ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

THE SOLUTIONS OF SOME SYSTEMS OF EXPONENTIAL DIFFERENCE EQUATIONS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

CANSU GÜĞERÇİN

JANUARY 2020

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Thesis Advisor: Assist. Prof. Melih GÖCEN

ZONGULDAK
JANUARY 2020

APPROVAL OF THE THESIS:

The thesis entitled "The Solutions Of Some Systems Of Exponential Difference Equations" and submitted by Cansu GÜĞERÇİN has been examined and accepted by the jury as a Master of Science in Department of Mathematics, Graduate School of Natural and Applied Sciences, Zonguldak Bülent Ecevit University. 23/01/2020

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly."

C.Gun

Cansu GÜĞERÇİN

ABSTRACT

Master of Science Thesis

THE SOLUTIONS OF SOME SYSTEMS OF EXPONENTIAL DIFFERENCE EQUATIONS

CANSU GÜĞERÇİN
Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assist. Prof. Melih GÖCEN
January 2020, 53 pages

Difference equations are applied to mathematical models within biology, genetics, population dynamics, probability theory, psychology, sociology and many other disciplines. For this reason, recently there has been a lot of interest in studying the difference equations.

This thesis consists of four chapters.

In the first chapter, general informations, basic definitions and theorems about difference equations are given.

In the second chapter, a literature review of the studies on difference equations of exponential form is presented.

In the third chapter, the equilibrium point and local asymptotic stability of positive solutions of some rational exponential difference equations are investigated.

ABSTRACT (continued)

In last chapter, the local asymptotic stability of positive solutions of some systems of rational exponential difference equations are studied.

Moreover, in the thesis, the local asymptotic stability of positive solutions are showed by graphs.

Key Words: Exponential difference equations, equilibrium point, local asymptotic stability.

Science Code: 403.03.01

ÖZET

Yüksek Lisans Tezi

BAZI ÜSTEL FARK DENKLEM SİSTEMLERİNİN ÇÖZÜMLERİ

Cansu GÜĞERÇİN

Zonguldak Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Ana Bilim Dalı

Tez Danışmanı: Dr. Öğr. Üyesi Melih GÖCEN
Ocak 2020, 53 sayfa

Fark denklemleri biyoloji, genetik, popülasyon dinamiği, olasılık teorisi, psikoloji, sosyoloji ve daha bir çok bilim dalının içindeki matematiksel modellere uygulanır. Bu nedenden dolayı, son zamanlarda fark denklem sistemlerinin çalışmasına çok büyük ilgi vardır.

Bu tez dört bölümden oluşmaktadır.

Birinci bölümde fark denklemleri ile ilgili genel bilgiler, temel tanımlar ve teoremler verilmiştir.

İkinci bölümde üstel fark denklemleri ile ilgili yapılan çalışmaların literatür taraması sunulmuştur.

Üçüncü bölümde bazı rasyonel üstel fark denklemlerinin pozitif çözümlerinin denge noktası ve lokal asimptotik kararlılığı incelenmiştir.

ÖZET (devam ediyor)

Son bölümde ise bazı rasyonel üstel fark denklem sistemlerinin pozitif çözümlerinin lokal asimptotik kararlılığı çalışılmıştır.

Ayrıca tezde, pozitif çözümlerin lokal asimptotik kararlılığı grafiklerle gösterilmiştir.

Anahtar Kelimeler: Üstel fark denklemleri, denge noktası, lokal asimptotik kararlılık

Bilim Kodu: 403.03.01

ACKNOWLEDGMENTS

First, I would like to express my sincere gratitude to my esteemed teacher, Assist. Prof. Melih GÖCEN who patiently guided me through the course of my master's degree and thesis study, who has supported me at every stage, who has devoted precious time to me and who has contributed a great deal to my thesis.

I would like to thank all my family, especially my father Abdülbaki GÖKÇEK and my mother Necla GÖKÇEK, who have always supported me thoughout my education life and have a big role in my success. I would like to thank Tenzile ARSLAN, the esteemed English teacher who has always been there for me, who has always supported me with regard to foreign language.

I also want to thank my husband Barış GÜĞERÇİN who has never left me alone in my thesis study, as in every aspect of my life and my precious daughter Meva.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

The difference equation is an algebraic connection that gives the relationship between finite differences of a function with one or more variables and independent variables of this function. In these equations the independent variable is defined on integers. Therefore, the difference equations contain differences of unknown function instead of differentiation terms. Difference equations are used in mathematical modeling of events that are not continuous and vary according to evenly spaced time.

Some of the events in the nature are not continous. For example, in the genetic field, genetic characteristics vary between generations. The variable that represents the generation is an independent variable and also a discrete variable. The price changes in economy is calculated annually, monthly, weekly or daily. In this case, time variable is an independent variable and also a discrete variable. In population dynamics, the variable that shows the age groups appears as a discrete independent variable in the problems of population change among the age groups. Economic problems, such as national income and government debts, that take the same values in a period and change as the period changes are analyzed by the difference equations. It is a common theme to process discrete subjects in noncontinuous stages, from clocks to computers and chromosomes.

The difference equations are used in many disciplines and in the fields such as probability theory, sequence problems, statistical problems, probabilistic time series, combinatorial analysis, number theory, geometry, electrical circuits, radiation, psychology, sociology, stock market movements in economics, the research of the number of live populations in medicine and biology, and more importantly the study of cell movements (the rate of increase in cancer cells).

The wide range of application has increased the interest in difference equations and it has attracted the attention of not only mathematicians but also the researchers working in science, engineering, health and social science. For example, in the 1950s, several environmental scientists used the simple nonlinear difference equation which included a logistic equation, to examine year-to-year changes in the behavior of populations. However, in the early 1970s, Robert May researched the types of complex behaviors exhibited by the logistic equation and he studied on the relationship between these behaviors and fluctations in real populations. Furthermore, more advanced models developed from this logistic model are used to identify the behaviors of HIV viruses, bacteria or cancer cells. In recent years, many studies have been carried out on the behaviors of difference equations, and the system of difference equations, in particular periodicity, stability, boundedness. In this study, equilibrium point and asymptotic stability of some nonlinear rational difference equation of exponential form and rational difference equation systems of exponential form are examined. In addition, some numerical examples are given to support our theoretical results.

In this section, we give basic definitions and theorems related to difference equations.

A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots,\ x_{n-k}),\ n = 0, 1, \ldots \tag{1.1}$$

where F is a function that maps some set I^{k+1} into I. The set I is usually an interval of real numbers, or a union of intervals, or a discrete set such as the set of integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$.

A solution of (1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$that satisfies Equation (1.1) for all $n > 0$.

A solution of (1.1) that is constant for all $n \geq -k$ is called an equilibrium solution of (1.1). If

$x_n = \overline{x}$, for all $n \geq -k$

is an equilibrium solution of (1.1), then $\overline{x}$ is called an equilibrium point.

Definition 1.1 *(Discrete Dynamical System) An m-dimensional discrete dynamical system is a system of the following form*

$$\begin{cases} x_{n+1} = f_1(x_n, x_{n-1}, \ldots, x_{n-m}, y_n, y_{n-1}, \ldots, y_{n-m}) \\ y_{n+1} = f_2(x_n, x_{n-1}, \ldots, x_{n-m}, y_n, y_{n-1}, \ldots, y_{n-m}) \end{cases} \tag{1.2}$$

where $f_1 : I_1^{m+1} \times I_2^{m+1} \to I_1$ *and* $f_2 : I_1^{m+1} \times I_2^{m+1} \to I_2$ *are continuously differentiable functions and* $I_1,\ I_2$ *are some intervals of real numbers. Also, a solution* $\{(x_n, y_n)\}_{n=-m}^{\infty}$ *of system* (1.2) *is uniquely determined by initial values* $(x_{-i}, y_{-i}) \in I_1 \times I_2$ *for* $i = 0, 1, \ldots, m$.

Definition 1.2 *An equilibrium point of Equation* (1.1) *is a point* $\overline{x}$ *that satisfies*

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

The point $\overline{x}$ *is also called to a fixed point of the function* f.

Definition 1.3 *An equilibrium point of system* (1.2) *is a point* $(\overline{x}, \overline{y})$ *that satisfies*

$$\begin{cases} \overline{x} = f_1(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}) \\ \overline{y} = f_2(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}) \end{cases}$$

if we assume that the associated vector map

$$F = (f_1, x_n, x_{n-1}, \ldots, x_{n-m}, f_2, y_n, y_{n-1}, \ldots, y_{n-m}),$$

then, the point $(\overline{x}, \overline{y})$ *is also called a fixed point of the vector map* F.

Definition 1.4 *Let* $\overline{x}$ *be a positive equilibrium of* (1.1).

***i*)** *An equilibrium point* $\overline{x}$ *of Equation* (1.1) *is called locally stable if, for every* $\epsilon > 0$; *there exists* $\delta > 0$ *such that if* $\{x_n\}_{n=-k}^{\infty}$ *is a solution of Equation* (1.1) *with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \epsilon, \quad for \;\; all \;\; n \geq 0.$$

***ii*)** *An equilibrium point* $\overline{x}$ *of Equation* (1.1) *is called locally asymptotically stable if,* $\overline{x}$ *is locally stable, and if in addition there exists* $\gamma > 0$ *such that if* $\{x_n\}_{n=-k}^{\infty}$ *is a solution of Equation* (1.1) *with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \overline{x}.$$

***iii*)** *An equilibrium point* $\overline{x}$ *of Equation* (1.1) *is called a global attractor if, for every solution* $\{x_n\}_{n=-k}^{\infty}$ *of Equation* (1.1) *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

iv) *An equilibrium point* $\overline{x}$ *of Equation* (1.1) *is called globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Equation* (1.1)*.*

v) *An equilibrium point* $\overline{x}$ *of Equation* (1.1) *is called unstable if* $\overline{x}$ *is not locally stable.*

Definition 1.5 *Let* $(\overline{x}, \overline{y})$ *be an equilibrium point of the system* (1.2)*.*

i) *An equilibrium point* $(\overline{x}, \overline{y})$ *is said to be stable if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that for every initial condition* (x_i, y_i)*,* $i \in \{-1, 0\}$*, with if* $\| \sum_{i=-1}^{0}(x_i, y_i) - (\overline{x}, \overline{y}) \| < \delta$ *implies* $\|(x_n, y_n) - (\overline{x}, \overline{y})\| < \epsilon$*, for all* $n > 0$*, where* $\| \cdot \|$ *is usual Euclidian norm in* R^2*.*

ii) *It is said to be unstable, if an equilibrium point* $(\overline{x}, \overline{y})$ *is not stable.*

iii) *If an equilibrium point* $(\overline{x}, \overline{y})$ *is stable and there exists* $\eta > 0$ *such that* $\| \sum_{i=-1}^{0}(x_i, y_i) - (\overline{x}, \overline{y}) \| < \eta$ *and* $(x_n, y_n) \to (\overline{x}, \overline{y})$ *as* $n \to \infty$*, then, it is called to be asymptotically stable.*

iv) *The equilibrium point* $(\overline{x}, \overline{y})$ *is said to be a global attractor if*

$$\lim_{n \to \infty} (x_n, y_n) = (\overline{x}, \overline{y}).$$

v) *If an equilibrium point* $(\overline{x}, \overline{y})$ *is both global attractor and stable, then, it is called to be globally asymptotically stable.*

Definition 1.6 *(Linearization Method) Suppose that the function* F *is continuously differentiable in some open neighborhood of an equilibrium point* $\overline{x}$*. Let*

$$q_i = \frac{\partial F}{\partial u_i}(\overline{x}, \overline{x}, \ldots, \overline{x}) \text{ for } i = 0, 1, ..., k$$

denote the partial derivative of $F(u_0, u_1, ..., u_k)$ *with respect to* u_i *evaluated at the equilibrium point* $\overline{x}$ *of Equation* (1.1)*. Then the equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + \ldots + q_k y_{n-k}, \quad n = 0, 1, ... \tag{1.3}$$

is called the linearized equation of Equation (1.1) *about the equilibrium point* $\overline{x}$*, and the equation*

$$\lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - \ldots - p_k = 0 \tag{1.4}$$

is called the characteristic equation of Equation (1.1) *about* $\overline{x}$*.*

Theorem 1.1 *(The Linearized Stability Theorem) Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:*

i) *The equilibrium point $\overline{x}$ of Equation* (1.1) *is locally asymptotically stable if all the roots of Equation* (1.4) *have absolute value less than one.*

ii) *The equilibrium point $\overline{x}$ of Equation* (1.1) *is unstable if at least one root of Equation* (1.4) *has absolute value greater than one.*

iii) *When there exists a root of Equation* (1.4) *with absolute value equal to one, then the equilibrium $\overline{x}$ is called nonhyperbolic, then the equilibrium point $\overline{x}$ of Equation* (1.1) *is called hyperbolic if no root of Equation* (1.4) *has absolute value equal to one.*

iv) *When it is hyperbolic and if there exists a root of Equation* (1.4) *with absolute value less than one and another root of Equation* (1.4) *with absolute value greater than one then equilibrium point $\overline{x}$ of Equation* (1.1) *is called a saddle point.*

v) *When all roots of Equation* (1.4) *have absolute value greater than one then equilibrium point $\overline{x}$ of Equation* (1.1) *is called a repeller.*

Theorem 1.2 *Assume that*

$$|q_0| + |q_1| + ... + |q_k| < 1.$$

Then all roots of Equation (1.4) *lie inside the unit disk.*

Definition 1.7 *(Linearization Method for a discrete dynamical system) If $(\overline{x}, \overline{y})$ be an equilibrium point of a map*

$$F = (f_1, x_n, ..., x_{n-m}, f_1, y_n, ..., y_{n-m})$$

where f_1 and f_2 are continuously differentiable functions at $(\overline{x}, \overline{y})$. The linearized system of system (1.2) *about the equilibrium point $(\overline{x}, \overline{y})$ is*

$$X_{n+1} = F(X_n) = F_J X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ \vdots \\ x_{n-m} \\ y_n \\ \vdots \\ y_{n-m} \end{pmatrix}$$

and F_J *is a Jacobian matrix of the system* (1.2) *about the equilibrium point* $(\overline{x}, \overline{y})$.

