NUMERICAL ANALYSIS OF A PROJECTION-BASED STABILIZATION METHOD FOR THE NATURAL CONVECTION PROBLEMS

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ABSTRACT

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In this thesis, we consider a projection-based stabilization method for solving buoyancy driven flows (natural convection problems). The method consists of adding global stabilization for all scales and then anti-diffusing these effects on the large scales defined by projections into appropriate function spaces. In this way, stabilization acts only on the small scales. We consider two different variations of buoyancy driven flows based on the projection-based stabilization.

First, we focus on the steady-state natural convection problem of heat transport through combined solid and fluid media in a classical enclosure. We present the mathematical analysis of the projection-based method and prove existence, uniqueness and convergence of the approximate solutions of the velocity, temperature and pressure. We also present some numerical tests to support theoretical findings.

Second, we consider a system of combined heat and mass transfer in a porous medium due to the natural convection. For the semi-discrete problem, a stability analysis of the projection-based method and a priori error estimate are given for the Darcy-Brinkman equations in

double-diffusive convection. Then we provide numerical assessments and a comparison with some benchmark data for the Darcy-Brinkman equations.

In the last part of the thesis, we present a fully discrete scheme with the linear extrapolation of convecting velocity terms for the Darcy-Brinkman equations.

Keywords: Projection-based stabilization, finite element method, natural convection equation, error analysis, double-diffusive convection.

ÖZ

DOĞAL KONVEKSİYON PROBLEMLERİ İÇİN PROJEKSİYON-ESASLI KARARLILAŞTIRMA YÖNTEMİNİN SAYISAL ANALİZİ

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Bu tezde, kaldırma tesirli akışları (doğal konveksiyon problemleri) çözebilmek için projeksiyon-esaslı kararlılaştırma yöntemini ele aldık. Bu yöntem global kararlılaştırmanın tüm ölçeklere eklenmesi ve ardından uygun fonksiyon uzaylarına projeksiyon vasıtasıyla tanımlanan kalın ölçeklerden bu etkinin geri çözünümünden ibarettir. Bu vasıta ile kararlılaştırma sadece küçük ölçeklere tesir eder. İki farklı kaldırma tesirli akışı projeksiyon-esaslı kararlılaştırma yöntemiyle ele aldık.

İlk olarak, klasik bir kapalı ortamda katıdan akışkana ısı transferinin durağan halli doğal konveksiyon problemi üzerinde yoğunlaştık. Projeksiyon-esaslı kararlılaştırma yönteminin matematiksel analizini verdik ve hız, sıcaklık ve basınç değişkenlerinin varlık, teklik ve yakınsama özelliklerini ispatladık. Ayrıca, teorik bulguları destekleyen sayısal testleride sunduk.

İkinci olarak, gözenekli bir ortamda doğal konveksiyon vasıtasıyla gerçekleşen birleşik ısı ve kütle transfer problemini ele aldık. Yarı-ayrık durumda projeksiyon-esaslı metodun kararlılık

ve öncül hata analizini çifte çözünümlü konveksiyonda Darcy-Brinkman denklemleri için verdik. Ardından Darcy-Brinkman denklemleri için bazı referans değerlerle kıyaslama yapan sayısal ölçümleri verdik.

Tezin son kısmında Darcy-Brinkman denklemleri için konvektif hız terimlerinin lineer dışdeğerlerini içeren tam ayrık bir şemayı sundu.

Anahtar Kelimeler: Projeksiyon-esaslı kararlılaştırma, sonlu elemanlar yöntemi, doğal konveksiyon denklemi, hata analizi, çifte dağılımlı konveksiyon.

To my precious wife and sweet daughter

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CHAPTER 1

INTRODUCTION

This thesis deals with the analysis of the natural convective flows inside enclosures. Such flows occur by the effect of body forces which are formed due to the density differences along with the gravitational impacts. In other words, as the fluid touches to a hot or cold surface, a density difference occurs due to the temperature gradients near the vicinity of the mentioned surfaces. Thus, the lighter fluid moves upward and the denser fluid moves downward. Finally, a natural convective flow occurs due to the gravity effect on a such kind of a density gradient. In contrast to the case of forced convection, in which the flow is driven by some external effects i.e. a pump or a fan, density differences causing the flow formation arise as a result of temperature changes in the system. Along with the temperature differences, some additional effects which change fluid density, like concentration differences, could also be seen on some natural convective systems. The mentioned body forces are referred as buoyancy forces and so the flows affected by these forces are called buoyancy driven flows.

We consider two different type of buoyancy driven flows in this thesis. The first one is a classical natural convection problem in a closed non-porous cavity. The buoyancy force forms by the effect of temperature differences. For the second one, we study on porous enclosures in which concentration difference accompanies temperature differences to change the fluid density. Those kind of natural convective flows are known as double-diffusive, thermohaline or thermosolutal convection. For both cases, the flows are convection dominated and coupling between velocity, temperature and concentration fields is pretty strong. So, various kinds of stabilization techniques are developed to approximate these flows. Departing from these forewords about the character of buoyancy driven flows, we separate this chapter into three main parts to make explanations for each topic in detail.

1.1 Classical Natural Convection in Non-Porous Media

1.1.1 Physical mechanism

The flow in natural convection is induced by the buoyancy force arising from the temperature differences. Although the flow motion due to the natural convection is slower than the forced convection, flow character formed near the heat transfer surfaces are similar. In general, there are two different configurations for natural convection occurring in an enclosure. In first one, the system is heated from below and cooled from above. Such kind of natural convection is called the Rayleigh-Benard convection. If one heats the system from above and cools from below, the character of the flow will change totally. In such kind of a system, a convective flow is not observed due to the gravity effect. The denser fluid will always be at the bottom and there is no flow motion opposing with the gravity effect. The heat transfer coefficient equals to unity and the heat transfer occurs only by conduction. In the case of Rayleigh-Benard convection, the heavier fluid will be on top of the lighter fluid, and there will be a tendency for the lighter fluid to topple the heavier fluid and rise to the top, where it will come in contact with the cooler surface and cool down [13]. When the Rayleigh number exceeds 1708, the buoyant force overcomes the fluid resistance and initiates natural convection currents, which are observed to be in the form of hexagonal cells called Benard cells [13]. In second configuration, horizontal boundaries of the enclosure are adiabatic, one of the vertical wall kept cold and the other wall kept hot. In contrast to the Rayleigh-Benard convection, altering the hot and cold walls do not change the flow character since the gravity will always be perpendicular to temperature gradients. Double pane window problem is a typical example which gives rise to such a system. Fluid motion which forms in natural convective heat transfer in pane cavities is driven by the buoyancy force. This force depends on the temperature difference between indoor and outdoor environment, and is characterized by a rising fluid at the cavity hot wall and a descending fluid at the cavity cold wall, [71]. Thus, the system models a rectangular enclosure in which vertical walls present indoor and outdoor panes, where the top and bottom sections are adiabatic parts. The effect of wall conductance has to be considered carefully for this kind of problems. Neglecting the thickness of the solid body outside the fluid results in different systems than the one considered here. If the solid is not included, subtracting the heated boundary adds an extra term on the right hand side of the energy equation. This term can be avoided only when the solid region is included [11].

1.1.2 Governing equations

The total system of equation consists of a Navier-Stokes system coupled with an energy equation. We stick on a steady-state formulation for natural convection flows. The Navier-Stokes system for a viscous incompressible fluid is build up from a combination of mass balance equation and a momentum equation [55]. The main difference in a natural convection system comes from the body force term and coupling of an energy equation. The key point in the system we study is the assumption of Boussinesq approximation. It states that, we neglect the density differences seen on the system unless they are multiplied with the gravity. We now explain the derivation of the equations of a natural convection system for a 2-D flow in detail. We use a control volume for the formulations we derive and they can be extended to general cases trivially.

Basically, the principle of mass conservation states that mass can neither be created nor destroyed. For a steady flow, the quantity of mass inside the control volume stays unchanged. Assume that the flow enter from left side of the volume and exit from right side of the volume with the velocity $\mathbf{v} = (v_x, v_y)$ and density ρ. Entering mass flow rate to the volume is $\rho v_x dy$ and $\rho v_y dx$ for x and y directions respectively. Exiting mass rate to the volume is $\rho\left(v_x + \frac{\partial v_x}{\partial x}\right)dy$ and $\rho\left(v_y + \frac{\partial v_y}{\partial y}\right)dx$ for x and y directions respectively. Thus, using the principle of mass conservation we have

$$\rho v_x dy + \rho v_y dx = \rho\left(v_x + \frac{\partial v_x}{\partial x}\right)dy + \rho\left(v_y + \frac{\partial v_y}{\partial y}\right)dx. \tag{1.1}$$

Simplifying (1.1) and dividing both sides with $dxdy$, one finally arrives,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

which is the equation of continuity.

We state the momentum equation next. Newton's second law of motion states that *"the net force acting on the control volume is equal to the mass times the acceleration of the fluid element within the control volume, which is also equal to the net rate of momentum outflow from the control volume"* [56]. One can express this statement in mathematical way as $M.a = F$, where M is the mass of the fluid, a is the acceleration and F is the net force. Let us redefine these quantities in means of our system now. Total mass M is given by $M = \rho\, dxdy$, total derivative becomes $dv_x = \frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy$, acceleration is $a_x = v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}$ in x direction.

The force contains viscous, pressure and body forces. These forces in our volume become, $\nu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial P}{\partial x} + \rho_f$ in x direction with pressure P, kinematic viscosity ν and body force ρ_f. Finally, reformulating the Newton's law with these new definitions gives us the momentum equation in x direction, which is

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = \nu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial P}{\partial x} + \rho_f. \tag{1.2}$$

Derivation of same equations in y direction, assumption of Boussinesq approximation, departing from control volume to general case and rewriting the system in a compact form gives

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{u} + \rho_f \tag{1.3}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1.4}$$

along with the continuity equation. The body force term ρ_f is of great importance. This buoyancy force could be expressed as $\rho_f = \rho \mathbf{g}$. The variation of the density of a fluid with respect to the temperature at a constant pressure, namely the thermal expansion coefficient β could be formulated in terms of density and temperature differences of the system as

$$\beta = \frac{-1}{\rho} \frac{\rho_0 - \rho}{T_0 - T} \tag{1.5}$$

where ρ_0 and T_0 denotes the reference density and temperature. Thus, one arrives $\rho_f = \beta \, \mathbf{g} \, (T - T_0)$ after some modifications on (1.5) and assumption of Boussinesq approximation.

The final equation to derive is the energy equation. The energy balance of a natural convection system could be expressed as: the total difference of energies that enters and exits the system is zero. In other words, the amount of the total energy convected by the fluid out of the system and the amount of the energy installed into the system by heat conduction are equal. Assuming a constant pressure and a negligible viscous dissipation, we give this fact in terms of a mathematical relation as

$$-c_p \, T \nabla \cdot (\rho \mathbf{u}) + \nabla \cdot \left(\rho \, c_p \, T \mathbf{u} \right) = \nabla \cdot (\kappa \, \nabla T) + \gamma \tag{1.6}$$

where c_p is the thermodynamics coefficient, κ is the thermal conductivity parameter and γ is a heat source. So, one gets a system of natural convection via combining (1.3), (1.4) and (1.6). There are various non-dimensionalizations of this kind of systems. We use the one given in [11] with a very detailed process of non-dimensionalization.

1.1.3 Real life applications and previous work

Due to the wide range of applications, many scientists are increasingly attracted to natural convection flows. Some of the commonly used buoyancy-driven flows are observed in nature; such as atmospheric fronts, katabatic winds etc., and in industry; such as dense gas dispersion, natural ventilation, solar collectors, insulation with double pane window, cooling of electronic equipments, cooling of nuclear reactors and so on. In solar collectors, heat exhaustion must be prevented to make the system work efficiently. Natural convection occurs between the solar heat absorber and the conductive covers. One should minimize such kind of natural convection to prevent the energy loss. Consideration of natural convection in a double pane window system is also crucial. The distance between the panes must be arranged appropriately to block the natural convection. Cooling of electronic equipment is due for making them work properly. Natural convection is preferred for cooling such kind of equipments since it is cheap, safe and simple. Alienation of circuit boards and chips is only allowed via natural convection and it defines the working configuration of the system [30]. Consideration of natural convective flows is also important for the arrangement of indoor air quality and selection of appropriate heating and cooling systems in houses.

Since the system we study has too many applications in many engineering branches, the number of publications concerning the natural convection equations is almost countless. We mention here which are of importance especially in point of a mathematical view. Publications concerning the computational results rely on various numerical techniques. The most outstanding study concerning the computational results on natural convection problem was published by de Vahl Davis [16]. In [16], the system is solved using a finite difference scheme based on an implicit alternating direction method (ADI) and his findings are still accepted as benchmark results for new studies. Hortmann et al. used the finite volume method for the problem and take a step further in terms of larger Rayleigh number in [27]. Using a streamfunction-vorticity approach, Shu and Xue performed a differential quadrature method in [65]. In noteworthy study of Massarotti et al. [51], a characteristic-based split (CBS) scheme was performed in a semi-implicit form. Manzari et al. [50] studied on application of an artificial viscosity based scheme concerning the turbulent thermal convective flows. As a benchmark computational study, the work of Wan et al. [70] deserves the attention in which the numerical solution of the system is studied both with a finite element method and a discrete singular convolution. From the aspect of finite element error analysis, the studies on natural convection equations are limited. Boland and Layton studied the finite element error analysis of the system in [11] and [10] for steady-state and time dependent cases respectively. In [9], Boland et al. studied the finite element error of the system along with the non-physical dynamics induced by the discretization.

1.2 Double-Diffusive Convection in a Porous Medium

1.2.1 Porous medium

A porous medium is a material which contains tiny spaces connected to each other inside a solid frame. A solid or a liquid may pass through these spaces. We observe so many examples of porous media in nature and our daily lives. Sea sand, human lungs, a piece of bread and wood are some well known instances that we encounter everyday. A typical porous media must possess two important properties. First, it must contain minute spaces compared to its own scale and these spaces may contain different kind of fluids and/or mixtures such as water, oil and air. Second, the material must be permeable. In other words, any fluid could enter from an edge of the frame and exit from the other edge. The structure of the pores in a porous medium leads the general behavior of the medium. The most important pore properties are the porosity, permeability and the flow channel.

Porosity is the ratio of the volume of total minute spaces to the volume of whole material. It takes values in between 0 and 1 depending on the material properties. For instance, porosity is bigger for heat insulation materials and fibre filters, whereas it is very small for metals and some volcanic rocks. Permeability characterizes the amenity of the fluid flow under a pressure gradient inside the porous medium. It was first stated by Henry Darcy in 1856 and the unit of permeability took after his name. Permeability is a macroscopic property of the porous medium and it is also related to the geometry of the medium. Another characteristic property of a porous medium is the flow channel. Flow channel inside medium are mostly not a straight line and it is longer than the length of the porous medium. The considered media may contain more then one flow channel.

1.2.2 Governing equations

Conservation equations for a porous media are derived by considering means for mediums that include so many spaces. If the space are saturated by a single fluid, the flow is said to be single phase and if there are two distinct fluids the flow is double phased. Darcy found a relation about the movement of fluid inside a porous medium after his experimental studies. This is known as Darcy's law and given by

$$\mathbf{v} = -\frac{K}{\nu}\left(\nabla P + \rho \mathbf{g}\right)$$

where K is the permeability. Brinkman extended the Darcy's law by consideration viscous diffusion effects and stated it as

$$\nabla P = -\frac{\nu}{K}\mathbf{v} + \nu \nabla^2 \mathbf{v}.$$

Today, it has still not been understood well that which model is more valid under what conditions. In a recent article of J-L Auriault, the concept of this validity was discussed [3]. According to his conclusion, the Brinkman model is valid when describing flows through swarms of fixed particles or fixed beds of fibres only, and under precise conditions. This restriction could be the answer of the question that, why the number of studies carried out assuming the Brinkman law is small than the Darcy's law. In this study, we assume the validity of the Brinkman's extension.

Conservation equations are derived in the same way as in the previous section. We only state the differences for the porous medium in this part. Continuity equation is exactly same. For the momentum equation, we have

$$\rho\left(\frac{1}{\epsilon}\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\epsilon^2}(\mathbf{v}\cdot\nabla\mathbf{v})\right) = -\nabla P + \nu\nabla^2\mathbf{v} - \frac{\nu}{K}\mathbf{v} + \rho\mathbf{g} \tag{1.7}$$

with porosity ϵ and permeability K. Note that this is a classical momentum equation as derived in previous chapter except the installation of the Brinkman extended Darcy's law.

For the energy balance, we assume an isotropic material in which the heat conduction between the solid and fluid phases are parallel and there is no conduction of heat from one phase to another. Thus heat balance equation becomes

$$\sigma\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \nabla(\gamma\nabla T) \tag{1.8}$$

where, σ is the specific heat ratio and γ is the thermal diffusivity. Concentration equations are formed in a very similar manner as in the temperature cases and it takes the form

$$\epsilon\frac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C = \nabla(D_c\nabla C) \tag{1.9}$$

with mass diffusivity D_c. Finally, since we are dealing with the combined heat and mass transfer in a porous enclosure, the body force term in (1.7) becomes

$$\rho_f = \rho\mathbf{g} = \rho_0\{1 - \beta_T\,(T - T_0) - \beta_C\,(C - C_0)\}\mathbf{g} \tag{1.10}$$

where the subscript 0 denotes a reference density, temperature or concentration. β_T and β_C stands for the thermal and solutal expansion coefficients respectively. Combining (1.7)-(1.10) with the usual continuity equation, one gets the system of double diffusive convection in a porous medium. We do not emphasize on non-dimensionalization again. We use the one given in [20] for the analysis we present.

1.2.3 Real life applications and previous work

Combined heat and mass transfer in a porous medium due to the free convection, which is also known as double-diffusive convection has begun to attract scientists from varying fields. Especially at last three decades, as the case of pure thermal convection in a porous medium becomes better understood, attention is now turning to systems in which the density differences causing natural convection are occurred owing to coupled thermal and solutal effects. The extensive interest to double-diffusive convection phenomena in porous media comes from its wide existence observed in nature and applications in industry. It usually forms in seawater flow and mantle flow in earth's crust naturally. Furthermore, examples in astrophysics, electrochemistry, geophysics and metallurgy are commonly encountered. Especially, it has crucial applications in geophysics such as extraction of oil confined by porous rocky materials. Double-diffusive flows are also related to contaminant transport in groundwater and development of geothermal sources [54].

Two different configurations are commonly seen in the literature for the system of combined heat and mass transfer in porous media. The first one consists of a porous enclosure with different temperature and concentration gradients at horizontal walls. For the second, adiabatic and impermeable horizontal walls are accompanied by vertical walls with different temperature and concentration gradients. In early studies [57] and [69], linear stability analysis of the first configuration was studied. Solution behaviors in a cubic porous cavity were given in [64] according to the first formulation. Studies based on second formulation is more common. Both analytical and numerical aspects of the flow with the Darcy formulation was analyzed in [68]. There are some other noteworthy studies with Darcy formulation which involves boundary layer flows [2, 6]. A very detailed numerical treatment of the Darcy-Brinkman model was given in [20]. The influence of boundary and inertial effects on double-diffusive convection was studied in [38] and the onset of convection along with the stability analysis was considered in [18] again using the Darcy-Brinkman formulation. In point of a mathematical view, number of studies on double-diffusive convection is rather limited compared to engineering considerations. In the early study of Siegmann and Rubenfeld [66], stability of conductive solutions of a double-diffusive system was considered. Bennacer et al. carried out a numerical study for a Darcy-Brinkman flow of double-diffusive convection within a vertical circular annulus in [7]. A Galerkin finite element method applied to a Darcy model was given in [49]. Mohamad and Bennacer provided a numerical treatment for a Darcy-Brinkman flow in both three and two space dimesions in [53]. Kramer et al. studied the boundary element solutions of a Darcy-Brinkman system in [41] which is very similar to one considered here. Although there are a wide range of publications concerning the continuous dependence and structural stability of solutions about the double-diffusive convection in a porous medium [59, 46, 47], topic of finite element error analysis on such kind of systems has not been considered yet. To do authors best knowledge, the error estimates of the finite element method with a projection-based stabilization idea applied to double-diffusive convection in porous media are not yet available.

1.3 Stabilization Methods

Classical natural convection problem in fluid mechanics occurs in an enclosed domain, [40]. For natural convection in enclosures, a boundary layer forms near the walls. Outside this layer, a rolling core is formed inside the enclosure. The boundary layer and the core could not be considered independent since the core is covered by the layer. There is a coupling between the core and the boundary layer. This coupling is the main reason of the difficulty in solving these systems analytically. Thus, numerical methods and experimental analysis are used frequently, [72].

The finite element method is one of the most popular and mathematically sound variants of numerical approximation [48]. Standard Galerkin finite element method for natural convection type problems with high Reynolds or Grashof numbers generally results with inaccurate approximate solutions and may display global counterfeit oscillations, [21, 52, 14]. This disappointing behavior occurs since such methods lose stability and cannot adequately approximate solutions inside layers due to the dominance of convection terms and the strong coupling between the unknown flow characteristics.

An interesting property of the flows with small viscosity is the diversity of scales. These are resolved small and large scales and unresolved scales. One of the main reason of failure of non-stabilized finite element methods for turbulent flows is to attempt to define overall flow character at once [33]. Thus the use of an appropriate stabilization mechanism for approximation of such flows is inevitable.

So many stabilization strategies are developed for finite element approximations in order to cure mentioned disadvantages for the flow problems that we interest [63]. Among them all, most popular ones are residual based techniques as Streamline-Upwind Petrov Galerkin (SUPG) and Pressure-Stabilization (PSPG) methods, Large Eddy Simulation (LES) and Variational Multiscale Method (VMS). Analysis of some well-known stabilization techniques applied on a convection diffusion system was given in [15] and a comprehensive comparison of various stabilization techniques applied on an Oseen problem was studied in [12].

In residual based techniques, numerical viscosity is added on all scales and this gives rise to some problems due to the richness of flow scales. We refer the reader to [63] for a comprehensive overview of such kind of stabilization mechanisms. Classical LES techniques attempt to model only the character of large scales. Thus, various drawbacks like definition of appropriate boundary conditions for large scales and commutations errors are encountered [48]. For a more general discussion of LES models, see e.g., [8]. Among other stabilization mechanisms, the emphasis of this study is on a rather new technique so called projection-based stabilization which is a variant of (VMS) [29, 22, 45, 32]. VMS was first proposed in [29] and an extension with a combination of Large Eddy simulation idea was given in [28] . A broad and comparative investigation of various kinds of VMS method were studied in [37]. In classical VMS approach, solution spaces are separated into coarse and fine scale spaces through an overlapping sum decomposition. Here the coarse scale space is finite dimensional and fine scale space is infinite dimensional. Then one rewrites the variational formulation of the problem with a set of two equations for each equation component of the system. One equation contains the test functions from coarse scale space and the other from fine scale space. So, the fine scale equation contains infinitely many equations and in order to approximate these, bubble functions which contains local higher order polynomials are used. An eddy viscosity model is used to take into account the effect of scales which are not resolved by bubble functions. This eddy viscosity model acts directly only on the bubble functions [36].

In this study, we consider a projection based VMS approach similar to the idea presented in [34]. A noteworthy Guermond's stabilization idea of subgrid viscosity concept makes the diffusion acts only on the finest resolved mesh scale, [22], with the definition of solution spaces via bubble functions. Based on the ideas developed in [22, 29], several multiscale decompositions have been proposed in the literature, [32, 35, 45]. Since then, considerable progress has been made for the use of projection-based stabilization method both in mathematical and computational analysis in past years, [25, 39]. The philosophy of the projection-based stabilization is to use projections into appropriate function spaces in order to decompose solution scales. In this way, the stabilization is added in different ways. In the method we use, finite element spaces for all variables are defined for all resolved scales first. Then the large scale spaces are defined via L^2 projections. For turbulent flows, the effect of unresolved scales onto resolved small scales are considered through a turbulence model. Examples regarding the projection-based stabilization and the VMS applied on the natural convection problems are very limited. The recent study of Löwe and Lube [48] discussed the error analysis of a variational multiscale scheme applied on a non-stationary system. A projection based stabilization idea performed for the stationary case was considered in [14].

The main aim of this thesis is to upgrade the numerical models developed for buoyancy driven flows in enclosures through making use of projection-based stabilization idea. Our another goal is the transmission of the successful variational multiscale ideas which are presented for turbulent Navier-Navier stokes equations in the literature and open a way to understand turbulent natural convection phenomena better in the future.

1.4 Chapter Descriptions

This thesis consists of four chapters.

Chapter 2 presents the analysis of classical natural convection problem in a non-porous enclosure. In particular, a stationary system of equations of heat transport through combined solid and fluid media is considered. After introducing the system, we present the mathematical preliminaries and the scheme. Existence, uniqueness and stability issues of the problem are discussed in the next section. After providing a priori error analysis for the velocity and temperature variables, we give the error estimations for the pressure. Numerical assessment of the problem will also be given by the end of this chapter.

Chapter 3 provides the finite element analysis of a natural convection problem, namely double-diffusive convection in a confined porous enclosure. We present the mathematical preliminaries and the projection-based stabilization scheme. We then investigate the stability and error estimation of the semi-discrete problem. We perform some numerical tests to verify both the theory and the effectiveness of the method next. A fully discrete scheme along with a comprehensive error analysis follows.

Chapter 4 is dedicated to address the conclusions and discuss some possibilities for future investigations.

CHAPTER 2

NATURAL CONVECTION PROBLEM

In this chapter, we provide a finite element error analysis of the projection-based stabilization method for solving steady-state natural convection equations. We consider the same type projection-based stabilization technique of the steady-state Navier Stokes equations, [39]. As in [39], we also define the large scale spaces on a coarser grid for the solution scales. Main difference in the present work comes from the technical point of view, which is the coupling of the Navier-Stokes equation to the energy equation. We first present stabilized finite element scheme and give comprehensive error analysis of this coupled problem. We derive error estimations for the velocity, temperature and pressure and show that these errors are optimal with respect to the mesh sizes along with the choices of viscosity parameters. To evaluate the performance and accuracy of the method, we provide numerical experiments.

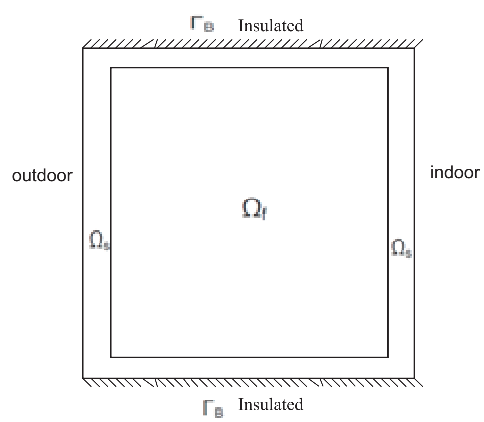

Figure 2.1: Typical geometry of a double pane window problem.

One of the most common uses of the fluids is to transfer heat to solid bodies. So, it is necessary to consider a coupled domain where the solid is included. We consider herein heat transport between solid and fluid media. This complex phenomena can be formulated as follows: let Ω_s, Ω_f be disjoint polyhedral domains in $\Omega \subset \mathbb{R}^d (d = 2, 3)$ where Ω is the regular bounded open set. The steady-state natural convection equations including solid media are governed by

$$\begin{aligned} -Pr\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= Pr\,Ra\,T\mathbf{e} \quad \text{in} \quad \Omega_f, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega_f, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega_f, \quad \mathbf{u}\equiv\mathbf{0} \quad \text{in} \quad \Omega-\Omega_f=\Omega_s, \\ -\nabla\cdot(\kappa\nabla T) + (\mathbf{u}\cdot\nabla)T &= \gamma \quad \text{in} \quad \Omega, \\ T &= 0 \quad \text{on} \quad \Gamma_T, \quad \frac{\partial T}{\partial\mathbf{n}} = 0 \quad \text{on} \quad \Gamma_B, \end{aligned} \tag{2.1}$$

where $\Gamma_T = \partial\Omega\backslash\Gamma_B$ where Γ_B is a regular open subset of $\partial\Omega$. $\mathbf{u}, p, T$ denote the velocity, pressure and temperature fields, respectively, γ is a forcing function, $\mathbf{e}$ is a unit vector in the direction of gravitational acceleration and $Pr, Ra, \kappa > 0$ refer to the Prandtl, Rayleigh numbers and thermal conductivity parameter, respectively. Furthermore, we consider the case $\kappa \equiv \kappa_f$ in Ω_f and $\kappa \equiv \kappa_s$ in Ω_s where κ_f and κ_s are positive constants. Figure 2.1 illustrates the geometry of the problem we study on.

System (2.1) presents severe computational problems for large Rayleigh numbers. It is well known that, the solution of (2.1) is unique under some restrictions on the Rayleigh and Prandtl numbers. Uniqueness is lost for high Rayleigh numbers, [61]. We use the rigorous finite element method for solving this system numerically.

2.1 Notation, Mathematical Preliminaries and Scheme

We use the standard notations used for Sobolev and Lebesgue spaces in Adams [1] throughout the entire thesis. The Sobolev space $W^{k,r}(\Omega)$ on a domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ is given as

$$W^{k,r}(\Omega) = \{\phi \in L^r(\Omega) : \forall|s| \leq k,\ \partial^s\phi \in L^r(\Omega)\}.$$

We denote usual inner product and norm in $L^2(\Omega)$ by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. The norm and semi-norm in a Sobolev space $W^{k,r}(\Omega)$ is also given by $\|\cdot\|_{k,r}$ and $|\cdot|_{k,r}$. For the special case $r = 2$, the norm in the space $W^{k,2}(\Omega) = H^k(\Omega)$ is shown by $\|\cdot\|_k$. The space $H^1(\Omega)$ is of special interest and we use it frequently throughout the thesis. The norm in $H^1(\Omega)$ is given by $\|\mathbf{u}\|_1 = (\|\mathbf{u}\| + \|\nabla\mathbf{u}\|)^{1/2}$.

We remark that the vector-valued functions are denoted with boldface character. Given a vector valued function ϕ, its gradient $\nabla\phi = \frac{\partial \phi_i}{\partial \phi_j}$ is called a tensor and the product of tensors $\mathbf{A}, \mathbf{B}$ is given by $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$.

The following well-known functional vector spaces are considered to define a variational formulation of (2.1) .

$$
\begin{aligned}
X := \mathbf{H}_0^1(\Omega_f) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega_f) : \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega_f\},\\
W : &= \{S \in H^1(\Omega) : S = 0 \quad \text{on} \quad \Gamma_B\},\\
Q : &= \{p \in L^2(\Omega) : \int_\Omega p\, d\mathbf{x} = 0\},\\
V := \mathbf{H}^1_{0,div}(\Omega_f) &= \{\mathbf{u} \in X : \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega_f\}
\end{aligned}
$$

We introduce the following bilinear and trilinear forms, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X,\ T, S \in W$ and $q \in Q$:

$$
\begin{aligned}
a_0(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_f} \nabla\mathbf{u} : \nabla\mathbf{v} d\mathbf{x} && (2.2)\\
a_1(T, S) &= \int_\Omega \kappa \nabla T \cdot \nabla S\, d\mathbf{x} && (2.3)\\
b(\mathbf{v}, q) &= -\int_{\Omega_f} q \nabla \cdot \mathbf{v} d\mathbf{x} && (2.4)\\
c_0(\mathbf{u}, \mathbf{w}, \mathbf{v}) &= \frac{1}{2}\int_{\Omega_f} \Big((\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\Big) d\mathbf{x} && (2.5)\\
c_1(\mathbf{u}, T, S) &= \frac{1}{2}\int_{\Omega_f} \Big((\mathbf{u}\cdot\nabla)T S - (\mathbf{u}\cdot\nabla)S\, T\Big) d\mathbf{x} && (2.6)\\
d(T, \mathbf{v}) &= \int_{\Omega_f} T\mathbf{e}\cdot\mathbf{v} d\mathbf{x}. && (2.7)
\end{aligned}
$$

The variational formulation of (2.1) reads as follows: seek $\mathbf{u} \in X$, $p \in Q$, $T \in W$ such that

$$
\begin{aligned}
Pr\ a_0(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= Pr\, Ra\, d(T, \mathbf{v})\\
b(\mathbf{u}, q) &= 0 \qquad\qquad (2.8)\\
a_1(T, S) + c_1(\mathbf{u}, T, S) &= (\gamma, S)
\end{aligned}
$$

for all $(\mathbf{v}, q, S) \in (X, Q, W)$. The notations in equations (2.8) are inspired by the work in [11], in which the standard Galerkin finite element method for (2.8) is studied.

The scheme introduces the addition of global stabilization and then subtracts its effect onto large scales of the coupled equations for both velocity and temperature spaces. In this way, stabilization acts only on the smallest resolved scales of both scales. Let $\mathcal{F}^H, \mathcal{G}^K$ be a conforming triangulation of Ω and let $\mathcal{F}^h, \mathcal{G}^k$ be a refinement of $\mathcal{F}^H, \mathcal{G}^K$, i.e. $H \geq h$ and $K \geq k$ respectively. Let $X^h \subset X, W^k \subset W$ and $Q^h \subset Q$ be conforming finite element spaces satisfying the discrete inf-sup condition (2.20) and L^H, M^K denote the finite element subspaces of $(L^2(\Omega))^d$. The discretization we investigate adds additional diffusion acting on all discrete velocity and temperature scales and then anti-diffuses on the scales resolvable on $\mathcal{F}^H, \mathcal{G}^K$ as follows: find $\mathbf{u}^h \in X^h$, $p^h \in Q^h$, $T^k \in W^k$, $\mathbf{F}^H \in L^H$ and $G^K \in M^K$ such that

$$
\begin{aligned}
Pr\, a_0(\mathbf{u}^h, \mathbf{v}^h) + (\alpha_1(\nabla \mathbf{u}^h - \mathbf{F}^H), \nabla \mathbf{v}^h) + c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) &= Pr\, Ra\, d(T^k, \mathbf{v}^h) \quad (2.9)\\
b(\mathbf{u}^h, q^h) &= 0\\
(\mathbf{F}^H - \nabla \mathbf{u}^h, \mathbf{l}^H) &= 0 \quad (2.10)\\
a_1(T^k, S^k) + \alpha_2(\nabla(T^k - G^K), \nabla S^k) + c_1(\mathbf{u}^h, T^k, S^k) &= (\gamma, S^k) \quad (2.11)\\
(G^K - \nabla T^k, m^K) &= 0, \quad (2.12)
\end{aligned}
$$

for all $(\mathbf{v}^h, q^h, \mathbf{l}^H, S^k, m^K) \in (X^h, Q^h, L^H, W^k, M^K)$ where $\alpha_1 := \alpha_1(h)$ and $\alpha_2 := \alpha_2(k)$ are non-negative constant functions and user selected stabilization parameters. These parameters can be thought of as an additional viscosity in the coarse space.

