FULLY COMPUTABLE CONVERGENCE ANALYSIS OF DISCONTINOUS GALERKIN FINITE ELEMENT APPROXIMATION WITH AN ARBITRARY NUMBER OF LEVELS OF HANGING NODES

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ABSTRACT

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In this thesis, we analyze an adaptive discontinuous finite element method for symmetric second order linear elliptic operators. Moreover, we obtain a fully computable convergence analysis on the broken energy seminorm in first order symmetric interior penalty discontinuous Galerkin finite element approximations of this problem. The method is formulated on nonconforming meshes made of triangular elements with first order polynomial in two dimension. We use an estimator which is completely free of unknown constants and provide a guaranteed numerical bound on the broken energy norm of the error. This estimator is also shown to provide a lower bound for the broken energy seminorm of the error up to a constant and higher order data oscillation terms. Consequently, the estimator yields fully reliable, quantitative error control along with efficiency.

As a second problem, explicit expression for constants of the inverse inequality are given in 1D, 2D and 3D. Increasing mathematical analysis of finite element methods is motivating the inclusion of mesh dependent terms in new classes of methods for a variety of applications.

Several inequalities of functional analysis are often employed in convergence proofs. Inverse estimates have been used extensively in the analysis of finite element methods. It is characterized as tools for the error analysis and practical design of finite element methods with terms that depend on the mesh parameter. Sharp estimates of the constants of this inequality is provided in this thesis.

Keywords: convergence analysis, Discontinuous Galerkin Method, Finite Element Method, inverse inequalities, orthogonal polynomials

ÖZ

SÜREKSİZ GALERKİN METODU İÇİN BİLİNMEYEN KATSAYILARDAN BAĞIMSIZ YAKINSAKLIK ANALİZİ

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Bu tezde, uyarlanabilir süreksiz Galerkin sonlu elemanlar yonteminin, ikinci dereceden eliptik kısmi turevlenebilir denklemler için yakınsaklık analizi yapıldı. Tamamı hesaplanabilir yakınsaklık analizinde birinci dereceden simetrik interior penaltı süreksiz Galerkin yaklaşımı kullanıldı ve norm olarak enerji normu seçildi. Kullanılan bu yöntem eş olmayan ve üçgenlerden oluşan ağ örgüsü üzerinde uygulandı. Uyarlanabilir bütün sonlu elemenlar yönteminde gerekli olan hata tahmincisi olarak şu ana kadar hiç bir calışmada kullanılmamış olan bir tahminci seçildi. Bu tahminci diğerlerinin aksine bilinmeyen katsayılardan bağımsız olduğu için bir indikator değil gerçek bir tahminci olarak kullanılabilir. Bu tahminci, hata için alt ve üst sınırlari sağlamaktadır. Sonuç olarak, kullanılan bu tahminci güvenilir ve etkili sayısal hata kontrolünü mümkün kılar.

İkinci bir calışma olarak, ters eşitsizliklerde kullanılan katsayılarin gercek değerleri hesaplandı. Bu değerler 1 boyutlu, 2 boyutlu ve 3 boyutlu uzaylar için üçgensel elemanlar kullanılarak bulundu. Sonlu elemanlar yönteminin artan matematiksel analizi, ağ örgüsüne bağlı terimlerin varlığını güdülemektedir. Fonksiyonel analizin bir kaç eşitsizliğide yakınsaklık ıspatlarında sık sık kullanılmaktadır. Ters eşitsizlikler, sonlu elemanlar yönteminin analizinde en yaygın kullanılan eşitsizliklerdendir. Bu eşitsizlikler, hata analizi ve ağ örgüsüne bağımlı sonlu elemanlar yönteminin pratik dizaynı için bir araç olarak karakterize edilir. Bu tezde, bu eşizliklerin katsayılarının keskin bir tahmini verilmiştir.

Anahtar Kelimeler: yakınsaklık analizi, süreksiz galerkin metodu, sonlu elemanlar metodu, ters eşitsizlikler, ortogonal polinomlar

To my mother and the rest of my lovely family

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CHAPTER 1

INTRODUTION

Many physical fundamental phenomena in nature, whether in the domain of fluid dynamics, electricity, or heat flow, can be described by equations that involve physical quantities together with their time and space derivatives. Equations involving time and space derivatives (partial derivatives) are called partial differential equations [43, 44, 78]. However, it may not always possible to obtain a closed form of the solution to these partial differential equations. Sometimes even, it is hard to know whether a unique solution exists or not. For these reasons, one can approach to the solution of these partial differential equations by using some numerical methods. One of the most widely used numerical methods to solve partial differential equations is finite element method, which is based on region discretization and each small region is called elements (often triangles or quadrilaterals in 2D and tetrahedral, or prisms in 3D). In this way, the original problem is transformed into a discrete problem for a finite number of unknown coefficients [24, 35, 78].

Principally, two types of finite element approximations are possible; conforming and nonconforming. If the approximated solution space is a subspace of weak solution space, the method is called conforming finite element. If this condition is not satisfied, we obtain a nonconforming element method [24].

In this thesis, we consider one of the non-conforming method, called discontinuous Galerkin (DG) and its convergence analysis. Specifically, we focus on:

The problem of convergence analysis of interior penalty DG method for elliptic problems for a residual type posteriori error estimation.

The second problem is to evaluate accurate constant for inverse inequalities in L_2-norm, which is very important to obtain a reliable error and convergence analysis of the finite element method.

1.1 On the Convergence of a Posteriori Error Estimation

One of the most common nonconforming method is Discontinuous Galerkin (DG) finite element method. It is known that DG finite element approximations are technically nonconforming finite element approximations, since an approximate solution is obtained from a finite dimensional space which is not a subspace of the weak solution space. This method was first proposed and analyzed in 1973 by Reed and Hill [70] to solve the hyperbolic neutron transport equation. DG methods for obtaining approximate solutions to elliptic partial differential equations have been around for several decades [8, 10, 11, 16, 36, 74, 84], and the use of these methods has become widespread. Some of the major advantage of DG methods are that they are locally conservative, stable, and high-order accurate methods which can easily handle complex geometries, irregular meshes with hanging nodes, and also approximations that have polynomials of different degrees in different elements [36, 37, 74].

Whether one uses conforming or nonconforming finite element method, error analysis is required to have confidence in the numerical approximation of the solution. Obviously, since the error is defined in terms of the unknown weak solution to the partial differential equation we cannot evaluate the error directly. However, the best one can do is to estimate error. Error estimators can be categorized as either a priori or a posteriori. A priori error estimation is that the error is estimated without employing the numerical solution. Indeed, the primary objective is to derive rates of convergence with respect to the discretization parameters in order to evaluate the performance of a given numerical method. It means that a priori error estimators involve the unknown solution to the partial differential equation. Hence it cannot actually be used to estimate the value of the error. However, they can be used to predict the asymptotic behavior of the error.

In contrast, a posteriori error estimators [2,3,13,82] do not involve the unknown weak solution and are given instead in terms of quantities like the known data in the partial differential equation, the discretization of the domain and the approximate solution. A posteriori error estimators estimate the value of the error in an appropriate norm. They are important not only for determining the accuracy of finite element approximations but also for implementing adaptive refinement strategies.

In addition, a posteriori error estimation plays a key role in the assessment of the accuracy of finite element simulations and in the control of adaptive refinement algorithms. One of the most common types of error estimator is residual-based estimators. Residual-based estimators involve terms which appear in the elementwise and edgewise terms in the residual equation.

