1}+x_n}{z_{n-1}} & \overset{(\text{from } 3.2)}{=} \frac{z_{n-1}+y_n}{z_{n-1}} & \overset{(\text{from } 3.2)}{=} \frac{y_{n-1}+z_n}{y_{n-1}} \\
=x_n & =y_n & =z_n
\end{array}
$$

Theorem 3.6 *Assume $A>1$. Then, every positive solution of system (3.1) is bounded.*

Proof. Let $\{(x_n, y_n, z_n)\}$ is a positive solution of system (3.1). Clearly,

$$x_n, y_n, z_n > A > 1, \quad \text{for } n \geq 1. \tag{3.18}$$

From (3.18), we have

$$x_{n+1} = A + \frac{x_{n-1}}{z_n} \leq A + \frac{x_{n-1}}{A}, \quad n \geq 1. \tag{3.19}$$

Set

$$u_{n+1} = A + \frac{u_{n-1}}{A}, \;\; n \geq 1 \tag{3.20}$$

such that

$$u_s = x_s, \;\; u_{s+1} = x_{s+1}, \;\; s \in \{-1, 0, 1, ...\}, \quad n \geq s. \tag{3.21}$$

Then, the solution u_n of the difference equation (3.20) is as follows:

$$u_n = c_1 \left(\frac{1}{\sqrt{A}}\right)^n + c_1 \left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1}. \tag{3.22}$$

Indeed, from (3.20), we get

$$
\begin{aligned}
u_{n+1} - \frac{1}{A}u_{n-1} &= 0 \Rightarrow \lambda^2 - \frac{1}{A} = 0 \\
&\Rightarrow \lambda_{1,2} = \pm\frac{1}{\sqrt{A}}.
\end{aligned}
$$

The homogen solution of difference equation (3.20) is given by

$$u_h = c_1 \left(\frac{1}{\sqrt{A}}\right)^n + c_2 \left(-\frac{1}{\sqrt{A}}\right)^n.$$

Also, from (3.20), the equilibrium solution of difference equation (3.20) is following

$$\overline{x} - \frac{1}{A}\overline{x} = A \Rightarrow \overline{x} = \frac{A^2}{A-1}.$$

In addition, the relations (3.19) and (3.22) imply that

$$x_{n+1} - u_{n+1} \leq \frac{x_{n-1} - u_{n-1}}{A}, \;\; n > s. \tag{3.23}$$

Then, by using (3.21) and (3.23) and induction, we have

$$x_n \leq u_n, \;\; n \geq s. \tag{3.24}$$

Therefore, from (3.18), (3.22), and (3.24), we obtain

$$A < x_n \leq c_1 \left(\frac{1}{\sqrt{A}}\right)^n + c_2 \left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1},$$

where

$$\begin{aligned} c_1 &= \frac{1}{2}\left(x_0 + \sqrt{A}x_1 - \frac{A^2}{A-1}\left(1+\sqrt{A}\right)\right), \\ c_2 &= \frac{1}{2}\left(x_0 - \sqrt{A}x_1 - \frac{A^2}{A-1}\left(1-\sqrt{A}\right)\right). \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} A &< y_n \leq c_3 \left(\frac{1}{\sqrt{A}}\right)^n + c_4 \left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1}, \\ A &< z_n \leq c_5 \left(\frac{1}{\sqrt{A}}\right)^n + c_6 \left(-\frac{1}{\sqrt{A}}\right)^n + \frac{A^2}{A-1}, \end{aligned}$$

where

$$\begin{aligned} c_3 &= \frac{1}{2}\left(y_0 + \sqrt{A}y_1 - \frac{A^2}{A-1}\left(1+\sqrt{A}\right)\right), \\ c_4 &= \frac{1}{2}\left(y_0 - \sqrt{A}y_1 - \frac{A^2}{A-1}\left(1-\sqrt{A}\right)\right), \\ c_5 &= \frac{1}{2}\left(z_0 + \sqrt{A}z_1 - \frac{A^2}{A-1}\left(1+\sqrt{A}\right)\right), \\ c_6 &= \frac{1}{2}\left(z_0 - \sqrt{A}z_1 - \frac{A^2}{A-1}\left(1-\sqrt{A}\right)\right). \end{aligned}$$

Theorem 3.7 *Suppose that $A > 1$. Then, the positive equilibrium point of system (3.1) is globally asymptotically stable.*

Proof. By the means of Theorem 3.6, we set

$$\begin{aligned} L_1 &= \lim_{n\to\infty} \sup x_n, \;\; L_2 = \lim_{n\to\infty} \sup y_n, \;\; L_3 = \lim_{n\to\infty} \sup z_n, \\ m_1 &= \lim_{n\to\infty} \inf x_n, \; m_2 = \lim_{n\to\infty} \inf y_n, \;\; m_3 = \lim_{n\to\infty} \inf z_n. \end{aligned} \tag{3.25}$$

Then, from (3.1) and (4.4) we have

$$\begin{aligned} L_1 &\leq A + \frac{L_1}{m_3}, \;\; L_2 \leq A + \frac{L_2}{m_3}, \;\; L_3 \leq A + \frac{L_3}{m_2}, \\ m_1 &\geq A + \frac{m_1}{L_3}, \;\; m_2 \geq A + \frac{m_2}{L_3}, \;\; m_3 \geq A + \frac{m_3}{L_2}. \end{aligned} \tag{3.26}$$

Relations (4.5) imply that

$$AL_2 + m_3 \leq m_3 L_2 \leq Am_3 + L_2, \;\; AL_3 + m_2 \leq m_2 L_3 \leq Am_2 + L_3,$$

from which we have

$$(A-1)(L_2 - m_3) \leq 0, \;\; (A-1)(L_3 - m_2) \leq 0.$$

Since $A > 1$, we get

$$L_2 \leq m_3 \leq L_3, \;\; L_3 \leq m_2 \leq L_2,$$

from this it is obvious that

$$L_2 = L_3 = m_2 = m_3. \tag{3.27}$$

Moreover, from (4.5) it follows that

$$L_1 m_3 \leq Am_3 + L_1, \;\; m_1 L_3 \leq AL_3 + m_1,$$

from which

$$L_1 (m_3 - 1) \leq Am_3, \;\; AL_3 \leq m_1 (L_3 - 1).$$

Using (4.6), we have

$$L_1 (L_3 - 1) \leq m_1 (L_3 - 1),$$

then

$$L_1 \leq m_1.$$

Since x_n is bounded, it implys that

$$L_1 = m_1.$$

Hence, every positive solution $\{(x_n, y_n, z_n)\}$ of system (3.1) tends to the positive equlibrium system (3.1). So, this completes the proof.

CHAPTER 4

GLOBAL ANALYSIS OF A SYSTEM OF HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS

We state that the results of this chapter are cited from [66] which has been published by us.

In this chapter, in the light of work in [65], we investigate the global asymptotic stability, boundedness character and oscillatory of positive solutions of the system of difference equations

$$x_{n+1} = A + \frac{x_{n-m}}{z_n}, \quad y_{n+1} = A + \frac{y_{n-m}}{z_n}, \quad z_{n+1} = A + \frac{z_{n-m}}{y_n}, \quad n = 0, 1, ..., \tag{4.1}$$

where A and the initial values x_{-i}, y_{-i}, z_{-i}, for $i = 0, 1, ..., m$, are positive real numbers and m is positive integer.

4.1 MAIN RESULTS

In this section, we prove our main results. We deal with the following cases of $0 < A < 1$, $A = 1$, and $A > 1$.

Theorem 4.1 *If $(\overline{x}, \overline{y}, \overline{z})$ is a positive equilibrium point of system (4.1), then*

$$(\overline{x}, \overline{y}, \overline{z}) = \begin{cases} (A+1, A+1, A+1), & \text{if } A \neq 1, \\ \left(\mu, \mu, \frac{\mu}{\mu-1}\right), \ \mu \in (1, \infty) & \text{if } A = 1. \end{cases}$$

Proof. It is easily seen from the definition of equilibrium point that the equilibrium points of system (4.1) are the nonnegative solution of the equations

$$\overline{x} = A + \frac{\overline{x}}{\overline{z}}, \overline{y} = A + \frac{\overline{y}}{\overline{z}}, \overline{z} = A + \frac{\overline{z}}{\overline{y}}.$$

From this, we get

$$\begin{aligned} \overline{xz} &= A\overline{z} + \overline{x}, \ \ \overline{yz} = A\overline{z} + \overline{y}, \ \ \overline{zy} = A\overline{y} + \overline{z} \\ &\Rightarrow \ \overline{xz} - \overline{x} = \overline{yz} - \overline{y}, \ \ A\overline{z} + \overline{y} = A\overline{y} + \overline{z} \\ &\Rightarrow \ \overline{x}(\overline{z} - 1) = \overline{y}(\overline{z} - 1), \ \ \overline{z}(A - 1) = \overline{y}(A - 1). \end{aligned}$$

From which it follows that if $A \neq 1$,

$$\overline{x} = \overline{y} = \overline{z} = A + 1 \Rightarrow (\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1).$$

Also, we have

$$\begin{aligned}
\frac{\overline{xz} - \overline{x}}{\overline{z}} &= A, \ \frac{\overline{yz} - \overline{y}}{\overline{z}} = A, \ \frac{\overline{zy} - \overline{z}}{\overline{y}} = A \\
&\Rightarrow \ \frac{\overline{xz} - \overline{x}}{\overline{z}} = \frac{\overline{yz} - \overline{y}}{\overline{z}}, \ \frac{\overline{yz} - \overline{y}}{\overline{z}} = \frac{\overline{zy} - \overline{z}}{\overline{y}} \\
&\Rightarrow \ \overline{xz} - \overline{x} = \overline{yz} - \overline{y}, \ \overline{y}^2\overline{z} - \overline{y}^2 = \overline{z}^2\overline{y} - \overline{z}^2 \\
&\Rightarrow \ \overline{x}(\overline{z} - 1) = \overline{y}(\overline{z} - 1), \ \overline{yz}(\overline{y} - \overline{z}) = (\overline{y} - \overline{z})(\overline{y} + \overline{z}).
\end{aligned}$$

From which it follows that if $A = 1$,

$$\overline{x} = \overline{y} \text{ and } \overline{yz} = \overline{y} + \overline{z} \Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(\mu, \mu, \frac{\mu}{\mu - 1}\right), \ \mu \in (1, \infty).$$

In that case, we have a continuous of positive equilibriums which lie on the hyperboloid $\overline{yz} = \overline{y} + \overline{z}$.

Theorem 4.2 *Assume that* $0 < A < 1$. *Let* $\{(x_n, y_n, z_n)\}$ *be an arbitrary positive solution of the system (4.1). Then, the following statements are true.*

(i) *If* m *is odd and* $0 < x_{2k-1} < 1$, $0 < y_{2k-1} < 1$, $0 < z_{2k-1} < 1$, $x_{2k} > \frac{1}{1-A}$, $y_{2k} > \frac{1}{1-A}$, $z_{2k} > \frac{1}{1-A}$ *for* $k = \frac{1-m}{2}, \frac{3-m}{2}, ..., 0,$ *then*

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \infty, \ \lim_{n\to\infty} y_{2n} = \infty, \ \lim_{n\to\infty} z_{2n} = \infty, \\
\lim_{n\to\infty} x_{2n+1} &= A, \ \lim_{n\to\infty} y_{2n+1} = A, \ \lim_{n\to\infty} z_{2n+1} = A.
\end{aligned}$$

(ii) *If* m *is odd and* $0 < x_{2k} < 1$, $0 < y_{2k} < 1$, $0 < z_{2k} < 1$, $x_{2k-1} > \frac{1}{1-A}$, $y_{2k-1} > \frac{1}{1-A}$, $z_{2k-1} > \frac{1}{1-A}$ *for* $k = \frac{1-m}{2}, \frac{3-m}{2}, ..., 0,$ *then*

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= A, \ \lim_{n\to\infty} y_{2n} = A, \ \lim_{n\to\infty} z_{2n} = A, \\
\lim_{n\to\infty} x_{2n+1} &= \infty, \ \lim_{n\to\infty} y_{2n+1} = \infty, \ \lim_{n\to\infty} z_{2n+1} = \infty.
\end{aligned}$$

(iii) *If* m *is even, we can not get some useful results.*

Proof.

(i) Clearly, we get

$$
\begin{aligned}
0 &< x_1 = A + \frac{x_{-m}}{z_0} < A + \frac{1}{z_0} < A + (1-A) = 1, \\
0 &< y_1 = A + \frac{y_{-m}}{z_0} < A + \frac{1}{z_0} < A + (1-A) = 1, \\
0 &< z_1 = A + \frac{z_{-m}}{y_0} < A + \frac{1}{y_0} < A + (1-A) = 1, \\
x_2 &= A + \frac{x_{1-m}}{z_1} > x_{1-m} > \frac{1}{1-A}, \\
y_2 &= A + \frac{y_{1-m}}{z_1} > y_{1-m} > \frac{1}{1-A}, \\
z_2 &= A + \frac{z_{1-m}}{y_1} > z_{1-m} > \frac{1}{1-A}.
\end{aligned}
$$

By induction for $n = 1, 2, ...,$ we obtain

$$
\begin{aligned}
0 &< x_{2n-1} < 1,\ 0 < y_{2n-1} < 1,\ \ 0 < z_{2n-1} < 1, \\
x_{2n} &> \frac{1}{1-A},\ y_{2n} > \frac{1}{1-A},\ z_{2n} > \frac{1}{1-A}.
\end{aligned} \tag{4.2}
$$

Thus, for $n \geq (m+2)/2$,

$$
\begin{aligned}
x_{2n} &= A + \frac{x_{2n-(m+1)}}{z_{2n-1}} > A + x_{2n-(m+1)} = 2A + \frac{x_{2n-(2m+2)}}{z_{2n-(m+2)}} \\
&> 2A + x_{2n-(2m+2)}, \\
y_{2n} &= A + \frac{y_{2n-(m+1)}}{z_{2n-1}} > A + y_{2n-(m+1)} = 2A + \frac{y_{2n-(2m+2)}}{z_{2n-(m+2)}} \\
&> 2A + y_{2n-(2m+2)}, \\
z_{2n} &= A + \frac{z_{2n-(m+1)}}{y_{2n-1}} > A + z_{2n-(m+1)} = 2A + \frac{z_{2n-(2m+2)}}{y_{2n-(m+2)}} \\
&> 2A + z_{2n-(2m+2)},
\end{aligned}
$$

from which we get

$$
\lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} z_{2n} = \infty.
$$

Noting that (4.2) and taking limits on the both sides of three equations

$$
x_{2n+1} = A + \frac{x_{2n-m}}{z_{2n}},\ y_{2n+1} = A + \frac{y_{2n-m}}{z_{2n}},\ z_{2n+1} = A + \frac{z_{2n-m}}{y_{2n}},
$$

we have

$$
\lim_{n\to\infty} x_{2n+1} = A, \quad \lim_{n\to\infty} y_{2n+1} = A, \quad \lim_{n\to\infty} z_{2n+1} = A.
$$

(ii) Obviously, we have

$$
\begin{aligned}
x_1 &= A + \frac{x_{-m}}{z_0} > x_{-m} > \frac{1}{1-A}, \\
y_1 &= A + \frac{y_{-m}}{z_0} > y_{-m} > \frac{1}{1-A}, \\
z_1 &= A + \frac{z_{-m}}{y_0} > z_{-m} > \frac{1}{1-A}, \\
0 &< x_2 = A + \frac{x_{1-m}}{z_1} < A + \frac{1}{z_1} < A + (1-A) = 1, \\
0 &< y_2 = A + \frac{y_{1-m}}{z_1} < A + \frac{1}{z_1} < A + (1-A) = 1, \\
0 &< z_2 = A + \frac{z_{1-m}}{y_1}. < A + \frac{1}{y_1} < A + (1-A) = 1.
\end{aligned}
$$

By induction for $n = 1, 2, ...,$ we obtain

$$
\begin{aligned}
x_{2n-1} &> \frac{1}{1-A}, \; y_{2n-1} > \frac{1}{1-A}, \; z_{2n-1} > \frac{1}{1-A}, \\
0 &< x_{2n} < 1, \;\; 0 < y_{2n} < 1, \;\; 0 < z_{2n} < 1.
\end{aligned}
\tag{4.3}
$$

So, for $n \geq (m+2)/2$,

$$
\begin{aligned}
x_{2n+1} &= A + \frac{x_{2n-m}}{z_{2n}} > A + x_{2n-m} = 2A + \frac{x_{(2n-2m)-1}}{z_{2n-(m+1)}} \\
&> 2A + x_{(2n-2m)-1}, \\
y_{2n+1} &= A + \frac{y_{2n-m}}{z_{2n}} > A + y_{2n-m} = 2A + \frac{y_{(2n-2m)-1}}{z_{2n-(m+1)}} \\
&> 2A + y_{(2n-2m)-1}, \\
z_{2n+1} &= A + \frac{z_{2n-m}}{y_{2n}} > A + z_{2n-m} = 2A + \frac{z_{(2n-2m)-1}}{y_{2n-(m+1)}} \\
&> 2A + z_{(2n-2m)-1},
\end{aligned}
$$

from which we get

$$
\lim_{n\to\infty} x_{2n+1} = \infty, \quad \lim_{n\to\infty} y_{2n+1} = \infty, \quad \lim_{n\to\infty} z_{2n+1} = \infty.
$$

Noting that (4.3) and taking limits on the both sides of three equations

$$
x_{2n} = A + \frac{x_{2n-(m+1)}}{z_{2n-1}}, \; y_{2n+1} = A + \frac{y_{2n-(m+1)}}{z_{2n-1}}, \; z_{2n+1} = A + \frac{z_{2n-(m+1)}}{y_{2n-1}},
$$

we have

$$
\lim_{n\to\infty} x_{2n} = A, \quad \lim_{n\to\infty} y_{2n} = A, \quad \lim_{n\to\infty} z_{2n+} = A.
$$

Theorem 4.3 *Suppose that $A = 1$. Then every positive solution of system (4.1) is bounded and persists.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a positive solution of the system (4.1).
Obviously, $x_n > 1$, $y_n > 1$, $z_n > 1$, for $n \geq 1$. So, we have

$$x_i, y_i, z_i \in \left[M, \frac{M}{M-1}\right], \quad i = 1, 2, ..., m+1,$$

where

$$M = \min\left\{\alpha, \frac{\beta}{\beta-1}\right\} > 1, \ \ \alpha = \min_{1\leq i\leq m+1} \{x_i, y_i, z_i\}, \ \ \beta = \max_{1\leq i\leq m+1} \{x_i, y_i, z_i\}.$$

Then, we obtain

$$\begin{aligned}
M &= 1 + \frac{M}{M/(M-1)} \leq x_{m+2} = 1 + \frac{x_1}{z_{m+1}} \leq 1 + \frac{M/(M-1)}{M} = \frac{M}{M-1}, \\
M &= 1 + \frac{M}{M/(M-1)} \leq y_{m+2} = 1 + \frac{y_1}{z_{m+1}} \leq 1 + \frac{M/(M-1)}{M} = \frac{M}{M-1}, \\
M &= 1 + \frac{M}{M/(M-1)} \leq z_{m+2} = 1 + \frac{z_1}{y_{m+1}} \leq 1 + \frac{M/(M-1)}{M} = \frac{M}{M-1}.
\end{aligned}$$

By induction, we get

$$x_i, y_i, z_i \in \left[M, \frac{M}{M-1}\right], \quad i = 1, 2, ...$$

Theorem 4.4 *Assume that $A = 1$. Let $\{(x_n, y_n, z_n)\}$ be a positive solution of the system (4.1). Then, either $\{(x_n, y_n, z_n)\}$ consists of a single semicycle or $\{(x_n, y_n, z_n)\}$ oscillates about the equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = \left(\mu, \mu, \frac{\mu}{\mu-1}\right)$ with semicycles having at most m terms.*

Proof. Suppose that $\{(x_n, y_n, z_n)\}$ has at least two semicycles Then, there exists $N \geq -m$ such that either $x_N < \overline{x} \leq x_{N+1}$ or $x_{N+1} < \overline{x} \leq x_N$ ($y_N < \overline{y} \leq y_{N+1}$ or $y_{N+1} < \overline{y} \leq y_N$ and $z_N < \overline{z} \leq z_{N+1}$ or $z_{N+1} < \overline{z} \leq z_N$). Firstly, we assume that the case $x_N < \overline{x} \leq x_{N+1}$, $y_N < \overline{y} \leq y_{N+1}$ and $z_N < \overline{z} \leq z_{N+1}$. Since the other case is similar, it will be omitted. Suppose that the positive semicycle beginning with the term $(x_{N+1}, y_{N+1}, z_{N+1})$ have m terms. Then we have

$$\begin{aligned}
x_{N+1} &< \overline{x} = \mu \leq x_{N+m}, \\
y_{N+1} &< \overline{y} = \mu \leq y_{N+m}, \\
z_{N+1} &< \overline{z} = \frac{\mu}{\mu-1} \leq z_{N+m}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned} x_{N+m+1} &= 1+\frac{x_N}{z_{N+m}} < 1+\frac{\overline{x}}{\overline{z}} = \mu, \\ y_{N+m+1} &= 1+\frac{y_N}{z_{N+m}} < 1+\frac{\overline{y}}{\overline{z}} = \mu, \\ z_{N+m+1} &= 1+\frac{z_N}{y_{N+m}} < 1+\frac{\overline{z}}{\overline{y}} = \frac{\mu}{\mu-1}. \end{aligned}$$

This completes the proof.