Remark 2.1 *Multiscale decomposition requires selection of large scale spaces for both velocity and temperature, L^H and M^K, respectively. If both of them are selected as zero subspaces, then Galerkin formulation is recovered in [11]. We employ $L^H = \nabla X^H$ and $M^K = \nabla W^K$ choices of [45] for the large scale spaces to obtain the bounds in this paper. Some other possible choices for these spaces are $L^H \subseteq \nabla X^h$ and $M^K \subseteq \nabla W^k$ (see [35]).*

Let $V^h = \{\mathbf{v}^h \in X^H : (q^h, \nabla \cdot \mathbf{v}^h) = 0$, for all $q^h \in Q^h\}$ be the space of discretely divergence free functions. It is easy to verify the following: (2.10) and (2.12) imply that F^H and G^K are L^2 projections of $\nabla \mathbf{u}^h$ and ∇T^k onto L^H and M^K, respectively. If we denote these projections with P_H and P_K, respectively, the properties of the projection operator give the reformulations of (2.9)-(2.12) in V^h as follows: find $\mathbf{u}^h \in V^h, T^k \in W^k$ such that

$$
\begin{aligned}
A_0(\mathbf{u}^h, \mathbf{v}^h) + c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) &= Pr\, Ra\, d(T^k, \mathbf{v}^h) \quad (2.13)\\
A_1(T^k, S^k) + c_1(\mathbf{u}^h, T^k, S^k) &= (\gamma, S^k) \quad (2.14)
\end{aligned}
$$

for all $(\mathbf{v}^h, S^k) \in (V^h, W^k)$ where

$$A_0(\mathbf{u}^h, \mathbf{v}^h) = Pr\, a_0(\mathbf{u}^h, \mathbf{v}^h) + \alpha_1((I - P_H)\nabla\mathbf{u}^h, (I - P_H)\nabla\mathbf{v}^h) \quad (2.15)$$

$$A_1(T^k, S^k) = a_1(T^k, S^k) + \alpha_2((I - P_K)\nabla T^k, (I - P_K)\nabla S^k). \quad (2.16)$$

For vanishing boundary values, we define $H_0^1(\Omega)$ and its dual space, $H^{-1}(\Omega)$ and its norm is defined by

$$||\mathbf{f}||_{-1} = \sup_{\mathbf{v}\in X} \frac{|(\mathbf{f}, \mathbf{v})|}{||\nabla\mathbf{v}||}$$

where $(\cdot, \cdot)$ denotes the duality pairing.

We make use of well-known Sobolev embedding theorem for the following spaces: if Ω is bounded and has a Lipschitz boundary then $H^1(\Omega) \hookrightarrow L^4(\Omega)$, that is

$$||\mathbf{u}||_4 \le C||\mathbf{u}||_1. \quad (2.17)$$

Inequalities which are used frequently are

Young's inequality,

$$ab \le \frac{t}{p}a^p + \frac{t^{-q/p}}{q}b^q, \quad a, b, p, q, t \in \mathbb{R}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q \in (1, \infty), \quad t > 0, \quad (2.18)$$

and Poincaré's inequality in X,

$$||\mathbf{v}|| \le C||\nabla\mathbf{v}|| \quad (2.19)$$

for all $\mathbf{v} \in X$ with $C = C(\Omega)$.

We assume that finite element spaces have the following properties. The discrete spaces X^h, Q^h satisfy the usual approximation theoretic conditions and the inf-sup condition or Babuška-Brezzi condition i.e. there is a constant β independent of the mesh size h such that

$$\inf_{q^h\in Q^h} \sup_{\mathbf{v}^h\in X^h} \frac{(q^h,\ \nabla\cdot\mathbf{v}^h)}{\|\nabla\mathbf{v}^h\|\,\|q^h\|} \ge \beta > 0. \quad (2.20)$$

For examples of such compatible spaces see e.g., [23] and [19].

Definition 2.2 *Let V and V^h denote respectively the divergence free subspaces of X and X^h:*

$$V := \{\mathbf{v} \in X : (q, \nabla\cdot\mathbf{v}) = 0, \forall q \in Q\},$$

$$V^h := \{\mathbf{v}^h \in X^h : (q^h, \nabla\cdot\mathbf{v}^h) = 0, \forall q^h \in Q^h\}.$$

Although typically $V^h \subsetneq V$, it is known that under the discrete inf-sup condition (2.20), functions in V are well approximated by ones in V^h, [19].

We consider X^h and W^k to be spaces of continuous piecewise polynomials of degree r and Q^h is the space of continuous piecewise polynomials of degree $r-1$. We also make the standard assumptions that the spaces X^h, Q^h and W^k satisfy the following approximation properties for a given integer $1 \leq s \leq r$:

$$\inf_{\mathbf{v}_h \in X^h, q^h \in Q^h} \left\{ \|(\mathbf{u}-\mathbf{v}^h)\| + h\|\nabla(\mathbf{u}-\mathbf{v}^h)\| + h\|p-q^h\| \right\} \leq Ch^{s+1}(\|\mathbf{u}\|_{s+1} + \|p\|_s), \tag{2.21}$$

$$\inf_{S^k \in W^k} \|T - S^k\| \leq k^{s+1}\|T\|_{s+1} \tag{2.22}$$

for $(\mathbf{u}, p, T) \in (X \cap H^{s+1}(\Omega), Q \cap H^s(\Omega), W \cap H^{s+1}(\Omega))$.

We also use the fact that L^2 orthogonal projections of L^H and M^K satisfy

$$\|G - P_\mu G\| \leq C\mu^s |G|_s, \quad \mu = H, K, \ 1 \leq s \leq r \tag{2.23}$$

for $G \in (L^2(\Omega) \cap H^s(\Omega))$.

We define the following weighted norms.

Definition 2.3 *For $\mathbf{u} \in X$, $T \in W$, the weighted norms of functions $\mathbf{u} : \Omega_f \to \mathbb{R}$, $T : \Omega \to \mathbb{R}$ are defined by*

$$\begin{aligned} \|\mathbf{u}\|^2_{a,b} &= a\|\mathbf{u}\|^2 + b\|\nabla\mathbf{u}\|^2 + \alpha_1 \|(I - P_H)\nabla\mathbf{u}\|^2 \\ \|T\|^2_{a,b} &= a\|T\|^2 + b\|\nabla T\|^2 + \alpha_2 \|(I - P_K)\nabla T\|^2 \end{aligned}$$

where $a, b > 0$ are constants and α_1, α_2 are stabilizing parameters.

From now on, we denote $\min(\kappa_f, \kappa_s)$ as κ_{min} and $\max(\kappa_f, \kappa_s)$ as κ_{max} for the sake of simplicity.

Lemma 2.4 *The bilinear forms $A_0(\cdot,\cdot), A_1(\cdot,\cdot)$ are continuous and coercive with respect to corresponding weighted norms. That is, for $\mathbf{u}, \mathbf{v} \in X$, $T, S \in W$, we have*

$$\begin{aligned} A_0(\mathbf{u}, \mathbf{v}) &\leq \|\mathbf{u}\|_{1,Pr} \|\mathbf{v}\|_{1,Pr} \ , \\ A_0(\mathbf{u}, \mathbf{u}) &\geq \|\mathbf{u}\|_{CPr, \frac{Pr}{2}} \ , \\ A_1(T, S) &\leq \|T\|_{1,\kappa_{max}} \|S\|_{1,\kappa_{max}} \ , \\ A_1(T, T) &\geq \|T\|^2_{C\kappa_{min}, \frac{\kappa_{min}}{2}} \ . \end{aligned}$$

Proof. We give the proof of continuity and coercivity of A_0 now. The results for A_1 follows analogously. Using Cauchy-Schwarz inequality, addition of some extra terms and Definition 2.3, one obtains the continuity relation for A_0 as follows.

$$A_0(\mathbf{u},\mathbf{v}) \leq \Pr \|\nabla\mathbf{u}\| \|\nabla\mathbf{v}\| + \alpha_1 \|(I-P_H)\nabla\mathbf{u}\| \|(I-P_H)\nabla\mathbf{v}\|$$
$$\leq \left(\|\mathbf{u}\| + \sqrt{\Pr}\|\nabla\mathbf{u}\| + \sqrt{\alpha_1}\|(I-P_H)\nabla\mathbf{u}\|\right)\left(\|\mathbf{v}\| + \sqrt{\Pr}\|\nabla\mathbf{v}\| + \sqrt{\alpha_1}\|(I-P_H)\nabla\mathbf{v}\|\right)$$
$$\leq \|\mathbf{u}\|_{1,Pr} \|\mathbf{v}\|_{1,Pr}.$$

For coercivity, making use of Poincaré's inequality gives

$$A_0(\mathbf{u},\mathbf{u}) = \Pr\|\nabla\mathbf{u}\|^2 + \alpha_1\|(I-P_H)\nabla\mathbf{u}\|^2 = \frac{\Pr}{2}\|\nabla\mathbf{u}\|^2 + \frac{\Pr}{2}\|\nabla\mathbf{u}\|^2 + \alpha_1\|(I-P_H)\nabla\mathbf{u}\|^2$$
$$\geq C\Pr\|\mathbf{u}\|^2 + \frac{\Pr}{2}\|\nabla\mathbf{u}\|^2 + \alpha_1\|(I-P_H)\nabla\mathbf{u}\|^2 = \|\mathbf{u}\|_{CPr,\frac{Pr}{2}}$$

which completes the proof. □

Throughout this chapter, the constant C is generic constant which depends on the domain Ω and independent from h, k, H, K, α_1 and α_2 unless stated otherwise.

We now emphasize on the trilinear forms defined by (2.5)-(2.6). In the continuous case, the standard form of the convective term and skew-symmetric form of trilinear form are identical if $\nabla\cdot\mathbf{u} = 0$ and if $\mathbf{u}$ vanishes on the boundary. Since standard convective terms are not divergence free on the finite element spaces, we use the modified ones, [19]. The following important properties of trilinear forms could be obtained by making use of integration by parts.

$$c_0(\mathbf{u},\mathbf{v},\mathbf{w}) = -c_0(\mathbf{u},\mathbf{w},\mathbf{v}), \quad c_1(\mathbf{u},T,S) = -c_1(\mathbf{u},S,T)$$

and

$$c_0(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad c_1(\mathbf{u},T,T) = 0 \tag{2.24}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in X$ $T,S\in W$

Lemma 2.5 *Let $\Omega\subset\mathbb{R}^d$ with $d = 2,3$. The skew-symmetric trilinear forms then satisfy the following estimation with finite constants $C_i(\Omega)(i = 1,2,3,4)$.*

$$c_0(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C_1\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\| \tag{2.25}$$
$$c_1(\mathbf{u},T,S) \leq C_2\|\nabla\mathbf{u}\|\|\nabla T\|\|\nabla S\| \tag{2.26}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ *and* $T, S \in W$. *Furthermore, if d=3 we have*

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C_3||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\nabla\mathbf{v}||\,||\nabla\mathbf{w}|| \tag{2.27}$$

$$c_1(\mathbf{u}, T, S) \leq C_4||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\nabla T||\,||\nabla S|| \tag{2.28}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ *and* $T, S \in W$.

Proof. For the first part, using Hölder's inequality we have

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C_1||\mathbf{u}||_4\,||\nabla\mathbf{v}||\,||\mathbf{w}||_4$$

and thus the the result is obtained by using (2.17). For the second part, we use the relation

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C\,||\mathbf{u}||_{1/2}\,\,||\mathbf{v}||_1\,\,||\mathbf{w}||_1 \tag{2.29}$$

which is given in [67]. An interpolation inequality implies

$$||\mathbf{u}||_{1/2} \leq C\,||\mathbf{u}||^{1/2}\,||\nabla\mathbf{u}||^{1/2}\,. \tag{2.30}$$

The final result is obtained through combining (2.29) and (2.30) and the use of Poincaré's inequality. The results for $c_1(\mathbf{u}, T, S)$ are obtained analogously. □

The following well-known theorems are useful for the proof of existence of solution. We present them here without proofs.

Theorem 2.6 *(Lax-Milgram) Given a Hilbert space* X *with its norm* $||\cdot||_X$, *its dual* X' *and the duality pairing* $< \cdot, \cdot >$, *consider the problem*

$$a(u, v) =< f, v > \quad \forall v \in X$$

in which $a(u, v)$ *is a bilinear form defined on* $X \times X$ *and* $f \in X'$.

Suppose that, the bilinear form a is continuous and coercive on X *with finite positive constants* C, β, *i.e.,*

i) $|a(u, v)| \leq C(\Omega)\,||u||_X\,||v||_X$

ii) $a(u, u) \geq \beta\,||u||_X^2$

for all $u, v \in X$. *Then the problem,* $a(u, v) =< f, v > \quad \forall v \in X$, *has a unique solution u in* X.

Theorem 2.7 *(Leray-Schauder fixed point theorem) For a Hilbert space Y, let $\phi : Y \to Y$ be a continuous and compact mapping such that the set*

$$\{y \in Y : y = \lambda\phi(y) \quad for\ some\ \lambda \in [0, 1]\}$$

is bounded. Then ϕ has a fixed point.

Now, we also define the finite constant N_h which used throughout the thesis frequently:

$$N_h = sup\left\{c_0(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h) : \|\nabla\mathbf{v}^h\| = \|\nabla\mathbf{u}^h\| = \|\nabla\mathbf{w}^h\| = 1, \mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h \in V^h\right\}.$$

2.2 Existence and uniqueness results of discrete problem

Throughout this section, we consider the existence, uniqueness and stability properties of the discrete projection-based natural convection problem. These results without extra stabilization terms of continuous natural convection problem have been established in [11]. Using Lemma 2.4, similar results for continuous problem with stabilization can be established in the same way. For completeness, we only state and prove the existence and uniqueness of the discrete problem.

Theorem 2.8 *(Existence) The problem (2.13)-(2.14) has at least one solution.*

Proof. The proof consists of applying Lax-Milgram Theorem and Leray-Schauder Principle. Lax-Milgram Theorem guarantees the existence and uniqueness of T^k in the solution of (2.14). Note that the approximate temperature T^k depends on the velocity field $\mathbf{u}^h$. Thus we may define a mapping $F^{hk} : V^h \to W^k$ by $F^{hk}\left(\mathbf{u}^h\right) = T^k$.

Now, we show that there is at least one $\mathbf{u}^h \in V^h$ satisfying,

$$A_0(\mathbf{u}^h, \mathbf{v}^h) + c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) = Pr\ Ra\ d(T^k, \mathbf{v}^h) \tag{2.31}$$

for all $\mathbf{v}^h \in V^h$. From Lemma 2.4, $A_0(\mathbf{u}^h, \mathbf{v}^h)$ is a continuous elliptic bilinear form on $V^h \times V^h$ and

$$|-c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + Pr\ Ra\, d(F^{hk}(\mathbf{u}^h)| \leq \left(C\|\nabla\mathbf{u}^h\|^2 + Pr\ Ra\|F^{hk}(\mathbf{u}^h)\|\right)\|\nabla\mathbf{v}^h\|$$

for all $\mathbf{v}^h \in V^h$. Thus, we may define a mapping $G^h : V^h \to V^h$ by

$$A_0(G^h\left(\mathbf{u}^h\right), \mathbf{v}^h) = -c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + Pr\ Ra\ d(F^{hk}\left(\mathbf{u}^h\right), \mathbf{v}^h).$$

Note that $\mathbf{u}^h$ is a solution of (2.31) if it is a solution of

$$G^h\left(\mathbf{u}^h\right) = \mathbf{u}^h. \tag{2.32}$$

Thus, it suffices to show that there exists at least one solution to the fixed point problem (2.32). Leray-Schauder Principle guarantees the existence of a fixed point under two conditions: (i) G^h should be completely continuous (ii) there exists $\theta > 0$ such that for every $\lambda \in [0,1]$ and $\mathbf{v}^h \in V^h$ with

$$\lambda G^h(\mathbf{v}^h) = \mathbf{v}^h, \tag{2.33}$$

$\mathbf{v}^h$ should satisfy $||\nabla \mathbf{v}^h|| \leq \theta$.

Since V^h is finite dimensional, G^h is continuous and compact and thus completely continuous. This proves part (i). To prove the second condition, we consider only $\lambda \in (0,1]$ with $\lambda G^h(\mathbf{v}^h) = \mathbf{v}^h$. Then, we have

$$\lambda^{-1}A_0(\mathbf{v}^h,\mathbf{v}^h) = -c_0(\mathbf{v}^h,\mathbf{v}^h,\mathbf{v}^h) + Pr\ Ra\ d(F^{hk}\left(\mathbf{v}^h\right),\mathbf{v}^h)$$

and

$$\lambda^{-1}Pr||\nabla\mathbf{v}^h||^2 + \lambda^{-1}\alpha_1||(I-P_H)\nabla\mathbf{v}^h||^2 \leq Pr\ Ra||\nabla F^{hk}(\mathbf{v}^h)||||\nabla\mathbf{v}^h|| \leq Pr\ Ra\,\kappa_{min}^{-1}||\gamma||_{-1}||\nabla\mathbf{v}^h||.$$

Hence

$$||\nabla\mathbf{v}^h|| \leq \lambda Ra\kappa_{min}^{-1}||\gamma||_{-1}$$

which completes the proof. □

Before considering the uniqueness issue, we present some stability results.

Lemma 2.9 *(Stability of the velocity, temperature and pressure) The finite element approximation of (2.13) - (2.14) is stable in the following sense:*

(i) $\kappa_{min}||\nabla T^k||^2 + 2\alpha_2||(I-P_K)\nabla T^k||^2 \leq \kappa_{min}^{-1}||\gamma||_{-1}^2,$

(ii) $Pr||\nabla\mathbf{u}^h||^2 + 2\alpha_1||(I-P_H)\nabla\mathbf{u}^h||^2 \leq Pr\,Ra^2||T^k||_{-1}^2,$

(iii) $Pr||\nabla\mathbf{u}^h||^2 + 2\alpha_1||(I-P_H)\nabla\mathbf{u}^h||^2 \leq Pr\,Ra^2\kappa_{min}^{-2}||\gamma||_{-1}^2,$

(iv) $||p^h|| \leq C\beta^{-1}\kappa_{min}^{-1}||\gamma||_{-1}\left(Pr\ Ra + \sqrt{Pr\alpha_1}Ra + Ra^2 N_h\kappa_{min}^{-1}||\gamma||_{-1}\right).$

Proof. To prove (i), we set $S^k = T^k$ in (2.14) and apply the Young's inequality. For (ii), we set $\mathbf{u}^h = \mathbf{v}^h$ in (2.13) and use a similar argument as in (i). Combination of the parts (i) and (ii) gives (iii).

To prove part (iv), consider the equation (2.13) in X^h:

$$(p^h, \nabla \cdot \mathbf{v}^h) = A_0(\mathbf{u}^h, \mathbf{v}^h) + c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - Pr\, Ra\;\, d(T^k, \mathbf{v}^h).$$

Cauchy-Schwarz inequality and (3.6) yield

$$\begin{aligned}(p^h, \nabla \cdot \mathbf{v}^h) \quad &\leq \quad Pr\|\nabla \mathbf{u}^h\|\|\nabla \mathbf{v}^h\| + \alpha_1\|(I - P_H)\nabla \mathbf{u}^h\|\|(I - P_H)\nabla \mathbf{v}^h\| + N_h\|\nabla \mathbf{u}^h\|^2\|\nabla \mathbf{v}^h\| \\ &\quad + Pr\, Ra\|T^k\|_{-1}\|\nabla \mathbf{v}^h\|.\end{aligned}$$

Making use of the stability bounds for the velocity and temperature gives,

$$\frac{(p^h, \nabla \cdot \mathbf{v}^h)}{\|\nabla \mathbf{v}^h\|} \leq Pr\, Ra\kappa_{min}^{-1}\|\gamma\|_{-1} + \sqrt{\frac{Pr\alpha_1}{2}}Ra\kappa_{min}^{-1}\|\gamma\|_{-1} + N_h Ra^2\kappa_{min}^{-2}\|\gamma\|_{-1}^2 + Pr\, Ra\kappa_{min}^{-1}\|\gamma\|_{-1}.$$

Taking supremum over $\mathbf{v}^h \in X^h$ and using the inf sup condition (2.20) yield the desired result. □

Corollary 2.10 *Existence and uniqueness of p^h is guaranteed by part (iv) of Lemma 2.9 and the inf-sup condition (2.20), [19].*

We are now in a position to prove the global uniqueness condition of the discrete solution, which is the same as with the continuous case in [11]. First, by using the solution operator F^{hk} in Theorem 3.5, we define the following constant:

$$M_{hk} = sup\left\{\frac{d(F^{hk}(\mathbf{u}^h) - F^{hk}(\mathbf{v}^h), \mathbf{u}^h - \mathbf{v}^h)}{\|\nabla(\mathbf{u}^h - \mathbf{v}^h)\|^2}, \mathbf{u}^h \neq \mathbf{v}^h, \mathbf{u}^h, \mathbf{v}^h \in V^h\right\}. \qquad (2.34)$$

Theorem 2.11 *Suppose $N_h\|\nabla \mathbf{u}^h\| + Pr\, RaM_{hk} < Pr$. Then, $\mathbf{u}^h$ and $F^{hk}(\mathbf{u}^h) = T^k$ are unique solutions.*

Proof. Let $\mathbf{u}^h, \mathbf{w}^h \in V^h$ and $\mathbf{u}^h \neq \mathbf{w}^h$ be two solutions. Writing the equation (2.13) for $\mathbf{u}^h$ and $\mathbf{w}^h$, and subtracting them give

$$A_0(\mathbf{u}^h - \mathbf{w}^h, \mathbf{v}^h) \quad = \quad c_0(\mathbf{w}^h, \mathbf{w}^h, \mathbf{v}^h) - c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + Pr\, Ra\, d(F^{hk}(\mathbf{u}^h) - F^{hk}(\mathbf{w}^h), \mathbf{v}^h) \qquad (2.35)$$

for all $\mathbf{v}^h \in V^h$. Setting $\mathbf{v}^h = \mathbf{u}^h - \mathbf{w}^h$ in (2.35), using Cauchy-Schwarz inequality, adding and subtracting terms, using (2.24) and (2.34) lead us to

$$Pr\|\nabla(\mathbf{u}^h-\mathbf{w}^h)\|^2+\alpha_1\|(I-P_H)\nabla(\mathbf{u}^h-\mathbf{w}^h)\|^2 \leq N_h\|\nabla(\mathbf{u}^h-\mathbf{w}^h)\|^2\|\nabla\mathbf{u}^h\|+Pr\ Ra\|\nabla(\mathbf{u}^h-\mathbf{w}^h)\|^2 M_{hk}.$$

So,

$$\left(Pr - (N_h\|\nabla\mathbf{u}^h\| + Pr\ RaM_{hk})\right)\|\nabla(\mathbf{u}^h - \mathbf{w}^h)\|^2 + \alpha_1\|(I - P_H)\nabla(\mathbf{u}^h - \mathbf{w}^h)\|^2 \leq 0$$

Since $(N_h\|\nabla\mathbf{u}^h\| + Pr\ RaM_{hk}) < Pr$, we have a contradiction. Therefore, $\mathbf{u}^h = \mathbf{v}^h$.

□

Remark 2.12 *If one uses the results of Lemma 2.9, global uniqueness condition,* $N_h\|\nabla\mathbf{u}^h\| + Pr\ RaM_{hk} < Pr$ *can be reformulated as* $Ra\kappa_{min}^{-1}\|\gamma\|_{-1}(N_h + Pr\ C_2\kappa_{min}^{-1}) < Pr$ *in terms problem data.*

Furthermore, global uniqueness condition of the discrete problem ensures $\mathbf{u}^h$ *to be a fixed point of a contractive map in* V^h*, [44].*

2.3 A priori error estimation

This section states a priori error estimation for the velocity and temperature. Before giving the main theorem, we define so-called *modified Stokes projection* operators. Lemma 2.4, hence Lax-Milgram theorem guarantees the existence of such projection operators for both velocity and temperature. When we split the errors into approximation terms and a finite element remainder for $\mathbf{u}$ and T, the use of such operators simplifies the approximation terms and so the error estimations. We first state the stability of these projections and give the related error bounds.

Definition 2.13 *(Modified Stokes projections for the velocity and temperature) The operator of the modified Stokes projection for the velocity and pressure,* P_S*, is defined by;* $P_S : (X, Q) \to (X^h, Q^h)$, $\quad P_S(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p})$ *where*

$$\begin{aligned} A_0(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}^h) + b(\mathbf{v}^h, p - \tilde{p}) &= 0, \\ b(\mathbf{u} - \tilde{\mathbf{u}}, q^h) &= 0 \end{aligned}$$

for all $(\mathbf{v}^h, q^h) \in (X^h, Q^h)$*. In the discretely divergence free space* V^h *and in the pressure space* Q^h*, this definition reduces to*

$$A_0(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}^h) + b(\mathbf{v}^h, p - q^h) = 0 \tag{2.36}$$

for all $\mathbf{v}^h \in V^h$*. The modified Stokes projection operator for the temperature,* P_T*, is defined by* $P_T : W \to W^k$*,* $P_T(T) = \tilde{T}$ *where*

$$A_1(T - \tilde{T}, S^k) \quad = \quad 0 \tag{2.37}$$

for all $S^k \in W^k$*.*

Lemma 2.14 *(Stability of modified Stokes projections) The modified Stokes projections defined by (2.36) and (2.37) are stable in the following sense:*

$$Pr||\nabla\tilde{\mathbf{u}}||^2 + \alpha_1||(I - P_H)\nabla\tilde{\mathbf{u}}||^2 \quad \leq \quad C(Pr||\nabla\mathbf{u}||^2 + \alpha_1||(I - P_H)\nabla\mathbf{u}||^2 + Pr^{-1}||p - q^h||^2) \tag{2.38}$$

$$\kappa_{min}||\nabla\tilde{T}||^2 + \alpha_2||(I - P_K)\nabla\tilde{T}||^2 \quad \leq \quad C(\frac{\kappa_{max}^2}{\kappa_{min}}||\nabla T||^2 + \alpha_2||(I - P_K)\nabla T||^2). \tag{2.39}$$

Proof. For the proof of (2.38), first set $\mathbf{v}^h = \tilde{\mathbf{u}}$ in (3.47) and use Cauchy-Schwarz inequality:

$$\begin{aligned} Pr||\nabla\tilde{\mathbf{u}}||^2 + \alpha_1||(I - P_H)\nabla\tilde{\mathbf{u}}||^2 \quad \leq \quad & Pr||\nabla\mathbf{u}||||\nabla\tilde{\mathbf{u}}|| + \alpha_1||(I - P_H)\nabla\mathbf{u}||||(I - P_H)\nabla\tilde{\mathbf{u}})|| \\ & + ||p - q^h||||\nabla \cdot \tilde{\mathbf{u}}||. \end{aligned}$$

Young's inequality and combining terms give the result.

The stability of the modified Stokes projection of temperature is established by writing $S^k = \tilde{T}$ in (2.37) and using similar arguments as in the first part. □

The next lemma states the error in those projection operators.

Lemma 2.15 *(Error in modified Stokes projections) Suppose the discrete inf-sup condition (2.20) holds. Then* $(\tilde{\mathbf{u}}, \tilde{T})$ *exists uniquely in* (X^h, Q^h, W^k) *and satisfies*

$$\begin{aligned} Pr||\nabla(\mathbf{u} - \tilde{\mathbf{u}})||^2 + \alpha_1||(I - P_H)\nabla(\mathbf{u} - \tilde{\mathbf{u}})||^2 \leq C(\inf_{\hat{\mathbf{u}} \in X^h} ||\nabla(\mathbf{u} - \hat{\mathbf{u}})||^2 \\ + \alpha_1 \inf_{\hat{\mathbf{u}} \in X^h} ||(I - P_H)\nabla(\mathbf{u} - \hat{\mathbf{u}})||^2 + Pr^{-1} \inf_{q^h \in Q^h} ||p - q^h||^2), \end{aligned} \tag{2.40}$$

$$\begin{aligned} \kappa_{min}||\nabla(T - \tilde{T})||^2 + \alpha_2||(I - P_K)\nabla(T - \tilde{T})||^2 \leq C(\kappa_{max}^2\kappa_{min}^{-1} \inf_{\hat{T} \in W^k} ||\nabla(T - \hat{T})||^2 \\ + \alpha_2 \inf_{\hat{T} \in W^k} ||(I - P_K)\nabla(T - \hat{T})||^2). \end{aligned} \tag{2.41}$$

Proof. To prove (2.40), let $\mathbf{e} = \mathbf{u} - \tilde{\mathbf{u}}$ and decompose the error $\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\phi}^h$, where $\boldsymbol{\eta} = \mathbf{u} - \hat{\mathbf{u}}$, $\boldsymbol{\phi} = \tilde{\mathbf{u}} - \hat{\mathbf{u}}$. Here $\hat{\mathbf{u}}$ is the approximation of $\mathbf{u}$ in V^h. Thus (2.36) reads as

$$\begin{aligned} &Pr(\nabla\boldsymbol{\phi}^h, \nabla\mathbf{v}^h) + \alpha_1((I-P_H)\nabla\boldsymbol{\phi}^h, (I-P_H)\nabla\mathbf{v}^h) \\ &= Pr(\nabla\boldsymbol{\eta}, \nabla\mathbf{v}^h) + \alpha_1((I-P_H)\nabla\boldsymbol{\eta}, (I-P_H)\nabla\mathbf{v}^h) + (p - q^h, \nabla.\mathbf{v}^h) \end{aligned} \tag{2.42}$$

Setting $\mathbf{v}^h = \boldsymbol{\phi}^h$ in (2.42) and applying Cauchy-Schwarz and Young's inequalities direct us to

$$\frac{Pr}{2}\|\nabla\boldsymbol{\phi}^h\|^2 + \frac{\alpha_1}{2}\|(I-P_H)\nabla\boldsymbol{\phi}^h\|^2 \leq C(Pr\|\nabla\boldsymbol{\eta}\|^2 + \frac{\alpha_1}{2}\|(I-P_H)\nabla\boldsymbol{\eta}\|^2 + Pr^{-1}\|p-q^h\|^2). \tag{2.43}$$

Since $\hat{\mathbf{u}}$ is an approximation of $\mathbf{u}$ in V^h, we can take infimum over V^h in (2.43). Recall that under the discrete inf-sup condition (2.20) and $\nabla \cdot \mathbf{u} = 0$, the infimum can be replaced by X^h, [19]. The stated error estimate now follows from the triangle inequality.

To prove (2.41), define $T - \tilde{T} = \tilde{e} = (T - \hat{T}) - (\tilde{T} - \hat{T}) = \chi - \xi^k$ where $\hat{T}$ approximates T in W^k. As in the first part, if one sets $S^k = \xi^k$ and uses Cauchy-Schwarz and Young's inequalities, the following estimation is obtained:

$$\kappa_{min}\|\nabla\xi^k\|^2 + \alpha_2\|(I-P_K)\nabla\xi^k\|^2 \leq C\kappa_{max}^2\kappa_{min}^{-1}\|\nabla\chi\|^2 + \alpha_2\|(I-P_K)\nabla\chi\|^2.$$

Taking infimum over W^k and applying the triangle inequality complete the proof. □

We now give our main theorems. Since the equations are coupled in (2.13)-(2.14), the error estimations are also coupled. Now we first state the error estimation for $T - T^k$ in terms of the error in $\mathbf{u} - \mathbf{u}^h$.

Theorem 2.16 *The error for $T - T^k$ satisfies*

$$\begin{aligned} \kappa_{min}\|\nabla(T-T^k)\|^2 + \alpha_2\|(I-P_K)\nabla(T-T^k)\|^2 \leq C(\kappa_{min}^{-1} \inf_{\tilde{T}\in W^k} \|\nabla\mathbf{u}\|^2\|\nabla(T-\tilde{T})\|^2 \\ +\alpha_2\|(I-P_K)\nabla T\|^2) + C_2^2\kappa_{min}^{-3}\|\gamma\|_{-1}^2 \left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|^2 \end{aligned}$$

Proof. Making use of (2.8) and (2.14) gives the error equation:

$$A_1(\tilde{e}, S^k) + c_1(\mathbf{u}, T, S^k) - c_1(\mathbf{u}^h, T^k, S^k) = \alpha_2((I-P_K)\nabla T, (I-P_K)\nabla S^k) \tag{2.44}$$

for all $S^k \in W^k$ where $\tilde{e} = T - T^k$. Decompose the error as an approximation term and a finite element remainder: $\tilde{e} = (T - \tilde{T}) - (T^k - \tilde{T}) = \chi - \xi^k$. Here, $\tilde{T}$ denotes the modified Stokes projection of T defined by (2.39). Now, set $S^k = \xi^k$ into the error equation (2.44). With a rearrangement of terms, we obtain

$$|A_1(\xi^k,\xi^k)| \leq |c_1(\mathbf{u},T,\xi^k) - c_1(\mathbf{u}^h,T^k,\xi^k)| + |\alpha_2((I-P_K)\nabla T,(I-P_K)\nabla\xi^k)|. \tag{2.45}$$

Note that $A_1(\chi,\xi^k) = 0$ due to the definition of the modified Stokes projection. Now, let us bound each term on the right hand side of (2.45):

$$\begin{aligned}
|c_1(\mathbf{u},T,\xi^k) - c_1(\mathbf{u}^h,T^k,\xi^k)| &= |c_1(\mathbf{u},\chi,\xi^k) - c_1(\mathbf{u}-\mathbf{u}^h,T^k,\xi^k)| \\
&\leq C\kappa_{min}^{-1}\|\nabla\mathbf{u}\|^2\|\nabla\chi\|^2 + \frac{\kappa_{min}}{4}\|\nabla\xi^k\|^2 + C_2^2\kappa_{min}^{-1}\|\nabla(\mathbf{u}-\mathbf{u}^h)\|^2\|\nabla T^k\|^2, \\
|\alpha_2((I-P_K)\nabla T,(I-P_K)\nabla\xi^k)| &\leq \frac{\alpha_2}{2}\|(I-P_K)\nabla T\|^2 + \frac{\alpha_2}{2}\|(I-P_K)\nabla\xi^k\|^2.
\end{aligned}$$

Thus, bounding the terms as shown above for (2.45) results in

$$\begin{aligned}
\frac{\kappa_{min}}{2}\|\nabla\xi^k\|^2 + \frac{\alpha_2}{2}\|(I-P_K)\nabla\xi^k\|^2 &\leq C\kappa_{min}^{-1}\|\nabla\mathbf{u}\|^2\|\nabla\chi\|^2 + C_2^2\kappa_{min}^{-1}\|\nabla(\mathbf{u}-\mathbf{u}^h)\|^2\|\nabla T^k\|^2 \\
&+\frac{\alpha_2}{2}\|(I-P_K)\nabla T\|^2
\end{aligned}$$

Combination of terms and application of the triangle inequality yield the stated error estimation. □

The error estimation for the velocity is proved next. This error estimation uses Theorem 2.16.