Proposition 1.3 *Assume that* $x_{n+1} = F(x_n)$, $n = 0, 1, \ldots$, *is a system of difference equations and* $\overline{x}$ *be a fixed point of* F. *If all eigenvalues of the Jacobian matrix* J_F *about* $\overline{x}$ *lie inside the open unit disk* $|\lambda| < 1$, *then* $\overline{x}$ *is locally asymptotically stable. If one of them has a modulus greater than one, then* $\overline{x}$ *is unstable.*

Theorem 1.4 *(Rouche's Theorem) Let two functions* $f(z)$ *and* $g(z)$ *be analytic inside and on a simple closed curve* C, *and suppose that* $|f(z)| > |g(z)|$ *at each point on* C. *Then* $f(z)$ *and* $f(z) + g(z)$ *have the same number of zeroes, inside* C.

CHAPTER 2

LITERATURE REVIEW

2.1 DIFFERENCE EQUATIONS OF EXPONENTIAL FORM

In this section, we give some information about difference equations of exponential form and difference equation systems of exponential form.
In [8] , *Metwally et al.* dealt with the global stability, the boundedness nature, and the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \alpha \ + \beta x_{n-1} e^{-x_n} \ , \ n = 0, 1, \ \ldots \tag{2.1}$$

where α, β are positive constants and the initial values x_{-1}, x_0 are positive numbers. As a result, they proved that:

- Equation (2.1) has a unique equilibrium solution $\overline{x}$ and $\overline{x} > \alpha$.

- The equilibrium $\overline{x}$ of Equation (2.1) is locally asymtotically stable if
  $\beta < \dfrac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} e^{\frac{\alpha+\sqrt{\alpha^2+4\alpha}}{2}}$,
  and is unstable (and in fact is a saddle point) if
  $\beta > \dfrac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} e^{\frac{\alpha+\sqrt{\alpha^2+4\alpha}}{2}}$.

- Every positive solution of Equation (2.1) is bounded if $\beta < e^{\alpha}$ and Equation (2.1) has positive unbounded solutions if $\beta > e^{\alpha}$.

- If $\beta \leq e^{\alpha}(\frac{-\alpha+\sqrt{\alpha^2+4}}{2})$, Equation (2.1) has no positive solutions of prime period two and the equilibrium $\overline{x}$ of Equation (2.1) is globally asymptotically stable.

In [9], *Fotiades and Papaschinopoulos* discussed the existence, uniqueness and attractivity of prime period two solution of the difference equation

$$x_{n+1} = a + b x_{n-1} e^{-x_n}, \tag{2.2}$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers. Moreover, they found

- Equation (2.2) has a periodic solution of prime period two if
  $\dfrac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}} e^{\frac{a+\sqrt{a^2+4a}}{2}} < b$ and $b < e^a$.

- When $b < e^a$ and $ab > 2b - 2$, Equation (2.2) has a unique prime period two solution.

In [22], *Ma et al.* investigated the boundedness and the asymptotic behavior of the positive solutions of the difference equation

$$x_{n+1} = a + bx_n e^{-x_{n-1}}, \tag{2.3}$$

where a, b are positive constants, and the initial values x_{-1}, x_0 are positive numbers. Then the following results were exhibited in their paper for the equation (2.3);

- If $b < e^{\mathfrak{a}}$, the equation (2.3) has a unique positive equilibrium $\overline{x}$ and every positive solution is bounded.

- If $b < e^{\mathfrak{a}}$, the equation (2.3) has a unique positive equilibrium $\overline{x}$ so that $\overline{x} \in [a, \frac{a}{1-be^{-a}}]$ and every positive solution of (2.3) tends to the unique positive equilibrium $\overline{x}$ as $n \to \infty$.

- If $a \geq 2$, $b < \frac{2}{a+\sqrt{a^2-4a}} e^{1+a}$, the equilibrium $\overline{x}$ of (2.3) is locally asymptotically stable.

- If $a \geq 2$, $b < \min\{e^a, \frac{2}{a+\sqrt{a^2-4a}} e^{1+a}\}$, the equilibrium $\overline{x}$ of (2.3) is globally asymptotically stable.

In [29], *Papaschinopoulos, Radin and Schinas* discussed the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential type

$$x_{n+1} = a + bx_{n-1} e^{-y_n}, \quad y_{n+1} = c + dy_{n-1} e^{-x_n} \tag{2.4}$$

where a, b, c, d are positive constants, and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real values.
They demonstrated that:

- If $be^{-c} < 1$, $de^{-a} < 1$, every positive solution of (2.4) is bounded and persists.

- If $c \geq a$

  $$b < e^c \frac{-a + \sqrt{a^2 + 4}}{2}, \; d < e^a \min\{\frac{-c + \sqrt{c^2 + 4}}{2}, \; \frac{c - \sqrt{c^2 - a^2}}{a}\}$$

  and if $a \geq c$,

  $$d < e^a \frac{-c + \sqrt{c^2 + 4}}{2}, \; b < e^c \min\{\frac{-a + \sqrt{a^2 + 4}}{2}, \; \frac{a - \sqrt{a^2 - c^2}}{c}\}$$

  the system (2.4) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of (2.4) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ as $n \to \infty$.

- Every positive solution of (2.4) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ as $n \to \infty$ when assume that the constants a, b, c, d satisfy

  $$b < \frac{e^c}{c+1}, \; d < e^a \min\{\frac{1 - be^{-c}}{1 - be^{-c} + a}, \; \frac{1 - b(c+1)e^{-c}}{1 - be^{-c}}\}$$

  and the system (2.4) has unique positive equilibrium $(\overline{x}, \overline{y})$ such that

  $$\overline{x} \in (a, \frac{a}{1 - be^{-c}}), \; \overline{y} \in (c, \frac{c}{1 - de^{-a}}).$$

- If

  $$0 < be^{-c} + de^{-a} + bde^{-a-c} + \frac{abcde^{-a-c}}{(1 - de^{-a})(1 - be^{-c})} < 1,$$

  the unique positive equilibrium $(\overline{x}, \overline{y})$ of (2.4) is globally asymptotically stable.

In [27], *Papaschinopoulos, Fotiades and Schinas* dealt with the asymptotic behaviour of the positive solutions of the system of two difference equations

$$x_{n+1} = ay_n + bx_{n-l}e^{-y_n}, \quad y_{n+1} = cx_n + dy_{n-l}e^{-x_n}, \quad n = 0, 1, \ldots \tag{2.5}$$

where a, b, c, d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

- They demonstrated that under the requirement that $a, b, c, d \in (0, 1)$ $a + b > 1, \; c + d > 1$;

  - Every positive solution of (2.5) is bounded and persists.

- ◦ Every positive solution of (2.5) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ of (2.5) as $n \to \infty$, assume that both relationship

  $c \leq a, \quad b \leq c, \quad d \leq c$

  or

  $a \leq c, \quad b \leq a, \quad d \leq a.$

- When $a + b \leq 1$, $c + d \leq 1$; every positive solution of (2.5) tends to the zero equilibrium $(0, 0)$ of (2.5) as $n \to \infty$.

- They proved that such that

  $a + b < 1, \; c + d < 1$

  or

  $a + b = 1, \; c + d = 1,$

  the zero equilibrium $(0, 0)$ of (2.5) is globally asymptotically stable.

In [5], *Din and Elsayed* studied

$$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n}, \quad y_{n-1} = \delta + \varepsilon y_n + \zeta y_{n-1} e^{-x_n}, \tag{2.6}$$

where parameters $\alpha, \beta, \gamma, \; \delta, \; \varepsilon, \; \zeta$ and initial conditions x_0, x_{-1}, y_0, y_{-1} are positive real numbers. They dealt with the existence and uniqueness of positive equilibrium point, boundedness character, persistence, local asymptotic stability, global behavior and rate of convergence of unique positive equilibrium point.
They obtained the following results:

- If

  $$e^{\delta}\beta + e^{\delta/2}\sqrt{e^{\delta}\beta^2 + 4\gamma} < 2e^{\delta}, \qquad e^{a}\varepsilon + e^{a/2}\sqrt{e^{a}\varepsilon^2 + 4\zeta} < 2e^{a},$$

  every positive solution $\{(x_n, y_n)\}$ of system (2.6) is bounded and persists.

- If

$$e^{\delta}\beta + e^{\delta/2}\sqrt{e^{\delta}\beta^2 + 4\gamma} < 2e^{\delta}, \quad e^{a}\varepsilon + e^{a/2}\sqrt{e^{a}\varepsilon^2 + 4\zeta} < 2e^{a}, \quad 0 < \beta, \quad \in < 1,$$

and

$$\zeta \exp\left(\frac{-\varepsilon\delta - \delta\zeta\exp(\frac{\alpha}{1-\beta-e^{-\delta}\gamma})}{1 - \varepsilon - \zeta\exp(\frac{\alpha}{1-\beta-e^{-\delta}\gamma})}\right) < \gamma < (1-\beta)\exp\left(\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}\right),$$

the system (2.6) has a unique positive equilibrium point $(\overline{x}, \overline{y})$ such that

$$\overline{x} \in [\alpha, \frac{a}{1-\beta-\gamma e^{-\delta}}] = I \text{ and } \overline{y} \in [\delta, \frac{\delta}{1-\varepsilon-\zeta e^{-\alpha}}] = J.$$

- If

$$\beta + \varepsilon + \beta\varepsilon + \zeta(1+\beta)e^{-a} + \gamma(1+\varepsilon)e^{-\delta} + \gamma\zeta e^{-a-\delta}\left(1 + \frac{a\delta}{(1-\beta-e^{-\delta}\gamma)(1-\varepsilon-e^{-\alpha}\zeta)}\right) < 1;$$

the unique positive equilibrium point $(\overline{x}, \overline{y})$ of system (2.6) is locally asymptotically stable.

- The unique positive equilibrium point $(\overline{x}, \overline{y})$ of system (2.6) is globally asymptotically stable.

- Under the condition that $\beta, \varepsilon \in (0,1)$ and $\{(x_n, y_n)\}$ be a positive solution of system (2.6);

  - When $\gamma > e^{\frac{\delta}{1-\varepsilon}}$, *then* $x_n \to \infty$, $y_n \to \delta + \varepsilon\ln(\gamma)$ as $n \to \infty$,
  - If $\zeta > e^{\frac{\alpha}{1-\gamma}}$, *then* $x_n \to \alpha + \beta\ln(\zeta)$, $y_n \to \infty$ *as* $n \to \infty$.

In [30], *Papaschinopoulos and Schinas* investigated the asymptotic behavior of the positive solutions of the systems of the two difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \quad y_{n+1} = c + dx_{n-1}e^{-y_n}, \tag{2.7}$$

$$x_{n+1} = a + by_{n-1}e^{-y_n}, \quad y_{n+1} = c + dx_{n-1}e^{-x_n}, \tag{2.8}$$

where the constants a, b, c, d are positive real numbers, and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive real numbers.

They proved that under the condition that:

- If

$\theta_1 = be^{-a} < 1, \quad \theta_2 = de^{-c} < 1,$

$(1 \ + a)p + c\theta_1 < 1, \quad (1 \ + c)p + a\theta_2 < 1$

and

$$\lambda = \frac{p(1-p)^2}{[1-(1+a)p-c\theta_1][1-(1+c)p-a\theta_2]} < 1;$$

the system (2.7) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every solution of (2.7) tends to the unique positive equilibrium of (2.7) as $n \to \infty$.

- If

$\zeta_1 = be^{-c} < 1, \quad \zeta_2 = de^{-a} < 1$

and

$$\mu = \frac{p(1-p+c+a\zeta_2)(1-p+a+c\zeta_1)}{(1-p)^2} < 1;$$

the system (2.8) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of (2.8) tends to the unique positive equilibrium of (2.8) as $n \to \infty$.

- If

$\theta_1 = be^{-a} < 1, \theta_2 = de^{-c} < 1$

$$\lambda = \frac{p(1-p)^2}{[1-(1+a)p-c\theta_1][1-(1+c)p-a\theta_2]} < 1,$$

and when

$$\kappa \ = \frac{c\theta_1 + a\theta_2 + (a+c)p}{1-p} + \frac{p(a+c\theta_1)(c+a\theta_2)}{(1-p)^2} + p < 1,$$

the unique positive equilibrium $(\overline{x}, \overline{y})$ of (2.7) is globally asymptotically stable.

- If

$\zeta_1 = be^{-c} < 1, \quad \zeta_2 = de^{-a} < 1$

and

$$\mu = \frac{p(1-p+c+a\zeta_2)(1-p+a+c\zeta_1)}{(1-p)^2} < 1;$$

the unique positive equilibrium $(\overline{x}, \overline{y})$ of (2.8) is globally asymptotically stable.

- When $\theta_1 > 1$, $\theta_2 > 1$, there exist unbounded solutions (x_n, y_n) of (2.7) so that

$$\lim_{n\to\infty} x_{2n+1} = \infty,\ \lim_{n\to\infty} x_{2n} = a,\ \lim_{n\to\infty} y_{2n+1} = \infty,\ \lim_{n\to\infty} y_{2n} = c \tag{2.9}$$

$$\lim_{n\to\infty} x_{2n+1} = a,\ \lim_{n\to\infty} x_{2n} = \infty,\ \lim_{n\to\infty} y_{2n+1} = c,\ \lim_{n\to\infty} y_{2n} = \infty. \tag{2.10}$$

- $\zeta_1 > 1$, $\zeta_2 > 1$, there exist unbounded solutions (x_n, y_n) of (2.8) such either relations (2.9) or (2.10) hold.

