ZONGULDAK BÜLENT ECEVİT ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

BAZI RASYONEL FARK DENKLEM SİSTEMLERİNİN PERİYODİK ÇÖZÜMLERİ

MATEMATİK ANABİLİM DALI

YÜKSEK LİSANS TEZİ

MÜJGAN KURU

TEMMUZ 2019

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DANIŞMAN: Dr. Öğr. Üyesi Melih GÖCEN

ZONGULDAK
Temmuz 2019

KABUL:

Müjgan KURU tarafından hazırlanan "Bazı Rasyonel Fark Denklem Sistemlerinin Periyodik Çözümleri" başlıklı bu çalışma jürimiz tarafından değerlendirilerek Zonguldak Bülent Ecevit Üniversitesi, Fen Bilimleri Enstitüsü, Matematik Anabilim Dalında Yüksek Lisans Tezi olarak oybirliğiyle kabul edilmiştir. 26/07/2019

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ONAY:

Yukarıdaki imzaların, adı geçen öğretim üyelerine ait olduğunu onaylarım. ...../…./20…

Prof. Dr. Ahmet ÖZARSLAN
Fen Bilimleri Enstitüsü Müdürü

"Bu tezdeki tüm bilgilerin akademik kurallara ve etik ilkelere uygun olarak elde edildiğini ve sunulduğunu; ayrıca bu kuralların ve ilkelerin gerektirdiği şekilde, bu çalışmadan kaynaklanmayan bütün atıfları yaptığımı beyan ederim."

Müjgan KURU

ÖZET

Yüksek Lisans Tezi

BAZI RASYONEL FARK DENKLEM SİSTEMLERİNİN PERİYODİK ÇÖZÜMLERİ

Müjgan KURU

Zonguldak Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Dr. Öğr. Üyesi Melih GÖCEN
Temmuz 2019, 61 sayfa

Bu tezde, bazı lineer olmayan rasyonel fark denklem sistemlerinin periyodik çözümleri incelenmiştir.

Bu tez dört bölümden oluşmaktadır.

Birinci bölümde, tez için gerekli olan bazı temel tanım ve teoremler verilmiştir.

İkinci bölümde, literatürdeki bazı rasyonel fark denklemleriyle ilgili çalışmalar sunulmuştur.

Üçüncü bölümde, ikinci mertebeden bazı rasyonel fark denklem sistemlerinin periyodikliği incelenmiştir.

Son bölümde ise, bazı özel rasyonel fark denklem sistemlerinin periyodik çözümleri elde edilmiştir.

ÖZET (devam ediyor)

Ayrıca tezde, teorik sonuçlarımızı desteklemek için bazı sayısal örnekler verilmiştir.

Anahtar Kelimeler: Fark denklemleri, periyodiklik, periyodik çözümler,denklem sistemleri.

Bilim Kodu: 403.03.01

ABSTRACT

M. Sc. Thesis

PERIODIC SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATION SYSTEMS

Müjgan KURU

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assist. Prof. Melih GÖCEN
July 2019, 61 pages

In this thesis, the periodic solutions of some nonlinear rational difference equation systems are investigated.

This thesis consists of four chapters.

In the first chapter, some basic definitions and theorems necessary for the thesis are given.

In the second chapter, the studies about some rational difference equations in the literature are presented.

In the third chapter, the periodicity of some second order rational difference equation systems are investigated.

In the last chapter, the periodic solutions of some special rational difference equation systems are obtained.

ABSTRACT (continued)

Furthermore, in the thesis, some numerical examples are given to support our theoretical results.

Keywords: Difference equations, periodicity, periodic solutions, systems of equations.

Science Code: 403.03.01

TEŞEKKÜR

Yüksek lisans çalışmam boyunca yakın ilgisini eksik etmeyen, bana daima inanan ve destek veren sözleriyle çalışma azmimi perçinleyen saygı değer danışman hocam sayın Dr. Öğr. Üyesi Melih GÖCEN'e öncelikli teşekkürlerimi sunarım.

Ayrıca bu süreçte maddi manevi desteğini hep hissettiğim aileme ve arkadaşlarıma da sonsuz teşekkürlerimi sunarım.

İÇİNDEKİLER

Sayfa

Sayfa

ÇİZELGELER DİZİNİ

BÖLÜM 1

TEMEL TANIM VE TEOREMLER

1.1 GİRİŞ

Bu bölümde tez boyunca kullanacağımız bazı temel tanım ve teoremler verilmiştir. Bu kısımda Kocic ve Ladas (1993), Elaydi (1995), Kulenovic ve Ladas (2002) kaynaklarından yararlanılmıştır.

Tanım 1.1.1 *n bağımsız değişken ve buna bağılı değişken de x olmak üzere, bağımlı ve bağımsız değişken ile bağımlı değişkenin $E(x), E^2(x), ..., E^n(x), ...$ gibi farklarını içine alan bağıntılara Fark Denklemi denir.Ayrıca n' nin sürekli olduğu halde Diferansiyel Denklemler ile arasında büyük benzerlikler vardır.*

Tanım 1.1.2 *Bir fark denkleminde bilinmeyen fonksiyonun en büyük ve en küçük argümentlerinin farkına o fark denkleminin mertebesi denir.*

Tanım 1.1.3 *$F(x_n, x_{n+1}, ..., x_{n+k}) = 0$ şeklinde k. mertebeden bir fark denkleminin genel ifadesinde eşitliğin sağ tarafı "0" ise bu fark denklemine homojen (otonom) fark denklemi denir. Sıfırdan farklı ise homojen olmayan fark denklemi denir.*

Tanım 1.1.4 *Eğer bir fark denklemi x_n ya da herhangi bir fark ifadesinin 2. ya da daha yüksek mertebeden kuvvetini içeriyorsa ya da x_n ile x_{n+m}'nin $(0 < m < k)$ çarpımını içeriyorsa bu fark denklemine lineer olmayan fark denklemi denir. Aksi durumda ise lineer fark denklemi denir.*

Genel olarak lineer fark denklemleri

$$a_k x_{n+k} + a_{k-1} x_{n+k-1} + ... + a_0 x_n = G(n)$$

şeklinde gösterilir ve lineer fark denklemleri katsayılarının durumuna göre isimlendirilir:

(a) Eğer $G(n) = 0$ ise denkleme Lineer Homojen Fark Denklemi denir.

(b) $a_0, a_1, a_2, ..., a_k$ katsayıları sabit iseler, denkleme Sabit Katsayılı Lineer Fark Denklemi denir.

(c) $a_0, a_1, a_2, ..., a_k$ katsayıları bağımsız değişkenin fonksiyonu iseler, denkleme Değişken Katsayılı Lineer Fark Denklemi denir.

Teorem 1.1.5 *I reel sayıların herhangi bir alt aralığı ve $f : I^{k+1} \to I$ sürekli türevlenebilir bir fonksiyon olsun.$(k+1)$. mertebeden bir fark denklemi*

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \;\; n = 0, 1, 2... \tag{1.1}$$

formunda bir denklemdir.

Lemma 1.1.6 *$x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ başlangıç koşullarının her kümesi için, (1.1) fark denklemi bir tek $\{x_n\}_{n=-k}^{\infty}$ çözümüne sahiptir.*

Yukarıdaki lemmanın bir özel durumu olarak, $x_0, x_{-1}, x_{-2} \in I$ başlangıç koşullarının her kümesi için

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \;\; n = 0, 1, 2... \tag{1.2}$$

üçüncü dereceden fark denklemi bir tek $\{x_n\}_{n=-2}^{\infty}$ çözümüne sahiptir.

Tanım 1.1.7 (1.1) *denkleminin bir çözümü yani her $n \geq -k$ için sabit olan* (1.1) *denkleminin bir çözümüne* (1.1) *denkleminin bir denge çözümü denir. Her $n \geq -k$ için*

$$x_n = \overline{x}$$

ise (1.1) *denkleminin bir denge çözümüdür, o zaman $\overline{x}$ bir denge noktası olarak adlandırılır.*
Ayrıca $\overline{x}$ noktasına f fonksiyonunun bir sabit noktası denir.
Dolayısıyla $\overline{x} \in I$ noktası

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$$

ise (1.1) *denkleminin bir denge noktası olarak adlandırılır yani $n \geq -k$ için*

$$x_n = \overline{x}$$

(1.1) *denkleminin bir çözümüdür.*

Örnek 1.1.1 $x_{n+1} = \dfrac{16}{x_n}$ *fark denkleminin denge noktasının* ± 4 *olduğunu gösteriniz.*

Çözüm 1 $f(\overline{x}, \overline{x}) = \overline{x} = \dfrac{16}{\overline{x}}$ *ise* $\overline{x} = \pm 4$ *dir*

Tanım 1.1.8 $\overline{x}$, (1.1) *denkleminin denge noktası olsun.*

(a) *Her* $\varepsilon > 0$ *için* $\{x_n\}_{n=-k}^{\infty}$, (1.1) *denkleminin bir çözümü olacak şekilde*

$$|x_0 - \overline{x}| + |x_{-1} - \overline{x}| + ... + |x_{-k} - \overline{x}| < \delta$$

olduğunda her $n \geq -k$ *için*

$$|x_n - \overline{x}| < \varepsilon$$

ifadesini sağlayan bir $\delta > 0$ *sayısı varsa,* $\overline{x}$ *denge noktasına kararlıdır denir.*

(b) $\{x_n\}_{n=-k}^{\infty}$, (1.1) *denkleminin bir çözümü olacak şekilde*

$$|x_0 - \overline{x}| + |x_{-1} - \overline{x}| + ... + |x_{-k} - \overline{x}| < \gamma$$

olduğunda

$$\lim_{n\to\infty} x_n = \overline{x}$$

ifadesini sağlayan $\gamma > 0$ *sayısı varsa,* $\overline{x}$ *denge noktasına lokal asimptotik kararlıdır denir.*

(c) (1.1) *denkleminin her* $\{x_n\}_{n=-k}^{\infty}$ *çözümü için*

$$\lim_{n\to\infty} x_n = \overline{x}$$

oluyorsa $\overline{x}$ *denge noktasına global çekicidir denir.*

(d) *Eğer* $\overline{x}$ *denge noktası kararlı ve bir global çekici ise,* $\overline{x}$ *denge noktasına global asimptotik kararlıdır denir.*

(e) *Eğer* $\overline{x}$ *denge noktası kararlı değil ise kararsızdır denir.*

Tanım 1.1.9 *Eğer* $\{x_n\}$ *dizisi için* $x_{n+p} = x_n$ *olacak şekilde bir* p *pozitif tam sayısı mevcut ise* $\{x_n\}$ *dizisine* p *periyotludur denir ve* p *sayısı bu şartı sağlayan en küçük pozitif tam sayıdır.*

Tanım 1.1.10 *Eğer* $\{x_n\}$ *dizisinde sonlu sayıda terim hariç tutulduğunda, geriye kalan sonsuz sayıdaki terim için* $x_{n+p} = x_n$ *olacak şekilde bir* p *pozitif tam sayısı mevcut ise* $\{x_n\}$ *dizisine er geç* p *periyotludur denir ve* p *sayısı bu şartı sağlayan en küçük pozitif tam sayıdır.*

Örnek 1.1.2 $x_{n+1} = \dfrac{16}{x_n}$ *denkleminin periyodunun* 2 *olduğunu gösterelim.* x_0 *başlangıç şartı* $n = 0, 1, 2, ...$ *için iterasyon yöntemiyle*

$$x_1 = \frac{16}{x_0}, x_2 = \frac{16}{x_1} = x_0, x_3 = \frac{16}{x_2} = \frac{16}{x_0} = x_1$$

olup, bu şekilde iterasyona devam edilirse;

$$x_n = \left\{\frac{16}{x_0}, x_{0,} \frac{16}{x_0}, x_{0,} ...\right\}$$

şeklinde çözümler elde edilir. Böylece söz konusu denklemin 2 *periyotlu olduğu gösterilir.*

Tanım 1.1.11 (1.1) *denkleminde,* $f(x_n, x_{n-1}, x_{n-2})$ *fonksiyonunu* $f(u, v, w)$ *şeklinde alalım:*

$$r = \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial u}\ , s = \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial v} ve\ t = \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial w}$$

olmak üzere;

$$y_{n+1} = ry_n + sy_{n-1} + ty_{n-2} \tag{1.3}$$

denklemi elde edilir. Bu denkleme (1.2) *denkleminin* $\overline{x}$ *denge noktası civarındaki lineer denklemi adı verilir.*
(1.3) *denkleminin karakteristik denklemi ise*

$$\lambda^3 - r\lambda^2 - s\lambda - t = 0 \tag{1.4}$$

dir.

(a) (1.4) denkleminin tüm kökleri mutlak değerce 1'den küçük olduğunda, $\overline{x}$ denge noktası lokal asimptotik kararlıdır.

(b) (1.4) denkleminin köklerinden en az biri mutlak değerce 1'den büyük olduğunda, $\overline{x}$ denge noktası kararsızdır.

BÖLÜM 2

FARK DENKLEMLERİ İLE İLGİLİ YAPILAN ÇALIŞMALAR

Çınar (2004) çalışmasında,

$$x_{n+1} = \frac{1}{y_n}, y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$$

fark denklem sisteminin pozitif çözümlerini ve periyodikliğini ele almıştır.

Çınar ve Yalçınkaya (2004) çalışmalarında,

$$x_{n+1} = \frac{1}{z_n}, y_{n+1} = \frac{1}{x_{n-1}y_{n-1}}, z_{n+1} = \frac{1}{x_{n-1}}$$

fark denklem sisteminin pozitif çözümlerinin periyodikliğini ele almışlardır.

Douraki ve arkadaşları (2006) çalışmalarında $A, B \in (0, \infty)$ olmak üzere,

$$x_{n+1} = \frac{A}{x_{n-k}} + \frac{B}{x_{n-3k}}$$

fark denkleminin çözümlerinin periyodikliğini incelemişlerdir.

Özban (2006) çalışmasında,

$$x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$$

denklem sisteminin pozitif çözümlerini incelemiştir.

Elabbasy ve arkadaşları (2008) çalışmalarında,

$$\begin{aligned} x_{n+1} &= \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n}, \quad y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}}, \\ z_{n+1} &= \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n} \end{aligned}$$

fark denklem sisteminin bir sınıfının çözümlerinin periyodikliğini ele almışlardır.

Stevic (2012) çalışmasında,

$$x_{n+1} = \frac{x_n y_{n-k}}{y_{n-k+1}(a_n + b_n x_n y_{n-k})}, \quad y_{n+1} = \frac{y_n x_{n-k}}{x_{n-k+1}(c_n + d_n y_n x_{n-k})}$$

fark denklem sistemlerinin çözümlerinin periyodikliğini ele almışlardır.

Touafek ve Elsayed (2012a) çalışmalarında,

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm y_{n-1}x_{n-3}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_{n-1}y_{n-3}}$$

fark denklem sisteminin çözümlerinin periyodikliğini incelemişlerdir.

Özkan ve Kurbanlı (2013) çalışmalarında,

$$x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}, \quad z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}$$

fark denklem sisteminin çözümlerinin periyodikliğini incelemişlerdir.

Din ve arkadaşları (2014) çalışmalarında,

$$x_{n+1} = \frac{y_{n-1}}{x_{n-3}(\alpha + y_n)}, \quad y_{n+1} = \frac{x_{n-1}}{y_{n-3}(\beta \pm y_n)}$$

fark denklem sisteminin çözümlerinin periyodikliğini ele almışlardır.

Yacine (2016) çalışmasında,

$$x_{n+1} = \frac{1}{1 - y_{n-k}}, \quad y_{n+1} = \frac{1}{1 - y_{n-k}}$$

fark denklem sisteminin çözümlerinin periyodikliğini incelemiştir.

Touafek ve Elsayed (2012b) çalışmalarında,

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}$$

fark denklem sisteminin çözümlerini incelemişlerdir.

T.F. İbrahim (2009) çalışmasında,

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}$$

fark denkleminin çözümlerinin davranışlarını incelemiştir.

BÖLÜM 3

İKİNCİ MERTEBEDEN BAZI RASYONEL FARK DENKLEM SİSTEMLERİNİN PERİYODİKLİĞİ

3.1 $x_{n+1} = \frac{x_n}{x_{n-1}(\pm 1 + x_n)}$ FARK DENKLEMLERİNİN PERİYODİKLİĞİ

Bu bölümde x_{-1}, x_0 başlangıç koşulları paydayı sıfır yapmayacak reel sayılar olmak üzere,

$$x_{n+1} = \frac{x_n}{x_{n-1}(1 + x_n)} \quad n = 0, 1, 2, ... \tag{3.1}$$

ve

$$x_{n+1} = \frac{x_n}{x_{n-1}(-1 + x_n)} \quad n = 0, 1, 2, ... \tag{3.2}$$

fark denklemlerinin çözümleri araştırılmıştır.

Teorem 3.1.1 (3.1) *denkleminin çözümlerinin* $\{x_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve beş periyotludur.*

(b)

$$\begin{aligned} x_{5n-1} &= x_{-1}, \\ x_{5n} &= x_0, \\ x_{5n+1} &= \tfrac{x_0}{x_{-1}(1+x_0)}, \\ x_{5n+2} &= \tfrac{1}{x_{-1}(1+x_0)+x_0}, \\ x_{5n+3} &= \tfrac{x_{-1}}{x_0(1+x_{-1})}. \end{aligned}$$

veya buna eş değer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} x_{-1}, x_0, \dfrac{x_0}{x_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{x_0(1+x_{-1})}, x_{-1}, x_0, \\ \dfrac{x_0}{x_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{x_0(1+x_{-1})}, x_{-1}, x_0, \\ \dfrac{x_0}{x_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{x_0(1+x_{-1})}, ... \end{array} \right\}$$

çözümleri elde edilir.

İspat.