Other types of a posteriori error estimators include hierarchical estimators and averaging based estimators. Hierarchical estimators [14, 15] involve bounding the norm of the error by the norm of the difference between the finite element approximation and an approximation to the problem. Averaging based estimators [1, 29, 30] make use of a continuous smoothed function obtained by averaging the approximate solution or its gradient. More precise details of the above mentioned estimators as well as information on other types of a posteriori error estimators can be found in [2, 3, 13, 82].

It is desirable for a posteriori estimators to provide two-sided bounds on the error up to unknown positive constants that are independent of the size of the elements in the mesh. This is important since it means that the estimator is efficient and reliable. By "reliable" we mean that the estimator behaves in the same way with the error as the mesh is refined. Efficiency is also important since it allows the estimator to be used to show that the approximation converges with respect to an adaptive refinement strategy.

In practice, however it is not always possible to show that the estimator provide lower bound (i.e.,estimator is less than some positive constant multiple of the norm of the error) since the data may belong to an infinite dimensional space. Instead we pose for the estimator being less than a positive constant multiple of the norm of the error plus terms which decrease at a rate faster than the error, provided that the data is sufficiently smooth, as the mesh is refined. The most common way of showing that estimators provide lower bounds on the error is to use bubble function arguments. By this, the quality of an a posteriori error estimator can be measured by its efficiency index, i.e.; the ratio of the estimated error and of the true error. These have been first used in [80] and developed in [81] into the way in which they are most commonly used now.

In order to realize the full flexibility of DG finite element methods, one generally wishes to perform local refinements of the mesh in the neighborhood of regions where the accuracy is poor. A posteriori error estimators are often used for this purpose, and to provide a stopping criterion for an adaptive feedback procedure.

However, because of the presence of unknown constants, none of the a posteriori error estimators provide actual computable upper bounds on the error. This type of estimator can be found in literature [21, 22, 29, 31–33, 38, 39, 51, 55, 68, 76] for nonconforming finite element approximations and [18, 28, 32, 53, 57, 59, 73, 75] for DG finite element approximations.

As stated in [73], the estimators commonly used in the above mentioned literature can only be used as a error indicator not as estimators of the actual value of the error. It also means that they cannot be used as a stopping criterion for an adaptive refinement procedure. Nevertheless, in practice the value of the unknown constant in the estimator is set equal to unity and the estimator is used as a stopping criterion. While such error indicators have a role to play, if one really wants to estimate the value of the error, then a new approach is required. The new approach should contain a fully computable estimator.

In [5], actual computable bounds were obtained for both the broken energy seminorm and the DG-norm of the error in the first order symmetric interior penalty DG finite element approximation of a linear second order elliptic problem with variable permeability on triangles. While it is more common to obtain error estimates for DG methods in the parameter and mesh dependent DG-norm it is also shown in [5] that the broken energy seminorm of the error was in fact equivalent to the DG-norm of the error provided the interior penalty parameter is sufficiently large. Recently, in [7], the approach of [5] is generalized to perform adaptivity with a constant free fully computable posteriori error estimators which are applicable to symmetric interior penalty DG, non-symmetric interior penalty DG and incomplete interior penalty DG finite element approximations of first order on meshes containing hanging nodes

Adaptive procedures for the numerical solution of partial differential equations started in the late 70's and are now standard tools in science and engineering. We may refer to [82] on adaptivity of elliptic partial differential equations. Adaptive finite element methods (AFEM) are indeed a meaningful approach for handling multiscale phenomena and making realistic computations feasible, especially in three dimensions.

A posteriori error estimators are an essential ingredient of the adaptivity. The ultimate purpose of adaptivity is to construct a sequence of meshes that would eventually equidistribute the approximation errors, and as a consequence the computational effort. To this end, a posteriori error estimators are split into element indicators which are then employed to make local mesh modifications by refinement and coarsening. The principal goal of an adaptive algorithm is to achieve a user specified error level in a finite number of cycles. A typical cycle consists of the following basic steps:

- **Solve:** For a given mesh, we calculate the approximation solution on this mesh.
- **Estimate:** Estimate the error of the approximation for each element by using error estimator.
- **Mark:** Mark the triangle which is error considerably larger by a specific marking strategy.
- **Refine:** Refine the given mesh using the information above to obtain a new refine mesh.

Experience strongly suggests that, starting from a coarse mesh, such an iteration always converges within any prescribed error tolerance in a finite number of steps.

The convergence of adaptive algorithms for elliptic problems started with the work of Babuska and Vogelius [12] where a detailed treatment of the one-dimensional case was given. A convergence proof is given in [41] for the two-dimensional case for the standard Galerkin method using linear elements while outlining an extension to quadratic elements. One of the highlights of this work is that bounds on the convergence rate were provided, which was not the case for [12]. Further studied is given in [64–66], whereas the issue of optimal order of convergence has been addressed in [20] and [79]. Non-standard finite element techniques such as mixed and nonconforming methods and edge element discretization of Maxwell's equations have been recently investigated in [25–27]. On the other hand, the initial mesh had to be fine enough to essentially get the solution. The latter issue is provided the starting point for the work of [65, 66], who introduced the concept of data oscillation. The nagging issue of calculating this quantity accurately on a coarse mesh is not resolved and should be treated within the larger and important framework of accounting for the quadrature errors arising from the implementation of the finite element formulation as well as from the calculation of certain terms in the a posteriori estimators. More recently, a modification of the algorithm of [66] is proposed in [20] that incorporates coarsening to prove optimal work for estimates.

In the recent works of [23, 52, 59], a convergence analysis of symmetric interior penalty DG finite element methods for elliptic problems have been obtained. They used the same estimator to analyze convergence of the method.

In [59] regularity conditions for datums were very restrictive. Solution is needed extra regularity conditions. Two successive subdivisions are not too far from each other. A relatively large amount of new nodes (12 vertices for piecewise linear elements in 2D) must be created by refining each marked element. Hoppe et al. [52] improved upon [59]: first the refinement procedure consists of just one bisection per marked element, and second regularity conditions for datum is more flexible. However, [52] assumes that the ensuing data oscillation contracts relative to itself between consecutive iterates, which is not guaranteed when marking only by the estimator. Also, the technique used in [52] for the error analysis is based on the Crouzeix-Raviart element, and thereby applies only to conforming meshes. To enforce the aforementioned contraction of data oscillation, one would need to mark also by oscillation. Unfortunately, this would lead to separate marking and, as discussed by [34], to the risk of getting sub-optimal meshes. However, [23] extended and improved the deficiencies of [52,59] in several respects. First, the less restrictive data regularity is assumed. Secondly, different types of nonconforming subdivisions are allowed such as tetrahedral or hexahedral meshes with hanging nodes. Also, the complexity of refine with fixed level of non-conformity is examined. Each marked element is refined using only one subdivision, either quad-refinement for hexahedral meshes, or red-refinement and bisection for tetrahedral meshes. Contraction property of the adaptive DG finite element method is proved, without further assumptions on refine, for the sum of energy error and scaled error estimator. Also, it is shown that the approximation classes consisting of continuous and discontinuous finite elements are equivalent. Quasi-optimal asymptotic rate of convergence for the adaptive DG finite element method is derived, which seems to be the first result of this type in the literature for DG methods. A quasi-optimal asymptotic rate of convergence for the continuous Galerkin method on (hexahedral and tetrahedral) nonconforming meshes is obtained. However, in [52, 59], mixed boundary conditions are considered, while in [23] homogeneous Dirichlet boundary condition is assumed to simplify given technical presentation.