In [33], *Phong* dealt with the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential type:

$$x_{n+1} = a + bx_{n-1} + cx_{n-1}e^{-y_n}, \quad y_{n+1} = \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-x_n}, \tag{2.11}$$

where $a, b, c, \alpha, \beta, \gamma \in (0, \infty)$, and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real values. They proved the following results:

- If $b + ce^{-\alpha} < 1$, $\beta + \gamma e^{-a} < 1$, every positive solution of (2.11) is bounded and persists.

- If $\alpha(1-b) \geq a(1-\beta)$ then

  $c < e^{\alpha}\frac{-a(1-2\beta)+\sqrt{a^2(1-2\beta)^2+4(1-b)^2}}{2}$,

  $\gamma < e^{a}\min\{\frac{\alpha(1-b)-\sqrt{\alpha^2(1-b)^2-a^2(1-\beta)^2}}{a}, \frac{-\alpha(1-2b)+\sqrt{\alpha^2(1-2b)^2+4(1-\beta)^2}}{2}\}$

  and if $c(1-b) \leq a(1-\beta)$ then

  $\gamma < e^{a}\frac{-\alpha(1-2b)+\sqrt{\alpha^2(1-2b)^2+4(1-\beta)^2}}{2}$,

  $c < e^{\alpha}$,

  system (2.11) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of (2.11) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ as $n \to \infty$.

- When the contants $a, b, c, \alpha, \beta, \gamma$ satisfy the following relations:

  $\gamma < e^{a}\min\{\frac{(1-\beta)(1-b-ce^{-\alpha})}{1+a-b-ce^{-\alpha}}, \frac{(1-b)(1-\beta)-c(1+\alpha-\beta)e^{-\alpha})}{1-b-ce^{-\alpha}}\}$,

  $c < e^{\alpha}\frac{(1-b)(1-\beta)}{1+\alpha-\beta}$,

  system (2.11) has unique positive equilibrium $(\overline{x}, \overline{y})$ such that $x \in (\ a, \frac{a}{1-b-ce^{-\alpha}})$, $y \in (\alpha, \frac{\alpha}{1-\beta-\gamma e^{-a}})$ every positive solution of (2.11) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ as $n \to \infty$.

- If $b+ce^{-\alpha}+\beta+\gamma e^{-a}+(b+ce^{-\alpha})(\beta+\gamma e^{-a})+\frac{ac\alpha\gamma e^{-a-\alpha}}{(1-b-oe^{-\alpha})(1-\beta-\gamma e^{-a})}<1$, the unique positive equilibrium $(\overline{x},\overline{y})$ of (2.11) is globally asymptotically stable.

In [26], *Papaschinopoulos et al.* studied the existence of a unique positive equilibrium, the boundedness, persistence and global attractivity of the positive solutions of a system of the following two difference equations

$$x_{n+1}=ax_n+by_{n-l}e^{-x_n},\quad y_{n+1}=cy_n+dx_{n-l}e^{-y_n},\quad n=0,1,\ldots \tag{2.12}$$

where a,b,c,d are positive contants and the initial values x_{-1},x_0,y_{-1},y_0 are positive real numbers.

They obtained the results which are listed below:

- When $a,b,c,d\in(0,1)$,
  - $\theta=\frac{bd}{(1-a)(1-c)}>1$, system (2.12) has a unique positive equilibrium $(\overline{x},\overline{y})$.
  - $\theta\leq 1$, the zero equilibrium $(0,0)$ is the unique equilibrium of system (2.12).
- If $a,b,c,d\in(0,1)$, every positive solution of (2.12) is bounded.
- If $a,b,c,d\in(0,1)$, $\frac{b}{1-a}>1$, $\frac{d}{1-c}>1$, every positive solution of (2.12) is bounded and persists.
- When $a,b,c,d\in(0,1), bd<(1-a)(1-c)$, every positive solution of (2.12) tends to the zero equilibrium.
- If $\max\left\{\frac{d}{1-c},\frac{b}{1-a}\right\}<\min\left\{e^{\frac{c}{d}},e^{\frac{a}{b}}\right\}$, every positive solution of (2.12) tends to the unique positive equilibrium of (2.12).

In [31] under some conditions on the constants $A,B\in(0,\infty)$, *Papaschinopoulos et al.* studied the existence of positive solutions, the existence of a unique nonnegative equilibrium and the convergence of the positive solutions to the nonnegative equilibrium of the system of difference equations

$$x_{n+1}=(1-y_n-y_{n-1})(1-e^{-Ay_n}),\quad y_{n+1}=(1-x_n-x_{n-1})(1-e^{-Bx_n}) \tag{2.13}$$

where $A, B \in (0, \infty)$ and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers which satisfy the relations $x_0 + x_{-1} < 1,\ y_0 + y_{-1} < 1, 1 - y_0 > (1 - x_0 - x_{-1})(1 - e^{-Bx_0}),\ 1 - x_0 > (1 - y_0 - y_{-1})(1 - e^{-Ay_0})$.
They proved the following results:

- Under the condition that the system of algebrations

$$x = (1 - 2y)(1 - e^{-Ay}), \qquad y = (1 - 2x)(1 - e^{-Bx}), \quad x, y \in (0,\ 0.5). \tag{2.14}$$

  - If $0 < A \leq 1,\ 0 < B \leq 1$, the system (2.14) has a unique nonnegative solution $(\overline{x}, \overline{y}) = (0, 0)$.
  - If $1 < A \leq 4,\ 1 < B \leq 4$, system (2.14) has a unique positive solution $(\overline{x}, \overline{y})$, $\overline{x}, \overline{y} \in (0, 0.5)$.

- If $0 < A \leq 1,\ 0 < B \leq 1$ are satisfied the solution (x_n, y_n) tends to the zero equilibrium $(0, 0)$ of (2.13) as $n \to \infty$.

- When $1 < A \leq 4,\ 1 < B \leq 4$ are satisfied and there exists a $m \in N$ so that for $n \geq m$ either $x_n < \overline{x}$, $y_n < \overline{y}$ or $x_n \geq \overline{x}$, $y_n \geq \overline{y}$; (x_n, y_n) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ of (2.13) as $n \to \infty$.

In [3], *Din* studied

$$x_{n+1} = \alpha x_n e^{-y_n} + \beta, \qquad y_{n+1} = \alpha x_n (1 - e^{-y_n}), \tag{2.15}$$

where $0 < \alpha < 1$ and $0 < \beta < \infty$. More precisely, he investigated boundedness character, existence and uniqueness of positive equilibrium point, local asymptotic stability and global asymptotic stability of unique positive equilibrium point, and the rate of convergence of positive solutions of a population model.
He determined the following results:

- If $0 < \alpha < 1$, then every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the system (2.15) is bounded.

- When $0 < \alpha < 1$ and $\alpha\beta > 1 - \alpha$, the system (2.15) has a unique positive equilibrium point $(\overline{x}, \overline{y}) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$.

- When $0 < \alpha < 1$, then following statements are true:
    - The equilibrium point $(\frac{\beta}{1-\alpha}, 0)$ of the system (2.15) is locally asymptotically stable if and only if $\alpha\beta < 1 - \alpha$.
    - If $\alpha\beta > 1 - \alpha$, the equilibrium point $(\frac{\beta}{1-\alpha}, 0)$ of the system (2.15) is unstable.
- If $\beta(1 + \alpha) + r < \frac{r}{1-r}$ where $r = \frac{\beta}{\overline{x}}$, the unique positive equilibrium point $(\overline{x}, \overline{y}) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$ of system 2.15 is locally asymptotically stable.
- If and only if $(1 \ - r)(r + \beta) < r + \alpha\beta(1 - r) < 2r$ where $r = \frac{\beta}{\overline{x}}$, the unique positive equilibrium point $(\overline{x}, \overline{y}) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$ of system (2.15) is locally asymptotically stable.
- When $0 < \alpha < 1$, then the unique positive equilibrium point $(\overline{x}, \overline{y}) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$ is a global attractor.
- When $0 < \alpha < 1$ and $(1 \ - r)(r + \beta) < r + \alpha\beta(1 - r) < 2r$, the unique positive equilibrium point $(\overline{x}, \overline{y}) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$ of the system (2.15) is globally asymptotically stable.
- When $\{(x_n, y_n)\}$ be a positive solution of the system (2.15) then

  $$\lim_{n\to\infty} x_n = \overline{x}, \quad \lim_{n\to\infty} y_n = \overline{y},$$

  where $\overline{x} \in [\beta, \frac{\beta}{1-\alpha}]$ and $\overline{y} \in [0, \frac{\alpha\beta}{1-\alpha}]$.

In [7], *Ding and Zhang* studied the following discrete delay *Mosquito population equation* [13]:

$$x_{n+1} = (ax_n + \beta x_{n-1})e^{-x_n}, \ x_0, x_1 > 0, \ n = 1, 2, 3, \ \dots \ , \tag{2.16}$$

where $a \in (0, 1), \beta \in (0, \infty)$. For $a + \beta > 1$, a unique nontrivial positive fixed point $E^* = (\overline{x}, \overline{x})^T$ appears.

They got the results are given below:

- The equilibrium points of (2.16) are solutions of the following equation $\overline{x} = (a\overline{x} + \beta\overline{x})e^{-\overline{x}}$.

- When $a+\beta > 1$, $\overline{x} = 0$ is always a equilibrium to (2.16), and (2.16) has an unique positive equilibrium $\overline{x} = \ln(a+\beta)$.

- When $a+\beta < 1$, the zero equilibrium of (2.16) is asymptotically stable, and unstable when $a+\beta > 1$, and a fold bifurcation takes place when $a+\beta = 1$.

- When $a+\beta > 1$,

  - If $\beta < \beta_0(a)$, E^* is asymptotically stable.
  - If $\beta > \beta_0(a)$, E^* is unstable.
  - The bifurcation of a period two solution occurs at $\beta = \beta_0(a)$, that is, system $u_{n+1} = (au_n + \beta v_n)e^{-u_n}$, $v_{n+1} = u_n$ has a unique period two solution bifurcating from the equilibrium E^*.

- A period two bifurcation of $u_{n+1} = (au_n + \beta v_n)e^{-u_n}$, $v_{n+1} = u_n$ at $\beta = \beta_0$ occurs, and the unique period two solution bifurcating from E^* is unstable.

In [21], Ma and Feng discussed the boundedness and the global asymptotic behavior of the positive solutions of the system of difference equation

$$\begin{aligned} x_{n+1} &= x_n + (\alpha_1 - \beta_1 x_n)x_n e^{-(x_n+y_n)} \\ y_{n+1} &= y_n + (\alpha_2 - \beta_2 y_n)y_n e^{-(x_n+y_n)} \end{aligned} \tag{2.17}$$

where $\alpha_i, \beta_i \in (0,\infty)$ with $\alpha_i > \beta_i$, $i = 1,2$, and the initial values x_0, y_0 are positive numbers.

They then achieved the following main results:

- The equilibrium point E_0 of system (2.17) is unstable.

- The equilibria points E_1 and E_2 of system (2.17) are saddle points.

- If either $0 < \alpha_i e^{-\left(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}\right)} < 1$ or $1 < \alpha_i e^{-\left(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}\right)} < 2$, $i = 1,2$, the Nash equilibrium point E_3 of system (2.17) is asymptotically stable.

- When $\alpha_i < 1$, the unique positive equilibrium point $(\overline{x},\overline{y})$ of system (2.17) is a global attractor of all positive solutions of system (2.17).

- If $\alpha < 1$, the unique positive equilibrium point $(\overline{x}, \overline{y})$ of system (2.17) is a global attractor of all positive solutions of system (2.17) for $\alpha_1 = \alpha_2 = \alpha,\ \beta_1 = \beta_2 = \beta$ and if $\beta > \frac{\alpha^2+4\beta^2}{4}$.

- When $\alpha_i \leq \beta_i,\ \alpha_i e^{-\frac{\alpha_i}{\beta_i}} < 1$, the unique positive equilibrium point $(\overline{x}, \overline{y})$ of system (2.17) is a global attractor of all positive solutions of system (2.17) for $\alpha_1 = \alpha_2 = \alpha,\ \beta_1 = \beta_2 = \beta$.

2.2 DIFFERENCE EQUATIONS OF RATIONAL EXPONENTIAL FORM

In this section, we give some information about the rational exponential type difference equations and systems.
In [25], *Ozturk, Bozkurt and Ozen* investigated the convergence, the boundedness and the periodic character of the positive solutions of the difference equation

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}},\ n = 0, 1, 2, \ldots \tag{2.18}$$

where the parameters α, β, γ are positive numbers and the initial conditions y_{-1}, y_0 are arbitrary nonnegative numbers.
They obtained the results are given below:

- The equilibrium point $\overline{y}$ is locally asymptotically stable if

  $$\beta < (2 + \sqrt{(\gamma-2)^2 + 4(\alpha+\gamma)})e^{\dfrac{-(\gamma-2) + \sqrt{(\gamma-2)^2 + 4(\alpha+\gamma)}}{2}}$$

  and is unstable if

  $$\beta > (2 + \sqrt{(\gamma-2)^2 + 4(\alpha+\gamma)})e^{\dfrac{-(\gamma-2) + \sqrt{(\gamma-2)^2 + 4(\alpha+\gamma)}}{2}},$$

  furthermore, it is a saddle point.

- The following items are correct;

  - When $\alpha < y_n$, every positive solution of Equation (2.18) is bounded.
  - When $\alpha < \overline{y}_1$, the positive equilibrium point of Equation (2.18) is bounded.

- Equation (2.18) has no positive solutions of prime period two.

- When $\beta < (2+\sqrt{(\gamma-2)^2+4(\alpha+\gamma)})e^{\frac{-(\gamma-2)+\sqrt{(\gamma-2)^2+4(\alpha+\gamma)}}{2}}$ and that $\beta < \gamma$, the equilibrium $\overline{y}_1$ of Equation (2.18) is global asymptotically stable.