Theorem 4.5 *Suppose that* $A > 1$. *Then every positive solution of system (4.1) is bounded and persists.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a positive solution of the system (4.1).
Obviously, $x_n > A > 1$, $y_n > A > 1$, $z_n > A > 1$, for $n \geq 1$. So, we have

$$x_i, y_i, z_i \in \left[M, \frac{M}{M-A}\right], \quad i = 1, 2, ..., m+1,$$

where

$$M = \min\left\{\alpha, \frac{\beta}{\beta-1}\right\} > 1, \quad \alpha = \min_{1\leq i\leq m+1}\{x_i, y_i, z_i\}, \quad \beta = \max_{1\leq i\leq m+1}\{x_i, y_i, z_i\}.$$

Then, we obtain

$$\begin{aligned} M &= A+\frac{M}{M/(M-A)} \leq x_{m+2} = 1+\frac{x_1}{z_{m+1}} \leq 1+\frac{M/(M-A)}{M} = \frac{M}{M-A}, \\ M &= A+\frac{M}{M/(M-A)} \leq y_{m+2} = 1+\frac{y_1}{z_{m+1}} \leq 1+\frac{M/(M-A)}{M} = \frac{M}{M-A}, \\ M &= A+\frac{M}{M/(M-A)} \leq z_{m+2} = 1+\frac{z_1}{y_{m+1}} \leq 1+\frac{M/(M-A)}{M} = \frac{M}{M-A}. \end{aligned}$$

By induction, we get

$$x_i, y_i, z_i \in \left[M, \frac{M}{M-A}\right], \quad i = 1, 2, ...$$

The proof is completed.

Before we give the following theorems about the stability of the equilibrium points, we consider the following transformation to build the corresponding linearized form of system (4.1) :

$$\begin{aligned} & (x_n, x_{n-1}, ..., x_{n-m}, y_n, y_{n-1}, ..., y_{n-m}, z_n, z_{n-1}, ..., z_{n-m}) \\ \rightarrow\ & (f, f_1, ..., f_m, g, g_1, ..., g_m, h, h_1, ..., h_m) \end{aligned}$$

where

$$
\begin{aligned}
f &= A+\frac{x_{n-m}}{z_n} \\
f_1 &= x_n \\
&\vdots \\
f_m &= x_{n-m} \\
g &= A+\frac{y_{n-m}}{z_n} \\
g_1 &= y_n \\
&\vdots \\
g_m &= y_{n-m} \\
h &= A+\frac{z_{n-m}}{y_n} \\
h_1 &= z_n \\
&\vdots \\
h_m &= z_{n-m}.
\end{aligned}
$$

The Jacobian matrix about the equilibrium point $(\overline{x},\overline{y},\overline{z})$ under the above transformation is given by

$$
B\left(\overline{x},\overline{y},\overline{z}\right)=\begin{pmatrix}
0 & \dots & 0 & \frac{1}{\overline{z}} & 0 & \dots & 0 & 0 & -\frac{\overline{x}}{\overline{z}^2} & \dots & 0 & 0 \\
1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\overline{z}} & -\frac{\overline{y}}{\overline{z}^2} & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & -\frac{\overline{z}}{\overline{y}^2} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\overline{y}} \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix},
$$

where $B=(b_{ij})$, $1\leq i,j\leq 3m+3$ is an $(3m+3)\times(3m+3)$ matrix.

Theorem 4.6 *If $A=1$, then the equilibrium point of system (4.1) is locally asymptotically stable.*

Proof. The linearized system of system (4.1) about the equilibrium point $\left(\mu, \mu, \frac{\mu}{\mu-1}\right)$ is

$$X_{n+1} = BX_n,$$

where $X_n = (x_n, x_{n-1}, ..., x_{n-m}, y_n, y_{n-1}, ..., y_{n-m}, z_n, z_{n-1}, ..., z_{n-m})^T$ and

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} 0 & \dots & 0 & \frac{\mu-1}{\mu} & 0 & \dots & 0 & 0 & -\frac{(\mu-1)^2}{\mu} & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\mu-1}{\mu} & -\frac{(\mu-1)^2}{\mu} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & -\frac{1}{\mu(\mu-1)^2} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\mu} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, ..., \lambda_{3m+3}$ denote the $3m+3$ eigenvalues of the matrix B and

$$D = diag(d_1, d_2, ..., d_{3m+3})$$

be a diagonal matrix, where

$$d_1 = d_{m+2} = d_{2m+3} = 1, \;\; d_{1+k} = d_{m+2+k} = d_{2m+3+k} = 1 - k\varepsilon, \;\; 1 \leq k \leq m$$

and

$$0 < \varepsilon < \left\{ \frac{\mu^2 - 2\mu + 2}{m\mu}, \frac{\mu^2 - 2\mu + 2}{m\mu(\mu - 1)} \right\}.$$

Obviously, D is invertible. Computing matrix DBD^{-1}, we have that

$$DBD^{-1} = \begin{pmatrix}
0 & \dots & 0 & p & 0 & \dots & 0 & 0 & q & \dots & 0 & 0 \\
\frac{d_2}{d_1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & \frac{d_{m+1}}{d_m} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & r & s & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \frac{d_{m+3}}{d_{m+2}} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & \frac{d_{2m+2}}{d_{2m+1}} & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & t & \dots & 0 & 0 & 0 & \dots & 0 & w \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \frac{d_{2m+4}}{d_{2m+3}} & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \frac{d_{3m+3}}{d_{3m+2}} & 0
\end{pmatrix},$$

where

$$\begin{aligned}
p &= \frac{\mu-1}{\mu}\frac{d_1}{d_{m+1}}, \\
q &= -\frac{(\mu-1)^2}{\mu}\frac{d_1}{d_{2m+3}}, \\
r &= \frac{\mu-1}{\mu}\frac{d_{m+2}}{d_{2m+2}}, \\
s &= -\frac{(\mu-1)^2}{\mu}\frac{d_{m+2}}{d_{2m+3}}, \\
t &= -\frac{1}{\mu(\mu-1)^2}\frac{d_{2m+3}}{d_{m+2}}, \\
w &= \frac{1}{\mu}\frac{d_{2m+3}}{d_{3m+3}}.
\end{aligned}$$

The three chains of inequalities

$$\begin{aligned}
1 &= d_1 > d_2 > ... > d_m > d_{m+1} > 0, \\
1 &= d_{m+2} > d_{m+3} > ... > d_{2m+1} > d_{2m+2} > 0, \\
1 &= d_{2m+3} > d_{2m+4} > ... > d_{3m+2} > d_{3m+3} > 0,
\end{aligned}$$

imply that

$$\begin{aligned}
d_2 d_1^{-1} &< 1,\ d_3 d_2^{-1} < 1, ...,\ d_{m+1} d_m^{-1} < 1, \\
d_{m+3} d_{m+2}^{-1} &< 1,\ d_{m+4} d_{m+3}^{-1} < 1, ...,\ d_{2m+2} d_{2m+1}^{-1} < 1, \\
d_{2m+4} d_{2m+3}^{-1} &< 1,\ d_{2m+5} d_{2m+4}^{-1} < 1, ...,\ d_{3m+3} d_{3m+2}^{-1} < 1.
\end{aligned}$$

Also,

$$\begin{aligned}
& \left(\frac{\mu-1}{\mu}\right) d_1 d_{m+1}^{-1} + \left(-\frac{(\mu-1)^2}{\mu}\right) d_1 d_{2m+3}^{-1} \\
= & \left(\frac{\mu-1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) - \frac{(\mu-1)^2}{\mu} \\
< & \left(\frac{\mu-1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) - \left(\frac{(\mu-1)^2}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) \\
= & \left(\frac{-\mu^2+3\mu-2}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) < 1,
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\mu-1}{\mu}\right) d_{m+2} d_{2m+2}^{-1} + \left(-\frac{(\mu-1)^2}{\mu}\right) d_{m+2} d_{2m+3}^{-1} \\
= & \left(\frac{\mu-1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) - \frac{(\mu-1)^2}{\mu} \\
= & \left(\frac{-\mu^2+3\mu-2}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) < 1,
\end{aligned}$$

$$\begin{aligned}
& \left(-\frac{1}{\mu(\mu-1)^2}\right) d_{2m+3} d_{m+2}^{-1} + \left(\frac{1}{\mu}\right) d_{2m+3} d_{3m+3}^{-1} \\
= & \left(-\frac{1}{\mu(\mu-1)^2}\right) + \left(\frac{1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) \\
< & \left(-\frac{1}{\mu(\mu-1)^2}\right)\left(\frac{1}{1-m\varepsilon}\right) + \left(\frac{1}{\mu}\right)\left(\frac{1}{1-m\varepsilon}\right) \\
= & \left(\frac{\mu-2}{\mu(\mu-1)}\right)\left(\frac{1}{1-m\varepsilon}\right) < 1.
\end{aligned}$$

Since B has the same eigenvalues as $DBD^{-1} = E = (e_{ij})$, we obtain that

$$\begin{aligned}
\max_{1\le i\le 3m+3} |\lambda_i| & \le \left\|DBD^{-1}\right\|_\infty \\
& = \max_{1\le i\le 3m+3} \left\{ \sum_{j=1}^{3m+3} |e_{ij}| \right\} \\
& = \max \left\{ \begin{array}{c}
d_2 d_1^{-1},\ d_3 d_2^{-1}, \ldots,\ d_{m+1} d_m^{-1}, \\
d_{m+3} d_{m+2}^{-1},\ d_{m+4} d_{m+3}^{-1}, \ldots,\ d_{2m+2} d_{2m+1}^{-1}, \\
d_{2m+4} d_{2m+3}^{-1},\ d_{2m+5} d_{2m+4}^{-1}, \ldots,\ d_{3m+3} d_{3m+2}^{-1}, \\
\left(\frac{\mu-1}{\mu}\right) d_1 d_{m+1}^{-1} - \left(\frac{(\mu-1)^2}{\mu}\right) d_1 d_{2m+3}^{-1}, \\
\left(\frac{\mu-1}{\mu}\right) d_{m+2} d_{2m+2}^{-1} - \left(\frac{(\mu-1)^2}{\mu}\right) d_{m+2} d_{2m+3}^{-1}, \\
\left(-\frac{1}{\mu(\mu-1)^2}\right) d_{2m+3} d_{m+2}^{-1} + \left(\frac{1}{\mu}\right) d_{2m+3} d_{3m+3}^{-1}
\end{array} \right\} \\
& < 1.
\end{aligned}$$

This implies that the equilibrium point of system (4.1) is locally asymptotically stable.

Theorem 4.7 *If $A > 1$, then the equilibrium point of system (4.1) is locally asymptotically stable.*

Proof. The linearized system of system (4.1) about the equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1)$ is

$$X_{n+1} = BX_n,$$

where $X_n = (x_n, x_{n-1}, ..., x_{n-m}, y_n, y_{n-1}, ..., y_{n-m}, z_n, z_{n-1}, ..., z_{n-m})^T$ and

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} 0 & \dots & 0 & c^{-1} & 0 & \dots & 0 & 0 & -c^{-1} & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & c^{-1} & -c^{-1} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & -c^{-1} & \dots & 0 & 0 & 0 & \dots & 0 & c^{-1} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $c = A + 1$.

Let $\lambda_1, \lambda_2, ..., \lambda_{3m+3}$ denote the $3m+3$ eigenvalues of the matrix B and

$$D = diag\left(d_1, d_2, ..., d_{3m+3}\right)$$

be a diagonal matrix, where

$$d_1 = d_{m+2} = d_{2m+3} = 1, \;\; d_{1+k} = d_{m+2+k} = d_{2m+3+k} = 1 - k\varepsilon, \;\; 1 \leq k \leq m$$

and

$$0 < \varepsilon < \left\{ \frac{1}{m}, \frac{c-2}{cm} \right\}.$$

Obviously, D is invertible. Computing matrix DBD^{-1}, we have that

$$DBD^{-1} =$$

$$\begin{pmatrix}
0 & \cdots & 0 & \frac{c^{-1}d_1}{d_{m+1}} & 0 & \cdots & 0 & 0 & \frac{-c^{-1}d_1}{d_{2m+3}} & \cdots & 0 & 0 \\
\frac{d_2}{d_1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \frac{d_{m+1}}{d_m} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{c^{-1}d_{m+2}}{d_{2m+2}} & \frac{-c^{-1}d_{m+2}}{d_{2m+3}} & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \frac{d_{m+3}}{d_{m+2}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \frac{d_{2m+2}}{d_{2m+1}} & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \frac{-c^{-1}d_{2m+3}}{d_{m+2}} & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{c^{-1}d_{2m+3}}{d_{3m+3}} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{d_{2m+4}}{d_{2m+3}} & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{d_{3m+3}}{d_{3m+2}} & 0
\end{pmatrix}.$$

The three chains of inequalities

$$\begin{aligned}
1 &= d_1 > d_2 > ... > d_m > d_{m+1} > 0, \\
1 &= d_{m+2} > d_{m+3} > ... > d_{2m+1} > d_{2m+2} > 0, \\
1 &= d_{2m+3} > d_{2m+4} > ... > d_{3m+2} > d_{3m+3} > 0,
\end{aligned}$$

imply that

$$\begin{aligned}
d_2 d_1^{-1} &< 1,\ d_3 d_2^{-1} < 1, ...,\ d_{m+1} d_m^{-1} < 1, \\
d_{m+3} d_{m+2}^{-1} &< 1,\ d_{m+4} d_{m+3}^{-1} < 1, ...,\ d_{2m+2} d_{2m+1}^{-1} < 1, \\
d_{2m+4} d_{2m+3}^{-1} &< 1,\ d_{2m+5} d_{2m+4}^{-1} < 1, ...,\ d_{3m+3} d_{3m+2}^{-1} < 1.
\end{aligned}$$

Also,

$$\begin{aligned}
c^{-1} d_1 d_{m+1}^{-1} + c^{-1} d_1 d_{2m+3}^{-1} &= c^{-1}\left(\frac{1}{1-m\varepsilon} + 1\right) \\
&< c^{-1}\frac{2}{1-m\varepsilon} < 1,
\end{aligned}$$

$$\begin{aligned}
c^{-1} d_{m+2} d_{2m+2}^{-1} + c^{-1} d_{m+2} d_{2m+3}^{-1} &= c^{-1}\left(\frac{1}{1-m\varepsilon} + 1\right) \\
&< c^{-1}\frac{2}{1-m\varepsilon} < 1,
\end{aligned}$$

$$\begin{aligned}
c^{-1} d_{2m+3} d_{m+2}^{-1} + c^{-1} d_{2m+3} d_{3m+3}^{-1} &= c^{-1}\left(1 + \frac{1}{1-m\varepsilon}\right) \\
&< c^{-1}\frac{2}{1-m\varepsilon} < 1.
\end{aligned}$$

Since B has the same eigenvalues as $DBD^{-1} = E = (e_{ij})$, we obtain that

$$
\begin{aligned}
\max_{1\le i\le 3m+3} |\lambda_i| &\le \left\|DBD^{-1}\right\|_\infty \\
&= \max_{1\le i\le 3m+3} \left\{ \sum_{j=1}^{3m+3} |e_{ij}| \right\} \\
&= \max \left\{ \begin{array}{c} d_2 d_1^{-1},\ d_3 d_2^{-1}, \ldots,\ d_{m+1} d_m^{-1}, \\ d_{m+3} d_{m+2}^{-1},\ d_{m+4} d_{m+3}^{-1}, \ldots,\ d_{2m+2} d_{2m+1}^{-1}, \\ d_{2m+4} d_{2m+3}^{-1},\ d_{2m+5} d_{2m+4}^{-1}, \ldots,\ d_{3m+3} d_{3m+2}^{-1}, \\ c^{-1} d_1 d_{m+1}^{-1} + c^{-1} d_1 d_{2m+3}^{-1}, \\ c^{-1} d_{m+2} d_{2m+2}^{-1} + c^{-1} d_{m+2} d_{2m+3}^{-1}, \\ c^{-1} d_{2m+3} d_{m+2}^{-1} + c^{-1} d_{2m+3} d_{3m+3}^{-1} \end{array} \right\} \\
&< 1.
\end{aligned}
$$

This implies that the equilibrium point of system (4.1) is locally asymptotically stable.

Theorem 4.8 *Assume that $A > 1$. Then, the positive equilibrium point of system (4.1) is globally asymptotically stable.*

Proof. Using Theorem 4.5, we have

$$
\begin{aligned}
L_1 &= \lim_{n\to\infty} \sup x_n,\ \ L_2 = \lim_{n\to\infty} \sup y_n,\ \ L_3 = \lim_{n\to\infty} \sup z_n, \\
m_1 &= \lim_{n\to\infty} \inf x_n,\ m_2 = \lim_{n\to\infty} \inf y_n,\ \ m_3 = \lim_{n\to\infty} \inf z_n.
\end{aligned} \tag{4.4}
$$

Then, from (4.1) and (4.4) we have

$$
\begin{aligned}
L_1 &\le A + \frac{L_1}{m_3},\ \ L_2 \le A + \frac{L_2}{m_3},\ \ L_3 \le A + \frac{L_3}{m_2}, \\
m_1 &\ge A + \frac{m_1}{L_3},\ \ m_2 \ge A + \frac{m_2}{L_3},\ \ m_3 \ge A + \frac{m_3}{L_2}.
\end{aligned} \tag{4.5}
$$

Relations (4.5) imply that

$$AL_2 + m_3 \le m_3 L_2 \le Am_3 + L_2,\ \ AL_3 + m_2 \le m_2 L_3 \le Am_2 + L_3,$$

from which we have

$$(A-1)(L_2 - m_3) \le 0,\ \ \ (A-1)(L_3 - m_2) \le 0.$$

Since $A > 1$, we get

$$L_2 \le m_3 \le L_3,\ \ L_3 \le m_2 \le L_2,$$

from which

$$L_2 = L_3 = m_2 = m_3. \tag{4.6}$$

Moreover, from (4.5) it follows that $L_1 m_3 \leq A m_3 + L_1,\ \ m_1 L_3 \leq A L_3 + m_1$, from which

$$L_1 (m_3 - 1) \leq A m_3,\ \ A L_3 \leq m_1 (L_3 - 1).$$

Using (4.6), we have

$$L_1 (L_3 - 1) \leq m_1 (L_3 - 1),$$

from which $L_1 \leq m_1$.
Since x_n is bounded, it implys that $L_1 = m_1$.
Hence, every positive solution $\{(x_n, y_n, z_n)\}$ of system (4.1) tends to the positive equilibrium system (4.1). So, the proof is completed.

CHAPTER 5

THE NATURE OF SOLUTIONS OF A SYSTEM OF SECOND ORDER RATIONAL DIFFERENCE EQUATIONS

We state that the results of this chapter are cited from [68] which was published by us. In the light of the works in [65] and [66], the aim of this chapter is to study local stability of the equilibrium points and oscillation behaviour of positive solutions of the following system of rational difference equations

$$x_{n+1} = A + \frac{x_{n-1}^p}{z_n^p},\ y_{n+1} = A + \frac{y_{n-1}^p}{z_n^p},\ z_{n+1} = A + \frac{z_{n-1}^p}{y_n^p},\ n = 0, 1, ..., \tag{5.1}$$

where $A \in (0, \infty)$, $p \in [1, \infty)$ and the initial values x_i, y_i, $z_i \in (0, \infty)$, $i = -1, 0$.

5.1 MAIN RESULTS

In this section, we prove our main results.

Theorem 5.1 *The following statements are true:*

(i) *The system (5.1) has a positive equilibrium point* $(\overline{x}, \overline{y}, \overline{z}) = (A + 1, A + 1, A + 1)$.

(ii) *If* $A > 2p - 1$*, then the equilibrium point of system (5.1) is locally asymptotically stable.*

(iii) *If* $A < 2p - 1$*, then the equilibrium point of system (5.1) is unstable.*

(iv) *Also, when* $A = 2p - 1$ *and* $p = 1$*, the results has been investigated in [65].*

Proof.