In this thesis, as different from [23,52,59], we use a new estimator which is introduced in [7]. The aforementioned estimator provides actual computable numerical bounds on the error in the broken energy seminorm and DG-norm. Using this estimator, we proved convergence of the adaptive DG method for symmetric second order linear elliptic operators with explicit fully computable constant.

1.2 On the Inverse Estimation

The process of designing finite element methods has become increasingly dependent in recent years on understanding the mathematical framework underlying this methodology. This is evident from the advent of a profusion of new classes of methods which are founded on the basis of error analysis. In such cases, additional quantities are introduced into the formulation in order to demonstrate convergence of numerical solutions to the exact solution, usually at optimal or quasi-optimal rates.

Accurate approximate values for the constants which appear in the convergence analysis are crucial for the correct derivation of a priori and a posteriori error estimations.

Convergence proofs frequently make use of well known inequalities of functional analysis. For the purpose of analysis it is sufficient to know that these inequalities hold for positive constants. In addition, for general-purpose definitions of the mesh parameter (e.g., the length of the longest element side in the mesh) for regular elements (i.e., aspect ratios and distortion are limited) on quasi-uniform meshes (in which elements are of essentially same size). However, these restrictions on the mesh are frequently violated in the computation of solutions to engineering problems. In contrast to the perspective of mathematical analysis, when constructing methods for practical implementation, engineers need to be concerned with precise contextual definitions of the element size and sharp estimation of the constants to determine the coefficients of the least-squares terms. Indeed, various techniques employed in estimating the coefficients in inverse and trace inequalities and many others. Those quantities are computed for many cases. In this thesis, we also deal with the coefficients of inverse inequalities.

Inverse inequalities (or Markov inequalities) play an important role in many areas of mathematical research. For instance, they are commonly used in the error analysis of variational methods such as finite element methods and DG methods for solving partial differential equations. Explicit constants for some inverse inequalities can be found in [46]. The classical

Markov inequality for univariate polynomials states that for any polynomial u total degree N

$$\|u'\|_{L_\infty([a,b])} \leqslant \frac{2N^2}{|b-a|}\|u\|_{L_\infty([a,b])}.$$

A discussion of the exact constant in the univariate Markov inequality is given in [19]. It is proved that for a polynomial degree at most N with real coefficients that have at most m distinct complex zeros,

$$\|u'\|_{L_\infty[-1,1]} \leqslant 32 \cdot 8^m N \|u\|_{L_\infty[-1,1]}.$$

Another discussion of the exact constant in the univariate Markov inequality in L_2-norm is given in [62]. In [62] it is proved that for a polynomial u total degree N

$$\|u'\|_{L_2([-1,1])} \leqslant M_N \|u\|_{L_2([-1,1])},$$

where M_N coincides with the largest positive root of the following equation

$$\sum_{k=0}^{\frac{N+1}{2}} (-1)^k x^{-2k} \frac{(N+1+2k)!}{2^{2k} 2k!(N+1-2k)!} = 0\,, \tag{1.1}$$

where a new simple elementary method is presented for finding M_N. By using spectral analysis methods, the special case of the L_2-norm has been previously studied in [50]. It is shown that M_N is the solution of a certain equation which is equivalent to (1.1).

In [77], it is proved that for a polynomial u total degree N on a finite interval, the following inequality holds

$$\|u'\|_{L_2([a,b])} \leqslant 2\sqrt{3}\frac{N^2}{|b-a|}\|u\|_{L_2([a,b])}.$$

In the last thirty years possible extensions of the above estimations for multivariate polynomials have been widely investigated.

In [40], the following result is proved: for a polynomial $\boldsymbol{u}$ of total degree N and a bounded convex set K

$$\left\|\frac{\partial \boldsymbol{u}}{\partial \xi}\right\|_{L_p(K)} \leqslant CN^2 \|u\|_{L_p(K)}$$

for $0 < N \leqslant \infty$, $\frac{\partial}{\partial \xi}$ an arbitrary unit directional derivative, and C a constant independent of N and $\boldsymbol{u}$.

In [47], it is shown that certain directional derivatives of polynomials in two variables have a unit bound at the Chebyshev nodes. A Markov-type estimate for an arbitrary convex body $K \subset \mathbb{R}^m$ is given in [85]. For a convex body $K \subset \mathbb{R}^m$, the minimal distance between two parallel supporting hyperplanes for K is denoted by $w(K)$. Then, in [85], for a polynomial of total degree N, it holds:

$$\|\nabla \boldsymbol{u}\|_{L_\infty(K)} \leqslant \frac{4N^2}{w(K)}\|\boldsymbol{u}\|_{L_\infty(K)}.$$

In [67], the above inequality is verified in the special case when K is a triangle in $\mathbb{R}^2$. A similar result was shown in [63] improving the constant as $4N^2 - 2N$ instead of $4N^2$.

In addition, in [77], it is proved that for a polynomial $\boldsymbol{u}$ total degree N, Markov inequality holds for a triangle K or quadrilateral Q with an unknown constant C in L_2 and L_∞-norm,

$$\|\nabla \boldsymbol{u}\|_{L_\infty(K)} \leqslant CN^2\|\boldsymbol{u}\|_{L_\infty(K)},$$

$$\|\nabla \boldsymbol{u}\|_{L_2(K)} \leqslant CN^2\|\boldsymbol{u}\|_{L_2(K)}.$$

In conclusion, using the above inequalities, one may get an exact constant for univariate Markov inequality in L_∞-norm and L_2-norm. Moreover, one can estimate the exact constant for multivariate Markov inequalities in L_∞-norm. We emphasize that by using these results, constants for univariate and multivariate Markov inequalities in L_2-norm for several dimensions can be obtained.

To summarize; obtaining efficient and reliable analysis is very important task in the numerical approximation of partial differential equations. The accomplishments of this thesis can be summarized as follows:

1. Fully computable convergence analysis is obtained to get error reduction property. One can realize that how large interior penalty parameter, one needs to get convergence result in adaptive strategy.

2. Inverse inequality constants which are very important in correct derivation of convergence analysis of the partial differential equations are given up to polynomial degree 10 for 1D, 2D and 3D problems.

The remainder of this thesis is organized as follows.

Chapter 2 of thesis defines some notations and preliminaries used through this thesis. We also discuss some ways of partitioning the domain over which the problem is posed. Chapter 2 is concluded with an important result which allows us to decompose the broken energy seminorm of the error into conforming and nonconforming components [4].

In Chapter 3, we consider fully computable convergence analysis, in which the error does not contain any unknown constants. Specifically, DG finite element approximation of first order on meshes is given with an arbitrary number of hanging nodes.

We then go on to give an exact constant for inverse estimates which is used in mathematical analysis frequently, in Chapter 4. For 1D, 2D and 3D problems, inverse estimates constant is given up to polynomial degree 10.

Summary and a novel contributions of our established results are discussed in Chapter 5.