Theorem 2.17 *Under the condition* $Ra\kappa_{min}^{-1}\|\gamma\|_{-1}(N_h + \frac{3}{2}C_2^2 Pr\, Ra\kappa_{min}^{-3}\|\gamma\|_{-1}^3) < Pr$, *the error satisfies*

$$\begin{aligned}
& Pr\|\nabla(\mathbf{u}-\mathbf{u}^h)\|^2 + \alpha_1\|(I-P_H)\nabla(\mathbf{u}-\mathbf{u}^h)\|^2 \\
\leq\ & C\Big\{M_1\Big[\inf_{\tilde{\mathbf{u}}\in X^h}\|\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^2 + Pr^{-1}\alpha_1\inf_{\tilde{\mathbf{u}}\in X^h}\|(I-P_H)\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^2 + Pr^{-2}\inf_{q^h\in Q^h}\|p-q^h\|^2\Big] \\
& +M_2\Big[\kappa_{min}^{-2}\inf_{\tilde{T}\in W^k}\|\nabla(T-\tilde{T})\|^2 + \kappa_{min}^{-1}\alpha_2\inf_{\tilde{T}\in W^k}\|(I-P_K)\nabla(T-\tilde{T})\|^2\Big] \\
& +\alpha_1\|(I-P_H)\nabla\mathbf{u}\|^2 + Pr\, Ra^2\kappa_{min}^{-1}\alpha_2\|(I-P_K)\nabla T\|^2\Big\}.
\end{aligned}$$

where C_2 *is as in (2.26) and* M_1 *and* M_2 *are also constants which are defined below explicitly:*

$$\begin{aligned}
M_1 &= C\left[Pr^{-1}\kappa_{min}^{-2}Ra^2\|\gamma\|_{-1}^2 + Pr\, Ra^2\kappa_{min}^{-4}\|\gamma\|_{-1}^2\right], \\
M_2 &= C\left[PrRa^4\frac{\kappa_{max}^2}{\kappa_{min}^6}\|\gamma\|_{-1}^2\right].
\end{aligned}$$

Proof. The use of (2.8) and (2.13) results with the error equation:

$$\begin{aligned} & A_0(\mathbf{e},\mathbf{v}^h)+c_0(\mathbf{u},\mathbf{u},\mathbf{v}^h)-c_0(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)+b(\mathbf{v}^h,p-q^h) \\ & = Pr\ Ra\ d(\tilde{e},\mathbf{v}^h)+\alpha_1((I-P_H)\nabla\mathbf{u},(I-P_H)\nabla\mathbf{v}^h) \end{aligned} \tag{2.46}$$

for all $(\mathbf{v}^h,q^h)\in(V^h,Q^h)$ where $\mathbf{e}=\mathbf{u}-\mathbf{u}^h$ and $\tilde{e}=T-T^k$. Split the errors as $\mathbf{e}=\boldsymbol{\eta}-\boldsymbol{\phi}^h$ where $\boldsymbol{\eta}=(\mathbf{u}-\tilde{\mathbf{u}})$, $\quad\boldsymbol{\phi}^h=(\mathbf{u}^h-\tilde{\mathbf{u}})$ and $\tilde{e}=\chi-\xi^k$ where $\chi=(T-\tilde{T})$, $\quad\xi^k=(T^k-\tilde{T})$. Note that $\tilde{\mathbf{u}}$ and $\tilde{T}$ denote the modified Stokes projections of $\mathbf{u}$ and T, respectively. Now, writing $\mathbf{v}^h$=$\boldsymbol{\phi}^h$ in (2.46) yields;

$$\begin{aligned} & A_0(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h)=A_0(\boldsymbol{\eta},\boldsymbol{\phi}^h)+b(\boldsymbol{\phi}^h,p-q^h)+c_0(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h)-c_0(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h) \\ & +\alpha_1((I-P_H)\nabla\mathbf{u},(I-P_H)\nabla\boldsymbol{\phi}^h)+Pr\ Ra\ d(\tilde{e},\boldsymbol{\phi}^h). \end{aligned} \tag{2.47}$$

Note that, $A_0(\boldsymbol{\eta},\boldsymbol{\phi}^h)+b(\boldsymbol{\phi}^h,p-q^h)=0$ by the definition of the modified Stokes projection. To bound the terms on the right hand side of (2.47), we first consider the nonlinear terms. Adding, subtracting terms and observing the skew-symmetry of convective term yield

$$|c_0(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h)-c_0(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h)|=|c_0(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\phi}^h)+c_0(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\phi}^h)-c_0(\boldsymbol{\phi}^h,\mathbf{u}^h,\boldsymbol{\phi}^h)|.$$

Inequalities Cauchy-Schwarz, (2.25) and Young's give

$$\begin{aligned} |c_0(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\phi}^h)| &\le CPr^{-1}||\nabla\mathbf{u}||^2||\nabla\boldsymbol{\eta}||^2+\frac{Pr}{6}||\nabla\boldsymbol{\phi}^h||^2, \\ |c_0(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\phi}^h)| &\le CPr^{-1}||\nabla\boldsymbol{\eta}||^2||\nabla\mathbf{u}^h||^2+\frac{Pr}{6}||\nabla\boldsymbol{\phi}^h||^2, \\ |c_0(\boldsymbol{\phi}^h,\mathbf{u}^h,\boldsymbol{\phi}^h)| &\le N_h||\nabla\boldsymbol{\phi}^h||^2||\nabla\mathbf{u}^h||. \end{aligned}$$

Similarly, consistency term and the last term on the right hand side of (2.47) are bounded with

$$|\alpha_1((I-P_H)\nabla\mathbf{u},(I-P_H)\nabla\boldsymbol{\phi}^h)|\le\frac{\alpha_1}{2}||(I-P_H)\nabla\mathbf{u}||^2+\frac{\alpha_1}{2}||(I-P_H)\nabla\boldsymbol{\phi}^h||^2$$

and

$$|Pr\ Ra\ d(T-T^k,\boldsymbol{\phi}^h)|\le\frac{3}{2}Pr\ Ra^2||T-T^k||_{-1}^2+\frac{Pr}{6}||\nabla\boldsymbol{\phi}^h||^2.$$

Combining all the terms involving $\boldsymbol{\phi}^h$ on the left hand side gives,

$$\begin{aligned} & \left(\frac{Pr}{2}-N_h||\nabla\mathbf{u}^h||\right)||\nabla\boldsymbol{\phi}^h||^2+\frac{\alpha_1}{2}||(I-P_H)\nabla\boldsymbol{\phi}^h||^2 \\ & \le C(Pr^{-1}||\nabla\boldsymbol{\eta}||^2\left(||\nabla\mathbf{u}||^2+||\nabla\mathbf{u}^h||^2\right)+\alpha_1||(I-P_H)\nabla\mathbf{u}||^2)+\frac{3}{2}Pr\ Ra^2||T-T^k||_{-1}^2 \end{aligned} \tag{2.48}$$

Clearly, the next step we should follow is to find a bound for the term $||T - T^k||_{-1}^2$. In order to do that, we write $\mathbf{u} - \mathbf{u}^h = \boldsymbol{\eta} - \boldsymbol{\phi}^h$ in the statement of Theorem 2.16 and plug in estimation in (2.48). Rearranging the terms yields

$$\left(\frac{Pr}{2} - N_h||\nabla \mathbf{u}^h|| - \frac{3}{2}Pr\,Ra^2 C_2^2 \kappa_{min}^{-4}||\gamma||_{-1}^2\right)||\nabla \boldsymbol{\phi}^h||^2 + \frac{\alpha_1}{2}||(I - P_H)\nabla \boldsymbol{\phi}^h||^2$$
$$\leq C(Pr^{-1}||\nabla \boldsymbol{\eta}||^2(||\nabla \mathbf{u}||^2 + ||\nabla \mathbf{u}^h||^2) + \alpha_1||(I - P_H)\nabla \mathbf{u}||^2 + Pr\,Ra^2\kappa_{min}^{-4}||\gamma||_{-1}^2||\nabla \boldsymbol{\eta}||^2$$
$$+Pr\,Ra^2\kappa_{min}^{-2}||\nabla \mathbf{u}||^2||\nabla \chi||^2 + Pr\,Ra^2\kappa_{min}^{-1}\alpha_2||(I - P_K)\nabla T||^2). \tag{2.49}$$

Let us consider the coefficient of the term $||\nabla \boldsymbol{\phi}^h||^2$. Making use of the uniqueness bound and the assumption of the theorem, we have

$$\frac{Pr}{2} < \frac{Pr}{2} - N_h Ra\kappa_{min}^{-1}||\gamma||_{-1} - \frac{3}{2}Pr\,Ra^2 C_2^2\kappa_{min}^{-4}||\gamma||_{-1}^2. \tag{2.50}$$

Plugging (2.50) into (2.49) and writing the stability bounds for the terms we have,

$$Pr||\nabla \boldsymbol{\phi}^h||^2 + \alpha_1||(I - P_H)\nabla \boldsymbol{\phi}^h||^2 \leq C(Pr^{-1}\kappa_{min}^{-2}Ra^2||\gamma||_{-1}^2||\nabla \boldsymbol{\eta}||^2$$
$$+Pr\,Ra^2\kappa_{min}^{-4}||\gamma||_{-1}^2||\nabla \boldsymbol{\eta}||^2 + Pr\,Ra^4\kappa_{min}^{-4}||\gamma||_{-1}^2||\nabla \chi||^2 + \alpha_1||(I - P_H)\nabla \mathbf{u}||^2$$
$$+Pr\,Ra^2\kappa_{min}^{-1}\alpha_2||(I - P_K)\nabla T||^2). \tag{2.51}$$

Substituting the error bounds of Lemma 2.15 into (2.51) and applying the triangle inequality complete the proof. □

One might see also that the addition of the extra term in (2.13)-(2.14) does not degrade the order of convergence. To see this, we give the following Remark.

Remark 2.18 *If we assume the regularity assumptions,* $(\mathbf{u}, p, T) \in (X \cap H^{s+1}(\Omega),\ Q \cap H^s(\Omega),\ W \cap H^{s+1}(\Omega))$ *and the use of the estimations (2.21), (2.22) and (2.23) yield*

$$Pr||\nabla(u - u^h)||^2 + \alpha_1||(I - P_H)\nabla(u - u^h)||^2 \leq M_1((h^{2s}|u|_{s+1}^2(1 + Pr^{-1}\alpha_1) + Pr^{-2}h^{2s}|p|_s^2)$$
$$+M_2(\kappa_{min}^{-1}k^{2s}|T|_{s+1}^2(\kappa_{min}^{-1} + \alpha_2)) + \alpha_1 H^{2s}|u|_{s+1}^2 + \alpha_2 K^{2s}|T|_{s+1}^2). \tag{2.52}$$

Here h, k are given and by equilibrating the orders of convergence, appropriate values for the mesh scales H, K and parameters α_1, α_2 are chosen. That is, the error is optimal for $\alpha_1 H^{2s} = h^{2s}$ and $\alpha_2 K^{2s} = k^{2s}$. For instance, let us consider the case for $s = 2$ and use Taylor-Hood finite element pairs, satisfying the inf-sup condition (2.20), which are given below explicitly *along with the choices of* $L^H = \nabla X^H$ *and* $M^K = \nabla W^K$ *:*

$$\begin{aligned}
X^h &= \{\mathbf{v} \in C^0(\overline{\Omega}) : \mathbf{v}|_\Delta \in P_2(\Delta), \forall \Delta \in \mathcal{F}^h\},\\
W^k &= \{S \in C^0(\overline{\Omega}) : S|_\Delta \in P_2(\Delta), \forall \Delta \in \mathcal{G}^k\},\\
Q^h &= \{\mathbf{v} \in C^0(\overline{\Omega}) : \mathbf{v}|_\Delta \in P_1(\Delta), \forall \Delta \in \mathcal{F}^h\},\\
L^H &= \{l^H \in L^2(\Omega) : l^H|_\Delta \in P_1(\Delta), \forall \Delta \in \mathcal{F}^H\},\\
M^K &= \{m^K \in L^2(\Omega) : m^K|_\Delta \in P_1(\Delta), \forall \Delta \in \mathcal{G}^K\},
\end{aligned}$$

If we make the same assumptions as in Theorem 2.17 and consider (2.52), one can imply that along with the choices of $(\alpha_1, H) = (h^2, h^{1/2})$ *the error*

$$||\nabla(\mathbf{u} - \mathbf{u}^h)|| \cong \mathcal{O}(h^2 + k^2)$$

is optimal for the velocity.

If we plug the results obtained for velocity into Theorem 2.16 and carry out similar computations, we have

$$||\nabla(T - T^k)|| \cong \mathcal{O}(h^2 + k^2).$$

Similarly, for the choices of $(\alpha_2, K) = (k^2, k^{1/2})$ *we have the optimal error for the temperature.*

2.4 Error estimation for pressure

This section deals with the estimation of the error for the discrete pressure in the L^2 norm.

Theorem 2.19 *(Error estimate for pressure) Suppose that the assumptions of the Theorem 2.17 hold. Then the error* $p - p^h$ *satisfies*

$$\begin{aligned}
||p - p^h|| \leq C\Big(&(Pr + ||\nabla\mathbf{u}||)||\nabla(\mathbf{u} - \mathbf{u}^h)|| + ||\nabla(\mathbf{u} - \mathbf{u}^h)||^2 + \alpha_1||(I - P_H)\nabla(\mathbf{u} - \mathbf{u}^h)||\\
&+ \inf_{q^h \in Q^h} ||p - q^h|| + Pr\, Ra||(T - T^k)||_{-1} + \alpha_1||(I - P_H)\nabla\mathbf{u}||\Big).
\end{aligned}$$

Proof. To prove this, we consider (2.46). Let $p - p^h = (p - \tilde{p}) - (p^h - \tilde{p})$, where $\tilde{p}$ is an approximation of the pressure in Q^h. (2.46) reads as;

$$\begin{aligned}
&b(\mathbf{v}^h, p^h - \tilde{p}) = A_0(\mathbf{e}, \mathbf{v}^h) + (c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h)) + b(\mathbf{v}^h, p - \tilde{p}) - Pr\, Ra\ d(\tilde{e}, \mathbf{v}^h)\\
&-\alpha_1((I - P_H)\nabla\mathbf{u}, (I - P_H)\nabla\mathbf{v}^h).
\end{aligned}$$

We first consider here the nonlinear terms. Adding and subtracting terms and using (2.25) yield

$$
\begin{aligned}
&|c_0(\mathbf{u},\mathbf{u},\mathbf{v}^h) - c_0(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)| = | - c_0(\mathbf{e},\mathbf{e},\mathbf{v}^h) + c_0(\mathbf{e},\mathbf{u},\mathbf{v}^h) + c_0(\mathbf{u},\mathbf{e},\mathbf{v}^h)|\\
&\le C(||\nabla\mathbf{e}|| + ||\nabla\mathbf{u}||)||\nabla\mathbf{e}||||\nabla\mathbf{v}^h||.
\end{aligned}
$$

Bounds for the other terms are obtained in a similar manner as in the estimation of the error $||\nabla(\mathbf{u} - \mathbf{u}^h)||$. Hence

$$
\begin{aligned}
|b(\mathbf{v}^h, p^h - \tilde{p})| \le C||\nabla\mathbf{v}^h||(Pr||\nabla\mathbf{e}|| + \alpha_1||(I - P_H)\nabla\mathbf{e}|| + (||\nabla\mathbf{e}|| + ||\nabla\mathbf{u}||)||\nabla\mathbf{e}||\\
+||p - \tilde{p}|| + Pr\, Ra||T - T^k||_{-1} + \alpha_1||(I - P_H)\nabla\mathbf{u}||) \qquad (2.53)
\end{aligned}
$$

Notice that (2.20) implies,

$$
(p^h - \tilde{p}, \nabla\cdot\mathbf{v}^h) \ge \beta||\nabla\mathbf{v}^h||||p^h - \tilde{p}||
$$

and using this relation yields,

$$
||p - p^h|| \le ||p - \tilde{p}|| + ||\tilde{p} - p^h|| \le ||p - \tilde{p}|| + \beta^{-1}\frac{|b(\mathbf{v}^h, p^h - \tilde{p})|}{||\nabla\mathbf{v}^h||}. \qquad (2.54)
$$

Substituting (2.53) into (2.54) taking infimum over Q^h give us the desired result. □

Remark 2.20 *Making use of Taylor-Hood elements as in Remark 2.18 with the choices* $(\alpha_1, H) = (h^2, h^{1/2})$ *and* $(\alpha_2, K) = (k^2, k^{1/2})$ *and using the approximation results (2.21)-(2.22) for the velocity and temperature errors, we have*

$$
||p - p^h|| \cong \mathcal{O}(h^2 + k^2)
$$

which is the optimal error.

2.5 Numerical Studies

In this section, numerical studies are given in order to show the effectiveness of the method and validate the obtained theoretical results. The projection-based stabilization method for steady natural convection problem has been assessed on two numerical examples in two dimensions. The first example is a well-known test case for the natural convection codes which is called buoyancy-driven cavity problem. For the other test case, we consider a known particular analytical solution in order to check the error rates.

All computations are carried out by using the software $FreeFem++$ [24]. In both examples, we use conforming Taylor-Hood finite element pairs. It is well known that these pairs fullfill the inf-sup condition (2.20) (see [23]). Finite element spaces are given in Remark 2.18 and Remark 2.20 with the algorithmic choices for the size of the meshes and the parameters: $H \sim h^{1/2}$ and $K \sim k^{1/2}$, $\alpha_1 = h^2$, $\alpha_2 = k^2$. Since we solve the problem on the same mesh, we let $h = k$ and $H = K$.

To handle the nonlinearity of the system, the Newton method of [23] is used. The algorithm consists of starting with an initial guess $(\mathbf{u}^{(0)}, T^{(0)})$ and then generate the sequence of iterates $(\mathbf{u}^{(m)} \in X^h, p^{(m)} \in Q^h$ and $T^{(m)} \in W^k)$ for $m \geq 1$ by solving the sequence of linear systems

$$
\begin{aligned}
&Pr\ a_0(\mathbf{u}^{(m)}, \mathbf{v}^h) + c_0(\mathbf{u}^{(m-1)}, \mathbf{u}^{(m)}, \mathbf{v}^h) + c_0(\mathbf{u}^{(m)}, \mathbf{u}^{(m-1)}, \mathbf{v}^h) + b(\mathbf{v}^h, p^{(m)}) = Pr\ Ra\ d(T^{(m)}, \mathbf{v}^h)\\
&+c_0(\mathbf{u}^{(m-1)}, \mathbf{u}^{(m-1)}, \mathbf{v}^h) - \alpha_1((I - P_H)\nabla \mathbf{u}^{(m-1)}, (I - P_H)\nabla \mathbf{v}^h)\\
&b(\mathbf{u}^{(m)}, q^h) = 0\\
&a_1(T^{(m)}, S^k) + c_1(\mathbf{u}^{(m)}, T^{(m-1)}, S^k) + c_1(\mathbf{u}^{(m-1)}, T^{(m)}, S^k) = (\gamma, S^k) + c_1(\mathbf{u}^{(m-1)}, T^{(m-1)}, S^k)\\
&-\alpha_2((I - P_H)\nabla T^{(m-1)}, (I - P_H)\nabla S^k)
\end{aligned}
$$

for all $(\mathbf{v}^h, q^h, S^k) \in (X^h, Q^h, W^k)$.

This scheme is known to be locally convergent at least either or both T and $\mathbf{u} \cdot \mathbf{n}$ are specified at every point of the boundary.

2.5.1 Buoyancy-Driven Cavity Problem

The problem of buoyancy-driven cavity is used as a suitable benchmark for testing the natural convection codes in the literature. The simplicity of geometry and clear boundary conditions make this problem attractive. The domain consists of a square cavity with differentially heated vertical walls where right and left walls are kept at T_C and T_H, respectively, with $T_H > T_C$. The remaining walls are insulated and there is no heat transfer through them. The boundary conditions are no-slip boundary conditions for the velocity at all four walls ($\mathbf{u} = 0$) and Dirichlet boundary conditions for the temperature at vertical walls. As the horizontal walls are adiabatic, we employ $\frac{\partial T}{\partial n} = 0$ here. Figure 2.2 shows the physical domain of the buoyancy-driven cavity flow problem. In the test case, we take $\kappa = 1$, $\gamma = 0$, $T_C = 0$ and $T_H = 1$. While we consider the air as the cavity filling fluid in our model, we take the fixed value $Pr = 0.71$. We have performed our computations for Rayleigh number varying from 10^3 to 10^6. The performance of the projection-based stabilization is compared with the famous benchmark solutions of de Vahl Davis [16] and some other authors such as Massarotti et al. [51], Manzari [50], and the more recent study of Wan et al. [70]. From these benchmark solutions, [16] used second-order central approximations to solve natural convection problem in a square cavity. [51] developed a semi-implicit form of the characteristic-based split scheme and [50] employed an explicit finite element algorithm. Recently, [70] used discrete singular convolution for the solution of the problem. We also include the results for the classical Galerkin Finite Element Method (GFEM) where we keep the same mesh sizes for the proposed method and GFEM. Numerical simulations are obtained for three uniform grids of $11 \times 11, 21 \times 21$ and 32×32.

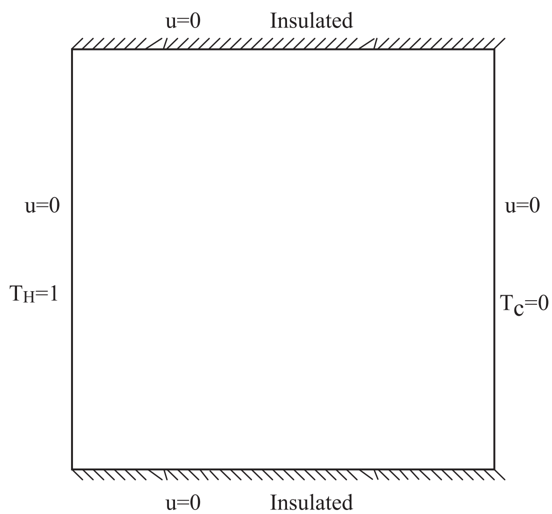

Figure 2.2: The physical domain with its boundary conditions

We start our illustrations by giving peak values of vertical velocity at $y = 0.5$ and horizontal velocity at $x = 0.5$. Table 2.1 and Table 2.2 summarize the maximum vertical velocity values at mid-height and at mid-width for different Rayleigh numbers. For quantitative assessment, we also include those velocity values obtained by [16], [51], [50] and [70].

Table 2.1: Comparison of maximum vertical velocity at $y = 0.5$ with mesh size used in computation.

Ra	GFEM	Present Study	Ref. [16]	Ref. [51]	Ref. [50]	Ref. [70]
10^4	16.41(11×11)	19.91(11×11)	19.51(41×41)	19.63(71×71)	19.90(71×71)	19.79(101×101)
10^5	51.22(21×21)	70.60(21×21)	68.22(81×81)	68.85(71×71)	70.00(71×71)	70.63(101×101)
10^6	201.20(32×32)	228.12(32×32)	216.75(81×81)	221.60(71×71)	228.00(71×71)	227.11(101×101)

As can be observed, the results of our computations are in an excellent agreement with the benchmark data even at coarser grid. We also see that as the Rayleigh number increases, GFEM yields results which are not so close to the benchmark solutions.

Table 2.2: Comparison of maximum horizontal velocity at $x = 0.5$ with mesh size used in computation.

Ra	GFEM	Present Study	Ref. [16]	Ref. [50]	Ref. [70]
10^4	15.70(11×11)	15.90 (11×11)	16.18(41×41)	16.10(71×71)	16.10(101×101)
10^5	41.00(21×21)	33.51(21×21)	34.81(81×81)	34.00(71×71)	34.00(101×101)
10^6	80.25(32×32)	65.52(32×32)	65.33(81×81)	65.40(71×71)	65.40(101×101)

We also present the vertical velocity distribution at the mid-height and horizontal velocity distribution at the mid-width in Figure 2.3, respectively, which are very popular graphical illustrations in the study of buoyancy-driven cavity type tests. These profiles are also comparable with the similar ones in [70]. It is obvious that as Rayleigh numbers increases, the differences in the profiles presented in Figure 2.3 and Figure 2.4 are getting larger. For higher Rayleigh number circulation gets more pronounced but with decreasing viscous effects, the flow becomes more and more restricted to the walls for higher velocity gradients established near the walls.

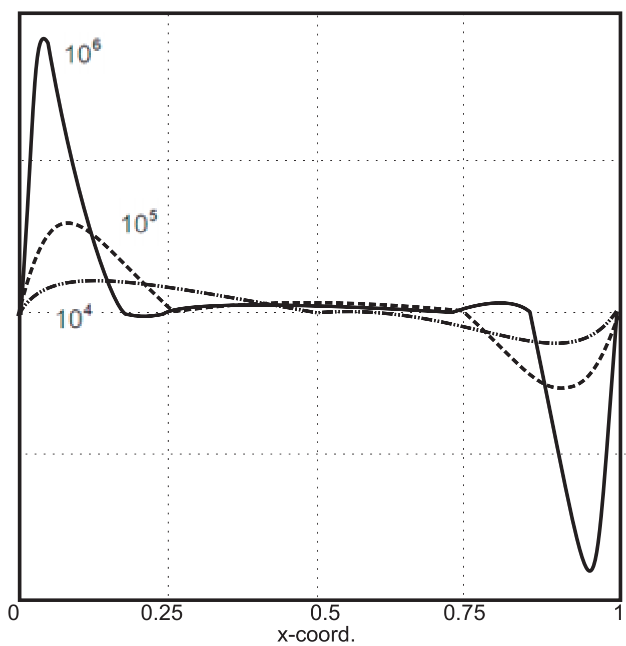

Figure 2.3: Variation of vertical velocity at mid-height for varying Rayleigh numbers.

A very important property of the natural convection flows, especially for engineers, is the rate

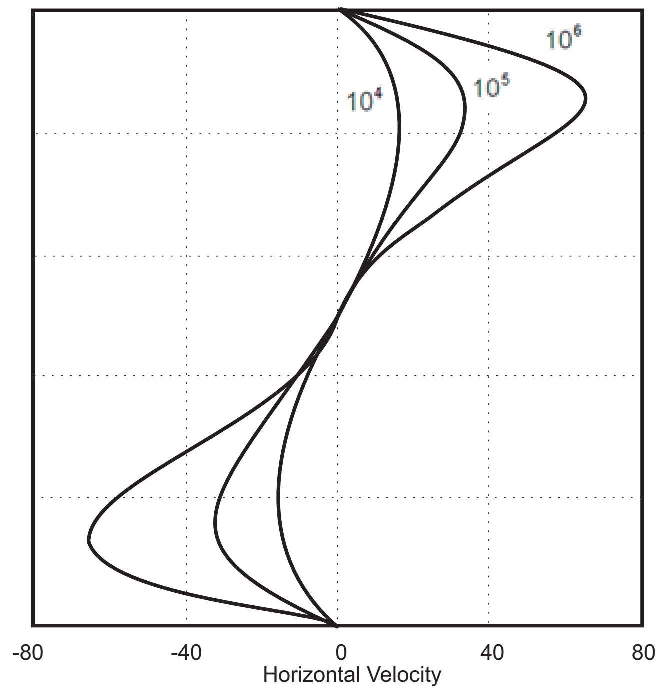

Figure 2.4: Variation of horizontal velocity at mid-width for varying Rayleigh numbers.

of heat transfer along the vertical walls of the cavity. The dimensionless parameter called Nusselt number stands for this quantity. The local Nusselt number can be calculated as

$$Nu_{local} = \pm\frac{\partial T}{\partial x}$$

The negative sign means heat transfer at the hot wall and the positive sign means heat transfer at the cold wall. The local Nusselt number at the cavity hot wall is used for comparison with benchmark problems in the literature frequently. As in the velocity components case, we calculate the average Nusselt numbers with GFEM and our method. The benchmark data results are also included to compare the average Nusselt numbers values with the presented study. The results are given in Table 2.3.

Table 2.3: Comparison of average Nusselt number on the vertical boundary of the cavity at $x = 0$ with mesh size used in computation.

Ra	GFEM	Present Study	Ref. [16]	Ref. [50]	Ref. [51]	Ref. [70]
10^4	2.40(11×11)	2.15 (11×11)	2.24(41×41)	2.08(71×71)	2.24(71×71)	2.25(101×101)
10^5	5.11(21×21)	4.35(21×21)	4.52(81×81)	4.30(71×71)	4.52(71×71)	4.59(101×101)
10^6	6.00(32×32)	8.83(32×32)	8.92(81×81)	8.74(71×71)	8.82(71×71)	8.97(101×101)

As we can understand from the Table 2.3, there is a very good agreement with the benchmark solutions and the present study, which can still capture reasonable results for rather coarser grid. The variation of the Nusselt number along the hot wall and cold wall of the cavity for different Rayleigh numbers are depicted in Figure 2.5 and Figure 2.6 respectively. These profiles are also look reasonable when compared with those reported in [16], [50], [51] and [70]. We can also observe from Figure 2.5 and Figure 2.6 that, largest Nusselt number is obtained at the range of highest temperature gradients naturally. That is, these are the ranges where the coolest fluid is first exposed to the hot surface or the warmest fluid first hits to the cold surface. Also high *Ra* causes higher *Nu* or increased heat transfer as expected.

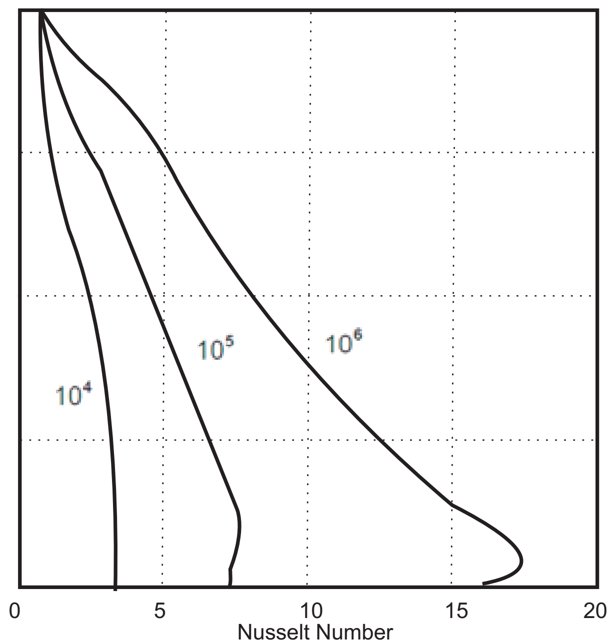

Figure 2.5: Variation of local Nusselt number at cavity hot wall.

Characters of the flow patterns for increasing Rayleigh numbers are seen very often in the study of natural convection problems. Diagrams showing the streamlines and temperature isolines are very popular among the convective heat transport illustrations. We present these patterns in the Figure 2.7. It is clear from the streamline patterns that, as Rayleigh number increases circular vortex at the cavity center begin to deform into an ellipse and then break up into two vortices tending to approach to the corners differentially heated sides of the cavity. So we can conclude that, the flow is swifter as the thermal convection is concentrated. Through the increase in Rayleigh number, parallel behavior of the temperature isolines is distorted and these lines seem to have a flat behavior in the central part of the region. Near the sides of the cavity, isolines tend to be vertical only. With $Ra = 10^6$, the temperature slopes at the

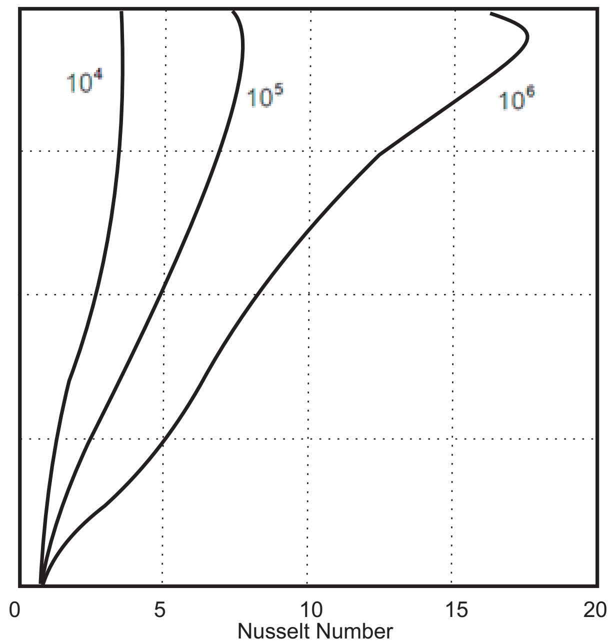

Figure 2.6: Variation of local Nusselt number at cavity cold wall.

corners of the differentially heated sides are more immersed then the cases of lower Rayleigh number. We also note that these graphics are also perfectly comparable with the ones given in the investigations of [16], [51], [50] and [70].