(i) It is easily seen from the definition of equilibrium point.

(ii) We consider the following transformation to build the corresponding linearized form of system (5.1):

$$(x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1}) \to (f, f_1, g, g_1, h, h_1)$$

where

$$\begin{aligned}
f &= A + \frac{x_{n-1}^p}{z_n^p} \\
f_1 &= x_n \\
g &= A + \frac{y_{n-1}^p}{z_n^p} \\
g_1 &= y_n \\
h &= A + \frac{z_{n-1}^p}{y_n^p} \\
h_1 &= z_n.
\end{aligned}$$

The Jacobian matrix about the equilibrium point $(\overline{x}, \overline{y}, \overline{z})$ under the above transformation is given by

$$B(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix}
0 & \frac{p\overline{x}^{p-1}}{\overline{z}^p} & 0 & 0 & -\frac{p\overline{x}^p}{\overline{z}^{p+1}} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{p\overline{y}^{p-1}}{\overline{z}^p} & -\frac{p\overline{y}^p}{\overline{z}^{p+1}} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\frac{p\overline{z}^p}{\overline{y}^{p+1}} & 0 & 0 & \frac{p\overline{z}^{p-1}}{\overline{y}^p} \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}. \tag{5.2}$$

Hence, the linearized system of system (5.1) about the equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (A+1, A+1, A+1)$$

is

$$X_{n+1} = B(\overline{x}, \overline{y}, \overline{z}) X_n,$$

where

$$X_n = (x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1})^T$$

and

$$B(\overline{x},\overline{y},\overline{z})=\begin{pmatrix} 0 & \frac{p}{A+1} & 0 & 0 & -\frac{p}{A+1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{A+1} & -\frac{p}{A+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p}{A+1} & 0 & 0 & \frac{p}{A+1} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic equation of $B(\overline{x},\overline{y},\overline{z})$ about

$(\overline{x},\overline{y},\overline{z})=(A+1,A+1,A+1)$

is

$$\lambda^6-\left(\frac{p^2}{(A+1)^2}+3\frac{p}{A+1}\right)\lambda^4+\left(\frac{p^3}{(A+1)^3}+3\frac{p^2}{(A+1)^2}\right)\lambda^2-\frac{p^3}{(A+1)^3}=0. \quad (5.3)$$

From this, the roots of characteristic equation (5.3) are

$$\begin{aligned}
\lambda_1 &= \sqrt{\frac{p}{A+1}},\\
\lambda_2 &= -\sqrt{\frac{p}{A+1}},\\
\lambda_3 &= -\frac{1}{2A+2}\left(p+\sqrt{p^2+4Ap+4p}\right),\\
\lambda_4 &= \frac{1}{2A+2}\left(-p+\sqrt{p^2+4Ap+4p}\right),\\
\lambda_5 &= \frac{1}{2A+2}\left(p+\sqrt{p^2+4Ap+4p}\right),\\
\lambda_6 &= \frac{1}{2A+2}\left(p-\sqrt{p^2+4Ap+4p}\right).
\end{aligned}$$

From the Linearized Stability Theorem, since $A>2p-1$, all roots of the characterictic equation lie inside the open unit disk $|\lambda|<1$. Therefore, the positive equilibrium point of system (5.1) is locally asymptotically stable.

(iii) From the proof of (ii), it is true.

Theorem 5.2 *Let $0<A<1$ and $\{(x_n,y_n,z_n)\}$ be an arbitrary positive solution of system (5.1). Then, the following statements are true.*

(i) *If*

$$\begin{aligned} 0 &< x_{-1} < 1,\ 0 < y_{-1} < 1,\ 0 < z_{-1} < 1, \\ x_0 &\geq \frac{1}{(1-A)^{1/p}},\ y_0 \geq \frac{1}{(1-A)^{1/p}},\ z_0 \geq \frac{1}{(1-A)^{1/p}}, \end{aligned} \tag{5.4}$$

then

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= A,\ \lim_{n\to\infty} y_{2n+1} = A,\ \lim_{n\to\infty} z_{2n+1} = A, \\ \lim_{n\to\infty} x_{2n} &= \infty,\ \lim_{n\to\infty} y_{2n} = \infty,\ \lim_{n\to\infty} z_{2n} = \infty. \end{aligned}$$

(ii) *If*

$$\begin{aligned} 0 &< x_0 < 1,\ 0 < y_0 < 1,\ 0 < z_0 < 1, \\ x_{-1} &\geq \frac{1}{(1-A)^{1/p}},\ y_{-1} \geq \frac{1}{(1-A)^{1/p}},\ z_{-1} \geq \frac{1}{(1-A)^{1/p}}, \end{aligned} \tag{5.5}$$

then

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= \infty,\ \lim_{n\to\infty} y_{2n+1} = \infty,\ \lim_{n\to\infty} z_{2n+1} = \infty, \\ \lim_{n\to\infty} x_{2n} &= A,\ \lim_{n\to\infty} y_{2n} = A,\ \lim_{n\to\infty} z_{2n} = A. \end{aligned}$$

Proof. **(i)** From system (5.1) and (5.4), we have

$$\begin{aligned} x_1 &= A + \frac{x_{-1}^p}{z_0^p} \leq A + \frac{1}{z_0^p} \leq A + (1-A) = 1, \\ y_1 &= A + \frac{y_{-1}^p}{z_0^p} \leq A + \frac{1}{z_0^p} \leq A + (1-A) = 1, \\ z_1 &= A + \frac{z_{-1}^p}{y_0^p} \leq A + \frac{1}{y_0^p} \leq A + (1-A) = 1, \\ x_1 &= A + \frac{x_{-1}^p}{z_0^p} > A, \\ y_1 &= A + \frac{y_{-1}^p}{z_0^p} > A, \\ z_1 &= A + \frac{z_{-1}^p}{y_0^p} > A. \end{aligned}$$

Hence,

$x_1 \in (A, 1]\,,\ y_1 \in (A, 1]\,,\ z_1 \in (A, 1]\,.$

Also,

$$\begin{aligned}
x_2 &= A+\frac{x_0^p}{z_1^p} \geq A+x_0^p,\\
y_2 &= A+\frac{y_0^p}{z_1^p} \geq A+y_0^p,\\
z_2 &= A+\frac{z_0^p}{y_1^p} \geq A+z_0^p.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
x_3 &= A+\frac{x_1^p}{z_2^p} \leq A+\frac{1}{(A+z_0^p)^p} \leq A+\frac{1}{A+z_0^p} \leq A+\frac{1}{z_0^p} \leq A+(1-A)=1,\\
y_3 &= A+\frac{y_1^p}{z_2^p} \leq A+\frac{1}{(A+z_0^p)^p} \leq A+\frac{1}{A+z_0^p} \leq A+\frac{1}{z_0^p} \leq A+(1-A)=1,\\
z_3 &= A+\frac{z_1^p}{y_2^p} \leq A+\frac{1}{(A+y_0^p)^p} \leq A+\frac{1}{A+y_0^p} \leq A+\frac{1}{y_0^p} \leq A+(1-A)=1.
\end{aligned}$$

Thus,

$$x_3 \in (A,1]\,,\ y_3 \in (A,1]\,,\ z_3 \in (A,1]\,.$$

Also,

$$\begin{aligned}
x_4 &= A+\frac{x_2^p}{z_3^p} \geq A+x_2^p \geq A+(A+x_0^p)^p \geq A+(A+x_0^p)=2A+x_0^p,\\
y_4 &= A+\frac{y_2^p}{z_3^p} \geq A+y_2^p \geq A+(A+y_0^p)^p \geq A+(A+y_0^p)=2A+y_0^p,\\
z_4 &= A+\frac{z_2^p}{y_3^p} \geq A+z_2^p \geq A+(A+z_0^p)^p \geq A+(A+z_0^p)=2A+z_0^p.
\end{aligned}$$

By induction for $n=1,2,...$, we obtain

$$\begin{aligned}
A &< x_{2n-1}<1,\ \ A<y_{2n-1}<1,\ A<z_{2n-1}<1, \qquad (5.6)\\
x_{2n} &\geq nA+x_0^p,\ y_{2n}\geq nA+y_0^p,\ \ z_{2n}\geq nA+z_0^p.
\end{aligned}$$

From system (5.1) and (5.6), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \infty,\ \lim_{n\to\infty} y_{2n}=\infty,\ \ \lim_{n\to\infty} z_{2n}=\infty,\\
\lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty}\left(A+\frac{x_{2n-1}^p}{z_{2n}^p}\right)=A,\\
\lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty}\left(A+\frac{y_{2n-1}^p}{z_{2n}^p}\right)=A,\\
\lim_{n\to\infty} z_{2n+1} &= \lim_{n\to\infty}\left(A+\frac{z_{2n-1}^p}{y_{2n}^p}\right)=A.
\end{aligned}$$

(ii) The proof is similar to the proof of (i), so we omit it. The proof is completed.

Theorem 5.3 *Let $\{(x_n, y_n, z_n)\}$ be a positive solution of system (5.1) which consists of at least two semicycles. Then $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ is oscillatory.*

Proof. Since $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ has at least two semicycles, there exists $N \geq 0$ such that either

$$\begin{aligned} x_{N-1} &< A+1 \leq x_N, \\ y_{N-1} &< A+1 \leq y_N, \\ z_{N-1} &< A+1 \leq z_N, \end{aligned} \tag{5.7}$$

or

$$\begin{aligned} x_N &< A+1 \leq x_{N-1}, \\ y_N &< A+1 \leq y_{N-1}, \\ z_N &< A+1 \leq z_{N-1}. \end{aligned} \tag{5.8}$$

First, we suppose the case (5.7). Then

$$\begin{aligned} x_{N+1} &= A + \frac{x_{N-1}^p}{z_N^p} < A+1, \\ y_{N+1} &= A + \frac{y_{N-1}^p}{z_N^p} < A+1, \\ z_{N+1} &= A + \frac{z_{N-1}^p}{y_N^p} < A+1, \\ x_{N+2} &= A + \frac{x_N^p}{z_{N+1}^p} > A+1, \\ y_{N+2} &= A + \frac{y_N^p}{z_{N+1}^p} > A+1, \\ z_{N+2} &= A + \frac{z_N^p}{y_{N+1}^p} > A+1. \end{aligned}$$

So, we have

$$\begin{aligned} x_{N+1} &< A+1 < x_{N+2}, \\ y_{N+1} &< A+1 < y_{N+2}, \\ z_{N+1} &< A+1 < z_{N+2}. \end{aligned}$$

Last, we suppose the case (5.8). The case is similar to the first case, so we leave it to readers.

CHAPTER 6

THE EXACT SOLUTIONS OF FOUR RATIONAL DIFFERENCE EQUATIONS ASSOCIATED TO TRIBONACCI NUMBERS

Initially, we express that the results of this chapter are cited from [101] which has been published by us.

In Chapter 6 and Chapter 7, we discuss eight cases (eight distinct difference equations) of the following difference equation

$$x_{n+1} = \frac{\pm 1}{x_n (x_{n-1} \pm 1) \pm 1}.$$

We study the four cases of eight cases in this chapter and the remaining four cases in Chapter 7.

As far as we examine, there is no paper dealing with the following difference equations. Hence, in this chapter, we investigate the form of solutions, stability character and asymptotic behavior of the following four rational difference equations

$$x_{n+1} = \frac{1}{x_n (x_{n-1} - 1) - 1}, \quad n = 0, 1, ..., \tag{6.1}$$

$$x_{n+1} = \frac{1}{x_n (x_{n-1} + 1) + 1}, \quad n = 0, 1, ..., \tag{6.2}$$

$$x_{n+1} = \frac{-1}{x_n (x_{n-1} - 1) + 1}, \quad n = 0, 1, ..., \tag{6.3}$$

$$x_{n+1} = \frac{-1}{x_n (x_{n-1} + 1) - 1}, \quad n = 0, 1, ..., \tag{6.4}$$

such that their solutions are associated with Tribonacci numbers.

Our aim in this chapter is to investigate some relationships both between Tribonacci numbers and solutions of above mentioned difference equations and between the Tribonacci constant and the equilibrium points of these difference equations.

6.1 MAIN RESULTS

In this section, we present our main results considering above mentioned difference equations. Our aim is to investigate the general solutions in explicit form of difference equations and the asymptotic behavior of solutions of difference equations.

6.1.1 The Difference Equation (6.1)

In this subsection, we consider the Eq.(6.1), that is,

$$x_{n+1} = \frac{1}{x_n\left(x_{n-1}-1\right)-1}, \quad n = 0, 1, ...,$$

and investigate the dynamical behavior of solutions of Eq.(6.1).

Theorem 6.1 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.1). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{T_{-n}x_{-1}x_0 + \left(T_{-(n+1)} + T_{-(n+2)}\right)x_0 + T_{-(n+1)}}{T_{-(n+1)}x_{-1}x_0 + \left(T_{-n} - T_{-(n+1)}\right)x_0 + T_{-(n+2)}}, \tag{6.5}$$

where T_n is the nth Tribonacci number and the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_1$, with F_1 is the forbidden set of Eq.(6.1) given by

$$F_1 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : T_{-(n+1)}x_{-1}x_0 + \left(T_{-n} - T_{-(n+1)}\right)x_0 + T_{-(n+2)} = 0\right\}.$$

Proof. We will prove this theorem by induction on k. For $k = 0$, from Eq.(6.1),

$$x_1 = \frac{1}{x_0\left(x_{-1}-1\right)-1} = \frac{1}{x_{-1}x_0 - x_0 - 1} = \frac{T_{-1}x_{-1}x_0 + \left(T_{-2}+T_{-3}\right)x_0 + T_{-2}}{T_{-2}x_{-1}x_0 + \left(T_{-1}-T_{-2}\right)x_0 + T_{-3}}.$$

Now, we assume that

$$x_k = \frac{T_{-k}x_{-1}x_0 + \left(T_{-(k+1)} + T_{-(k+2)}\right)x_0 + T_{-(k+1)}}{T_{-(k+1)}x_{-1}x_0 + \left(T_{-k} - T_{-(k+1)}\right)x_0 + T_{-(k+2)}} \tag{6.6}$$

is true for all $1 \leq n \leq k$. Hence, we have to prove that it is true for $k+1$. Taking into account (2.12) and (6.6), we have

$$\begin{aligned}
& x_{k+1} \\
= & \frac{1}{x_k\left(x_{k-1}-1\right)-1} \\
= & \frac{1}{\left(\frac{T_{-k}x_{-1}x_0+\left(T_{-(k+1)}+T_{-(k+2)}\right)x_0+T_{-(k+1)}}{T_{-(k+1)}x_{-1}x_0+\left(T_{-k}-T_{-(k+1)}\right)x_0+T_{-(k+2)}}\right)\left(\frac{T_{-(k-1)}x_{-1}x_0+\left(T_{-k}+T_{-(k+1)}\right)x_0+T_{-k}}{T_{-k}x_{-1}x_0+\left(T_{-(k-1)}-T_{-k}\right)x_0+T_{-(k+1)}}-1\right)-1} \\
= & \frac{T_{-(k+1)}x_{-1}x_0+\left(T_{-k}-T_{-(k+1)}\right)x_0+T_{-(k+2)}}{\left(T_{-(k-1)}-T_{-k}-T_{-(k+1)}\right)x_{-1}x_0+\left(T_{-(k+1)}-T_{-(k+2)}\right)x_0+T_{-k}-T_{-(k+1)}-T_{-(k+2)}} \\
= & \frac{T_{-(k+1)}x_{-1}x_0+\left(T_{-(k+2)}+T_{-(k+3)}\right)x_0+T_{-(k+2)}}{T_{-(k+2)}x_{-1}x_0+\left(T_{-(k+1)}-T_{-(k+2)}\right)x_0+T_{-(k+3)}},
\end{aligned}$$

which ends the induction and the proof.

Theorem 6.2 *Eq.(6.1) has unique positive equilibrium point $\overline{x}=\alpha$ and α is saddle point.*

Proof. Equilibrium point of Eq.(6.1) satisfy the equation

$$\overline{x}=\frac{1}{\overline{x}\left(\overline{x}-1\right)-1}.$$

After simplification, we get the following cubic equation

$$\overline{x}^3-\overline{x}^2-\overline{x}-1=0. \tag{6.7}$$

The cubic equation (6.7) is the characteristic equation of the recurrence relation of the Tribonacci numbers in (2.13) having the unique real root α. Therefore, the unique positive equilibrium point of Eq.(6.1) is $\overline{x}=\alpha$.

Now, we indicate that the equilibrium point of Eq.(6.1) is saddle point.

Let I be an interval of real numbers and

$$f: I^2 \rightarrow I$$

be a continuous function defined by

$$f\left(x,y\right)=\frac{1}{x\left(y-1\right)-1}.$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f\left(x,y\right)}{\partial x} &= \frac{-\left(y-1\right)}{\left(x\left(y-1\right)-1\right)^2}, \\
\frac{\partial f\left(x,y\right)}{\partial y} &= \frac{-x}{\left(x\left(y-1\right)-1\right)^2}.
\end{aligned}$$

Then, from (6.7)

$$
\begin{aligned}
\frac{\partial f(\overline{x},\overline{x})}{\partial x} &= \frac{-(\alpha-1)}{(\alpha(\alpha-1)-1)^2} \\
&= \frac{1-\alpha}{(\alpha^2-\alpha-1)^2} \\
&= \frac{1-\alpha}{\left(\frac{1}{\alpha}\right)^2} \\
&= \alpha^2-\alpha^3 \\
&= -(\alpha+1), \\
\frac{\partial f(\overline{x},\overline{x})}{\partial y} &= \frac{-\alpha}{(\alpha(\alpha-1)-1)^2} \\
&= \frac{-\alpha}{(\alpha^2-\alpha-1)^2} \\
&= \frac{-\alpha}{\left(\frac{1}{\alpha}\right)^2} \\
&= -\alpha^3,
\end{aligned}
$$

and the linearized equation of Eq.(6.1) about $\overline{x}=\alpha$ is

$$z_{n+1} = -(\alpha+1)z_n + \left(-\alpha^3\right)z_{n-1}$$

or equivalently

$$z_{n+1} + (\alpha+1)z_n + \alpha^3 z_{n-1} = 0.$$

Therefore, the corresponding characteristic polynomial is

$$\lambda^2 + (\alpha+1)\lambda + \alpha^3 = 0.$$

Then, from Theorem (2.1), it is clearly seen that

$$\lambda_{1,2} = \frac{-(\alpha+1) \pm \sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{2}$$

and numerically

$$
\begin{aligned}
|\lambda_1| &= \left|\frac{-(\alpha+1) + \sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{2}\right| = 0,11228 < 1 \\
|\lambda_2| &= \left|\frac{-(\alpha+1) - \sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{2}\right| = 1,4314 > 1.
\end{aligned}
$$

So, the equilibrium point α is a saddle point. This completes the proof.

6.1.2 The Difference Equation (6.2)

In this subsection, we study the Eq.(6.2), that is,

$$x_{n+1} = \frac{1}{x_n\left(x_{n-1}+1\right)+1}, \quad n = 0, 1, ...,$$

and examine the dynamical behavior of solutions of Eq.(6.2).

Theorem 6.3 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.2). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{T_{n-1}x_{-1}x_0 + \left(T_{n+1} - T_n\right)x_0 + T_n}{T_n x_{-1}x_0 + \left(T_{n-1} + T_n\right)x_0 + T_{n+1}}, \tag{6.8}$$

where T_n is the nth Tribonacci number and the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_2$, with F_2 is the forbidden set of Eq.(6.2) given by

$$F_2 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : T_n x_{-1}x_0 + \left(T_{n-1} + T_n\right)x_0 + T_{n+1} = 0\right\}.$$

Proof. (**First proof**) Now, we give the proof with an analytic approach. We make the substitution

$$x_n = \frac{t_{n-1}}{t_n} \tag{6.9}$$

in Eq.(6.2) to get the linear difference equation. Then, we have

$$t_{n+1} = t_n + t_{n-1} + t_{n-2}.$$

By using same operations in Theorem 2.1 in [86] such that $a = b = c = 1$, we obtain the initial values of three sequences are defined

$$\begin{aligned} a_n &= aa_{n-1} + ba_{n-2} + ca_{n-3}, \\ b_n &= a_{n+1} - aa_n, \\ c_n &= ca_{n_{-1}}, \end{aligned}$$

such that

$$\begin{array}{lll} a_0 = 1, & a_{-1} = 0, & a_{-2} = 0, \\ b_0 = 0, & b_{-1} = 1, & b_{-2} = 0, \\ c_0 = 0, & c_{-1} = 0, & c_{-2} = 1. \end{array} \tag{6.10}$$

Next, we get

$$t_n = a_n t_0 + (a_{n+1} - a a_n)\, t_{-1} + c a_{n_{-1}} t_{-2}.$$

So $a = b = c = 1$ and from (6.9), we obtain

$$x_n = \frac{a_{n-2} x_{-1} x_0 + (a_n - a_{n-1})\, x_0 + a_{n-1}}{a_{n-1} x_{-1} x_0 + (a_{n+1} - a_n)\, x_0 + a_n}$$

or equivalently

$$x_n = \frac{a_{n-2} x_{-1} x_0 + (a_n - a_{n-1})\, x_0 + a_{n-1}}{a_{n-1} x_{-1} x_0 + (a_{n-1} + a_{n-2})\, x_0 + a_n}.$$

From initial values (6.10) and definitions of sequences a_n and T_n, we have

$$a_n = T_{n+1},$$

with the backward shifted initial values of the sequence a_n. Hence, we obtain

$$x_n = \frac{T_{n-1} x_{-1} x_0 + (T_{n+1} - T_n)\, x_0 + T_n}{T_n x_{-1} x_0 + (T_{n-1} + T_n)\, x_0 + T_{n+1}}.$$

So, the proof is complete.