CHAPTER 2

NOTATIONS and PRELIMINARIES

2.1 The Model Problem

Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary denoted by Γ. Let $\overline{\Omega}$ denote the closure of a region Ω. Let $\boldsymbol{x} = (x\,,\, y)^T$ denote the position vector of a point on $\overline{\Omega}$ where the superscript T denotes the transpose. Also, the L_2 inner product over a region Ω denoted by $(\cdot, \cdot)_\Omega$ and its norm is given with

$$\| \cdot \|_\Omega = (\cdot, \cdot)_\Omega^{1/2},$$

where the L_2-inner product space

$$L_2(\Omega) = \left\{ v : \|v\|_{L_2(\Omega)} < \infty \right\}.$$

The Sobolev spaces are

$$\begin{aligned} H^1(\Omega) &= \{v : v \in L_2(\Omega), \nabla v \in L_2(\Omega) \times L_2(\Omega)\} \text{ and} \\ H^1_D(\Omega) &= \{v : v \in H^1(\Omega), v = 0 \text{ on } \Gamma_D\} \end{aligned}$$

with the operator ∇ being such that

$$\nabla = \left(\frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \right)^T.$$

Also, we use the following space:

$$H(\mathrm{div}; \Omega) = \{\boldsymbol{v} : \boldsymbol{v} \in L_2(\Omega) \times L_2(\Omega), \mathrm{div}\boldsymbol{v} \in L_2(\Omega)\}$$

with the operator div being such that

$$\mathrm{div}\, \boldsymbol{v} = \frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 \text{ for } \boldsymbol{v} = (v_1\,,\, v_2)^T.$$

Then, consider the following model problem

$$\begin{aligned} -\mathrm{div}(\boldsymbol{A}\nabla u) &= f \quad \text{in} \quad \Omega, \\ u &= q \quad \text{on the Dirichlet boundary} \;\; \Gamma_D, \\ \boldsymbol{A}\nabla u \cdot n_{\Gamma_N} &= g, \quad \text{on the Neumann boundary} \;\; \Gamma_N, \end{aligned}$$

where the disjoint sets Γ_D and Γ_N form a partitioning of the boundary Γ of the domain Ω and n_{Γ_N} is the outward unit normal vector to Γ_N. The data satisfy $f \in L_2(\Omega)$, $g \in L_2(\Gamma_N)$ and $q \in L_2(\Gamma_D)$, and $\boldsymbol{A}$ is symmetric and positive definite and satisfies the condition that we can decompose Ω into polygons such that $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ on each of these polygons, i.e., if we start with a coarse mesh called $\mathcal{P}_{h,0}$, then $\boldsymbol{A}$ is constant on that mesh only.

The standard weak formulation of problem (2.1) is: Find $u \in H^1(\Omega)$ such that

$$(\boldsymbol{A}\nabla u, \nabla v) = (f, v) + (g, v)_{\Gamma_N} \qquad \forall v \in H^1_D(\Omega). \tag{2.1}$$

2.1.1 The Partitioning of the Domain and Some Standard Notation

Let $\mathcal{P}_h$ be a partition of a domain Ω and K denotes an individual triangle. The boundary of triangle K is denoted by ∂K.

The set containing the individual edges of triangle K is denoted by $\mathcal{E}_K$. Likewise, $\mathcal{E}_h^I$, $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$ stand for the sets of edges defined by

$$\begin{aligned} \mathcal{E}_h^I &= \left\{\gamma : \gamma = \partial K \cap \partial K^{'}, K, K^{'} \in \mathcal{P}_h\right\}, \\ \mathcal{E}_h^D &= \left\{\gamma \subset \overline{\Gamma_D} : \gamma \in \mathcal{E}_K \text{ for some } \; K \in \mathcal{P}_h\right\}, \\ \mathcal{E}_h^N &= \left\{\gamma \subset \overline{\Gamma_N} : \gamma \in \mathcal{E}_K \text{ for some } \; K \in \mathcal{P}_h\right\}, \end{aligned}$$

and let $\partial\mathcal{P}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D \cup \mathcal{E}_h^N$. We use $|K|$ and $|\gamma|$ to refer the area of triangle K and the length of edge γ, respectively. The size and shape of an element K of $\mathcal{P}_h$ are measured in terms of two quantities, h_K and ρ_K, defined as:

$$\begin{aligned} h_K &:= \text{Longest edge of an element } \; K, \\ \rho_K &:= \sup\{\mathrm{diam}(B); \text{B is a ball contained in } \; K\}. \end{aligned}$$

Definition 2.1.1 *A family of partitions $\mathcal{P}_h$ is said to be shape regular if there exists a number $\varrho > 0$, independent of h_K and K such that*

$$\frac{h_K}{\rho_K} \leqslant \varrho, \qquad \forall K \in \mathcal{P}_h. \tag{2.2}$$

We assume all partitions $\mathcal{P}_h$ are shape regular in this thesis.

2.2 Projection Operators

For $m \in \mathbb{N}_0$, $\mathbb{P}_m(K)$ signifies the space of polynomials on $K \in \mathcal{P}_h$ of total degree at most m. Similarly, $\mathbb{P}_m(\gamma)$ denotes the space of polynomials on $\gamma \in \partial\mathcal{P}_h$ of total degree at most m, with respect to the arc length parameter on edge γ.

For $v \in L_2(K)$, let $\Pi_K^{(m)} v \in \mathbb{P}_m(K)$ be the function satisfying

$$(v - \Pi_K^{(m)} v, p)_K = 0 \quad \text{for all} \;\; p \in \mathbb{P}_m(K).$$

Similarly, for $v \in L_2(\gamma)$ and $\gamma \in \partial\mathcal{P}_h$, let $\Pi_\gamma^{(m)} v \in \mathbb{P}_m(\gamma)$ be the function satisfying

$$(v - \Pi_\gamma^{(m)} v, p)_\gamma = 0 \quad \text{for all} \;\; p \in \mathbb{P}_m(\gamma).$$

2.3 Jumps and Average

For each element $K \in \mathcal{P}_h$, let $v_{|K}$ refer the restriction of v to $K \in \mathcal{P}_h$. Let $\boldsymbol{n}_\gamma$ be a fixed unit normal vector for each edge $\gamma \in \mathcal{E}_h^I$ shared by two adjacent elements K and K'. For a function v such that $v_{|K} \in H^2(K)$ for all $K \in \mathcal{P}_h$, average and jumps are defined by

$$\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_\gamma = \begin{cases} \frac{1}{2}\left(\boldsymbol{n}_\gamma \cdot \boldsymbol{A}_K \nabla v_{|K} + \boldsymbol{n}_\gamma \cdot \boldsymbol{A}_{K'} \nabla v_{|K'}\right) & \text{on } \gamma = \partial K \cap \partial K', \\ \boldsymbol{n}_\gamma \cdot \boldsymbol{A}_K \nabla v_{|K} & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_h^D; \end{cases}$$

$$[v]_\gamma = \begin{cases} v_{|K} - v_{|K'} & \text{on } \partial K \cap \partial K' \subseteq \gamma, \\ v_{|K} & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_h^D; \end{cases}$$

$$[\boldsymbol{n} \cdot \boldsymbol{A}\nabla v]_\gamma = \begin{cases} \boldsymbol{n}_\gamma \cdot \boldsymbol{A}_K \nabla v_{|K} - \boldsymbol{n}_\gamma \cdot \boldsymbol{A}_{K'} \nabla v_{|K'} & \text{on } \partial K \cap \partial K' \subseteq \gamma, \\ \boldsymbol{n}_\gamma \cdot \boldsymbol{A}_K \nabla v_{|K} - \Pi_\gamma^{(0)} g & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_h^N; \\ 0 & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_h^D; \end{cases}$$

where $\boldsymbol{A}_K = \boldsymbol{A}_{|K} \in \mathbb{R}^{2\times 2}$.

2.4 Data Oscillation

The oscillation of the datum f on an element $K \in \mathcal{P}_h$ is defined to be

$$osc(f, K, m) = |K|^{1/2} \|f - \Pi_K^{(m)} f\|_K. \tag{2.3}$$

Likewise, the oscillation of the Neumann datum g on an edge $\gamma \in \mathcal{E}_h^N \cap \mathcal{E}_K$ is defined to be

$$osc(g, \gamma, m) = |\gamma|^{1/2} \|g - \Pi_\gamma^{(m)} g\|_\gamma. \tag{2.4}$$

The oscillation of the Dirichlet data q on an edge $\gamma \in \mathcal{E}_K \cap \mathcal{E}_h^D$ is defined to be

$$osc(q, \gamma) = |\gamma|^{1/2} \|\frac{\partial q}{\partial s_\gamma} - \Pi_\gamma^{(0)} \frac{\partial q}{\partial s_\gamma}\|_\gamma, \tag{2.5}$$

where s_γ is the arc length parameter on edge γ.