2.5.2 Numerical Convergence Study

An assessment of the convergence of the numerical simulation is presented in this subsection. We consider the problem (2.1) in the domain $\Omega = [-1, 1] \times [-1, 1]$. The forcing function γ and boundary values of the temperature are given so that the prescribed solution of the system is given by:

$$\mathbf{u} = ((x^2-1)^2 y(y^2-1), -(x^2-1)x(y^2-1)^2)$$

$$p = \frac{1}{8}\left(y^4 - y^6 + y^2 - 1\right)x^8 + \frac{1}{2}\left(y^6 - y^4 - y^2 + 1\right)x^6 + \frac{6}{5}yx^5 + \frac{3}{4}\left(y^4 - y^6 + y^2 - 1\right)x^4$$
$$+\left(4y^3 - 8y\right)x^3 + \frac{1}{2}\left(y^6 - y^4 - y^2 + 1\right)x^2 + \left(10y - 4y^3\right)x$$

$$T = \frac{1}{400}\left(2y^3 - 3y^5 + y\right)x^8 + \frac{1}{100}\left(3y^5 - 2y^3 - y\right)x^6 + \frac{3}{250}x^5 + \frac{3}{100}\left(y^3 - \frac{3}{2}y^5 + \frac{1}{2}y\right)x^4$$
$$+\frac{1}{25}\left(3y^2 - 2\right)x^3 + \frac{1}{100}\left(3y^5 - 2y^3 - y\right)x^2 + \frac{1}{50}\left(3y^4 - 12y^2 + 8\right)x$$

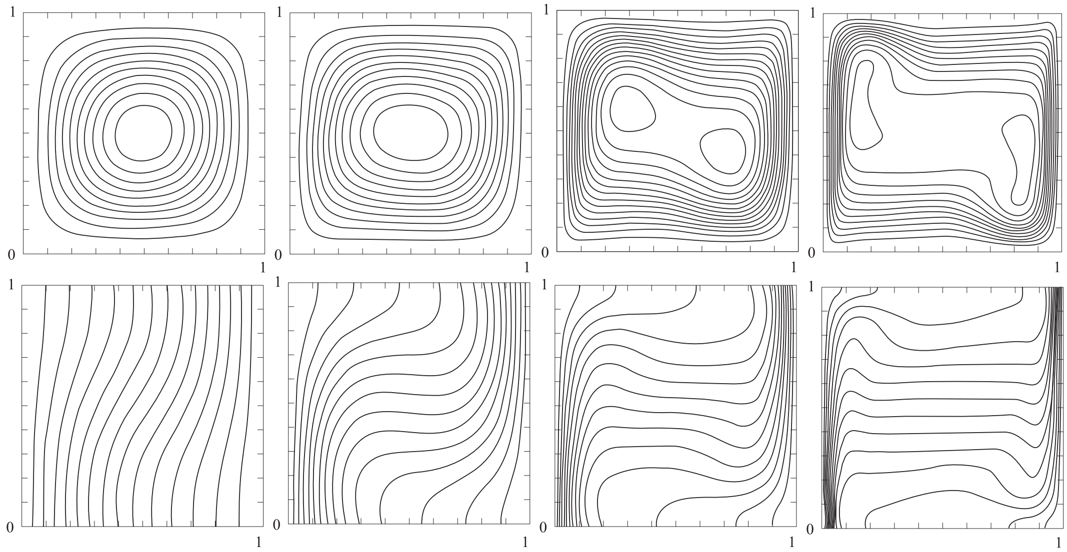

Figure 2.7: Streamlines (upper left to right) and isotherms (lower left to right) for with $Ra = 10^3, 10^4, 10^5, 10^6$, respectively

In (2.1), nonzero Neumann boundary condition for T on Γ_B and Dirichlet boundary condition for $\mathbf{u}$ are chosen so that $(\mathbf{u}, p, T)$ is the solution of the system. Note that, using the non-zero Neumann boundary condition for the variable T affect the stability bounds given in Lemma 2.9 and hence the main theorems. Although this replacement changes some terms and constants in the error analysis, it does not degrade the order of errors given in Remark 2.18 and Remark 2.20.

We use the same settings as in Remark 2.18 and Remark 2.20 with $Pr = 1$, $Ra = 100$ and $\kappa = 1$. We compute the errors between exact solution and computed numerical solution for the variables $\mathbf{u}$, p and T. Then, we compare error rates with the theoretical expectations. Table 2.4 presents the corresponding total degree of freedoms for $\mathbf{u}$, T and p, errors and convergence rates for different mesh sizes. We first compute the errors for the coarsest mesh of $h = 1/4$ and then refine the mesh to obtain finer ones. The theory predicts the error rates in Table 2.4,

Table 2.4: Total degree of freedoms, numerical errors and convergence rates for each variable.

Mesh	# of d.o.f.	$\|\|\mathbf{u} - \mathbf{u}^h\|\|$	Rate	$\|\|\nabla(\mathbf{u} - \mathbf{u}^h)\|\|$	Rate	$\|\|p - p^h\|\|$	Rate	$\|\|\nabla(T - T^k)\|\|$	Rate
$h = 1/4$	374	0.0170	–	0.3712	–	0.3521	–	0.2922	
$h = 1/8$	1318	0.0021	2.85	0.0905	2.02	0.0951	1.92	0.0767	1.95
$h = 1/16$	4934	2.434e-04	2.92	0.0222	2.01	0.0215	1.94	0.0187	2.02
$h = 1/32$	19078	2.722e-05	2.99	0.0054	2.01	0.0054	1.98	0.0042	2.10

$O(h^3)$ for the L^2 norm for $\mathbf{u}$, and $O(h^2)$ for the L^2 norm for p and $O(h^2)$ in energy norm for the temperature. Note that the behavior of the error is exactly as anticipated by the theory. Thus we can conclude that the projection-based stabilization does not degrade the order of the errors for all variables.

CHAPTER 3

DOUBLE-DIFFUSIVE CONVECTION IN POROUS MEDIA

In this section, we consider a projection-based stabilized finite element method for the double-diffusive convection in porous media modeled by Darcy-Brinkman formulation. Stabilization idea introduced in Chapter 2 is applied to a new system here. Although this new system of equations seem similar to one dealt with in previous chapter, the main difference comes from the dependency to the time and coupling of one more equation, namely the concentration equation. We present the scheme first and then consider the stability issues. After emphasizing the existence and uniqueness results of the problem, we pass to the semi-discrete a priori error analysis. We give a fully discrete scheme for the system and a detailed stability and error analysis are presented. As in previous chapter, we perform some numerical tests to measure the effectiveness of the method.

Double-diffusion phenomena in a confined porous enclosure is modeled by the non-linear time dependent Darcy-Brinkman equations which read in dimensionless form

$$\begin{aligned}
\partial_t \mathbf{u} - 2\nu\nabla\cdot\mathbb{D}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + Da^{-1}\mathbf{u} + \nabla p &= (\beta_T T + \beta_C C)\mathbf{g} \quad \text{in} \quad (0,t^*]\times\Omega,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad (0,t^*]\times\Omega,\\
\mathbf{u} &= 0 \quad \text{on} \quad (0,t^*]\times\partial\Omega,\\
\partial_t T + \mathbf{u}\cdot\nabla T &= \gamma\Delta T \quad \text{in} \quad (0,t^*]\times\Omega,\\
\partial_t C + \mathbf{u}\cdot\nabla C &= D_c\Delta C \quad \text{in} \quad (0,t^*]\times\Omega,\\
T, C &= 0 \quad \text{on} \quad \Gamma_T, \quad \frac{\partial T}{\partial\mathbf{n}}, \frac{\partial C}{\partial\mathbf{n}} = 0 \quad \text{on} \quad \Gamma_B,\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0, \quad T(0,\mathbf{x}) = T_0, \quad C(0,\mathbf{x}) = C_0 \quad \text{in} \quad \Omega.
\end{aligned} \tag{3.1}$$

Here Ω be a regular bounded open set in $\mathbb{R}^d$ with $(d = 2,3)$, $\partial\Omega = \Gamma_T \cup \Gamma_B$ is a polygonal boundary, and $\mathbf{u}, p, T, C$ denote the velocity, pressure, temperature and concentration fields, respectively. The kinematic viscosity is $\nu > 0$, the thermal diffusivity $\gamma > 0$, the mass diffusivity $D_c > 0$, the Darcy number Da, the gravitational acceleration vector is $\mathbf{g}$, the velocity deformation tensor is $\mathbb{D}\mathbf{u} = \left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)/2$ and the thermal and solutal expansion coefficients are β_T, β_C, respectively. Some important dimensionless parameters which could be seen frequently in relevant publications are the thermal Grashof number $Gr_T = \frac{\mathbf{g}\beta_T \triangle T H^3}{\nu^2}$ and the solutal Grashof number $Gr_C = \frac{\mathbf{g}\beta_C \triangle C H^3}{\nu^2}$, the buoyancy ratio $N = \frac{\beta_C \triangle C}{\beta_T \triangle T}$, the Prandtl number $Pr = \frac{\nu}{\gamma}$, the Schmidt number $Sc = \frac{\nu}{D_c}$, the Lewis number $Le = \frac{Sc}{Pr}$ and the Darcy number $Da = \frac{K}{H^2}$ with given cavity height H, a permeability K, $\triangle T$ and $\triangle C$ are the characteristics temperature and concentration differences along the enclosure, respectively.

As in the cases of thermal natural convection in porous and non-porous enclosures, formation of boundary layers near the boundaries of the enclosure is seen on double-diffusive case too. In contrast with the thermal case, three kinds of layers are formed. These are thermal, solutal and hydrodynamic layers and thickness of each layer leads the rate of heat and mass transfer and the dynamics of the overall flow [54, 58, 68]. The coupling of different boundary layers and circulating main core inside the enclosure is the fundamental difficulty in solving these systems analytically. Hence, numerical methods have to be developed. The dominance of the convection and coupling of the variables in the system leads the general behavior of the flow and approximation of the solution between mentioned boundary layers fails.

3.1 Notation, Mathematical Preliminaries and Semi-Discrete Scheme

In this section, we only give notations and mathematical results which are not given in section 2 of Chapter 2. We define norms or other additional properties here only if they are not defined before.

Let Y be a Banach space and for $0 < t < \infty$, the space $L^p(0,t;Y)$ consists of all functions $\mathbf{u}(0,t) \times X$ for which the norm

$$||\mathbf{u}||_{L^p(0,t;Y)} := \left(\int_0^t ||\mathbf{u}||_Y^p\right)^{1/p}, \quad p \in [1,\infty),$$

is finite and with the usual modification in the definition of this space for $p = \infty$. Throughout this chapter, symbol K stands for generic positive constant and may have different values at different places, but it does not depend on mesh sizes and other important parameters unless stated otherwise.

We consider the following functional vector spaces associated with the boundary conditions for the variational formulation of (3.1):

$$\begin{aligned} X := \mathbf{H}_0^1(\Omega_f) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\}, \qquad Q := \{p \in L^2(\Omega) : \int_\Omega p\, d\mathbf{x} = 0\}, \\ W : &= \{S \in H^1(\Omega) : S = 0 \quad \text{on} \quad \Gamma_B\}, \qquad \Psi := \{\Phi \in H^1(\Omega) : \Phi = 0 \quad \text{on} \quad \Gamma_B\}, \end{aligned}$$

The variational form of the system (3.1) is as follows : find $\mathbf{u} : [0, t^*] \to X, p : (0, t^*] \to Q, T : [0, t^*] \to W$ and $C : [0, t^*] \to \Psi$ satisfying

$$(\partial_t \mathbf{u}, \mathbf{v}) + (2\nu\mathbb{D}\mathbf{u}, \mathbb{D}\mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (Da^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = \beta_T(\mathbf{g}T, \mathbf{v}) + \beta_C(\mathbf{g}C, \mathbf{v}) \tag{3.2}$$

$$(q, \nabla \cdot \mathbf{u}) = 0, \tag{3.3}$$

$$(\partial_t T, S) + c_1(\mathbf{u}, T, S) + (\gamma\nabla T, \nabla S) = 0 \tag{3.4}$$

$$(\partial_t C, \Phi) + c_2(\mathbf{u}, C, \Phi) + (D_c\nabla C, \nabla\Phi) = 0 \tag{3.5}$$

for all $(\mathbf{v}, q, S, \Phi) \in (X, Q, W, \Psi)$. Here the trilinear skew-symmetric forms of convective terms are

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}\int_\Omega \Big((\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} - (\mathbf{u} \cdot \nabla)\mathbf{w} \cdot \mathbf{v}\Big) d\mathbf{x} \tag{3.6}$$

$$c_1(\mathbf{u}, T, S) = \frac{1}{2}\int_\Omega \Big((\mathbf{u} \cdot \nabla)TS - (\mathbf{u} \cdot \nabla)S\,T\Big) d\mathbf{x} \tag{3.7}$$

$$c_2(\mathbf{u}, C, \Phi) = \frac{1}{2}\int_\Omega \Big((\mathbf{u} \cdot \nabla)C\Phi - (\mathbf{u} \cdot \nabla)\Phi C\Big) d\mathbf{x}. \tag{3.8}$$

An alternative and useful definition of these forms are also given as follows

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega \Big((\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{v} \cdot \mathbf{w}\Big) d\mathbf{x} \tag{3.9}$$

$$c_1(\mathbf{u}, T, S) = \int_\Omega \Big((\mathbf{u} \cdot \nabla)TS + \frac{1}{2}(\nabla \cdot \mathbf{u})TS\Big) d\mathbf{x} \tag{3.10}$$

$$c_2(\mathbf{u}, C, \Phi) = \int_\Omega \Big((\mathbf{u} \cdot \nabla)C\Phi + \frac{1}{2}(\nabla \cdot \mathbf{u})C\Phi\Big) d\mathbf{x} \tag{3.11}$$

With the use of integration by parts, one can show $c_0(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{u}, \mathbf{v} \in X$, $c_1(\mathbf{u}, S, S) = 0$ for all $(\mathbf{u}, S) \in X \times W$, and $c_2(\mathbf{u}, \Phi, \Phi) = 0$ for all $(\mathbf{u}, \Phi) \in X \times \Psi$.

We study semi-discretization approach in space first i.e., only the continuous-in-time is considered. Let τ^h be an admissible triangulation of the domain Ω and the mesh τ^h results from a coarser mesh τ^H by refinement for which $H \geq h$. We introduce the conforming finite element spaces $X^h \subset X, Q^h \subset Q, W^h \subset W$ and $\Psi^h \subset \Psi$. We also assume that a pair of finite element spaces (X^h, Q^h) satisfies the discrete inf-sup condition. Furthermore, we again introduce the discretely divergence-free subspace of X^h,

$$V^h = \{\mathbf{v}^h \in X^h : (q^h, \nabla \cdot \mathbf{v}^h) = 0, \forall q^h \in Q^h\}.$$

The projection-based stabilization method uses the following coarse or large scale spaces. Let L^H, M^H, K^H are the coarse finite spaces of the deformation tensor, the temperature gradient and the concentration gradient, respectively, i.e. with

$$\begin{aligned} L^H \subseteq \mathbb{D}X^h \subseteq L &:= \{l_{ij} \in [L^2(\Omega)]^{d\times d} | l_{ij} = l_{ji}\} \\ M^H \subseteq \nabla W^h \subseteq M &:= \{m_i \in [L^2(\Omega)]^d\} \\ K^H \subseteq \nabla \Psi^h \subseteq K &:= \{k_i \in [L^2(\Omega)]^d\}. \end{aligned}$$

Remark 3.1 *In the limit case, the choices, $L^H = \mathbb{D}X^h$, $M^H = \nabla W^h$ and $K^H = \nabla \Psi^h$, yield a standard Galerkin finite element formulation. In addition the choices $L^H = \{0\}, M^H = \{0\}$ and $K^H = \{0\}$ yield an artificial viscosity method.*

There are two natural ways to define the coarse finite element spaces: on a coarser grid or by low order finite elements on the finest grid. For the error analysis, choose L^H to be a particular subspace for the velocity, namely, $L^H = \mathbb{D}X^H$, so that it contains the discontinuous piecewise polynomials on the coarse mesh finite space X^H.

Let the relevant L^2 orthogonal projection operators for those coarse finite element spaces are

$$\begin{aligned} P_L^u &: L \to L^H \\ P_M^T &: M \to M^H \\ P_K^C &: K \to K^H. \end{aligned}$$

By using the same ideas in Chapter 2, the projection-based stabilized method for Darcy-Brinkman equations with the additional viscosities $\alpha_1 := \alpha_1(h)$, $\alpha_2 := \alpha_2(h)$, $\alpha_3 := \alpha_3(h)$ becomes: find $\mathbf{u}^h : [0, t^*] \to V^h, T^h : [0, t^*] \to W^h, C^h : [0, t^*] \to \Psi^h$

$$(\partial_t \mathbf{u}^h, \mathbf{v}^h) + (2\nu \mathbb{D}\mathbf{u}^h, \mathbb{D}\mathbf{v}^h) + \alpha_1((I - P_L^u)\mathbb{D}\mathbf{u}^h, (I - P_L^u)\mathbb{D}\mathbf{v}^h) + c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + (Da^{-1}\mathbf{u}^h, \mathbf{v}^h)$$
$$= \beta_T(\mathbf{g}T^h, \mathbf{v}^h) + \beta_C(\mathbf{g}C^h, \mathbf{v}^h) \quad (3.12)$$
$$(\partial_t T^h, S^h) + c_1(\mathbf{u}^h, T^h, S^h) + (\gamma \nabla T^h, \nabla S^h) + \alpha_2((I - P_M^T)\nabla T^h, (I - P_M^T)\nabla S^h) = 0 \quad (3.13)$$
$$(\partial_t C^h, \Phi^h) + c_2(\mathbf{u}^h, C^h, \Phi^h) + (D_c \nabla C^h, \nabla \Phi^h) + \alpha_3((I - P_K^C)\nabla C^h, (I - P_K^C)\nabla \Phi^h) = 0 \quad (3.14)$$

for all $(\mathbf{v}^h, q^h, S^h, \Phi^h) \in (V^h, Q^h, W^h, \Psi^h)$. Here $\alpha_1, \alpha_2, \alpha_3$ are non-negative constant functions and user selected stabilization parameters. Note that the operators $(I-P_L^u)$, $(I-P_M^T)$ and $(I-P_K^C)$ represent small scales or fluctuations of the deformation $\mathbb{D}\mathbf{u}^h$, the temperature gradient ∇T and the concentration gradient ∇C, respectively.

Convective terms defined via (3.6)-(3.8) satisfy the same properties as in Chapter 2. The only different term is $c_2(\mathbf{u}, C, \Phi)$ and it is treated exactly as $c_1(\mathbf{u}, T, S)$. So Lemma 2.5 could be extended with $c_2(\mathbf{u}, C, \Phi)$ directly.

We make several common assumptions about the finite element spaces that we will use. We assume that the finite element spaces X^h, W^h and Ψ^h rely on quasiuniform triangulations of Ω and contain piecewise continuous polynomials of degree m and the space Q^h contains continuous piecewise polynomials of degree $m-1$. In addition, the spaces satisfy the following approximation properties: for a given integer $1 \leq s \leq m$:

$$\inf_{\mathbf{v}_h \in X^h, q^h \in Q^h} \left\{ \|(\mathbf{u} - \mathbf{v}^h)\| + h\|\nabla(\mathbf{u} - \mathbf{v}^h)\| + h\|p - q^h\| \right\} \leq Ch^{s+1}(\|\mathbf{u}\|_{s+1} + \|p\|_s), \tag{3.15}$$

$$\inf_{S^h \in W^h} \|T - S^h\| \leq h^{s+1}\|T\|_{s+1} \tag{3.16}$$

$$\inf_{\Phi^h \in \Psi^h} \|C - \Phi^h\| \leq h^{s+1}\|C\|_{s+1} \tag{3.17}$$

for $(\mathbf{u}, p, T, C) \in (X \cap H^{s+1}(\Omega), Q \cap H^s(\Omega), W \cap H^{s+1}(\Omega), \Psi \cap H^{s+1}(\Omega))$.

We choose the coarse finite element spaces $L^H = \mathbb{D}X^H$, M^H and Ψ^H containing the space of discontinuous polynomials of degree $m - 1$ on a coarse mesh.

The L^2 orthogonal projections of L^H, M^H and K^H satisfy

$$\|G - P_\mu^\kappa G\| \leq CH^s|G|_s, \quad \mu = L, M, K;\ \kappa = u, T, C;\ 1 \leq s \leq m \tag{3.18}$$

for all $G \in (L^2(\Omega) \cap H^s(\Omega))$. The following lemma is used in the error analysis of the semi-discrete problem.

Lemma 3.2 *Assume that the domain Ω has a quasiuniform triangulation and velocity coarse scale space L^H satisfies $L^H = \mathbb{D}X^H$. Then we have*

$$\|P_L^u \nabla \phi^h\| \leq CH^{-1}\|\phi^h\| \tag{3.19}$$

for all $\phi^h \in V^h$.

Proof. Defining the elliptic projection $P_H : X \to X^H$ for velocity large scales, it is shown in [32] that $P_L^u \mathbb{D} v = \mathbb{D}(P_H v) \quad \forall\, v \in X$. Thus we have,

$$||P_L^u \nabla \phi^h|| = ||\mathbb{D}(P_H v)||\,.$$

Assumption on quasiuniform triangulation allows us to use inverse estimates for the finite element function ϕ, so we reach

$$||P_L^u \nabla \phi^h|| = ||\mathbb{D}(P_H v)|| \leq C H^{-1} ||P_H \phi^h|| \tag{3.20}$$

Finally, assuming the L^2 stability of elliptic projection for functions in V^h i.e., $||P_H \phi^h|| \leq C||\phi^h||, \quad \forall\, \phi^h \in V^h$ and combining this result with (3.20), the final estimate is obtained. □

Since the velocity deformation tensor is used in this section, we need Korn's inequality which states

$$||\nabla \mathbf{v}|| \leq K ||\mathbb{D}\mathbf{v}|| \tag{3.21}$$

for all $\mathbf{v} \in X$ with $K = K(\Omega)$

Throughout the finite element error analysis of this section, we use the following weighted norms.

Definition 3.3 *For $\mathbf{u} \in X$, $T \in W$, $C \in \Psi$, the weighted norms of functions $\mathbf{u} : \Omega_f \to \mathbb{R}$, $T : \Omega \to \mathbb{R}$ and $C : \Omega \to \mathbb{R}$ are defined by*

$$\begin{aligned}
||\mathbf{u}||_{a,b}^2 &= a\,||\mathbf{u}||^2 + b\nu\,||\mathbb{D}\mathbf{u}||^2 + \alpha_1 \left\|(I - P_L^u)\mathbb{D}\mathbf{u}\right\|^2 \\
||T||_{a,b}^2 &= a\,||T||^2 + b\,||\nabla T||^2 + \alpha_2 \left\|(I - P_M^T)\nabla T\right\|^2 \\
||C||_{a,b}^2 &= a\,||C||^2 + b\,||\nabla C||^2 + \alpha_3 \left\|(I - P_K^C)\nabla C\right\|^2
\end{aligned}$$

where $a, b, c > 0$ are constants and $\alpha_1, \alpha_2, \alpha_3$ are stabilizing parameters.

We now give the following version of Gronwall's Lemma which is given in [62].

Lemma 3.4 *(Gronwall's Lemma) Let λ be a non-negative function and $F_i \in L^1(0, t^*)$ with $\int_0^t F_i(s)ds$ continuous and non decreasing on $[0, t^*]$ for $i = 1, 2, 3$. Then*

$$\partial_t \lambda + F_1(t) \leq F_2(t) + F_3(t)\lambda(t) \quad a.e. \quad in \quad [0, t^*]$$

implies for almost all $t \in [0, t^*]$

$$\lambda(t) + \int_0^t F_1(s)ds \leq \exp\left(\int_0^t F_3(s)ds\right)\left(\lambda(0) + \int_0^t F_2(s)ds\right)$$

Proof. We refer [60] for the proof. □

3.2 Semi-Discrete A Priori Error Analysis

This section gives a finite element error analysis for the discrete solutions of (3.12)-(3.14). We state the stability and then give the main convergence result. Finite element approximation results follow.

3.2.1 Stability of the method

We first prove the stability results for discrete solutions of (3.12)-(3.14), i.e., the solutions $\mathbf{u}^h, T^h$ and C^h are bounded a priori by the data of the problem.

Lemma 3.5 *(Stability) Let* $\mathbf{u}_0 \in (L^2(\Omega))^d$, $T_0, C_0 \in L^2(\Omega)$. *Then the solution* $(\mathbf{u}^h, T^h, C^h) \in (V^h, W^h, \Psi^h)$ *of (3.12)-(3.14) fulfills* $\mathbf{u}^h \in (L^\infty(0,t^*;L^2))^d, \mathbb{D}\mathbf{u}^h \in (L^2(0,t^*;L^2))^{d\times d}, T^h \in L^\infty(0,t^*;L^2)$, $\nabla T^h \in (L^2(0,t^*;L^2))^{d\times d}, C^h \in L^2(0,t^*;L^2), \nabla C^h \in (L^2(0,t^*;L^2))^{d\times d}$.

Proof. Setting $S^h = T^h$ in (3.13), $\Phi^h = C^h$ in (3.14) and skew-symmetry of $c_1(\cdot,\cdot,\cdot)$ and $c_2(\cdot,\cdot,\cdot)$ imply that

$$\begin{aligned}
\frac{1}{2}\partial_t\|T^h\|^2 + \gamma\|\nabla T^h\|^2 + \alpha_2\|(I-P_M^T)\nabla T^h\|^2 &= 0\\
\frac{1}{2}\partial_t\|C^h\|^2 + D_c\|\nabla C^h\|^2 + \alpha_3\|(I-P_K^C)\nabla C^h\|^2 &= 0.
\end{aligned}$$

Integrating the above two equations over from $(0,t)$ with $t \le t^*$ yield

$$\|T^h\|^2_{L^\infty(0,t,L^2)} + 2D_c\|\nabla T^h\|^2_{L^2(0,t,L^2)} + 2\alpha_2\|(I-P_M^T)\nabla T^h\|^2_{L^2(0,t,L^2)} \le \|T_0\|^2 \tag{3.22}$$

$$\|C^h\|^2_{L^\infty(0,t,L^2)} + 2\gamma\|\nabla C^h\|^2_{L^2(0,t,L^2)} + 2\alpha_3\|(I-P_K^C)\nabla C^h\|^2_{L^2(0,t,L^2)} \le \|C_0\|^2. \tag{3.23}$$

For the velocity equation set $\mathbf{v}^h$=$\mathbf{u}^h$ in (3.12), then we have

$$\begin{aligned}
\frac{1}{2}\partial_t\|\mathbf{u}^h\|^2 + 2\nu\|\mathbb{D}\mathbf{u}^h\|^2 + \alpha_1\|(I-P_L^u)\mathbb{D}\mathbf{u}^h\|^2 + Da^{-1}\|\mathbf{u}^h\|^2\\
\le \|\mathbf{g}\|_\infty\left(\beta_T\|T^h\|\|\mathbf{u}^h\| + \beta_C\|C^h\|\|\mathbf{u}^h\|\right)
\end{aligned}$$

Making use of Young's inequality, integration over $(0,t)$ and the stability estimations of T^h and C^h in (3.22)-(3.23) yield

$$\|\mathbf{u}^h\|^2_{L^\infty(0,t,L^2)} + 4\nu\|\mathbb{D}\mathbf{u}^h\|^2_{L^2(0,t,L^2)} + 2\alpha_1\|(I-P_L^u)\mathbb{D}\mathbf{u}^h\|^2_{L^2(0,t,L^2)} + Da^{-1}\|\mathbf{u}^h\|^2_{L^2(0,t,L^2)}$$
$$\le Da\|\mathbf{g}\|^2_\infty\left(\beta_T^2\|T_0\|^2 + \beta_C^2\|C_0\|^2\right) + \|u_0\|^2,$$

from which the statement of the result follows. □

Remark 3.6 *The continuous solution of (3.1) is stable in the same sense that of discrete case. The proof of the stability of* $\mathbf{u}, T, C$ *can be established by using the same idea of Lemma 3.5.*

For the error analysis, we now state the regularity assumptions:

$$\nabla\mathbf{u} \in L^4(0,t^*;L^2(\Omega)),\ \partial_t\mathbf{u} \in L^2(0,t^*;H^{-1}(\Omega)),\ \nabla T \in L^4(0,t;L^2(\Omega)), \tag{3.24}$$
$$\partial_t T \in L^2(0,t^*;H^{-1}(\Omega)),\ \nabla C \in L^4(0,t;L^2(\Omega)),\quad \partial_t C \in L^2(0,t^*;H^{-1}(\Omega)) \tag{3.25}$$

Note that the assumptions $\nabla\mathbf{u}$ in (3.24) are natural regularity assumptions for the Navier-Stokes equations. These assumptions imply that Serrin's condition is fulfilled. From these assumptions, the uniqueness is guaranteed, [67], [17]. This type of existence-uniqueness results are well known for the Navier-Stokes type problems which could be easily adapted to the uncoupled Navier-Stokes case.

3.2.2 A Priori Error Estimation

In this part, we state and prove main error estimation theorem. The proof contains same strategy by Rannacher and Heywood [26].

A priori error analysis starts in the usual way by deriving the error equations with the subtraction of (3.12) from (3.2), (3.13) from (3.4) and (3.14) from (3.5) for all test functions from V^h, W^h and Ψ^h, respectively. Then we obtain

$$\begin{aligned}
&(\partial_t(\mathbf{u}-\mathbf{u}^h),\mathbf{v}^h) + (2\nu\mathbb{D}(\mathbf{u}-\mathbf{u}^h),\mathbb{D}\mathbf{v}^h) + \alpha_1((I-P_L^u)\mathbb{D}(\mathbf{u}-\mathbf{u}^h),(I-P_L^u)\mathbb{D}\mathbf{v}^h)\\
&+(Da^{-1}(\mathbf{u}-\mathbf{u}^h),\mathbf{v}^h) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}^h) - c_0(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) - (p-q^h,\nabla\cdot\mathbf{v}^h)\\
&=\beta_T(\mathbf{g}(T-T^h),\mathbf{v}^h) + \beta_C(\mathbf{g}(C-C^h),\mathbf{v}^h) + \alpha_1((I-P_L^u)\mathbb{D}\mathbf{u},(I-P_L^u)\mathbb{D}\mathbf{v}^h)
\end{aligned} \tag{3.26}$$

for all $(\mathbf{v}^h, q^h) \in (V^h, Q^h)$,

$$(\partial_t(T - T^h), S^h) + (\gamma\nabla(T - T^h), \nabla S^h) + \alpha_2((I - P_M^T)\nabla(T - T^h), (I - P_M^T)\nabla S^h)$$
$$c_1(\mathbf{u}, T, S^h) - c_1(\mathbf{u}^h, T^h, S^h) = \alpha_2((I - P_M^T)\nabla T, (I - P_M^T)\nabla S^h) \tag{3.27}$$

for all $S^h \in W^h$ and

$$(\partial_t(C - C^h), \Phi^h) + (D_c\nabla(C - C^h), \nabla\Phi^h) + \alpha_3((I - P_K^C)\nabla(C - C^h), (I - P_K^C)\nabla\Phi^h)$$
$$+c_2(\mathbf{u}, C, \Phi^h) - c_2(\mathbf{u}^h, C^h, \Phi^h) = \alpha_3((I - P_M^T)\nabla C, (I - P_M^T)\nabla\Phi^h) \tag{3.28}$$

for all $\Phi^h \in \Psi^h$. We split the error terms in two parts: the approximation errors, $\boldsymbol{\eta}_u, \eta_T, \eta_C$ and the finite element remainders $\boldsymbol{\phi}_u^h$, ϕ_T^h and ϕ_C^h

$$\mathbf{u} - \mathbf{u}^h = (\mathbf{u} - \tilde{\mathbf{u}}) - (\mathbf{u}^h - \tilde{\mathbf{u}}) = \boldsymbol{\eta}_u - \boldsymbol{\phi}_u^h$$
$$T - T^h = (T - \tilde{T}) - (T^h - \tilde{T}) = \eta_T - \phi_T^h$$
$$C - C^h = (C - \tilde{C}) - (C^h - \tilde{C}) = \eta_C - \phi_C^h.$$

where $\tilde{\mathbf{u}} \in V^h, \tilde{T} \in W^h$ and $\tilde{C} \in \Psi^h$ are the approximations of $\mathbf{u}, T$ and C, respectively, which fulfill certain interpolation estimations. Revising the error equations according to these decompositions and writing $\mathbf{v}^h = \boldsymbol{\phi}_u^h$, $S^h = \phi_T^h$ and $\Phi^h = \phi_C^h$, respectively, one obtains

$$\frac{1}{2}\partial_t\|\boldsymbol{\phi}_u^h\|^2 + 2\nu\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + \alpha_1\|(I - P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + Da^{-1}\|\boldsymbol{\phi}_u^h\|^2$$
$$= (\partial_t\boldsymbol{\eta}_u, \boldsymbol{\phi}_u^h) + (2\nu\mathbb{D}\boldsymbol{\eta}_u, \mathbb{D}\boldsymbol{\phi}_u^h) + \alpha_1((I - P_L^u)\mathbb{D}\boldsymbol{\eta}_u, (I - P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h) + (Da^{-1}\boldsymbol{\eta}_u, \boldsymbol{\phi}_u^h)$$
$$-\alpha_1((I - P_L^u)\mathbb{D}\mathbf{u}, (I - P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h) - (p - q^h, \nabla\cdot\boldsymbol{\phi}_u^h)) - \beta_T(\mathbf{g}(T - T^h), \boldsymbol{\phi}_u^h)$$
$$-\beta_C(\mathbf{g}(C - C^h), \boldsymbol{\phi}_u^h) + c_0(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - c_0(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) \tag{3.29}$$
$$\frac{1}{2}\partial_t\|\phi_T^h\|^2 + \gamma\|\nabla\phi_T^h\|^2 + \alpha_2\|(I - P_M^T)\nabla\phi_T^h\|^2$$
$$= (\partial_t\eta_T, \phi_T^h) + (\gamma\nabla\eta_T, \nabla\phi_T^h) + \alpha_2((I - P_M^T)\nabla\eta_T, (I - P_M^T)\nabla\phi_T^h)$$
$$-\alpha_2((I - P_M^T)\nabla T, (I - P_M^T)\nabla\phi_T^h) + c_1(\mathbf{u}, T, S^h) - c_1(\mathbf{u}^h, T^h, S^h) \tag{3.30}$$

and

$$\frac{1}{2}\partial_t\|\phi_C^h\|^2 + D_c\|\nabla\phi_C^h\|^2 + \alpha_3\|(I - P_K^C)\nabla\phi_C^h\|^2$$
$$= (\partial_t\eta_C, \phi_C^h) + (D_c\nabla\eta_C, \nabla\phi_C^h) + \alpha_3((I - P_K^C)\nabla\eta_C, (I - P_K^C)\nabla\phi_C^h)$$
$$-\alpha_3((I - P_M^T)\nabla C, (I - P_K^C)\nabla\phi_C^h) + c_2(\mathbf{u}, C, \phi_C^h) - c_2(\mathbf{u}^h, C^h, \phi_C^h) \tag{3.31}$$

The next step is to estimate the terms on the right hand sides of (3.30), (3.31) and (3.29), respectively. In the following estimations we basically use Cauchy-Schwarz (or duality pairing), Korn's, Young's and Poincaré's inequalities. Then, the terms in the temperature equation

(3.30) are estimated as follows:

$$|(\partial_t \eta_T, \phi_T^h)| \leq \|\partial_t \eta_T\|_{-1} \|\nabla \phi_T^h\| \leq K\gamma^{-1} \|\partial_t \eta_T\|_{-1}^2 + \frac{\gamma}{16} \|\nabla \phi_T^h\|^2$$
$$|(\nabla \eta_T, \nabla \phi_T^h)| \leq K\gamma^{-1} \|\nabla \eta_T\|^2 + \frac{\gamma}{16} \|\nabla \phi_T^h\|^2$$
$$|\alpha_2((I - P_M^T)\nabla \eta_T, (I - P_M^T)\nabla \phi_T^h)| \leq K\alpha_2 \|(I - P_M^T)\nabla \eta_T\|^2 + \frac{\alpha_2}{4} \|(I - P_M^T)\nabla \phi_T^h\|^2$$
$$|\alpha_2((I - P_M^T)\nabla T, (I - P_M^T)\nabla \phi_T^h)| \leq K\alpha_2 \|(I - P_M^T)\nabla T\|^2 + \frac{\alpha_2}{4} \|(I - P_M^T)\nabla \phi_T^h\|^2$$