Proof. (**Second proof**) We will prove this theorem by induction on k. For $k = 0$, from Eq.(6.2),

$$x_1 = \frac{1}{x_0 (x_{-1} + 1) + 1} = \frac{1}{x_{-1} x_0 + x_0 + 1} = \frac{T_0 x_{-1} x_0 + (T_2 - T_1)\, x_0 + T_1}{T_1 x_{-1} x_0 + (T_0 + T_1)\, x_0 + T_2}.$$

Now, we assume that

$$x_k = \frac{T_{k-1} x_{-1} x_0 + (T_{k+1} - T_k)\, x_0 + T_k}{T_k x_{-1} x_0 + (T_{k-1} + T_k)\, x_0 + T_{k+1}}, \tag{6.11}$$

is true for all $1 \leq n \leq k$. Hence, we have to prove that it is true for $k+1$. Taking into account (2.11) and (6.11), we have

$$\begin{aligned}
x_{k+1} &= \frac{1}{x_k (x_{k-1} + 1) + 1} \\
&= \frac{1}{\left(\frac{T_{k-1} x_{-1} x_0 + (T_{k+1} - T_k) x_0 + T_k}{T_k x_{-1} x_0 + (T_{k-1} + T_k) x_0 + T_{k+1}}\right)\left(\frac{T_{k-2} x_{-1} x_0 + (T_k - T_{k-1}) x_0 + T_{k-1}}{T_{k-1} x_{-1} x_0 + (T_{k-2} + T_{k-1}) x_0 + T_k} + 1\right) + 1} \\
&= \frac{T_k x_{-1} x_0 + (T_{k-1} + T_k)\, x_0 + T_{k+1}}{(T_{k-2} + T_{k-1} + T_k)\, x_{-1} x_0 + (T_k + T_{k+1})\, x_0 + T_{k-1} + T_k + T_{k+1}} \\
&= \frac{T_k x_{-1} x_0 + (T_{k+2} - T_{k+1})\, x_0 + T_{k+1}}{T_{k+1} x_{-1} x_0 + (T_k + T_{k+1})\, x_0 + T_{k+2}},
\end{aligned}$$

which ends the induction and the proof.

Proof. (**Third proof**) Consider Eq.(6.2) by taking $n = 0, 1, 2, ...$ as follows:

$$\begin{aligned}
n = 0 \quad &\Rightarrow \quad x_1 = \tfrac{1}{x_{-1}x_0 + x_0 + 1}, \\
n = 1 \quad &\Rightarrow \quad x_2 = \tfrac{x_{-1}x_0 + x_0 + 1}{x_{-1}x_0 + 2x_0 + 2}, \\
n = 2 \quad &\Rightarrow \quad x_3 = \tfrac{x_{-1}x_0 + 2x_0 + 2}{2x_{-1}x_0 + 3x_0 + 4}, \\
n = 3 \quad &\Rightarrow \quad x_4 = \tfrac{2x_{-1}x_0 + 3x_0 + 4}{4x_{-1}x_0 + 6x_0 + 7}, \\
n = 4 \quad &\Rightarrow \quad x_5 = \tfrac{4x_{-1}x_0 + 6x_0 + 7}{7x_{-1}x_0 + 11x_0 + 13}, \\
n = 5 \quad &\Rightarrow \quad x_6 = \tfrac{7x_{-1}x_0 + 11x_0 + 13}{13x_{-1}x_0 + 20x_0 + 24}, \\
&\quad \ldots
\end{aligned}$$

If we keep on this process and also regard (2.11), then the solution in (9.6) directly follows from a simple induction.

Theorem 6.4 *Eq.(6.2) has unique positive equilibrium point $\overline{x} = a$ and a is locally asymptotically stable.*

Proof. Equilibrium point of Eq.(6.2) is the real roots of the equation

$$\overline{x} = \frac{1}{\overline{x}\left(\overline{x} + 1\right) + 1}.$$

After simplification, we get the following cubic equation

$$\overline{x}^3 + \overline{x}^2 + \overline{x} - 1 = 0. \tag{6.12}$$

Then, the roots of the cubic equation (6.12) are given by

$$\begin{aligned}
a &= \frac{-1 + \sqrt[3]{3\sqrt{33} + 17} - \sqrt[3]{3\sqrt{33} - 17}}{3}, \\
b &= \frac{-1 + \omega\sqrt[3]{3\sqrt{33} + 17} - \omega^2\sqrt[3]{3\sqrt{33} - 17}}{3}, \\
c &= \frac{-1 + \omega^2\sqrt[3]{3\sqrt{33} + 17} - \omega\sqrt[3]{3\sqrt{33} - 17}}{3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(2\pi i/3\right)$$

is a primitive cube root of unity. So, the root a is only real number. Therefore, the unique positive equilibrium point of Eq.(6.2) is $\overline{x} = a$.

Now, we show that the equilibrium point of Eq.(6.2) is locally asymptotically stable.

Let I be an interval of real numbers and consider the function

$f : I^2 \rightarrow I$

defined by

$$f(x, y) = \frac{1}{x(y+1)+1}.$$

The linearized equation of Eq.(6.2) about the equilibrium point $\overline{x} = a$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$\begin{aligned} p &= \frac{\partial f(\overline{x}, \overline{x})}{\partial x} = \frac{-(a+1)}{(a(a+1)+1)^2} = a - 1, \\ q &= \frac{\partial f(\overline{x}, \overline{x})}{\partial y} = \frac{-a}{(a(a+1)+1)^2} = -a^3, \end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 + (1-a)\lambda + a^3 = 0.$$

Therefore, from Theorem 2.1, it is easily seen that

$$\lambda_{1,2} = \frac{a - 1 \pm \sqrt{-4a^3 + a^2 - 2a + 1}}{2}$$

and numerically

$$|\lambda_1| = |\lambda_2| = 0,40089 < 1.$$

This completes the proof.

Theorem 6.5 *The equilibrium point of Eq.(6.2) is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n \geq -1}$ be a solution of Eq.(6.2). By Theorem 6.4, we need only to prove that the equilibrium point a is global attractor, that is

$$\lim_{n\to\infty} x_n = a.$$

From Theorem 6.3 and (2.13) and (2.14), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{T_{n-1}x_{-1}x_0 + (T_{n+1} - T_n)\, x_0 + T_n}{T_n x_{-1}x_0 + (T_{n-1} + T_n)\, x_0 + T_{n+1}} \\
&= \lim_{n\to\infty} \frac{T_{n-1}\left(x_{-1}x_0 + \left(\frac{T_{n+1}}{T_{n-1}} - \frac{T_n}{T_{n-1}}\right) x_0 + \frac{T_n}{T_{n-1}}\right)}{T_n\left(x_{-1}x_0 + \left(\frac{T_{n-1}}{T_n} + 1\right) x_0 + \frac{T_{n+1}}{T_n}\right)} \\
&= \left(\frac{x_{-1}x_0 + (\alpha^2 - \alpha)\, x_0 + \alpha}{x_{-1}x_0 + \left(\frac{1}{\alpha} + 1\right) x_0 + \alpha}\right) \lim_{n\to\infty} \frac{T_{n-1}}{T_n} \\
&= \lim_{n\to\infty} \frac{T_{n-1}}{T_n} \\
&= \frac{1}{\alpha} \\
&= a.
\end{aligned}$$

The proof is complete.

6.1.3 The Difference Equation (6.3)

In this subsection, we take into account the Eq.(6.3), that is,

$$x_{n+1} = \frac{-1}{x_n\,(x_{n-1} - 1) + 1}, \quad n = 0, 1, ...,$$

and analyze the dynamical behavior of solutions of Eq.(6.3).

Theorem 6.6 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.3). Then, for $n = 0, 1, 2, ...,$ the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{-\left(T_{n-1}x_{-1}x_0 + (T_n - T_{n+1})\, x_0 + T_n\right)}{T_n x_{-1}x_0 - (T_{n-1} + T_n)\, x_0 + T_{n+1}}, \tag{6.13}$$

where T_n is the nth Tribonacci number and the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_3$, with F_3 is the forbidden set of Eq.(6.3) given by

$$F_3 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : T_n x_{-1}x_0 - (T_{n-1} + T_n)\, x_0 + T_{n+1} = 0\right\}.$$

Proof. (**First proof**) We will prove this theorem by induction on k. For $k = 0$, from Eq.(6.3),

$$x_1 = \frac{-1}{x_0\,(x_{-1} - 1) + 1} = \frac{-1}{x_{-1}x_0 - x_0 + 1} = \frac{-\left(T_0 x_{-1}x_0 + (T_1 - T_2)\, x_0 + T_1\right)}{T_1 x_{-1}x_0 - (T_0 + T_1)\, x_0 + T_2}.$$

Now, we assume that

$$x_k = \frac{-\left(T_{k-1}x_{-1}x_0 + (T_k - T_{k+1})\, x_0 + T_k\right)}{T_k x_{-1}x_0 - (T_{k-1} + T_k)\, x_0 + T_{k+1}}, \tag{6.14}$$

is true for all $1 \leq n \leq k$. Hence, we have to prove that it is true for $k+1$. Taking into account (2.11) and (6.14), we have

$$\begin{aligned} x_{k+1} &= \frac{-1}{x_k\left(x_{k-1}-1\right)+1} \\ &= \frac{-1}{\left(\frac{-(T_{k-1}x_{-1}x_0+(T_k-T_{k+1})x_0+T_k)}{T_kx_{-1}x_0-(T_{k-1}+T_k)x_0+T_{k+1}}\right)\left(\frac{-(T_{k-2}x_{-1}x_0+(T_{k-1}-T_k)x_0+T_{k-1})}{T_{k-1}x_{-1}x_0-(T_{k-2}+T_{k-1})x_0+T_k}-1\right)+1} \\ &= \frac{-\left(T_kx_{-1}x_0-\left(T_{k-1}+T_k\right)x_0+T_{k+1}\right)}{\left(T_{k-2}+T_{k-1}+T_k\right)x_{-1}x_0-\left(T_k+T_{k+1}\right)x_0+T_{k-1}+T_k+T_{k+1}} \\ &= \frac{-\left(T_kx_{-1}x_0+\left(T_{k+1}-T_{k+2}\right)x_0+T_{k+1}\right)}{T_{k+1}x_{-1}x_0-\left(T_k+T_{k+1}\right)x_0+T_{k+2}}, \end{aligned}$$

which ends the induction and the proof.

Proof. (**Second proof**) Consider Eq.(6.3) by taking $n = 0, 1, 2, ...$ as follows:

$$\begin{aligned} n=0 \quad &\Rightarrow \quad x_1 = \frac{-1}{x_{-1}x_0-x_0+1}, \\ n=1 \quad &\Rightarrow \quad x_2 = \frac{-(x_{-1}x_0-x_0+1)}{x_{-1}x_0-2x_0+2}, \\ n=2 \quad &\Rightarrow \quad x_3 = \frac{-(x_{-1}x_0-2x_0+2)}{2x_{-1}x_0-3x_0+4}, \\ n=3 \quad &\Rightarrow \quad x_4 = \frac{-(2x_{-1}x_0-3x_0+4)}{4x_{-1}x_0-6x_0+7}, \\ n=4 \quad &\Rightarrow \quad x_5 = \frac{-(4x_{-1}x_0-6x_0+7)}{7x_{-1}x_0-11x_0+13}, \\ n=5 \quad &\Rightarrow \quad x_6 = \frac{-(7x_{-1}x_0-11x_0+13)}{13x_{-1}x_0-20x_0+24}, \\ &\cdots \end{aligned}$$

If we keep on this process and also regard (2.11), then the solution in (6.13) directly follows from a simple induction.

Theorem 6.7 *Eq.(6.3) has unique negative equilibrium point* $\overline{x} = d$ *and* d *is locally asymptotically stable.*

Proof. Equilibrium point of Eq.(6.3) is the real roots of the equation

$$\overline{x} = \frac{-1}{\overline{x}\left(\overline{x}-1\right)+1}.$$

After simplification, we get the following cubic equation

$$\overline{x}^3 - \overline{x}^2 + \overline{x} + 1 = 0. \tag{6.15}$$

Then, the roots of the cubic equation (6.15) are given by

$$\begin{aligned} d &= \frac{1+\sqrt[3]{3\sqrt{33}-17}-\sqrt[3]{3\sqrt{33}+17}}{3}, \\ e &= \frac{1+\omega\sqrt[3]{3\sqrt{33}-17}-\omega^2\sqrt[3]{3\sqrt{33}+17}}{3}, \\ f &= \frac{1+\omega^2\sqrt[3]{3\sqrt{33}-17}-\omega\sqrt[3]{3\sqrt{33}+17}}{3}, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(2\pi i/3\right)$$

is a primitive cube root of unity. So, the root d is only real number. Therefore, the unique negative equilibrium point of Eq.(6.3) is $\overline{x} = d$.

Now, we show that the unique negative equilibrium point of Eq.(6.3) is locally asymptotically stable.

Let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f(x, y) = \frac{-1}{x(y-1)+1}.$$

The linearized equation of Eq.(6.3) about the equilibrium point $\overline{x} = d$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where, from (6.15),

$$\begin{aligned} p &= \frac{\partial f(\overline{x}, \overline{x})}{\partial x} = \frac{d-1}{\left(d(d-1)+1\right)^2} = \frac{d-1}{\left(d^2-d+1\right)^2} = \frac{d-1}{\left(-\frac{1}{d}\right)^2} = -(d+1), \\ q &= \frac{\partial f(\overline{x}, \overline{x})}{\partial y} = \frac{d}{\left(d(d-1)+1\right)^2} = \frac{d}{\left(d^2-d+1\right)^2} = \frac{d}{\left(-\frac{1}{d}\right)^2} = d^3, \end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 + (d+1)\lambda - d^3 = 0.$$

Therefore, from Theorem 2.1, it is easily seen that

$$\lambda_{1,2} = \frac{-(d+1) \pm \sqrt{4d^3 + d^2 + 2d + 1}}{2}$$

and numerically

$$|\lambda_1| = |\lambda_2| = 0,40089 < 1.$$

So, this completes the proof.

Theorem 6.8 *The equilibrium point of Eq.(6.3) is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n\geq -1}$ be a solution of Eq.(6.3). By Theorem 6.7, we need only to prove that the equilibrium point d is global attractor, that is

$$\lim_{n\to\infty} x_n = d.$$

From Theorem 6.6 and (2.13) and (2.14), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{-\left(T_{n-1}x_{-1}x_0 + \left(T_n - T_{n+1}\right)x_0 + T_n\right)}{T_n x_{-1}x_0 - \left(T_{n-1}+T_n\right)x_0 + T_{n+1}} \\
&= \lim_{n\to\infty} \frac{-\left(T_{n-1}\left(x_{-1}x_0 + \left(\frac{T_n}{T_{n-1}} - \frac{T_{n+1}}{T_{n-1}}\right)x_0 + \frac{T_n}{T_{n-1}}\right)\right)}{T_n\left(x_{-1}x_0 - \left(\frac{T_{n-1}}{T_n}+1\right)x_0 + \frac{T_{n+1}}{T_n}\right)} \\
&= \left(\frac{x_{-1}x_0 + \left(\alpha-\alpha^2\right)x_0 + \alpha}{x_{-1}x_0 - \left(\frac{1}{\alpha}+1\right)x_0 + \alpha}\right)\lim_{n\to\infty}\frac{-T_{n-1}}{T_n} \\
&= \lim_{n\to\infty}\frac{-T_{n-1}}{T_n} \\
&= -\frac{1}{\alpha} \\
&= d.
\end{aligned}$$

The proof is complete.

6.1.4 The Difference Equation (6.4)

In this subsection, we take into account the Eq.(6.4), that is,

$$x_{n+1} = \frac{-1}{x_n\left(x_{n-1}+1\right)-1}, \quad n = 0, 1, ...,$$

and analyze the dynamical behavior of solutions of Eq.(6.4).

Theorem 6.9 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(6.4). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=0-1}^{\infty}$ is given by*

$$x_n = \frac{-\left(T_{-n}x_{-1}x_0 - \left(T_{-(n+1)} + T_{-(n+2)}\right)x_0 + T_{-(n+1)}\right)}{T_{-(n+1)}x_{-1}x_0 + \left(T_{-(n+1)} - T_{-n}\right)x_0 + T_{-(n+2)}}, \tag{6.16}$$

where T_n is the nth Tribonacci number and the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_4$, with F_4 is the forbidden set of Eq.(6.4) given by

$$F_4 = \bigcup_{n=-1}^{\infty}\left\{(x_{-1}, x_0) : T_{-(n+1)}x_{-1}x_0 + \left(T_{-(n+1)} - T_{-n}\right)x_0 + T_{-(n+2)} = 0\right\}.$$

Proof. We will prove this theorem by induction on k. For $k=0$, from Eq.(6.4),

$$x_1 = \frac{-1}{x_0\left(x_{-1}+1\right)-1} = \frac{-1}{x_{-1}x_0+x_0-1} = \frac{-\left(T_{-1}x_{-1}x_0\left(T_{-2}+T_{-3}\right)x_0+T_{-2}\right)}{T_{-2}x_{-1}x_0+\left(T_{-2}-T_{-1}\right)x_0+T_{-3}}.$$

Now, we assume that

$$x_k = \frac{-\left(T_{-k}x_{-1}x_0-\left(T_{-(k+1)}+T_{-(k+2)}\right)x_0+T_{-(k+1)}\right)}{T_{-(k+1)}x_{-1}x_0+\left(T_{-(k+1)}-T_{-k}\right)x_0+T_{-(k+2)}} \tag{6.17}$$

is true for all $1 \leq n \leq k$. Hence, we have to prove that it is true for $k+1$. Taking into account (2.12) and (6.17), we have

$$\begin{aligned}
& x_{k+1} \\
= & \frac{-1}{x_k\left(x_{k-1}+1\right)-1} \\
= & \frac{-1}{\left(\frac{-\left(T_{-k}x_{-1}x_0-\left(T_{-(k+1)}+T_{-(k+2)}\right)x_0+T_{-(k+1)}\right)}{T_{-(k+1)}x_{-1}x_0+\left(T_{-(k+1)}-T_{-k}\right)x_0+T_{-(k+2)}}\right)\left(\frac{-\left(T_{-(k-1)}x_{-1}x_0-\left(T_{-k}+T_{-(k+1)}\right)x_0+T_{-k}\right)}{T_{-k}x_{-1}x_0+\left(T_{-k}-T_{-(k-1)}\right)x_0+T_{-(k+1)}}+1\right)-1} \\
= & \frac{-\left(T_{-(k+1)}x_{-1}x_0-\left(T_{-(k+2)}+T_{-(k+3)}\right)x_0+T_{-(k+2)}\right)}{\left(T_{-(k-1)}-T_{-k}-T_{-(k+1)}\right)x_{-1}x_0+\left(T_{-(k+2)}-T_{-(k+1)}\right)x_0+T_{-k}-T_{-(k+1)}-T_{-(k+2)}} \\
= & \frac{-\left(T_{-(k+1)}x_{-1}x_0-\left(T_{-(k+2)}+T_{-(k+3)}\right)x_0+T_{-(k+2)}\right)}{T_{-(k+2)}x_{-1}x_0+\left(T_{-(k+2)}-T_{-(k+1)}\right)x_0+T_{-(k+3)}},
\end{aligned}$$

which ends the induction and the proof.