2.5 DG Finite Element Approximations

We assume that, the solution of this problem is a first order DG finite element approximation obtained on a mesh where there is an arbitrary, but bounded, number of hanging nodes. The DG finite element space on $\mathcal{P}_h$ is defined by

$$X_h = \{v : \Omega \longrightarrow \mathbb{R} : v_{|K} \in \mathbb{P}_1(K) \;\; \forall K \in \mathcal{P}_h\}.$$

Let $\tau \in \{-1, 1\}$ be fixed and, for $w, v \in X_h$, define the bilinear form $B_h : X_h \times X_h \longrightarrow \mathbb{R}$ by

$$\begin{aligned} B_h(w, v) \;\; = \;\; & \sum_{K \in \mathcal{P}_h} (\boldsymbol{A}\nabla w, \nabla v)_K - \sum_{\gamma \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla w \rangle_\gamma, [v]_\gamma \right)_\gamma \\ & - \;\; \tau \sum_{\gamma \in \mathcal{E}_I \cup \mathcal{E}_h^D} \left([w]_\gamma, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_\gamma \right)_\gamma + \sum_{\gamma \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left(\frac{\kappa}{|\gamma|} [w]_\gamma, [v]_\gamma \right)_\gamma \end{aligned} \tag{2.6}$$

and linear form $L : X_h \longrightarrow \mathbb{R}$ by

$$\begin{aligned} L(v) \;\; = \;\; & \sum_{K \in \mathcal{P}_h} (f, v)_K + \sum_{\gamma \in \mathcal{E}_h^N} (g, v)_\gamma \\ & + \sum_{\gamma \in \mathcal{E}_h^D} \left(\frac{\kappa}{|\gamma|} q, v \right)_\gamma - \tau \sum_{\gamma \in \mathcal{E}_h^D} \left(q, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_\gamma \right)_\gamma, \end{aligned}$$

where the $\kappa > 0$ are the usual interior penalty parameters. We can obtain the DG finite element approximation of the solution by finding $u_h \in X_h$ such that

$$B_h(u_h, v) = L(v) \;\; \forall v \in X_h. \tag{2.7}$$

2.6 The Broken Energy Seminorm and DG-norm of the Error

For functions v such that $v_{|K} \in H^1(K)$ for all $K \in \mathcal{P}_h$, the operator $\nabla_{\mathcal{P}_h}$ is defined by

$$(\nabla_{\mathcal{P}_h} v)_{|K} = (\nabla v)_{|K} \text{ for } K \in \mathcal{P}_h.$$

We also define the ***curl*** operator with

$$\mathbf{curl} = \left(\frac{\partial}{\partial y}\, ,\, -\frac{\partial}{\partial x} \right)^T .$$

The broken energy seminorm over a region w is denoted by

$$|||\cdot|||_w = (\boldsymbol{A}\nabla_{\mathcal{P}_h}\cdot, \nabla_{\mathcal{P}_h}\cdot)_w^{1/2} \tag{2.8}$$

where we omit the subscript in the case where $w = \Omega$. We shall decompose the broken energy seminorm of the error into conforming and nonconforming components. This is done by using a generalization of the Helmholtz decomposition of the space $L_2(\Omega) \times L_2(\Omega)$ into the gradient of a function in the space $H^1_D(\Omega)$ plus the ***curl*** of a function in the space

$$\mathcal{H} = \{w \in H^1(\Omega) : (w, 1)_\Omega = 0 \text{ and } \boldsymbol{t}_{\Gamma_N} \cdot \nabla w = 0 \text{ on } \Gamma_N\}, \tag{2.9}$$

where $\boldsymbol{t}_{\Gamma_N}$ is a tangent vector to Γ_N. We shall use the result proved in [4] to decompose the broken energy seminorm of the error e_h in discontinuous Galerkin finite element approximations of the solution to problem (2.1) as follows.

Theorem 2.6.1 *The error e_h defined by $e_h = u - u_h$ may be decomposed into the form*

$$\boldsymbol{A}\nabla_{\mathcal{P}_h} e_h = \boldsymbol{A}\nabla\phi_h + \boldsymbol{curl}\psi_h,$$

where the conforming error $\phi_h \in H^1_D(\Omega)$ satisfies

$$(\boldsymbol{A}\nabla\phi_h, \nabla v) = (\boldsymbol{A}\nabla_{\mathcal{P}_h} e_h, \nabla v) \qquad \forall v \in H^1_D(\Omega), \tag{2.10}$$

and the nonconforming error $\psi_h \in \mathcal{H}$ satisfies

$$(\boldsymbol{A}^{-1}\boldsymbol{curl}\psi_h, \boldsymbol{curl}w) = (\nabla_{\mathcal{P}_h} e_h, \boldsymbol{curl}w) \qquad \forall w \in \mathcal{H}. \tag{2.11}$$

Moreover,

$$|||e_h|||^2 = |||\phi_h|||^2 + (\boldsymbol{A}^{-1}\boldsymbol{curl}\psi_h, \boldsymbol{curl}\psi_h). \tag{2.12}$$

The importance of this theorem is that it allows us to write $|||e_h|||^2$ as the sum of a conforming part $|||\phi_h|||^2$ and a nonconforming part $(\boldsymbol{A}^{-1}\boldsymbol{curl}\psi_h, \boldsymbol{curl}\psi_h)$, which reduces the task of obtaining an estimator for $|||e_h|||$ to that of obtaining separate estimators for each of the two terms in this decomposition. Let the DG-norm over a region ω be denoted by

$$|||\cdot|||^2_{DG,\omega} = |||\cdot|||^2_\omega + \sum_{\substack{\gamma \in \mathcal{E}^I_h \cup \mathcal{E}^D_h \\ \gamma \subset \overline{\omega}}} \frac{\kappa}{|\gamma|} \left\| [\cdot]_\gamma \right\|^2_\gamma,$$

with $|||\cdot|||_{DG} = |||\cdot|||_{DG,\Omega}$.

2.7 Trace and Inverse Estimates

We shall make frequent use of the following trace and inverse inequalities

Theorem 2.7.1 *For an interval $K = [a, b]$ the following result holds :*

$$\forall u \in \mathbb{P}_1(K) \qquad |u(a)| \leqslant 2|a-b|^{-1/2}\|u\|_K. \tag{2.13}$$

Proof. It was proved in [83], Theorem 2. ■

Theorem 2.7.2 *For a planar triangle, K, the following result holds :*

$$\forall u \in \mathbb{P}_1(K) \qquad \|u\|_{\partial K} \leqslant 2\sqrt{3\varrho}\, h_K^{-1/2}\|u\|_K$$

and

$$\forall u \in \mathbb{P}_0(K) \qquad \|u\|_{\partial K} \leqslant 2\sqrt{\varrho}\, h_K^{-1/2}\|u\|_K.$$

Proof. The proof is given in [83], Theorem 3. ■

Theorem 2.7.3 *Let K is a planar triangle. For all $u \in \mathbb{P}_1(K)$, the following multivariate Markov inequality holds*

$$\|\nabla u\|_K \leqslant C_i h_K^{-1}\|u\|_K \tag{2.14}$$

with $C_i = 4\sqrt{6}\varrho$.