(a) (3.1) *denklemi yardımıyla aşağıdaki eşitlikler elde edilir:*

$$\begin{aligned} x_{n+1} &= \tfrac{x_n}{x_{n-1}(1+x_n)}, \\ x_{n+2} &= \tfrac{1}{x_{n-1}(1+x_n)+x_n}, \\ x_{n+3} &= \tfrac{x_{n-1}}{x_n(1+x_{n-1})}, \\ x_{n+4} &= x_{n-1}, \\ x_{n+5} &= x_n. \end{aligned}$$

(b) $n = 0$ *için sonuçlar sağlanır.* $n > 0$ *olduğunu ve iddiamızın* $(n-1)$ *için sağlandığını varsayalım. Yani,*

$$\begin{aligned} x_{5n-6} &= x_{-1}, \\ x_{5n-5} &= x_0, \\ x_{5n-4} &= \tfrac{x_0}{x_{-1}(1+x_0)}, \\ x_{5n-3} &= \tfrac{1}{x_{-1}(1+x_0)+x_0}, \\ x_{5n-2} &= \tfrac{x_{-1}}{x_0(1+x_{-1})}. \end{aligned}$$

elde edilir ve denklem (3.1)*' den,*

$$x_{5n-1} = \frac{x_{5n-2}}{x_{5n-3}(1+x_{5n-2})} = \frac{\frac{x_{-1}}{x_0(1+x_{-1})}}{\frac{1}{x_{-1}(1+x_0)+x_0}\left(1+\frac{x_{-1}}{x_0(1+x_{-1})}\right)} = x_{-1},$$

$$x_{5n} = \tfrac{x_{5n-1}}{x_{5n-2}(1+x_{5n-1})} = \frac{x_{-1}}{\frac{x_{-1}}{x_0(1+x_{-1})}(1+x_{-1})} = x_0,$$

ve benzer şekilde

$$x_{5n+1} = x_1$$
$$\vdots$$

çözümleri elde edilerek ispat tamamlanır.

■

Teorem 3.1.2 (3.2) *denkleminin çözümlerinin* $\{x_n\}_{n=-1}^{+\infty}$ *olduğunu farz edelim. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$ *çözümleri beş periyotlu periyodiktir.*

(b)

$$\begin{aligned} x_{5n-1} &= x_{-1}, \\ x_{5n} &= x_0, \\ x_{5n+1} &= \tfrac{x_0}{x_{-1}(-1+x_0)}, \\ x_{5n+2} &= \tfrac{1}{-x_{-1}(-1+x_0)+x_0}, \\ x_{5n+3} &= \tfrac{x_{-1}}{x_0(-1+x_{-1})}. \end{aligned}$$

veya buna eş değer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} x_{-1}, x_0, \dfrac{x_0}{x_{-1}(-1+x_0)}, \dfrac{1}{-x_{-1}(-1+x_0)+x_0}, \dfrac{x_{-1}}{x_0(-1+x_{-1})}, \\ x_{-1}, x_0, \dfrac{x_0}{x_{-1}(-1+x_0)}, \dfrac{1}{-x_{-1}(-1+x_0)+x_0}, \dfrac{x_{-1}}{x_0(-1+x_{-1})}, \\ x_{-1}, x_0, \dfrac{x_0}{x_{-1}(-1+x_0)}, \dfrac{1}{-x_{-1}(-1+x_0)+x_0}, \dfrac{x_{-1}}{x_0(-1+x_{-1})}, \ldots \end{array} \right\}$$

çözümleri elde edilir.

İspat. Teorem 3.1.1 ile benzer yolla ispatı görülür. ■

Örnek 3.1.1

$$x_{n+1} = \frac{x_n}{x_{n-1}(+1+x_n)} \tag{3.1}$$

denkleminin $x_0 = 0.1$ *ve* $x_{-1} = 0.2$ *başlangıç koşullarındaki çözümleri aşağıda verilmiş ve 5 periyotlu periyodik olduğu görülmüştür.*

Çizelge 3.1 (3.1) denkleminin periyodik çözümleri

n	x_n
0	0.1
1	0.454
2	3.125
3	1.666
4	0.2
5	0.1
6	0.454
⋮	⋮

3.2 $x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 + y_n)}$, $y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 + x_n)}$ FARK DENKLEM SİSTEMLERİNİN PERİYODİKLİĞİ

Bu bölümde $x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ başlangıç koşulları paydayı sıfır yapmayacak reel sayılar olmak üzere

$$x_{n+1} = \frac{y_n}{x_{n-1}(1 + y_n)}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(1 + x_n)} \tag{3.3}$$

ve

$$x_{n+1} = \frac{y_n}{x_{n-1}(-1 + y_n)}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(-1 + x_n)} \tag{3.4}$$

fark denklem sistemlerinin çözümleri araştırılmıştır.

Teorem 3.2.1 (3.3) *denklem sisteminin çözümlerinin* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $n \geqslant -1$ *için* $\{x_n\}_{n=-1}^{+\infty}$ *ve* $\{y_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve on periyotludur.*

(b) $n \geqslant -1$ *için* $x_{n+5} = y_n$ *ve* $y_{n+5} = x_n$ *dir.*

(c)

$$\begin{aligned}
x_{10n-1} &= x_{-1}, \\
x_{10n} &= x_0, \\
x_{10n+1} &= \tfrac{y_0}{x_{-1}(1+y_0)}, \\
x_{10n+2} &= \tfrac{1}{y_{-1}(1+x_0)+x_0}, \\
x_{10n+3} &= \tfrac{x_{-1}}{y_0(1+x_{-1})}. \\
x_{10n+4} &= y_{-1}, \\
x_{10n+5} &= y_0, \\
x_{10n+6} &= \tfrac{x_0}{y_{-1}(1+x_0)}, \\
x_{10n+7} &= \tfrac{1}{x_{-1}(1+y_0)+y_0}, \\
x_{10n+8} &= \tfrac{y_{-1}}{x_0(1+y_{-1})}.
\end{aligned}$$

ve

$$
\begin{aligned}
y_{10n-1} &= y_{-1}, \\
y_{10n} &= y_0, \\
y_{10n+1} &= \tfrac{x_0}{y_{-1}(1+x_0)}, \\
y_{10n+2} &= \tfrac{1}{x_{-1}(1+y_0)+y_0}, \\
y_{10n+3} &= \tfrac{y_{-1}}{x_0(1+y_{-1})}. \\
y_{10n+4} &= x_{-1}, \\
y_{10n+5} &= x_0, \\
y_{10n+6} &= \tfrac{y_0}{x_{-1}(1+y_0)}, \\
y_{10n+7} &= \tfrac{1}{y_{-1}(1+x_0)+x_0}, \\
y_{10n+8} &= \tfrac{x_{-1}}{y_0(1+x_{-1})}.
\end{aligned}
$$

veya buna eşdeğer olarak,

$$
\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l}
x_{-1}, x_0, \dfrac{y_0}{x_{-1}(1+y_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{y_0(1+x_{-1})}, y_{-1}, \\
y_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+y_0)+y_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, x_{-1}, x_0, \\
\dfrac{y_0}{x_{-1}(1+y_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{y_0(1+x_{-1})}, y_{-1}, y_0, \\
\dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+y_0)+y_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, x_{-1}, \\
x_0, \dfrac{y_0}{x_{-1}(1+y_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{y_0(1+x_{-1})}, y_{-1}, \\
y_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+y_0)+y_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \ldots
\end{array} \right\}
$$

$$
\{y_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l}
y_{-1}, y_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+y_0)+y_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, x_{-1}, \\
x_0, \dfrac{y_0}{x_{-1}(1+y_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{y_0(1+x_{-1})}, y_{-1}, y_0, \\
\dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+y_0)+y_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, x_{-1}, x_0, \\
\dfrac{y_0}{x_{-1}(1+y_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{y_0(1+x_{-1})}, y_{-1}, \\
y_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{x_{-1}(1+y_0)+y_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, x_{-1}, \\
x_0, \dfrac{y_0}{x_{-1}(1+y_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{x_{-1}}{y_0(1+x_{-1})}, \ldots
\end{array} \right\}
$$

çözümleri elde edilir.

İspat.

(a) (3.3) denklem sistemi kullanılarak aşağıdaki eşitlikler elde edilir:

$$\begin{aligned} x_{n+1} &= \tfrac{y_n}{x_{n-1}(1+y_n)}, \\ y_{n+1} &= \tfrac{x_n}{y_{n-1}(1+x_n)}. \end{aligned}$$

$$\begin{aligned} x_{n+2} &= \tfrac{y_{n+1}}{x_n(1+y_{n+1})} = \frac{\frac{x_n}{y_{n-1}(1+x_n)}}{x_n(1+\frac{x_n}{y_{n-1}(1+x_n)})} = \tfrac{1}{y_{n-1}(1+x_n)+x_n}, \\ y_{n+2} &= \tfrac{x_{n+1}}{y_n(1+x_{n+1})} = \frac{\frac{y_n}{x_{n-1}(1+y_n)}}{y_n(1+\frac{y_n}{x_{n-1}(1+y_n)})} = \tfrac{1}{x_{n-1}(1+y_n)+y_n}. \end{aligned}$$

$$\begin{aligned} x_{n+3} &= \tfrac{y_{n+2}}{x_{n+1}(1+y_{n+2})} = \frac{\frac{1}{x_{n-1}(1+y_n)+y_n}}{\frac{y_n}{x_{n-1}(1+y_n)}(1+\frac{1}{x_{n-1}(1+y_n)+y_n})} = \tfrac{x_{n-1}}{y_n(1+x_{n-1})}, \\ y_{n+3} &= \tfrac{x_{n+2}}{y_{n+1}(1+x_{n+2})} = \frac{\frac{1}{y_{n-1}(1+x_n)+x_n}}{\frac{x_n}{y_{n-1}(1+x_n)}(1+\frac{1}{y_{n-1}(1+x_n)+x_n})} = \tfrac{y_{n-1}}{x_n(1+y_{n-1})}. \end{aligned}$$

$$\begin{aligned} x_{n+4} &= \tfrac{y_{n+3}}{x_{n+2}(1+y_{n+3})} = \frac{\frac{y_{n-1}}{x_n(1+y_{n-1})}}{\frac{1}{y_{n-1}(1+x_n)+x_n}(1+\frac{y_{n-1}}{x_n(1+y_{n-1})})} = y_{n-1}, \\ y_{n+4} &= \tfrac{x_{n+3}}{y_{n+2}(1+x_{n+3})} = \frac{\frac{x_{n-1}}{y_n(1+x_{n-1})}}{\frac{1}{x_{n-1}(1+y_n)+y_n}(1+\frac{x_{n-1}}{y_n(1+x_{n-1})})} = x_{n-1}. \end{aligned}$$

$$\begin{aligned} x_{n+5} &= \tfrac{y_{n+4}}{x_{n+3}(1+y_{n+4})} = \frac{x_{n-1}}{\frac{x_{n-1}}{y_n(1+x_{n-1})}(1+x_{n-1})} = y_n, \\ y_{n+5} &= \tfrac{x_{n+4}}{y_{n+3}(1+x_{n+4})} = \frac{y_{n-1}}{\frac{y_{n-1}}{x_n(1+y_{n-1})}(1+y_{n-1})} = x_n. \end{aligned}$$

$$\begin{aligned} x_{n+6} &= \tfrac{y_{n+5}}{x_{n+4}(1+y_{n+5})} = \tfrac{x_n}{y_{n-1}(1+x_n)}, \\ y_{n+6} &= \tfrac{x_{n+5}}{y_{n+4}(1+x_{n+5})} = \tfrac{y_n}{x_{n-1}(1+y_n)}. \end{aligned}$$

$$\begin{aligned} x_{n+7} &= \tfrac{y_{n+6}}{x_{n+5}(1+y_{n+6})} = \frac{\frac{y_n}{x_{n-1}(1+y_n)}}{y_n(1+\frac{y_n}{x_{n-1}(1+y_n)})} = \tfrac{1}{x_{n-1}(1+y_n)+y_n}, \\ y_{n+7} &= \tfrac{x_{n+6}}{y_{n+5}(1+x_{n+6})} = \frac{\frac{x_n}{y_{n-1}(1+x_n)}}{x_n(1+\frac{x_n}{y_{n-1}(1+x_n)})} = \tfrac{1}{y_{n-1}(1+x_n)+x_n}. \end{aligned}$$

$$x_{n+8} = \frac{y_{n+7}}{x_{n+6}(1+y_{n+7})} = \frac{\frac{1}{y_{n-1}(1+x_n)+x_n}}{\frac{x_n}{y_{n-1}(1+x_n)}(1+\frac{1}{y_{n-1}(1+x_n)+x_n})} = \frac{y_{n-1}}{x_n(1+y_{n-1})},$$

$$y_{n+8} = \frac{x_{n+7}}{y_{n+6}(1+x_{n+7})} = \frac{\frac{1}{x_{n-1}(1+y_n)+y_n}}{\frac{y_n}{x_{n-1}(1+y_n)}(1+\frac{1}{x_{n-1}(1+y_n)+y_n})} = \frac{x_{n-1}}{y_n(1+x_{n-1})}.$$

$$x_{n+9} = \frac{y_{n+8}}{x_{n+7}(1+y_{n+8})} = \frac{\frac{x_{n-1}}{y_n(1+x_{n-1})}}{\frac{1}{x_{n-1}(1+y_n)+y_n}(1+\frac{x_{n-1}}{y_n(1+x_{n-1})})} = x_{n-1},$$

$$y_{n+9} = \frac{x_{n+8}}{y_{n+7}(1+x_{n+8})} = \frac{\frac{y_{n-1}}{x_n(1+y_{n-1})}}{\frac{1}{y_{n-1}(1+x_n)+x_n}(1+\frac{y_{n-1}}{x_n(1+y_{n-1})})} = y_{n-1}.$$

$$x_{n+10} = \frac{y_{n+9}}{x_{n+8}(1+y_{n+9})} = \frac{y_{n-1}}{\frac{y_{n-1}}{x_n(1+y_{n-1})}(1+y_{n-1})} = x_n,$$

$$y_{n+10} = \frac{x_{n+9}}{y_{n+8}(1+x_{n+9})} = \frac{x_{n-1}}{\frac{x_{n-1}}{y_n(1+x_{n-1})}(1+x_{n-1})} = y_n.$$

ve böylece sistemin on periyotlu olduğu görülür.

(b) (a) 'daki eşitliklerden,

$x_{n+5} = y_n$

ve

$y_{n+5} = x_n$

olduğu görülür.

(c) $n = 0$ için sonuçlar sağlanır. İddiamızın $n > 0$ iken $(n-1)$ için sağlandığını varsayalım.

Yani,

$$
\begin{aligned}
x_{10n-11} &= x_{-1}, \\
x_{10n-10} &= x_0, \\
x_{10n-9} &= \tfrac{y_0}{x_{-1}(1+y_0)}, \\
x_{10n-8} &= \tfrac{1}{y_{-1}(1+x_0)+x_0}, \\
x_{10n-7} &= \tfrac{x_{-1}}{y_0(1+x_{-1})}, \\
x_{10n-6} &= y_{-1}, \\
x_{10n-5} &= y_0, \\
x_{10n-4} &= \tfrac{x_0}{y_{-1}(1+x_0)}, \\
x_{10n-3} &= \tfrac{1}{x_{-1}(1+y_0)+y_0}, \\
x_{10n-2} &= \tfrac{y_{-1}}{x_0(1+y_{-1})}.
\end{aligned}
$$

ve

$$
\begin{aligned}
y_{10n-11} &= y_{-1}, \\
y_{10n-10} &= y_0, \\
y_{10n-9} &= \tfrac{x_0}{y_{-1}(1+x_0)}, \\
y_{10n-8} &= \tfrac{1}{x_{-1}(1+y_0)+y_0}, \\
y_{10n-7} &= \tfrac{y_{-1}}{x_0(1+y_{-1})}, \\
y_{10n-6} &= x_{-1}, \\
y_{10n-5} &= x_0, \\
y_{10n-4} &= \tfrac{y_0}{x_{-1}(1+y_0)}, \\
y_{10n-3} &= \tfrac{1}{y_{-1}(1+x_0)+x_0}, \\
y_{10n-2} &= \tfrac{x_{-1}}{y_0(1+x_{-1})}.
\end{aligned}
$$

eşitlikleri elde edilir ve (3.3) denklem sistemi yardımıyla

$$
\begin{aligned}
x_{10n-1} &= \tfrac{y_{10n-2}}{x_{10n-3}(1+y_{10n-2})} = \frac{\dfrac{x_{-1}}{y_0(1+x_{-1})}}{\dfrac{1}{x_{-1}(1+y_0)+y_0}^{(1+\frac{x_{-1}}{y_0(1+x_{-1})})}} = x_{-1}, \\
y_{10n-1} &= \tfrac{x_{10n-2}}{y_{10n-3}(1+x_{10n-2})} = \frac{\dfrac{y_{-1}}{x_0(1+y_{-1})}}{\dfrac{1}{y_{-1}(1+x_0)+x_0}^{(1+\frac{y_{-1}}{x_0(1+y_{-1})})}} = y_{-1}, \\
x_{10n} &= \tfrac{y_{10n-1}}{x_{10n-2}(1+y_{10n-1})} = \frac{y_{-1}}{\dfrac{y_{-1}}{x_0(1+y_{-1})}^{(1+y_{-1})}} = x_0, \\
y_{10n} &= \tfrac{x_{10n-1}}{y_{10n-2}(1+x_{10n-1})} = \frac{x_{-1}}{\dfrac{x_{-1}}{y_0(1+x_{-1})}^{(1+x_{-1})}} = y_0,
\end{aligned}
$$

$$
\begin{aligned}
x_{10n+1} &= x_1, \\
y_{10n+1} &= y_1, \\
&\vdots
\end{aligned}
$$

çözümleri elde edilerek ispat tamamlanır.