Theorem 6.10 *Eq.(6.4) has unique negative equilibrium point $\overline{x}=a$ and a unstable.*

Proof. Equilibrium point of Eq.(6.4) satisfy the equation

$$\overline{x} = \frac{-1}{\overline{x}\left(\overline{x}+1\right)-1}.$$

After simplification, we get the following cubic equation

$$\overline{x}^3+\overline{x}^2-\overline{x}+1=0. \tag{6.18}$$

Then, the roots of the cubic equation (6.18) are given by

$$\begin{aligned}
g &= \frac{-1-\sqrt[3]{19+3\sqrt{33}}-\sqrt[3]{19-3\sqrt{33}}}{3}, \\
h &= \frac{-1-\omega\sqrt[3]{19+3\sqrt{33}}-\omega^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\
k &= \frac{-1-\omega^2\sqrt[3]{19+3\sqrt{33}}-\omega\sqrt[3]{19-3\sqrt{33}}}{3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(2\pi i/3\right)$$

is a primitive cube root of unity. So, the root g is only real number. Therefore, the unique negative equilibrium point of Eq.(6.4) is $\overline{x} = g$.

Now, we indicate that the negative equilibrium point of Eq.(6.4) is unstable.

Let I be an interval of real numbers and

$$f : I^2 \to I$$

be a continuous function defined by

$$f\left(x, y\right) = \frac{-1}{x\left(y+1\right) - 1}.$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f\left(x, y\right)}{\partial x} &= \frac{y+1}{\left(x\left(y+1\right)-1\right)^2}, \\
\frac{\partial f\left(x, y\right)}{\partial y} &= \frac{x}{\left(x\left(y+1\right)-1\right)^2}.
\end{aligned}$$

Then, from (6.18)

$$\begin{aligned}
\frac{\partial f\left(\overline{x}, \overline{x}\right)}{\partial x} &= \frac{g+1}{\left(g\left(g+1\right)-1\right)^2} \\
&= \frac{g+1}{\left(g^2+g-1\right)^2} \\
&= \frac{g+1}{\left(-\frac{1}{g}\right)^2} \\
&= g^3 + g^2 \\
&= g - 1, \\
\frac{\partial f\left(\overline{x}, \overline{x}\right)}{\partial y} &= \frac{g}{\left(g\left(g+1\right)-1\right)^2} \\
&= \frac{g}{\left(g^2+g-1\right)^2} \\
&= \frac{g}{\left(-\frac{1}{g}\right)^2} \\
&= g^3,
\end{aligned}$$

and the linearized equation of Eq.(6.4) about $\overline{x} = g$ is

$$z_{n+1} = \left(g-1\right) z_n + g^3 z_{n-1}$$

or equivalently

$$z_{n+1} - (g-1)\, z_n - g^3 z_{n-1} = 0.$$

Therefore, the corresponding characteristic polynomial is

$$\lambda^2 - (g-1)\,\lambda - g^3 = 0.$$

Then, from Theorem 2.1, it is clearly seen that

$$\lambda_{1,2} = \frac{(g-1) \pm \sqrt{4g^3 + g^2 - 2g + 1}}{2}$$

and numerically

$$|\lambda_1| = |\lambda_2| = 2,4944 > 1.$$

So, the equilibrium point g is unstable. This completes the proof.

CHAPTER 7

THE DYNAMICS OF EXACT SOLUTIONS OF SECOND ORDER NONLINEAR FOUR DIFFERENCE EQUATIONS

We mention that the results of this chapter are cited from [102] which has been published by us.

The purpose of this chapter is to determine the forms of solutions, the stability character of equilibrium points, the periodicity of solutions and global behavior of solutions of the following four difference equations

$$x_{n+1} = \frac{1}{x_n\left(x_{n-1}+1\right)-1}, \quad n = 0, 1, ..., \tag{7.1}$$

$$x_{n+1} = \frac{-1}{x_n\left(x_{n-1}-1\right)-1}, \quad n = 0, 1, ..., \tag{7.2}$$

$$x_{n+1} = \frac{1}{x_n\left(x_{n-1}-1\right)+1}, \quad n = 0, 1, ..., \tag{7.3}$$

$$x_{n+1} = \frac{-1}{x_n\left(x_{n-1}+1\right)+1}, \quad n = 0, 1, ... \tag{7.4}$$

7.1 MAIN RESULTS

In this section, we present our main results for the above mentioned difference equations. Our aim is to investigate the general solutions in explicit form of the above mentioned difference equations and the asymptotic behavior of solutions of these difference equations.

7.1.1 The Difference Equation (7.1)

Theorem 7.1 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.1). Then, for $n = 0, 1, 2, ...,$ the forms of solutions $\{x_n\}_{n=-1}^{\infty}$ are given by*

$$x_{2n-1} = \frac{(1-n)\,x_{-1}x_0+n}{nx_{-1}x_0+x_0-n} \tag{7.5}$$

$$x_{2n} = \frac{nx_{-1}x_0+x_0-n}{-nx_{-1}x_0+n+1} \tag{7.6}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_1$, with F_1 is the forbidden set of Eq.(7.1) given by

$$F_1 = \bigcup_{n=-1}^{\infty} \{(x_{-1}, x_0) : nx_{-1}x_0 + x_0 - n = 0 \text{ or } -nx_{-1}x_0 + n + 1 = 0\}.$$

Proof. For $n = 0$ the result holds. Assume that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$x_{2n-3} = \frac{(2-n)\,x_{-1}x_0 + n - 1}{(n-1)\,x_{-1}x_0 + x_0 - (n-1)}$$

and

$$x_{2n-2} = \frac{(n-1)\,x_{-1}x_0 + x_0 - (n-1)}{(1-n)\,x_{-1}x_0 + n}.$$

From this and from Eq.(7.1), it follows that

$$\begin{aligned} x_{2n-1} &= \frac{1}{x_{2n-2}\,(x_{2n-3}+1) - 1} \\ &= \frac{1}{\frac{(n-1)x_{-1}x_0+x_0-(n-1)}{(1-n)x_{-1}x_0+n}\left(\frac{(2-n)x_{-1}x_0+n-1}{(n-1)x_{-1}x_0+x_0-(n-1)}+1\right)-1} \\ &= \frac{(1-n)\,x_{-1}x_0+n}{nx_{-1}x_0+x_0-n}. \end{aligned}$$

Hence, similarly, we obtain

$$\begin{aligned} x_{2n} &= \frac{1}{x_{2n-1}\,(x_{2n-2}+1) - 1} \\ &= \frac{1}{\frac{(1-n)x_{-1}x_0+n}{nx_{-1}x_0+x_0-n}\left(\frac{(n-1)x_{-1}x_0+x_0-(n-1)}{(1-n)x_{-1}x_0+n}+1\right)-1} \\ &= \frac{nx_{-1}x_0+x_0-n}{-nx_{-1}x_0+n+1}. \end{aligned}$$

Theorem 7.2 *The following statements are true.*

(i) *The equilibrium points of Eq.(7.1) are $\overline{x}_1 = 1$ and $\overline{x}_2 = -1$.*

(ii) *The positive equilibrium point of Eq.(7.1), $\overline{x}_1 = 1$, is nonhyperbolic point.*

(iii) *The negative equilibrium point of Eq.(7.1), $\overline{x}_2 = -1$, is nonhyperbolic point.*

Proof.

(i) Equilibrium points of Eq.(7.1) satisfy the equation

$$\overline{x} = \frac{1}{\overline{x}\left(\overline{x}+1\right)-1}.$$

After simplification, we have the following cubic equation

$$\overline{x}^3 + \overline{x}^2 - \overline{x} - 1 = 0. \tag{7.7}$$

The roots of the cubic equation (7.7) are -1, -1, 1. Therefore, Eq.(7.1) has two equilibra, one positive and one negative, such that

$$\overline{x}_1 = 1, \ \overline{x}_2 = -1.$$

(ii) Now, let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f\left(x, y\right) = \frac{1}{x\left(y+1\right)-1}. \tag{7.8}$$

Then, it follows that

$$\begin{aligned} \frac{\partial f\left(x,y\right)}{\partial x} &= \frac{-\left(y+1\right)}{\left(x\left(y+1\right)-1\right)^2}, \\ \frac{\partial f\left(x,y\right)}{\partial y} &= \frac{-x}{\left(x\left(y+1\right)-1\right)^2}. \end{aligned}$$

Therefore, the linearized equation of Eq.(7.1) about the equilibrium point $\overline{x}_1 = 1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$\begin{aligned} p &= \frac{\partial f\left(\overline{x}_1, \overline{x}_1\right)}{\partial x} = -2, \\ q &= \frac{\partial f\left(\overline{x}_1, \overline{x}_1\right)}{\partial y} = -1, \end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0.$$

Therefore, from Theorem 2.1, it is clearly seen that

$\lambda_{1,2} = -1$

and

$|\lambda_1| = |\lambda_2| = 1.$

So, $\overline{x}_1$ is nonhyperbolic point.

(iii) Similarly, from (7.8), the linearized equation of Eq.(7.1) about the equilibrium point $\overline{x}_2 = -1$ is

$z_{n+1} = pz_n + qz_{n-1},$

where

$$\begin{aligned} p &= \frac{\partial f(\overline{x}_2, \overline{x}_2)}{\partial x} = 0, \\ q &= \frac{\partial f(\overline{x}_2, \overline{x}_2)}{\partial y} = 1, \end{aligned}$$

and its characteristic equation is

$\lambda^2 - 1 = 0.$

Thus, it follows that

$\lambda_{1,2} = \pm 1$

and

$|\lambda_1| = |\lambda_2| = 1.$

So, $\overline{x}_2$ is nonhyperbolic point.

Theorem 7.3 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.1). Then, the negative equilibrium point of Eq.(7.1), $\overline{x}_2$, is a global attractor.*

Proof. From Theorem 7.1, we have

$$\begin{aligned}
\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty} \frac{(1-n)\,x_{-1}x_0+n}{nx_{-1}x_0+x_0-n} \\
&= \lim_{n\to\infty} \frac{(1-n)\left(x_{-1}x_0+\frac{n}{1-n}\right)}{n\left(x_{-1}x_0+\frac{x_0}{n}-1\right)} \\
&= \lim_{n\to\infty} \frac{(1-n)\left(x_{-1}x_0-1+\frac{1}{1-n}\right)}{n\left(x_{-1}x_0+\frac{x_0}{n}-1\right)} \\
&= -1,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{nx_{-1}x_0+x_0-n}{-nx_{-1}x_0+n+1} \\
&= \lim_{n\to\infty} \frac{n\left(x_{-1}x_0+\frac{x_0}{n+1}-1\right)}{-n\left(x_{-1}x_0-1-\frac{1}{n}\right)} \\
&= -1.
\end{aligned}$$

Hereby, it implies

$$\lim_{n\to\infty} x_n = -1.$$

7.1.2 The Difference Equation (7.2)

Theorem 7.4 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.2). Then, for $n = 0, 1, 2, ...,$ the forms of solutions $\{x_n\}_{n=-1}^{\infty}$ are given by*

$$x_{2n-1} = \frac{-\left((1-n)\,x_{-1}x_0+n\right)}{nx_{-1}x_0-x_0-n} \tag{7.9}$$

$$x_{2n} = \frac{-\left(nx_{-1}x_0-x_0-n\right)}{-nx_{-1}x_0+n+1} \tag{7.10}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R}-F_2$, with F_2 is the forbidden set of Eq.(7.2) given by

$$F_2 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1},x_0) : nx_{-1}x_0-x_0-n=0 \ \ or \ -nx_{-1}x_0+n+1=0\right\}.$$

Proof. For $n = 0$ the result holds. Assume that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{2n-3} = \frac{-\left((2-n)\,x_{-1}x_0+n-1\right)}{(n-1)\,x_{-1}x_0-x_0-(n-1)}$$

and

$$x_{2n-2} = \frac{-\left((n-1)\,x_{-1}x_0-x_0-(n-1)\right)}{-(n-1)\,x_{-1}x_0+n}.$$

From this and from Eq.(7.2), it follows that

$$\begin{aligned} x_{2n-1} &= \frac{-1}{x_{2n-2}\left(x_{2n-3}-1\right)-1} \\ &= \frac{-1}{\frac{-((n-1)x_{-1}x_0-x_0-(n-1))}{-(n-1)x_{-1}x_0+n}\left(\frac{-((2-n)x_{-1}x_0+n-1)}{(n-1)x_{-1}x_0-x_0-(n-1)}-1\right)-1} \\ &= \frac{-\left(\left(1-n\right)x_{-1}x_0+n\right)}{nx_{-1}x_0-x_0-n}. \end{aligned}$$

Hence, similarly, we obtain

$$\begin{aligned} x_{2n} &= \frac{-1}{x_{2n-1}\left(x_{2n-2}-1\right)-1} \\ &= \frac{-1}{\frac{-((1-n)x_{-1}x_0+n)}{nx_{-1}x_0-x_0-n}\left(\frac{-((n-1)x_{-1}x_0-x_0-(n-1))}{-(n-1)x_{-1}x_0+n}-1\right)-1} \\ &= \frac{-\left(nx_{-1}x_0-x_0-n\right)}{-nx_{-1}x_0+n+1}. \end{aligned}$$

Theorem 7.5 *The following statements are true.*

(i) *The equilibrium points of Eq.(7.2) are* $\overline{x}_1 = 1$ *and* $\overline{x}_2 = -1$.

(ii) *The positive equilibrium point of Eq.(7.2),* $\overline{x}_1 = 1$, *is nonhyperbolic point.*

(iii) *The negative equilibrium point of Eq.(7.2),* $\overline{x}_2 = -1$, *is nonhyperbolic point.*

Proof.

(i) Equilibrium points of Eq.(7.2) satisfy the equation

$$\overline{x} = \frac{-1}{\overline{x}\left(\overline{x}-1\right)-1}.$$

After simplification, we have the following cubic equation

$$\overline{x}^3 - \overline{x}^2 - \overline{x} + 1 = 0. \tag{7.11}$$

The roots of the cubic equation (7.11) are -1, 1, 1. Therefore, Eq.(7.2) has two equilibra, one positive and one negative, such that

$$\overline{x}_1 = 1,\ \overline{x}_2 = -1.$$

(ii) Now, let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f(x, y) = \frac{-1}{x(y-1) - 1}. \tag{7.12}$$

Then, it follows that

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \frac{(y-1)}{(x(y-1)-1)^2}, \\ \frac{\partial f(x,y)}{\partial y} &= \frac{x}{(x(y-1)-1)^2}. \end{aligned}$$

Therefore, the linearized equation of Eq.(7.2) about the equilibrium point $\overline{x}_1 = 1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$\begin{aligned} p &= \frac{\partial f(\overline{x}_1, \overline{x}_1)}{\partial x} = 0, \\ q &= \frac{\partial f(\overline{x}_1, \overline{x}_1)}{\partial y} = 1, \end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 - 1 = 0.$$

Therefore, from Theorem 2.1, it is clearly seen that

$$\lambda_{1,2} = \pm 1$$

and

$$|\lambda_1| = |\lambda_2| = 1.$$

So, $\overline{x}_1$ is nonhyperbolic point.

(iii) Similarly, from (7.12), the linearized equation of Eq.(7.2) about the equilibrium point $\overline{x}_2 = -1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$
\begin{aligned}
p &= \frac{\partial f(\overline{x}_2, \overline{x}_2)}{\partial x} = -2, \\
q &= \frac{\partial f(\overline{x}_2, \overline{x}_2)}{\partial y} = -1,
\end{aligned}
$$

and its characteristic equation is

$\lambda^2 + 2\lambda + 1 = 0.$

Thus, it follows that

$\lambda_{1,2} = -1$

and

$|\lambda_1| = |\lambda_2| = 1.$

So, $\overline{x}_2$ is nonhyperbolic point.

Theorem 7.6 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.2). Then, the positive equilibrium point of Eq.(7.2), $\overline{x}_1$, is a global attractor.*

Proof. From Theorem 7.4, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty} \frac{-\left((1-n)\, x_{-1}x_0 + n\right)}{nx_{-1}x_0 - x_0 - n} \\
&= \lim_{n\to\infty} \frac{(n-1)\left(x_{-1}x_0 + \frac{n}{1-n}\right)}{n\left(x_{-1}x_0 - \frac{x_0}{n} - 1\right)} \\
&= \lim_{n\to\infty} \frac{(n-1)\left(x_{-1}x_0 - 1 + \frac{1}{1-n}\right)}{n\left(x_{-1}x_0 - \frac{x_0}{n} - 1\right)} \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{-\left(nx_{-1}x_0 - x_0 - n\right)}{-nx_{-1}x_0 + n + 1} \\
&= \lim_{n\to\infty} \frac{-n\left(x_{-1}x_0 - \frac{x_0}{n} - 1\right)}{-n\left(x_{-1}x_0 - 1 - \frac{1}{n}\right)} \\
&= 1.
\end{aligned}
$$

Herewith, it implies

$\lim_{n\to\infty} x_n = 1.$

So, the proof is complete.

7.1.3 The Difference Equation (7.3)

Lemma 7.7 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.3). Then, $\{x_n\}_{n=-1}^{\infty}$ is periodic with period four.*

Proof. From Eq.(7.3),

$$\begin{aligned}
x_{n+4} &= \frac{1}{x_{n+3}\left(x_{n+2}-1\right)+1} \\
&= \frac{1}{\left(\frac{1}{x_{n+2}(x_{n+1}-1)+1}\right)\left(\frac{1}{x_{n+1}(x_n-1)+1}-1\right)+1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\left(\frac{1}{\left(\frac{1}{x_{n+1}(x_n-1)+1}\right)\left(\frac{1}{x_n(x_{n-1}-1)+1}-1\right)+1}\right)\left(\frac{1}{\left(\frac{1}{x_n(x_{n-1}-1)+1}\right)(x_n-1)+1}-1\right)+1} \\
&= \frac{1}{\left(\frac{1}{\left(\frac{1}{\frac{1}{x_n(x_{n-1}-1)+1}(x_n-1)+1}\right)\left(\frac{x_n(1-x_{n-1})}{x_n x_{n-1}-x_n+1}\right)+1}\right)\left(\frac{1-x_n}{x_n x_{n-1}}\right)+1} \\
&= \frac{1}{x_{n-1}\left(\frac{1-x_n}{x_n x_{n-1}}\right)+1} \\
&= x_n.
\end{aligned}$$

Hence, the result holds.

Theorem 7.8 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.3). Then, for $n = 1, 2, \ldots,$*

$$\begin{aligned}
x_{4n-3} &= \frac{1}{x_{-1}x_0 - x_0 + 1} \\
x_{4n-2} &= \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0} \\
x_{4n-1} &= x_{-1} \\
x_{4n} &= x_0
\end{aligned} \tag{7.13}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_3$, with F_3 is the forbidden set of Eq.(7.3) given by

$$F_3 = \left\{(x_{-1}, x_0) : x_{-1}x_0 = 0 \text{ or } x_{-1} = \frac{x_0 - 1}{x_0}\right\}.$$

Proof. From (7.3), for $n = 0$, the result holds. Suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned} x_{4n-7} &= \frac{1}{x_{-1}x_0 - x_0 + 1}, \\ x_{4n-6} &= \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0}, \\ x_{4n-5} &= x_{-1}, \\ x_{4n-4} &= x_0. \end{aligned}$$

Now, from Eq.(7.3), it follows that

$$x_{4n-3} = \frac{1}{x_{4n-4}\left(x_{4n-5} - 1\right) + 1} = \frac{1}{x_{-1}x_0 - x_0 + 1}.$$

From this and from Eq.(7.3), it follows that

$$x_{4n-2} = \frac{1}{x_{4n-3}\left(x_{4n-4} - 1\right) + 1} = \frac{1}{\frac{1}{x_{-1}x_0 - x_0 + 1}\left(x_0 - 1\right) + 1} = \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0}.$$

Again from Eq.(7.3), we get

$$x_{4n-1} = \frac{1}{x_{4n-2}\left(x_{4n-3} - 1\right) + 1} = \frac{1}{\frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0}\left(\frac{1}{x_{-1}x_0 - x_0 + 1} - 1\right) + 1} = \frac{x_{-1}x_0}{x_0} = x_{-1}.$$

Similarly, from Eq.(7.3), we have

$$\begin{aligned} x_{4n-4} &= \frac{1}{x_{4n-1}\left(x_{4n-2} - 1\right) + 1} = \frac{1}{x_{-1}\left(\frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0} - 1\right) + 1} \\ &= \frac{1}{x_{-1} - 1 + \frac{1}{x_0} - x_{-1} + 1} = x_0. \end{aligned}$$

Thus, the proof is complete.

Theorem 7.9 *Eq.(7.3) has unique positive equilibrium point $\overline{x} = 1$ and 1 is nonhyperbolic point.*

Proof. Equilibrium point of Eq.(7.3) satisfy the equation

$$\overline{x} = \frac{1}{\overline{x}\left(\overline{x} - 1\right) + 1}.$$

After simplification, we have the following cubic equation

$$\overline{x}^3 - \overline{x}^2 + \overline{x} - 1 = 0. \tag{7.14}$$

The roots of the cubic equation (7.14) are $-i$, i, 1. Therefore, the unique positive equilibrium point of Eq.(7.3) is $\overline{x} = 1$.