In [28] *Papaschinopoulos et al.* studied the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following systems of two difference equations of exponential form:

$$x_{n+1} = \frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}}, \; y_{n+1} = \frac{\delta+\varepsilon e^{-x_n}}{\zeta+x_{n-1}}, \tag{2.19}$$

$$x_{n+1} = \frac{\alpha+\beta e^{-y_n}}{\gamma+x_{n-1}}, y_{n+1} = \frac{\delta+\varepsilon e^{-x_n}}{\zeta+y_{n-1}}, \tag{2.20}$$

$$x_{n+1} = \frac{\alpha+\beta e^{-x_n}}{\gamma+y_{n-1}}, y_{n+1} = \frac{\delta+\varepsilon e^{-y_n}}{\zeta+x_{n-1}}, \tag{2.21}$$

where $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$ are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive constants.

Consequently, the following statements are obtained:

- For (2.19) system
  - Every positive solution of Equation (2.19) is bounded and persists.
  - When $\varepsilon < \gamma$, $\beta < \zeta$, system Equation (2.19) has a unique positive equilibrium $(\overline{x},\overline{y})$ and every positive solution of Equation (2.19) tends to the unique positive equilibrium of Equation (2.19) as $n \to \infty$.
  - Consider system Equation (2.19) where the condition Equation $\varepsilon < \gamma$, $\beta < \zeta$ holds true and suppose that

    $$\frac{\beta\varepsilon+(\beta+\varepsilon)e^{-1}}{\gamma\zeta} + \frac{(\alpha+\beta)(\delta+\varepsilon)}{\gamma^2\zeta^2} < 1,$$

    the unique positive equilibrium $(\overline{x},\overline{y})$ of Equation (2.19) is globally asymptotically stable.

- For (2.20) system
  - Every positive solution of Equation (2.20) is bounded and persists.
  - When $\beta\varepsilon < \gamma\zeta$, Equation (2.20) has a unique positive equilibrium $(\overline{x},\overline{y})$ and every positive solution of Equation (2.20) tends to the unique positive equilibrium of Equation (2.20) as $n \to \infty$.

- Assume that system Equation (2.20) where $\beta\varepsilon < \gamma\zeta$ holds true and suppose that

  $$\frac{\alpha+\beta}{\gamma^2}+\frac{\delta+\varepsilon}{\zeta^2}+\frac{\beta\varepsilon}{\gamma\zeta}+\frac{(\alpha+\beta)(\delta+\varepsilon)}{\gamma^2\zeta^2}<1,$$

  the unique positive equilibrium $(\overline{x}, \overline{y})$ of Equation (2.20) is globally asymptotically stable.

- For (2.21) system

  - Every positive solution of Equation (2.21) is bounded and persists.
  - When $\beta < \gamma$, $\varepsilon < \zeta$, system Equation (2.21) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of Equation (2.21) tends to the unique positive equilibrium of Equation (2.21) as $n \to \infty$.
  - Assume that system Equation (2.21) where $\beta < \gamma$, $\varepsilon < \zeta$ holds true and suppose that

    $$\frac{\beta}{\gamma}+\frac{\varepsilon}{\zeta}+\frac{\beta\varepsilon}{\gamma\zeta}+\frac{(\alpha+\beta)(\delta+\varepsilon)}{\gamma^2\zeta^2}<1,$$

    the unique positive equilibrium $(\overline{x}, \overline{y})$ of Equation (2.21) is globally asymptotically stable.

In [6] *Din, Khan and Nosheen* studied the boundedness character and persistence, existence and uniqueness of positive equilibrium, local and global behavior, and rate of convergence of positive solutions of the following system of exponential difference equations:

$$x_{n+1}=\frac{\alpha_1+\beta_1 e^{-x_n}+\gamma_1 e^{-x_{n-1}}}{a_1+b_1 y_n+c_1 y_{n-1}}, \quad y_{n+1}=\frac{\alpha_2+\beta_2 e^{-y_n}+\gamma_2 e^{-y_{n-1}}}{a_2+b_2 x_n+c_2 x_{n-1}}, \tag{2.22}$$

where the parameters $\alpha_i, \beta_i, \gamma_i, a_i, b_i$ and c_i for $i \in \{1, 2\}$ and initial conditions x_0, x_{-1}, y_0 and y_{-1} are positive real numbers.
They demonstrated that:

- Every positive solution $\{(x_n, y_n)\}$ of system (2.22) is bounded and persists.

- If the following condition is satisfied:

  $$a_2 + L_1(b_2 + c_2) < \frac{\alpha_2 + (\beta_2 + \gamma_2)e^{-K}}{K}$$

where

$$K = \frac{\alpha_1 + e^{-L_1}(\beta_1+\gamma_1) - a_1 L_1}{L_1(b_1+c_1)},$$

system (2.22) has a unique positive equilibrium point $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$.

- When

$$\frac{\beta_1 e^{-L_1}}{a_1+(b_1+c_1)L_2} + (1 + \frac{\beta_1 e^{-L_1}}{a_1+(b_1+c_1)L_2})$$
$$\times(\frac{c_2(\alpha_2+(\beta_2+\gamma_2)e^{-L_2})}{(a_2+(b_2+c_2)L_1)^2} + \frac{(\beta_2+\gamma_2)e^{-L_2}}{a_2+(b_2+c_2)L_1})$$
$$+\frac{b_2(\alpha_2+(\beta_2+\gamma_2)e^{-L_2})}{(a_2+(b_2+c_2)L_1)^2}$$
$$\times(\frac{(b_1+c_1)(\alpha_1+(\beta_1+\gamma_1)e^{-L_1})}{(a_1+(b_1+c_1)L_2)^2}$$
$$+\frac{\gamma_1 e^{-L_1}}{a_1+(b_1+c_1)L_2}) < 1,$$

the unique positive equilibrium point of system (2.22) is locally asymptotically stable.

- If

$$\alpha_1 + (\beta_1+\gamma_1)e^{-L_1} < \overline{x}(a_1+(b_1+c_1)L_2),$$

$$\alpha_2 + (\beta_2+\gamma_2)e^{-L_2} < \overline{y}(a_2+(b_2+c_2)L_1),$$

the unique positive equilibrium point $(\overline{x}, \overline{y})$ of system (2.22) is globally asymptotically stable.

In [18] *Khan and Qureshi* studied

$$x_{n+1} = \frac{bx_n e^{-ay_n}}{1+dx_n}, \qquad y_{n+1} = cx_n(1-e^{-ay_n}), \tag{2.23}$$

where a, b, c, d and the initial conditions x_0, y_0 are positive real numbers. More precisely, they investigated the boundedness character, existence and uniqueness of a positive equilibrium point, local asymptotic stability and global stability of the unique equilibrium point, and the rate of convergence of equilibrium solutions of the system.
They demonstrated that:

- Every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (2.23) is bounded.

- If $b > 1$ and $d < \frac{ac}{b\ln(7)}$ then system (2.23) has a unique positive equilibrium point $(\overline{x}, \overline{y})$ in $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$.

- For the unique positive equilibrium point $(\overline{x}, \overline{y})$ in $[0, \frac{b}{d}] \times [0, \frac{bc}{d}]$ of system (2.23) following statements hold true:

  - If and only if

    $$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1\ +bdr)^2}$$

    $$< 1 - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2} < 2,$$

    the unique positive equilibrium point of system (2.23) is locally asymptotically stable.

  - If and only if

    $$|\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}| > 1$$

    and

    $$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1\ +bdr)^2}$$

    $$< |1 - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}|,$$

    the unique positive equilibrium point is a repeller.

  - If and only if

    $$(\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2})^2$$

    $$+4(\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}) > 0$$

    and

    $$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1\ +bdr)^2}$$

    $$> |1 - \frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}|,$$

    the unique positive equilibrium point is a saddle point.

- If and only if

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1\ +bdr)^2}$$
$$=|1-\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}|,$$

the unique positive equilibrium point is nonhyperbolic.

- If $ac+d>abc$, then the unique positive equilibrium point $(\overline{x},\overline{y})$ in $[0,\frac{b}{d}]\times[0,\frac{bc}{d}]$ of system (2.23) is a global attractor.

- If and only if

$$\frac{be^{acr(bdr+1-b)}(acr(1+bdr)^2+1)}{(1+bdr)^2}$$
$$<\ 1\ -\frac{ab^2cre^{acr(bdr+1-b)}(bdr(e^{acr(bdr+1-b)}-1)-1)}{(1+bdr)^2}<2,$$

the unique positive equilibrium point $(\overline{x},\overline{y})$ in $[0,\frac{b}{d}]\times[0,\frac{bc}{d}]$ of system (2.23) is globally asymptotically stable.

In [17] *Khan and Qureshi* studied the qualitative behavior of the following exponential system of rational difference equations:

$$x_{n+1}=\frac{\alpha e^{-y_n}+\beta e^{-y_{n-1}}}{\gamma+\alpha x_n+\beta x_{n-1}},\ y_{n+1}=\frac{\alpha_1 e^{-x_n}+\beta_1 e^{-x_{n-1}}}{\gamma_1+\alpha_1 y_n+\beta_1 y_{n-1}},\ n=0,1,\ \ldots, \tag{2.24}$$

where $\alpha,\beta,\gamma,\alpha_1,\beta_1$ and γ_1 and initial conditions x_0,x_{-1},y_0 and y_{-1} are positive real numbers. They investigated the boundedness character and persistence, existence and uniqueness of positive equilibrium, local and global behavior, and rate of convergence of positive solutions that converges to unique positive equilibrium point of the system. Later, the following results were exhibited in their paper:

- Every positive solution $\{(x_n,y_n)\}$ of the system (2.24) is bounded and persists.

- If

$$\zeta<(\gamma_1+(\alpha_1+\beta_1)\ln[\frac{\alpha+\beta}{(\gamma+(\alpha+\beta)L_1)L_1}])$$
$$\times\ln[\frac{\alpha+\beta}{(\gamma+(\alpha+\beta)L_1)L_1}],$$

where

$\zeta = \gamma_1(((\gamma + 2(\alpha+\beta)U_1)/((\gamma + (\alpha+\beta)L_1)L_1))$

$-\ln[(\alpha+\beta)/((\gamma + (\alpha+\beta)L_1)L_1)])$

$+2(\alpha_1+\beta_1)((\gamma + 2(\alpha+\beta)U_1)/((\gamma + (\alpha+\beta)L_1)L_1))$

$\times \ln[(\alpha+\beta)/((\gamma + (\alpha+\beta)L_1)L_1)],$

and so the system (2.24) has a unique positive equilibrium point $(\overline{x}, \overline{y})$ in $[L_1, U_1] \times [L_2, U_2]$.

- If

  $\mu \ < (\gamma + (\alpha+\beta)L_1)(\gamma_1 + (\alpha_1+\beta_1)L_2),$

  where

  $\mu \ = (\alpha+\beta)U_1(\gamma_1 + (\alpha_1+\beta_1)L_2)$

  $+(\alpha_1+\beta_1)U_2(\gamma + (\alpha+\beta)L_1)$

  $+(\alpha+\beta)(\alpha_1+\beta_1)(e^{-L_1-L_2} + U_1U_2),$

  and so, the unique positive equilibrium point $(\overline{x}, \overline{y})$ of the system (2.24) is locally asymptotically stable.

- The unique positive equilibrium point $(\overline{x}, \overline{y})$ of the system (2.24) is a global attractor.

In [19] *Khuong and Phong* investigated the boundedness, the continuity and the asymptotic behavior of the positive solutions of the system of difference equations of exponential type:

$$x_{n+1} = \frac{a + be^{-x_n}}{c + y_n} \quad , \quad y_{n+1} = \frac{a + be^{-y_n}}{c + x_n}, \tag{2.25}$$

where a, b, c are positive constants and the initial values x_0, y_0 are positive real values. Also, they determined the rate of convergence of a solution that converges to the equilibrium $E = (\overline{x}, \overline{y})$ of this system. Moreover, they found,

- Every positive solution of (2.25) is bounded and persists.

- If $b < c$, system (2.25) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of (2.25) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ as $n \to \infty$. Additively, the equilibrium $(\overline{x}, \overline{y})$ is globally asymptotically stable.

In [1] *Bozkurt*, the local and global behavior of the positive solutions of the difference equation

$$y_{n+1} = \frac{\alpha e^{y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \ n = 0, 1, \cdots \tag{2.26}$$

was investigated, where the parameters α, β and γ and the initial conditions are arbitrary positive numbers. Furthermore, the characterization of the stability was studied with a basin that depends on the conditions of the coefficients.

- If $0 < y_n$, every solution of Equation (2.26) is bounded.
- If $0 < \overline{y}$, the equilibrium point of Equation (2.26) is bounded.
- If $\alpha > \beta$ and $(\alpha + \beta) < \gamma e^{\frac{-\gamma + \sqrt{\gamma^2 + 4(\alpha+\beta)\gamma}}{2(\alpha+\beta)}}$;
  - The positive equilibrium point of Equation (2.26) is locally asymptotically stable.
  - $\overline{y}_1$ and $\overline{y}_2$ are the equilibrium points of Equation (2.26), which parameters have the conditions $\gamma_2 < \gamma_1 < \frac{(\alpha+\beta)\beta}{\alpha}$. If the parameter γ decreases, then the local stability of the positive equilibrium point

    $$\overline{y} = \frac{-(\alpha\gamma - (\alpha + \beta)\beta) + \sqrt{(\alpha\gamma - (\alpha + \beta)\beta)^2 + 4\alpha(\alpha + \beta)^2}}{2\alpha(\alpha + \beta)}$$

    decreases also.
  - If $\{y_n\}_{-1}^{\infty}$ is a monoton decreasing solution of Equation (2.26) and $y_n > 2\overline{y}$, the positive equilibrium point of Equation (2.26) is globally asymptotically stable.
- If $f(x, y) = \frac{\alpha e^{-x} + \beta e^{-y}}{\gamma + \alpha x + \beta y}$ be a function such that $f \in C[(0, \infty)x(0, \infty), (0, \infty)]$, every oscillatory solution of Equation (2.26) has semicycle of length at most two.