The critical error estimation is the convective terms. We first split this term as follows

$$c_2(\mathbf{u}, T, \phi_T^h) - c_2(\mathbf{u}^h, T^h, \phi_T^h) = c_2(\boldsymbol{\eta}_u, T, \phi_T^h) - c_2(\boldsymbol{\phi}_u^h, T, \phi_T^h) + c_2(\mathbf{u}^h, \eta_T, \phi_T^h). \tag{3.32}$$

By using Lemma 2.5, we estimate each term on the right hand side of (3.32) separately:

$$|c_2(\boldsymbol{\eta}_u, T, \phi_T^h)| \leq K\|\boldsymbol{\eta}_u\|^{1/2} \|\nabla \boldsymbol{\eta}_u\|^{1/2} \|\nabla C\| \|\nabla \phi_T^h\| \leq \frac{\gamma}{16} \|\nabla \phi_T^h\|^2 + K\gamma^{-1} \|\mathbb{D}\boldsymbol{\eta}_u\|^2 \|\nabla T\|^2$$
$$|c_2(\boldsymbol{\phi}_u^h, T, \phi_T^h)| \leq K\|\boldsymbol{\phi}_u^h\|^{1/2} \|\nabla \boldsymbol{\phi}_u^h\|^{1/2} \|\nabla T\| \|\nabla \phi_T^h\| \leq \frac{\gamma}{4} \|\nabla \phi_T^h\|^2 + \frac{\nu}{8} \|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\nu^{-1}\gamma^{-2} \|\nabla T\|^4 \|\boldsymbol{\phi}_u^h\|^2$$
$$|c_2(\mathbf{u}^h, \eta_T, \phi_T^h)| \leq K\|\mathbf{u}^h\|^{1/2} \|\nabla \mathbf{u}^h\|^{1/2} \|\nabla \eta_T\| \|\nabla \phi_T^h\| \leq \frac{\gamma}{16} \|\nabla \phi_T^h\|^2 + K\gamma^{-1} \|\mathbf{u}^h\| \|\mathbb{D}\mathbf{u}^h\| \|\nabla \eta_T\|^2$$

Plugging these estimation in (3.30) and arranging the terms yield

$$\begin{aligned} &\frac{1}{2}\partial_t \|\phi_T^h\|^2 + \frac{\gamma}{2} \|\nabla \phi_T^h\|^2 + \frac{\alpha_2}{2} \|(I - P_M^T)\nabla \phi_T^h\|^2 \leq K\{\gamma^{-1}(\|\partial_t \eta_T\|_{-1}^2 + \|\nabla \eta_T\|^2 \\ &+ \|\mathbb{D}\boldsymbol{\eta}_u\|^2 \|\nabla T\|^2 + \|\mathbb{D}\boldsymbol{\eta}_u\|^2 \|\nabla T\|^2 + \|\mathbf{u}^h\| \|\mathbb{D}\mathbf{u}^h\| \|\nabla \eta_T\|^2 + \nu^{-1}\gamma^{-1} \|\nabla T\|^4 \|\boldsymbol{\phi}_u^h\|^2) \\ &+ \alpha_2 \|(I - P_M^T)\nabla \eta_T\|^2 + \alpha_2 \|(I - P_M^T)\nabla T\|^2\} + \frac{\nu}{8} \|\mathbb{D}\boldsymbol{\phi}_u^h\|^2. \end{aligned} \tag{3.33}$$

Similar estimations are used to bound the terms in (3.31). Then for the concentration equation (3.31), one obtains

$$\begin{aligned} &\frac{1}{2}\partial_t \|\phi_C^h\|^2 + \frac{D_c}{2} \|\nabla \phi_C^h\|^2 + \frac{\alpha_3}{2} \|(I - P_K^C)\nabla \phi_C^h\|^2 \leq K\{D_c^{-1}(\|\partial_t \eta_C\|_{-1}^2 + \|\nabla \eta_C\|^2 \\ &+ \|\mathbb{D}\boldsymbol{\eta}_u\|^2 \|\nabla C\|^2 + \|\mathbf{u}^h\| \|\mathbb{D}\mathbf{u}^h\| \|\nabla \eta_C\|^2 + \nu^{-1} D_c^{-1} \|\nabla C\|^4 \|\boldsymbol{\phi}_u^h\|^2) \\ &+ \alpha_3 \|(I - P_K^C)\nabla \eta_C\|^2 + \alpha_3 \|(I - P_K^C)\nabla C\|^2\} + \frac{\nu}{8} \|\mathbb{D}\boldsymbol{\phi}_u^h\|^2. \end{aligned} \tag{3.34}$$

The most important part of the analysis is for the velocity variable. We use (3.2) and the result given in [33] for the L^2 projection via

$$\|\mathbb{D}\boldsymbol{\phi}_u^h\| \leq \|(I - P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\| + CH^{-1} \|\boldsymbol{\phi}_u^h\| \tag{3.35}$$

for $\boldsymbol{\phi}_u^h \in V^h$. The estimations for velocity equation (3.29) yield

$$|(\partial_t \boldsymbol{\eta}_u, \boldsymbol{\phi}_u^h)| \le K\|\partial_t \boldsymbol{\eta}_u\|_{-1}\|\mathbb{D}\boldsymbol{\phi}_u^h\| \le K\|\partial_t \boldsymbol{\eta}_u\|_{-1}\left(\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\| + H^{-1}\|\boldsymbol{\phi}_u^h\|\right),$$

$$\le K\left(H^{-2}Da\|\partial_t \boldsymbol{\eta}_u\|_{-1}^2 + \alpha_1^{-1}\|\partial_t \boldsymbol{\eta}_u\|_{-1}^2\right) + \frac{Da^{-1}}{4}\|\boldsymbol{\phi}_u^h\|^2 + \frac{\alpha_1}{8}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2,$$

$$|\nu(\mathbb{D}\boldsymbol{\eta}_u, \mathbb{D}\boldsymbol{\phi}_u^h)| \le \frac{3\nu}{16}\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\nu^{-1}\|\mathbb{D}\boldsymbol{\eta}_u\|^2,$$

$$Da^{-1}|(\boldsymbol{\eta}_u, \boldsymbol{\phi}_u^h)| \le K\left(H^{-2}Da^{-1}\|\boldsymbol{\eta}_u\|_{-1}^2 + \alpha_1^{-1}Da^{-2}\|\boldsymbol{\eta}_u\|_{-1}^2\right) + \frac{Da^{-1}}{4}\|\boldsymbol{\phi}_u^h\|^2, + \frac{\alpha_1}{8}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2$$

$$|\alpha_1((I-P_L^u)\mathbb{D}\boldsymbol{\eta}_u, (I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h)| \le \frac{\alpha_1}{8}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\alpha_1\|(I-P_L^u)\mathbb{D}\boldsymbol{\eta}_u\|^2,$$

$$|\alpha_1((I-P_L^u)\mathbb{D}\mathbf{u}, (I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h)| \le \frac{\alpha_1}{8}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\alpha_1\|(I-P_L^u)\mathbb{D}\mathbf{u}\|^2,$$

$$|(p-q^h, \nabla\cdot\boldsymbol{\phi}_u^h)| \le \frac{3\nu}{16}\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\nu^{-1}\|p-q^h\|^2,$$

$$|\beta_T(\mathbf{g}(T-T^h), \boldsymbol{\phi}_u^h)| \le \beta_T\|\mathbf{g}\|_\infty\left(\|\eta_T\| + \|\phi_T^h\|\right)\|\boldsymbol{\phi}_u^h\| \le K\beta_T^2\|\mathbf{g}\|_\infty^2\left(\|\eta_T\|^2 + \|\phi_T^h\|^2 + \|\boldsymbol{\phi}_u^h\|^2\right),$$

$$|\beta_C(\mathbf{g}(C-C^h), \boldsymbol{\phi}_u^h)| \le \beta_C\|\mathbf{g}\|_\infty\left(\|\eta_C\| + \|\phi_C^h\|\right)\|\boldsymbol{\phi}_u^h\| \le K\beta_C^2\|\mathbf{g}\|_\infty^2\left(\|\eta_C\|^2 + \|\phi_C^h\|^2 + \|\boldsymbol{\phi}_u^h\|^2\right).$$

To estimate non-linear terms in (3.29), we first rewrite the trilinear forms

$$|c_0(\mathbf{u},\mathbf{u},\boldsymbol{\phi}_u^h) - c_0(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}_u^h)| \le |c_0(\boldsymbol{\eta}_u,\mathbf{u},\boldsymbol{\phi}_u^h)| + |c_0(\boldsymbol{\phi}_u^h,\mathbf{u},\boldsymbol{\phi}_u^h)| + |c_0(\mathbf{u}^h,\boldsymbol{\eta}_u,\boldsymbol{\phi}_u^h)|$$

and we bound them as

$$|c_0(\mathbf{u}^h,\boldsymbol{\eta}_u,\boldsymbol{\phi}_u^h)| \le \frac{3\nu}{16}\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\nu^{-1}\|\mathbb{D}\mathbf{u}^h\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2,$$

$$|c_0(\boldsymbol{\eta}_u,\mathbf{u},\boldsymbol{\phi}_u^h)| \le \frac{3\nu}{16}\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + K\nu^{-1}\|\mathbb{D}\mathbf{u}\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2,$$

$$|c_0(\boldsymbol{\phi}_u^h,\mathbf{u},\boldsymbol{\phi}_u^h)| \le K\left((\alpha_1^{-3}\|\mathbb{D}\mathbf{u}\|^4 + H^{-3/2}\|\mathbb{D}\mathbf{u}\|)\|\boldsymbol{\phi}_u^h\|^2 + \alpha_1\|(I-P_L^u)\mathbb{D}\mathbf{u}^h\|^2.\right)$$

So, from (3.29) we have

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{\phi}_u^h\|^2 + \frac{5\nu}{4}\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + \frac{\alpha_1}{2}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + \frac{Da^{-1}}{2}\|\boldsymbol{\phi}_u^h\|^2 \le \\
&K\{H^{-2}\left(Da\|\partial_t\boldsymbol{\eta}_u\|_{-1}^2 + Da^{-1}\|\boldsymbol{\eta}_u\|_{-1}^2\right) + \alpha_1^{-1}\left(\|\partial_t\boldsymbol{\eta}_u\|_{-1}^2 + Da^{-2}\|\boldsymbol{\eta}_u\|_{-1}^2\right) + \nu^{-1}\|\mathbb{D}\boldsymbol{\eta}_u\|^2 \\
&+\alpha_1\left(\|(I-P_L^u)\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|(I-P_L^u)\mathbb{D}\mathbf{u}\|^2\right) + \nu^{-1}\left(\|p-q^h\|^2 + \|\mathbb{D}\mathbf{u}^h\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|\mathbb{D}\mathbf{u}\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2\right) \\
&+\left(\alpha_1^{-3}\|\mathbb{D}\mathbf{u}\|^4 + H^{-3/2}\|\mathbb{D}\mathbf{u}\| + (\beta_T^2+\beta_C^2)\|\mathbf{g}\|_\infty^2\right)\|\boldsymbol{\phi}_u^h\|^2 + \beta_T^2\|\mathbf{g}\|_\infty^2\left(\|\eta_T\|^2 + \|\phi_T^h\|^2\right) \\
&+\beta_C^2\|\mathbf{g}\|_\infty^2\left(\|\eta_C\|^2 + \|\phi_C^h\|^2\right)\}. \qquad (3.36)
\end{aligned}$$

After this point we combine (3.33), (3.34) and (3.36) and apply Lemma 3.4. To apply Gronwall's Lemma, the $L^1(0,t^*)$- regularity of the appearing terms has to be studied.

First, we combine (3.33), (3.34) and (3.36) and multiply both sides with 2,

$$
\begin{aligned}
&\partial_t \left(\|\boldsymbol{\phi}_u^h\|^2 + \|\phi_T^h\|^2 + \|\phi_C^h\|^2\right) + \left(2\nu\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + \gamma\|\nabla\phi_T^h\|^2 + D_c\|\nabla\phi_C^h\|^2\right) + \left(\alpha_1\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2\right.\\
&\left.+\alpha_2\|(I-P_M^T)\nabla\phi_T^h\|^2 + \alpha_3\|(I-P_K^C)\nabla\phi_C^h\|^2\right) + Da^{-1}\|\boldsymbol{\phi}_u^h\|^2\\
&\le K\Big\{H^{-2}\left(Da\|\partial_t\boldsymbol{\eta}_u\|_{-1}^2 + Da^{-1}\|\boldsymbol{\eta}_u\|_{-1}^2\right) + \alpha_1^{-1}\left(\|\partial_t\boldsymbol{\eta}_u\|_{-1}^2 + Da^{-2}\|\boldsymbol{\eta}_u\|_{-1}^2\right) + \alpha_1\left(\|(I-P_L^u)\mathbb{D}\boldsymbol{\eta}_u\|^2\right.\\
&\left.+\|(I-P_L^u)\mathbb{D}\mathbf{u}\|^2\right) + \nu^{-1}\left(\|\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|p-q^h\|^2 + \|\mathbb{D}\mathbf{u}^h\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|\mathbb{D}\mathbf{u}\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2\right)\\
&+\|\mathbf{g}\|_\infty^2\left(\beta_T^2\|\eta_T\|^2 + \beta_C^2\|\eta_C\|^2\right) + \gamma^{-1}(\|\partial_t\eta_T\|_{-1}^2 + \|\nabla\eta_T\|^2 + \|\mathbb{D}\boldsymbol{\eta}_u\|^2\|\nabla T\|^2 + \|\mathbf{u}^h\|\|\mathbb{D}\mathbf{u}^h\|\|\nabla\eta_T\|^2)\\
&+\alpha_2\|(I-P_M^T)\nabla\eta_T\|^2 + \alpha_2\|(I-P_M^T)\nabla T\|^2 + D_c^{-1}(\|\partial_t\eta_C\|_{-1}^2 + \|\nabla\eta_C\|^2 + \|\mathbb{D}\boldsymbol{\eta}_u\|^2\|\nabla C\|^2\\
&+\|\mathbf{u}^h\|\|\mathbb{D}\mathbf{u}^h\|\|\nabla\eta_C\|^2) + \alpha_3\|(I-P_K^C)\nabla\eta_C\|^2 + \alpha_3\|(I-P_K^C)\nabla C\|^2 + \Big\{\alpha_1^{-3}\|\mathbb{D}\mathbf{u}\|^4 + H^{-3/2}\|\mathbb{D}\mathbf{u}\|\\
&+\|\mathbf{g}\|_\infty^2(\beta_T^2 + \beta_C^2) + \nu^{-1}\left(\gamma^{-1}\|\nabla T\|^4 + D_c^{-1}\|\nabla C\|^4\right)\Big\}\|\boldsymbol{\phi}_u^h\|^2\Big\}
\end{aligned}
$$

To apply the Lemma 3.4, we now define and appropriate λ and F_i, $i = 1,2,3$ functions. Clearly,

$$\lambda = \|\boldsymbol{\phi}_u^h\|^2 + \|\phi_T^h\|^2 + \|\phi_C^h\|^2.$$

For the others, we set

$$
\begin{aligned}
F_1(t) &= 2\nu\|\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + \alpha_1\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}_u^h\|^2 + \gamma\|\nabla\phi_T^h\|^2 + \alpha_2\|(I-P_M^T)\nabla\phi_T^h\|^2 + D_c\|\nabla\phi_C^h\|^2\\
&\quad+\alpha_3\|(I-P_K^C)\nabla\phi_C^h\|^2 + Da^{-1}\|\boldsymbol{\phi}_u^h\|^2, \qquad (3.37)\\
F_2(t) &= K\Big\{\left(H^{-2}+\alpha_1^{-1}\right)\left(\|\partial_t\boldsymbol{\eta}_u\|_{-1}^2 + \|\boldsymbol{\eta}_u\|^2\right) + \alpha_1\left(\|(I-P_L^u)\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|(I-P_L^u)\mathbb{D}\mathbf{u}\|^2\right)\\
&\quad+\nu^{-1}\left(\|\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|p-q^h\|^2 + \|\mathbb{D}\mathbf{u}^h\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2 + \|\mathbb{D}\mathbf{u}\|^2\|\mathbb{D}\boldsymbol{\eta}_u\|^2\right)\\
&\quad+\|\mathbf{g}\|_\infty^2\left(\beta_T^2\|\eta_T\|^2 + \beta_C^2\|\eta_C\|^2\right) + \|\partial_t\eta_T\|_{-1}^2 + \|\nabla\eta_T\|^2 + \alpha_2\|(I-P_M^T)\nabla\eta_T\|^2\\
&\quad+\alpha_2\|(I-P_M^T)\nabla T\|^2 + \|\mathbb{D}\boldsymbol{\eta}_u\|^2\|\nabla T\|^2 + \|\mathbf{u}^h\|\|\mathbb{D}\mathbf{u}^h\|\|\nabla\eta_T\|^2 + \|\partial_t\eta_C\|_{-1}^2 + \|\nabla\eta_C\|^2\\
&\quad+\alpha_3\|(I-P_K^C)\nabla\eta_C\|^2 + \alpha_3\|(I-P_K^C)\nabla C\|^2 + \|\mathbb{D}\boldsymbol{\eta}_u\|^2\|\nabla C\|^2 + \|\mathbf{u}^h\|\|\mathbb{D}\mathbf{u}^h\|\|\nabla\eta_C\|^2\Big\}
\end{aligned}
\qquad (3.38)
$$

and

$$F_3(t) = K\left(\alpha_1^{-3}\|\mathbb{D}\mathbf{u}\|^4 + H^{-3/2}\|\mathbb{D}\mathbf{u}\| + \|\mathbf{g}\|_\infty^2(\beta_T^2+\beta_C^2) + \nu^{-1}\left(\gamma^{-1}\|\nabla T\|^4 + D_c^{-1}\|\nabla C\|^4\right)\right). \qquad (3.39)$$

It is straightforward to show that $F_i(t)$, $i = 1,2,3$ functions satisfy the assumptions of Gronwall's Lemma, but for convenience let us show non-trivial parts only.

$$\int_0^t ||\mathbb{D}\mathbf{u}^h||^2||\mathbb{D}\boldsymbol{\eta}_u||^2 \leq ||\mathbb{D}\mathbf{u}^h||^2_{L^4(0,t;L^2)}||\mathbb{D}\boldsymbol{\eta}_u||^2_{L^4(0,t;L^2)} < \infty,$$

$$\int_0^t ||\mathbb{D}\mathbf{u}||^2||\mathbb{D}\boldsymbol{\eta}_u||^2 \leq ||\mathbb{D}\mathbf{u}||^2_{L^4(0,t;L^2)}||\mathbb{D}\boldsymbol{\eta}_u||^2_{L^4(0,t;L^2)} < \infty,$$

$$\int_0^t ||\mathbb{D}\boldsymbol{\eta}_u||^2||\nabla T||^2 \leq ||\mathbb{D}\boldsymbol{\eta}_u||^2_{L^4(0,t;L^2)}||\nabla T||^2_{L^4(0,t;L^2)} < \infty,$$

$$\int_0^t ||\mathbf{u}^h||\,||\mathbb{D}\mathbf{u}^h||\,||\nabla\eta_T||^2 \leq ||\mathbf{u}^h||_{L^\infty(0,t;L^2)}||\mathbb{D}\mathbf{u}^h||_{L^2(0,t;L^2)}||\nabla\eta_T||^2_{L^4(0,t;L^2)} < \infty,$$

$$\int_0^t ||\mathbb{D}\boldsymbol{\eta}_u||^2||\nabla C||^2 \leq ||\mathbb{D}\boldsymbol{\eta}_u||^2_{L^4(0,t;L^2)}||\nabla C||^2_{L^4(0,t;L^2)} < \infty,$$

$$\int_0^t ||\mathbf{u}^h||\,||\mathbb{D}\mathbf{u}^h||\,||\nabla\eta_C||^2 \leq ||\mathbf{u}^h||_{L^\infty(0,t;L^2)}||\mathbb{D}\mathbf{u}^h||_{L^2(0,t;L^2)}||\nabla\eta_C||^2_{L^4(0,t;L^2)} < \infty.$$

Note that we omit the positive finite constants in front of the integrals. These estimations are obtained by using the Hölder inequality, Poincaré's inequality, stability results and assumptions (3.24)-(3.25). Once we guarantee to apply the Gronwall's lemma , one can obtain the final error estimate by using Definition 3.3 and the triangle inequality. As a conclusion, we get the following convergence theorem.

Theorem 3.7 *Let $(\mathbf{u}^h, T^h, C^h)$ be the solution of (3.12)-(3.14). Suppose that (3.24), (3.25) hold and let $F_2(t)$ and $F_3(t)$ be given as in (3.38) and (3.39) respectively. Then the error satisfies for $0 \leq t^* < \infty$:*

$$\begin{aligned}
&||(\mathbf{u}-\mathbf{u}^h)||^2_{(Da^{-1},2)} + ||(T-T^h)||^2_{(1,\gamma)} + ||(C-C^h)||^2_{(1,D_c)} \\
&\leq K\Big\{ \inf_{\substack{\tilde{\mathbf{u}}^h\in L^2(0,t^*;V^h)\\ \tilde{T}^h\in L^2(0,t^*;W^h)\tilde{C}^h\in L^2(0,t^*;\Phi^h)}} \Big(||(\mathbf{u}-\tilde{\mathbf{u}}^h)||^2_{(Da^{-1},2)} + ||(T-\tilde{T}^h)||^2_{(1,\gamma)} + ||(C-\tilde{C}^h)||^2_{(1,D_c)}\Big) \\
&+ \exp\left(\int_0^t F_3(s)\mathrm{d}s\right) \inf_{\substack{\tilde{\mathbf{u}}^h\in L^2(0,t^*;V^h)\\ \tilde{T}^h\in L^2(0,t^*;W^h),\tilde{C}^h\in L^2(0,t^*;\Phi^h)}} \Big(||(\mathbf{u}^h-\tilde{\mathbf{u}}^h)(0)||^2 + ||(T^h-\tilde{T}^h)(0)||^2 \\
&+||(C^h-\tilde{C}^h)(0)||^2 + \int_0^t F_2(s)\mathrm{d}s\Big)\Big\}
\end{aligned}$$

with constant K which may depend on domain Ω and independent from $h, \alpha_1, \alpha_2, \alpha_3, \nu, \gamma, D_c$.

Remark 3.8 *Let us study the convergence of the Theorem 3.7. The right hand side of (3.38) includes the crucial terms $\alpha_1||(I-P^u_L)\nabla u||^2, \alpha_2||(I-P^T_M)\nabla T||^2, \alpha_3||(I-P^C_K)\nabla C||^2$. Except, these three terms, all other terms contain interpolation errors. The crucial terms does not have a factor of interpolation error terms. However, they tend to zero as mesh width $h \to 0$*

if $\alpha_1, \alpha_2, \alpha_3 \to 0$. *In this case, standard Galerkin finite element discretization is recovered asymptotically.*

In the next corollary, we present a typical example of the finite element spaces and the parameter choices of $(\alpha_i, H) = (h^2, h^{1/2})$, for $i = 1, 2, 3$. We note that the parameters are chosen in such a way that the crucial terms behave at least as interpolation error.

Corollary 3.9 *Assume that* $(\mathbf{u}, p, T, C) \in (X \cap H^{s+1}(\Omega), Q \cap H^s(\Omega), W \cap H^{s+1}(\Omega), \Psi \cap H^{s+1}(\Omega))$ *and let the finite element spaces are chosen as*

$$\begin{aligned}
X^h &= \{\mathbf{v} \in C^0(\overline{\Omega}) : \mathbf{v}|_\Delta \in P_2(\Delta), \forall \Delta \in \tau^h\},\\
W^h &= \{S \in C^0(\overline{\Omega}) : S|_\Delta \in P_2(\Delta), \forall \Delta \in \tau^h\},\\
\Psi^h &= \{\Phi \in C^0(\overline{\Omega}) : \Phi|_\Delta \in P_2(\Delta), \forall \Delta \in \tau^h\},\\
Q^h &= \{\mathbf{v} \in C^0(\overline{\Omega}) : \mathbf{v}|_\Delta \in P_1(\Delta), \forall \Delta \in \tau^h\},\\
L^H &= \{l^H \in L^2(\Omega) : l^H|_\Delta \in P_1(\Delta), \forall \Delta \in \tau^H\},\\
M^H &= \{m^H \in L^2(\Omega) : m^H|_\Delta \in P_1(\Delta), \forall \Delta \in \tau^H\},\\
K^H &= \{k^H \in L^2(\Omega) : k^H|_\Delta \in P_1(\Delta), \forall \Delta \in \tau^H\}.
\end{aligned}$$

Then, the error becomes

$$||\mathbb{D}(\mathbf{u} - \mathbf{u}^h)|| + ||\nabla(T - T^h)|| + ||\nabla(C - C^h)|| \cong O(h^2)$$

along with the choices of $(\alpha_i, H) = (h^2, h^{1/2})$ *for* $i = 1, 2, 3$.

3.3 Numerical Studies

In this section, we perform some numerical tests to validate the effectiveness of our method. We carry out these analysis in two different steps depending on the buoyancy ratio N. We first consider the case $N = 0$ which corresponds to a system of pure thermal convection in a porous medium. Then we investigate the performance of the method under different buoyancy ratios with $N \neq 0$.

In our computations, we particularly select conforming Taylor-Hood finite elements which are known to satisfy the inf-sup condition (2.20). Parameter and mesh scalings are arranged as $H \sim h^{1/2}$ and $\alpha_i \sim h^2$ for $i = 1, 2, 3$. The use free finite element software *FreeFem++* as in previous chapter for making these computations [24]. The following iterative scheme is utilized for solving the system

$$
\begin{aligned}
&\frac{1}{\tau}(\mathbf{u}^{(m)} - \mathbf{u}^{(m-1)}, \mathbf{v}^h) + 2\nu(\mathbb{D}\mathbf{u}^{(m)}, \mathbb{D}\mathbf{v}^h) + c_0(\mathbf{u}^{(m)}, \mathbf{u}^{(m-1)}, \mathbf{v}^h) + c_0(\mathbf{u}^{(m-1)}, \mathbf{u}^{(m)}, \mathbf{v}^h) \\
&+(Da^{-1}\mathbf{u}^{(m)}, \mathbf{v}^h) = Gr_T(T^{(m)}, \mathbf{v}^h) + Gr_C(C^{(m)}, \mathbf{v}^h) + c_0(\mathbf{u}^{(m-1)}, \mathbf{u}^{(m-1)}, \mathbf{v}^h) \\
&-\alpha_1((I - P_L^u)\mathbb{D}\mathbf{u}^{(m-1)}, (I - P_L^u)\mathbb{D}\mathbf{v}^h) \\
&(q^h, \nabla \cdot \mathbf{u}^{(m)}) = 0 \\
&\frac{1}{\tau}(T^{(m)} - T^{(m-1)}, S^h) + \frac{1}{Pr}(\nabla T^{(m)}, \nabla S^h) + c_1(\mathbf{u}^{(m-1)}, T^{(m)}, S^h) + c_1(\mathbf{u}^{(m)}, T^{(m-1)}, S^h) \\
&= c_1(\mathbf{u}^{(m-1)}, T^{(m-1)}, S^h) - \alpha_2((I - P_M^T)\nabla T^{(m-1)}, (I - P_M^T)\nabla S^h) \\
&\frac{1}{\tau}(C^{(m)} - C^{(m-1)}, \Phi^h) + \frac{1}{Sc}(\nabla C^{(m)}, \nabla \Phi^h) + c_2(\mathbf{u}^{(m-1)}, C^{(m)}, \Phi^h) + c_2(\mathbf{u}^{(m)}, C^{(m-1)}, \Phi^h) \\
&= c_1(\mathbf{u}^{(m-1)}, C^{(m-1)}, \Phi^h) - \alpha_3((I - P_K^C)\nabla C^{(m-1)}, (I - P_K^C)\nabla \Phi^h) \qquad (3.40)
\end{aligned}
$$

for all $(\mathbf{v}^h, q^h, S^h, \Phi^h) \in (X^h, Q^h, W^h, \Psi^h)$. The scheme initializes by an initial guess $(\mathbf{u}^0, T^0, C^0)$ and generates $(\mathbf{u}^m, p^m, T^m, C^m)$. Note that we put some dimensionless numbers as Gr_T, Gr_C, Pr, Sc in the computational scheme in order to make comparisons with published results. Explicit definition of these numbers were given in the beginning of this chapter. We also use another dimensionless parameter namely, the thermal Rayleigh number, $Ra = Gr_T Pr Da$ for comparison issues which is not put on scheme (3.40) directly.

The computational domain we use is a classical rectangle with an aspect ratio of $A = H/L$. We mostly prefer the case $A = 1$. Figure 3.1 illustrates the domain with its boundary conditions. We employ no-slip velocity boundary conditions for whole boundary. Horizontal walls are kept adiabatic and impermeable i.e. $\partial T/\partial \mathbf{n} = \partial C/\partial \mathbf{n} = 0$ at these walls. Temperature and concentration are kept at T_0, C_0 for right and T_1, C_1 for left vertical walls with $T_0 < T_1$ and $C_0 < C_1$ respectively. We pick $T_0 = C_0 = -1$ and $T_1 = C_1 = 1$ here. In the computations, besides the other dimenionless parameters given before, we use another dimensionless parameter namely, the thermal Rayleigh number, $Ra = Gr_T Pr Da$ for comparison issues.

We carry out numerical tests in two different steps depending on the buoyancy ratio N. We first consider the case $N = 0$ which corresponds to a system of pure thermal convection in a porous medium. Then we also investigate the performance of the method under different buoyancy ratios with $N \neq 0$.

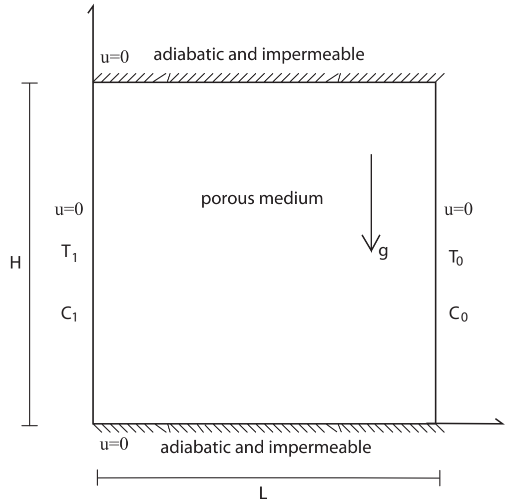

Figure 3.1: The computational domain with its boundary conditions.

3.3.1 Case I: The Buoyancy ratio $N = 0$

In the case of $N = 0$, which is a purely thermal natural convection in a porous cavity, the flow is solely driven by the thermal natural convection. The accuracy of the numerical results are checked by comparing Nusselt and Sherwood numbers. We compare our results with the well-known benchmark studies of Lauriat et al. [43], Trevisan et al. [68] and Goyeu et al. [20]. Although this configuration does not involve solutal buoyancy force, mass transfer still occurs due to the density differences led by thermal forces. For all values of N, the Darcy number (Da) has a very crucial role in the analysis of such kind of flows. Even with making use of a Brinkman extended formulation as in the present study, the model is said to be in Darcy regime provided that the Darcy number is less than or near 10^{-7}.

For $N = 0$ case, we concentrate on comparing our Nusselt and Sherwood numbers with benchmark data. In engineering, the Nusselt and Sherwood numbers are of great importance. Roughly, average Nusselt number is the dimensionless heat flux and average Sherwood number is dimensionless mass flux in the system which are given explicitly as

$$Nu = \frac{1}{A}\int_0^A \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \qquad Sh = \frac{1}{A}\int_0^A \left(\frac{\partial C}{\partial x}\right)_{x=0} dy.$$

We first give values of the Nusselt number at $A = 5$ for the different Da in which, both Darcy-

Brinkman and Darcy regimes are taken into account. Table 3.1 summarizes these results along with the results obtained by [43, 20] which enable us to make a clear comparison. A similar computation is carried out for the case $A = 1$ in which, the Sherwood numbers are also calculated additionally. Such kind of analysis was presented in benchmark studies of [68] and [20] and we introduce our results in comparison with those studies in Table 3.2. For the values in both tables, one can observe the excellent agreement with previously published data even with coarser grids. We use a uniform grid of maximum 24×24 for $A = 1$ and 24×44 for $A = 5$ where a sinusoidal grid of 64×64, 145×95 was used in [20] and uniform 41×41, 41×81 was used in [43] for $A = 1$ and $A = 5$ respectively. A uniform grid of 42×42 was used in [68] also. This is a significant advantage of the proposed method.

Table 3.1: Comparison of average Nusselt numbers for $N = 0$ at $A = 5$ with different Da and thermal Rayleigh numbers.

Da		10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-7}
Ra=500	Present Study	7.30	9.11	10.01	10.30	10.42
	Ref.[43]	7.25	9.15	9.95	10.25	10.40
	Ref.[20]	7.29	9.13	10.00	10.34	10.39
Ra=1000	Present Study	9.45	12.49	14.32	15.06	15.17
	Ref.[43]	9.44	12.55	14.28	14.99	15.19
	Ref.[20]	9.45	12.60	14.30	14.90	15.15

Table 3.2: Comparison of average Nusselt and Sherwood numbers for $N = 0$, $Le = 10$ at $A = 1$ with different thermal Rayleigh numbers (Darcy Regime).