■

Teorem 3.2.2 (3.4) *denklem sisteminin çözümlerinin* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $n \geqslant -1$ *için* $\{x_n\}_{n=-1}^{+\infty}$ *ve* $\{y_n\}_{n=-1}^{+\infty}$ *çözümleri on periyotlu periyodiktir*

(b) $n \geqslant -1$ *için* $x_{n+5} = y_n$ *ve* $y_{n+5} = x_n$ *dir.*

(c)

$$
\begin{aligned}
x_{10n-1} &= x_{-1}, \\
x_{10n} &= x_0, \\
x_{10n+1} &= \tfrac{y_0}{x_{-1}(-1+y_0)}, \\
x_{10n+2} &= \tfrac{1}{-y_{-1}(-1+x_0)+x_0}, \\
x_{10n+3} &= \tfrac{x_{-1}}{y_0(-1+x_{-1})}, \\
x_{10n+4} &= y_{-1}, \\
x_{10n+5} &= y_0, \\
x_{10n+6} &= \tfrac{x_0}{y_{-1}(-1+x_0)}, \\
x_{10n+7} &= \tfrac{1}{-x_{-1}(-1+y_0)+y_0}, \\
x_{10n+8} &= \tfrac{y_{-1}}{x_0(-1+y_{-1})}.
\end{aligned}
$$

ve

$$\begin{aligned}
y_{10n-1} &= y_{-1}, \\
y_{10n} &= y_0, \\
y_{10n+1} &= \tfrac{x_0}{y_{-1}(-1+x_0)}, \\
y_{10n+2} &= \tfrac{1}{-x_{-1}(-1+y_0)+y_0}, \\
y_{10n+3} &= \tfrac{y_{-1}}{x_0(-1+y_{-1})}. \\
y_{10n+4} &= x_{-1}, \\
y_{10n+5} &= x_0, \\
y_{10n+6} &= \tfrac{y_0}{x_{-1}(-1+y_0)}, \\
y_{10n+7} &= \tfrac{1}{-y_{-1}(-1+x_0)+x_0}, \\
y_{10n+8} &= \tfrac{x_{-1}}{y_0(-1+x_{-1})}.
\end{aligned}$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{\begin{array}{l}
x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)}, \frac{1}{-y_{-1}(-1+x_0)+x_0}, \frac{x_{-1}}{y_0(-1+x_{-1})}, \\
y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)}, \frac{1}{-x_{-1}(-1+y_0)+y_0}, \frac{y_{-1}}{x_0(-1+y_{-1})}, \\
x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)}, \frac{1}{-y_{-1}(-1+x_0)+x_0}, \frac{x_{-1}}{y_0(-1+x_{-1})}, \\
y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)}, \frac{1}{-x_{-1}(-1+y_0)+y_0}, \frac{y_{-1}}{x_0(-1+y_{-1})}, \\
x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)}, \frac{1}{-y_{-1}(-1+x_0)+x_0}, \frac{x_{-1}}{y_0(-1+x_{-1})}, \\
y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)}, \frac{1}{-x_{-1}(1+y_0)+y_0}, \frac{y_{-1}}{x_0(-1+y_{-1})}, \\
\ldots
\end{array}\right\}$$

$$\{y_n\}_{n=-1}^{+\infty} = \left\{\begin{array}{l}
y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)}, \frac{1}{-x_{-1}(-1+y_0)+y_0}, \frac{y_{-1}}{x_0(-1+y_{-1})}, \\
x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)}, \frac{1}{-y_{-1}(-1+x_0)+x_0}, \frac{x_{-1}}{y_0(-1+x_{-1})}, \\
y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)}, \frac{1}{-x_{-1}(-1+y_0)+y_0}, \frac{y_{-1}}{x_0(-1+y_{-1})}, \\
x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)}, \frac{1}{-y_{-1}(-1+x_0)+x_0}, \frac{x_{-1}}{y_0(-1+x_{-1})}, \\
y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)}, \frac{1}{-x_{-1}(-1+y_0)+y_0}, \frac{y_{-1}}{x_0(-1+y_{-1})}, \\
x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)}, \frac{1}{-y_{-1}(-1+x_0)+x_0}, \frac{x_{-1}}{y_0(-1+x_{-1})}, \\
\ldots
\end{array}\right\}$$

çözümleri elde edilir.

İspat. Teorem 3.2.1 ile benzer yolla ispatı görülür. ■

Örnek 3.2.1

$$x_{n+1} = \frac{y_n}{x_{n-1}(1+y_n)}, y_{n+1} = \frac{x_n}{y_{n-1}(1+x_n)} \tag{3.3}$$

denklem sisteminin $x_0 = 0.1$, $x_{-1} = 0.2$, $y_0 = -0.5$ *ve* $y_{-1} = 1.2$ *başlangıç koşullarındaki çözümleri aşağıda verilmiş ve 10 periyotlu periyodik olduğu görülmüştür.*

Çizelge 3.2 (3.3) denkleminin periyodik çözümleri

n	x_n	y_n
0	0.1	−0.5
1	−5	0.075
2	0.704	−2.5
3	0.333	5.454
4	1.2	0.2
5	−0.5	0.1
6	0.075	−5
7	−2.5	0.704
8	5.454	0.333
9	0.2	1.2
10	0.1	−0.5
11	−5	0.075
12	0.704	−2.5
13	0.333	5.454
14	1.2	0.2
⋮	⋮	⋮

3.3 $x_{n+1} = \frac{y_n}{z_{n-1}(\pm 1+y_n)}$, $y_{n+1} = \frac{z_n}{x_{n-1}(\pm 1+z_n)}$, $z_{n+1} = \frac{x_n}{y_{n-1}(\pm 1+x_n)}$ FARK DENKLEM SİSTEMLERİNİN PERİYODİKLİĞİ

Bu bölümde $x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0$ başlangıç koşulları paydayı sıfır yapmayacak reel sayılar olmak üzere,

$$x_{n+1} = \frac{y_n}{z_{n-1}(1+y_n)}, y_{n+1} = \frac{z_n}{x_{n-1}(1+z_n)}, z_{n+1} = \frac{x_n}{y_{n-1}(1+x_n)} \tag{3.5}$$

ve

$$x_{n+1}=\frac{y_n}{z_{n-1}(-1+y_n)}, y_{n+1}=\frac{z_n}{x_{n-1}(-1+z_n)}, z_{n+1}=\frac{x_n}{y_{n-1}(-1+x_n)} \tag{3.6}$$

fark denklem sistemlerinin çözümleri araştırılmıştır.

Teorem 3.3.1 (3.5) *denklem sisteminin çözümlerinin* $\{x_n, y_n, z_n\}_{n=-1}^{+\infty}$ *olduğunu farz edelim. Bu durumda;*

(a) $n \geqslant -1$ *için* $\{x_n\}_{n=-1}^{+\infty}$, $\{y_n\}_{n=-1}^{+\infty}$ *ve* $\{z_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve on beş periyotludur.*

(b) $n \geqslant -1$ *için* $x_{n+5} = z_n$, $y_{n+5} = x_n$ *ve* $z_{n+5} = y_n$ *dir.*

(c) $n \geqslant -1$ *için* $x_{n+10} = y_n$, $y_{n+10} = z_n$ *ve* $z_{n+10} = x_n$ *dir.*

(d)

$$\begin{aligned}
x_{15n-1} &= x_{-1},\\
x_{15n} &= x_0,\\
x_{15n+1} &= \tfrac{y_0}{z_{-1}(1+y_0)},\\
x_{15n+2} &= \tfrac{1}{x_{-1}(1+z_0)+z_0},\\
x_{15n+3} &= \tfrac{y_{-1}}{x_0(1+y_{-1})},\\
x_{15n+4} &= z_{-1},\\
x_{15n+5} &= z_0,\\
x_{15n+6} &= \tfrac{x_0}{y_{-1}(1+x_0)},\\
x_{15n+7} &= \tfrac{1}{z_{-1}(1+y_0)+y_0},\\
x_{15n+8} &= \tfrac{x_{-1}}{z_0(1+x_{-1})},\\
x_{15n+9} &= y_{-1},\\
x_{15n+10} &= y_0,\\
x_{15n+11} &= \tfrac{z_0}{x_{-1}(1+z_0)},\\
x_{15n+12} &= \tfrac{1}{y_{-1}(1+x_0)+x_0},\\
x_{15n+13} &= \tfrac{z_{-1}}{y_0(1+z_{-1})}.
\end{aligned}$$

ve

$$
\begin{aligned}
y_{15n-1} &= y_{-1},\\
y_{15n} &= y_0,\\
y_{15n+1} &= \tfrac{z_0}{x_{-1}(1+z_0)},\\
y_{15n+2} &= \tfrac{1}{y_{-1}(1+x_0)+x_0},\\
y_{15n+3} &= \tfrac{z_{-1}}{y_0(1+z_{-1})},\\
y_{15n+4} &= x_{-1},\\
y_{15n+5} &= x_0,\\
y_{15n+6} &= \tfrac{y_0}{z_{-1}(1+y_0)},\\
y_{15n+7} &= \tfrac{1}{x_{-1}(1+z_0)+z_0},\\
y_{15n+8} &= \tfrac{y_{-1}}{x_0(1+y_{-1})},\\
y_{15n+9} &= z_{-1},\\
y_{15n+10} &= z_0,\\
y_{15n+11} &= \tfrac{x_0}{y_{-1}(1+x_0)},\\
y_{15n+12} &= \tfrac{1}{z_{-1}(1+y_0)+y_0},\\
y_{15n+13} &= \tfrac{x_{-1}}{z_0(1+x_{-1})}.
\end{aligned}
$$

ve

$$
\begin{aligned}
z_{15n-1} &= z_{-1},\\
z_{15n} &= z_0,\\
z_{15n+1} &= \tfrac{x_0}{y_{-1}(1+x_0)},\\
z_{15n+2} &= \tfrac{1}{z_{-1}(1+y_0)+y_0},\\
z_{15n+3} &= \tfrac{x_{-1}}{z_0(1+x_{-1})},\\
z_{15n+4} &= y_{-1},\\
z_{15n+5} &= y_0,\\
z_{15n+6} &= \tfrac{z_0}{x_{-1}(1+z_0)},\\
z_{15n+7} &= \tfrac{1}{y_{-1}(1+x_0)+x_0},\\
z_{15n+8} &= \tfrac{z_{-1}}{y_0(1+z_{-1})},\\
z_{15n+9} &= x_{-1},\\
z_{15n+10} &= x_0,\\
z_{15n+11} &= \tfrac{y_0}{z_{-1}(1+y_0)},\\
z_{15n+12} &= \tfrac{1}{x_{-1}(1+z_0)+z_0},\\
z_{15n+13} &= \tfrac{y_{-1}}{x_0(1+y_{-1})}
\end{aligned}
$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ \ldots \end{array} \right\}$$

$$\{y_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ \ldots \end{array} \right\}$$

$$\{z_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(1+x_0)}, \dfrac{1}{z_{-1}(1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(1+z_0)}, \dfrac{1}{y_{-1}(1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(1+y_0)}, \dfrac{1}{x_{-1}(1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(1+y_{-1})}, \\ \ldots \end{array} \right\}$$

çözümleri elde edilir.

İspat.

(a) (3.5) denklem sistemi kullanılarak aşağıdaki eşitliklerin sağlandığı görülür:

$$\begin{aligned} x_{n+1} &= \frac{y_n}{z_{n-1}(1+y_n)}, \\ y_{n+1} &= \frac{z_n}{x_{n-1}(1+z_n)}, \\ z_{n+1} &= \frac{x_n}{y_{n-1}(1+x_n)}. \end{aligned}$$

$$\begin{aligned} x_{n+2} &= \frac{y_{n+1}}{z_n(1+y_{n+1})} = \frac{\dfrac{z_n}{x_{n-1}(1+z_n)}}{z_n\left(1+\dfrac{z_n}{x_{n-1}(1+z_n)}\right)} = \frac{1}{x_{n-1}(1+z_n)+z_n}, \\ y_{n+2} &= \frac{z_{n+1}}{x_n(1+z_{n+1})} = \frac{\dfrac{x_n}{y_{n-1}(1+x_n)}}{x_n\left(1+\dfrac{x_n}{y_{n-1}(1+x_n)}\right)} = \frac{1}{y_{n-1}(1+x_n)+x_n}, \\ z_{n+2} &= \frac{x_{n+1}}{y_n(1+x_{n+1})} = \frac{\dfrac{y_n}{z_{n-1}(1+y_n)}}{y_n\left(1+\dfrac{y_n}{z_{n-1}(1+y_n)}\right)} = \frac{1}{z_{n-1}(1+y_n)+y_n}. \end{aligned}$$

$$x_{n+3} = \frac{y_{n+2}}{z_{n+1}(1+y_{n+2})} = \frac{\frac{1}{y_{n-1}(1+x_n)+x_n}}{\frac{x_n}{y_{n-1}(1+x_n)}(1+\frac{1}{y_{n-1}(1+x_n)+x_n})} = \frac{y_{n-1}}{x_n(1+y_{n-1})},$$

$$y_{n+3} = \frac{z_{n+2}}{x_{n+1}(1+z_{n+2})} = \frac{\frac{1}{z_{n-1}(1+y_n)+y_n}}{\frac{y_n}{z_{n-1}(1+y_n)}(1+\frac{1}{z_{n-1}(1+y_n)+y_n})} = \frac{z_{n-1}}{y_n(1+x_{n-1})},$$

$$z_{n+3} = \frac{x_{n+2}}{y_{n+1}(1+x_{n+2})} = \frac{\frac{1}{x_{n-1}(1+z_n)+z_n}}{\frac{z_n}{x_{n-1}(1+z_n)}(1+\frac{1}{x_{n-1}(1+z_n)+z_n})} = \frac{x_{n-1}}{z_n(1+x_{n-1})}.$$

$$x_{n+4} = \frac{y_{n+3}}{z_{n+2}(1+y_{n+3})} = \frac{\frac{z_{n-1}}{y_n(1+x_{n-1})}}{\frac{1}{z_{n-1}(1+y_n)+y_n}(1+\frac{z_{n-1}}{y_n(1+x_{n-1})})} = z_{n-1},$$

$$y_{n+4} = \frac{z_{n+3}}{x_{n+2}(1+z_{n+3})} = \frac{\frac{x_{n-1}}{z_n(1+x_{n-1})}}{\frac{1}{x_{n-1}(1+z_n)+z_n}(1+\frac{x_{n-1}}{z_n(1+x_{n-1})})} = x_{n-1},$$

$$z_{n+4} = \frac{x_{n+3}}{y_{n+2}(1+x_{n+3})} = \frac{\frac{y_{n-1}}{x_n(1+y_{n-1})}}{\frac{1}{y_{n-1}(1+x_n)+x_n}(1+\frac{y_{n-1}}{x_n(1+y_{n-1})})} = y_{n-1}.$$

$$x_{n+5} = \frac{y_{n+4}}{z_{n+3}(1+y_{n+4})} = \frac{x_{n-1}}{\frac{x_{n-1}}{z_n(1+x_{n-1})}(1+x_{n-1})} = z_n,$$

$$y_{n+5} = \frac{z_{n+4}}{x_{n+3}(1+z_{n+4})} = \frac{y_{n-1}}{\frac{y_{n-1}}{x_n(1+y_{n-1})}(1+y_{n-1})} = x_n,$$

$$z_{n+5} = \frac{x_{n+4}}{y_{n+3}(1+x_{n+4})} = \frac{z_{n-1}}{\frac{z_{n-1}}{y_n(1+x_{n-1})}(1+z_{n-1})} = y_n.$$

$$x_{n+6} = \frac{y_{n+5}}{z_{n+4}(1+y_{n+5})} = \frac{x_n}{y_{n-1}(1+x_n)},$$

$$y_{n+6} = \frac{z_{n+5}}{x_{n+4}(1+z_{n+5})} = \frac{y_n}{z_{n-1}(1+y_n)},$$

$$z_{n+6} = \frac{x_{n+5}}{y_{n+4}(1+x_{n+5})} = \frac{z_n}{x_{n-1}(1+z_n)}.$$

$$x_{n+7} = \frac{y_{n+6}}{z_{n+5}(1+y_{n+6})} = \frac{\frac{y_n}{z_{n-1}(1+y_n)}}{y_n(1+\frac{y_n}{z_{n-1}(1+y_n)})} = \frac{1}{z_{n-1}(1+y_n)+y_n},$$

$$y_{n+7} = \frac{z_{n+6}}{x_{n+5}(1+z_{n+6})} = \frac{\frac{z_n}{x_{n-1}(1+z_n)}}{z_n(1+\frac{z_n}{x_{n-1}(1+z_n)})} = \frac{1}{x_{n-1}(1+z_n)+z_n},$$

$$z_{n+7} = \frac{x_{n+6}}{y_{n+5}(1+x_{n+6})} = \frac{\frac{x_n}{y_{n-1}(1+x_n)}}{x_n(1+\frac{x_n}{y_{n-1}(1+x_n)})} = \frac{1}{y_{n-1}(1+x_n)+x_n}.$$

$$x_{n+8} = \frac{y_{n+7}}{z_{n+6}(1+y_{n+7})} = \frac{\frac{1}{x_{n-1}(1+z_n)+z_n}}{\frac{z_n}{x_{n-1}(1+z_n)}(1+\frac{1}{x_{n-1}(1+z_n)+z_n})} = \frac{x_{n-1}}{z_n(1+x_{n-1})},$$

$$y_{n+8} = \frac{z_{n+7}}{x_{n+6}(1+z_{n+7})} = \frac{\frac{1}{y_{n-1}(1+x_n)+x_n}}{\frac{x_n}{y_{n-1}(1+x_n)}(1+\frac{1}{y_{n-1}(1+x_n)+x_n})} = \frac{y_{n-1}}{x_n(1+y_{n-1})},$$

$$z_{n+8} = \frac{x_{n+7}}{y_{n+6}(1+x_{n+7})} = \frac{\frac{1}{z_{n-1}(1+y_n)+y_n}}{\frac{y_n}{z_{n-1}(1+y_n)}(1+\frac{1}{z_{n-1}(1+y_n)+y_n})} = \frac{z_{n-1}}{y_n(1+z_{n-1})}.$$

$$x_{n+9} = \frac{y_{n+8}}{z_{n+7}(1+y_{n+8})} = \frac{\frac{y_{n-1}}{x_n(1+y_{n-1})}}{\frac{1}{y_{n-1}(1+x_n)+x_n}(1+\frac{y_{n-1}}{x_n(1+y_{n-1})})} = y_{n-1},$$

$$y_{n+9} = \frac{z_{n+8}}{x_{n+7}(1+z_{n+8})} = \frac{\frac{z_{n-1}}{y_n(1+z_{n-1})}}{\frac{1}{z_{n-1}(1+y_n)+y_n}(1+\frac{z_{n-1}}{y_n(1+z_{n-1})})} = z_{n-1},$$

$$z_{n+9} = \frac{x_{n+8}}{y_{n+7}(1+x_{n+8})} = \frac{\frac{x_{n-1}}{z_n(1+x_{n-1})}}{\frac{1}{x_{n-1}(1+z_n)+z_n}(1+\frac{x_{n-1}}{z_n(1+x_{n-1})})} = x_{n-1}.$$

$$x_{n+10} = \frac{y_{n+9}}{z_{n+8}(1+y_{n+9})} = \frac{z_{n-1}}{\frac{z_{n-1}}{y_n(1+z_{n-1})}(1+z_{n-1})} = y_n,$$

$$y_{n+10} = \frac{z_{n+9}}{x_{n+8}(1+z_{n+9})} = \frac{x_{n-1}}{\frac{x_{n-1}}{z_n(1+x_{n-1})}(1+x_{n-1})} = z_n,$$