Proof. Proof is given in Chapter 4. ■

Using the above results for trace and inverse inequalities, we get the following estimates, for $u \in X_h$, and $\gamma \in \partial K_1 \cap \partial K_2$ with $\gamma \in \mathcal{E}_h^I$:

$$\begin{aligned} \|\langle \boldsymbol{n}\cdot \boldsymbol{A}\nabla u\rangle_\gamma\|_\gamma &= \frac{1}{2}\|\boldsymbol{n}_\gamma \cdot (\boldsymbol{A}\nabla u)_{|K_1} + \boldsymbol{n}_\gamma \cdot (\boldsymbol{A}\nabla u)_{|K_2}\|_\gamma \\ &\leqslant \frac{1}{2}\|\boldsymbol{n}_\gamma \cdot (\boldsymbol{A}\nabla u)_{|K_1}\|_\gamma + \frac{1}{2}\|\boldsymbol{n}_\gamma \cdot (\boldsymbol{A}\nabla u)_{|K_2}\|_\gamma \\ &\leqslant \sqrt{\varrho}\, h_{K_1}^{-1/2}\|\boldsymbol{A}\nabla u\|_{K_1} + \sqrt{\varrho}\, h_{K_2}^{-1/2}\|\boldsymbol{A}\nabla u\|_{K_2} \end{aligned} \tag{2.15}$$

and for $\gamma \in \mathcal{E}_h^D \cup \mathcal{E}_h^N$ and $\gamma \in \mathcal{E}_K$

$$\|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u\|_\gamma \leqslant 2\sqrt{\varrho}\, h_K^{-1/2}\|\boldsymbol{A}\nabla u\|_K \tag{2.16}$$

and for $\gamma \in \partial K_1 \cap \partial K_2$ and $\gamma \in \mathcal{E}_h^I$

$$\begin{aligned}
\|[u]_\gamma\|_\gamma &= \|u_{|K_1} - u_{|K_2}\|_\gamma \\
&\leqslant \|u_{|K_1}\|_\gamma + \|u_{|K_2}\|_\gamma \\
&\leqslant 2\sqrt{3\varrho}h_{K_1}^{-1/2}\|u\|_{K_1} + 2\sqrt{3\varrho}h_{K_2}^{-1/2}\|u\|_{K_2}, \qquad (2.17)
\end{aligned}$$

where h_{K_1} and h_{K_2} are the lengths of the longest edge of the elements K_1 and K_2, respectively.

By using (2.14), for all $u \in \mathbb{P}_1(K)$

$$\|\boldsymbol{A}\nabla u\|_K \leqslant \|\boldsymbol{A}\|_K \|\nabla u\|_K \leqslant 4\sqrt{6}\varrho\rho(\boldsymbol{A}_K)h_K^{-1}\|u\|_K = C_i\rho(\boldsymbol{A}_K)h_K^{-1}\|u\|_K. \qquad (2.18)$$

It is known that for symmetric and positive definite matrices

$$\|\boldsymbol{A}\|_K = \rho(\boldsymbol{A}_K), \qquad (2.19)$$

where $\rho(\cdot)$ denotes spectral radius of a matrix.

CHAPTER 3

FULLY COMPUTABLE CONVERGENCE ANALYSIS of DISCONTINOUS GALERKIN FINITE ELEMENT APPROXIMATION WITH an ARBITRARY NUMBER of LEVELS of HANGING NODES

We obtain a fully computable convergence analysis on the broken energy seminorm in first order symmetric interior penalty DG finite element approximations of a linear second order elliptic problem. Our mesh contain an arbitrary number of levels of hanging nodes and is comprised of triangular elements. We use an estimator which is completely free of unknown constants and provide a guaranteed numerical bound on the broken energy norm of the error. This residual-type a posteriori error estimator is introduced and analyzed for a DG formulation of a model second-order elliptic problem with Dirichlet-Neumann-type boundary conditions in [69]. An adaptive algorithm using this estimator together with specific marking and refinement strategies is constructed and shown to achieve any specified error level in the energy norm in a finite number of cycles. The convergence rate is linear with a guaranteed error reduction at every cycle.

3.1 The Convergence Result

Rankin's fully computable upper bound on broken energy seminorm of the error e_h in the first order DG finite element approximation [69] can be summarized as follows:

Let $\mathcal{V}$ index the set of the vertices of the elements in $\mathcal{P}_h$. For $K \in \mathcal{P}_h$, let $\mathcal{V}(K)$ index the set of the vertices of element K and, for $\gamma \in \partial\mathcal{P}_h$, let $\mathcal{V}(\gamma)$ index the vertices at the endpoints of

γ.

For any element $K \in \mathcal{P}_h$, let $\mathcal{T}_K$ be a sub-partitioning of K which is created by performing uniform refinements of K such that every vertex in $\mathcal{V}$ which lies on ∂K is located at a vertex of a triangle in $\mathcal{T}_K$. We shall use $\partial\mathcal{T}_K$ to denote the set of the edges of the triangles in $\mathcal{T}_K$.
Upper Bound for Conforming Part : Let Φ_K be conforming part error estimator defined as follow:

$$\Phi_K = (\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K^{1/2} + \mathfrak{C}_K \left\| f - \Pi_K^{(0)} f \right\|_K + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_h^N} \mathfrak{C}_\gamma \|g - \Pi_\gamma^{(0)} g\|_\gamma, \tag{3.1}$$

where $\boldsymbol{\sigma}_{\mathcal{T}_K}$ is fully computable and bounded by

$$\left(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K}\right)_K \leqslant \mathfrak{C}_\sigma \sum_{\gamma' \in \mathcal{E}_K} \sum_{\substack{\gamma \in \partial\mathcal{P}_h \\ \gamma \subset \gamma'}} |\gamma| \|R_{\gamma',K}\|_\gamma^2, \tag{3.2}$$

where

$$R_{\gamma',K} = -[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_h]_\gamma - \left(\frac{\kappa}{|\gamma|^2}, [u_h]_\gamma\right)_\gamma$$

on $\gamma \in \partial\mathcal{P}_h$ such that $\gamma \subset \gamma' \in \mathcal{E}_K$. The constants $\mathfrak{C}_K$, $\mathfrak{C}_\gamma$ and $\mathfrak{C}_\sigma$ are the fully computable constants defined as follows:

$$\mathfrak{C}_K = \frac{h_K}{\pi} \rho(\boldsymbol{A}_K^{-1})^{1/2}. \tag{3.3}$$

For $\gamma \in \mathcal{E}_K \cap \mathcal{E}_h^N$, let

$$\mathfrak{C}_\gamma = \left(\frac{|\gamma|}{|K|} \frac{h_K}{\pi} \Big(\frac{h_K}{\pi} + \max_{\substack{\gamma' \in \mathcal{E}_K \\ \gamma' \neq \gamma}} |\gamma'| \Big) \right)^{1/2} \rho(\boldsymbol{A}_K^{-1})^{1/2} \tag{3.4}$$

and

$$\mathfrak{C}_\sigma = \frac{3}{2} \varrho \rho(\boldsymbol{A}_K^{-1})$$

where h_K is the length of the longest edge of element K, ϱ is the shape regularity parameter defined in (2.2) and $\rho(\boldsymbol{A}_K^{-1})$ and $\rho(\boldsymbol{A}_K)$ are the spectral radius of the matrices $\boldsymbol{A}_K^{-1}$ and $\boldsymbol{A}_K$ respectively, defined in (2.19). Then, it was proved in [69] that

$$|||\phi_h|||^2 \leqslant \sum_{K \in \mathcal{P}_h} \Phi_K^2. \tag{3.5}$$

An explicit expression for $(A^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K$

An explicit formula was given in [69] for $(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K$ which can be used in the cases when there are no hanging nodes per edge of element K or at most one hanging node per edge of element K and at most two levels of refinement per edge of element K. Here, we give the result for the case when there are no hanging nodes per edge of element K.