In [16] *Khan*, studied boundedness character and persistence, existence and uniqueness of the positive equilibrium, local and global behavior, and rate of convergence of positive solutions of the following two systems of exponential rational difference equations:

$$x_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \ y_{n+1} = \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}, \ n = 0, 1, \ \ldots \tag{2.27}$$

and

$$x_{n+1} = \frac{\alpha e^{-x_n} + \beta e^{-x_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \ y_{n+1} = \frac{\alpha_1 e^{-y_n} + \beta_1 e^{-y_{n-1}}}{\gamma_1 + \alpha_1 x_n + \beta_1 x_{n-1}}, \ n = 0, 1, \ \ldots \tag{2.28}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and the initial conditions are positive real numbers. He obtained the following results:

- For (2.27) system
  - Every positive solution $\{(x_n, y_n)\}$ of system (2.27) is bounded and persists.
  - If

    $$\eta < (\gamma(\gamma_1 + (\alpha_1 + \beta_1)L_1) + (\alpha + \beta)(\alpha_1 + \beta_1)e^{-L_1})^2(\gamma_1 + (\alpha_1 + \beta_1)L_1)$$

    where

    $$\eta = (\alpha + \beta)(\alpha_1 + \beta_1)e^{-\frac{(\alpha_1+\beta_1)e^{-L_1}}{\gamma_1+(\alpha_1+\beta_1)L_1}L_1}((\gamma + \alpha + \beta)(\gamma_1 + (\alpha_1 + \beta_1)U_1)$$

    $$+(\alpha + \beta)(\alpha_1 + \beta_1)e^{-L_1})(\gamma_1 + (1 \ + U_1)(\alpha_1 + \beta_1)),$$

    system (2.27) has a unique positive equilibrium point $(\overline{x}, \overline{y})$ in $[L_1, U_1] \times [L_2, U_2]$.
  - If $(\alpha + \beta)(\alpha_1 + \beta_1)(e^{-L_2} + U_1)(e^{-L_1} + U_2) < (\gamma + (\alpha + \beta)L_2)(\gamma_1 + (\alpha_1 + \beta_1)L_1)$, the unique positive equilibrium point $(\overline{x}, \overline{y})$ in $[L_1, U_1] \times [L_2, U_2]$ system (2.27) is locally asymptotically stable.
  - If $(\alpha + \beta)e^{-L_2} < \overline{x}(\gamma + (\alpha + \beta)L_2)$ and $(\alpha_1 + \beta_1)e^{-L_1} < \overline{y}(\gamma_1 + (\alpha_1 + \beta_1)L_1)$, the unique positive equilibrium point $(\overline{x}, \overline{y})$ of system (2.27) is globally asymptotically stable.

- For (2.28) system
  - Every positive solution $\{(x_n, y_n)\}$ of system (2.28) is bounded and persists.
  - If

    $$(U_1 + 1)e^{-(L_1 + \frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha+\beta})}(\frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha + \beta} + 1) < L_1^2(\frac{e^{-L_1}}{L_1} - \frac{\gamma}{\alpha + \beta})^2,$$

    system (2.28) has a unique positive equilibrium point $(\overline{x}, \overline{y})$ in $[L_1, U_1] \times [L_2, U_2]$.

- If

$$(\alpha+\beta)(\alpha_1+\beta_1)(e^{-L_1-L_2}+U_1U_2)$$
$$<(1-U_1-U_2)(\gamma+(\alpha+\beta)L_2)(\gamma_1+(\alpha_1+\beta_1)L_1),$$

the unique positive equilibrium point $(\overline{x},\overline{y})$ of system (2.28) is locally asymptotically stable.

- When $(\alpha+\beta)e^{-L_1}<\overline{x}(\gamma+(\alpha+\beta)L_2)$ and $(\alpha_1+\beta_1)e^{-L_2}<\overline{y}(\gamma_1+(\alpha_1+\beta_1)L_1)$, the unique positive equilibrium point $(\overline{x},\overline{y})$ of system (2.28) is globally asymptotically stable.

In [20] *Khuong and Thai* studied the boundedness, the persistence, and the asymptotic behavior of the positive solutions of the system of difference equations of exponential form:

$$x_{n+1}=\frac{a+be^{-y_n}+ce^{-x_n}}{d+hy_n},\quad y_{n+1}=\frac{a+be^{-x_n}+ce^{-y_n}}{d+hx_n}, \tag{2.29}$$

where a,b,c,d and h are positive constants and the initial values x_0,y_0 are positive real values. After, they achieved the following main results:

- Every positive solution of (2.29) is bounded and persists.
- In the case $b+c<d$, system (2.29) has a unique positive equilibrium $(\overline{x},\overline{y})$ and every positive solution of (2.29) tends to the unique positive equilibrium $(\overline{x},\overline{y})$ as $n\to\infty$. Additively, the equilibrium $(\overline{x},\overline{y})$ is globally asymptotically stable.

In [32] *Papaschinopoulos et al.* studied the boundedness and the persistence of the positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium and the existence of penodic solutions conceming the biological model given by

$$x_{n+1}=\frac{\alpha x_n^2}{x_n+b}+c\frac{e^{k-dx_n}}{1+e^{k-dx_n}}, \tag{2.30}$$

where $0<a<1,b,c,d,k$ are positive constants and x_0 is a positive real number.
They obtained the results are given below:

- All positive solution of (2.30) is bounded and persists.

- Equation (2.30) has a unique positive equilibrium.

- If $cd < 2(1-a)$,

  - Every positive solution of (2.30) tends to the unique positive equilibrium $\overline{x}$ of (2.30).

  - The unique positive equilibrium $\overline{x}$ of (2.30) is globally asymptotically stable.

- If ε be a positive real number and

  $$d > \frac{(1+a)(1+e^k)}{\varepsilon(1-a)},\ c > \frac{((1-a)\varepsilon^2 + b\varepsilon)(1+e^{k-d\varepsilon})}{(\varepsilon+b)e^{k-d\varepsilon}},$$

  equation (2.30) has periodic solutions of prime period two.

In [4] *Din* studied, the qualitative behavior the following two dimensional discrete dynamical system of exponential form:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-y_n} + \gamma_1 e^{-y_{n-1}}}{a_1 + b_1 y_n + c_1 y_{n-1}} \ , \qquad y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-x_n} + \gamma_2 e^{-x_{n-1}}}{a_2 + b_2 x_n + c_2 x_{n-1}}, \tag{2.31}$$

where the parameters $\alpha_i, \beta_i, \gamma_i, a_i, b_i, c_i$ for $i \in \{1,2\}$ and initial conditions x_0, x_{-1}, y_0, y_{-1} are positive real numbers.
He proved the following results:

- Every positive solution $\{(x_n, y_n)\}$ of system (2.31) is bounded and persists.

- If $\beta_2 + \gamma_2 < a_1$, $\beta_1 + \gamma_1 < a_2$, $b_1 + c_1 = b_2 + c_2$,

  - System (2.31) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of system (2.31) converges to the unique positive equilibrium $(\overline{x}, \overline{y})$ as $n \to \infty$.

  - The unique positive equilibrium of system (2.31) is globally asymptotically stable.

CHAPTER 3

SOME DIFFERENCE EQUATIONS OF EXPONENTIAL FORM

In this section we consider the following rational difference equation

$$x_{n+1} = \frac{x_n + x_{n-k}}{1 + x_n e^{x_{n-k}}}$$

and its special cases which was studied by Gocen [10].

3.1 THE EQUATION $x_{n+1} = \frac{x_n + x_{n-1}}{1 + x_n e^{x_{n-1}}}$

In this section, we consider the equilibrium point and local asymptotic stability of the following rational exponential difference equation

$$x_{n+1} = \frac{x_n + x_{n-1}}{1 + x_n e^{x_{n-1}}} \tag{3.1}$$

where the initial values x_{-1}, x_0 are arbitrary nonnegative numbers.
Firstly, we demonstrate that Equation (3.1) has a unique positive equilibrium *point* $\overline{x}$.
The equilibrium points of Equation (3.1) are the solutions of the equation

$$\overline{x} = \frac{\overline{x} + \overline{x}}{1 + \overline{x} e^{\overline{x}}} \tag{3.2}$$

Set

$$f(x) = \frac{x + x}{1 + xe^x} - x$$

then we have

$$f(\frac{1}{2}) > 0$$

and

$$\lim_{x \to \infty} f(x) = -\infty.$$

Moreover, it follows from the

$$f'(x) = \frac{2 - 2x^2 e^x}{(1 + xe^x)^2} - 1 < 0$$

that (3.1) has a unique positive equilibrium point $\overline{x}$.

The equilibrium point $\overline{x}$ of Equation (3.1) is the solution of the equation

$$\overline{x} = \frac{\overline{x} + \overline{x}}{1 + \overline{x}e^{\overline{x}}}. \tag{3.3}$$

It is easy to see that the first root of the Equation (3.3) is $\overline{x} = 0$ and the other root is evaluated numerically as $\overline{x} = 0.56714$.

The linearized equation of (3.1) is

$$x_{n+1} - \frac{1}{1 + \overline{x}e^{\overline{x}}} x_n - \frac{(1 - \overline{x}e^{\overline{x}} - 2\overline{x}^2 e^{\overline{x}})}{(1 + \overline{x}e^{\overline{x}})^2} x_{n-1} = 0 \tag{3.4}$$

Theorem 3.1 *The positive equilibrium point $\overline{x}$ of Equation (3.1) is locally asymptotically stable.*

Proof. *If we set $\overline{x} = 0.56714$ in the lineariazed Equation (3.4), we have the following characteristic equation*

$$\lambda^2 - (0.5)\lambda + 0.28357 = 0. \tag{3.5}$$

The roots of Equation (3.5) are

$$\lambda_{1,2} = 0.25 \pm 0.47018i$$

and the absolute value of each root is less than one. It follows from the Theorem (1.1) that the positive equilibrium point $\overline{x}$ of Equation (3.1), is locally asymptotically stable. ■

Example 3.1 *Consider the system (3.1) with the initial conditions $x_{-1} = 0.2$, $x_0 = 0.4$ to verify our results.*

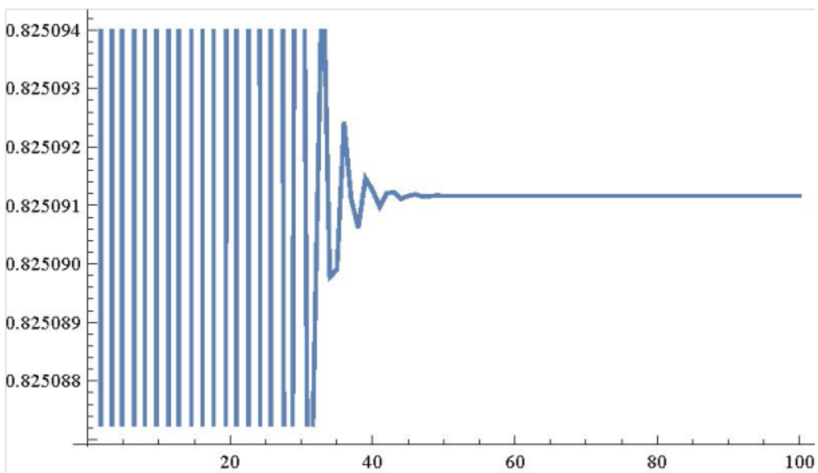

Figure 3.1. Plot of the equation $x_{n+1} = \frac{x_n + x_{n-1}}{1 + x_n e^{x_{n-1}}}$

3.2 THE EQUATION $x_{n+1} = \frac{x_n + x_{n-2}}{1 + x_n e^{x_{n-2}}}$

We study the equilibrium point and local asymptotic stability of the following rational exponential difference equation

$$x_{n+1} = \frac{x_n + x_{n-2}}{1 + x_n e^{x_{n-2}}} \tag{3.6}$$

where the initial values x_{-2}, x_{-1}, x_0 are arbitrary nonnegative numbers.

Theorem 3.2 *The positive equilibrium point $\overline{x}$ of Equation* (3.6) *is locally asymptotically stable.*

Proof. *The linearized equation of Equation* (3.6)

$$x_{n+1} - \frac{1}{1 + \overline{x}e^{\overline{x}}} x_n - \frac{(1 - \overline{x}e^{\overline{x}} - 2\overline{x}^2 e^{\overline{x}})}{(1 + \overline{x}e^{\overline{x}})^2} x_{n-2} = 0 \tag{3.7}$$

the characteristic equation of Equation (3.7) *is as follows:*

$$\lambda^3 - (0.5)\lambda^2 + 0.28357 = 0. \tag{3.8}$$

The roots of Equation (3.8) *are*

$$\begin{aligned} \lambda_{1,2} &= 0.51289 + 0.52562i, \\ \lambda_3 &= 0.52578 \end{aligned}$$

and the absolute value of each root is less than one. So the positive equilibrium point $\overline{x}$ of Equation (3.6) *is locally asymptotically stable.* ■

3.3 THE EQUATION $x_{n+1} = \frac{x_n + x_{n-3}}{1 + x_n e^{x_{n-3}}}$

In this part, we investigate the equilibrium point and local asymptotic stability of the following rational exponential difference equation

$$x_{n+1} = \frac{x_n + x_{n-3}}{1 + x_n e^{x_{n-3}}} \tag{3.9}$$

where the initial values $x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonnegative numbers.