$$z_{n+10} = \frac{x_{n+9}}{y_{n+8}(1+x_{n+9})} = \frac{y_{n-1}}{\frac{y_{n-1}}{x_n(1+y_{n-1})}(1+y_{n-1})} = x_n.$$

$$x_{n+11} = \frac{y_{n+10}}{z_{n+9}(1+y_{n+10})} = \frac{z_n}{x_{n-1}(1+z_n)},$$

$$y_{n+11} = \frac{z_{n+10}}{x_{n+9}(1+z_{n+10})} = \frac{x_n}{y_{n-1}(1+x_n)},$$

$$z_{n+11} = \frac{x_{n+10}}{y_{n+9}(1+x_{n+10})} = \frac{y_n}{z_{n-1}(1+y_n)}.$$

$$x_{n+12} = \frac{y_{n+11}}{z_{n+10}(1+y_{n+11})} = \frac{\frac{x_n}{y_{n-1}(1+x_n)}}{x_n(1+\frac{x_n}{y_{n-1}(1+x_n)})} = \frac{1}{y_{n-1}(1+x_n)+x_n},$$

$$y_{n+12} = \frac{z_{n+11}}{x_{n+10}(1+z_{n+11})} = \frac{\frac{y_n}{z_{n-1}(1+y_n)}}{y_n(1+\frac{y_n}{z_{n-1}(1+y_n)})} = \frac{1}{z_{n-1}(1+y_n)+y_n},$$

$$z_{n+12} = \frac{x_{n+11}}{y_{n+10}(1+x_{n+11})} = \frac{\frac{z_n}{x_{n-1}(1+z_n)}}{z_n(1+\frac{z_n}{x_{n-1}(1+z_n)})} = \frac{1}{x_{n-1}(1+z_n)+z_n}.$$

$$x_{n+13} = \frac{y_{n+12}}{z_{n+11}(1+y_{n+12})} = \frac{\frac{1}{z_{n-1}(1+y_n)+y_n}}{\frac{y_n}{z_{n-1}(1+y_n)}(1+\frac{1}{z_{n-1}(1+y_n)+y_n})} = \frac{z_{n-1}}{y_n(1+z_{n-1})},$$

$$y_{n+13} = \frac{z_{n+12}}{x_{n+11}(1+z_{n+12})} = \frac{\frac{1}{x_{n-1}(1+z_n)+z_n}}{\frac{z_n}{x_{n-1}(1+z_n)}(1+\frac{1}{x_{n-1}(1+z_n)+z_n})} = \frac{x_{n-1}}{z_n(1+x_{n-1})},$$

$$z_{n+13} = \frac{x_{n+12}}{y_{n+11}(1+x_{n+12})} = \frac{\frac{1}{y_{n-1}(1+x_n)+x_n}}{\frac{x_n}{y_{n-1}(1+x_n)}(1+\frac{1}{y_{n-1}(1+x_n)+x_n})} = \frac{y_{n-1}}{x_n(1+y_{n-1})}.$$

$$x_{n+14} = \frac{y_{n+13}}{z_{n+12}(1+y_{n+13})} = \frac{\frac{x_{n-1}}{z_n(1+x_{n-1})}}{\frac{1}{x_{n-1}(1+z_n)+z_n}(1+\frac{x_{n-1}}{z_n(1+x_{n-1})})} = x_{n-1},$$

$$y_{n+14} = \frac{z_{n+13}}{x_{n+12}(1+z_{n+13})} = \frac{\frac{y_{n-1}}{x_n(1+y_{n-1})}}{\frac{1}{y_{n-1}(1+x_n)+x_n}(1+\frac{y_{n-1}}{x_n(1+y_{n-1})})} = y_{n-1},$$

$$z_{n+14} = \frac{x_{n+13}}{y_{n+12}(1+x_{n+13})} = \frac{\frac{z_{n-1}}{y_n(1+z_{n-1})}}{\frac{1}{z_{n-1}(1+y_n)+y_n}(1+\frac{z_{n-1}}{y_n(1+z_{n-1})})} = z_{n-1}.$$

ve böylece sistemin on beş periyotlu olduğu elde edilir.

(b) (a)' daki eşitlikler yardımıyla

$x_{n+5} = z_n, y_{n+5} = x_n \ ve \ z_{n+5} = y_n$

sağlandığı görülür.

(c) (a)' daki eşitliklerden,

$x_{n+10} = y_n, y_{n+10} = z_n \ ve \ z_{n+10} = x_n$

elde edilir.

(d) $n = 0$ için sonuçlar sağlanır. $n > 0$ olduğunu ve iddiamızın $(n-1)$ için sağlandığını varsayalım. Yani,

$$x_{15n-16} = x_{-1},$$
$$x_{15n-15} = x_0,$$
$$x_{15n-14} = \frac{y_0}{z_{-1}(1+y_0)},$$
$$x_{15n-13} = \frac{1}{x_{-1}(1+z_0)+z_0},$$
$$x_{15n-12} = \frac{y_{-1}}{x_0(1+y_{-1})},$$
$$x_{15n-11} = z_{-1},$$
$$x_{15n-10} = z_0,$$
$$x_{15n-9} = \frac{x_0}{y_{-1}(1+x_0)},$$
$$x_{15n-8} = \frac{1}{z_{-1}(1+y_0)+y_0},$$
$$x_{15n-7} = \frac{x_{-1}}{z_0(1+x_{-1})},$$
$$x_{15n-6} = y_{-1},$$
$$x_{15n-5} = y_0,$$
$$x_{15n-4} = \frac{z_0}{x_{-1}(1+z_0)},$$
$$x_{15n-3} = \frac{1}{y_{-1}(1+x_0)+x_0},$$
$$x_{15n-2} = \frac{z_{-1}}{y_0(1+z_{-1})}.$$

ve

$$y_{15n-16} = y_{-1},$$
$$y_{15n-15} = y_0,$$
$$y_{15n-14} = \frac{z_0}{x_{-1}(1+z_0)},$$
$$y_{15n-13} = \frac{1}{y_{-1}(1+x_0)+x_0},$$
$$y_{15n-12} = \frac{z_{-1}}{y_0(1+z_{-1})},$$
$$y_{15n-11} = x_{-1},$$
$$y_{15n-10} = x_0,$$
$$y_{15n-9} = \frac{y_0}{z_{-1}(1+y_0)},$$
$$y_{15n-8} = \frac{1}{x_{-1}(1+z_0)+z_0},$$
$$y_{15n-7} = \frac{y_{-1}}{x_0(1+y_{-1})},$$
$$y_{15n-6} = z_{-1},$$
$$y_{15n-5} = z_0,$$
$$y_{15n-4} = \frac{x_0}{y_{-1}(1+x_0)},$$
$$y_{15n-3} = \frac{1}{z_{-1}(1+y_0)+y_0},$$
$$y_{15n-2} = \frac{x_{-1}}{z_0(1+x_{-1})}.$$

ve

$$\begin{aligned}
z_{15n-16} &= z_{-1},\\
z_{15n-15} &= z_0,\\
z_{15n-14} &= \frac{x_0}{y_{-1}(1+x_0)},\\
z_{15n-13} &= \frac{1}{z_{-1}(1+y_0)+y_0},\\
z_{15n-12} &= \frac{x_{-1}}{z_0(1+x_{-1})},\\
z_{15n-11} &= y_{-1},\\
z_{15n-10} &= y_0,\\
z_{15n-9} &= \frac{z_0}{x_{-1}(1+z_0)},\\
z_{15n-8} &= \frac{1}{y_{-1}(1+x_0)+x_0},\\
z_{15n-7} &= \frac{z_{-1}}{y_0(1+z_{-1})},\\
z_{15n-6} &= x_{-1},\\
z_{15n-5} &= x_0,\\
z_{15n-4} &= \frac{y_0}{z_{-1}(1+y_0)},\\
z_{15n-3} &= \frac{1}{x_{-1}(1+z_0)+z_0},\\
z_{15n-2} &= \frac{y_{-1}}{x_0(1+y_{-1})}.
\end{aligned}$$

olduğu görülür ve (3.5) denklem sistemi yardımıyla

$$\begin{aligned}
x_{15n-1} &= \frac{y_{15n-2}}{z_{15n-3}(1+y_{15n-2})} = \frac{\frac{x_{-1}}{z_0(1+x_{-1})}}{\frac{1}{x_{-1}(1+z_0)+z_0}(1+\frac{x_{-1}}{z_0(1+x_{-1})})} = x_{-1},\\
y_{15n-1} &= \frac{z_{15n-2}}{x_{15n-3}(1+z_{15n-2})} = \frac{\frac{y_{-1}}{x_0(1+y_{-1})}}{\frac{1}{y_{-1}(1+x_0)+x_0}(1+\frac{y_{-1}}{x_0(1+y_{-1})})} = y_{-1},\\
z_{15n-1} &= \frac{x_{15n-2}}{y_{15n-3}(1+x_{15n-2})} = \frac{\frac{z_{-1}}{y_0(1+z_{-1})}}{\frac{1}{z_{-1}(1+y_0)+y_0}(1+\frac{z_{-1}}{y_0(1+z_{-1})})} = z_{-1}.
\end{aligned}$$

$$\begin{aligned}
x_{15n} &= \frac{y_{15n-1}}{z_{15n-2}(1+y_{15n-1})} = \frac{y_{-1}}{\frac{y_{-1}}{x_0(1+y_{-1})}(1+y_{-1})} = x_0,\\
y_{15n} &= \frac{z_{15n-1}}{x_{15n-2}(1+z_{15n-1})} = \frac{z_{-1}}{\frac{z_{-1}}{y_0(1+z_{-1})}(1+z_{-1})} = y_0,\\
z_{15n} &= \frac{x_{15n-1}}{y_{15n-2}(1+x_{15n-1})} = \frac{x_{-1}}{\frac{x_{-1}}{z_0(1+x_{-1})}(1+x_{-1})} = z_0
\end{aligned}$$

ve benzer şekilde

$$\begin{aligned}
x_{15n+1} &= x_1,\\
y_{15n+1} &= y_1,\\
z_{15n+1} &= z_1.\\
&\vdots
\end{aligned}$$

çözümleri elde edilerek ispat tamamlanır.

■

Teorem 3.3.2 (3.6) *denklem sisteminin çözümlerinin* $\{x_n, y_n, z_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $n \geqslant -1$ *için* $\{x_n\}_{n=-1}^{+\infty}$, $\{y_n\}_{n=-1}^{+\infty}$ *ve* $\{z_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve on beş periyotludur.*

(b) $n \geqslant -1$ *için* $x_{n+5} = z_n$, $y_{n+5} = x_n$ *ve* $z_{n+5} = y_n$ *dir.*

(c) $n \geqslant -1$ *için* $x_{n+10} = y_n$, $y_{n+10} = z_n$ *ve* $z_{n+10} = x_n$ *dir.*

(d)

$$\begin{aligned}
x_{15n-1} &= x_{-1}, \\
x_{15n} &= x_0, \\
x_{15n+1} &= \tfrac{y_0}{z_{-1}(-1+y_0)}, \\
x_{15n+2} &= \tfrac{1}{-x_{-1}(-1+z_0)+z_0}, \\
x_{15n+3} &= \tfrac{y_{-1}}{x_0(-1+y_{-1})}, \\
x_{15n+4} &= z_{-1}, \\
x_{15n+5} &= z_0, \\
x_{15n+6} &= \tfrac{x_0}{y_{-1}(-1+x_0)}, \\
x_{15n+7} &= \tfrac{1}{-z_{-1}(-1+y_0)+y_0}, \\
x_{15n+8} &= \tfrac{x_{-1}}{z_0(-1+x_{-1})}, \\
x_{15n+9} &= y_{-1}, \\
x_{15n+10} &= y_0, \\
x_{15n+11} &= \tfrac{z_0}{x_{-1}(-1+z_0)}, \\
x_{15n+12} &= \tfrac{1}{-y_{-1}(-1+x_0)+x_0}, \\
x_{15n+13} &= \tfrac{z_{-1}}{y_0(-1+z_{-1})}.
\end{aligned}$$

ve

$$
\begin{aligned}
y_{15n-1} &= y_{-1},\\
y_{15n} &= y_0,\\
y_{15n+1} &= \tfrac{z_0}{x_{-1}(-1+z_0)},\\
y_{15n+2} &= \tfrac{1}{-y_{-1}(-1+x_0)+x_0},\\
y_{15n+3} &= \tfrac{z_{-1}}{y_0(-1+z_{-1})},\\
y_{15n+4} &= x_{-1},\\
y_{15n+5} &= x_0,\\
y_{15n+6} &= \tfrac{y_0}{z_{-1}(-1+y_0)},\\
y_{15n+7} &= \tfrac{1}{-x_{-1}(-1+z_0)+z_0},\\
y_{15n+8} &= \tfrac{y_{-1}}{x_0(-1+y_{-1})},\\
y_{15n+9} &= z_{-1},\\
y_{15n+10} &= z_0,\\
y_{15n+11} &= \tfrac{x_0}{y_{-1}(-1+x_0)},\\
y_{15n+12} &= \tfrac{1}{-z_{-1}(-1+y_0)+y_0},\\
y_{15n+13} &= \tfrac{x_{-1}}{z_0(-1+x_{-1})}.
\end{aligned}
$$

ve

$$
\begin{aligned}
z_{15n-1} &= z_{-1},\\
z_{15n} &= z_0,\\
z_{15n+1} &= \tfrac{x_0}{y_{-1}(-1+x_0)},\\
z_{15n+2} &= \tfrac{1}{-z_{-1}(-1+y_0)+y_0},\\
z_{15n+3} &= \tfrac{x_{-1}}{z_0(-1+x_{-1})},\\
z_{15n+4} &= y_{-1},\\
z_{15n+5} &= y_0,\\
z_{15n+6} &= \tfrac{z_0}{x_{-1}(-1+z_0)},\\
z_{15n+7} &= \tfrac{1}{-y_{-1}(-1+x_0)+x_0},\\
z_{15n+8} &= \tfrac{z_{-1}}{y_0(-1+z_{-1})},\\
z_{15n+9} &= x_{-1},\\
z_{15n+10} &= x_0,\\
z_{15n+11} &= \tfrac{y_0}{z_{-1}(-1+y_0)},\\
z_{15n+12} &= \tfrac{1}{-x_{-1}(-1+z_0)+z_0},\\
z_{15n+13} &= \tfrac{y_{-1}}{x_0(-1+y_{-1})}.
\end{aligned}
$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l}
x_{-1},x_0,\dfrac{y_0}{z_{-1}(-1+y_0)},\dfrac{1}{-x_{-1}(-1+z_0)+z_0},\dfrac{y_{-1}}{x_0(-1+y_{-1})},\\
z_{-1},z_0,\dfrac{x_0}{y_{-1}(-1+x_0)},\dfrac{1}{-z_{-1}(-1+y_0)+y_0},\dfrac{x_{-1}}{z_0(-1+x_{-1})},\\
y_{-1},y_0,\dfrac{z_0}{x_{-1}(-1+z_0)},\dfrac{1}{-y_{-1}(-1+x_0)+x_0},\dfrac{z_{-1}}{y_0(-1+z_{-1})},\\
x_{-1},x_0,\dfrac{y_0}{z_{-1}(-1+y_0)},\dfrac{1}{-x_{-1}(-1+z_0)+z_0},\dfrac{y_{-1}}{x_0(-1+y_{-1})},\\
z_{-1},z_0,\dfrac{x_0}{y_{-1}(-1+x_0)},\dfrac{1}{-z_{-1}(-1+y_0)+y_0},\dfrac{x_{-1}}{z_0(-1+x_{-1})},\\
y_{-1},y_0,\dfrac{z_0}{x_{-1}(-1+z_0)},\dfrac{1}{-y_{-1}(-1+x_0)+x_0},\dfrac{z_{-1}}{y_0(-1+z_{-1})},\\
x_{-1},x_0,\dfrac{y_0}{z_{-1}(-1+y_0)},\dfrac{1}{-x_{-1}(-1+z_0)+z_0},\dfrac{y_{-1}}{x_0(-1+y_{-1})},\\
z_{-1},z_0,\dfrac{x_0}{y_{-1}(-1+x_0)},\dfrac{1}{-z_{-1}(-1+y_0)+y_0},\dfrac{x_{-1}}{z_0(-1+x_{-1})},\\
y_{-1},y_0,\dfrac{z_0}{x_{-1}(-1+z_0)},\dfrac{1}{-y_{-1}(-1+x_0)+x_0},\dfrac{z_{-1}}{y_0(-1+z_{-1})},\\
\ldots
\end{array}\right\}$$

$$\{y_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l}
y_{-1},y_0,\dfrac{z_0}{x_{-1}(-1+z_0)},\dfrac{1}{-y_{-1}(-1+x_0)+x_0},\dfrac{z_{-1}}{y_0(-1+z_{-1})},\\
x_{-1},x_0,\dfrac{y_0}{z_{-1}(-1+y_0)},\dfrac{1}{-x_{-1}(-1+z_0)+z_0},\dfrac{y_{-1}}{x_0(-1+y_{-1})},\\
z_{-1},z_0,\dfrac{x_0}{y_{-1}(-1+x_0)},\dfrac{1}{-z_{-1}(-1+y_0)+y_0},\dfrac{x_{-1}}{z_0(-1+x_{-1})},\\
y_{-1},y_0,\dfrac{z_0}{x_{-1}(-1+z_0)},\dfrac{1}{-y_{-1}(-1+x_0)+x_0},\dfrac{z_{-1}}{y_0(-1+z_{-1})},\\
x_{-1},x_0,\dfrac{y_0}{z_{-1}(-1+y_0)},\dfrac{1}{-x_{-1}(-1+z_0)+z_0},\dfrac{y_{-1}}{x_0(-1+y_{-1})},\\
z_{-1},z_0,\dfrac{x_0}{y_{-1}(-1+x_0)},\dfrac{1}{-z_{-1}(-1+y_0)+y_0},\dfrac{x_{-1}}{z_0(-1+x_{-1})},\\
y_{-1},y_0,\dfrac{z_0}{x_{-1}(-1+z_0)},\dfrac{1}{-y_{-1}(-1+x_0)+x_0},\dfrac{z_{-1}}{y_0(-1+z_{-1})},\\
x_{-1},x_0,\dfrac{y_0}{z_{-1}(-1+y_0)},\dfrac{1}{-x_{-1}(-1+z_0)+z_0},\dfrac{y_{-1}}{x_0(-1+y_{-1})},\\
z_{-1},z_0,\dfrac{x_0}{y_{-1}(-1+x_0)},\dfrac{1}{-z_{-1}(-1+y_0)+y_0},\dfrac{x_{-1}}{z_0(-1+x_{-1})},\\
\ldots
\end{array}\right\}$$

$$\{z_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} z_{-1}, z_0, \dfrac{x_0}{y_{-1}(-1+x_0)}, \dfrac{1}{-z_{-1}(-1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(-1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(-1+z_0)}, \dfrac{1}{-y_{-1}(-1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(-1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(-1+y_0)}, \dfrac{1}{-x_{-1}(-1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(-1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(-1+x_0)}, \dfrac{1}{-z_{-1}(-1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(-1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(-1+z_0)}, \dfrac{1}{-y_{-1}(-1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(-1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(-1+y_0)}, \dfrac{1}{-x_{-1}(-1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(-1+y_{-1})}, \\ z_{-1}, z_0, \dfrac{x_0}{y_{-1}(-1+x_0)}, \dfrac{1}{-z_{-1}(-1+y_0)+y_0}, \dfrac{x_{-1}}{z_0(-1+x_{-1})}, \\ y_{-1}, y_0, \dfrac{z_0}{x_{-1}(-1+z_0)}, \dfrac{1}{-y_{-1}(-1+x_0)+x_0}, \dfrac{z_{-1}}{y_0(-1+z_{-1})}, \\ x_{-1}, x_0, \dfrac{y_0}{z_{-1}(-1+y_0)}, \dfrac{1}{-x_{-1}(-1+z_0)+z_0}, \dfrac{y_{-1}}{x_0(-1+y_{-1})}, \\ \ldots \end{array} \right\}$$

çözümleri elde edilir.