Theorem 3.1.1 *Let $K \in \mathcal{P}_h$ be an element without any hanging nodes on its edges and suppose the vertices and edges of element $K \in \mathcal{P}_h$ are labeled as following,*

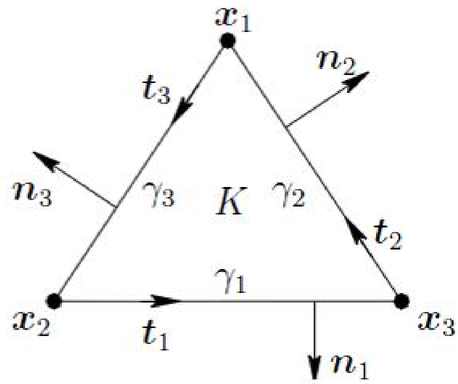


Figure 3.1: Direction and enumeration of the vertices, edges, unit tangent vectors and unit normal vectors of triangle K.

For $\vec{R} \in \mathbb{R}^3$ define discrete norm $\|\vec{R}\|_{\mathcal{R}_K}$ by the rule

$$\|\vec{R}\|^2_{\mathcal{R}_K} = \vec{R}\mathcal{R}_K\vec{R}^T$$

with $\mathcal{R}_K$ being a 3×3 matrix with entries

$$[\mathcal{R}_K]_{ij} = \frac{|\gamma_i|\,|\gamma_j|}{4|K|^2}\int_K (\boldsymbol{x} - \boldsymbol{x}_i)^T \boldsymbol{A}^{-1}(\boldsymbol{x} - \boldsymbol{x}_j)dx$$

Then

$$(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K = \|[R_{\gamma_1,K},\ R_{\gamma_2,K},\ R_{\gamma_3,K}]\|^2_{\mathcal{R}_K}.$$

Upper Bound for Nonconforming Part : For a point $x_m \subseteq \overline{\Omega}$, let Ω_m denote the set of elements in $\mathcal{P}_h$ whose closure contains the point x_m and $\partial\Omega_m$ denote the set of edges in $\partial\mathcal{P}_h$ which lie on the boundary of Ω_m. Also, $\#\Omega_m$ denotes the number of elements of $\mathcal{P}_h$ contained within the set Ω_m.

For $K \in \mathcal{P}_h$ and $\gamma \in \mathcal{E}_K \cap \mathcal{E}_h^D$, we define the space

$$H^1_\gamma(K) = \{v : v \in H^1(K), v = 0 \text{ on } \partial K \backslash \gamma\}.$$

Let $K \in \mathcal{P}_h$ be given. Let $\mathcal{V}(\mathcal{T}_K)$ index the set of position vectors x_m of the vertices of the triangles in $\mathcal{T}_K$ and let $\mathcal{V}_{\partial K}(\mathcal{T}_K)$ denote the restriction of this set to the vertices which lie on the boundary of element K. Similarly, $\mathcal{V}_D(\mathcal{T}_K)$ denotes the restriction of $\mathcal{V}_{\partial K}(\mathcal{T}_K)$ to the vertices which lie on the closure of the Dirichlet boundary Γ_D. We can define the function $q_{I,h}$ be such that $q_{I,h|\gamma} \in \mathbb{P}_1(\gamma)$ for all $\gamma \in \mathcal{E}_h^D$ and $q_{I,h}(x_m) = q(x_m)$ for all $m \in \mathcal{V}_D(\mathcal{T}_K)$.

Let $\mathcal{S}(u_h)$ be a continuous function on Ω satisfying $\mathcal{S}(u_h)|_{\mathcal{K}} \in \mathbb{P}_1(\mathcal{K})$ for all $\mathcal{K} \in \mathcal{T}_K$ for all $K \in \mathcal{P}_h$, such that

$$\mathcal{S}(u_h) = \begin{cases} q_{I,h}(x_m) & \text{if } m \in \mathcal{V}_D(\mathcal{T}_K) \\ u_{h|K}(x_m) & \text{if } m \in \mathcal{V}(\mathcal{T}_K) \setminus \mathcal{V}_{\partial K}(\mathcal{T}_K) \\ \frac{1}{\#\Omega_m} \sum_{K' \in \Omega_m} u_{h|K'}(x_m) & \text{if } m \in \mathcal{V} \cap \mathcal{V}_{\partial K}(\mathcal{T}_K) \setminus \mathcal{V}_D(\mathcal{T}_K). \end{cases}$$

The estimator Ψ_K of the nonconforming part of the error is defined as follows,

$$\Psi_K = |||u_h - \mathcal{S}(u_h)|||_K + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_h^D} \inf_{\substack{v \in H^1_\gamma(K): \\ v_{|\gamma} = q - q_{I,h}}} |||v|||_K$$

Then, it was proved in [69]

$$(\boldsymbol{A}^{-1}\boldsymbol{curl}\psi_h, \boldsymbol{curl}\psi_h) \leqslant \sum_{K \in \mathcal{P}_h} \Psi_K^2 \tag{3.6}$$

Then by (2.12), (3.5)and (3.6), the broken energy seminorm of the total error $e_h = u - u_h$ can be bounded as

$$|||e_h|||^2 \leqslant \sum_{K \in \mathcal{P}_h} \left(\Phi_K^2 + \Psi_K^2\right).$$

3.1.1 The Adaptive Algorithm

An adaptive finite element method for the DG consists of the successive loops of the following sequence:

1. **Solve:** Given a mesh $\mathcal{P}_H$, a DG approximation u_H is constructed by solving the equation

$$B_H(u_H, v) = L(v) \ \ \forall v \in X_H.$$

2. **Estimate:** A posteriori error estimation of the error e_H is obtained by calculating $\left(\Phi_K^2 + \Psi_K^2\right)$ for all $K \in \mathcal{P}_H$.

3. **Mark:** Based on the information supplied by the a posteriori error estimate certain triangles of $\mathcal{P}_H$ are marked for refinement.

4. **Refine:** The triangles marked for refinement in step 3 are refined in a specific way. This refinement strategy defines the new mesh $\mathcal{P}_h$.

By using the above adaptive strategy, our main goal is to show following inequality

$$B_h(e_h, e_h) \leqslant \eta B_H(e_H, e_H) \tag{3.7}$$

where $0 < \eta < 1$ is a fully computable constant and u_H and u_h are the DG solutions in X_H and X_h, respectively.

The following orthogonality relation is essential in the proof of (3.7).