Now, we prove that the equilibrium point of Eq.(7.3) is nonhyperbolic.

Let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f(x, y) = \frac{1}{x(y-1)+1}.$$

The linearized equation of Eq.(7.3) about the equilibrium point $\overline{x} = 1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$\begin{aligned} p &= \frac{\partial f(\overline{x}, \overline{x})}{\partial x} = 0, \\ q &= \frac{\partial f(\overline{x}, \overline{x})}{\partial y} = -1, \end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 + 1 = 0.$$

Therefore, from Theorem 2.1, it is clearly seen that

$$\lambda_{1,2} = \pm i$$

and

$$|\lambda_1| = |\lambda_2| = 1.$$

So, this completes the proof.

7.1.4 The Difference Equation (7.4)

Lemma 7.10 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.4). Then, $\{x_n\}_{n=-1}^{\infty}$ is periodic with periods four.*

Proof. From Eq.(7.4),

$$\begin{aligned}
x_{n+4} &= \frac{-1}{x_{n+3}\left(x_{n+2}+1\right)+1} \\
&= \frac{-1}{\left(\frac{-1}{x_{n+2}(x_{n+1}+1)+1}\right)\left(\frac{-1}{x_{n+1}(x_n+1)+1}+1\right)+1} \\
&= \frac{-1}{\left(\frac{-1}{\left(\frac{-1}{x_{n+1}(x_n+1)+1}\right)\left(\frac{-1}{x_n(x_{n-1}+1)+1}+1\right)+1}\right)\left(\frac{-1}{\left(\frac{-1}{x_n(x_{n-1}+1)+1}\right)(x_n+1)+1}+1\right)+1} \\
&= \frac{-1}{\left(\frac{-1}{\left(\frac{-1}{\frac{-1}{x_n(x_{n-1}+1)+1}(x_n+1)+1}\right)\left(\frac{x_n(x_{n-1}+1)}{x_nx_{n-1}+x_n+1}\right)+1}\right)\left(-\frac{x_n+1}{x_nx_{n-1}}\right)+1} \\
&= \frac{-1}{x_{n-1}\left(-\frac{x_n+1}{x_nx_{n-1}}\right)+1} \\
&= x_n.
\end{aligned}$$

Hence, the result holds.

Theorem 7.11 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(7.4). Then, for $n = 1, 2, ...,$*

$$\begin{aligned}
x_{4n-3} &= \frac{-1}{x_{-1}x_0+x_0+1} \\
x_{4n-2} &= \frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0} \\
x_{4n-1} &= x_{-1} \\
x_{4n} &= x_0
\end{aligned} \tag{7.15}$$

where the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F_4$, with F_4 is the forbidden set of Eq.(7.4) given by

$$F_4 = \left\{(x_{-1}, x_0) : x_{-1}x_0 = 0 \text{ or } x_{-1} = \frac{-\left(x_0+1\right)}{x_0}\right\}.$$

Proof. From (7.4), for $n = 0$, the result holds. Suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{4n-7} &= \frac{-1}{x_{-1}x_0+x_0+1}, \\
x_{4n-6} &= \frac{-\left(x_{-1}x_0+x_0+1\right)}{x_{-1}x_0}, \\
x_{4n-5} &= x_{-1}, \\
x_{4n-4} &= x_0.
\end{aligned}$$

Now, from Eq.(7.4), it follows that

$$x_{4n-3} = \frac{-1}{x_{4n-4}\left(x_{4n-5}+1\right)+1} = \frac{-1}{x_{-1}x_0 + x_0 + 1}.$$

From this and from Eq.(7.4), it follows that

$$x_{4n-2} = \frac{-1}{x_{4n-3}\left(x_{4n-4}+1\right)+1} = \frac{-1}{\frac{-1}{x_{-1}x_0+x_0+1}\left(x_0+1\right)+1} = \frac{-\left(x_{-1}x_0 + x_0 + 1\right)}{x_{-1}x_0}.$$

Again from Eq.(7.4), we get

$$x_{4n-1} = \frac{-1}{x_{4n-2}\left(x_{4n-3}+1\right)+1} = \frac{-1}{\frac{-(x_{-1}x_0+x_0+1)}{x_{-1}x_0}\left(\frac{-1}{x_{-1}x_0+x_0+1}+1\right)+1} = \frac{-x_{-1}x_0}{-x_0} = x_{-1}.$$

Similarly, from Eq.(7.4), we have

$$\begin{aligned} x_{4n} &= \frac{-1}{x_{4n-1}\left(x_{4n-2}+1\right)+1} = \frac{-1}{x_{-1}\left(\frac{-(x_{-1}x_0+x_0+1)}{x_{-1}x_0}+1\right)+1} \\ &= \frac{-1}{-x_{-1} - 1 - \frac{1}{x_0} + x_{-1} + 1} = x_0. \end{aligned}$$

Thus, the proof is complete.

Theorem 7.12 *Eq.(7.4) has unique positive equilibrium point $\overline{x} = 1$ and the equilibrium point 1 is locally assymptotically stable.*

Proof. Equilibrium point of Eq.(7.4) satisfy the equation

$$\overline{x} = \frac{-1}{\overline{x}\left(\overline{x}+1\right)+1}.$$

After simplification, we have the following cubic equation

$$\overline{x}^3 + \overline{x}^2 + \overline{x} + 1 = 0. \qquad (7.16)$$

The roots of the cubic equation (7.16) are $-i$, i, 1. Therefore, the unique positive equilibrium point of Eq.(7.4) is $\overline{x} = 1$.

Now, we demonstrate that the equilibrium point of Eq.(7.4) is locally asymptotically stable.

Let $I = (0, \infty)$ and consider the function

$$f : I^2 \to I$$

defined by

$$f(x,y) = \frac{-1}{x(y+1)+1}.$$

The linearized equation of Eq.(7.4) about the equilibrium point $\overline{x} = 1$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$\begin{aligned} p &= \frac{\partial f(\overline{x},\overline{x})}{\partial x} = \frac{2}{9}, \\ q &= \frac{\partial f(\overline{x},\overline{x})}{\partial y} = \frac{1}{9}, \end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 - \frac{2}{9}\lambda - \frac{1}{9} = 0.$$

Therefore, from Theorem 2.1, it is clearly seen that

$$\lambda_{1,2} = \frac{1}{9} \pm \frac{1}{9}\sqrt{10}$$

and

$$|\lambda_{1,2}| < 1.$$

Thus, the proof is complete.

CHAPTER 8

GLOBAL ANALYSIS OF SOLUTIONS OF SYSTEMS OF DIFFERENCE EQUATIONS VIA TRIBONACCI NUMBERS

Primarily, we mean that the results of this chapter are cited from [103] which has been published by us.

The main objective of this chapter is to investigate the explicit form, stability character and global behavior of solutions of the following two systems of rational difference equations

$$x_{n+1} = \frac{1}{y_n\left(x_{n-1}+1\right)+1}, \quad y_{n+1} = \frac{1}{x_n\left(y_{n-1}+1\right)+1}, \quad n = 0, 1, ..., \tag{8.1}$$

$$x_{n+1} = \frac{-1}{y_n\left(x_{n-1}-1\right)+1}, \quad y_{n+1} = \frac{-1}{x_n\left(y_{n-1}-1\right)+1}, \quad n = 0, 1, ..., \tag{8.2}$$

such that their solutions are associated with Tribonacci numbers.

Our aim in this chapter is to determine some relationships both between Tribonacci numbers and and solutions of the aforementioned systems of difference equations and between the Tribonacci constant and the equilibrium points of these systems of difference equations.

8.1 MAIN RESULTS

In this section, we introduce our results.

8.1.1 The System (8.1)

In this subsection, we present our main results related to the system (8.1). Our aim is to investigate the general solution in exact form of system (8.1) and the asymptotic behavior of solutions of system (8.1).

Theorem 8.1 *Let $\{x_n, y_n\}_{n=-1}^{\infty}$ be a solution system (8.1). Then, for $n = 0, 1, 2, ...,$ the form of solutions $\{x_n, y_n\}_{n=-1}^{\infty}$ is given by*

$$\begin{aligned}
x_{2n-1} &= \frac{T_{2n-2}x_{-1}y_0 + (T_{2n} - T_{2n-1})\, y_0 + T_{2n-1}}{T_{2n-1}x_{-1}y_0 + (T_{2n-2} + T_{2n-1})\, y_0 + T_{2n}}, \\
x_{2n} &= \frac{T_{2n-1}y_{-1}x_0 + (T_{2n+1} - T_{2n})\, x_0 + T_{2n}}{T_{2n}y_{-1}x_0 + (T_{2n-1} + T_{2n})\, x_0 + T_{2n+1}}, \\
y_{2n-1} &= \frac{T_{2n-2}y_{-1}x_0 + (T_{2n} - T_{2n-1})\, x_0 + T_{2n-1}}{T_{2n-1}y_{-1}x_0 + (T_{2n-2} + T_{2n-1})\, x_0 + T_{2n}}, \\
y_{2n} &= \frac{T_{2n-1}x_{-1}y_0 + (T_{2n+1} - T_{2n})\, y_0 + T_{2n}}{T_{2n}x_{-1}y_0 + (T_{2n-1} + T_{2n})\, y_0 + T_{2n+1}},
\end{aligned}$$

where T_n is the nth Tribonacci number and the initial conditions x_{-1}, y_{-1}, x_0, $y_0 \in \mathbb{R} - F_1$, with F_1 is the forbidden set of system (8.1) given by

$$F_1 = \bigcup_{n=-1}^{\infty} \{(x_{-1}, y_{-1}, x_0, y_0) : A_n = 0 \text{ or } B_n = 0 \text{ or } C_n = 0 \text{ or } D_n = 0\}$$

where

$$\begin{aligned}
A_n &= T_{2n-1}x_{-1}y_0 + (T_{2n-2} + T_{2n-1})\, y_0 + T_{2n}, \\
B_n &= T_{2n}y_{-1}x_0 + (T_{2n-1} + T_{2n})\, x_0 + T_{2n+1}, \\
C_n &= T_{2n-1}y_{-1}x_0 + (T_{2n-2} + T_{2n-1})\, x_0 + T_{2n}, \\
D_n &= T_{2n}x_{-1}y_0 + (T_{2n-1} + T_{2n})\, y_0 + T_{2n+1}.
\end{aligned}$$

Proof. We use the induction on k. For $k = 0$, the result holds. Suppose that $k > 0$ and that our assumption holds for $k - 1$. That is,

$$\begin{aligned}
x_{2k-3} &= \frac{T_{2k-4}x_{-1}y_0 + (T_{2k-2} - T_{2k-3})\, y_0 + T_{2k-3}}{T_{2k-3}x_{-1}y_0 + (T_{2k-4} + T_{2k-3})\, y_0 + T_{2k-2}}, \\
x_{k-2} &= \frac{T_{2k-3}y_{-1}x_0 + (T_{2k-1} - T_{2k-2})\, x_0 + T_{2k-2}}{T_{2k-2}y_{-1}x_0 + (T_{2k-3} + T_{2k-2})\, x_0 + T_{2k-1}}, \\
y_{2k-3} &= \frac{T_{2k-4}y_{-1}x_0 + (T_{2k-2} - T_{2k-3})\, x_0 + T_{2k-3}}{T_{2k-3}y_{-1}x_0 + (T_{2k-4} + T_{2k-3})\, x_0 + T_{2k-2}}, \\
y_{2k-2} &= \frac{T_{2k-3}x_{-1}y_0 + (T_{2k-1} - T_{2k-2})\, y_0 + T_{2k-2}}{T_{2k-2}x_{-1}y_0 + (T_{2k-3} + T_{2k-2})\, y_0 + T_{2k-1}}.
\end{aligned}$$

From system (8.1) and (2.11), it follows that

$$\begin{aligned}
x_{2k-1} &= \frac{1}{y_{2k-2}\,(x_{2k-3} + 1) + 1} \\
&= \frac{1}{\frac{T_{2k-3}x_{-1}y_0 + (T_{2k-1} - T_{2k-2})y_0 + T_{2k-2}}{T_{2k-2}x_{-1}y_0 + (T_{2k-3} + T_{2k-2})y_0 + T_{2k-1}} \left(\frac{T_{2k-4}x_{-1}y_0 + (T_{2k-2} - T_{2k-3})y_0 + T_{2k-3}}{T_{2k-3}x_{-1}y_0 + (T_{2k-4} + T_{2k-3})y_0 + T_{2k-2}} + 1 \right) + 1} \\
&= \frac{T_{2k-2}x_{-1}y_0 + (T_{2k-3} + T_{2k-2})\, y_0 + T_{2k-1}}{(T_{2k-4} + T_{2k-3} + T_{2k-2})\, x_{-1}y_0 + (T_{2k-2} + T_{2k-1})\, y_0 + T_{2k-3} + T_{2k-2} + T_{2k-1}}.
\end{aligned}$$

Therefore, we have

$$x_{2k-1} = \frac{T_{2k-2}x_{-1}y_0 + (T_{2k} - T_{2k-1})\, y_0 + T_{2k-1}}{T_{2k-1}x_{-1}y_0 + (T_{2k-2} + T_{2k-1})\, y_0 + T_{2k}}.$$

And also, it follows that

$$\begin{aligned} y_{2k-1} &= \frac{1}{x_{2k-2}\,(y_{2k-3}+1)+1} \\ &= \frac{1}{\frac{T_{2k-3}y_{-1}x_0+(T_{2k-1}-T_{2k-2})x_0+T_{2k-2}}{T_{2k-2}y_{-1}x_0+(T_{2k-3}+T_{2k-2})x_0+T_{2k-1}}\left(\frac{T_{2k-4}y_{-1}x_0+(T_{2k-2}-T_{2k-3})x_0+T_{2k-3}}{T_{2k-3}y_{-1}x_0+(T_{2k-4}+T_{2k-3})x_0+T_{2k-2}}+1\right)+1} \\ &= \frac{T_{2k-2}y_{-1}x_0 + (T_{2k-3}+T_{2k-2})\,x_0 + T_{2k-1}}{(T_{2k-4}+T_{2k-3}+T_{2k-2})\,y_{-1}x_0 + (T_{2k-2}+T_{2k-1})\,x_0 + T_{2k-3}+T_{2k-2}+T_{2k-1}}. \end{aligned}$$

So, we obtain

$$y_{2k-1} = \frac{T_{2k-2}y_{-1}x_0 + (T_{2k} - T_{2k-1})\, x_0 + T_{2k-1}}{T_{2k-1}y_{-1}x_0 + (T_{2k-2} + T_{2k-1})\, x_0 + T_{2k}}.$$

Similarly, from system (8.1) and (2.11), it follows that

$$\begin{aligned} x_{2k} &= \frac{1}{y_{2k-1}\,(x_{2k-2}+1)+1} \\ &= \frac{1}{\frac{T_{2k-2}y_{-1}x_0+(T_{2k}-T_{2k-1})x_0+T_{2k-1}}{T_{2k-1}y_{-1}x_0+(T_{2k-2}+T_{2k-1})x_0+T_{2k}}\left(\frac{T_{2k-3}y_{-1}x_0+(T_{2k-1}-T_{2k-2})x_0+T_{2k-2}}{T_{2k-2}y_{-1}x_0+(T_{2k-3}+T_{2k-2})x_0+T_{2k-1}}+1\right)+1} \\ &= \frac{T_{2k-1}y_{-1}x_0 + (T_{2k-2}+T_{2k-1})\,x_0 + T_{2k}}{(T_{2k-3}+T_{2k-2}+T_{2k-1})\,y_{-1}x_0 + (T_{2k-1}+T_{2k})\,x_0 + T_{2k-2}+T_{2k-1}+T_{2k}}. \end{aligned}$$

Thus, we get

$$x_{2k} = \frac{T_{2k-1}y_{-1}x_0 + (T_{2k-2} + T_{2k-1})\, x_0 + T_{2k}}{T_{2k}y_{-1}x_0 + (T_{2k-1} + T_{2k})\, x_0 + T_{2k+1}}.$$

And also, it follows that

$$\begin{aligned} y_{2k} &= \frac{1}{x_{2k-1}\,(y_{2k-2}+1)+1} \\ &= \frac{1}{\frac{T_{2k-2}x_{-1}y_0+(T_{2k}-T_{2k-1})y_0+T_{2k-1}}{T_{2k-1}x_{-1}y_0+(T_{2k-2}+T_{2k-1})y_0+T_{2k}}\left(\frac{T_{2k-3}x_{-1}y_0+(T_{2k-1}-T_{2k-2})y_0+T_{2k-2}}{T_{2k-2}x_{-1}y_0+(T_{2k-3}+T_{2k-2})y_0+T_{2k-1}}+1\right)+1} \\ &= \frac{T_{2k-1}x_{-1}y_0 + (T_{2k-2}+T_{2k-1})\,y_0 + T_{2k}}{(T_{2k-3}+T_{2k-2}+T_{2k-1})\,x_{-1}y_0 + (T_{2k-1}+T_{2k})\,y_0 + T_{2k-2}+T_{2k-1}+T_{2k}}. \end{aligned}$$

Herefrom, we have

$$y_{2k} = \frac{T_{2k-1}x_{-1}y_0 + (T_{2k-2} + T_{2k-1})\, y_0 + T_{2k}}{T_{2k}x_{-1}y_0 + (T_{2k-1} + T_{2k})\, y_0 + T_{2k+1}}.$$

Theorem 8.2 *The system (8.1) has unique positive equilibrium point* $(\overline{x}, \overline{y}) = (a, a)$ *and* (a, a) *is locally asymptotically stable.*

Proof. Clearly, equilibrium point of system (8.1) is the real roots of the equations

$$\overline{x} = \frac{1}{\overline{x}\left(\overline{y}+1\right)+1},\ \ \overline{y} = \frac{1}{\overline{y}\left(\overline{x}+1\right)+1}. \tag{8.3}$$

In (8.3), after some operations, we obtain

$$\overline{x} = \overline{y}.$$

As a result, we obtain the following equation

$$\overline{x}^3 + \overline{x}^2 + \overline{x} - 1 = 0. \tag{8.4}$$

Then, the roots of the cubic equation (8.4) are given by

$$\begin{aligned} a &= \frac{-1 + \sqrt[3]{3\sqrt{33}+17} - \sqrt[3]{3\sqrt{33}-17}}{3}, \\ b &= \frac{-1 + \omega\sqrt[3]{3\sqrt{33}+17} - \omega^2\sqrt[3]{3\sqrt{33}-17}}{3}, \\ c &= \frac{-1 + \omega^2\sqrt[3]{3\sqrt{33}+17} - \omega\sqrt[3]{3\sqrt{33}-17}}{3}, \end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp\left(2\pi i/3\right)$$

is a primitive cube root of unity. So, the root a is only real number. Therefore, the unique positive equilibrium point of system (8.1) is $(\overline{x}, \overline{y}) = (a, a)$.

Now, we show that the unique positive equilibrium point of system (8.1) is locally asymptotically stable.

Let I and J are some intervals of real numbers and consider the functions

$$f : I^2 \times J^2 \to I \text{ and } g : I^2 \times J^2 \to J$$

defined by

$$f\left(x_n, x_{n-1}, y_n, y_{n-1}\right) = \frac{1}{y_n\left(x_{n-1}+1\right)+1},\ \ g\left(x_n, x_{n-1}, y_n, y_{n-1}\right) = \frac{1}{x_n\left(y_{n-1}+1\right)+1}.$$

We consider the following transformation to build corresponding linearized form of system (8.1)

$$\left(x_n, x_{n-1}, y_n, y_{n-1}\right) \to \left(f, f_1, g, g_1\right),$$

where

$$\begin{aligned}
f\left(x_n, x_{n-1}, y_n, y_{n-1}\right) &= \frac{1}{y_n\left(x_{n-1}+1\right)+1},\\
f_1\left(x_n, x_{n-1}, y_n, y_{n-1}\right) &= x_n,\\
g\left(x_n, x_{n-1}, y_n, y_{n-1}\right) &= \frac{1}{x_n\left(y_{n-1}+1\right)+1},\\
g_1\left(x_n, x_{n-1}, y_n, y_{n-1}\right) &= y_n.
\end{aligned}$$

Then, the linearized system of system (8.1) about the equilibrium point (a, a) under the above transformation is given as

$$X_{n+1} = BX_n,$$

where $X_n = \left(x_n, x_{n-1}, y_n, y_{n-1}\right)^T$ and B is a Jacobian matrix of system (8.1) about the equilibrium point (a, a) and given by

$$\begin{aligned}
B &= \begin{pmatrix}
0 & \frac{-a}{(a(a+1)+1)^2} & \frac{-(1+a)}{(a(a+1)+1)^2} & 0\\
1 & 0 & 0 & 0\\
\frac{-(1+a)}{(a(a+1)+1)^2} & 0 & 0 & \frac{-a}{(a(a+1)+1)^2}\\
0 & 0 & 1 & 0
\end{pmatrix}\\
&= \begin{pmatrix}
0 & -a^3 & a-1 & 0\\
1 & 0 & 0 & 0\\
a-1 & 0 & 0 & -a^3\\
0 & 0 & 1 & 0
\end{pmatrix}.
\end{aligned}$$

Thus, we obtain the characteristic equation of the Jacobian matrix B as

$$\left(a^3 + \lambda^2\right)^2 - (a-1)^2 \lambda^2 = 0,$$

or

$$\left(\lambda^2 + (a-1)\lambda + a^3\right)\left(\lambda^2 - (a-1)\lambda + a^3\right) = 0.$$

Hence, it is clearly seen that numerically

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = 0.40089 < 1.$$

Consequently, the equilibrium point (a, a) is locally asymptotically stable. So, this completes the proof.