Theorem 3.3 *The positive equilibrium point $\overline{x}$ of Equation* (3.9) *is locally asymptotically stable.*

Proof. *The linearized equation of Equation* (3.9) *is*

$$x_{n+1} - \frac{1}{1+\overline{x}e^{\overline{x}}}x_n - \frac{(1-\overline{x}e^{\overline{x}}-2\overline{x}^2e^{\overline{x}})}{(1+\overline{x}e^{\overline{x}})^2}x_{n-3} = 0 \tag{3.10}$$

and the corresponding characteristic equation of Equation (3.10) *is*

$$\lambda^4 - (0.5)\lambda^3 + 0.28357 = 0. \tag{3.11}$$

The roots of Equation (3.11) *are*

$$\begin{aligned} \lambda_{1,2} &= 0.66294 \pm 0.48501i \\ \lambda_{3,4} &= -0.41294 \pm 0.49976i. \end{aligned}$$

It is easy to see that absolute value of each root is less than one. This implies that the positive equilibrium point $\overline{x}$ of Equation (3.9), *is locally asymptotically stable.* ■

3.4 THE EQUATION $x_{n+1} = \frac{x_n+x_{n-k}}{1+x_ne^{x_{n-k}}}$

Finally, we consider the equilibrium point and local asymptotic stability of the following higher order exponential difference equation

$$x_{n+1} = \frac{x_n + x_{n-k}}{1+x_ne^{x_{n-k}}} \tag{3.12}$$

where the initial values $x_{-k}, ..., x_{-1}, x_0$ are arbitrary nonnegative numbers.

Theorem 3.4 *The positive equilibrium point $\overline{x}$ of Equation* (3.12) *is locally asymptotically stable.*

Proof. *The linearized equation associated with Equation* (3.12) *about the equilibrium point $\overline{x}$ is*

$$x_{n+1} - \frac{1}{1+\overline{x}e^{\overline{x}}}x_n - \frac{(1-\overline{x}e^{\overline{x}}-2\overline{x}^2e^{\overline{x}})}{(1+\overline{x}e^{\overline{x}})^2}x_{n-k} = 0. \tag{3.13}$$

Hence, for all $k \in \mathbb{N}$ values, we obtain the characteristic equation of Equation (3.13) *about the equilibrium point $\overline{x}$ is*

$$\lambda^{k+1} - (0.5)\lambda^k + 0.28357 = 0. \tag{3.14}$$

Set

$f(\lambda) = \lambda^{k+1}$

and

$g(\lambda) = -(0.5)\lambda^k + 0.28357.$

Then, by Rouche's theorem, $f(\lambda)$ *and* $f(\lambda) + g(\lambda)$ *have same number zeroes in an open unit disc* $|\lambda| < 1$*. Hence, all the roots of* (3.14) *satisfies* $|\lambda| < 1$*, and it follows from Theorem* (1.1) *that the unique positive equilibrium point* $\overline{x}$ *of Equation* (3.12) *is locally asymptotically stable. Therefore, one can obtain the desired results and the proof is complete.* ■

Remark 3.1 *The zero equilibrium point of Equation* (3.1), (3.6), (3.9) *and* (3.12) *is unstable. Because the corresponding characteristic equations of these equations is respectively*

$\lambda^2 - \lambda - 1 = 0$

$\lambda^3 - \lambda^2 - 1 = 0$

$\lambda^4 - \lambda^3 - 1 = 0$

$\vdots$

$\lambda^{k+1} - \lambda^k - 1 = 0.$

At least one root of the all above characteristic equations is greater than one, so we obtain the zero equilibrium point of Equation (3.1), (3.6), (3.9) *and* (3.12) *is unstable.*
Now, we give a numerical example to show the positive equilibrium point $\overline{x}$ of Equation (3.1) is locally asymptotically stable.

CHAPTER 4

SOME SYSTEMS OF EXPONENTIAL DIFFERENCE EQUATIONS

4.1 THE EQUATION SYSTEM $x_{n+1} = \frac{x_n + y_{n-1}}{1+y_{n-1}e^{x_{n-1}}}$, $y_{n+1} = \frac{y_n + x_{n-1}}{1+x_{n-1}e^{y_{n-1}}}$

In this section, we consider the equilibrium point and local asymptotic stability of the second order difference equation system

$$x_{n+1} = \frac{x_n + y_{n-1}}{1 + y_{n-1}e^{x_{n-1}}}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{1 + x_{n-1}e^{y_{n-1}}} \tag{4.1}$$

where the initial values x_{-1}, x_0, y_{-1}, y_0 are arbitrary nonnegative numbers.
Let $(\overline{x}, \overline{y})$ be the equilibrium point of system (4.1)

$$\overline{x} = \frac{\overline{x} + \overline{y}}{1 + \overline{y}e^{\overline{x}}}, \quad \overline{y} = \frac{\overline{y} + \overline{x}}{1 + \overline{x}e^{\overline{y}}}. \tag{4.2}$$

Set

$$\overline{x} = \frac{\overline{x} + \frac{\overline{x}+\overline{y}}{1+\overline{x}e^{\overline{y}}}}{1 + \frac{\overline{x}+\overline{y}}{1+\overline{x}e^{\overline{y}}}e^{\overline{x}}} = \frac{\overline{x} + \overline{x}^2 e^{\overline{y}} + \overline{x} + \overline{y}}{1 + \overline{x}e^{\overline{y}} + \overline{x}e^{\overline{x}} + \overline{y}e^{\overline{x}}}$$

Then

$$\overline{x}^2 e^{\overline{x}} + \overline{x}\ \overline{y}e^{\overline{x}} - \overline{x} - \overline{y} = 0$$

and we obtain that

$$(\overline{x} + \overline{y})(1 - \overline{x}e^{\overline{x}}) = 0 \tag{4.3}$$

from which it follows that system (4.2) has solution

$$(\overline{x}, \overline{y}) = (0, 0) \quad \text{or} \quad (\overline{x}, \overline{y}) = (0.56714, 0.56714)\,.$$

Let us consider the four dimensional discrete dynamical system of the form

$$(x_n, y_n, x_{n-1}, y_{n-1}) \to (f, g, f_1, g_1), \tag{4.4}$$

where

$$
\begin{aligned}
f &= \frac{x_n + y_{n-1}}{1 + y_{n-1}e^{x_{n-1}}}, \quad f_1 = x_n \\
g &= \frac{y_n + x_{n-1}}{1 + x_{n-1}e^{y_{n-1}}}, \quad g_1 = y_n.
\end{aligned}
$$

The linearized system of (4.1) about $(\overline{x}, \overline{y})$ is given by

$$Z_{n+1} = F_J(\overline{x}, \overline{y})Z_n, \tag{4.5}$$

where $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and the Jacobian matrix about the fixed point $(\overline{x}, \overline{y})$ under transformation (4.4) is given by

$$F_J(\overline{x}, \overline{y}) = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} & \frac{\partial x_{n+1}}{\partial x_{n-1}} & \frac{\partial x_{n+1}}{\partial y_{n-1}} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} & \frac{\partial y_{n+1}}{\partial x_{n-1}} & \frac{\partial y_{n+1}}{\partial y_{n-1}} \\ \frac{\partial x_n}{\partial x_n} & \frac{\partial x_n}{\partial y_n} & \frac{\partial x_n}{\partial x_{n-1}} & \frac{\partial x_n}{\partial y_{n-1}} \\ \frac{\partial y_n}{\partial x_n} & \frac{\partial y_n}{\partial y_n} & \frac{\partial y_n}{\partial x_{n-1}} & \frac{\partial y_n}{\partial y_{n-1}} \end{pmatrix}$$

and it follows that

$$F_J(\overline{x}, \overline{y}) = \begin{pmatrix} A_1 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & B_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where

$$A_1 = \frac{1}{1 + \overline{y}e^{\overline{x}}}, \quad A_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1 + \overline{y}e^{\overline{x}}},$$

$$B_1 = \frac{1}{1 + \overline{x}e^{\overline{y}}}, \quad B_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1 + \overline{x}e^{\overline{y}}}.$$

The characteristic equation of $F_J(\overline{x}, \overline{y})$ is

$$\begin{cases} P(\lambda) = \lambda^4 - (A_1 + B_1)\lambda^3 - (A_2 + B_2 - A_1B_1)\lambda^2 \\ +(A_2B_1 + A_1B_2)\lambda + A_2B_2. \end{cases} \tag{4.6}$$

Then we take A_1, A_2, B_1, B_2 as above, we have

$$\begin{aligned}
P(\lambda) &= \lambda^4 - \left(\frac{1}{1+\overline{y}e^{\overline{x}}} + \frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^3 \\
&\quad - \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}} + \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}} - \frac{1}{1+\overline{y}e^{\overline{x}}}\frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^2 \\
&\quad + \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}}\frac{1}{1+\overline{x}e^{\overline{y}}} + \frac{1}{1+\overline{y}e^{\overline{x}}}\frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\right)\lambda \\
&\quad + \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}}\frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\right).
\end{aligned}$$

Theorem 4.1 **i)** *The zero equilibrium point* $(\overline{x}, \overline{y})$ *of system* (4.1) *is nonhyperbolic point.*

ii) *The positive equilibrium* $(\overline{x}, \overline{y})$ *of system* (4.1) *is locally asymptotically stable.*

Proof.

i) *For the zero equilibrium point* $(\overline{x}, \overline{y})$ *of system* (4.1), *using* (4.4), (4.5) *and* (4.6) *we have*

$$P(\lambda) = \lambda^4 - 2\lambda^3 + \lambda^2 = 0. \tag{4.7}$$

Obviously, the roots of characteristic equation of $F_J(\overline{x}, \overline{y})$ *are given by* $\lambda = 0$ *(multiple root) and* $\lambda = 1$ *(multiple root). From this result, the equilibrium point* $(\overline{x}, \overline{y}) = (0, 0)$ *is a nonhyperbolic point since the modulus of one of the roots of the Equation* (4.7) *is equal to one.*

ii) *For the positive equilibrium point* $(\overline{x}, \overline{y})$ *of system* (4.1), *using* (4.4), (4.5) *and* (4.6) *we have*

$$\begin{aligned}
P(\lambda) &= \lambda^4 - \left(\frac{1}{1+0.56714e^{0.56714}} + \frac{1}{1+0.56714e^{0.56714}}\right)\lambda^3 \\
&\quad - \left(\begin{array}{c} \left.\dfrac{-(0.56714)^2 e^{0.56714}}{1+0.56714e^{0.56714}}\right) + \left.\dfrac{-(0.56714)^2 e^{0.56714}}{1+0.56714e^{0.56714}}\right) \\ -\dfrac{1}{1+0.56714e^{0.56714}}\dfrac{1}{1+0.56714e^{0.56714}} \end{array}\right)\lambda^2 \\
&\quad + \left(\begin{array}{c} \dfrac{1}{1+0.56714e^{0.56714}}\left(\dfrac{-(0.56714)^2 e^{0.56714}}{1+0.56714e^{0.56714}}\right) \\ +\left(\dfrac{-(0.56714)^2 e^{0.56714}}{1+0.56714e^{0.56714}}\right)\dfrac{1}{1+0.56714e^{0.56714}} \end{array}\right)\lambda \\
&\quad + \left(\left(\frac{-(0.56714)^2 e^{0.56714}}{1+0.56714e^{0.56714}}\right)\left(\frac{-(0.56714)^2 e^{0.56714}}{1+0.56714e^{0.56714}}\right)\right)
\end{aligned}$$

and we obtain

$$P(\lambda) = \lambda^4 - \lambda^3 + (0.25)\lambda^2 + (0.56714)\lambda^2 - (0.28376)\lambda + 0.08041. \tag{4.8}$$

Obviously, the roots of characteristic equation of $F_J(\overline{x}, \overline{y})$ *are given by*

$\lambda_{1,2} = 0.24454 \mp 0.47938i$

$\lambda_{3,4} = 0.25546 \mp 0.46086i$

Hence, all the roots of Equation (4.8) *are of modulus less than one which implies that* $(\overline{x}, \overline{y}) = (0.56714, 0.56714)$ *is locally asymptotically stable.*

■

Now, we can give an example to verify our results.

Example 4.1 *Consider the system* (4.1) *with the initial conditions* $x_{-1} = 0.3$, $x_0 = 0.1$, $y_{-1} = 0.7$, $y_0 = 0.5$ *to support our results.*

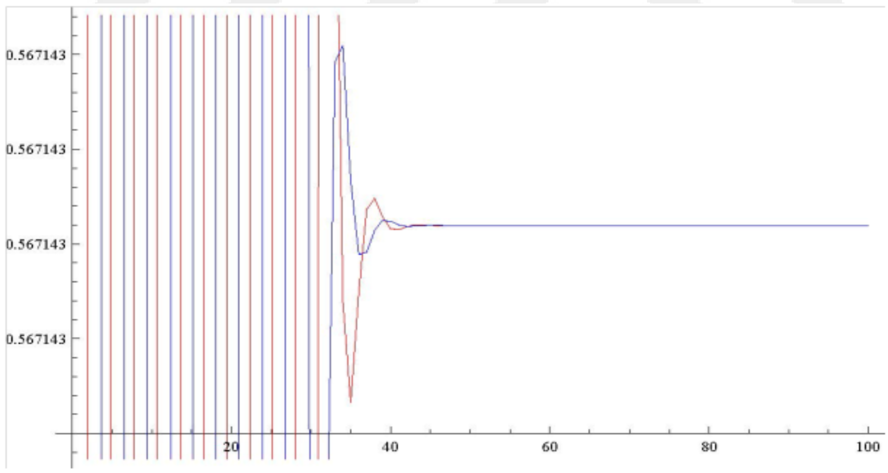

Figure 4.1. Plot of the equation $x_{n+1} = \frac{x_n + y_{n-1}}{1 + y_{n-1}e^{x_{n-1}}}, y_{n+1} = \frac{y_n + x_{n-1}}{1 + x_{n-1}e^{y_{n-1}}}$.