Ra		100	200	400	1000	2000
Nu	Present Study	3.15	5.02	7.83	14.01	20.00
	Ref.[68]	3.27	5.61	9.69	–	–
	Ref.[20]	3.11	4.96	7.77	13.47	19.90
Sh	Present Study	13.54	20.11	27.96	48.01	71.25
	Ref.[68]	15.61	23.23	30.73	–	–
	Ref.[20]	13.25	19.86	28.41	48.32	69.29

3.3.2 Case II: The Buoyancy ratio $N \neq 0$

The case of $N \neq 0$ is identified as mass driven flow by [68] in the flow configuration. We take N in between 0 and 35, Ra in 100 and 1000 and Le in 1 and 300 throughout our computations. As in the case of $N = 0$, we calculate average Nusselt numbers for evaluating the performance of the method. However, the effect of Lewis number and Darcy number on Nusselt Number for different N values are also considered. Other parameters, A and Pr are kept at $A = 1$, $Pr = 10$ throughout all simulations. The effect of Darcy number on flow patterns is illustrated by streamline, isotherm and isoconcentration lines in Figure 3.2. For displaying these patterns, we take $N = 10, Ra = 100$ and $Le = 10$. As could be concluded from the figure, when the Darcy number increases and the flow character turns into Darcy-Brinkman regime from Darcy regime, boundary layers become thicker in streamlines and concentration gradients get smaller. This is because the Brinkman term in the system of equations turns out to be more significant as Darcy number increases. Displayed flow patterns match perfectly with ones given in [20] via using the coarser meshes stated in previous part. Lewis number has a

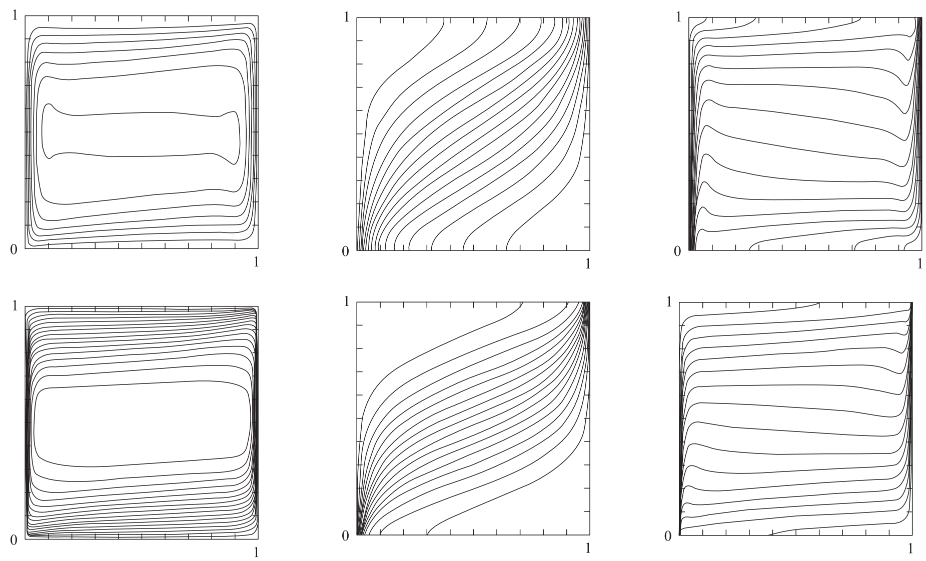

Figure 3.2: Streamlines, isotherms and isoconcentration lines for $Da = 10^{-3}$ (upper left to right) and streamlines, isotherms and isoconcentration lines for $Da = 10^{-7}$(lower left to right), respectively.

crucial role on average heat and mass transfer in the system also. We know that, Nusselt and Sherwood numbers are identical in the classical case $Le = 1$. We now investigate the effect of Lewis number on average heat transfer for different Le and N. In order to understand stated effects, we display the Nusselt number as a function of Le in Figure 3.3. The scale analysis in [68] states that Nu satisfies

$$Nu \sim \left(\frac{RaN}{Le}\right)^{1/2}.$$

One can read this situation as, the average heat transfer is directly proportional to Ra, N and it decreases as Le increases. Clearly, Figure 3.3 supports this scaling and the same ideas stated in [68]. Since $Le = \frac{\gamma}{D_c}$, increasing Le restricts the mass transfer. So the effect of N also gets restricted. At a given Le, N favors inflation which also increases Nu. In order to make

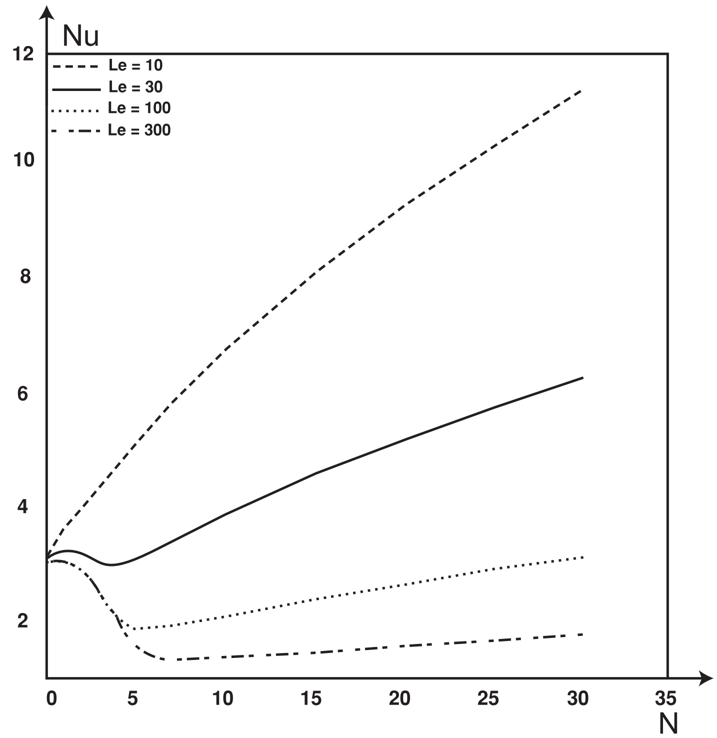

Figure 3.3: Nusselt number as a function of Lewis number.

distinctions about Darcy and Darcy-Brinkman regimes, we present the results about the effect of Darcy number on Nusselt number. We can conclude from Figure 3.4 that, as N increases, Nusselt number tends to a pure conduction limit, $Nu \cong 1$, through passing to Darcy-Brinkman regime with a high Lewis number of 100 [20].

As a last graphical interpretation, we give the variation of vertical velocity, temperature and concentration at mid-height in Figure 3.5. These profiles are perfectly comparable with ones given in [20], which is the only available reference presenting such kind of illustration.

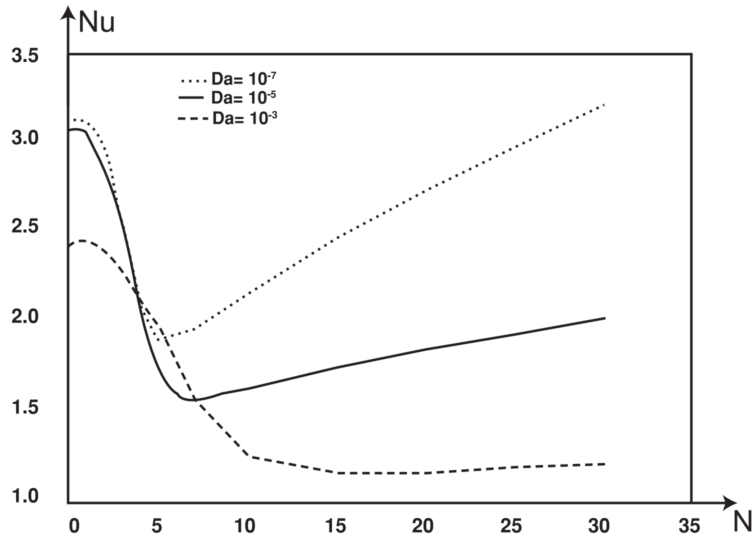

Figure 3.4: Nusselt number as a function of N with varying Darcy numbers.

3.4 A fully discrete scheme

We consider the discretization of the system (3.1) for finite element in space and Crank-Nicolson in time in this section. Instead of usual Crank-Nicolson method, we prefer a more accurate fully implicit version which is obtained via linear extrapolation of convecting velocity terms [4, 5, 42, 31]. Before giving the algorithm, we note here that since the method we study is a two-step method, we specify the first step individually along with the initial conditions.

Definition 3.10 *(Crank-Nicolson with Linear Extrapolation-CNLE) The CNLE scheme for (3.1) for $n \geq 1$ reads: given* $(\mathbf{u}_n^h, p_n^h, T_n^h, C_n^h) \in (X^h, Q^h, W^h, \Psi^h)$ *find* $(\mathbf{u}_{n+1}^h, p_{n+1}^h, T_{n+1}^h, C_{n+1}^h)$

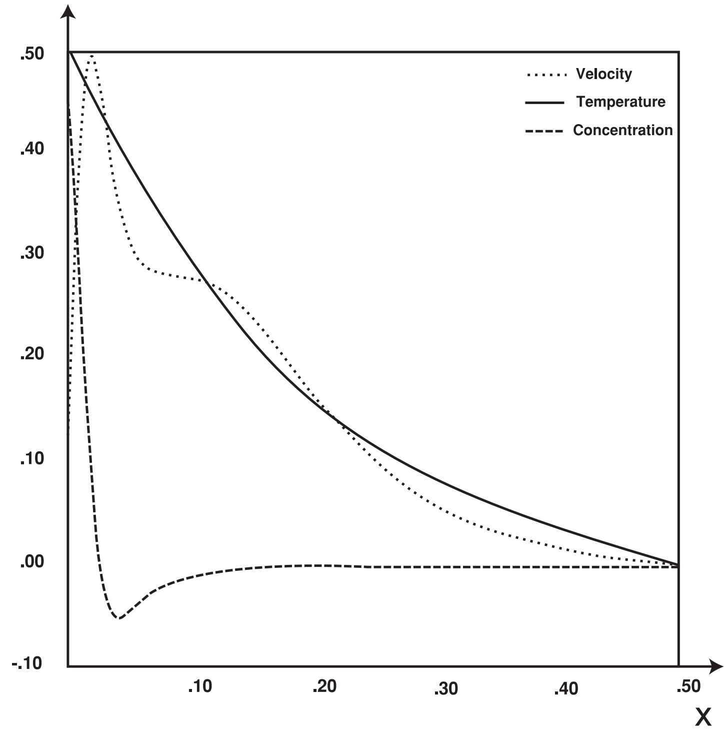

Figure 3.5: Vertical velocity, temperature and concentration profiles at mid-height for $Da = 10^{-3}$, $Ra = 100$, $Le = 100$ at $A = 1$.

satisfying

$$
\begin{aligned}
&\left(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{\tau}, \mathbf{v}^h\right) + 2\nu\left(\mathbb{D}\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbb{D}\mathbf{v}^h\right) + \alpha_1\left((I - P_L^u)\mathbb{D}\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, (I - P_L^u)\mathbb{D}\mathbf{v}^h\right)\\
&+Da^{-1}\left(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbf{v}^h\right) - \left(\frac{p_{n+1}^h + p_n^h}{2}, \nabla\cdot\mathbf{v}^h\right) + c_0\left(\chi(\mathbf{u}_n^h), \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbf{v}^h\right)\\
&= \beta_T\left(\mathbf{g}\frac{T_{n+1}^h + T_n^h}{2}, \mathbf{v}^h\right) + \beta_C\left(\mathbf{g}\frac{C_{n+1}^h + C_n^h}{2}, \mathbf{v}^h\right) \qquad (3.41)
\end{aligned}
$$

$$
\left(q^h, \nabla\cdot\mathbf{u}_{n+1}^h\right) = 0,
$$

$$
\begin{aligned}
&\left(\frac{T_{n+1}^h - T_n^h}{\tau}, S^h\right) + \gamma\left(\nabla\left(\frac{T_{n+1}^h + T_n^h}{2}\right), \nabla S^h\right) + \alpha_2\left((I - P_M^T)\nabla\left(\frac{T_{n+1}^h + T_n^h}{2}\right), (I - P_M^T)\nabla S^h\right)\\
&+c_1\left(\chi(\mathbf{u}_n^h), \frac{T_{n+1}^h + T_n^h}{2}, S^h\right) = 0 \qquad (3.42)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{C_{n+1}^h - C_n^h}{\tau}, \Phi^h\right) + D_c\left(\nabla\left(\frac{C_{n+1}^h + C_n^h}{2}\right), \nabla\Phi^h\right) + \alpha_3\left((I - P_K^C)\nabla\left(\frac{C_{n+1}^h + C_n^h}{2}\right), (I - P_K^C)\nabla\Phi^h\right)\\
&+c_1\left(\chi(\mathbf{u}_n^h), \frac{C_{n+1}^h + C_n^h}{2}, \Phi^h\right) = 0 \qquad (3.43)
\end{aligned}
$$

where $\tau > 0$ is a given time step and $\chi(\mathbf{u}_n^h) = \frac{1}{2}\left(3\mathbf{u}_n^h - \mathbf{u}_{n-1}^h\right)$.

For the first time step find $(\mathbf{u}_1^h, p_1^h, T_1^h, C_1^h)$ satisfying

$$\left(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{\tau}, \mathbf{v}^h\right) + 2\nu\left(\mathbb{D}\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbb{D}\mathbf{v}^h\right) + \alpha_1\left((I - P_L^u)\mathbb{D}\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, (I - P_L^u)\mathbb{D}\mathbf{v}^h\right)$$
$$+Da^{-1}\left(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h\right) - \left(\frac{p_1^h + p_0^h}{2}, \nabla\cdot\mathbf{v}^h\right) + c_0\left(\mathbf{u}_0^h, \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h\right)$$
$$= \beta_T\left(\mathbf{g}\frac{T_1^h + T_0^h}{2}, \mathbf{v}^h\right) + \beta_C\left(\mathbf{g}\frac{C_1^h + C_0^h}{2}, \mathbf{v}^h\right) \tag{3.44}$$
$$\left(q^h, \nabla\cdot\mathbf{u}_1^h\right) = 0,$$
$$\left(\frac{T_1^h - T_0^h}{\tau}, S^h\right) + \gamma\left(\nabla\left(\frac{T_1^h + T_0^h}{2}\right), \nabla S^h\right) + \alpha_2\left((I - P_M^T)\nabla\left(\frac{T_1^h + T_0^h}{2}\right), (I - P_M^T)\nabla S^h\right)$$
$$+c_1\left(\mathbf{u}_0^h, \frac{T_1^h + T_0^h}{2}, S^h\right) = 0 \tag{3.45}$$
$$\left(\frac{C_1^h - C_0^h}{\tau}, \Phi^h\right) + D_c\left(\nabla\left(\frac{C_1^h + C_0^h}{2}\right), \nabla\Phi^h\right) + \alpha_3\left((I - P_K^C)\nabla\left(\frac{C_1^h + C_0^h}{2}\right), (I - P_K^C)\nabla\Phi^h\right)$$
$$+c_1\left(\mathbf{u}_0^h, \frac{C_1^h + C_0^h}{2}, \Phi^h\right) = 0 \tag{3.46}$$

with the modified stokes projection $\mathbf{u}_0^h, T_0^h, C_0^h$ of $\mathbf{u}_0, T_0, C_0$ into V^h, W^h and Ψ^h respectively.

We now give the definition of so-called *modified Stokes projection* operators for our new system. In Chapter 2, *modified Stokes projection* is established for velocity and temperature. For the new system, we give it for also the concentration of the flow field. Throughout the error analysis, choosing the *modified Stokes projection* of each variable for the approximation terms in the splitting of these variables simplifies the error analysis. We first state the stability of these projections and give the related error bounds. From now on, we denote $\zeta_{n+1/2} = \frac{\zeta_{n+1}+\zeta_n}{2}$ for any function or variable ζ.

Definition 3.11 *(Modified Stokes projections) The operator of the modified Stokes projection for the velocity and pressure, P_S, is defined by; $P_S : (X, Q) \to (X^h, Q^h)$, $\quad P_S(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p})$ where*

$$2\nu(\mathbb{D}(\mathbf{u} - \tilde{\mathbf{u}}), \mathbb{D}\mathbf{v}^h) + \alpha_1((I - P_L^u)\mathbb{D}(\mathbf{u} - \tilde{\mathbf{u}}), (I - P_L^u)\mathbb{D}\mathbf{v}^h) + Da^{-1}(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}^h) - (p - \tilde{p}, \nabla\cdot\mathbf{v}^h) = 0,$$
$$(q^h, \nabla\cdot(\mathbf{u} - \tilde{\mathbf{u}})) = 0$$

for all $(\mathbf{v}^h, q^h) \in (X^h, Q^h)$. In the discretely divergence free space V^h and in the pressure space

Q^h, this definition reduces to

$$2\nu(\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}),\mathbb{D}\mathbf{v}^h)+\alpha_1((I-P_L^u)\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}),(I-P_L^u)\mathbb{D}\mathbf{v}^h)+Da^{-1}(\mathbf{u}-\tilde{\mathbf{u}},\mathbf{v}^h)-(p-q^h,\nabla\cdot\mathbf{v}^h)=0 \tag{3.47}$$

for all $\mathbf{v}^h \in V^h$. The modified Stokes projection operator for the temperature, P_T, is defined by $P_T : W \to W^h$, $P_T(T) = \tilde{T}$ where

$$\gamma(\nabla(T-\tilde{T}),\nabla S^h)+\alpha_2((I-P_M^T)\nabla(T-\tilde{T}),(I-P_M^T)\nabla S^h)=0 \tag{3.48}$$

for all $S^h \in W^h$.

Finally for concentration P_C is given by $P_C : \Psi \to \Psi^h$, $P_C(C) = \tilde{C}$ where

$$D_c(\nabla(C-\tilde{C}),\nabla\Phi^h)+\alpha_3((I-P_K^C)\nabla(C-\tilde{C}),(I-P_K^C)\nabla\Phi^h)=0 \tag{3.49}$$

for all $\Phi^h \in \Psi^h$.

Lemma 3.12 *The modified Stokes projections defined by (3.47)-(3.49) are stable in the following sense:*

$$\begin{aligned}
2\nu\|\mathbb{D}\tilde{\mathbf{u}}\|^2+\alpha_1\|(I-P_L^u)\mathbb{D}\tilde{\mathbf{u}}\|^2+Da^{-1}\|\tilde{\mathbf{u}}\|^2 &\le K\Big(\nu\|\mathbb{D}\mathbf{u}\|^2+\alpha_1\|(I-P_L^u)\mathbb{D}\mathbf{u}\|^2\\
&+ Da^{-1}\|\mathbf{u}\|^2+\nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2\Big)\\
\gamma\|\nabla\tilde{T}\|^2+\alpha_2\|(I-P_M^T)\nabla\tilde{T}\|^2 &\le K\Big(\gamma\|\nabla T\|^2+\alpha_2\|(I-P_M^T)\nabla T\|^2\Big)\\
D_c\|\nabla\tilde{C}\|^2+\alpha_3\|(I-P_K^C)\nabla\tilde{C}\|^2 &\le K\Big(D_c\|\nabla C\|^2+\alpha_3\|(I-P_K^C)\nabla C\|^2\Big)
\end{aligned}$$

Proof. Setting $\mathbf{v}^h = \tilde{\mathbf{u}}$ in (3.47) and using the Cauchy-Schwarz inequality yield

$$\begin{aligned}
2\nu\|\mathbb{D}\tilde{\mathbf{u}}\|^2+\alpha_1\|(I-P_L^u)\mathbb{D}\tilde{\mathbf{u}}\|^2+Da^{-1}\|\tilde{\mathbf{u}}\|^2 \le\ & 2\nu\|\mathbb{D}\tilde{\mathbf{u}}\|\|\mathbb{D}\mathbf{u}\|+\alpha_1\|(I-P_L^u)\mathbb{D}\tilde{\mathbf{u}}\|\|(I-P_L^u)\mathbb{D}\mathbf{u}\|\\
&+Da^{-1}\|\tilde{\mathbf{u}}\|\|\mathbf{u}\|+\|p-q^h\|\|\nabla\cdot\tilde{\mathbf{u}}\|
\end{aligned}$$

thanks to the homogeneous boundary conditions, we have $\|\nabla\cdot\tilde{\mathbf{u}}\| \le \|\nabla\tilde{\mathbf{u}}\|$ and the result is directly obtained through the application of Young's and Korn's inequalities. Setting $S^h = \tilde{T}$ in (3.48) and following similar steps as above, one gets

$$\gamma\|\nabla\tilde{T}\|^2+\alpha_2\|(I-P_M^T)\nabla\tilde{T}\|^2 \le \gamma\|\nabla\tilde{T}\|\|\nabla T\|+\alpha_2\|(I-P_M^T)\nabla\tilde{T}\|\|(I-P_M^T)\nabla T\|$$

and an application of Young's inequality gives the desired result for temperature variable. The result for the concentration part could easily be obtained by setting $\Phi^h = \tilde{C}$ in (3.49) and employing exactly same manipulations performed for the temperature variable. □

The next lemma states the error in those projection operators.

Lemma 3.13 *(Error in modified Stokes projections) Suppose the discrete inf-sup condition (2.20) holds. Then* $(\tilde{\mathbf{u}}, \tilde{T}, \tilde{C})$ *exists uniquely in* (X^h, W^h, Ψ^h) *and satisfies*

$$\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}})\|^2 + \alpha_1\|(I-P_L^u)\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}})\|^2 + Da^{-1}\|(\mathbf{u}-\tilde{\mathbf{u}})\|^2 \leq K \inf_{\hat{\mathbf{u}}\in X^h, q^h\in Q^h}(\nu\|\mathbb{D}(\mathbf{u}-\hat{\mathbf{u}})\|^2$$
$$+\alpha_1\|(I-P_L^u)\mathbb{D}(\mathbf{u}-\hat{\mathbf{u}})\|^2 + Da^{-1}\|(\mathbf{u}-\hat{\mathbf{u}})\|^2 + \nu^{-1}\|p-q^h\|^2), \quad (3.50)$$

$$\gamma\|\nabla(T-\tilde{T})\|^2 + \alpha_2\|(I-P_M^T)\nabla(T-\tilde{T})\|^2 \leq K \inf_{\hat{T}\in W^h}(\gamma\|\nabla(T-\hat{T})\|^2$$
$$+\alpha_2\|(I-P_M^T)\nabla(T-\hat{T})\|^2). \quad (3.51)$$

$$D_c\|\nabla(C-\tilde{C})\|^2 + \alpha_3\|(I-P_K^C)\nabla(C-\tilde{C})\|^2 \leq K \inf_{\hat{C}\in\Psi^h}(D_c\|\nabla(C-\hat{C})\|^2$$
$$+\alpha_3\|(I-P_K^C)\nabla(C-\hat{C})\|^2). \quad (3.52)$$

Proof. To prove (3.50), decompose the error $\mathbf{u}-\tilde{\mathbf{u}} = \boldsymbol{\eta}-\boldsymbol{\phi}^h$, where $\boldsymbol{\eta} = \mathbf{u}-\hat{\mathbf{u}}$, $\boldsymbol{\phi} = \tilde{\mathbf{u}}-\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ approximates $\mathbf{u}$ in V^h. Thus (3.47) becomes

$$2\nu(\mathbb{D}\boldsymbol{\phi}^h, \mathbb{D}\mathbf{v}^h) + \alpha_1((I-P_L^u)\mathbb{D}\boldsymbol{\phi}^h, (I-P_L^u)\mathbb{D}\mathbf{v}^h) + Da^{-1}(\boldsymbol{\phi}^h, \mathbf{v}^h)$$
$$= 2\nu(\mathbb{D}\boldsymbol{\eta}, \mathbb{D}\mathbf{v}^h) + \alpha_1((I-P_L^u)\mathbb{D}\boldsymbol{\eta}, (I-P_L^u)\mathbb{D}\mathbf{v}^h) + Da^{-1}(\boldsymbol{\eta}, \mathbf{v}^h) + (p-q^h, \nabla.\mathbf{v}^h) \quad (3.53)$$

Setting $\mathbf{v}^h = \boldsymbol{\phi}^h$ in (3.53) and applying Cauchy-Schwarz, Young's and Korn's inequalities yield

$$\begin{aligned}\|\mathbb{D}\boldsymbol{\phi}^h\|^2 + \alpha_1\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}^h\|^2 + Da^{-1}\|\boldsymbol{\phi}^h\|^2 &\leq K \inf_{\hat{\mathbf{u}}\in X^h, q^h\in Q^h}(\|\mathbb{D}(\mathbf{u}-\hat{\mathbf{u}})\|^2 \\ &+ \alpha_1\|(I-P_L^u)\mathbb{D}(\mathbf{u}-\hat{\mathbf{u}})\|^2 \\ &+ Da^{-1}\|(\mathbf{u}-\hat{\mathbf{u}})\|^2 + \nu^{-1}\|p-q^h\|^2). \quad (3.54)\end{aligned}$$

Since $\hat{\mathbf{u}}$ is an approximation of $\mathbf{u}$ in V^h, we can take infimum over V^h in (3.54). The final error estimate for velocity now follows from the triangle inequality. In order to prove (3.51) and (3.52), split the relevant errors for each variable as $(T-\tilde{T}) = (T-\hat{T})-(\tilde{T}-\hat{T}) = \eta_T-\phi_T$ where $\hat{T}$ approximates T in W^h and $(C-\tilde{C}) = (C-\hat{C})-(\tilde{C}-\hat{C}) = \eta_C-\phi_C$ where $\hat{C}$ approximates C in Ψ^h. As in the first part, we have

$$\begin{aligned}\gamma(\nabla\phi_T, \nabla S^h) + \alpha_2((I-P_M^T)\nabla\phi_T, (I-P_M^T)\nabla S^h) &\leq \gamma(\nabla\eta_T, \nabla S^h) \\ &+ \alpha_2((I-P_M^T)\nabla\eta_T, (I-P_M^T)\nabla S^h)\end{aligned}$$
$$(3.55)$$

for the temperature and

$$\begin{aligned} D_c(\nabla\phi_C, \nabla\Phi^h) + \alpha_3((I-P_K^C)\nabla\phi_C, (I-P_K^C)\nabla\Phi^h) &\le \gamma(\nabla\eta_C, \nabla\Phi^h) \\ &+ \alpha_3((I-P_K^C)\nabla\eta_C, (I-P_K^C)\nabla\Phi^h) \end{aligned} \tag{3.56}$$

for the concentration parts. We now let $S^h = \phi_T$ and $\Phi^h = \phi_C$ in (3.55) and (3.56) respectively. Final results are obtained via following the same steps in the velocity part. □

Corollary 3.14 *Let the regularity assumptions,* $(\mathbf{u}, p, T, C) \in (X \cap H^{s+1}(\Omega), Q \cap H^s(\Omega), W \cap H^{s+1}(\Omega), \Psi \cap H^{s+1}(\Omega))$ *holds. Then the use of the estimations (3.15)-(3.18) in (3.50)-(3.52) yield*

$$\begin{aligned} &2\nu\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}})\|^2 + \alpha_1\|(I-P_L^u)\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}})\|^2 + Da^{-1}\|(\mathbf{u}-\tilde{\mathbf{u}})\|^2 \\ &\le K\left(\nu h^{2s}\|\mathbf{u}\|_{s+1}^2 + \alpha_1 h^{2s}\|\mathbf{u}\|_{s+1}^2 + Da^{-1}h^{2s+2}\|\mathbf{u}\|_{s+1}^2 + \nu^{-1}h^{2s}\|p\|_s^2\right) \end{aligned} \tag{3.57}$$

$$\gamma\|\nabla(T-\tilde{T})\|^2 + \alpha_2\|(I-P_M^T)\nabla(T-\tilde{T})\|^2 \le K(\gamma+\alpha_2)(h^{2s}\|T\|_{s+1}^2) \tag{3.58}$$

$$D_c\|\nabla(C-\tilde{C})\|^2 + \alpha_3\|(I-P_K^C)\nabla(C-\tilde{C})\|^2 \le K(D_c+\alpha_3)(h^{2s}\|C\|_{s+1}^2) \tag{3.59}$$

Before beginning the analysis of the method, we give some preliminary lemmas which we use frequently during this section.

Lemma 3.15 *Let* $\lambda(\cdot, t)$ *be a function, the time step* $\tau = t_{n+1} - t_n$ *and* $t_{n+1/2} = \frac{t_{n+1}+t_n}{2}$*. The following estimates hold true under the stated conditions.*

1. *If* $(\partial_t\lambda) \in C^0(0, t^*; L^2(\Omega))$ *then* $\left\|\frac{\lambda(\cdot,t_{n+1})-\lambda(\cdot,t_n)}{\tau}\right\| \le K\left\|\partial_t\lambda(\cdot,\tilde{t})\right\|$ *for* $\tilde{t} \in (t_0, t^*)$.
2. *If* $(\partial_t^2\lambda) \in C^0(0, t^*; L^2(\Omega))$ *then* $\left\|\frac{\lambda(\cdot,t_{n+1})+\lambda(\cdot,t_n)}{2} - \lambda(\cdot, t_{n+1/2})\right\| \le K\tau^2\left\|\partial_t^2\lambda(\cdot,\tilde{t})\right\|$ *and* $\left\|\frac{3}{2}\lambda(\cdot,t_n) - \frac{1}{2}\lambda(\cdot,t_{n-1}) - \lambda(\cdot,t_{n+1/2})\right\| \le K\tau^2\left\|\partial_t^2\lambda(\cdot,\tilde{t})\right\|$ *for* $\tilde{t} \in (t_0, t^*)$.
3. *If* $(\partial_t^3\lambda) \in C^0(0, t^*; L^2(\Omega))$ *then* $\left\|\frac{\lambda(\cdot,t_{n+1})-\lambda(\cdot,t_n)}{\tau} - \partial_t\lambda(\cdot,t_{n+1/2})\right\| \le K\tau^2\left\|\partial_t^3\lambda(\cdot,\tilde{t})\right\|$ *for* $\tilde{t} \in (t_0, t^*)$

Proof. These estimates are direct results of Taylor series expansion of the function $\lambda(\cdot, t)$. □

The following lemma on skew-symmetric trilinear forms is useful on some parts of the proof of main theorem of the fully discrete error analysis.

Lemma 3.16 *Let $\mathbf{v}^h, \nabla\mathbf{v}^h, T^h, \nabla T^h, C^h, \nabla C^h \in L^\infty(\Omega)$. Then the skew-symmetric trilinear forms satisfy the following estimations with finite constants K which depend on Ω*

$$c_0(\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h) \leq K\left(\|\mathbf{v}^h\|_{L^\infty(\Omega)} + \|\nabla\mathbf{v}^h\|_{L^\infty(\Omega)}\right)\|\mathbf{u}^h\|\,\|\nabla\mathbf{w}^h\| \tag{3.60}$$

$$c_1(\mathbf{u}^h, T^h, S^h) \leq K\left(\|T^h\|_{L^\infty(\Omega)} + \|\nabla T^h\|_{L^\infty(\Omega)}\right)\|\mathbf{u}^h\|\,\|\nabla S^h\| \tag{3.61}$$

$$c_2(\mathbf{u}^h, C^h, \Phi^h) \leq K\left(\|C^h\|_{L^\infty(\Omega)} + \|\nabla C^h\|_{L^\infty(\Omega)}\right)\|\mathbf{u}^h\|\,\|\nabla\Phi^h\| \tag{3.62}$$

for all $\mathbf{u}^h, \mathbf{v}^h, \mathbf{w}^h \in X^h, \quad T^h, S^h \in \Psi^h$ and $C^h, \Phi^h \in W^h$.

Lastly, we state the so called discrete Gronwall's Lemma.

Lemma 3.17 *(Discrete Gronwall). Let $x_n, y_n, z_n, w_n, P, \tau$ be non-negative integers for $n \geq 0$ such that for $N \geq 1$, if*

$$x_N + \tau\sum_{n=0}^{N} y_n \leq \tau\sum_{n=0}^{N-1} w_n x_n + \tau\sum_{n=0}^{N} z_n + P$$

then we have

$$x_N + \tau\sum_{n=0}^{N} y_n \leq \exp(\tau\sum_{n=0}^{N-1} w_n)\left(\tau\sum_{n=0}^{N} z_n + P\right)$$

for all $\tau \geq 0$.

Proof. See e.g.,[26] for a proof. □

We state the unconditional stability of the scheme in the next theorem. By mentioning unconditionally, we mean that there is no restriction on time step for our scheme to be stable. Also, we have no restrictions on problem data to obtain stability.