Theorem 8.3 *The equilibrium point of system (8.1) is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n\geq -1}$ be a solution system (8.1). By Theorem 8.2, we need only to prove that the equilibrium point (a,a) is global attractor, that is

$$\lim_{n\to\infty}(x_n, y_n) = (a,a).$$

From Theorem 8.1 and (2.13) and (2.14), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty}\frac{T_{2n-2}x_{-1}y_0 + (T_{2n}-T_{2n-1})\,y_0 + T_{2n-1}}{T_{2n-1}x_{-1}y_0 + (T_{2n-2}+T_{2n-1})\,y_0 + T_{2n}} \\
&= \lim_{n\to\infty}\frac{T_{2n-2}\left(x_{-1}y_0 + \left(\frac{T_{2n}}{T_{2n-2}} - \frac{T_{2n-1}}{T_{2n-2}}\right)y_0 + \frac{T_{2n-1}}{T_{2n-2}}\right)}{T_{2n-1}\left(x_{-1}y_0 + \left(\frac{T_{2n-2}}{T_{2n-1}} + 1\right)y_0 + \frac{T_{2n}}{T_{2n-1}}\right)} \\
&= \left(\frac{x_{-1}y_0 + (\alpha^2-\alpha)\,y_0 + \alpha}{x_{-1}y_0 + \left(\frac{1}{\alpha}+1\right)y_0 + \alpha}\right)\lim_{n\to\infty}\frac{T_{2n-2}}{T_{2n-1}} \\
&= \lim_{n\to\infty}\frac{T_{2n-2}}{T_{2n-1}} \\
&= \frac{1}{\alpha} \\
&= a,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty}\frac{T_{2n-1}y_{-1}x_0 + (T_{2n+1}-T_{2n})\,x_0 + T_{2n}}{T_{2n}y_{-1}x_0 + (T_{2n-1}+T_{2n})\,x_0 + T_{2n+1}} \\
&= \lim_{n\to\infty}\frac{T_{2n-1}\left(y_{-1}x_0 + \left(\frac{T_{2n+1}}{T_{2n-1}} - \frac{T_{2n}}{T_{2n-1}}\right)x_0 + \frac{T_{2n}}{T_{2n-1}}\right)}{T_{2n}\left(y_{-1}x_0 + \left(\frac{T_{2n-1}}{T_{2n}} + 1\right)x_0 + \frac{T_{2n+1}}{T_{2n}}\right)} \\
&= \left(\frac{y_{-1}x_0 + (\alpha^2-\alpha)\,x_0 + \alpha}{y_{-1}x_0 + \left(\frac{1}{\alpha}+1\right)x_0 + \alpha}\right)\lim_{n\to\infty}\frac{T_{2n-1}}{T_{2n}} \\
&= \lim_{n\to\infty}\frac{T_{2n-1}}{T_{2n}} \\
&= \frac{1}{\alpha} \\
&= a.
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty} x_n = a.$$

Similarly, we obtain

$$\lim_{n\to\infty} y_n = a.$$

Therefore, we get

$$\lim_{n\to\infty}(x_n, y_n) = (a, a).$$

The proof is completed.

8.1.2 The System (8.2)

In this subsection, we introduce our main results related to the system (8.2). Our aim is to investigate the general solution in explicit form of system (8.2) and the asymptotic behavior of solutions of system (8.2).

Theorem 8.4 *Let $\{x_n, y_n\}_{n=-1}^{\infty}$ be a solution system (8.2). Then, for $n = 0, 1, 2, ...,$ the form of solutions $\{x_n, y_n\}_{n=-1}^{\infty}$ is given by*

$$\begin{aligned}
x_{2n-1} &= \frac{-\left(T_{2n-2}x_{-1}y_0 + \left(T_{2n-1} - T_{2n}\right)y_0 + T_{2n-1}\right)}{T_{2n-1}x_{-1}y_0 - \left(T_{2n-2} + T_{2n-1}\right)y_0 + T_{2n}}, \\
x_{2n} &= \frac{-\left(T_{2n-1}y_{-1}x_0 + \left(T_{2n} - T_{2n+1}\right)x_0 + T_{2n}\right)}{T_{2n}y_{-1}x_0 - \left(T_{2n-1} + T_{2n}\right)x_0 + T_{2n+1}}, \\
y_{2n-1} &= \frac{-\left(T_{2n-2}y_{-1}x_0 + \left(T_{2n-1} - T_{2n}\right)x_0 + T_{2n-1}\right)}{T_{2n-1}y_{-1}x_0 - \left(T_{2n-2} + T_{2n-1}\right)x_0 + T_{2n}}, \\
y_{2n-1} &= \frac{-\left(T_{2n-1}x_{-1}y_0 + \left(T_{2n} - T_{2n+1}\right)y_0 + T_{2n}\right)}{T_{2n}x_{-1}y_0 - \left(T_{2n-1} + T_{2n}\right)y_0 + T_{2n+1}}
\end{aligned}$$

where initial conditions x_{-1}, y_{-1}, x_0, $y_0 \in \mathbb{R} - F_2$, with F_2 is the forbidden set of system (8.2) given by

$$F_2 = \bigcup_{n=-1}^{\infty}\left\{(x_{-1}, y_{-1}, x_0, y_0) : A_n = 0 \text{ or } B_n = 0 \text{ or } C_n = 0 \text{ or } D_n = 0\right\}$$

where

$$\begin{aligned}
A_n &= T_{2n-1}x_{-1}y_0 - \left(T_{2n-2} + T_{2n-1}\right)y_0 + T_{2n}, \\
B_n &= T_{2n}y_{-1}x_0 - \left(T_{2n-1} + T_{2n}\right)x_0 + T_{2n+1}, \\
C_n &= T_{2n-1}y_{-1}x_0 - \left(T_{2n-2} + T_{2n-1}\right)x_0 + T_{2n}, \\
D_n &= T_{2n}x_{-1}y_0 - \left(T_{2n-1} + T_{2n}\right)y_0 + T_{2n+1}.
\end{aligned}$$

Proof. Consider system (8.2) by taking $n = 0, 1, 2, ...$ as follows:

$$\begin{array}{lll}
n=0 \;\Rightarrow & x_1 = \frac{-1}{x_{-1}y_0-y_0+1}, & y_1 = \frac{-1}{y_{-1}x_0-x_0+1}, \\
n=1 \;\Rightarrow & x_2 = \frac{-(y_{-1}x_0-x_0+1)}{y_{-1}x_0-2x_0+2}, & y_2 = \frac{-(x_{-1}y_0-y_0+1)}{x_{-1}y_0-2y_0+2}, \\
n=2 \;\Rightarrow & x_3 = \frac{-(x_{-1}y_0-2y_0+2)}{2x_{-1}y_0-3y_0+4}, & y_3 = \frac{-(y_{-1}x_0-2x_0+2)}{2y_{-1}x_0-3x_0+4}, \\
n=3 \;\Rightarrow & x_4 = \frac{-(2y_{-1}x_0-3x_0+4)}{4y_{-1}x_0-6x_0+7}, & y_4 = \frac{-(2x_{-1}y_0-3y_0+4)}{4x_{-1}y_0-6y_0+7}, \\
n=4 \;\Rightarrow & x_5 = \frac{-(4x_{-1}y_0-y_0+7)}{7x_{-1}y_0-11y_0+13}, & y_5 = \frac{-(4y_{-1}x_0-6x_0+7)}{7y_{-1}x_0-11x_0+13}, \\
n=5 \;\Rightarrow & x_6 = \frac{-(7y_{-1}x_0-11x_0+13)}{13y_{-1}x_0-20x_0+24}, & x_6 = \frac{-(7x_{-1}y_0-11y_0+13)}{13x_{-1}y_0-20y_0+24}, \\
\vdots & &
\end{array}$$

If we keep on this process and also regard (2.11), then the result directly follows from a simple induction.

Theorem 8.5 *The system (8.2) has unique negative equilibrium point $(\overline{x}, \overline{y}) = (d, d)$ and (d, d) is locally asymptotically stable.*

Proof. Clearly, equilibrium point of system (8.2) is the real roots of the equations

$$\overline{x} = \frac{-1}{\overline{x}\,(\overline{y}-1)+1}, \;\; \overline{y} = \frac{-1}{\overline{y}\,(\overline{x}-1)+1}. \tag{8.5}$$

In (8.5), after some operations, we get

$$\overline{x} = \overline{y}.$$

As a result, we obtain the following equation

$$\overline{x}^3 - \overline{x}^2 + \overline{x} + 1 = 0. \tag{8.6}$$

Then, the roots of the cubic equation (8.6) are given by

$$\begin{aligned}
d &= \frac{1 + \sqrt[3]{3\sqrt{33}-17} - \sqrt[3]{3\sqrt{33}+17}}{3}, \\
e &= \frac{1 + \omega\sqrt[3]{3\sqrt{33}-17} - \omega^2\sqrt[3]{3\sqrt{33}+17}}{3}, \\
f &= \frac{1 + \omega^2\sqrt[3]{3\sqrt{33}-17} - \omega\sqrt[3]{3\sqrt{33}+17}}{3},
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp\left(2\pi i/3\right)$$

is a primitive cube root of unity. So, the root d is only real number. Therefore, the unique negative equilibrium point of system (8.2) is $(\overline{x},\overline{y})=(d,d)$.

Now, we show that the unique negative equilibrium point of system (8.2) is locally asymptotically stable.

Let I and J are some intervals of real numbers and consider the functions

$f:I^2\times J^2\to I$ and $g:I^2\times J^2\to J$

defined by

$$f\left(x_n,x_{n-1},y_n,y_{n-1}\right)=\frac{-1}{y_n\left(x_{n-1}-1\right)+1},\ g\left(x_n,x_{n-1},y_n,y_{n-1}\right)=\frac{-1}{x_n\left(y_{n-1}-1\right)+1}.$$

We consider the following transformation to build corresponding linearized form of system (8.2)

$$\left(x_n,x_{n-1},y_n,y_{n-1}\right)\to\left(f,f_1,g,g_1\right),$$

where

$$\begin{aligned}
f\left(x_n,x_{n-1},y_n,y_{n-1}\right) &= \frac{-1}{y_n\left(x_{n-1}-1\right)+1},\\
f_1\left(x_n,x_{n-1},y_n,y_{n-1}\right) &= x_n,\\
g\left(x_n,x_{n-1},y_n,y_{n-1}\right) &= \frac{-1}{x_n\left(y_{n-1}-1\right)+1},\\
g_1\left(x_n,x_{n-1},y_n,y_{n-1}\right) &= y_n.
\end{aligned}$$

The linearized system of system (8.2) about the equilibrium point (d,d) under the above transformation is given as

$$X_{n+1}=BX_n,$$

where $X_n = (x_n, x_{n-1}, y_n, y_{n-1})^T$ and B is a Jacobian matrix of system (8.2) about the equilibrium point (d, d) and given by

$$B = \begin{pmatrix} 0 & \frac{d}{(d(d-1)+1)^2} & \frac{d-1}{(d(d-1)+1)^2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{d-1}{(d(d-1)+1)^2} & 0 & 0 & \frac{d}{(d(d-1)+1)^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & d^3 & -(1+d) & 0 \\ 1 & 0 & 0 & 0 \\ -(1+d) & 0 & 0 & d^3 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, we obtain the characteristic equation of the Jacobian matrix B as

$$\left(d^3 - \lambda^2\right)^2 - (1+d)^2 \lambda^2 = 0,$$

or

$$\left(\lambda^2 - (1+d)\lambda - d^3\right)\left(\lambda^2 + (1+d)\lambda - d^3\right) = 0.$$

Hence, it is clearly seen that numerically

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = 0.40089 < 1.$$

Consequently, the equilibrium point (d, d) is locally asymptotically stable.

Theorem 8.6 *The equilibrium point of system (8.2) is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n\geq -1}$ be a solution system (8.2). By Theorem 8.5, we need only to prove that the equilibrium point (d, d) is global attractor, that is

$$\lim_{n\to\infty} (x_n, y_n) = (d, d).$$

From Theorem 8.4 and (2.13) and (2.14), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty} \frac{-\left(T_{2n-2}x_{-1}y_0 + \left(T_{2n-1} - T_{2n}\right) y_0 + T_{2n-1}\right)}{T_{2n-1}x_{-1}y_0 - \left(T_{2n-2} + T_{2n-1}\right) y_0 + T_{2n}} \\
&= \lim_{n\to\infty} \frac{-T_{2n-2}\left(x_{-1}y_0 + \left(\frac{T_{2n-1}}{T_{2n-2}} - \frac{T_{2n}}{T_{2n-2}}\right) y_0 + \frac{T_{2n-1}}{T_{2n-2}}\right)}{T_{2n-1}\left(x_{-1}y_0 - \left(\frac{T_{2n-2}}{T_{2n-1}} + 1\right) y_0 + \frac{T_{2n}}{T_{2n-1}}\right)} \\
&= \left(\frac{x_{-1}y_0 + \left(\alpha - \alpha^2\right) y_0 + \alpha}{x_{-1}y_0 - \left(\frac{1}{\alpha} + 1\right) y_0 + \alpha}\right) \lim_{n\to\infty} \frac{-T_{2n-2}}{T_{2n-1}} \\
&= \lim_{n\to\infty} \frac{-T_{2n-2}}{T_{2n-1}} \\
&= -\frac{1}{\alpha} \\
&= d,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{-\left(T_{2n-1}y_{-1}x_0 + \left(T_{2n} - T_{2n+1}\right) x_0 + T_{2n}\right)}{T_{2n}y_{-1}x_0 - \left(T_{2n-1} + T_{2n}\right) x_0 + T_{2n+1}} \\
&= \lim_{n\to\infty} \frac{-T_{2n-1}\left(y_{-1}x_0 + \left(\frac{T_{2n}}{T_{2n-1}} - \frac{T_{2n+1}}{T_{2n-1}}\right) x_0 + \frac{T_{2n}}{T_{2n-1}}\right)}{T_{2n}\left(y_{-1}x_0 - \left(\frac{T_{2n-1}}{T_{2n}} + 1\right) x_0 + \frac{T_{2n+1}}{T_{2n}}\right)} \\
&= \left(\frac{y_{-1}x_0 + \left(\alpha - \alpha^2\right) x_0 + \alpha}{y_{-1}x_0 - \left(\frac{1}{\alpha} + 1\right) x_0 + \alpha}\right) \lim_{n\to\infty} \frac{-T_{2n-1}}{T_{2n}} \\
&= \lim_{n\to\infty} \frac{-T_{2n-1}}{T_{2n}} \\
&= -\frac{1}{\alpha} \\
&= d.
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty} x_n = d.$$

Similarly, we obtain

$$\lim_{n\to\infty} y_n = d.$$

Therefore, we get

$$\lim_{n\to\infty} \left(x_n, y_n\right) = \left(d, d\right),$$

which completes the proof.

CHAPTER 9

THE DYNAMICS OF SOLUTIONS OF A RATIONAL DIFFERENCE EQUATION VIA GENERALIZED TRIBONACCI NUMBERS

Initially, we state that the results of this chapter are cited from [104] which has been published by us.

In this chapter, in the light of Chapter 6 and 7, we study the following difference equation

$$x_{n+1} = \frac{\gamma}{x_n\left(x_{n-1}+\alpha\right)+\beta}, \quad n = 0, 1, ..., \tag{9.1}$$

where the inital values x_{-1} and x_0 are arbitrary nonzero real and the parameters α, β and γ are nonnegative real numbers with $\gamma \neq 0$.

9.1 Introduction

First, from [121], consider the generalized Tribonacci sequence $\{V_n\}_{n=0}^{\infty}$ defined by the recurrent relation

$$V_{n+3} = rV_{n+2} + sV_{n+1} + tV_n, \;\; n \in \mathbb{N}, \tag{9.2}$$

where the constant coefficients r, s, t are real numbers and the special initial conditions

$$V_0 = 0, \, V_1 = 1, \, V_2 = r.$$

The sequence $\{V_n\}_{n=0}^{\infty}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{s}{t}V_{-(n-1)} - \frac{r}{t}V_{-(n-2)} + \frac{1}{t}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Hereby, recurrence (9.2) holds for all integer n.

As $\{V_n\}_{n=0}^{\infty}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0, \tag{9.3}$$

whose roots are

$$\begin{aligned}
\varphi &= \varphi(r,s,t) = \frac{r}{3} + A + B \\
\chi &= \chi(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\
\psi &= \psi(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\
\Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \\
\omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}$$

Notice that we get the following identities

$$\begin{aligned}
\varphi + \chi + \psi &= r, \\
\varphi\chi + \varphi\psi + \chi\psi &= -s, \\
\varphi\chi\psi &= t.
\end{aligned}$$

From now on, we assume that $\Delta(r,s,t) > 0$, so that the Eq.(9.2) has one real φ and two non-real solutions with the latter being conjugate complex. Therefore, in this case, it is widely known that generalized Tribonacci numbers can be stated, for all integers n, using Binet's formula

$$V_n = \frac{\varphi^{n+1}}{(\varphi - \chi)(\varphi - \psi)} + \frac{\chi^{n+1}}{(\chi - \varphi)(\chi - \psi)} + \frac{\psi^{n+1}}{(\psi - \varphi)(\psi - \chi)}. \tag{9.4}$$

We can present Binet's formula of the generalized Tribonacci numbers for the negative subscripts: for $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned}
V_{-n} &= \frac{\varphi^2 - r\varphi - s}{t} \frac{\varphi^{2-n}}{(\varphi - \chi)(\varphi - \psi)} + \frac{\chi^2 - r\chi - s}{t} \frac{\chi^{2-n}}{(\chi - \varphi)(\chi - \psi)} \\
&\quad + \frac{\psi^2 - r\psi - s}{t} \frac{\psi^{2-n}}{(\psi - \varphi)(\psi - \chi)}.
\end{aligned}$$

Lemma 9.1 *Let φ, χ and ψ be the roots of Eq.(9.3), suppose that φ is a real root with* $\max\{|\varphi|;|\chi|;|\psi|\} = |\varphi|$. *Then,*

$$\lim_{n\to\infty} \frac{V_{n+1}}{V_n} = \varphi. \tag{9.5}$$

Proof. Note that there are three cases of the roots, that is when the roots are all real and distinct, all roots are equal or two roots are equal, complex conjugate. We will only proof the first case. The proof of the other two cases of the roots is similar to first one, so it will be omitted.