4.2 THE EQUATION SYSTEM $x_{n+1} = \frac{x_n + y_{n-2}}{1 + y_{n-2}e^{x_{n-2}}}$, $y_{n+1} = \frac{y_n + x_{n-2}}{1 + x_{n-2}e^{y_{n-2}}}$

We study on the equilibrium point and local asymptotic stability of the third order difference equation system

$$x_{n+1} = \frac{x_n + y_{n-2}}{1 + y_{n-2}e^{x_{n-2}}}, \quad y_{n+1} = \frac{y_n + x_{n-2}}{1 + x_{n-2}e^{y_{n-2}}} \tag{4.9}$$

where the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary nonnegative numbers.

Let us consider the six dimensional discrete dynamical system of the form

$$(x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}) \to (f, g, f_1, g_1, f_2, g_2), \tag{4.10}$$

where

$$\begin{aligned} f &= \frac{x_n + y_{n-2}}{1 + y_{n-2}e^{x_{n-2}}}, \ \ f_1 = x_n, \ f_2 = x_{n-1} \\ g &= \frac{y_n + x_{n-2}}{1 + x_{n-2}e^{y_{n-2}}}, \ g_1 = y_n, \ g_2 = y_{n-1}. \end{aligned}$$

The linearized system of (4.9) evaluated at positive equilibrium $(\overline{x}, \overline{y})$ is given by

$$Z_{n+1} = F_J(\overline{x}, \overline{y})Z_n, \tag{4.11}$$

where $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \\ x_{n-2} \\ y_{n-2} \end{pmatrix}$ and the Jacobian matrix about the fixed point $(\overline{x}, \overline{y})$ under transformation (4.10) is given by

$$F_J(\overline{x}, \overline{y}) = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} & \frac{\partial x_{n+1}}{\partial x_{n-1}} & \frac{\partial x_{n+1}}{\partial y_{n-1}} & \frac{\partial x_{n+1}}{\partial x_{n-2}} & \frac{\partial x_{n+1}}{\partial y_{n-2}} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} & \frac{\partial y_{n+1}}{\partial x_{n-1}} & \frac{\partial y_{n+1}}{\partial y_{n-1}} & \frac{\partial y_{n+1}}{\partial x_{n-2}} & \frac{\partial y_{n+1}}{\partial y_{n-2}} \\ \frac{\partial x_n}{\partial x_n} & \frac{\partial x_n}{\partial y_n} & \frac{\partial x_n}{\partial x_{n-1}} & \frac{\partial x_n}{\partial y_{n-1}} & \frac{\partial x_n}{\partial x_{n-2}} & \frac{\partial x_n}{\partial y_{n-2}} \\ \frac{\partial y_n}{\partial x_n} & \frac{\partial y_n}{\partial y_n} & \frac{\partial y_n}{\partial x_{n-1}} & \frac{\partial y_n}{\partial y_{n-1}} & \frac{\partial x_n}{\partial x_{n-2}} & \frac{\partial x_n}{\partial y_{n-2}} \\ \frac{\partial x_{n-1}}{\partial x_n} & \frac{\partial x_{n-1}}{\partial y_n} & \frac{\partial x_{n-1}}{\partial x_{n-1}} & \frac{\partial x_{n-1}}{\partial y_{n-1}} & \frac{\partial x_{n-1}}{\partial x_{n-2}} & \frac{\partial x_{n-1}}{\partial x_{n-2}} \\ \frac{\partial y_{n-1}}{\partial x_n} & \frac{\partial y_{n-1}}{\partial y_n} & \frac{\partial y_{n-1}}{\partial x_{n-1}} & \frac{\partial y_{n-1}}{\partial y_{n-1}} & \frac{\partial y_{n-1}}{\partial x_{n-2}} & \frac{\partial y_{n-1}}{\partial x_{n-2}} \end{pmatrix}$$

and

$$F_J(\overline{x}, \overline{y}) = \begin{pmatrix} A_1 & 0 & 0 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & 0 & 0 & B_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where

$$A_1 = \frac{1}{1 + \overline{y}e^{\overline{x}}}, \quad A_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1 + \overline{y}e^{\overline{x}}},$$

$$B_1 = \frac{1}{1+\overline{x}e^{\overline{y}}}, \quad B_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}.$$

The characteristic equation of $F_J(\overline{x},\overline{y})$ is

$$\begin{cases} P(\lambda) = \lambda^6 - (A_1+B_1)\lambda^5 + A_1B_1\lambda^4 - (A_2+B_2)\lambda^3 \\ +(A_2B_1+A_1B_2)\lambda^2 + A_2B_2. \end{cases} \tag{4.12}$$

When we set A_1, A_2, B_1, B_2 as above, we have

$$\begin{aligned} P(\lambda) \quad = \quad & \lambda^6 - \left(\frac{1}{1+\overline{x}e^{\overline{y}}} + \frac{1}{1+\overline{y}e^{\overline{x}}}\right)\lambda^5 + \left(\frac{1}{1+\overline{y}e^{\overline{x}}}\frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^4 \\ & - \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}} + \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\right)\lambda^3 \\ & + \left(\frac{1}{1+\overline{y}e^{\overline{x}}}\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}} + \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^2 \\ & + \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}}\frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\right). \end{aligned}$$

It follows from Equation (4.3) that the equilibrium points are $(\overline{x},\overline{y}) = (0,0)$ and $(\overline{x},\overline{y}) = (0.56714, 0.56714)$.

Theorem 4.2 **i)** *The zero equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.9) *is nonhyperbolic point.*

ii) *The positive equilibrium* $(\overline{x},\overline{y})$ *of system* (4.9) *is locally asymptotically stable.*

Proof.

i) *For the zero equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.9), *using* (4.10), (4.11) *and* (4.12) *we have*

$$P(\lambda) = \lambda^6 - 2\lambda^5 + \lambda^4 = 0. \tag{4.13}$$

Obviously, the roots of characteristic equation of $F_J(\overline{x},\overline{y})$ *are given by* $\lambda = 0$ *(multiple root) and* $\lambda = 1$ *(multiple root). From this result, the equilibrium point* $(\overline{x},\overline{y}) = (0,0)$ *is a nonhyperbolic point since the modulus of one of the roots of the Equation* (4.13) *is equal to one.*

ii) *For the positive equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.9), *using* (4.10), (4.11) *and*

(4.12) *we have*

$$\begin{aligned}
P(\lambda) &= \lambda^6-\left(\frac{1}{1+0.56714e^{0.56714}}+\frac{1}{1+0.56714e^{0.56714}}\right)\lambda^5\\
&+\left(\frac{1}{1+0.56714e^{0.56714}}\frac{1}{1+0.56714e^{0.56714}}\right)\lambda^4\\
&-\left(\left(\frac{-(0.56714)^2e^{0.56714}}{1+0.56714e^{0.56714}}\right)+\left(\frac{-(0.56714)^2e^{0.56714}}{1+0.56714e^{0.56714}}\right)\right)\lambda^3\\
&+\left(\begin{array}{c}\frac{1}{1+0.56714e^{0.56714}}\left(\frac{-(0.56714)^2e^{0.56714}}{1+0.56714e^{0.56714}}\right)\\
+\left(\frac{-(0.56714)^2e^{0.56714}}{1+0.56714e^{0.56714}}\right)\frac{1}{1+0.56714e^{0.56714}}\end{array}\right)\lambda^2\\
&+\left(\left(\frac{-(0.56714)^2e^{0.56714}}{1+0.56714e^{0.56714}}\right)\left(\frac{-(0.56714)^2e^{0.56714}}{1+0.56714e^{0.56714}}\right)\right)
\end{aligned}$$

and so

$$P(\lambda)=\lambda^6-\lambda^5+(0.25)\lambda^4+(0.56714)\lambda^3-(0.28376)\lambda^2+0.08041. \tag{4.14}$$

Obviously, the roots of characteristic equation of $F_J(\overline{x},\overline{y})$ *are given by*

$$\begin{aligned}
\lambda_{1,2} &= 0.51549\mp 0.51774i,\\
\lambda_{3,4} &= 0.51029\mp 0.53344i,\\
\lambda_5 &= -0.52034,\\
\lambda_6 &= -0.53123.
\end{aligned}$$

Thus, all the roots of Equation (4.14) *are of modulus less than one which implies that* $(\overline{x},\overline{y})=(0.56714,0.56714)$ *is locally asymptotically stable.*

■

4.3 THE EQUATION SYSTEM $x_{n+1}=\frac{x_n+y_{n-3}}{1+y_{n-3}e^{x_{n-3}}}$, $y_{n+1}=\frac{y_n+x_{n-3}}{1+x_{n-3}e^{y_{n-3}}}$

In this part, we consider the equilibrium point and local asymptotic stability of the following system of fourth order rational exponential difference equation

$$x_{n+1}=\frac{x_n+y_{n-3}}{1+y_{n-3}e^{x_{n-3}}},\quad y_{n+1}=\frac{y_n+x_{n-3}}{1+x_{n-3}e^{y_{n-3}}} \tag{4.15}$$

where the initial values $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary nonnegative numbers.

Let us consider the eight dimensional discrete dynamical system of the form

$$(x_n, y_n, x_{n-1}, y_{n-1}, x_{n-2}, y_{n-2}, x_{n-3}, y_{n-3}) \to (f, g, f_1, g_1, f_2, g_2, f_3, g_3), \tag{4.16}$$

where

$$\begin{aligned} f &= \frac{x_n + y_{n-3}}{1 + y_{n-3}e^{x_{n-3}}}, \quad f_1 = x_n, \quad f_2 = x_{n-1}, \quad f_3 = x_{n-2} \\ g &= \frac{y_n + x_{n-3}}{1 + x_{n-3}e^{y_{n-3}}}, \quad g_1 = y_n, \quad g_2 = y_{n-1}, \quad g_3 = y_{n-2}. \end{aligned}$$

The linearized system of (4.15) about $(\overline{x}, \overline{y})$ is given by

$$Z_{n+1} = F_J(\overline{x}, \overline{y}) Z_n, \tag{4.17}$$

where $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \\ x_{n-2} \\ y_{n-2} \\ x_{n-3} \\ y_{n-3} \end{pmatrix}$ and the Jacobian matrix $F_J(\overline{x}, \overline{y})$ evaluated at $(\overline{x}, \overline{y})$ of system (4.16) is given by

$$F_J(\overline{x}, \overline{y}) = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} & \frac{\partial x_{n+1}}{\partial x_{n-1}} & \frac{\partial x_{n+1}}{\partial y_{n-1}} & \frac{\partial x_{n+1}}{\partial x_{n-2}} & \frac{\partial x_{n+1}}{\partial y_{n-2}} & \frac{\partial x_{n+1}}{\partial x_{n-3}} & \frac{\partial x_{n+1}}{\partial y_{n-3}} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} & \frac{\partial y_{n+1}}{\partial x_{n-1}} & \frac{\partial y_{n+1}}{\partial y_{n-1}} & \frac{\partial y_{n+1}}{\partial x_{n-2}} & \frac{\partial y_{n+1}}{\partial y_{n-2}} & \frac{\partial y_{n+1}}{\partial x_{n-3}} & \frac{\partial y_{n+1}}{\partial y_{n-3}} \\ \frac{\partial x_n}{\partial x_n} & \frac{\partial x_n}{\partial y_n} & \frac{\partial x_n}{\partial x_{n-1}} & \frac{\partial x_n}{\partial y_{n-1}} & \frac{\partial x_n}{\partial x_{n-2}} & \frac{\partial x_n}{\partial y_{n-2}} & \frac{\partial x_n}{\partial x_{n-3}} & \frac{\partial x_n}{\partial y_{n-3}} \\ \frac{\partial y_n}{\partial x_n} & \frac{\partial y_n}{\partial y_n} & \frac{\partial y_n}{\partial x_{n-1}} & \frac{\partial y_n}{\partial y_{n-1}} & \frac{\partial y_n}{\partial x_{n-2}} & \frac{\partial y_n}{\partial y_{n-2}} & \frac{\partial y_n}{\partial x_{n-3}} & \frac{\partial y_n}{\partial y_{n-3}} \\ \frac{\partial x_{n-1}}{\partial x_n} & \frac{\partial x_{n-1}}{\partial y_n} & \frac{\partial x_{n-1}}{\partial x_{n-1}} & \frac{\partial x_{n-1}}{\partial y_{n-1}} & \frac{\partial x_{n-1}}{\partial x_{n-2}} & \frac{\partial x_{n-1}}{\partial y_{n-2}} & \frac{\partial x_{n-1}}{\partial x_{n-3}} & \frac{\partial x_{n-1}}{\partial y_{n-3}} \\ \frac{\partial y_{n-1}}{\partial x_n} & \frac{\partial y_{n-1}}{\partial y_n} & \frac{\partial y_{n-1}}{\partial x_{n-1}} & \frac{\partial y_{n-1}}{\partial y_{n-1}} & \frac{\partial y_{n-1}}{\partial x_{n-2}} & \frac{\partial y_{n-1}}{\partial y_{n-2}} & \frac{\partial y_{n-1}}{\partial x_{n-3}} & \frac{\partial y_{n-1}}{\partial y_{n-3}} \\ \frac{\partial x_{n-2}}{\partial x_n} & \frac{\partial x_{n-2}}{\partial y_n} & \frac{\partial x_{n-2}}{\partial x_{n-1}} & \frac{\partial x_{n-2}}{\partial y_{n-1}} & \frac{\partial x_{n-2}}{\partial x_{n-2}} & \frac{\partial x_{n-2}}{\partial y_{n-2}} & \frac{\partial x_{n-2}}{\partial x_{n-3}} & \frac{\partial x_{n-2}}{\partial y_{n-3}} \\ \frac{\partial y_{n-2}}{\partial x_n} & \frac{\partial y_{n-2}}{\partial y_n} & \frac{\partial y_{n-2}}{\partial x_{n-1}} & \frac{\partial y_{n-2}}{\partial y_{n-1}} & \frac{\partial y_{n-2}}{\partial x_{n-2}} & \frac{\partial y_{n-2}}{\partial y_{n-2}} & \frac{\partial y_{n-2}}{\partial x_{n-3}} & \frac{\partial y_{n-2}}{\partial y_{n-3}} \end{pmatrix}$$

and

$$F_J(\overline{x},\overline{y}) = \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & B_2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where

$$A_1 = \frac{1}{1+\overline{y}e^{\overline{x}}}, \quad A_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}},$$

$$B_1 = \frac{1}{1+\overline{x}e^{\overline{y}}}, \quad B_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}.$$

We obtain the characteristic equation of $F_J(\overline{x},\overline{y})$ is as follows:

$$\begin{cases} P(\lambda) = \lambda^8 - (A_1+B_1)\lambda^7 + A_1B_1\lambda^6 - (A_2+B_2)\lambda^4 \\ +(A_2B_1 + A_1B_2)\lambda^3 + A_2B_2 \end{cases} \tag{4.18}$$

Then we get A_1, A_2, B_1, B_2 as above, it follows that

$$\begin{aligned} P(\lambda) &= \lambda^8 - \left(\frac{1}{1+\overline{y}e^{\overline{x}}} + \frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^7 \\ &+ \left(\frac{1}{1+\overline{y}e^{\overline{x}}}\frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^6 - \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}} + \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\right)\lambda^4 \\ &+ \left(\frac{1}{1+\overline{y}e^{\overline{x}}}\frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}} + \frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}}\frac{1}{1+\overline{x}e^{\overline{y}}}\right)\lambda^3 \\ &+ \left(\frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}}\frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}\right). \end{aligned}$$

From Equation (4.3), we have the some equilibrium points $(\overline{x},\overline{y}) = (0,0)$ and $(\overline{x},\overline{y}) = (0.56714, 0.56714)$.