İspat. Teorem 3.3.1 ile benzer yolla ispatı görülür. ■

Örnek 3.3.1

$$x_{n+1} = \frac{y_n}{z_{n-1}(-1+y_n)}, y_{n+1} = \frac{z_n}{x_{n-1}(-1+z_n)}, z_{n+1} = \frac{x_n}{y_{n-1}(-1+x_n)} \tag{3.6}$$

denklem sisteminin $x_0 = 1.2$, $x_{-1} = -0.3$, $y_0 = 0.4$, $y_{-1} = 1.4$, $z_0 = -0.1$ *ve* $z_{-1} = 0.5$ *başlangıç koşullarındaki çözümleri aşağıda verilmiş ve 15 periyotlu periyodik olduğu görülmüştür.*

Çizelge 3.3 (3.6) denkleminin periyodik çözümleri

n	x_n	y_n	z_n
0	1.2	0.4	-0.1
1	-1.333	-0.303	4.285
2	-2.325	1.086	1.428
3	2.916	-2.5	-2.307
4	0.5	-0.3	1.4
5	-0.1	1.2	0.4
6	4.285	-1.333	-0.303
7	1.428	-2.325	1.086
8	-2.307	2.916	-2.5
9	1.4	0.5	-0.3
10	0.4	-0.1	1.2
11	-0.303	4.285	-1.333
12	1.086	1.428	-2.325
13	-2.5	-2.307	2.916
14	-0.3	1.4	0.5
15	1.2	0.4	-0.1
16	-1.333	-0.303	4.285
17	-2.325	1.086	1.428
18	2.916	-2.5	-2.307
$\vdots$	$\vdots$	$\vdots$	$\vdots$

BÖLÜM 4

BAZI ÖZEL RASYONEL FARK DENKLEM SİSTEMLERİNİN PERİYODİKLİĞİ

4.1 $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1 \pm x_n x_{n-2})}$ FARK DENKLEMLERİNİN PERİYODİKLİĞİ

Bu bölümde x_{-2}, x_{-1}, x_0 başlangıç koşulları paydayı sıfır yapmayacak reel sayılar olmak üzere,

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1 + x_n x_{n-2})} \tag{4.1}$$

ve

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1 - x_n x_{n-2})} \tag{4.2}$$

fark denklemlerinin çözümleri araştırılmıştır.

Teorem 4.1.1 (4.1) *denkleminin çözümlerinin* $\{x_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$ *çözümleri dört periyotlu periyodiktir.*

(b)

$$\begin{aligned} x_{4n-1} &= x_{-1} \\ x_{4n} &= x_0 \\ x_{4n+1} &= \tfrac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})} \\ x_{4n+2} &= x_{-2} \end{aligned}$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} x_{-1}, x_0, \dfrac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})}, x_{-2}, x_{-1}, x_0, \dfrac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})}, \\ x_{-2}, x_{-1}, x_0, \dfrac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})}, x_{-2}, ... \end{array} \right\}$$

çözümleri elde edilir.

İspat.

(a) (4.1) denklemi kullanılarak aşağıdaki eşitliklerin sağlandığı görülür:

$$
\begin{aligned}
x_{n+1} &= \tfrac{x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})},\\
x_{n+2} &= \tfrac{x_{n+1} x_{n-1}}{x_n(-1+x_{n+1} x_{n-1})} = \frac{\frac{x_n x_{n-2} x_{n-1}}{x_{n-1}(-1+x_n x_{n-2})}}{x_n(-1+\frac{x_n x_{n-2} x_{n-1}}{x_{n-1}(-1+x_n x_{n-2})})} = x_{n-2},\\
x_{n+3} &= \tfrac{x_{n+2} x_n}{x_{n+1}(-1+x_{n+2} x_n)} = \frac{x_{n-2} x_n}{\frac{x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})}(-1+x_{n-2} x_n)} = x_{n-1},\\
x_{n+4} &= \tfrac{x_{n+3} x_{n+1}}{x_{n+2}(-1+x_{n+3} x_{n+1})} = \frac{\frac{x_{n-1} x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})}}{x_{n-2}(-1+\frac{x_{n-1} x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})})} = x_n.
\end{aligned}
$$

(b) $n = 0$ için sonuçlar sağlanır. $n > 0$ iken iddiamızın $(n-1)$ için sağlandığını varsayalım. Yani,

$$
\begin{aligned}
x_{4n-5} &= x_{-1},\\
x_{4n-4} &= x_0,\\
x_{4n-3} &= \tfrac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})},\\
x_{4n-2} &= x_{-2}
\end{aligned}
$$

eşitlikleri elde edilir ve denklem (4.1) yardımıyla

$$
x_{4n-1} = \frac{x_{4n-2} x_{4n-4}}{x_{4n-3}(-1+x_{4n-2} x_{4n-4})} = \frac{x_{-2} x_0}{\frac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})}(-1+x_{-2} x_0)} = x_{-1},
$$

$$
x_{4n} = \frac{x_{4n-1} x_{4n-3}}{x_{4n-2}(-1+x_{4n-1} x_{4n-3})} = \frac{x_{-1}\frac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})}}{x_{-2}(-1+x_{-1}\frac{x_0 x_{-2}}{x_{-1}(-1+x_0 x_{-2})})} = x_0
$$

ve benzer şekilde

$$
\begin{aligned}
x_{4n+1} &= x_1,\\
&\vdots
\end{aligned}
$$

çözümleri elde edilerek ispat tamamlanır.

■

Teorem 4.1.2 (4.2) *denkleminin çözümlerinin* $\{x_n\}_{n=-1}^{+\infty}$ *olduğunu farz edelim. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$ çözümleri periyodiktir ve dört periyotludur.

(b)

$$\begin{aligned} x_{4n-1} &= x_{-1} \\ x_{4n} &= x_0 \\ x_{4n+1} &= \tfrac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})} \\ x_{4n+2} &= x_{-2} \end{aligned}$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} x_{-1}, x_0, \dfrac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})}, x_{-2}, x_{-1}, x_0, \dfrac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})}, \\ x_{-2}, x_{-1}, x_0, \dfrac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})}, x_{-2}, \ldots \end{array} \right\}$$

çözümleri elde edilir.

İspat.

(a) (4.2) denklemi kullanılarak aşağıdaki eşitlikler elde edilir:

$$\begin{aligned} x_{n+1} &= \tfrac{x_nx_{n-2}}{x_{n-1}(-1-x_nx_{n-2})}, \\ x_{n+2} &= \tfrac{x_{n+1}x_{n-1}}{x_n(-1-x_{n+1}x_{n-1})} = \frac{\dfrac{x_nx_{n-2}x_{n-1}}{x_{n-1}(-1-x_nx_{n-2})}}{x_n(-1-\dfrac{x_nx_{n-2}x_{n-1}}{x_{n-1}(-1-x_nx_{n-2})})} = x_{n-2}, \\ x_{n+3} &= \tfrac{x_{n+2}x_n}{x_{n+1}(-1-x_{n+2}x_n)} = \frac{x_{n-2}x_n}{\dfrac{x_nx_{n-2}}{x_{n-1}(-1-x_nx_{n-2})}(-1-x_{n-2}x_n)} = x_{n-1}, \\ x_{n+4} &= \tfrac{x_{n+3}x_{n+1}}{x_{n+2}(-1-x_{n+3}x_{n+1})} = \frac{\dfrac{x_{n-1}x_nx_{n-2}}{x_{n-1}(-1-x_nx_{n-2})}}{x_{n-2}(-1-\dfrac{x_{n-1}x_nx_{n-2}}{x_{n-1}(-1-x_nx_{n-2})})} = x_n. \end{aligned}$$

(b) $n = 0$ için sonuçlar sağlanır. $n > 0$ olduğunu ve iddiamızın $(n-1)$ için sağlandığını varsayalım. Yani,

$$\begin{aligned} x_{4n-5} &= x_{-1}, \\ x_{4n-4} &= x_0, \\ x_{4n-3} &= \tfrac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})}, \\ x_{4n-2} &= x_{-2}. \end{aligned}$$

ve denklem (4.2)' den,

$$x_{4n-1} = \frac{x_{4n-2}x_{4n-4}}{x_{4n-3}(-1-x_{4n-2}x_{4n-4})} = \frac{x_{-2}x_0}{\frac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})}(-1-x_{-2}x_0)} = x_{-1},$$

$$x_{4n} = \frac{x_{4n-1}x_{4n-3}}{x_{4n-2}(-1-x_{4n-1}x_{4n-3})} = \frac{x_{-1}\frac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})}}{x_{-2}(-1-x_{-1}\frac{x_0x_{-2}}{x_{-1}(-1-x_0x_{-2})})} = x_0,$$

ve benzer şekilde

$$x_{4n+1} = x_1,$$
$$\vdots$$

çözümleri elde edilerek ispat tamamlanır.

■

Örnek 4.1.1

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(-1+x_n x_{n-2})} \tag{4.1}$$

denkleminin $x_0 = 0.1$, $x_{-1} = 0.2$ ve $x_{-2} = 0.3$ başlangıç koşullarındaki çözümleri aşağıda verilmiş ve 4 periyotlu periyodik olduğu görülmüştür.

Çizelge 4.1 (4.1) denkleminin periyodik çözümleri

n	x_n
0	0.1
1	− 0.154
2	0.3
3	0.2
4	0.1
5	− 0.154
⋮	⋮

4.2 $x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1}(-1\pm y_n y_{n-2})}$, $y_{n+1} = \frac{x_n x_{n-2}}{y_{n-1}(-1\pm x_n x_{n-2})}$ FARK DENKLEM SİSTEMLERİNİN PERİYODİKLİĞİ

Bu bölümde $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ başlangıç koşulları paydayı sıfır yapmayacak reel sayılar olmak üzere,

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})}, y_{n+1} = \frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})} \tag{4.3}$$

ve

$$x_{n+1}=\frac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})},y_{n+1}=\frac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})} \quad (4.4)$$

fark denklem sistemlerinin çözümleri araştırılmıştır.

Teorem 4.2.1 (4.3) *denklem sisteminin çözümlerinin* $\{x_n,y_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$ *ve* $\{y_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve dört periyotludur.*

(b)

$$\begin{aligned} x_{4n-1} &= x_{-1} \\ x_{4n} &= x_0 \\ x_{4n+1} &= \tfrac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})} \\ x_{4n+2} &= x_{-2} \end{aligned}$$

ve

$$\begin{aligned} y_{4n-1} &= y_{-1} \\ y_{4n} &= y_0 \\ y_{4n+1} &= \tfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})} \\ y_{4n+2} &= y_{-2} \end{aligned}$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l} x_{-1},x_0,\dfrac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})},x_{-2},x_{-1},x_0,\dfrac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}, \\ x_{-2},x_{-1},x_0,\dfrac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})},x_{-2},... \end{array}\right\}$$

$$\{y_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l} y_{-1},y_0,\dfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})},y_{-2},y_{-1},y_0,\dfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}, \\ y_{-2},y_{-1},y_0,\dfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})},y_{-2},... \end{array}\right\}$$

çözümleri elde edilir.

İspat.

(a) (4.3) denklem sistemi yardımıyla aşağıdaki eşitlikler elde edilir:

$$
\begin{aligned}
x_{n+1} &= \tfrac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})}, \\
y_{n+1} &= \tfrac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+2} &= \tfrac{y_{n+1} y_{n-1}}{x_n(-1+y_{n+1} y_{n-1})} = \frac{\frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})} y_{n-1}}{x_n\left(-1+\frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})} y_{n-1}\right)} = x_{n-2}, \\
y_{n+2} &= \tfrac{x_{n+1} x_{n-1}}{y_n(-1+x_{n+1} x_{n-1})} = \frac{\frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})} x_{n-1}}{y_n\left(-1+\frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})} x_{n-1}\right)} = y_{n-2}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+3} &= \tfrac{y_{n+2} y_n}{x_{n+1}(-1+y_{n+2} y_n)} = \frac{y_{n-2} y_n}{\frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})}(-1+y_{n-2} y_n)} = x_{n-1}, \\
y_{n+3} &= \tfrac{x_{n+2} x_n}{y_{n+1}(-1+x_{n+2} x_n)} = \frac{x_{n-2} x_n}{\frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})}(-1+x_{n-2} x_n)} = y_{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+4} &= \tfrac{y_{n+3} y_{n+1}}{x_{n+2}(-1+y_{n+3} y_{n+1})} = \frac{y_{n-1}\frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})}}{x_{n-2}\left(-1+y_{n-1}\frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})}\right)} = x_n, \\
y_{n+4} &= \tfrac{x_{n+3} x_{n+1}}{y_{n+2}(-1+x_{n+3} x_{n+1})} = \frac{x_{n-1}\frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})}}{y_{n-2}\left(-1+x_{n-1}\frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})}\right)} = y_n.
\end{aligned}
$$

(b) $n = 0$ için sonuçlar sağlanır. $n > 0$ olduğunu ve iddiamızın $(n-1)$ için sağlandığını varsayalım. Yani,

$$
\begin{aligned}
x_{4n-5} &= x_{-1} \\
x_{4n-4} &= x_0 \\
x_{4n-3} &= \tfrac{y_0 y_{-2}}{x_{-1}(-1+y_0 y_{-2})} \\
x_{4n-2} &= x_{-2}
\end{aligned}
$$

ve

$$
\begin{aligned}
y_{4n-5} &= y_{-1} \\
y_{4n-4} &= y_0 \\
y_{4n-3} &= \tfrac{x_0 x_{-2}}{y_{-1}(-1+x_0 x_{-2})} \\
y_{4n-2} &= y_{-2}
\end{aligned}
$$

ve (4.3) denklem sistemi kullanılarak

$$\begin{aligned}
x_{4n-1} &= \frac{y_{4n-2}y_{4n-4}}{x_{4n-3}(-1+y_{4n-2}y_{4n-4})} = \frac{y_{-2}y_0}{\frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}(-1+y_{-2}y_0)} = x_{-1}, \\
y_{4n-1} &= \frac{x_{4n-2}x_{4n-4}}{y_{4n-3}(-1+x_{4n-2}x_{4n-4})} = \frac{x_{-2}x_0}{\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}(-1+x_{-2}x_0)} = y_{-1}.
\end{aligned}$$

$$\begin{aligned}
x_{4n} &= \frac{y_{4n-1}y_{4n-3}}{x_{4n-2}(-1+y_{4n-1}y_{4n-3})} = \frac{y_{-1}\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}}{x_{-2}(-1+y_{-1}\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})})} = x_0, \\
y_{4n} &= \frac{x_{4n-1}x_{4n-3}}{y_{4n-2}(-1+x_{4n-1}x_{4n-3})} = \frac{x_{-1}\frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}}{y_{-2}(-1+x_{-1}\frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})})} = y_0.
\end{aligned}$$

ve benzer şekilde

$$\begin{aligned}
x_{4n+1} &= x_1, \\
y_{4n+1} &= y_1. \\
&\vdots
\end{aligned}$$

çözümleri elde edilerek ispat tamamlanır.

■

Teorem 4.2.2 (4.4) *denklem sisteminin çözümlerinin* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$ *ve* $\{y_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve dört periyotludur.*

(b)

$$\begin{aligned}
x_{4n-1} &= x_{-1} \\
x_{4n} &= x_0 \\
x_{4n+1} &= \frac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})} \\
x_{4n+2} &= x_{-2}
\end{aligned}$$

ve

$$\begin{aligned}
y_{4n-1} &= y_{-1} \\
y_{4n} &= y_0 \\
y_{4n+1} &= \frac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})} \\
y_{4n+2} &= y_{-2}
\end{aligned}$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l}\{x_{-1},x_0,\dfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})},x_{-2},x_{-1},x_0,\dfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})},\\ x_{-2},x_{-1},x_0,\dfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})},x_{-2},...\end{array}\right\}$$

$$\{y_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l}y_{-1},y_0,\dfrac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})},y_{-2},y_{-1},y_0,\dfrac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})},\\ y_{-2},y_{-1},y_0,\dfrac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})},y_{-2},...\end{array}\right\}$$

çözümleri elde edilir.

İspat.