If φ, χ and ψ are real and distinct then, from Binet's formula

$$\begin{aligned}
\lim_{n\to\infty}\frac{V_{n+1}}{V_n} &= \lim_{n\to\infty}\frac{\frac{\varphi^{n+2}}{(\varphi-\chi)(\varphi-\psi)}+\frac{\chi^{n+2}}{(\chi-\varphi)(\chi-\psi)}+\frac{\psi^{n+2}}{(\psi-\varphi)(\psi-\chi)}}{\frac{\varphi^{n+1}}{(\varphi-\chi)(\varphi-\psi)}+\frac{\chi^{n+1}}{(\chi-\varphi)(\chi-\psi)}+\frac{\psi^{n+1}}{(\psi-\varphi)(\psi-\chi)}} \\
&= \lim_{n\to\infty}\frac{\varphi^{n+1}\left(\frac{\varphi}{(\varphi-\chi)(\varphi-\psi)}+\frac{\chi}{(\chi-\varphi)(\chi-\psi)}\frac{\chi^{n+1}}{\varphi^{n+1}}+\frac{\psi}{(\psi-\varphi)(\psi-\chi)}\frac{\psi^{n+1}}{\varphi^{n+1}}\right)}{\varphi^{n}\left(\frac{\varphi}{(\varphi-\chi)(\varphi-\psi)}+\frac{\chi}{(\chi-\varphi)(\chi-\psi)}\frac{\chi^{n}}{\varphi^{n}}+\frac{\psi}{(\psi-\varphi)(\psi-\chi)}\frac{\psi^{n}}{\varphi^{n}}\right)} \\
&= \lim_{n\to\infty}\frac{\varphi^{n+1}}{\varphi^{n}}\frac{\left(\frac{\varphi}{(\varphi-\chi)(\varphi-\psi)}+\frac{\chi}{(\chi-\varphi)(\chi-\psi)}\left(\frac{\chi}{\varphi}\right)^{n+1}+\frac{\psi}{(\psi-\varphi)(\psi-\chi)}\left(\frac{\psi}{\varphi}\right)^{n+1}\right)}{\left(\frac{\varphi}{(\varphi-\chi)(\varphi-\psi)}+\frac{\chi}{(\chi-\varphi)(\chi-\psi)}\left(\frac{\chi}{\varphi}\right)^{n}+\frac{\psi}{(\psi-\varphi)(\psi-\chi)}\left(\frac{\psi}{\varphi}\right)^{n}\right)} \\
&= \varphi.
\end{aligned}$$

9.2 Main Results

In this section, we present our main results related to the difference equation (9.1). Our aim is to investigate the general solution in explicit form of Eq.(9.1) and the asymptotic behavior of solutions of Eq.(9.1).

Theorem 9.2 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(9.1). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{tV_{n-1}x_{-1}x_0 + (V_{n+1} - rV_n)\,x_0 + V_n}{tV_n x_{-1}x_0 + (V_{n+2} - rV_{n+1})\,x_0 + V_{n+1}}, \tag{9.6}$$

where V_n is the nth generalized-Tribonacci number and the initial conditions x_{-1}, $x_0 \in \mathbb{R} - F$, with F is the forbidden set of Eq.(9.1) given by

$$F = \bigcup_{n=-1}^{\infty}\{(x_{-1}, x_0) : tV_n x_{-1}x_0 + (V_{n+2} - rV_{n+1})\,x_0 + V_{n+1} = 0\}.$$

Proof. First, by using the change of variables

$$x_n = \frac{w_{n-1}}{w_n}, \tag{9.7}$$

Eq.(9.1) is reduced to linear third order difference equation

$$w_{n+1} = \frac{\beta}{\gamma}w_n + \frac{\alpha}{\gamma}w_{n-1} + \frac{1}{\gamma}w_{n-2}.$$

Set

$$r := \frac{\beta}{\gamma},\ s := \frac{\alpha}{\gamma},\ \ t := \frac{1}{\gamma},$$

so we have

$$w_{n+1} = rw_n + sw_{n-1} + tw_{n-2}.$$

Now, as done in [86], we describe initial values of three sequences which will be repetitively defined and used in the rest of the proof. Let

$$a_1 := r,\ \ b_1 := s,\ \ c_1 := t.$$

We use an recurrent (iterative) method. Thus, we get

$$\begin{aligned} w_n &= a_1 w_{n-1} + b_1 w_{n-2} + c_1 w_{n-3} \\ &= a_1 \left(rw_{n-2} + sw_{n-3} + tw_{n-4} \right) + b_1 w_{n-2} + c_1 w_{n-3} \\ &= \left(ra_1 + b_1 \right) w_{n-2} + \left(sa_1 + c_1 \right) w_{n-3} + ta_1 w_{n-4} \\ &= a_2 w_{n-2} + b_2 w_{n-3} + c_2 w_{n-4}, \end{aligned} \tag{9.8}$$

where

$$a_2 := ra_1 + b_1,\ \ b_2 := sa_1 + c_1,\ \ c_2 := ta_1. \tag{9.9}$$

By continuing iteration, it implies that

$$\begin{aligned} w_n &= a_2 w_{n-2} + b_2 w_{n-3} + c_2 w_{n-4} \\ &= a_2 \left(rw_{n-3} + sw_{n-4} + tw_{n-5} \right) + b_2 w_{n-3} + c_2 w_{n-4} \\ &= \left(ra_2 + b_2 \right) w_{n-3} + \left(sa_2 + c_2 \right) w_{n-4} + ta_2 w_{n-5} \\ &= a_3 w_{n-3} + b_3 w_{n-4} + c_3 w_{n-5}, \end{aligned} \tag{9.10}$$

where

$$a_3 := ra_2 + b_2,\ \ b_3 := sa_2 + c_2,\ \ c_3 := ta_2. \tag{9.11}$$

Based on relations (9.8)-(9.11), we suppose that for some $k \in \mathbb{N}$ such that $2 \leq k \leq n-1$, we have

$$w_n = a_k w_{n-k} + b_k w_{n-k-1} + c_k w_{n-k-2}, \tag{9.12}$$

and

$$a_k := ra_{k-1} + b_{k-1}, \;\; b_k := sa_{k-1} + c_{k-1}, \;\; c_k := ta_{k-1}. \tag{9.13}$$

Next, by continuing iteration, it follows that

$$\begin{aligned} w_n &= a_k w_{n-k} + b_k w_{n-k-1} + c_k w_{n-k-2} \\ &= a_k \left(r w_{n-k-1} + s w_{n-k-2} + t w_{n-k-3} \right) + b_k w_{n-k-1} + c_k w_{n-k-2} \\ &= \left(r a_k + b_k \right) w_{n-k-1} + \left(s a_k + c_k \right) w_{n-k-2} + t a_k w_{n-k-3} \\ &= a_{k+1} w_{n-k-1} + b_{k+1} w_{n-k-2} + c_{k+1} w_{n-k-3}, \end{aligned}$$

where

$$a_{k+1} := ra_k + b_k, \;\; b_{k+1} := sa_k + c_k, \;\; c_{k+1} := ta_k.$$

Now, we maintain sequences a_k, b_k and c_k for some nonpositive values of index k. Notice that since $\gamma \neq 0$, the recurrent relations in (9.13) can be really used for computing values of sequences a_k, b_k and c_k for every $k \leq 0$.

Using the recurrent relations with the indices $k = 1$, $k = 0$ and $k = -1$, respectively, after some computations, it implies that

$$\begin{aligned} a_0 &= \frac{c_1}{c} = 1 \\ b_0 &= a_1 - a a_0 = a - a.1 = 0 \\ c_0 &= b_1 - b a_0 = b - b.1 = 0 \\ a_{-1} &= \frac{c_0}{c} = 0 \\ b_{-1} &= a_0 - a a_{-1} = 1 - a.0 = 1 \\ c_{-1} &= b_0 - b a_{-1} = 0 - b.0 = 0 \\ a_{-2} &= \frac{c_{-1}}{c} = 0 \\ b_{-2} &= a_{-1} - a a_{-2} = 0 - a.0 = 0 \\ c_{-2} &= b_{-1} - b a_{-2} = 1 - b.0 = 1. \end{aligned}$$

Thus, we obtain

$$\begin{array}{lll} a_0 = 1 & a_{-1} = 0 & a_{-2} = 0 \\ b_0 = 0 & b_{-1} = 1 & b_{-2} = 0 \\ c_0 = 0 & c_{-1} = 0 & c_{-2} = 1. \end{array} \tag{9.14}$$

From (9.13), we get

$$a_n = ra_{n-1} + sa_{n-2} + ta_{n-3}, \tag{9.15}$$

$$b_n = a_{n+1} - ra_n, \tag{9.16}$$

$$c_n = ta_{n-1}, \tag{9.17}$$

for $n \in \mathbb{N}$.

If we get $k = n$ in (9.12), we have

$$w_n = a_n w_0 + b_n w_{-1} + c_n w_{-2},$$

for $n \in \mathbb{N}_0$.

From (9.15)-(9.17), we obtain

$$w_n = a_n w_0 + (a_{n+1} - ra_n) w_{-1} + ta_{n-1} w_{-2}, \tag{9.18}$$

for $n \in \mathbb{N}_0$.

Using (9.18) in (9.7), we get

$$x_n = \frac{a_{n-1} w_0 + (a_n - ra_{n-1}) w_{-1} + ta_{n-2} w_{-2}}{a_n w_0 + (a_{n+1} - ra_n) w_{-1} + ta_{n-1} w_{-2}},$$

it follows that

$$x_n = \frac{ta_{n-2} x_{-1} x_0 + (a_n - ra_{n-1}) x_0 + a_{n-1}}{ta_{n-1} x_{-1} x_0 + (a_{n+1} - ra_n) x_0 + a_n}$$

or equivalently

$$x_n = \frac{ta_{n-2} x_{-1} x_0 + (a_n - ra_{n-1}) x_0 + a_{n-1}}{ta_{n-1} x_{-1} x_0 + (sa_{n-1} + ta_{n-2}) x_0 + a_n}.$$

From initial values (9.14) and definitions of sequences a_n and V_n, we have

$$a_n = V_{n+1},$$

with the backward shifted initial values of the sequence a_n. Then, it follows

$$x_n = \frac{tV_{n-1} x_{-1} x_0 + (V_{n+1} - rV_n) x_0 + V_n}{tV_n x_{-1} x_0 + (V_{n+2} - rV_{n+1}) x_0 + V_{n+1}},$$

or

$$x_n = \frac{tV_{n-1} x_{-1} x_0 + (V_{n+1} - rV_n) x_0 + V_n}{tV_n x_{-1} x_0 + (sV_n + tV_{n-1}) x_0 + V_{n+1}}.$$

The proof is complete.

Now, we will analyze five special cases of the above theorem according to the states of r, s, t.

Case 1: $r = s = t = 1$

In this case, the (a_n) sequence has the following recurrence relation

$$a_n = a_{n-1} + a_{n-2} + a_{n-3},$$

such that a few terms of this sequence are

$$a_{-2} = 0,\ a_{-1} = 0,\ a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 4. \tag{9.19}$$

Then, from initial values (9.19) and definitions of sequences a_n and T_n which is Tribonacci numbers, we have

$$a_n = T_{n+1},$$

with the backward shifted initial values of the sequence a_n.

Hence, we obtain

$$x_n = \frac{T_{n-1}x_{-1}x_0 + (T_{n+1} - T_n)\,x_0 + T_n}{T_n x_{-1}x_0 + (T_n + T_{n-1})\,x_0 + T_{n+1}}.$$

Case 2: $r = 0$, $s = t = 1$

In this case, the (a_n) sequence has the following recurrence relation

$$a_n = a_{n-2} + a_{n-3},$$

such that a few terms of this sequence are

$$a_{-2} = 0,\ a_{-1} = 0,\ a_0 = 1,\ a_1 = 0,\ a_2 = 1,\ a_3 = 1, a_4 = 1. \tag{9.20}$$

Then, from initial values (9.20) and definitions of sequences a_n and P_n which is Padovan numbers, we get

$$a_{n+2} = P_n,$$

with the forward shifted initial values of the sequence a_n.

Therefore, we have

$$x_n = \frac{P_{n-4}x_{-1}x_0 + P_{n-2}x_0 + P_{n-3}}{P_{n-3}x_{-1}x_0 + P_{n-1}x_0 + P_{n-2}}.$$

Case 3: $r = 0,\ s = t = 1$

In this case, the (a_n) sequence has the following recurrence relation

$$a_n = a_{n-2} + a_{n-3},$$

such that a few terms of this sequence are

$$a_{-2} = 0,\ a_{-1} = 0,\ a_0 = 1,\ a_1 = 0,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1. \tag{9.21}$$

Then, from initial values (9.21) and definitions of sequences a_n and S_n which is Padovan-Perrin numbers, we have

$$a_n = S_{n+2},$$

with the backward shifted initial values of the sequence a_n.

Thus, we obtain

$$x_n = \frac{S_n x_{-1} x_0 + S_{n+2} x_0 + S_{n+1}}{S_{n+1} x_{-1} x_0 + S_{n+3} x_0 + S_{n+2}}.$$

Case 4: $r = 1,\ s = 0,\ t = 1$

In this case, the (a_n) sequence has the following recurrence relation

$$a_n = a_{n-1} + a_{n-3},$$

such that a few terms of this sequence are

$$a_{-2} = 0,\ a_{-1} = 0,\ a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3. \tag{9.22}$$

Then, from initial values (9.22) and definitions of sequences a_n and N_n which is Narayana numbers, we have

$$a_n = N_{n+1},$$

with the backward shifted initial values of the sequence a_n.

Fromhere, we have

$$x_n = \frac{N_{n-1} x_{-1} x_0 + N_{n-2} x_0 + N_n}{N_n x_{-1} x_0 + N_{n-1} x_0 + N_{n+1}}.$$

Case 5: $r = s = 1,\ t = 2$

In this case, the (a_n) sequence has the following recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 2a_{n-3},$$

such that a few terms of this sequence are

$$a_{-2} = 0,\ a_{-1} = 0,\ a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 5, a_4 = 9. \tag{9.23}$$

Next, from initial values (9.23) and definitions of sequences a_n and J_n which is third order Jacobsthal numbers, we have

$$a_n = J_{n+1},$$

with the backward shifted initial values of the sequence a_n.
Herefrom, we get

$$x_n = \frac{2J_{n-1}x_{-1}x_0 + (J_{n+1} - J_n)\,x_0 + J_n}{2J_n x_{-1}x_0 + (J_{n+2} - J_{n+1})\,x_0 + J_{n+1}}.$$

Theorem 9.3 *Eq.(9.1) has unique equilibrium point $\overline{x} = \mu$ and μ is locally asymptotically stable.*

Proof. Equilibrium point of Eq.(9.1) is the real roots of the equation

$$\overline{x} = \frac{\gamma}{\overline{x}\,(\overline{x} + \alpha) + \beta}.$$

After simplification, we get the following cubic equation

$$\overline{x}^3 + \alpha\overline{x}^2 + \beta\overline{x} - \gamma = 0. \tag{9.24}$$

Then, the roots of the cubic equation (9.24) are given by

$$\begin{aligned}
\mu &= \mu\,(\alpha, \beta, \gamma) = -\frac{\alpha}{3} + C + D,\\
\sigma &= \sigma\,(\alpha, \beta, \gamma) = -\frac{\alpha}{3} + \omega C + \omega^2 D,\\
\phi &= \phi\,(\alpha, \beta, \gamma) = -\frac{\alpha}{3} + \omega^2 C + \omega D,
\end{aligned}$$

where

$$\begin{aligned}
C &= \left(\frac{-\alpha^3}{27} + \frac{\alpha\beta}{6} + \frac{\gamma}{2} + \sqrt{\Delta}\right)^{1/3}, D = \left(\frac{-\alpha^3}{27} + \frac{\alpha\beta}{6} + \frac{\gamma}{2} - \sqrt{\Delta}\right)^{1/3}\\
\Delta &= \Delta(r, s, t) = -\frac{\alpha^3\gamma}{27} - \frac{\alpha^2\beta^2}{108} + \frac{\alpha\beta\gamma}{6} + \frac{\beta^3}{27} + \frac{\gamma^2}{4}
\end{aligned}$$

and

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(2\pi i/3\right)$$

is a primitive cube root of unity. So, the root μ is only real number. So, the unique equilibrium point of Eq.(9.1) is $\overline{x} = \mu$.

Now, we demonstrate that the equilibrium point of Eq.(9.1) is locally asymptotically stable.

Let I be an interval of real numbers and consider the function

$$f : I^2 \to I$$

defined by

$$f(x, y) = \frac{\gamma}{x(y + \alpha) + \beta}.$$

The linearized equation of Eq.(9.1) about the equilibrium point $\overline{x} = \mu$ is

$$z_{n+1} = p z_n + q z_{n-1},$$

where

$$\begin{aligned}
p &= \frac{\partial f(\overline{x}, \overline{x})}{\partial x} = \frac{-\gamma(\mu + \alpha)}{(\mu(\mu + \alpha) + \beta)^2} \\
&= \frac{-\gamma(\mu + \alpha)}{(\mu^2 + \alpha\mu + \beta)^2} \\
&= \frac{-\gamma(\mu + \alpha)}{\left(\frac{\gamma}{\mu}\right)^2} \\
&= \frac{-(\mu^3 + \alpha\mu^2)}{\gamma} \\
&= \frac{\beta\mu - \gamma}{\gamma}, \\
q &= \frac{\partial f(\overline{x}, \overline{x})}{\partial y} = \frac{-\gamma\mu}{(\mu(\mu + \alpha) + \beta)^2} \\
&= \frac{-\gamma\mu}{(\mu^2 + \alpha\mu + \beta)^2} \\
&= \frac{-\gamma\mu}{\left(\frac{\gamma}{\mu}\right)^2} \\
&= -\frac{\mu^3}{\gamma},
\end{aligned}$$

and the corresponding characteristic equation is

$$\lambda^2 - \left(\frac{\beta\mu - \gamma}{\gamma}\right)\lambda + \frac{\mu^3}{\gamma} = 0.$$

Consider two functions defined by

$$a(\lambda) = \lambda^2, \;\; b(\lambda) = \left(\frac{\beta\mu - \gamma}{\gamma}\right)\lambda - \frac{\mu^3}{\gamma}.$$

We have

$$\left|\frac{\beta\mu - \gamma}{\gamma} - \frac{\mu^3}{\gamma}\right| < 1.$$

Then, it follows that

$$|b(\lambda)| < |a(\lambda)|, \;\; \text{for all } \lambda : |\lambda| = 1.$$

Therefore, by Rouche's Theorem, all zeros of $P(\lambda) = a(\lambda) - b(\lambda) = 0$ lie in $|\lambda| < 1$. Hereby, by Theorem 2.4, we have that the unique equilibrium point of Eq.(9.1) $\overline{x} = \mu$ is locally asymptotically stable.

Theorem 9.4 *Assume that* $\mu\varphi = 1$. *Then, the equilibrium point of Eq.(9.1) is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n\geq -1}$ be a solution of Eq.(9.1). By Theorem 9.3, we need only to prove that the equilibrium point μ is global attractor, that is

$$\lim_{n\to\infty} x_n = \mu.$$

From Theorem 9.2 and (9.3) and (9.5), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{tV_{n-1}x_{-1}x_0 + (V_{n+1} - rV_n)x_0 + V_n}{tV_n x_{-1}x_0 + (V_{n+2} - rV_{n+1})x_0 + V_{n+1}} \\
&= \lim_{n\to\infty} \frac{tV_{n-1}\left(x_{-1}x_0 + \left(\frac{1}{t}\frac{V_{n+1}}{V_{n-1}} - \frac{r}{t}\frac{V_n}{V_{n-1}}\right)x_0 + \frac{V_n}{V_{n-1}}\right)}{tV_n\left(x_{-1}x_0 + \left(\frac{1}{t}\frac{V_{n+2}}{V_n} - \frac{r}{t}\frac{V_{n+1}}{V_n}\right)x_0 + \frac{V_{n+1}}{V_n}\right)} \\
&= \left(\frac{x_{-1}x_0 + \left(\frac{1}{t}\varphi^2 - \frac{r}{t}\varphi\right)x_0 + \varphi}{x_{-1}x_0 + \left(\frac{1}{t}\varphi^2 - \frac{r}{t}\varphi\right)x_0 + \varphi}\right)\lim_{n\to\infty}\frac{V_{n-1}}{V_n} \\
&= \lim_{n\to\infty}\frac{V_{n-1}}{V_n} \\
&= \frac{1}{\varphi} \\
&= \mu
\end{aligned}$$

This completes the proof.

Note that when $\alpha = \beta = \gamma = 1$, our assumption in Theorem 9.4 is immediately seen. Indeed,

$\mu\varphi = \frac{1}{3\gamma}\left(\frac{1}{3\sqrt[3]{2}}\sqrt[3]{27\gamma - 2\alpha^3 + 9\alpha\beta - 3\sqrt{3}S} - \frac{1}{3}\alpha + \frac{1}{3\sqrt[3]{2}}\sqrt[3]{27\gamma - 2\alpha^3 + 9\alpha\beta + 3\sqrt{3}S}\right)$
$\left(\beta + \frac{1}{\sqrt[3]{2}}\sqrt[3]{2\beta^3 + 27\gamma^2 - 3\sqrt{3}\gamma S + 9\alpha\beta\gamma} + \frac{1}{\sqrt[3]{2}}\sqrt[3]{2\beta^3 + 27\gamma^2 + 3\sqrt{3}\gamma S + 9\alpha\beta\gamma}\right)$
$S = \sqrt{-4\alpha^3\gamma - \alpha^2\beta^2 + 18\alpha\beta\gamma + 4\beta^3 + 27\gamma^2}.$

Then, in the case $\alpha = \beta = \gamma = 1$, it follows that $\mu\varphi = 1$.

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