Theorem 4.3 i) *The zero equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.15) *is nonhyperbolic point.*

ii) *The positive equilibrium* $(\overline{x},\overline{y})$ *of system* (4.15) *is locally asymptotically stable.*

Proof.

i) *For the zero equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.15), *using* (4.16), (4.17) *and* (4.18) *we have*

$$P(\lambda)=\lambda^{8}-2\lambda^{7}+\lambda^{6}=0. \tag{4.19}$$

Obviously, the roots of characteristic equation of $F_J(\overline{x},\overline{y})$ *are given by* $\lambda=0$ *(multiple root) and* $\lambda=1$ *(multiple root). From this result, the equilibrium point* $(\overline{x},\overline{y})=(0,0)$ *is a nonhyperbolic point since the modulus of one of the roots of the Equation* (4.19) *is equal to one.*

ii) *For the positive equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.15), *using* (4.16), (4.17) *and* (4.18) *we have*

$$\begin{aligned}
P(\lambda) &= \lambda^{8}-\left(\frac{1}{1+0.56714e^{0.56714}}+\frac{1}{1+0.56714e^{0.56714}}\right)\lambda^{7}\\
&+\left(\frac{1}{1+0.56714e^{0.56714}}\frac{1}{1+0.56714e^{0.56714}}\right)\lambda^{6}\\
&-\left(\left(\frac{-(0.56714)^{2}e^{0.56714}}{1+0.56714e^{0.56714}}\right)+\left(\frac{-(0.56714)^{2}e^{0.56714}}{1+0.56714e^{0.56714}}\right)\right)\lambda^{4}\\
&+\left(\begin{array}{c}\frac{1}{1+0.56714e^{0.56714}}\left(\frac{-(0.56714)^{2}e^{0.56714}}{1+0.56714e^{0.56714}}\right)\\ +\left(\frac{-(0.56714)^{2}e^{0.56714}}{1+0.56714e^{0.56714}}\right)\frac{1}{1+0.56714e^{0.56714}}\end{array}\right)\lambda^{3}\\
&+\left(\left(\frac{-(0.56714)^{2}e^{0.56714}}{1+0.56714e^{0.56714}}\right)\left(\frac{-(0.56714)^{2}e^{0.56714}}{1+0.56714e^{0.56714}}\right)\right)
\end{aligned}$$

and

$$P(\lambda)=\lambda^{8}-\lambda^{7}+(0.25)\lambda^{6}+(0.56714)\lambda^{4}-(0.28376)\lambda^{3}+0.08041. \tag{4.20}$$

Obviously, the roots of characteristic equation of $F_J(\overline{x},\overline{y})$ *are given by*

$$\begin{aligned}
\lambda_{1,2} &= -0.40903\mp 0.49723i,\\
\lambda_{3,4} &= -0.41686\mp 0.50229i,\\
\lambda_{5,6} &= 0.66435\mp 0.47842i,\\
\lambda_{7,8} &= 0.66153\mp 0.49157i.
\end{aligned}$$

So, all the roots of Equation (4.20) *are of modulus less than one which implies that* $(\overline{x},\overline{y})=(0.56714,0.56714)$ *is locally asymptotically stable.*

■

4.4 THE EQUATION SYSTEM $x_{n+1} = \frac{x_n+y_{n-k}}{1+y_{n-k}e^{x_{n-k}}}$, $y_{n+1} = \frac{y_n+x_{n-k}}{1+x_{n-k}e^{y_{n-k}}}$

In this section, we consider the equilibrium point and local asymptotic stability of some systems of higher order exponential difference equation

$$x_{n+1} = \frac{x_n + y_{n-k}}{1 + y_{n-k}e^{x_{n-k}}}, \quad y_{n+1} = \frac{y_n + x_{n-k}}{1 + x_{n-k}e^{y_{n-k}}} \tag{4.21}$$

where the initial values $x_{-k}, ..., x_{-1}, x_0, y_{-k}, ..., y_{-1}, y_0$ are arbitrary nonnegative numbers. Let us consider the $(2k+2)$ dimensional discrete dynamical system of the form

$$(x_n, y_n, x_{n-1}, y_{n-1}, ..., x_{n-k}, y_{n-k}) \to (f, g, f_1, g_1, ..., f_k, g_k), \tag{4.22}$$

where

$$\begin{aligned} f &= \frac{x_n + y_{n-k}}{1 + y_{n-k}e^{x_{n-k}}}, \quad f_1 = x_n, \quad f_2 = x_{n-1}, ..., f_k = x_{n-(k-1)} \\ g &= \frac{y_n + x_{n-k}}{1 + x_{n-k}e^{y_{n-k}}}, \quad g_1 = y_n, \; g_2 = y_{n-1}, ..., g_k = y_{n-(k-1)}. \end{aligned}$$

Futhermore, the linearized system of (4.21) about $(\overline{x}, \overline{y})$ is

$$Z_{n+1} = F_J(\overline{x}, \overline{y})Z_n, \tag{4.23}$$

where $Z_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \\ \vdots \\ x_{n-k} \\ y_{n-k} \end{pmatrix}$ and the Jacobian matrix about the fixed point $(\overline{x}, \overline{y})$ under transformation (4.22) is given by

$$F_J(\overline{x}, \overline{y}) = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} & \frac{\partial x_{n+1}}{\partial x_{n-1}} & \frac{\partial x_{n+1}}{\partial y_{n-1}} & \cdots & \frac{\partial x_{n+1}}{\partial x_{n-k}} & \frac{\partial x_{n+1}}{\partial y_{n-k}} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} & \frac{\partial y_{n+1}}{\partial x_{n-1}} & \frac{\partial y_{n+1}}{\partial y_{n-1}} & \cdots & \frac{\partial y_{n+1}}{\partial x_{n-k}} & \frac{\partial y_{n+1}}{\partial y_{n-k}} \\ \frac{\partial x_n}{\partial x_n} & \frac{\partial x_n}{\partial y_n} & \frac{\partial x_n}{\partial x_{n-1}} & \frac{\partial x_n}{\partial y_{n-1}} & \cdots & \frac{\partial x_n}{\partial x_{n-k}} & \frac{\partial x_n}{\partial y_{n-k}} \\ \frac{\partial y_n}{\partial x_n} & \frac{\partial y_n}{\partial y_n} & \frac{\partial y_n}{\partial x_{n-1}} & \frac{\partial y_n}{\partial y_{n-1}} & \cdots & \frac{\partial y_n}{\partial x_{n-k}} & \frac{\partial y_n}{\partial y_{n-k}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x_{n-(k-1)}}{\partial x_n} & \frac{\partial x_{n-(k-1)}}{\partial y_n} & \frac{\partial x_{n-(k-1)}}{\partial x_{n-1}} & \frac{\partial x_{n-(k-1)}}{\partial y_{n-1}} & \cdots & \frac{\partial x_{n-(k-1)}}{\partial x_{n-3}} & \frac{\partial x_{n-(k-1)}}{\partial y_{n-3}} \\ \frac{\partial y_{n-(k-1)}}{\partial x_n} & \frac{\partial y_{n-(k-1)}}{\partial y_n} & \frac{\partial y_{n-(k-1)}}{\partial x_{n-1}} & \frac{\partial y_{n-(k-1)}}{\partial y_{n-1}} & \cdots & \frac{\partial y_{n-(k-1)}}{\partial x_{n-3}} & \frac{\partial y_{n-(k-1)}}{\partial y_{n-3}} \end{pmatrix}$$

and

$$F_J(\overline{x},\overline{y}) = \begin{pmatrix} A_1 & 0 & 0 & 0 & \dots & A_2 & 0 \\ 0 & B_1 & 0 & 0 & \dots & 0 & B_2 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where

$$A_1 = \frac{1}{1+\overline{y}e^{\overline{x}}}, \quad A_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{x}}}{1+\overline{y}e^{\overline{x}}},$$

$$B_1 = \frac{1}{1+\overline{x}e^{\overline{y}}}, \quad B_2 = \frac{-\overline{x}\ \overline{y}e^{\overline{y}}}{1+\overline{x}e^{\overline{y}}}.$$

The characteristic polynomial of the $F_J(\overline{x},\overline{y})$ is as follows:

$$\begin{cases} P(\lambda) = \lambda^{2k+2} - (A_1+B_1)\lambda^{2k+1} + A_1B_1\lambda^{2k} - (A_2+B_2)\lambda^{k+1} \\ +(A_2B_1 + A_1B_2)\lambda^k + A_2B_2. \end{cases} \tag{4.24}$$

It follows from Equation (4.3) that the equilibrium points are $(\overline{x},\overline{y}) = (0,0)$ and $(\overline{x},\overline{y}) = (0.56714, 0.56714)$.

Theorem 4.4 **i)** *The zero equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.21) *is nonhyperbolic point.*

ii) *The characteristic Equation of system* (4.21) *about the positive equilibrium point* $(\overline{x},\overline{y})$ *is*

$$P(\lambda) = \lambda^{2k+2} - \lambda^{2k+1} + (0.25)\lambda^{2k} + (0.56714)\lambda^{k+1} - (0.28376)\lambda^k + 0.08041.$$

Proof.

i) *For the zero equilibrium point* $(\overline{x},\overline{y})$ *of system* (4.21), *using* (4.22), (4.23) *and* (4.24) *we have*

$$P(\lambda) = \lambda^{2k+2} - 2\lambda^{2k+1} + \lambda^{2k} = 0. \tag{4.25}$$

Obviously, the roots of characteristic Equation of $F_J(\overline{x},\overline{y})$ *are given by* $\lambda = 0$ *(multiple root) and* $\lambda = 1$ *(multiple root). From this result, the equilibrium point*

$(\overline{x}, \overline{y}) = (0, 0)$ *is a nonhyperbolic point since the modulus of one of the roots of the Equation* (4.25) *is equal to one.*

ii) *For the zero equilibrium point* $(\overline{x}, \overline{y})$ *of system* (4.21) , *using* (4.22), (4.23) *and* (4.24), *it follows that*

$$
\begin{aligned}
P(\lambda) &= \lambda^{2k+2} - \left(\frac{1}{1 + 0.56714 e^{0.56714}} + \frac{1}{1 + 0.56714 e^{0.56714}} \right) \lambda^{2k+1} \\
&+ \left(\frac{1}{1 + 0.56714 e^{0.56714}} \frac{1}{1 + 0.56714 e^{0.56714}} \right) \lambda^{2k} \\
&- \left(\left(\frac{-(0.56714)^2 e^{0.56714}}{1 + 0.56714 e^{0.56714}} \right) + \left(\frac{-(0.56714)^2 e^{0.56714}}{1 + 0.56714 e^{0.56714}} \right) \right) \lambda^{k+1} \\
&+ \left(\begin{array}{c} \frac{1}{1 + 0.56714 e^{0.56714}} \left(\frac{-(0.56714)^2 e^{0.56714}}{1 + 0.56714 e^{0.56714}} \right) \\ + \left(\frac{-(0.56714)^2 e^{0.56714}}{1 + 0.56714 e^{0.56714}} \right) \frac{1}{1 + 0.56714 e^{0.56714}} \end{array} \right) \lambda^{k} \\
&+ \left(\left(\frac{-(0.56714)^2 e^{0.56714}}{1 + 0.56714 e^{0.56714}} \right) \left(\frac{-(0.56714)^2 e^{0.56714}}{1 + 0.56714 e^{0.56714}} \right) \right).
\end{aligned}
$$

Hence, we obtain the characteristic equation of system of higher order rational exponential difference equation $F_J(\overline{x}, \overline{y})$ *is as follows*

$$P(\lambda) = \lambda^{2k+2} - \lambda^{2k+1} + (0.25)\lambda^{2k} + (0.56714)\lambda^{k+1} - (0.28376)\lambda^{k} + 0.08041. \quad (4.26)$$

■

Furthermore, intuitively we can see that all the roots of the higher order polynomial Equation (4.26) satisy $|\lambda| < 1$ by numerical methods and graphs. Therefore, it can be said that the positive equilibrium point $(\overline{x}, \overline{y})$ of system (4.21) is locally asymptotically stable but here we could not show it theoretically. Thus, in this section we discussed the equilibrium point and the local asymptotic stability of some systems of higher order rational exponential difference equations.

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