(a) (4.4) denklem sistemi yardımıyla aşağıdaki eşitlikler elde edilir:

$$\begin{aligned}
x_{n+1} &= \tfrac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})},\\
y_{n+1} &= \tfrac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})}.\\
x_{n+2} &= \tfrac{y_{n+1}y_{n-1}}{x_n(-1-y_{n+1}y_{n-1})} = \frac{\dfrac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})}y_{n-1}}{x_n(-1-\dfrac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})}y_{n-1})} = x_{n-2},\\
y_{n+2} &= \tfrac{x_{n+1}x_{n-1}}{y_n(-1-x_{n+1}x_{n-1})} = \frac{\dfrac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})}x_{n-1}}{y_n(-1-\dfrac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})}x_{n-1})} = y_{n-2}.\\
x_{n+3} &= \tfrac{y_{n+2}y_n}{x_{n+1}(-1-y_{n+2}y_n)} = \frac{y_{n-2}y_n}{\dfrac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})}(-1-y_{n-2}y_n)} = x_{n-1},\\
y_{n+3} &= \tfrac{x_{n+2}x_n}{y_{n+1}(-1-x_{n+2}x_n)} = \frac{x_{n-2}x_n}{\dfrac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})}(-1-x_{n-2}x_n)} = y_{n-1}.\\
x_{n+4} &= \tfrac{y_{n+3}y_{n+1}}{x_{n+2}(-1-y_{n+3}y_{n+1})} = \frac{y_{n-1}\dfrac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})}}{x_{n-2}(-1-y_{n-1}\dfrac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})})} = x_n,\\
y_{n+4} &= \tfrac{x_{n+3}x_{n+1}}{y_{n+2}(-1-x_{n+3}x_{n+1})} = \frac{x_{n-1}\dfrac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})}}{y_{n-2}(-1-x_{n-1}\dfrac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})})} = y_n.
\end{aligned}$$

(b) $n=0$ için sonuçlar sağlanır. $n>0$ olduğunu ve iddiamızın $(n-1)$ için sağlandığını varsayalım. Yani,

$$\begin{aligned}
x_{4n-5} &= x_{-1}\\
x_{4n-4} &= x_0\\
x_{4n-3} &= \tfrac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}\\
x_{4n-2} &= x_{-2}
\end{aligned}$$

ve

$$
\begin{aligned}
y_{4n-5} &= y_{-1} \\
y_{4n-4} &= y_0 \\
y_{4n-3} &= \tfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})} \\
y_{4n-2} &= y_{-2}
\end{aligned}
$$

eşitlikleri görülür ve (4.4) denklem sistemi yardımıyla

$$
\begin{aligned}
x_{4n-1} &= \tfrac{y_{4n-2}y_{4n-4}}{x_{4n-3}(-1+y_{4n-2}y_{4n-4})} = \frac{y_{-2}y_0}{\frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}(-1+y_{-2}y_0)} = x_{-1}, \\
y_{4n-1} &= \tfrac{x_{4n-2}x_{4n-4}}{y_{4n-3}(-1+x_{4n-2}x_{4n-4})} = \frac{x_{-2}x_0}{\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}(-1+x_{-2}x_0)} = y_{-1}.
\end{aligned}
$$

$$
\begin{aligned}
x_{4n} &= \tfrac{y_{4n-1}y_{4n-3}}{x_{4n-2}(-1+y_{4n-1}y_{4n-3})} = \frac{y_{-1}\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}}{x_{-2}(-1+y_{-1}\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})})} = x_0, \\
y_{4n} &= \tfrac{x_{4n-1}x_{4n-3}}{y_{4n-2}(-1+x_{4n-1}x_{4n-3})} = \frac{x_{-1}\frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}}{y_{-2}(-1+x_{-1}\frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})})} = y_0.
\end{aligned}
$$

ve benzer şekilde

$$
\begin{aligned}
x_{4n+1} &= x_1, \\
y_{4n+1} &= y_1. \\
&\vdots
\end{aligned}
$$

çözümleri elde edilerek ispat tamamlanır.

■

Örnek 4.2.1

$$
x_{n+1} = \frac{y_ny_{n-2}}{x_{n-1}(-1-y_ny_{n-2})}, y_{n+1} = \frac{x_nx_{n-2}}{y_{n-1}(-1-x_nx_{n-2})} \tag{4.4}
$$

denklem sisteminin $x_0 = 0.3$, $x_{-1} = 1.2$, $x_{-2} = 0.1$, $y_0 = 0.7$, $y_{-1} = -1.4$ *ve* $y_{-2} = 1.1$ *başlangıç koşullarındaki çözümleri aşağıda verilmiş ve 4 periyotlu periyodik olduğu görülmüştür.*

Çizelge 4.2 (4.4) denkleminin periyodik çözümleri

n	x_n	y_n
0	-0.3	0.7
1	-0.362	-0.022
2	0.1	1.1
3	1.2	-1.4
4	-0.3	0.7
5	-0.362	-0.022
6	0.1	1.1
7	1.2	-1.4
$\vdots$	$\vdots$	$\vdots$

4.3 $x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1}(-1 \pm y_n y_{n-2})}, y_{n+1} = \frac{z_n z_{n-2}}{y_{n-1}(-1 \pm z_n z_{n-2})}, z_{n+1} = \frac{x_n x_{n-2}}{z_{n-1}(-1 \pm x_n x_{n-2})}$ FARK DENKLEM SİSTEMLERİNİN PERİYODİKLİĞİ

Bu bölümde $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0$ başlangıç koşulları paydayı sıfır yapmayacak reel sayılar olmak üzere,

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1}(-1 + y_n y_{n-2})}, y_{n+1} = \frac{z_n z_{n-2}}{y_{n-1}(-1 + z_n z_{n-2})}, z_{n+1} = \frac{x_n x_{n-2}}{z_{n-1}(-1 + x_n x_{n-2})} \quad (4.5)$$

ve

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1}(-1 - y_n y_{n-2})}, y_{n+1} = \frac{z_n z_{n-2}}{y_{n-1}(-1 - z_n z_{n-2})}, z_{n+1} = \frac{x_n x_{n-2}}{z_{n-1}(-1 - x_n x_{n-2})} \quad (4.6)$$

fark denklem sistemlerinin çözümleri araştırılmıştır.

Teorem 4.3.1 (4.5) *denklem sisteminin çözümlerinin* $\{x_n, y_n, z_n\}_{n=-1}^{+\infty}$ *olduğunu varsayalım. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$, $\{y_n\}_{n=-1}^{+\infty}$ *ve* $\{z_n\}_{n=-1}^{+\infty}$ *çözümleri periyodiktir ve on iki periyotludur.*

(b)

$$
\begin{aligned}
x_{12n-1} &= x_{-1}, \\
x_{12n} &= x_0, \\
x_{12n+1} &= \tfrac{y_0 y_{-2}}{x_{-1}(-1+y_0 y_{-2})}, \\
x_{12n+2} &= \tfrac{z_0 z_{-2}}{x_0}, \\
x_{12n+3} &= \tfrac{x_0 x_{-1} x_{-2}(-1+y_0 y_{-2})}{y_0 y_{-2}(-1+x_0 x_{-2})}, \\
x_{12n+4} &= \tfrac{x_0 y_0 y_{-2}}{z_0 z_{-2}}, \\
x_{12n+5} &= \tfrac{z_0 z_{-2} y_0 y_{-2}(-1+x_0 x_{-2})}{x_0 x_{-1} x_{-2}(-1+y_0 y_{-2})(-1+z_0 z_{-2})}, \\
x_{12n+6} &= \tfrac{z_0 z_{-2} x_{-2}}{y_0 y_{-2}}, \\
x_{12n+7} &= \tfrac{x_0 x_{-1} x_{-2}(-1+z_0 z_{-2})}{z_0 z_{-2}(-1+x_0 x_{-2})}, \\
x_{12n+8} &= \tfrac{y_0 y_{-2}}{x_{-2}}, \\
x_{12n+9} &= \tfrac{z_0 z_{-2}}{x_{-1}(-1+z_0 z_{-2})}, \\
x_{12n+10} &= x_{-2},
\end{aligned}
$$

ve

$$
\begin{aligned}
y_{12n-1} &= y_{-1}, \\
y_{12n} &= y_0, \\
y_{12n+1} &= \tfrac{z_0 z_{-2}}{y_{-1}(-1+z_0 z_{-2})}, \\
y_{12n+2} &= \tfrac{x_0 x_{-2}}{y_0}, \\
y_{12n+3} &= \tfrac{y_0 y_{-1} y_{-2}(-1+z_0 z_{-2})}{z_0 z_{-2}(-1+y_0 y_{-2})}, \\
y_{12n+4} &= \tfrac{y_0 z_0 z_{-2}}{x_0 x_{-2}}, \\
y_{12n+5} &= \tfrac{x_0 x_{-2} z_0 z_{-2}(-1+y_0 y_{-2})}{y_0 y_{-1} y_{-2}(-1+z_0 z_{-2})(-1+x_0 x_{-2})}, \\
y_{12n+6} &= \tfrac{x_0 x_{-2} y_{-2}}{z_0 z_{-2}}, \\
y_{12n+7} &= \tfrac{y_0 y_{-1} y_{-2}(-1+x_0 x_{-2})}{x_0 x_{-2}(-1+y_0 y_{-2})}, \\
y_{12n+8} &= \tfrac{z_0 z_{-2}}{y_{-2}}, \\
y_{12n+9} &= \tfrac{x_0 x_{-2}}{y_{-1}(-1+x_0 x_{-2})}, \\
y_{12n+10} &= y_{-2},
\end{aligned}
$$

ve

$$
\begin{aligned}
z_{12n-1} &= z_{-1},\\
z_{12n} &= z_0,\\
z_{12n+1} &= \tfrac{x_0x_{-2}}{z_{-1}(-1+x_0x_{-2})},\\
z_{12n+2} &= \tfrac{y_0y_{-2}}{z_0},\\
z_{12n+3} &= \tfrac{z_0z_{-1}z_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+z_0z_{-2})},\\
z_{12n+4} &= \tfrac{z_0x_0x_{-2}}{y_0y_{-2}},\\
z_{12n+5} &= \tfrac{y_0y_{-2}x_0x_{-2}(-1+z_0z_{-2})}{z_0z_{-1}z_{-2}(-1+x_0x_{-2})(-1+y_0y_{-2})},\\
z_{12n+6} &= \tfrac{y_0y_{-2}z_{-2}}{x_0x_{-2}},\\
z_{12n+7} &= \tfrac{z_0z_{-1}z_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+z_0z_{-2})},\\
z_{12n+8} &= \tfrac{x_0x_{-2}}{z_{-2}},\\
z_{12n+9} &= \tfrac{y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})},\\
z_{12n+10} &= z_{-2},
\end{aligned}
$$

veya buna eşdeğer olarak,

$$
\{x_n\}_{n=-1}^{+\infty} = \left\{
\begin{array}{l}
x_{-1}, x_0, \frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}, \frac{z_0z_{-2}}{x_0}, \frac{x_0x_{-1}x_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+x_0x_{-2})},\\
\frac{x_0y_0y_{-2}}{z_0z_{-2}}, \frac{z_0z_{-2}y_0y_{-2}(-1+x_0x_{-2})}{x_0x_{-1}x_{-2}(-1+y_0y_{-2})(-1+z_0z_{-2})}, \frac{z_0z_{-2}x_{-2}}{y_0y_{-2}},\\
\frac{x_0x_{-1}x_{-2}(-1+z_0z_{-2})}{z_0z_{-2}(-1+x_0x_{-2})}, \frac{y_0y_{-2}}{x_{-2}}, \frac{z_0z_{-2}}{x_{-1}(-1+z_0z_{-2})}, x_{-2}, x_{-1},\\
x_0, \frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}, \frac{z_0z_{-2}}{x_0}, \frac{x_0x_{-1}x_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+x_0x_{-2})},\\
\frac{x_0y_0y_{-2}}{z_0z_{-2}}, \frac{z_0z_{-2}y_0y_{-2}(-1+x_0x_{-2})}{x_0x_{-1}x_{-2}(-1+y_0y_{-2})(-1+z_0z_{-2})},\\
\frac{z_0z_{-2}x_{-2}}{y_0y_{-2}}, \frac{x_0x_{-1}x_{-2}(-1+z_0z_{-2})}{z_0z_{-2}(-1+x_0x_{-2})}, \frac{y_0y_{-2}}{x_{-2}}, \frac{z_0z_{-2}}{x_{-1}(-1+z_0z_{-2})},\\
x_{-2}, x_{-1}, x_0, \frac{y_0y_{-2}}{x_{-1}(-1+y_0y_{-2})}, \frac{z_0z_{-2}}{x_0}, \frac{x_0x_{-1}x_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+x_0x_{-2})},\\
\frac{x_0y_0y_{-2}}{z_0z_{-2}}, \frac{z_0z_{-2}y_0y_{-2}(-1+x_0x_{-2})}{x_0x_{-1}x_{-2}(-1+y_0y_{-2})(-1+z_0z_{-2})}, \frac{z_0z_{-2}x_{-2}}{y_0y_{-2}},\\
\frac{x_0x_{-1}x_{-2}(-1+z_0z_{-2})}{z_0z_{-2}(-1+x_0x_{-2})}, \frac{y_0y_{-2}}{x_{-2}}, \frac{z_0z_{-2}}{x_{-1}(-1+z_0z_{-2})}, x_{-2}, \ldots
\end{array}
\right\}
$$

$$\{y_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l} y_{-1}, y_0, \dfrac{z_0z_{-2}}{y_{-1}(-1+z_0z_{-2})}, \dfrac{x_0x_{-2}}{y_0}, \dfrac{y_0y_{-1}y_{-2}(-1+z_0z_{-2})}{z_0z_{-2}(-1+y_0y_{-2})}, \\ \dfrac{y_0z_0z_{-2}}{x_0x_{-2}}, \dfrac{x_0x_{-2}z_0z_{-2}(-1+y_0y_{-2})}{y_0y_{-1}y_{-2}(-1+z_0z_{-2})(-1+x_0x_{-2})}, \dfrac{x_0x_{-2}y_{-2}}{z_0z_{-2}}, \\ \dfrac{y_0y_{-1}y_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+y_0y_{-2})}, \dfrac{z_0z_{-2}}{y_{-2}}, \dfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}, y_{-2}, y_{-1}, \\ y_0, \dfrac{z_0z_{-2}}{y_{-1}(-1+z_0z_{-2})}, \dfrac{x_0x_{-2}}{y_0}, \dfrac{y_0y_{-1}y_{-2}(-1+z_0z_{-2})}{z_0z_{-2}(-1+y_0y_{-2})}, \\ \dfrac{y_0z_0z_{-2}}{x_0x_{-2}}, \dfrac{x_0x_{-2}z_0z_{-2}(-1+y_0y_{-2})}{y_0y_{-1}y_{-2}(-1+z_0z_{-2})(-1+x_0x_{-2})}, \\ \dfrac{x_0x_{-2}y_{-2}}{z_0z_{-2}}, \dfrac{y_0y_{-1}y_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+y_0y_{-2})}, \dfrac{z_0z_{-2}}{y_{-2}}, \dfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}, \\ y_{-2}, y_{-1}, y_0, \dfrac{z_0z_{-2}}{y_{-1}(-1+z_0z_{-2})}, \dfrac{x_0x_{-2}}{y_0}, \dfrac{y_0y_{-1}y_{-2}(-1+z_0z_{-2})}{z_0z_{-2}(-1+y_0y_{-2})}, \\ \dfrac{y_0z_0z_{-2}}{x_0x_{-2}}, \dfrac{x_0x_{-2}z_0z_{-2}(-1+y_0y_{-2})}{y_0y_{-1}y_{-2}(-1+z_0z_{-2})(-1+x_0x_{-2})}, \dfrac{x_0x_{-2}y_{-2}}{z_0z_{-2}}, \\ \dfrac{y_0y_{-1}y_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+y_0y_{-2})}, \dfrac{z_0z_{-2}}{y_{-2}}, \dfrac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}, y_{-2}, \ldots \end{array}\right\}$$

$$\{z_n\}_{n=-1}^{+\infty}=\left\{\begin{array}{l} z_{-1}, z_0, \dfrac{x_0x_{-2}}{z_{-1}(-1+x_0x_{-2})}, \dfrac{y_0y_{-2}}{z_0}, \dfrac{z_0z_{-1}z_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+z_0z_{-2})}, \\ \dfrac{z_0x_0x_{-2}}{y_0y_{-2}}, \dfrac{y_0y_{-2}x_0x_{-2}(-1+z_0z_{-2})}{z_0z_{-1}z_{-2}(-1+x_0x_{-2})(-1+y_0y_{-2})}, \dfrac{y_0y_{-2}z_{-2}}{x_0x_{-2}}, \\ \dfrac{z_0z_{-1}z_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+z_0z_{-2})}, \dfrac{x_0x_{-2}}{z_{-2}}, \dfrac{y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})}, z_{-2}, z_{-1}, \\ z_0, \dfrac{x_0x_{-2}}{z_{-1}(-1+x_0x_{-2})}, \dfrac{y_0y_{-2}}{z_0}, \dfrac{z_0z_{-1}z_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+z_0z_{-2})}, \\ \dfrac{z_0x_0x_{-2}}{y_0y_{-2}}, \dfrac{y_0y_{-2}x_0x_{-2}(-1+z_0z_{-2})}{z_0z_{-1}z_{-2}(-1+x_0x_{-2})(-1+y_0y_{-2})}, \\ \dfrac{y_0y_{-2}z_{-2}}{x_0x_{-2}}, \dfrac{z_0z_{-1}z_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+z_0z_{-2})}, \dfrac{x_0x_{-2}}{z_{-2}}, \dfrac{y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})}, \\ z_{-2}, z_{-1}, z_0, \dfrac{x_0x_{-2}}{z_{-1}(-1+x_0x_{-2})}, \dfrac{y_0y_{-2}}{z_0}, \dfrac{z_0z_{-1}z_{-2}(-1+x_0x_{-2})}{x_0x_{-2}(-1+z_0z_{-2})}, \\ \dfrac{z_0x_0x_{-2}}{y_0y_{-2}}, \dfrac{y_0y_{-2}x_0x_{-2}(-1+z_0z_{-2})}{z_0z_{-1}z_{-2}(-1+x_0x_{-2})(-1+y_0y_{-2})}, \dfrac{y_0y_{-2}z_{-2}}{x_0x_{-2}}, \\ \dfrac{z_0z_{-1}z_{-2}(-1+y_0y_{-2})}{y_0y_{-2}(-1+z_0z_{-2})}, \dfrac{x_0x_{-2}}{z_{-2}}, \dfrac{y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})}, z_{-2}, \ldots \end{array}\right\}$$

çözümleri elde edilir.

İspat.