Theorem 3.18 *(Stability). The scheme (3.44)-(3.46) and (3.41)-(3.43) are unconditionally stable in the sense that,*

$$\|\mathbf{u}^h_{n+1}\|^2 + \tau\sum_{i=0}^{n}\left(4\nu\|\mathbb{D}\left(\mathbf{u}^h_{i+1/2}\right)\|^2 + 2\alpha_1\|(I - P^u_L)\mathbb{D}\left(\mathbf{u}^h_{i+1/2}\right)\|^2 + Da^{-1}\|\mathbf{u}^h_{i+1/2}\|^2\right)$$
$$\leq \|\mathbf{u}^h_0\|^2 + KDa\|\mathbf{g}\|^2_\infty\left(\beta_T^2\gamma^{-1}\|T^h_0\|^2 + \beta_C^2 D_c^{-1}\|C^h_0\|^2\right),$$
$$\|T^h_{n+1}\|^2 + 2\tau\gamma\sum_{i=1}^{n}\left(\|\nabla\left(T^h_{i+1/2}\right)\|^2 + \alpha_2\|(I - P^T_M)\nabla\left(T^h_{i+1/2}\right)\|^2\right) \leq \|T^h_0\|^2,$$
$$\|C^h_{n+1}\|^2 + 2\tau D_c\sum_{i=1}^{n}\left(\|\nabla\left(C^h_{i+1/2}\right)\|^2 + \alpha_3\|(I - P^C_K)\nabla\left(C^h_{i+1/2}\right)\|^2\right) \leq \|C^h_0\|^2$$

Proof. We start with the first time step. Setting $S^h = T^h_{1/2}$ in (3.45) gives

$$\frac{||T^h_1||^2 - ||T^h_0||^2}{2\tau} + \gamma||\nabla(T^h_{1/2})||^2 + \alpha_2||(I - P^T_M)\nabla\left(T^h_{1/2}\right)||^2 \leq 0 \tag{3.63}$$

so we have

$$||T^h_1||^2 + 2\tau\gamma||\nabla\left(T^h_{1/2}\right)||^2 + 2\tau\alpha_2||(I - P^T_M)\nabla\left(T^h_{1/2}\right)||^2 \leq ||T^h_0||^2 \tag{3.64}$$

through multiplying both sides of (3.63) with 2τ, which is the stability result of the temperature equation for the first time step. Similarly setting $\Phi^h = C^h_{1/2}$ in (3.46) gives the stability result

$$||C^h_1||^2 + 2\tau D_c||\nabla\left(C^h_{1/2}\right)||^2 + 2\tau\alpha_3||(I - P^C_K)\nabla\left(C^h_{1/2}\right)||^2 \leq ||C^h_0||^2 \tag{3.65}$$

for the concentration part. Finally for the velocity equation set $\mathbf{v}^h = \mathbf{u}^h_{1/2}$ in (3.44) and obtain the inequality

$$\begin{aligned}&\frac{||\mathbf{u}^h_1||^2 - ||\mathbf{u}^h_0||^2}{2\tau} + 2\nu||\mathbb{D}\left(\mathbf{u}^h_{1/2}\right)||^2 + \alpha_1||(I - P^u_L)\mathbb{D}\left(\mathbf{u}^h_{1/2}\right)||^2 + Da^{-1}||\mathbf{u}^h_{1/2}||^2 \\ &\leq \beta_T||\mathbf{g}||_\infty||T^h_{1/2}||||\mathbf{u}^h_{1/2}|| + \beta_C||\mathbf{g}||_\infty||C^h_{1/2}||||\mathbf{u}^h_{1/2}||.\end{aligned} \tag{3.66}$$

We now employ the Young's inequality for terms at the right-hand side of (3.66) as follows.

$$\begin{aligned}\beta_T||\mathbf{g}||_\infty||T^h_{1/2}||||\mathbf{u}^h_{1/2}|| + \beta_C||\mathbf{g}||_\infty||C^h_{1/2}||||\mathbf{u}^h_{1/2}|| \quad &\leq \quad KDa\beta_T^2||\mathbf{g}||^2_\infty||T^h_{1/2}||^2 + \frac{Da^{-1}}{4}||\mathbf{u}^h_{1/2}||^2 \\ &+ \quad KDa\beta_C^2||\mathbf{g}||^2_\infty||C^h_{1/2}||^2 + \frac{Da^{-1}}{4}||\mathbf{u}^h_{1/2}||^2\end{aligned} \tag{3.67}$$

Using Poincaré's inequality and previous stability bounds obtained for the temperature and concentration parts give

$$KDa||T^h_{1/2}||^2 \leq K(2\tau\gamma)^{-1}||T^h_0||^2, \qquad KDa||C^h_{1/2}||^2 \leq K(2\tau D_c)^{-1}||C^h_0||^2. \tag{3.68}$$

Updating the right-hand side of (3.66) through using (3.67) and (3.68) and multiplying both sides with 2τ gives

$$\begin{aligned}&||\mathbf{u}^h_1||^2 + 4\tau\nu||\mathbb{D}(\mathbf{u}^h_{1/2})||^2 + 2\tau|(I - P^u_L)\mathbb{D}(\mathbf{u}^h_{1/2})||^2 + \tau Da^{-1}||\mathbf{u}^h_{1/2}||^2 \\ &\leq ||\mathbf{u}^h_0||^2 + KDa||\mathbf{g}||^2_\infty\left(\beta_T^2\gamma^{-1}||T^h_0||^2 + \beta_C^2 D_c^{-1}||C^h_0||^2\right)\end{aligned} \tag{3.69}$$

which completes the proof of the stability for the first time step. Now consider the case $n \geq 1$. We set $S^h = T^h_{n+1/2}$ in (3.42) and $\Phi^h = C^h_{n+1/2}$ in (3.43) to have

$$||T^h_{n+1}||^2 + 2\tau\gamma\sum_{i=1}^{n}\left(||\nabla(T^h_{i+1/2})||^2 + \alpha_2||(I - P^T_M)\nabla(T^h_{i+1/2})||^2\right) \leq ||T^h_1||^2 \leq ||T^h_0||^2 \tag{3.70}$$

and

$$\|C_{n+1}^h\|^2 + 2\tau D_c \sum_{i=1}^{n} \left(\|\nabla(C_{i+1/2}^h)\|^2 + \alpha_3\|(I - P_K^C)\nabla(C_{i+1/2}^h)\|^2\right) \leq \|C_1^h\|^2 \leq \|C_0^h\|^2 \tag{3.71}$$

which prove the stability for T and C. For $\mathbf{u}$, we $\mathbf{v}^h = \mathbf{u}_{n+1/2}^h$ in (3.41) then use similar estimations done for the first time step and get the stability result

$$\begin{aligned}
&\|\mathbf{u}_{n+1}^h\|^2 + \tau \sum_{i=0}^{n} \left(4\nu\|\mathbb{D}(\mathbf{u}_{i+1/2}^h)\|^2 + 2\alpha_1\|(I - P_L^u)\mathbb{D}(\mathbf{u}_{i+1/2}^h)\|^2 + Da^{-1}\|\mathbf{u}_{i+1/2}^h\|^2\right) \\
&\leq \|\mathbf{u}_1^h\|^2 + KDa\|\mathbf{g}\|_\infty^2 \left(\beta_T^2\gamma^{-1}\|T_0^h\|^2 + \beta_C^2 D_c^{-1}\|C_0^h\|^2\right) \\
&\leq \|\mathbf{u}_0^h\|^2 + KDa\|\mathbf{g}\|_\infty^2 \left(\beta_T^2\gamma^{-1}\|T_0^h\|^2 + \beta_C^2 D_c^{-1}\|C_0^h\|^2\right).
\end{aligned}$$

□

We are now in a position to give the main theorem of this section.

Theorem 3.19 *Assume that* $\mathbf{u}, \partial_t\mathbf{u} \in L^2\left(0, t^*; H^{s+1}(\Omega)^2\right)$, $T, C, \partial_t T, \partial_t C \in L^2\left(0, t^*; H^{s+1}(\Omega)\right)$, $\partial_t^2\mathbf{u} \in L^2\left(0, t^*; H^2(\Omega)^2\right)$, $\partial_t^2 T, \partial_t^2 C \in L^2\left(0, t^*; H^2(\Omega)\right)$ $\partial_t^3\mathbf{u}, \partial_t^3 T, \partial_t^3 C \in C^0(0, t^*; L^2(\Omega))$ $p \in L^2\left(0, t^*; H^s(\Omega)\right)$ $\partial_t^2 p \in L^2\left(0, t^*; L^2(\Omega)\right)$ *and let* $K_1 Da\beta_T^2\|\mathbf{g}\|_\infty^2\tau \leq \frac{1}{2}$, $K_2 Da\beta_C^2\|\mathbf{g}\|_\infty^2\tau \leq \frac{1}{2}$. *Then the error satisfies*

$$\begin{aligned}
&\left\|\mathbf{u}(t_{n+1}) - \mathbf{u}_{n+1}^h\right\|^2 + \left\|T(t_{n+1}) - T_{n+1}^h\right\|^2 + \left\|C(t_{n+1}) - C_{n+1}^h\right\|^2 \\
&+\tau\left(\sum_{i=1}^{n}\left(2\nu\|\mathbb{D}\left(\mathbf{u}(t_{i+1}) - \mathbf{u}_{i+1}^h\right)\|^2 + \gamma\|\nabla\left(T(t_{i+1}) - T_{i+1}^h\right)\|^2 + D_c\|\nabla\left(C(t_{i+1}) - C_{i+1}^h\right)\|^2\right)\right) \\
&+\tau\sum_{i=1}^{n}(\alpha_1\|(I - P_L^u)\mathbb{D}\left(\mathbf{u}(t_{i+1}) - \mathbf{u}_{i+1}^h\right)\|^2 + \alpha_2\|(I - P_M^T)\nabla\left(T(t_{i+1}) - T_{i+1}^h\right)\|^2 \\
&+\alpha_3\|(I - P_K^C)\nabla\left(C(t_{i+1}) - C_{i+1}^h\right)\|^2) + \tau\sum_{i=1}^{n} Da^{-1}\|\left(\mathbf{u}(t_{i+1}) - \mathbf{u}_{i+1}^h\right)\|^2 \\
&\leq K(h^{2s} + (\alpha_1 + \alpha_2 + \alpha_3)H^{2s} + \tau^4)
\end{aligned}$$

with constants K, K_1, K_2 *depending on* $\mathbf{u}, p, T, C, \nu, \alpha_1, \alpha_2, \alpha_3, \beta_T, \beta_C, \mathbf{g}, Da, \gamma, D_c$.

Proof. We have to construct error equations for each variable first. We split each error term as shown below.

$$\mathbf{e}_n^u = \mathbf{u}(t_n) - \mathbf{u}_n^h = (\mathbf{u}(t_n) - \tilde{\mathbf{u}}) - (\mathbf{u}_n^h - \tilde{\mathbf{u}}) = \boldsymbol{\eta}_n^u - \boldsymbol{\phi}_n^u \tag{3.72}$$

$$e_n^T = T(t_n) - T_n^h = (T(t_n) - \tilde{T}) - (T_n^h - \tilde{T}) = \eta_n^T - \phi_n^T \tag{3.73}$$

$$e_n^C = C(t_n) - C_n^h = (C(t_n) - \tilde{C}) - (C_n^h - \tilde{C}) = \eta_n^C - \phi_n^C \tag{3.74}$$

where $\tilde{\mathbf{u}}, \tilde{T}, \tilde{C}$ are Modified Stokes projections of $\mathbf{u}, T, C$ respectively. Recall here that for any function or variable ζ, $\zeta_{n+1/2}$ is defined as $\zeta_{n+1/2} = \frac{\zeta_{n+1}+\zeta_n}{2}$. Subtracting (3.41) from (3.2), (3.42) from (3.4) and (3.43) from (3.5) at time level $t = t_{n+1/2}$ for test functions $\mathbf{v}^h \in X^h, S^h \in \Psi^h, \Phi^h \in W^h$ gives respectively

$$\begin{aligned}
&\left(\partial_t \mathbf{u}(t_{n+1/2}) - \frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{\tau}, \mathbf{v}^h\right) + 2\nu\left(\mathbb{D}\mathbf{u}(t_{n+1/2}) - \mathbb{D}\mathbf{u}^h_{n+1/2}, \mathbb{D}\mathbf{v}^h\right) \\
&+\alpha_1\left((I - P^u_L)\mathbb{D}(\mathbf{u}(t_{n+1/2}) - \mathbf{u}^h_{n+1/2}), (I - P^u_L)\mathbb{D}\mathbf{v}^h\right) + Da^{-1}\left(\mathbf{u}(t_{n+1/2}) - \mathbf{u}^h_{n+1/2}, \mathbf{v}^h\right) \\
&-\left(p(t_{n+1/2}) - p^h_{n+1/2}, \nabla\cdot\mathbf{v}^h\right) + c_0\left(\mathbf{u}(t_{n+1/2}), \mathbf{u}(t_{n+1/2}), \mathbf{v}^h\right) - c_0\left(\chi(\mathbf{u}^h_n), \mathbf{u}^h_{n+1/2}, \mathbf{v}^h\right) \\
&= \beta_T\left(\mathbf{g}\left(T(t_{n+1/2}) - T^h_{n+1/2}\right), \mathbf{v}^h\right) + \beta_C\left(\mathbf{g}\left(C(t_{n+1/2}) - C^h_{n+1/2}\right), \mathbf{v}^h\right) \\
&+\alpha_1\left((I - P^u_L)\mathbb{D}\mathbf{u}_{n+1/2}, (I - P^u_L)\mathbb{D}\mathbf{v}^h\right) \qquad (3.75)
\end{aligned}$$

$$\begin{aligned}
&\left(\partial_t T(t_{n+1/2}) - \frac{T^h_{n+1} - T^h_n}{\tau}, S^h\right) + \gamma\left(\nabla(T(t_{n+1/2}) - T^h_{n+1/2}), \nabla S^h\right) \\
&+\alpha_2\left((I - P^T_M)\nabla(T(t_{n+1/2}) - T^h_{n+1/2}), (I - P^T_M)\nabla S^h\right) + c_1\left(\mathbf{u}(t_{n+1/2}), T(t_{n+1/2}), S^h\right) \\
&-c_1(\chi(\mathbf{u}^n_h), T^h_{n+1/2}, S^h) = \alpha_2\left((I - P^T_M)\nabla T_{n+1/2}, (I - P^T_M)\nabla S^h\right) \qquad (3.76)
\end{aligned}$$

$$\begin{aligned}
&\left(\partial_t C(t_{n+1/2}) - \frac{C^h_{n+1} - C^h_n}{\tau}, \Phi^h\right) + D_c\left(\nabla(C(t_{n+1/2}) - C^h_{n+1/2}), \nabla\Phi^h\right) \\
&+\alpha_3\left((I - P^C_K)\nabla(C(t_{n+1/2}) - C^h_{n+1/2}), (I - P^C_K)\nabla\Phi^h\right) + c_2\left(\mathbf{u}(t_{n+1/2}), C(t_{n+1/2}), \Phi^h\right) \\
&-c_1\left(\chi(\mathbf{u}^n_h), C^h_{n+1/2}, \Phi^h\right) = \alpha_3\left((I - P^C_K)\nabla C_{n+1/2}, (I - P^C_K)\nabla\Phi^h\right) \qquad (3.77)
\end{aligned}$$

we add add and subtract

$$\begin{aligned}
&\left(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\tau}, \mathbf{v}^h\right) + 2\nu\left(\mathbb{D}\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbb{D}\mathbf{v}^h\right) + Da^{-1}\left(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h\right) \\
&+\alpha_1\left((I - P^u_L)\mathbb{D}\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, (I - P^u_L)\mathbb{D}\mathbf{v}^h\right) \\
&+c_0\left(\mathbf{u}(t_{n+1/2}) + \chi(\mathbf{u}(t_n)) + \chi(\mathbf{u}^h_n), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h\right) - \left(\frac{p(t_{n+1}) + p(t_n)}{2}, \nabla\cdot\mathbf{v}^h\right) \\
&-\beta_T\left(\mathbf{g}\left(\frac{T(t_{n+1}) + T(t_n)}{2}\right), \mathbf{v}^h\right) - \beta_C\left(\mathbf{g}\left(\frac{C(t_{n+1}) + C(t_n)}{2}\right), \mathbf{v}^h\right)
\end{aligned}$$

to (3.75),

$$\begin{aligned}
&\left(\frac{T(t_{n+1}) - T(t_n)}{\tau}, S^h\right) + \gamma\left(\nabla\frac{T(t_{n+1}) + T(t_n)}{2}, \nabla S^h\right) \\
&+\alpha_2\left((I - P^T_M)\nabla\frac{T(t_{n+1}) + T(t_n)}{2}, (I - P^T_M)\nabla S^h\right) \\
&+c_1\left(\mathbf{u}(t_{n+1/2}) + \chi(\mathbf{u}(t_n)) + \chi(\mathbf{u}^h_n), \frac{T(t_{n+1}) + T(t_n)}{2}, S^h\right)
\end{aligned}$$

to (3.76) and

$$\left(\frac{C(t_{n+1})-C(t_n)}{\tau},\Phi^h\right)+D_c\left(\nabla\frac{C(t_{n+1})+C(t_n)}{2},\nabla\Phi^h\right)$$
$$+\alpha_3\left((I-P_K^C)\nabla\frac{C(t_{n+1})+C(t_n)}{2},(I-P_K^C)\nabla\Phi^h\right)$$
$$+c_2\left(\mathbf{u}(t_{n+1/2})+\chi(\mathbf{u}(t_n))+\chi(\mathbf{u}_n^h),\frac{C(t_{n+1})+C(t_n)}{2},\Phi^h\right)$$

to (3.77) and finally obtain the error equations

$$\begin{aligned}&\left(\frac{\mathbf{e}^u_{n+1}-\mathbf{e}^u_n}{\tau},\mathbf{v}^h\right)+2\nu\left(\mathbb{D}\mathbf{e}^u_{n+1/2},\mathbb{D}\mathbf{v}^h\right)+\alpha_1\left((I-P_L^u)\mathbb{D}\mathbf{e}^u_{n+1/2},\mathbb{D}\mathbf{v}^h\right)+Da^{-1}\left(\mathbf{e}^u_{n+1/2},\mathbf{v}^h\right)\\&-\left(\frac{p(t_{n+1})+p(t_n)}{2}-q^h,\nabla\cdot\mathbf{v}^h\right)+c_0\left(\chi(\mathbf{u}_n^h),\mathbf{e}^u_{n+1/2},\mathbf{v}^h\right)-c_0\left(\chi(\mathbf{e}^u_n),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h\right)\\&=\beta_T\left(\mathbf{g}(e^T_{n+1/2}),\mathbf{v}^h\right)+\beta_C\left(\mathbf{g}(e^C_{n+1/2}),\mathbf{v}^h\right)+\alpha_1\left((I-P_L^u)\mathbb{D}\mathbf{u}_{n+1/2},(I-P_L^u)\mathbb{D}\mathbf{v}^h\right)+G_1\end{aligned}\tag{3.78}$$

$$\begin{aligned}&\left(\frac{e^T_{n+1}-e^T_n}{\tau},S^h\right)+\gamma\left(\nabla e^T_{n+1/2},\nabla S^h\right)+\alpha_2\left((I-P_M^T)\nabla e^T_{n+1/2},(I-P_M^T)\nabla S^h\right)\\&+c_1\left(\chi(\mathbf{u}_n^h),e^T_{n+1/2},S^h\right)-c_1\left(\chi(\mathbf{e}^u_n),\frac{T(t_{n+1})+T(t_n)}{2},S^h\right)\\&=\alpha_2\left((I-P_M^T)\nabla T_{n+1/2},(I-P_M^T)\nabla S^h\right)+G_2\end{aligned}\tag{3.79}$$

$$\begin{aligned}&\left(\frac{e^C_{n+1}-e^C_n}{\tau},\Phi^h\right)+D_c\left(\nabla e^C_{n+1/2},\nabla\Phi^h\right)+\alpha_3\left((I-P_K^C)\nabla e^C_{n+1/2},(I-P_K^C)\nabla\Phi^h\right)\\&+c_2\left(\chi(\mathbf{u}_n^h),e^C_{n+1/2},\Phi^h\right)-c_2\left(\chi(\mathbf{e}^u_n),\frac{C(t_{n+1})+C(t_n)}{2},\Phi^h\right)\\&=\alpha_3\left((I-P_K^C)\nabla T_{n+1/2},(I-P_K^C)\nabla\Phi^h\right)+G_3\end{aligned}\tag{3.80}$$

for $\mathbf{u},T,C$ respectively. Here the functions G_1,G_2,G_3 are given as

$$\begin{aligned}G_1=&\left(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\tau}-\partial_t\mathbf{u}(t_{n+1/2}),\mathbf{v}^h\right)+2\nu\left(\mathbb{D}\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbb{D}\mathbf{u}(t_{n+1/2}),\mathbb{D}\mathbf{v}^h\right)\\&+\alpha_1\left((I-P_L^u)\mathbb{D}\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+1/2})\right),(I-P_L^u)\mathbb{D}\mathbf{v}^h\right)\\&+Da^{-1}\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+1/2}),\mathbf{v}^h\right)+c_0\left(\mathbf{u}(t_{n+1/2}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+1/2}),\mathbf{v}^h\right)\\&-c_0\left(\chi(\mathbf{u}(t_n))-\mathbf{u}(t_{n+1/2}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h\right)-\left(\frac{p(t_{n+1})+p(t_n)}{2}-p(t_{n+1/2}),\nabla\cdot\mathbf{v}^h\right)\\&-\beta_T\left(\mathbf{g}\left(\frac{T(t_{n+1})+T(t_n)}{2}-T(t_{n+1/2})\right),\mathbf{v}^h\right)-\beta_C\left(\mathbf{g}\left(\frac{C(t_{n+1})+C(t_n)}{2}-C(t_{n+1/2})\right),\mathbf{v}^h\right)\end{aligned}$$

$$G_2 = \left(\frac{T(t_{n+1}) - T(t_n)}{\tau} - \partial_t T(t_{n+1/2}), S^h\right) + \gamma\left(\nabla\left(\frac{T(t_{n+1}) + T(t_n)}{2} - T(t_{n+1/2})\right), \nabla S^h\right)$$
$$+\alpha_2\left((I - P_M^T)\nabla\left(\frac{T(t_{n+1}) + T(t_n)}{2} - T(t_{n+1/2})\right), (I - P_M^T)\nabla S^h\right)$$
$$+c_1\left(\mathbf{u}(t_{n+1/2}), \frac{T(t_{n+1}) + T(t_n)}{2} - T(t_{n+1/2}), S^h\right)$$
$$-c_1\left(\chi(\mathbf{u}(t_n)) - \mathbf{u}(t_{n+1/2}), \frac{T(t_{n+1}) + T(t_n)}{2}, S^h\right)$$

$$G_3 = \left(\frac{C(t_{n+1}) - C(t_n)}{\tau} - \partial_t C(t_{n+1/2}), \Phi^h\right) + D_c\left(\nabla\left(\frac{C(t_{n+1}) + C(t_n)}{2} - C(t_{n+1/2})\right), \nabla\Phi^h\right)$$
$$+\alpha_3\left((I - P_K^C)\nabla\left(\frac{C(t_{n+1}) + T(t_n)}{2} - C(t_{n+1/2})\right), (I - P_M^T)\nabla\Phi^h\right)$$
$$+c_2\left(\mathbf{u}(t_{n+1/2}), \frac{C(t_{n+1}) + C(t_n)}{2} - C(t_{n+1/2}), \Phi^h\right)$$
$$-c_2\left(\chi(\mathbf{u}(t_n)) - \mathbf{u}(t_{n+1/2}), \frac{C(t_{n+1}) + C(t_n)}{2}, \Phi^h\right).$$

We start our analysis by the temperature error equation. Decomposing the error term as in (3.73) and writing $S^h = \phi^T_{n+1/2}$ in (3.79) lead us to

$$\frac{\|\phi^T_{n+1}\|^2 - \|\phi^T_n\|^2}{2\tau} + \gamma\|\nabla\phi^T_{n+1/2}\|^2 + \alpha_2\|(I - P_M^T)\nabla\phi^T_{n+1/2}\|^2 \leq$$
$$\left|\left(\frac{\eta^T_{n+1} - \eta^T_n}{\tau}, \phi^T_{n+1/2}\right)\right| + \left|\gamma\left(\nabla\eta^T_{n+1/2}, \nabla\phi^T_{n+1/2}\right) + \alpha_2\left((I - P_M^T)\nabla\eta^T_{n+1/2}, (I - P_M^T)\nabla\phi^T_{n+1/2}\right)\right|$$
$$+\left|c_1\left(\chi(\mathbf{u}^h_n), e^T_{n+1/2}, \phi^T_{n+1/2}\right) - c_1\left(\chi(\mathbf{e}^u_n), \frac{T(t_{n+1}) + T(t_n)}{2}, \phi^T_{n+1/2}\right)\right|$$
$$+\left|\alpha_2\left((I - P_M^T)\nabla T_{n+1/2}, (I - P_M^T)\nabla\phi^T_{n+1/2}\right)\right| + \left|G_2(\phi^T_{n+1/2})\right|. \tag{3.81}$$

We have to bound each term at the right-hand side of (3.81). Firstly, note that due to the definition of Modified Stokes projection $\tilde{T}$, the term

$$\left|\gamma\left(\nabla\eta^T_{n+1/2}, \nabla\phi^T_{n+1/2}\right) + \alpha_2\left((I - P_M^T)\nabla\eta^T_{n+1/2}, (I - P_M^T)\nabla\phi^T_{n+1/2}\right)\right|$$

vanishes. For the other linear terms we have

$$\left|\left(\frac{\eta^T_{n+1} - \eta^T_n}{\tau}, \phi^T_{n+1/2}\right)\right| \leq K\gamma^{-1}\|\frac{\eta^T_{n+1} - \eta^T_n}{\tau}\|^2 + \frac{\gamma}{18}\|\nabla\phi^T_{n+1/2}\|^2$$

$$\left|\alpha_2\left((I-P_M^T)\nabla T_{n+1/2},(I-P_M^T)\nabla\phi^T_{n+1/2}\right)\right|\leq K\alpha_2\left\|(I-P_M^T)\nabla T_{n+1/2}\right\|^2+\frac{\alpha_2}{4}\left\|(I-P_M^T)\nabla\phi^T_{n+1/2}\right\|^2$$

which are obtained by using Cauchy-Schwarz, Young's and Poincaré's inequality. Bounding nonlinear terms is the most challenging part of this analysis. We first write the difference as

$$\begin{aligned}&\left|c_1\left(\chi(\mathbf{u}_n^h),e^T_{n+1/2},\phi^T_{n+1/2}\right)-c_1\left(\chi(\mathbf{e}_n^u),\frac{T(t_{n+1})+T(t_n)}{2},\phi^T_{n+1/2}\right)\right|\\ \leq\ &\left|c_1\left(\chi(\mathbf{u}_n^h),e^T_{n+1/2},\phi^T_{n+1/2}\right)\right|+\left|c_1\left(\chi(\mathbf{e}_n^u),\frac{T(t_{n+1})+T(t_n)}{2},\phi^T_{n+1/2}\right)\right|.\end{aligned}\tag{3.82}$$

For the first term in (3.82) we have $c_1\left(\chi(\mathbf{u}_n^h),e^T_{n+1/2},\phi^T_{n+1/2}\right)=c_1\left(\chi(\mathbf{u}_n^h),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)$, since the term $c_1\left(\chi(\mathbf{u}_n^h),\phi^T_{n+1/2},\phi^T_{n+1/2}\right)$ vanishes. Adding and subtracting $c_1\left(\chi(\mathbf{u}(t_n),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)$ to this remaining term we have

$$\begin{aligned}\left|c_1\left(\chi(\mathbf{u}_n^h),e^T_{n+1/2},\phi^T_{n+1/2}\right)\right|\leq\ &\left|c_1\left(\chi(\boldsymbol{\eta}_n^u),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)\right|+\left|c_1\left(\chi(\boldsymbol{\phi}_n^u),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)\right|\\ +\ &\left|c_1\left(\chi(\mathbf{u}(t_n),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)\right|.\end{aligned}$$

So we go now term by term. The first one is bounded in a standard way with the help of the definition of χ as follows

$$\begin{aligned}\left|c_1\left(\chi(\boldsymbol{\eta}_n^u),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)\right|\leq\ &K\left\|\mathbb{D}\chi(\boldsymbol{\eta}_n^u)\right\|\|\mathbb{D}\eta^T_{n+1/2}\|\|\nabla\phi^T_{n+1/2}\|\leq\frac{\gamma}{18}\|\nabla\phi^T_{n+1/2}\|^2\\ &+K\gamma^{-1}\left(\|\mathbb{D}\boldsymbol{\eta}_n^u\|^2+\|\mathbb{D}\boldsymbol{\eta}_{n-1}^u\|^2\right)\|\nabla\eta^T_{n+1/2}\|^2.\end{aligned}$$

For the second one we assume an inverse inequality holds i.e. for all $\mathbf{v}\in X^h$ there exists a constant K independent from h satisfying

$$\|\nabla\mathbf{v}\|\leq Kh^{-1}\|\mathbf{v}\|.$$

So we have

$$\begin{aligned}&\left|c_1\left(\chi(\boldsymbol{\phi}_n^u),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)\right|\leq K\|\chi(\boldsymbol{\phi}_n^u)\|^{1/2}\|\nabla\chi(\boldsymbol{\phi}_n^u)\|^{1/2}\|\nabla\eta^T_{n+1/2}\|\|\nabla\phi^T_{n+1/2}\|\\ &\leq K\left(h^{-1}\|\chi(\boldsymbol{\phi}_n^u)\|^2\right)^{1/2}\|\nabla\eta^T_{n+1/2}\|\|\nabla\phi^T_{n+1/2}\|\\ &\leq Kh^{-1}\gamma^{-1}\left(\|\boldsymbol{\phi}_n^u\|^2+\|\boldsymbol{\phi}_{n-1}^u\|^2\right)\|\nabla\eta^T_{n+1/2}\|^2+\frac{\gamma}{18}\|\nabla\phi^T_{n+1/2}\|^2.\end{aligned}$$

The third term is bounded using the regularity assumption on $\mathbf{u}$ as

$$\left|c_1\left(\chi(\mathbf{u}(t_n)),\eta^T_{n+1/2},\phi^T_{n+1/2}\right)\right|\leq\frac{\gamma}{18}\|\nabla\phi^T_{n+1/2}\|^2+K\gamma^{-1}\|\nabla\eta^T_{n+1/2}\|^2.$$

The second term in (3.82) could be written as

$$\begin{aligned}\left|c_1\left(\chi(\mathbf{e}_n^u),\frac{T(t_{n+1})+T(t_n)}{2},\phi^T_{n+1/2}\right)\right|\ &\leq\left|c_1\left(\chi(\boldsymbol{\eta}_n^u),\frac{T(t_{n+1})+T(t_n)}{2},\phi^T_{n+1/2}\right)\right|\\ &+\left|c_1\left(\chi(\boldsymbol{\phi}_n^u),\frac{T(t_{n+1})+T(t_n)}{2},\phi^T_{n+1/2}\right)\right|\end{aligned}$$

and these are bounded as

$$\left|c_1\left(\chi(\boldsymbol{\eta}_n^u), \frac{T(t_{n+1})+T(t_n)}{2}, \phi_{n+1/2}^T\right)\right| \leq K\gamma^{-1}\left(\|\mathbb{D}\boldsymbol{\eta}_n^u\|^2 + \|\mathbb{D}\boldsymbol{\eta}_{n-1}^u\|^2\right) + \frac{\gamma}{18}\|\nabla\phi_{n+1/2}^T\|^2$$

$$\left|c_1\left(\chi(\boldsymbol{\phi}_n^u), \frac{T(t_{n+1})+T(t_n)}{2}, \phi_{n+1/2}^T\right)\right| \leq K\gamma^{-1}\left(\|\boldsymbol{\phi}_n^u\|^2 + \|\boldsymbol{\phi}_{n-1}^u\|^2\right) + \frac{\gamma}{18}\|\nabla\phi_{n+1/2}^T\|^2$$

through the regularity of T, (3.62) and the definition of χ. It only remains the bound G_2 for the analysis of the temperature part. Making use of Lemma 3.15 and usual estimations, we bound each component of G_2 as follows.

$$\left(\frac{T(t_{n+1})-T(t_n)}{\tau} - \partial_t T(t_{n+1/2}), \phi_{n+1/2}^T\right) \leq K\gamma^{-1}\tau^4\|\partial_t^3 T(\tilde{t})\|^2 + \frac{\gamma}{18}\|\nabla\phi_{n+1/2}^T\|^2$$

$$\left(\nabla\left(\frac{T(t_{n+1})+T(t_n)}{2} - T(t_{n+1/2})\right), \nabla\phi_{n+1/2}^T\right) \leq K\gamma^{-1}\tau^4\|\partial_t^2\nabla T(\tilde{t})\|^2 + \frac{\gamma}{18}\|\nabla\phi_{n+1/2}^T\|^2$$

$$\alpha_2\left((I-P_M^T)\nabla\left(\frac{T(t_{n+1})+T(t_n)}{2} - T(t_{n+1/2})\right), (I-P_M^T)\nabla\phi_{n+1/2}^T\right)$$
$$\leq K\alpha_2\tau^4\|(I-P_M^T)\partial_t^2\nabla T(\tilde{t})\|^2 + \frac{\alpha_2}{4}\alpha_2\|(I-P_M^T)\nabla\phi_{n+1/2}^T\|^2$$

$$c_1\left(\mathbf{u}(t_{n+1/2}), \frac{T(t_{n+1})+T(t_n)}{2} - T(t_{n+1/2}), \phi_{n+1/2}^T\right)$$
$$+c_1\left(\chi(\mathbf{u}(t_n)) - \mathbf{u}(t_{n+1/2}), \frac{T(t_{n+1})+T(t_n)}{2}, \phi_{n+1/2}^T\right) \leq K\gamma^{-1}\tau^4\|\partial_t^2\nabla T(\tilde{t})\|^2 + \frac{\gamma}{18}\|\nabla\phi_{n+1/2}^T\|^2$$

Putting all the bounds in (3.81) results with

$$\begin{aligned}
&\frac{\|\phi_{n+1}^T\|^2 - \|\phi_n^T\|^2}{2\tau} + \frac{\gamma}{2}\|\nabla\phi_{n+1/2}^T\|^2 + \frac{\alpha_2}{2}\|(I-P_M^T)\nabla\phi_{n+1/2}^T\|^2 \leq K\Bigg\{\gamma^{-1}\Bigg(\left\|\frac{\eta_{n+1}^T - \eta_n^T}{\tau}\right\|^2 \\
&+\left(1 + \|\mathbb{D}\boldsymbol{\eta}_n^u\|^2 + \|\mathbb{D}\boldsymbol{\eta}_{n-1}^u\|^2\right)\|\nabla\eta_{n+1/2}^T\|^2 + h^{-1}\left(\|\boldsymbol{\phi}_n^u\|^2 + \|\boldsymbol{\phi}_{n-1}^u\|^2\right)\|\nabla\eta_{n+1/2}^T\|^2 \\
&+\left(\|\mathbb{D}\boldsymbol{\eta}_n^u\|^2 + \|\mathbb{D}\boldsymbol{\eta}_{n-1}^u\|^2\right) + \left(\|\boldsymbol{\phi}_n^u\|^2 + \|\boldsymbol{\phi}_{n-1}^u\|^2\right) + \tau^4\left(\|\partial_t^3 T(\tilde{t})\|^2 + \|\partial_t^2\nabla T(\tilde{t})\|^2\right)\Bigg) \\
&+\tau^4\alpha_2\|(I-P_M^T)\partial_t^2\nabla T(\tilde{t})\|^2 + \alpha_2\|(I-P_M^T)\nabla T_{n+1/2}\|^2\Bigg\}.
\end{aligned} \tag{3.83}$$

One gets the following estimate for C by writing $\Phi^h = \phi_{n+1/2}^C$ in (3.80) and performing exactly same analysis done for T.

$$\begin{aligned}
&\frac{\|\phi_{n+1}^C\|^2 - \|\phi_n^C\|^2}{2\tau} + \frac{D_c}{2}\|\nabla\phi_{n+1/2}^C\|^2 + \frac{\alpha_3}{2}\|(I-P_K^C)\nabla\phi_{n+1/2}^C\|^2 \leq K\Bigg\{D_c^{-1}\Bigg(\left\|\frac{\eta_{n+1}^C - \eta_n^C}{\tau}\right\|^2 \\
&+\left(1 + \|\mathbb{D}\boldsymbol{\eta}_n^u\|^2 + \|\mathbb{D}\boldsymbol{\eta}_{n-1}^u\|^2\right)\|\nabla\eta_{n+1/2}^C\|^2 + h^{-1}\left(\|\boldsymbol{\phi}_n^u\|^2 + \|\boldsymbol{\phi}_{n-1}^u\|^2\right)\|\nabla\eta_{n+1/2}^C\|^2 \\
&+\left(\|\mathbb{D}\boldsymbol{\eta}_n^u\|^2 + \|\mathbb{D}\boldsymbol{\eta}_{n-1}^u\|^2\right) + \left(\|\boldsymbol{\phi}_n^u\|^2 + \|\boldsymbol{\phi}_{n-1}^u\|^2\right) + \tau^4\left(\|\partial_t^3 C(\tilde{t})\|^2 + \|\partial_t^2\nabla C(\tilde{t})\|^2\right)\Bigg) \\
&+\tau^4\alpha_3\|(I-P_K^C)\partial_t^2\nabla T(\tilde{t})\|^2 + \alpha_3\|(I-P_K^C)\nabla C_{n+1/2}\|^2\Bigg\}.
\end{aligned} \tag{3.84}$$