(a) (4.5) denklem sistemi yardımıyla aşağıdaki eşitliklerin sağlandığı görülür:

$$\begin{aligned}
x_{n+1} &= \frac{y_n y_{n-2}}{x_{n-1}(-1+y_n y_{n-2})},\\
y_{n+1} &= \frac{z_n z_{n-2}}{y_{n-1}(-1+z_n z_{n-2})},\\
z_{n+1} &= \frac{x_n x_{n-2}}{z_{n-1}(-1+x_n x_{n-2})}.\\
x_{n+2} &= \frac{z_n z_{n-2}}{x_n},\\
y_{n+2} &= \frac{x_n x_{n-2}}{y_{,n}},\\
z_{n+2} &= \frac{y_n y_{n-2}}{z_n}.\\
x_{n+3} &= \frac{x_n x_{n-1} x_{n-2}(-1+y_n y_{n-2})}{y_n y_{n-2}(-1+x_n x_{n-2})},\\
y_{n+3} &= \frac{y_n y_{n-1} y_{n-2}(-1+z_n z_{n-2})}{z_n z_{n-2}(-1+y_n y_{n-2})},\\
z_{n+3} &= \frac{z_n z_{n-1} z_{n-2}(-1+x_n x_{n-2})}{x_n x_{n-2}(-1+z_n z_{n-2})}.\\
x_{n+4} &= \frac{x_n y_n y_{n-2}}{z_n z_{n-2}},\\
y_{n+4} &= \frac{y_n z_n z_{n-2}}{x_n x_{n-2}},\\
z_{n+4} &= \frac{z_n x_n x_{n-2}}{y_n y_{n-2}}.\\
x_{n+5} &= \frac{z_n z_{n-2} y_n y_{n-2}(-1+x_n x_{n-2})}{x_n x_{n-1} x_{n-2}(-1+y_n y_{n-2})(-1+z_n z_{n-2})},\\
y_{n+5} &= \frac{x_n x_{n-2} z_n z_{n-2}(-1+y_n y_{n-2})}{y_n y_{n-1} y_{n-2}(-1+z_n z_{n-2})(-1+x_n x_{n-2})},\\
z_{n+5} &= \frac{y_n y_{n-2} x_n x_{n-2}(-1+z_n z_{n-2})}{z_n z_{n-1} z_{n-2}(-1+x_n x_{n-2})(-1+y_n y_{n-2})}.\\
x_{n+6} &= \frac{z_n z_{n-2} x_{n-2}}{y_n y_{n-2}},\\
y_{n+6} &= \frac{y_n y_{n-2} z_{n-2}}{z_n z_{n-2}},\\
z_{n+6} &= \frac{y_n y_{n-2} z_{n-2}}{x_n x_{n-2}}.\\
x_{n+7} &= \frac{x_n x_{n-1} x_{n-2}(-1+z_n z_{n-2})}{z_n z_{n-2}(-1+x_n x_{n-2})},\\
y_{n+7} &= \frac{y_n y_{n-1} y_{n-2}(-1+x_n x_{n-2})}{x_n x_{n-2}(-1+y_n y_{n-2})},\\
z_{n+7} &= \frac{z_n z_{n-1} z_{n-2}(-1+y_n y_{n-2})}{y_n y_{n-2}(-1+z_n z_{n-2})}.\\
x_{n+8} &= \frac{y_n y_{n-2}}{x_{n-2}},\\
y_{n+8} &= \frac{z_n z_{n-2}}{y_{n-2}},\\
z_{n+8} &= \frac{x_n x_{n-2}}{z_{n-2}}.
\end{aligned}$$

$$
\begin{aligned}
x_{n+9} &= \frac{z_n z_{n-2}}{x_{n-1}(-1+z_n z_{n-2})},\\
y_{n+9} &= \frac{x_n x_{n-2}}{y_{n-1}(-1+x_n x_{n-2})},\\
z_{n+9} &= \frac{y_n y_{n-2}}{z_{n-1}(-1+y_n y_{n-2})}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+10} &= x_{n-2},\\
y_{n+10} &= y_{n-2},\\
z_{n+10} &= z_{n-2}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+11} &= x_{n-1},\\
y_{n+11} &= y_{n-1},\\
z_{n+11} &= z_{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+12} &= x_n,\\
y_{n+12} &= y_n,\\
z_{n+12} &= z_n.
\end{aligned}
$$

(b) $n=0$ için sonuçlar sağlanır. $n>0$ iken iddiamızın $(n-1)$ için sağlandığını farz edelim. Yani,

$$
\begin{aligned}
x_{12n-13} &= x_{-1}\\
x_{12n-12} &= x_0\\
x_{12n-11} &= \frac{y_0 y_{-2}}{x_{-1}(-1+y_0 y_{-2})}\\
x_{12n-10} &= \frac{z_0 z_{-2}}{x_0},\\
x_{12n-9} &= \frac{x_0 x_{-1} x_{-2}(-1+y_0 y_{-2})}{y_0 y_{-2}(-1+x_0 x_{-2})}\\
x_{12n-8} &= \frac{x_0 y_0 y_{-2}}{z_0 z_{-2}}\\
x_{12n-7} &= \frac{z_0 z_{-2} y_0 y_{-2}(-1+x_0 x_{-2})}{x_0 x_{-1} x_{-2}(-1+y_0 y_{-2})(-1+z_0 z_{-2})}\\
x_{12n-6} &= \frac{z_0 z_{-2} x_{-2}}{y_0 y_{-2}}\\
x_{12n-5} &= \frac{x_0 x_{-1} x_{-2}(-1+z_0 z_{-2})}{z_0 z_{-2}(-1+x_0 x_{-2})}\\
x_{12n-4} &= \frac{y_0 y_{-2}}{x_{-2}}\\
x_{12n-3} &= \frac{z_0 z_{-2}}{x_{-1}(-1+z_0 z_{-2})}\\
x_{12n-2} &= x_{-2}
\end{aligned}
$$

ve

$$
\begin{aligned}
y_{12n-13} &= y_{-1}\\
y_{12n-12} &= y_0\\
y_{12n-11} &= \frac{z_0 z_{-2}}{y_{-1}(-1+z_0 z_{-2})}\\
y_{12n-10} &= \frac{x_0 x_{-2}}{y_0}\\
y_{12n-9} &= \frac{y_0 y_{-1} y_{-2}(-1+z_0 z_{-2})}{z_0 z_{-2}(-1+y_0 y_{-2})}\\
y_{12n-8} &= \frac{y_0 z_0 z_{-2}}{x_0 x_{-2}}\\
y_{12n-7} &= \frac{x_0 x_{-2} z_0 z_{-2}(-1+y_0 y_{-2})}{y_0 y_{-1} y_{-2}(-1+z_0 z_{-2})(-1+x_0 x_{-2})}\\
y_{12n-6} &= \frac{x_0 x_{-2} y_{-2}}{z_0 z_{-2}}\\
y_{12n-5} &= \frac{y_0 y_{-1} y_{-2}(-1+x_0 x_{-2})}{x_0 x_{-2}(-1+y_0 y_{-2})}\\
y_{12n-4} &= \frac{z_0 z_{-2}}{y_{-2}}\\
y_{12n-3} &= \frac{x_0 x_{-2}}{y_{-1}(-1+x_0 x_{-2})}\\
y_{12n-2} &= y_{-2}
\end{aligned}
$$

ve

$$
\begin{aligned}
z_{12n-13} &= z_{-1}\\
z_{12n-12} &= z_0\\
z_{12n-11} &= \frac{x_0 x_{-2}}{z_{-1}(-1+x_0 x_{-2})}\\
z_{12n-10} &= \frac{y_0 y_{-2}}{z_0}\\
z_{12n-9} &= \frac{z_0 z_{-1} z_{-2}(-1+x_0 x_{-2})}{x_0 x_{-2}(-1+z_0 z_{-2})}\\
z_{12n-8} &= \frac{z_0 x_0 x_{-2}}{y_0 y_{-2}}\\
z_{12n-7} &= \frac{y_0 y_{-2} x_0 x_{-2}(-1+z_0 z_{-2})}{z_0 z_{-1} z_{-2}(-1+x_0 x_{-2})(-1+y_0 y_{-2})}\\
z_{12n-6} &= \frac{y_0 y_{-2} z_{-2}}{x_0 x_{-2}}\\
z_{12n-5} &= \frac{z_0 z_{-1} z_{-2}(-1+y_0 y_{-2})}{y_0 y_{-2}(-1+z_0 z_{-2})}\\
z_{12n-4} &= \frac{x_0 x_{-2}}{z_{-2}}\\
z_{12n-3} &= \frac{y_0 y_{-2}}{z_{-1}(-1+y_0 y_{-2})}\\
z_{12n-2} &= z_{-2}
\end{aligned}
$$

elde edilir ve (4.5) denklem sisteminden,

$$
\begin{aligned}
x_{12n-1} &= \frac{y_{12n-2}y_{12n-4}}{x_{12n-3}(-1+y_{12n-2}y_{12n-4})} = \frac{y_{-2}\frac{z_0z_{-2}}{y_{-2}}}{\frac{z_0z_{-2}}{x_{-1}(-1+z_0z_{-2})}(-1+y_{-2}\frac{z_0z_{-2}}{y_{-2}})} = x_{-1},\\
y_{12n-1} &= \frac{z_{12n-2}z_{12n-4}}{y_{12n-3}(-1+z_{12n-2}z_{12n-4})} = \frac{z_{-2}\frac{x_0x_{-2}}{z_{-2}}}{\frac{x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}(-1+z_{-2}\frac{x_0x_{-2}}{z_{-2}})} = y_{-1},\\
z_{12n-1} &= \frac{x_{12n-2}x_{12n-4}}{z_{12n-3}(-1+x_{12n-2}x_{12n-4})} = \frac{x_{-2}\frac{y_0y_{-2}}{x_{-2}}}{\frac{y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})}(-1+x_{-2}\frac{y_0y_{-2}}{x_{-2}})} = z_{-1},
\end{aligned}
$$

$$
\begin{aligned}
x_{12n} &= \frac{y_{12n-1}y_{12n-3}}{x_{12n-2}(-1+y_{12n-1}y_{12n-3})} = \frac{\frac{y_{-1}x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})}}{x_{-2}(-1+\frac{y_{-1}x_0x_{-2}}{y_{-1}(-1+x_0x_{-2})})} = x_0,\\
y_{12n} &= \frac{z_{12n-1}z_{12n-3}}{y_{12n-2}(-1+z_{12n-1}z_{12n-3})} = \frac{\frac{z_{-1}y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})}}{y_{-2}(-1+\frac{z_{-1}y_0y_{-2}}{z_{-1}(-1+y_0y_{-2})})} = y_0,\\
z_{12n} &= \frac{x_{12n-1}x_{12n-3}}{z_{12n-2}(-1+x_{12n-1}x_{12n-3})} = \frac{\frac{x_{-1}z_0z_{-2}}{x_{-1}(-1+z_0z_{-2})}}{z_{-2}(-1+\frac{x_{-1}z_0z_{-2}}{x_{-1}(-1+z_0z_{-2})})} = z_0,
\end{aligned}
$$

ve benzer şekilde

$$
\begin{aligned}
x_{12n+1} &= x_1,\\
y_{12n+1} &= y_1,\\
z_{12n+1} &= z_1,\\
&\vdots
\end{aligned}
$$

çözümleri elde edilerek ispat tamamlanır.

■

Teorem 4.3.2 (4.6) *denklem sisteminin çözümlerinin* $\{x_n, y_n, z_n\}_{n=-1}^{+\infty}$ *olduğunu farz edelim. Bu durumda;*

(a) $\{x_n\}_{n=-1}^{+\infty}$, $\{y_n\}_{n=-1}^{+\infty}$ *ve* $\{z_n\}_{n=-1}^{+\infty}$ *çözümleri on iki periyotlu periyodiktir*

(b)

$$\begin{aligned}
x_{12n-1} &= x_{-1},\\
x_{12n} &= x_0,\\
x_{12n+1} &= \tfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})},\\
x_{12n+2} &= \tfrac{z_0z_{-2}}{x_0},\\
x_{12n+3} &= \tfrac{x_0x_{-1}x_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-x_0x_{-2})},\\
x_{12n+4} &= \tfrac{x_0y_0y_{-2}}{z_0z_{-2}},\\
x_{12n+5} &= \tfrac{z_0z_{-2}y_0y_{-2}(-1-x_0x_{-2})}{x_0x_{-1}x_{-2}(-1-y_0y_{-2})(-1-z_0z_{-2})},\\
x_{12n+6} &= \tfrac{z_0z_{-2}x_{-2}}{y_0y_{-2}},\\
x_{12n+7} &= \tfrac{x_0x_{-1}x_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-x_0x_{-2})},\\
x_{12n+8} &= \tfrac{y_0y_{-2}}{x_{-2}},\\
x_{12n+9} &= \tfrac{z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})},\\
x_{12n+10} &= x_{-2},
\end{aligned}$$

ve

$$\begin{aligned}
y_{12n-1} &= y_{-1},\\
y_{12n} &= y_0,\\
y_{12n+1} &= \tfrac{z_0z_{-2}}{y_{-1}(-1-z_0z_{-2})},\\
y_{12n+2} &= \tfrac{x_0x_{-2}}{y_0},\\
y_{12n+3} &= \tfrac{y_0y_{-1}y_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-y_0y_{-2})},\\
y_{12n+4} &= \tfrac{y_0z_0z_{-2}}{x_0x_{-2}},\\
y_{12n+5} &= \tfrac{x_0x_{-2}z_0z_{-2}(-1-y_0y_{-2})}{y_0y_{-1}y_{-2}(-1-z_0z_{-2})(-1-x_0x_{-2})},\\
y_{12n+6} &= \tfrac{x_0x_{-2}y_{-2}}{z_0z_{-2}},\\
y_{12n+7} &= \tfrac{y_0y_{-1}y_{-2}(-1-x_0x_{-2})}{x_0x_{-2}(-1-y_0y_{-2})},\\
y_{12n+8} &= \tfrac{z_0z_{-2}}{y_{-2}},\\
y_{12n+9} &= \tfrac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})},\\
y_{12n+10} &= y_{-2},
\end{aligned}$$

ve

$$\begin{aligned}
z_{12n-1} &= z_{-1}, \\
z_{12n} &= z_0, \\
z_{12n+1} &= \tfrac{x_0x_{-2}}{z_{-1}(-1-x_0x_{-2})}, \\
z_{12n+2} &= \tfrac{y_0y_{-2}}{z_0}, \\
z_{12n+3} &= \tfrac{z_0z_{-1}z_{-2}(-1-x_0x_{-2})}{x_0x_{-2}(-1-z_0z_{-2})}, \\
z_{12n+4} &= \tfrac{z_0x_0x_{-2}}{y_0y_{-2}}, \\
z_{12n+5} &= \tfrac{y_0y_{-2}x_0x_{-2}(-1-z_0z_{-2})}{z_0z_{-1}z_{-2}(-1-x_0x_{-2})(-1-y_0y_{-2})}, \\
z_{12n+6} &= \tfrac{y_0y_{-2}z_{-2}}{x_0x_{-2}}, \\
z_{12n+7} &= \tfrac{z_0z_{-1}z_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-z_0z_{-2})}, \\
z_{12n+8} &= \tfrac{x_0x_{-2}}{z_{-2}}, \\
z_{12n+9} &= \tfrac{y_0y_{-2}}{z_{-1}(-1-y_0y_{-2})}, \\
z_{12n+10} &= z_{-2},
\end{aligned}$$

veya buna eşdeğer olarak,

$$\{x_n\}_{n=-1}^{+\infty} = \left\{\begin{array}{l}
x_{-1}, x_0, \dfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})}, \dfrac{z_0z_{-2}}{x_0}, \dfrac{x_0x_{-1}x_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-x_0x_{-2})}, \\
\dfrac{x_0y_0y_{-2}}{z_0z_{-2}}, \dfrac{z_0z_{-2}y_0y_{-2}(-1-x_0x_{-2})}{x_0x_{-1}x_{-2}(-1-y_0y_{-2})(-1-z_0z_{-2})}, \dfrac{z_0z_{-2}x_{-2}}{y_0y_{-2}}, \\
\dfrac{x_0x_{-1}x_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-x_0x_{-2})}, \dfrac{y_0y_{-2}}{x_{-2}}, \dfrac{z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})}, x_{-2}, x_{-1}, \\
x_0, \dfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})}, \dfrac{z_0z_{-2}}{x_0}, \dfrac{x_0x_{-1}x_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-x_0x_{-2})}, \\
\dfrac{x_0y_0y_{-2}}{z_0z_{-2}}, \dfrac{z_0z_{-2}y_0y_{-2}(-1-x_0x_{-2})}{x_0x_{-1}x_{-2}(-1-y_0y_{-2})(-1-z_0z_{-2})}, \dfrac{z_0z_{-2}x_{-2}}{y_0y_{-2}}, \\
\dfrac{x_0x_{-1}x_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-x_0x_{-2})}, \dfrac{y_0y_{-2}}{x_{-2}}, \dfrac{z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})}, x_{-2}, x_{-1}, \\
x_0, \dfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})}, \dfrac{z_0z_{-2}}{x_0}, \dfrac{x_0x_{-1}x_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-x_0x_{-2})}, \\
\dfrac{x_0y_0y_{-2}}{z_0z_{-2}}, \dfrac{z_0z_{-2}y_0y_{-2}(-1-x_0x_{-2})}{x_0x_{-1}x_{-2}(-1-y_0y_{-2})(-1-z_0z_{-2})}, \dfrac{z_0z_{-2}x_{-2}}{y_0y_{-2}}, \\
\dfrac{x_0x_{-1}x_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-x_0x_{-2})}, \dfrac{y_0y_{-2}}{x_{-2}}, \dfrac{z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})}, x_{-2}, \ldots
\end{array}\right\}$$

$$\{y_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} y_{-1}, y_0, \dfrac{z_0 z_{-2}}{y_{-1}(-1 - z_0 z_{-2})}, \dfrac{x_0 x_{-2}}{y_0}, \dfrac{y_0 y_{-1} y_{-2}(-1 - z_0 z_{-2})}{z_0 z_{-2}(-1 - y_0 y_{-2})}, \\ \dfrac{y_0 z_0 z_{-2}}{x_0 x_{-2}}, \dfrac{x_0 x_{-2} z_0 z_{-2}(-1 - y_0 y_{-2})}{y_0 y_{-1} y_{-2}(-1 - z_0 z_{-2})(-1 - x_0 x_{-2})}, \dfrac{x_0 x_{-2} y_{-2}}{z_0 z_{-2}}, \\ \dfrac{y_0 y_{-1} y_{-2}(-1 - x_0 x_{-2})}{x_0 x_{-2}(-1 - y_0 y_{-2})}, \dfrac{z_0 z_{-2}}{y_{-2}}, \dfrac{x_0 x_{-2}}{y_{-1}(-1 - x_0 x_{-2})}, y_{-2}, y_{-1}, \\ y_0, \dfrac{z_0 z_{-2}}{y_{-1}(-1 - z_0 z_{-2})}, \dfrac{x_0 x_{-2}}{y_0}, \dfrac{y_0 y_{-1} y_{-2}(-1 - z_0 z_{-2})}{z_0 z_{-2}(-1 - y_0 y_{-2})}, \\ \dfrac{y_0 z_0 z_{-2}}{x_0 x_{-2}}, \dfrac{x_0 x_{-2} z_0 z_{-2}(-1 - y_0 y_{-2})}{y_0 y_{-1} y_{-2}(-1 - z_0 z_{-2})(-1 - x_0 x_{-2})}, \dfrac{x_0 x_{-2} y_{-2}}{z_0 z_{-2}}, \\ \dfrac{y_0 y_{-1} y_{-2}(-1 - x_0 x_{-2})}{x_0 x_{-2}(-1 - y_0 y_{-2})}, \dfrac{z_0 z_{-2}}{y_{-2}}, \dfrac{x_0 x_{-2}}{y_{-1}(-1 - x_0 x_{-2})}, y_{-2}, y_{-1}, \\ y_0, \dfrac{z_0 z_{-2}}{y_{-1}(-1 - z_0 z_{-2})}, \dfrac{x_0 x_{-2}}{y_0}, \dfrac{y_0 y_{-1} y_{-2}(-1 - z_0 z_{-2})}{z_0 z_{-2}(-1 - y_0 y_{-2})}, \\ \dfrac{y_0 z_0 z_{-2}}{x_0 x_{-2}}, \dfrac{x_0 x_{-2} z_0 z_{-2}(-1 - y_0 y_{-2})}{y_0 y_{-1} y_{-2}(-1 - z_0 z_{-2})(-1 - x_0 x_{-2})}, \dfrac{x_0 x_{-2} y_{-2}}{z_0 z_{-2}}, \\ \dfrac{y_0 y_{-1} y_{-2}(-1 - x_0 x_{-2})}{x_0 x_{-2}(-1 - y_0 y_{-2})}, \dfrac{z_0 z_{-2}}{y_{-2}}, \dfrac{x_0 x_{-2}}{y_{-1}(-1 - x_0 x_{-2})}, y_{-2}, \ldots \end{array} \right\}$$

$$\{z_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{l} z_{-1}, z_0, \dfrac{x_0 x_{-2}}{z_{-1}(-1 - x_0 x_{-2})}, \dfrac{y_0 y_{-2}}{z_0}, \dfrac{z_0 z_{-1} z_{-2}(-1 - x_0 x_{-2})}{x_0 x_{-2}(-1 - z_0 z_{-2})}, \\ \dfrac{z_0 x_0 x_{-2}}{y_0 y_{-2}}, \dfrac{y_0 y_{-2} x_0 x_{-2}(-1 - z_0 z_{-2})}{z_0 z_{-1} z_{-2}(-1 - x_0 x_{-2})(-1 - y_0 y_{-2})}, \dfrac{y_0 y_{-2} z_{-2}}{x_0 x_{-2}}, \\ \dfrac{z_0 z_{-1} z_{-2}(-1 - y_0 y_{-2})}{y_0 y_{-2}(-1 - z_0 z_{-2})}, \dfrac{x_0 x_{-2}}{z_{-2}}, \dfrac{y_0 y_{-2}}{z_{-1}(-1 - y_0 y_{-2})}, z_{-2}, z_{-1}, \\ z_0, \dfrac{x_0 x_{-2}}{z_{-1}(-1 - x_0 x_{-2})}, \dfrac{y_0 y_{-2}}{z_0}, \dfrac{z_0 z_{-1} z_{-2}(-1 - x_0 x_{-2})}{x_0 x_{-2}(-1 - z_0 z_{-2})}, \\ \dfrac{z_0 x_0 x_{-2}}{y_0 y_{-2}}, \dfrac{y_0 y_{-2} x_0 x_{-2}(-1 - z_0 z_{-2})}{z_0 z_{-1} z_{-2}(-1 - x_0 x_{-2})(-1 - y_0 y_{-2})}, \dfrac{y_0 y_{-2} z_{-2}}{x_0 x_{-2}}, \\ \dfrac{z_0 z_{-1} z_{-2}(-1 - y_0 y_{-2})}{y_0 y_{-2}(-1 - z_0 z_{-2})}, \dfrac{x_0 x_{-2}}{z_{-2}}, \dfrac{y_0 y_{-2}}{z_{-1}(-1 - y_0 y_{-2})}, z_{-2}, z_{-1}, \\ z_0, \dfrac{x_0 x_{-2}}{z_{-1}(-1 - x_0 x_{-2})}, \dfrac{y_0 y_{-2}}{z_0}, \dfrac{z_0 z_{-1} z_{-2}(-1 - x_0 x_{-2})}{x_0 x_{-2}(-1 - z_0 z_{-2})}, \\ \dfrac{z_0 x_0 x_{-2}}{y_0 y_{-2}}, \dfrac{y_0 y_{-2} x_0 x_{-2}(-1 - z_0 z_{-2})}{z_0 z_{-1} z_{-2}(-1 - x_0 x_{-2})(-1 - y_0 y_{-2})}, \dfrac{y_0 y_{-2} z_{-2}}{x_0 x_{-2}}, \\ \dfrac{z_0 z_{-1} z_{-2}(-1 - y_0 y_{-2})}{y_0 y_{-2}(-1 - z_0 z_{-2})}, \dfrac{x_0 x_{-2}}{z_{-2}}, \dfrac{y_0 y_{-2}}{z_{-1}(-1 - y_0 y_{-2})}, z_{-2}, \ldots \end{array} \right\}$$

çözümleri elde edilir.

İspat.

(a) (4.6) denklem sistemi kullanılarak aşağıdaki eşitliklerin sağlandığı görülür:

$$x_{n+1} = \tfrac{y_n y_{n-2}}{x_{n-1}(-1 - y_n y_{n-2})},$$
$$y_{n+1} = \tfrac{z_n z_{n-2}}{y_{n-1}(-1 - z_n z_{n-2})},$$
$$z_{n+1} = \tfrac{x_n x_{n-2}}{z_{n-1}(-1 - x_n x_{n-2})}.$$

$$
\begin{aligned}
x_{n+2} &= \frac{z_n z_{n-2}}{x_n}, \\
y_{n+2} &= \frac{x_n x_{n-2}}{y_n}, \\
z_{n+2} &= \frac{y_n y_{n-2}}{z_n}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+3} &= \frac{x_n x_{n-1} x_{n-2}(-1-y_n y_{n-2})}{y_n y_{n-2}(-1-x_n x_{n-2})}, \\
y_{n+3} &= \frac{y_n y_{n-1} y_{n-2}(-1-z_n z_{n-2})}{z_n z_{n-2}(-1-y_n y_{n-2})}, \\
z_{n+3} &= \frac{z_n z_{n-1} z_{n-2}(-1-x_n x_{n-2})}{x_n x_{n-2}(-1-z_n z_{n-2})}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+4} &= \frac{x_n y_n y_{n-2}}{z_n z_{n-2}}, \\
y_{n+4} &= \frac{y_n z_n z_{n-2}}{x_n x_{n-2}}, \\
z_{n+4} &= \frac{z_n x_n x_{n-2}}{y_n y_{n-2}}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+5} &= \frac{z_n z_{n-2} y_n y_{n-2}(-1-x_n x_{n-2})}{x_n x_{n-1} x_{n-2}(-1-y_n y_{n-2})(-1-z_n z_{n-2})}, \\
y_{n+5} &= \frac{x_n x_{n-2} z_n z_{n-2}(-1-y_n y_{n-2})}{y_n y_{n-1} y_{n-2}(-1-z_n z_{n-2})(-1-x_n x_{n-2})}, \\
z_{n+5} &= \frac{y_n y_{n-2} x_n x_{n-2}(-1-z_n z_{n-2})}{z_n z_{n-1} z_{n-2}(-1-x_n x_{n-2})(-1-y_n y_{n-2})}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+6} &= \frac{z_n z_{n-2} x_{n-2}}{y_n y_{n-2}}, \\
y_{n+6} &= \frac{y_n y_{n-2} z_{n-2}}{z_n z_{n-2}}, \\
z_{n+6} &= \frac{y_n y_{n-2} z_{n-2}}{x_n x_{n-2}}
\end{aligned}
$$

$$
\begin{aligned}
x_{n+7} &= \frac{x_n x_{n-1} x_{n-2}(-1-z_n z_{n-2})}{z_n z_{n-2}(-1-x_n x_{n-2})}, \\
y_{n+7} &= \frac{y_n y_{n-1} y_{n-2}(-1-x_n x_{n-2})}{x_n x_{n-2}(-1-y_n y_{n-2})}, \\
z_{n+7} &= \frac{z_n z_{n-1} z_{n-2}(-1-y_n y_{n-2})}{y_n y_{n-2}(-1-z_n z_{n-2})}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+8} &= \frac{y_n y_{n-2}}{x_{n-2}}, \\
y_{n+8} &= \frac{z_n z_{n-2}}{y_{n-2}}, \\
z_{n+8} &= \frac{x_n x_{n-2}}{z_{n-2}}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+9} &= \frac{z_n z_{n-2}}{x_{n-1}(-1-z_n z_{n-2})}, \\
y_{n+9} &= \frac{x_n x_{n-2}}{y_{n-1}(-1-x_n x_{n-2})}, \\
z_{n+9} &= \frac{y_n y_{n-2}}{z_{n-1}(-1-y_n y_{n-2})}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+10} &= x_{n-2}, \\
y_{n+10} &= y_{n-2}, \\
z_{n+10} &= z_{n-2}.
\end{aligned}
$$

$$
\begin{aligned}
x_{n+11} &= x_{n-1}, \\
y_{n+11} &= y_{n-1}, \\
z_{n+11} &= z_{n-1}.
\end{aligned}
$$

$$x_{n+12} = x_n,$$
$$y_{n+12} = y_n,$$
$$z_{n+12} = z_n.$$

(b) $n = 0$ için sonuçlar sağlanır. $n > 0$ olduğunu ve iddiamızın $(n-1)$ için sağlandığını varsayalım. Yani,

$$\begin{aligned}
x_{12n-13} &= x_{-1}\\
x_{12n-12} &= x_0\\
x_{12n-11} &= \tfrac{y_0y_{-2}}{x_{-1}(-1-y_0y_{-2})}\\
x_{12n-10} &= \tfrac{z_0z_{-2}}{x_0},\\
x_{12n-9} &= \tfrac{x_0x_{-1}x_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-x_0x_{-2})}\\
x_{12n-8} &= \tfrac{x_0y_0y_{-2}}{z_0z_{-2}}\\
x_{12n-7} &= \tfrac{z_0z_{-2}y_0y_{-2}(-1-x_0x_{-2})}{x_0x_{-1}x_{-2}(-1-y_0y_{-2})(-1-z_0z_{-2})}\\
x_{12n-6} &= \tfrac{z_0z_{-2}x_{-2}}{y_0y_{-2}}\\
x_{12n-5} &= \tfrac{x_0x_{-1}x_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-x_0x_{-2})}\\
x_{12n-4} &= \tfrac{y_0y_{-2}}{x_{-2}}\\
x_{12n-3} &= \tfrac{z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})}\\
x_{12n-2} &= x_{-2}
\end{aligned}$$

ve

$$\begin{aligned}
y_{12n-13} &= y_{-1}\\
y_{12n-12} &= y_0\\
y_{12n-11} &= \tfrac{z_0z_{-2}}{y_{-1}(-1-z_0z_{-2})}\\
y_{12n-10} &= \tfrac{x_0x_{-2}}{y_0}\\
y_{12n-9} &= \tfrac{y_0y_{-1}y_{-2}(-1-z_0z_{-2})}{z_0z_{-2}(-1-y_0y_{-2})}\\
y_{12n-8} &= \tfrac{y_0z_0z_{-2}}{x_0x_{-2}}\\
y_{12n-7} &= \tfrac{x_0x_{-2}z_0z_{-2}(-1-y_0y_{-2})}{y_0y_{-1}y_{-2}(-1-z_0z_{-2})(-1-x_0x_{-2})}\\
y_{12n-6} &= \tfrac{x_0x_{-2}y_{-2}}{z_0z_{-2}}\\
y_{12n-5} &= \tfrac{y_0y_{-1}y_{-2}(-1-x_0x_{-2})}{x_0x_{-2}(-1-y_0y_{-2})}\\
y_{12n-4} &= \tfrac{z_0z_{-2}}{y_{-2}}\\
y_{12n-3} &= \tfrac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})}\\
y_{12n-2} &= y_{-2}
\end{aligned}$$

ve

$$
\begin{aligned}
z_{12n-13} &= z_{-1}\\
z_{12n-12} &= z_0\\
z_{12n-11} &= \frac{x_0x_{-2}}{z_{-1}(-1-x_0x_{-2})}\\
z_{12n-10} &= \frac{y_0y_{-2}}{z_0}\\
z_{12n-9} &= \frac{z_0z_{-1}z_{-2}(-1-x_0x_{-2})}{x_0x_{-2}(-1-z_0z_{-2})}\\
z_{12n-8} &= \frac{z_0x_0x_{-2}}{y_0y_{-2}}\\
z_{12n-7} &= \frac{y_0y_{-2}x_0x_{-2}(-1-z_0z_{-2})}{z_0z_{-1}z_{-2}(-1-x_0x_{-2})(-1-y_0y_{-2})}\\
z_{12n-6} &= \frac{y_0y_{-2}z_{-2}}{x_0x_{-2}}\\
z_{12n-5} &= \frac{z_0z_{-1}z_{-2}(-1-y_0y_{-2})}{y_0y_{-2}(-1-z_0z_{-2})}\\
z_{12n-4} &= \frac{x_0x_{-2}}{z_{-2}}\\
z_{12n-3} &= \frac{y_0y_{-2}}{z_{-1}(-1-y_0y_{-2})}\\
z_{12n-2} &= z_{-2}
\end{aligned}
$$

eşitlikleri elde edilir ve (4.6) denklem sistemi yardımıyla

$$
\begin{aligned}
x_{12n-1} &= \frac{y_{12n-2}y_{12n-4}}{x_{12n-3}(-1-y_{12n-2}y_{12n-4})} = \frac{y_{-2}\dfrac{z_0z_{-2}}{y_{-2}}}{\dfrac{z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})}(-1-y_{-2}\dfrac{z_0z_{-2}}{y_{-2}})} = x_{-1},\\
y_{12n-1} &= \frac{z_{12n-2}z_{12n-4}}{y_{12n-3}(-1-z_{12n-2}z_{12n-4})} = \frac{z_{-2}\dfrac{x_0x_{-2}}{z_{-2}}}{\dfrac{x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})}(-1-z_{-2}\dfrac{x_0x_{-2}}{z_{-2}})} = y_{-1},\\
z_{12n-1} &= \frac{x_{12n-2}x_{12n-4}}{z_{12n-3}(-1-x_{12n-2}x_{12n-4})} = \frac{x_{-2}\dfrac{y_0y_{-2}}{x_{-2}}}{\dfrac{y_0y_{-2}}{z_{-1}(-1-y_0y_{-2})}(-1-x_{-2}\dfrac{y_0y_{-2}}{x_{-2}})} = z_{-1},
\end{aligned}
$$

$$
\begin{aligned}
x_{12n} &= \frac{y_{12n-1}y_{12n-3}}{x_{12n-2}(-1-y_{12n-1}y_{12n-3})} = \frac{\dfrac{y_{-1}x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})}}{x_{-2}(-1-\dfrac{y_{-1}x_0x_{-2}}{y_{-1}(-1-x_0x_{-2})})} = x_0,\\
y_{12n} &= \frac{z_{12n-1}z_{12n-3}}{y_{12n-2}(-1-z_{12n-1}z_{12n-3})} = \frac{\dfrac{z_{-1}y_0y_{-2}}{z_{-1}(-1-y_0y_{-2})}}{y_{-2}(-1-\dfrac{z_{-1}y_0y_{-2}}{z_{-1}(-1-y_0y_{-2})})} = y_0,\\
z_{12n} &= \frac{x_{12n-1}x_{12n-3}}{z_{12n-2}(-1-x_{12n-1}x_{12n-3})} = \frac{\dfrac{x_{-1}z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})}}{z_{-2}(-1-\dfrac{x_{-1}z_0z_{-2}}{x_{-1}(-1-z_0z_{-2})})} = z_0,
\end{aligned}
$$

ve benzer şekilde

$$x_{12n+1} = x_1,$$
$$y_{12n+1} = y_1,$$
$$z_{12n+1} = z_1,$$
$$\vdots$$

çözümleri elde edilerek ispat tamamlanır.

■

Örnek 4.3.1

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1}(-1 + y_n y_{n-2})}, y_{n+1} = \frac{z_n z_{n-2}}{y_{n-1}(-1 + z_n z_{n-2})}, z_{n+1} = \frac{x_n x_{n-2}}{z_{n-1}(-1 + x_n x_{n-2})} \quad (4.5)$$

denklem sisteminin $x_0 = 0.1$, $x_{-1} = 0.2$, $x_{-2} = 0.3$, $y_0 = -0.5$, $y_{-1} = 1.2$, $y_{-2} = 1.3$, $z_0 = 0.8$, $z_{-1} = 1.1$ *ve* $z_{-2} = -0.1$ *başlangıç koşullarındaki çözümleri aşağıda verilmiş ve 12 periyotlu periyodik olduğu görülmüştür.*

Çizelge 4.3 (4.5) denkleminin periyodik çözümleri

n	x_n	y_n	z_n
0	0.1	-0.5	1.1
1	1.969	0.082	-0.038
2	-1.1	-0.06	-0.590
3	-0.015	4.770	-2.563
4	0.590	1.833	-0.050
5	-6.311	-0.006	-0.153
6	0.050	-0.354	2.166
7	-0.062	-15.284	0.201
8	-2.166	-0.846	-0.3
9	0.495	-0.025	0.492
10	0.3	1.3	-0.1
11	0.2	1.2	0.8
12	0.1	-0.5	1.1
13	1.969	0.082	-0.038
14	-1.1	-0.06	-0.590
15	-0.015	4.770	-2.563
$\vdots$	$\vdots$	$\vdots$	$\vdots$

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