We now proceed to the analysis for u. Setting $\mathbf{v}^h = \phi^u_{n+1/2}$ in (3.78) gives

$$\frac{\|\boldsymbol{\phi}^u_{n+1}\|^2 - \|\boldsymbol{\phi}^u_n\|^2}{2\tau} + 2\nu\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 + \alpha_1\|(I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 + Da^{-1}\|\boldsymbol{\phi}^u_{n+1/2}\| \leq$$
$$\left|\left(\frac{\boldsymbol{\eta}^u_{n+1}-\boldsymbol{\eta}^u_n}{\tau}, \boldsymbol{\phi}^u_{n+1/2}\right)\right| + \left|2\nu\left(\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}, \mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right) + \alpha_1\left((I-P^u_L)\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}, (I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right)\right.$$
$$\left. +Da^{-1}\left(\boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right) - \left(\frac{p(t_{n+1})+p(t_n)}{2} - q^h, \nabla\cdot\mathbf{v}^h\right)\right| + \left|c_0\left(\chi(\mathbf{u}^h_n), \boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right|$$
$$+\left|c_0\left(\chi(\boldsymbol{\eta}^u_n), \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right| + \left|c_0\left(\chi(\boldsymbol{\phi}^u_n), \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right|$$
$$+\beta_T\left(\mathbf{g}(\eta^T_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right) + \beta_C\left(\mathbf{g}(\eta^C_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right) + \beta_T\left(\mathbf{g}(\phi^T_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right)$$
$$+\beta_C\left(\mathbf{g}(\phi^C_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right) + \left|\alpha_1\left((I-P^u_L)\mathbb{D}\mathbf{u}_{n+1/2}, (I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right)\right| + \left|G_1(\phi^T_{n+1/2})\right|. \quad (3.85)$$

We first bound the linear terms again. Clearly,

$$\left|2\nu\left(\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}, \mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right) + \alpha_1\left((I-P^u_L)\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}, (I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right) + Da^{-1}\left(\boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right.$$
$$\left. - \left(\frac{p(t_{n+1})+p(t_n)}{2} - q^h, \nabla\cdot\mathbf{v}^h\right)\right| = 0$$

by the property of Modified Stokes projection. For the others,

$$\left|\left(\frac{\boldsymbol{\eta}^u_{n+1}-\boldsymbol{\eta}^u_n}{\tau}\right)\right| \leq K\nu^{-1}\left\|\frac{\boldsymbol{\eta}^u_{n+1}-\boldsymbol{\eta}^u_n}{\tau}\right\|^2 + \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^T_{n+1/2}\|^2$$

$$\left|\alpha_1\left((I-P^u_L)\mathbb{D}\mathbf{u}_{n+1/2}, (I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right)\right| \leq K\alpha_1\left\|(I-P^u_L)\mathbb{D}\mathbf{u}_{n+1/2}\right\|^2 + \frac{\alpha_1}{4}\|(I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\left|\beta_T\left(\mathbf{g}(\eta^T_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right)\right| \leq KDa\beta_T^2\|\mathbf{g}\|^2_\infty\|\eta^T_{n+1/2}\|^2 + \frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\left|\beta_T\left(\mathbf{g}(\phi^T_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right)\right| \leq KDa\beta_T^2\|\mathbf{g}\|^2_\infty\|\phi^T_{n+1/2}\|^2 + \frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\left|\beta_C\left(\mathbf{g}(\eta^C_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right)\right| \leq KDa\beta_C^2\|\mathbf{g}\|^2_\infty\|\eta^C_{n+1/2}\|^2 + \frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\left|\beta_C\left(\mathbf{g}(\phi^C_{n+1/2}), \boldsymbol{\phi}^u_{n+1/2}\right)\right| \leq KDa\beta_C^2\|\mathbf{g}\|^2_\infty\|\phi^C_{n+1/2}\|^2 + \frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

are obtained through standard estimates. We now pass to nonlinear terms. The first term is decomposed into three terms as in the temperature case as

$$\begin{aligned}\left|c_0\left(\chi(\mathbf{u}^h_n), \boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right| \leq{} & \left|c_0\left(\chi(\mathbf{u}(t_n)), \boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right| + \left|c_0\left(\chi(\boldsymbol{\eta}^u_n), \boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right| \\ & + \left|c_0\left(\chi(\boldsymbol{\phi}^u_n), \boldsymbol{\eta}^u_{n+1/2}, \boldsymbol{\phi}^u_{n+1/2}\right)\right| \end{aligned} \quad (3.86)$$

and

$$\left|c_0\left(\chi(\mathbf{u}(t_n)),\boldsymbol{\eta}^u_{n+1/2},\boldsymbol{\phi}^u_{n+1/2}\right)\right| \le K\nu^{-1}\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2 + \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 \tag{3.87}$$

$$\left|c_0\left(\chi(\boldsymbol{\eta}^u_n),\boldsymbol{\eta}^u_{n+1/2},\boldsymbol{\phi}^u_{n+1/2}\right)\right| \le \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 + K\nu^{-1}\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2\left(\|\mathbb{D}\boldsymbol{\eta}^u_n\|^2 + \|\mathbb{D}\boldsymbol{\eta}^u_{n-1}\|^2\right) \tag{3.88}$$

$$\left|c_0\left(\chi(\boldsymbol{\phi}^u_n),\boldsymbol{\eta}^u_{n+1/2},\boldsymbol{\phi}^u_{n+1/2}\right)\right| \le \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 + K\nu^{-1}h^{-1}\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2\left(\|\boldsymbol{\phi}^u_n\|^2 + \|\boldsymbol{\phi}^u_{n-1}\|^2\right) \tag{3.89}$$

are obtained similarly as done before. The bounds for remaining nonlinear terms are obtained via the help of (3.60) and regularity assumptions on **u** and are given below.

$$\left|c_0\left(\chi(\boldsymbol{\phi}^u_n),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\boldsymbol{\phi}^u_{n+1/2}\right)\right| \le \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 + K\nu^{-1}\left(\|\boldsymbol{\phi}^u_n\|^2 + \|\boldsymbol{\phi}^u_{n-1}\|^2\right) \tag{3.90}$$

$$\left|c_0\left(\chi(\boldsymbol{\eta}^u_n),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\boldsymbol{\phi}^u_{n+1/2}\right)\right| \le \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2 + K\nu^{-1}\left(\|\mathbb{D}\boldsymbol{\eta}^u_n\|^2 + \|\mathbb{D}\boldsymbol{\eta}^u_{n-1}\|^2\right). \tag{3.91}$$

Hence the last term we should bound is G_1. We bound it term by term as follows

$$\left(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\tau} - \partial_t\mathbf{u}(t_{n+1/2}),\boldsymbol{\phi}^u_{n+1/2}\right) \le K\nu^{-1}\tau^4\|\partial_t^3\mathbf{u}(\tilde{t})\|^2 + \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$2\nu\left(\mathbb{D}\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2} - \mathbb{D}\mathbf{u}(t_{n+1/2}),\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right) \le K\tau^4\nu\|\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2 + \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\begin{aligned}\alpha_1\left((I-P^u_L)\left(\mathbb{D}\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2} - \mathbb{D}\mathbf{u}(t_{n+1/2})\right),(I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\right)\\ \le K\alpha_1\tau^4\|(I-P^u_L)\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2 + \frac{\alpha_1}{4}\|(I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2\end{aligned}$$

$$Da^{-1}\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+1/2}),\boldsymbol{\phi}^u_{n+1/2}\right) \le KDa^{-1}\tau^4\|\partial_t^2\mathbf{u}(\tilde{t})\|^2 + \frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\left(\frac{p(t_{n+1})+p(t_n)}{2} - p(t_{n+1/2}),\nabla\cdot\boldsymbol{\phi}^u_{n+1/2}\right) \le K\tau^4\nu^{-1}\|\partial_t^2 p(\tilde{t})\|^2 + \frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\beta_T\left(\mathbf{g}\left(\frac{T(t_{n+1})+T(t_n)}{2} - T(t_{n+1/2})\right),\boldsymbol{\phi}^u_{n+1/2}\right) \le KDa\beta_T^2\|\mathbf{g}\|_\infty^2\tau^4\|\partial_t^2 T(\tilde{t})\|^2 + \frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$\beta_C\left(\mathbf{g}\left(\frac{C(t_{n+1})+C(t_n)}{2}-C(t_{n+1/2})\right),\boldsymbol{\phi}^u_{n+1/2}\right)\le KDa\beta_C^2\|\mathbf{g}\|_\infty^2\tau^4\|\partial_t^2C(\tilde{t})\|^2+\frac{Da^{-1}}{14}\|\boldsymbol{\phi}^u_{n+1/2}\|^2$$

$$+c_0\left(\mathbf{u}(t_{n+1/2}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+1/2}),\boldsymbol{\phi}^u_{n+1/2}\right)$$
$$+c_0\left(\chi(\mathbf{u}(t_n))-\mathbf{u}(t_{n+1/2}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\boldsymbol{\phi}^u_{n+1/2}\right)\le K\tau^4\nu^{-1}\|\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2+\frac{\nu}{20}\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2.$$

Now rearranging the obtained bounds give

$$\frac{\|\boldsymbol{\phi}^u_{n+1}\|^2-\|\boldsymbol{\phi}^u_n\|^2}{2\tau}+\nu\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2+\frac{\alpha_1}{2}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2+\frac{Da^{-1}}{2}\|\boldsymbol{\phi}^u_{n+1/2}\|^2\le$$
$$K\Bigg\{\nu^{-1}\left\|\frac{\boldsymbol{\eta}^u_{n+1}-\boldsymbol{\eta}^u_n}{\tau}\right\|^2+\nu^{-1}\,(\,\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2\left(1+\|\mathbb{D}\boldsymbol{\eta}^u_n\|^2+\|\mathbb{D}\boldsymbol{\eta}^u_{n-1}\|^2\right)+\left(\|\mathbb{D}\boldsymbol{\eta}^u_n\|^2+\|\mathbb{D}\boldsymbol{\eta}^u_{n-1}\|^2\right)$$
$$+\left(\|\boldsymbol{\phi}^u_n\|^2+\|\boldsymbol{\phi}^u_{n-1}\|^2\right)+h^{-1}\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2\left(\|\boldsymbol{\phi}^u_n\|^2+\|\boldsymbol{\phi}^u_{n-1}\|^2\right)\,)+\alpha_1\left\|(I-P_L^u)\mathbb{D}\mathbf{u}_{n+1/2}\right\|^2$$
$$+Da\beta_T^2\|\mathbf{g}\|_\infty^2\left(\|\eta^T_{n+1/2}\|^2+\|\phi^T_{n+1/2}\|^2\right)+Da\beta_C^2\|\mathbf{g}\|_\infty^2\left(\|\eta^C_{n+1/2}\|^2+\|\phi^C_{n+1/2}\|^2\right)$$
$$+\tau^4\left(\nu^{-1}\|\partial_t^3\mathbf{u}(\tilde{t})\|^2+(\nu^{-1}+\nu)\|\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2+\alpha_1\|(I-P_L^u)\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2+Da^{-1}\|\partial_t^2\mathbf{u}(\tilde{t})\|^2\right.$$
$$\left.+\nu^{-1}\|\partial_t^2p(\tilde{t})\|^2+Da\beta_T^2\|\mathbf{g}\|_\infty^2\|\partial_t^2T(\tilde{t})\|^2+Da\beta_C^2\|\mathbf{g}\|_\infty^2\|\partial_t^2C(\tilde{t})\|^2\right)\Bigg\} \tag{3.92}$$

Now, we combine the resulting inequalities by adding (3.83) and (3.84) to (3.92) side by side and finally get

$$\left(\frac{\|\boldsymbol{\phi}^u_{n+1}\|^2-\|\boldsymbol{\phi}^u_n\|^2}{2\tau}+\frac{\|\phi^T_{n+1}\|^2-\|\phi^T_n\|^2}{2\tau}+\frac{\|\phi^C_{n+1}\|^2-\|\phi^C_n\|^2}{2\tau}\right)+\nu\|\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2+\frac{\gamma}{2}\|\nabla\phi^T_{n+1/2}\|^2$$
$$+\frac{D_c}{2}\|\nabla\phi^C_{n+1/2}\|^2+\frac{Da^{-1}}{2}\|\boldsymbol{\phi}^u_{n+1/2}\|+\frac{\alpha_1}{2}\|(I-P_L^u)\mathbb{D}\boldsymbol{\phi}^u_{n+1/2}\|^2+\frac{\alpha_2}{2}\|(I-P_M^T)\nabla\phi^T_{n+1/2}\|^2$$
$$+\frac{\alpha_3}{2}\|(I-P_M^T)\nabla\phi^T_{n+1/2}\|^2\le K\Bigg\{\nu^{-1}\left\|\frac{\boldsymbol{\eta}^u_{n+1}-\boldsymbol{\eta}^u_n}{\tau}\right\|^2+\gamma^{-1}\left\|\frac{\eta^T_{n+1}-\eta^T_n}{\tau}\right\|^2+D_c^{-1}\left\|\frac{\eta^C_{n+1}-\eta^C_n}{\tau}\right\|^2$$
$$+\left(\nu^{-1}\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2+\gamma^{-1}\|\nabla\eta^T_{n+1/2}\|^2+D_c^{-1}\|\nabla\eta^C_{n+1/2}\|^2\right)\left(1+\|\mathbb{D}\boldsymbol{\eta}^u_n\|^2+\|\mathbb{D}\boldsymbol{\eta}^u_{n-1}\|^2\right)$$
$$+(\nu^{-1}+\gamma^{-1}+D_c^{-1})\left(\|\mathbb{D}\boldsymbol{\eta}^u_n\|^2+\|\mathbb{D}\boldsymbol{\eta}^u_{n-1}\|^2+\|\boldsymbol{\phi}^u_n\|^2+\|\boldsymbol{\phi}^u_{n-1}\|^2\right)+h^{-1}\left(\|\boldsymbol{\phi}^u_n\|^2+\|\boldsymbol{\phi}^u_{n-1}\|^2\right)$$
$$\times\left(\nu^{-1}\|\mathbb{D}\boldsymbol{\eta}^u_{n+1/2}\|^2+\gamma^{-1}\|\nabla\eta^T_{n+1/2}\|^2+D_c^{-1}\|\nabla\eta^C_{n+1/2}\|^2\right)+\tau^4\left(\nu^{-1}\|\partial_t^3\mathbf{u}(\tilde{t})\|^2\right.$$
$$+(\nu+\nu^{-1})\|\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2+\alpha_1\|(I-P_L^u)\partial_t^2\mathbb{D}\mathbf{u}(\tilde{t})\|^2+Da^{-1}\|\partial_t^2\mathbf{u}(\tilde{t})\|^2+\nu^{-1}\|\partial_t^2p(\tilde{t})\|^2$$
$$\left.+Da\beta_T^2\|\mathbf{g}\|_\infty^2\|\partial_t^2T(\tilde{t})\|^2+Da\beta_C^2\|\mathbf{g}\|_\infty^2\|\partial_t^2C(\tilde{t})\|^2\,\right)+\tau^4\,(\,\gamma^{-1}\|\partial_t^3T(\tilde{t})\|^2+\gamma^{-1}\|\partial_t^2\nabla T(\tilde{t})\|^2$$
$$+\alpha_2\|(I-P_M^T)\partial_t^2\nabla T(\tilde{t})\|^2\,)+\tau^4\left(D_c^{-1}\|\partial_t^3C(\tilde{t})\|^2+D_c^{-1}\|\partial_t^2\nabla C(\tilde{t})\|^2+\alpha_3\|(I-P_K^C)\partial_t^2\nabla C(\tilde{t})\|^2\right)$$
$$+Da\beta_T^2\|\mathbf{g}\|_\infty^2\left(\|\eta^T_{n+1/2}\|^2+\|\phi^T_{n+1/2}\|^2\right)+Da\beta_C^2\|\mathbf{g}\|_\infty^2\left(\|\eta^C_{n+1/2}\|^2+\|\phi^C_{n+1/2}\|^2\right)$$
$$+\alpha_1\left\|(I-P_L^u)\mathbb{D}\mathbf{u}_{n+1/2}\right\|^2+\alpha_2\|(I-P_M^T)\nabla T_{n+1/2}\|^2+\alpha_3\|(I-P_K^C)\nabla C_{n+1/2}\|^2\Bigg\}. \tag{3.93}$$

After this point, we use the approximation properties (3.15)-(3.18) to finalize the convergence result. We should point out that, first three term at the right-hand side of (3.93) need special treatment. We only show the first one since the others will follow analogously.

Clearly

$$\left\| \frac{\boldsymbol{\eta}^u_{n+1} - \boldsymbol{\eta}^u_n}{\tau} \right\|^2 = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \partial_t \boldsymbol{\eta}^u d\hat{t} \right\|^2 .$$

Using the Cauchy-Schhwarz and generalized triangle inequality, one gets

$$\left\| \frac{\boldsymbol{\eta}^u_{n+1} - \boldsymbol{\eta}^u_n}{\tau} \right\|^2 \leq \left(\frac{1}{\tau} \int_{t_n}^{t_{n+1}} 1 \cdot \|\partial_t \boldsymbol{\eta}^u\|^2 d\hat{t} \right)^2 \leq \frac{\int_{t_n}^{t_{n+1}} 1^2 d\hat{t}}{\tau} \left(\frac{1}{\tau} \int_{t_n}^{t_{n+1}} \|\partial_t \boldsymbol{\eta}^u\|^2 d\hat{t} \right).$$

Next, using the standard interpolation estimates as in [42] we have

$$\nu^{-1} \left\| \frac{\boldsymbol{\eta}^u_{n+1} - \boldsymbol{\eta}^u_n}{\tau} \right\|^2 \leq K \nu^{-1} h^{2s+2} \|\partial_t \mathbf{u}\|^2_{L^2\left(0,t^*;H^{s+1}(\Omega)\right)}. \tag{3.94}$$

Similarly, for η^T and η^C we have

$$\gamma^{-1} \left\| \frac{\eta^T_{n+1} - \eta^T_n}{\tau} \right\|^2 \leq K \gamma^{-1} h^{2s+2} \|\partial_t T\|^2_{L^2\left(0,t^*;H^{s+1}(\Omega)\right)} \tag{3.95}$$

and

$$D_c^{-1} \left\| \frac{\eta^C_{n+1} - \eta^C_n}{\tau} \right\|^2 \leq K D_c^{-1} h^{2s+2} \|\partial_t C\|^2_{L^2\left(0,t^*;H^{s+1}(\Omega)\right)}. \tag{3.96}$$

Now, multiplying both sides of (3.93) with 2τ, summation over the time levels from 1 to n, making use of approximation properties and inserting (3.57)-(3.59) and (3.94)- (3.96) yield

$$
\begin{aligned}
&\left(\|\boldsymbol{\phi}^u_{n+1}\|^2 + \|\phi^T_{n+1}\|^2 + \|\phi^C_{n+1}\|^2\right) + \tau\left(\sum_{i=1}^{n}(2\nu\|\mathbb{D}\boldsymbol{\phi}^u_{i+1/2}\|^2 + \gamma\|\nabla\phi^T_{i+1/2}\|^2 + D_c\|\nabla\phi^C_{i+1/2}\|^2)\right)\\
&+\tau\sum_{i=1}^{n}(\alpha_1\|(I-P^u_L)\mathbb{D}\boldsymbol{\phi}^u_{i+1/2}\|^2 + \alpha_2\|(I-P^T_M)\nabla\phi^T_{i+1/2}\|^2 + \alpha_3\|(I-P^C_K)\nabla\phi^C_{i+1/2}\|^2)\\
&+\tau\sum_{i=1}^{n} Da^{-1}\|\boldsymbol{\phi}^u_{i+1/2}\|^2\\
&\le K\Bigg\{\left(\|\boldsymbol{\phi}^u_1\|^2 + \|\phi^T_1\|^2 + \|\phi^C_1\|^2\right) + h^{2s+2}\left(\nu^{-1}\|\partial_t\mathbf{u}\|^2_{L^2(0,t^*;H^{s+1}(\Omega))} + \gamma^{-1}\|\partial_t T\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\right.\\
&\left.+D_c^{-1}\|\partial_t C\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\right) + h^{2s}\,(\,\nu^{-1}\|\mathbf{u}\|^2_{L^2(0,t^*;H^{s+1}(\Omega))} + \gamma^{-1}\|T\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\\
&+D_c^{-1}\|C\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\,)\left(1 + h^{2s}\|\mathbf{u}\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\right) + (\nu^{-1}+\gamma^{-1}+D_c^{-1})\\
&\times\left(h^{2s}\|\mathbf{u}\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\right) + \tau^4\left(\nu^{-1}\|\partial_t^3\mathbf{u}\|^2_{L^2(0,t^*;L^2(\Omega))} + (\nu+\nu^{-1})\|\partial_t^2\mathbb{D}\mathbf{u}\|^2_{L^2(0,t^*;L^2(\Omega))}\right.\\
&+\alpha_1\|(I-P^u_L)\partial_t^2\mathbb{D}\mathbf{u}\|^2_{L^2(0,t^*;L^2(\Omega))} + Da^{-1}\|\partial_t^2\mathbf{u}\|^2_{L^2(0,t^*;L^2(\Omega))} + \nu^{-1}\|\partial_t^2 p\|^2_{L^2(0,t^*;L^2(\Omega))}\\
&\left.+Da\beta_T^2\|\mathbf{g}\|^2_\infty\|\partial_t^2 T\|^2_{L^2(0,t^*;L^2(\Omega))} + Da\beta_C^2\|\mathbf{g}\|^2_\infty\|\partial_t^2 C\|^2_{L^2(0,t^*;L^2(\Omega))}\,\right) + \tau^4\,(\,\gamma^{-1}\|\partial_t^3 T\|^2_{L^2(0,t^*;L^2(\Omega))}\\
&+\gamma^{-1}\|\partial_t^2\nabla T\|^2_{L^2(0,t^*;L^2(\Omega))} + \alpha_2\|(I-P^T_M)\partial_t^2\nabla T\|^2_{L^2(0,t^*;L^2(\Omega))}\,) + \tau^4\,(\,D_c^{-1}\|\partial_t^3 C\|^2_{L^2(0,t^*;L^2(\Omega))}\\
&+D_c^{-1}\|\partial_t^2\nabla C\|^2_{L^2(0,t^*;L^2(\Omega))} + \alpha_3\|(I-P^C_K)\partial_t^2\nabla C\|^2_{L^2(0,t^*;L^2(\Omega))}\,) + Da\beta_T^2\|\mathbf{g}\|^2_\infty(h^{2s}\|T\|^2_{L^2(0,t^*;H^{s+1})})\\
&+Da\beta_C^2\|\mathbf{g}\|^2_\infty(h^{2s}\|C\|^2_{L^2(0,t^*;H^{s+1})}) + H^{2s}\left(\alpha_1\|\mathbf{u}\|^2_{L^2(0,t^*;H^{s+1}(\Omega))} + \alpha_2\|T\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\right.\\
&\left.+\alpha_3\|C\|^2_{L^2(0,t^*;H^{s+1}(\Omega))}\right) + h^{-1}\tau\sum_{i=1}^{n}\left(\nu^{-1}|\mathbf{u}(t_{i+1/2})|^2_{s+1} + \gamma^{-1}|T(t_{i+1/2})|^2_{s+1} + D_c^{-1}|C(t_{i+1/2})|^2_{s+1}\right)\\
&\times\left(\|\boldsymbol{\phi}^u_{i-1}\|^2 + \|\boldsymbol{\phi}^u_i\|^2\right) + (\nu^{-1}+\gamma^{-1}+D_c^{-1})\tau\sum_{i=1}^{n}\left(\|\boldsymbol{\phi}^u_{i-1}\|^2 + \|\boldsymbol{\phi}^u_i\|^2\right)\\
&+Da\beta_T^2\|\mathbf{g}\|^2_\infty\tau\sum_{i=1}^{n}\left(\|\phi^T_i\|^2 + \|\phi^T_{i+1}\|^2\right) + Da\beta_C^2\|\mathbf{g}\|^2_\infty\tau\sum_{i=1}^{n}\left(\|\phi^C_i\|^2 + \|\phi^C_{i+1}\|^2\right)\Bigg\}
\end{aligned}
\tag{3.97}
$$

Using the regularity of $\mathbf{u}, T, C$, i.e. $\mathbf{u}, T, C \in L^\infty\left(0, t^*; H^{s+1}(\Omega)\right)$, we can combine last three summation terms as follows.

$$
\begin{aligned}
&h^{-1}\tau\sum_{i=1}^{n}\left(\nu^{-1}|\mathbf{u}(t_{i+1/2})|^2_{s+1} + \gamma^{-1}|T(t_{i+1/2})|^2_{s+1} + D_c^{-1}|C(t_{i+1/2})|^2_{s+1}\right)\left(\|\boldsymbol{\phi}^u_{i-1}\|^2 + \|\boldsymbol{\phi}^u_i\|^2\right)\\
&+(\nu^{-1}+\gamma^{-1}+D_c^{-1})\tau\sum_{i=1}^{n}\left(\|\boldsymbol{\phi}^u_{i-1}\|^2 + \|\boldsymbol{\phi}^u_i\|^2\right) \le K\tau(\nu^{-1}+\gamma^{-1}+D_c^{-1}+h^{2s-1})\sum_{i=1}^{n}\|\boldsymbol{\phi}^u_i\|^2
\end{aligned}
\tag{3.98}
$$

$$
Da\beta_T^2\|\mathbf{g}\|^2_\infty\tau\sum_{i=1}^{n}\left(\|\phi^T_i\|^2 + \|\phi^T_{i+1}\|^2\right) \le Da\beta_T^2\|\mathbf{g}\|^2_\infty\tau\|\phi^T_{n+1}\|^2 + Da\beta_T^2\|\mathbf{g}\|^2_\infty\tau\sum_{i=1}^{n}\|\phi^T_i\|^2 \tag{3.99}
$$

and

$$Da\beta_C^2\|\mathbf{g}\|_\infty^2\tau\sum_{i=1}^{n}\left(\|\phi_i^C\|^2+\|\phi_{i+1}^C\|^2\right)\le Da\beta_C^2\|\mathbf{g}\|_\infty^2\tau\|\phi_{n+1}^C\|^2+Da\beta_C^2\|\mathbf{g}\|_\infty^2\tau\sum_{i=1}^{n}\|\phi_i^C\|^2. \quad (3.100)$$

So the only remaining thing we have to do is to estimate $\|\boldsymbol{\phi}_1^u\|^2+\|\phi_1^T\|^2+\|\phi_1^C\|^2$. We point out here that $\tilde{\mathbf{u}}_0$ is chosen to be $\mathbf{u}_0^h$, $\tilde{T}_0 = T_0^h$ and $\tilde{C}_0 = C_0^h$ to give $\boldsymbol{\phi}_0^u = \phi_0^T = \phi_0^C = 0$. Note that the error equations and estimations for $\boldsymbol{\phi}_1^u, \phi_1^T$ and ϕ_1^C are same as for time step n except the nonlinear terms. Illustratively, we bound nonlinear terms of velocity equation and the others will follow analogously.

To begin the estimation, we add and subtract $c_0\left(\mathbf{u}_0^h-\mathbf{u}(t_0),\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h\right)$ to have

$$\begin{aligned}&\Big|c_0\left(\mathbf{u}(t_{1/2}),\mathbf{u}(t_{1/2}),\mathbf{v}^h\right)+c_0\left(\mathbf{u}_0^h,\mathbf{e}_{1/2}^u,\mathbf{v}^h\right)+c_0\left(\mathbf{e}_0^u,\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h\right)\\&-c_0\left(\mathbf{u}(t_0),\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h\right)\Big|\le\left|c_0\left(\mathbf{u}(t_{1/2}),\mathbf{u}(t_{1/2}),\mathbf{v}^h\right)-c_0\left(\mathbf{u}(t_0),\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h\right)\right|\\&+\left|c_0\left(\mathbf{u}_0^h,\mathbf{e}_{1/2}^u,\mathbf{v}^h\right)+c_0\left(\mathbf{e}_0^u,\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h\right)\right|.\end{aligned}$$

and set $\mathbf{v}^h=\boldsymbol{\phi}_{1/2}^u$. The second term in (3.101) is treated exactly as in (3.87)- (3.91). For the first term we write

$$\begin{aligned}\left|c_0\left(\mathbf{u}(t_{1/2}),\mathbf{u}(t_{1/2}),\boldsymbol{\phi}_{1/2}^u\right)-c_0\left(\mathbf{u}(t_0),\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\boldsymbol{\phi}_{1/2}^u\right)\right| &\le K\tau^2\left|c_0\left(\mathbf{u}(t_0),\partial_t^2\mathbf{u}(t_0),\boldsymbol{\phi}_{1/2}^u\right)\right|\\&+K\tau\left|c_0\left(\partial_t\mathbf{u}(t_0),\mathbf{u}(t_{1/2}),\boldsymbol{\phi}_{1/2}^u\right)\right|\end{aligned}$$

through the help of a Taylor expansion. Clearly,

$$K\tau^2\left|c_0\left(\mathbf{u}(t_0),\partial_t^2\mathbf{u}(t_0),\boldsymbol{\phi}_{1/2}^u\right)\right|\le K\left(\nu\|\boldsymbol{\phi}_{1/2}^u\|^2+\nu^{-1}\tau^4\right).$$

For the other term we use (3.9) and the third part of Lemma 3.16 to have

$$\begin{aligned}\tau\left|c_0\left(\partial_t\mathbf{u}(t_0),\mathbf{u}(t_{1/2}),\boldsymbol{\phi}_{1/2}^u\right)\right| &\le \tau\left(\|\partial_t\mathbf{u}(\tilde{t})\|\|\mathbb{D}\mathbf{u}(t_{1/2})\|_{L^\infty(\Omega)}+\|\partial_t\mathbb{D}\mathbf{u}(\tilde{t})\|\|\mathbf{u}(t_{1/2})\|_{L^\infty(\Omega)}\right)\|\boldsymbol{\phi}_{1/2}^u\|\\&\le K\left(\tau^3+\|\boldsymbol{\phi}_1^u\|^2\right)\end{aligned}$$

thanks to the fact that $\boldsymbol{\phi}_{1/2}^u=\frac{1}{2}\boldsymbol{\phi}_1^u$. Putting these bounds obtained for nonlinear terms into the error equations for the first time level and combining the inequalities as in time level n yield

$$\|\boldsymbol{\phi}_1^u\|^2+\|\phi_1^T\|^2+\|\phi_1^C\|^2\le K(h^{2s}+(\alpha_1+\alpha_2+\alpha_3)H^{2s}+\tau^4) \quad (3.101)$$

where K depend on $\mathbf{u}$, p, T, C, ν, α_1, α_2, α_3, β_T, β_C, $\mathbf{g}$.

Finally, using the theorem assumptions $KDa\beta_T^2||\mathbf{g}||_\infty^2\tau \le \frac{1}{2}$, $KDa\beta_C^2||\mathbf{g}||_\infty^2\tau \le \frac{1}{2}$ and writing (3.101) back into (3.97) give us

$$
\begin{aligned}
&\left(||\boldsymbol{\phi}_{n+1}^u||^2 + ||\phi_{n+1}^T||^2 + ||\phi_{n+1}^C||^2\right) + \tau\left(\sum_{i=1}^{n}(2\nu||\mathbb{D}\boldsymbol{\phi}_{i+1/2}^u||^2 + \gamma||\nabla\phi_{i+1/2}^T||^2 + D_c||\nabla\phi_{i+1/2}^C||^2)\right)\\
&+\tau\sum_{i=1}^{n}(\alpha_1||(I - P_L^u)\mathbb{D}\boldsymbol{\phi}_{i+1/2}^u||^2 + \alpha_2||(I - P_M^T)\nabla\phi_{i+1/2}^T||^2 + \alpha_3||(I - P_K^C)\nabla\phi_{i+1/2}^C||^2)\\
&+\tau\sum_{i=1}^{n} Da^{-1}||\boldsymbol{\phi}_{i+1/2}^u||^2 \le K(h^{2s} + (\alpha_1 + \alpha_2 + \alpha_3)H^{2s} + \tau^4)\\
&+K\tau\sum_{i=1}^{n}\left(||\boldsymbol{\phi}_i^u||^2 + ||\phi_i^T||^2 + ||\phi_i^C||^2\right).
\end{aligned}
$$

The result is thus obtained via the application of discrete Gronwall's Lemma and a triangle inequality. □

CHAPTER 4

CONCLUSIONS AND FUTURE RESEARCH

This thesis studied the finite element analysis of projection-based stabilization method for the steady-state natural convection equations in a classical enclosed domain including the solid media and the Darcy-Brinkman model of time dependent double-diffusive convection equations in a confined porous medium. By means of this method, global stabilizations are added for both velocity variable, temperature variable for both systems and additionally for concentration variable for double-diffusive convection equation and these effects are subtracted from the large scales. We established the rigorous finite element error analysis of the scheme for the velocity, temperature, concentration and pressure and proved that with the appropriate choices of mesh scales and the stabilization parameters, the optimal errors can be obtained. We examined performance and accuracy of the method and compared the results with other published data. The numerical results revealed excellent agreement with other published data and validation of theoretical results.

There are some possible research directions that could be inspired from this thesis. Firstly, the stabilization idea proposed here could be applied on some other buoyancy driven systems such as natural convective flows under a magnetic field or thermal convective flows under influence of mechanical vibrations in porous media. One could adopt the method for a natural convection system of two immiscible fluids in an appropriate enclosure.

Also, domains considered here for both systems should be extended and new numerical tests should be carried out. L-shaped domains, backward facing steps and rectangular domains with infinite length are some examples of interest.

We carry out all the estimations with continuous finite elements in this study. A combination of Discontinuous Galerkin (DG) methods, which are known to have several advantages on continuous finite element schemes, with the projection based stabilization idea applied on buoyancy driven flows could be more effective than the one studied here. Lastly, a very interesting and effective method, namely the defect correction method should be tried on natural convection systems presented in this study.

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PUBLICATIONS

2. **A. Çıbık** and S. Kaya *A Projection Based Stabilized Finite Element Method for Steady-State Natural Convection Problem*, J. Math. Anal. Appl., **381**, (2011) 469-484.

1. **A. Çıbık** and S. Kaya , *Finite Element Analysis of a Projection-Based Stabilization Method for the Darcy-Brinkman Equations in Double-Diffusive Convection*, Submitted to Appl. Numer. Math.,(2011)

FIELD OF STUDY

Major Field: Numerical Analysis of Partial Differential Equations, Finite Element Method, Computational Fluid Dynamics, Variational multiscale methods.