Lemma 3.1.2 *Let $\mathcal{P}_h$ be a local refinement of $\mathcal{P}_H$, such that $X_H \subset X_h$ with the DG solutions u_H and u_h, then the following relation holds for symmetric DG,*

$$B_h(e_H, e_H) = B_h(e_h, e_h) + B_h(u_h - u_H, u_h - u_H). \tag{3.8}$$

Proof. By Galerkin orthogonality $B_h(e_h, v) = 0$ for all $v \in X_h$. Hence $u_h - u_H \in X_h$ is perpendicular to $u - u_h$. Therefore

$$B_h(u - u_h, u_h - u_H) = B_h(e_h, u_h - u_H) = 0.$$

Since, bilinear form is symmetric, one can rewrite as

$$B_h(u_h - u_H, e_h) = B_h(e_h, u_h - u_H) = 0.$$

The decomposition $u - u_H = (u - u_h) + (u_h - u_H)$ yields

$$\begin{aligned} B_h(e_H, e_H) &= B_h(e_h + (u_h - u_H), e_h + (u_h - u_H)) \\ &= B_h(e_h, e_h) + \underbrace{B_h(e_h, u_h - u_H)}_{=0} + \underbrace{B_h(u_h - u_H, e_h)}_{=0} + B_h(u_h - u_H, u_h - u_H) \\ &= B_h(e_h, e_h) + B_h(u_h - u_H, u_h - u_H). \end{aligned}$$

■

Before engaging in the proof of the Theorem 3.7, we immediately notice a difficulty presented by the fact that we have $B_h(e_H, e_H)$ on the left-hand side of the last equality instead of $B_H(e_H, e_H)$. However we can tackle this difficulty by using the fact that $B_h(e_H, e_H)$ is bounded by $B_H(e_H, e_H)$, following the argument in [59],

Lemma 3.1.3 (Karakashian, Pascal, 2007) *Suppose $\mathcal{P}_h$ be a local refinement of $\mathcal{P}_H$. Then*

$$\begin{aligned} B_h(e_H, e_H) &\leqslant B_H(e_H, e_H) + \kappa \sum_{\gamma \in \mathcal{E}_H^I} (\delta(\gamma) - 1)|\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\ &+ \kappa \sum_{\gamma \in \mathcal{E}_H^D} (\delta(\gamma) - 1)|\gamma|^{-1} \|q - u_H\|_\gamma^2 \end{aligned} \tag{3.9}$$

where $\delta(\gamma) = \max\{\frac{|\gamma|}{|\gamma'|} \mid \gamma' \in \mathcal{E}_h \cup \mathcal{E}_h^D, \gamma' \in \gamma\}$.

Proof. By (2.6) we have

$$B_h(e_H, e_H) = \sum_{K' \in \mathcal{P}_h} \|\!|e_H|\!\|_{K'}^2 - (1+\tau) \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_{\gamma'}, [e_H]_{\gamma'}\right)_{\gamma'} + \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \frac{\kappa}{|\gamma'|} \|[e_H]_{\gamma'}\|_{\gamma'}^2$$

Since $u \in H^1(\Omega)$ and u_H is a polynomial on each $K \in \mathcal{P}_H$, we have

$$\sum_{K' \in \mathcal{P}_h} \|\!|e_H|\!\|_{K'}^2 = \sum_{K \in \mathcal{P}_H} \|\!|e_H|\!\|_K^2.$$

If $\gamma' \in \mathcal{E}_h^I$ is a completely new edge in other words if $\gamma' \in \mathcal{E}_h^I \cap \mathring{K}$ where $\mathring{K}$ refers the interior edge of some $K \in \mathcal{P}_H$, then $[e_H]_{\gamma'} = 0$. Also, the edges $\gamma' \in \mathcal{E}_h^D$ are parts of edges in $\mathcal{E}_H^D$, then

$$\sum_{\gamma' \in \mathcal{E}_h^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_{\gamma'}, [e_H]_{\gamma'}\right)_{\gamma'} = \sum_{\gamma' \in \mathcal{E}_h^D} (\boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H, e_H)_{\gamma'} = \sum_{\gamma \in \mathcal{E}_H^D} (\boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H, e_H)_\gamma.$$

So,one obtains

$$\sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_{\gamma'}, [e_H]_{\gamma'}\right)_{\gamma'} = \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma\right)_\gamma.$$

For the term $\sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \frac{\kappa}{|\gamma'|} \|[e_H]_{\gamma'}\|_{\gamma'}^2$, we can define $\delta(\gamma) = \max\{\frac{|\gamma|}{|\gamma'|} \mid \gamma' \in \mathcal{E}_h \cup \mathcal{E}_h^D, \gamma' \in \gamma\}$ which is a finite number. Then, it can be written that

$$\sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} \frac{\kappa}{|\gamma'|} \|[e_H]_{\gamma'}\|_{\gamma'}^2 \leqslant \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \frac{\delta(\gamma)\,\kappa}{|\gamma|} \|[e_H]_\gamma\|_\gamma^2.$$

We conclude,

$$\begin{aligned} B_h(e_H, e_H) &\leqslant \sum_{K \in \mathcal{P}_H} \|\!|e_H|\!\|_K^2 - (1+\tau) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma\right)_\gamma + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \frac{\delta(\gamma)\,\kappa}{|\gamma|} \|[e_H]_\gamma\|_\gamma^2 \\ &= B_H(e_H, e_H) + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} (\delta(\gamma) - 1)\kappa|\gamma|^{-1} \|[e_H]_\gamma\|_\gamma^2. \end{aligned}$$

Since $\|[e_H]_\gamma\|_\gamma = \|[u_H]_\gamma\|_\gamma$ for all $\gamma \in \mathcal{E}_H^I$ and $\|[e_H]_\gamma\|_\gamma = \|q - u_H\|_\gamma$ for all $\gamma \in \mathcal{E}_H^D$, we get the desired result. ■

To prove(3.7), we use the following key identity

Lemma 3.1.4 *Let u_H and u_h denote the DG solutions in X_H and X_h. Then,*

$$\begin{aligned}
&\sum_{K\in\mathcal{P}_h}(f+div(\boldsymbol{A}\nabla u_H),v)_K-\sum_{\gamma\in\mathcal{E}_h^I}\left([\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma,\ \langle v\rangle_\gamma\right)+\sum_{\gamma\in\mathcal{E}_h^N}(g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H,v)_\gamma\\
=\;& B_h(u_h-u_H,v)-\tau\sum_{\gamma\in\mathcal{E}_h^I}\left([u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma+\tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma\\
=\;& \sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla(u_h-u_H),\nabla v)_K-\tau\sum_{\gamma\in\mathcal{E}_h^I}\left([u_h]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma+\tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_h,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \qquad (3.10)
\end{aligned}$$

for all $v\in X_h\cap H_0^1(\Omega)$

Proof.: For the proof, apply the integration by parts to

$$\sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla e_H,\nabla v)_K.$$

Then,

$$\begin{aligned}
\sum_{K\in\mathcal{P}_h}\int_K\boldsymbol{A}\nabla e_H\cdot\nabla v &= -\sum_{K\in\mathcal{P}_h}\int_K\mathrm{div}(\boldsymbol{A}\nabla e_H)v+\sum_{K\in\mathcal{P}_h}\int_{\partial K}(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)\,v\\
&= -\sum_{K\in\mathcal{P}_h}\int_K\mathrm{div}(\boldsymbol{A}\nabla(u-u_H))v+\sum_{K\in\mathcal{P}_h}\int_{\partial K}(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)\,v\\
&= \sum_{K\in\mathcal{P}_h}\int_K(f+\mathrm{div}(\boldsymbol{A}\nabla u_H))v+\sum_{K\in\mathcal{P}_h}\int_{\partial K}(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)\,v,\qquad\forall v\in X_h.
\end{aligned}$$

We observe that the boundary integrals are defined on each element boundary as:

$$\begin{aligned}
\sum_{K\in\mathcal{P}_h}\int_{\partial K}(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)\,v &= \sum_{\gamma\in\mathcal{E}_h^D}\int_\gamma\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H v+\sum_{\gamma\in\mathcal{E}_h^N}\int_\gamma\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H v\\
&+\sum_{\gamma\in\mathcal{E}_h^I}\int_\gamma(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)_{|K}v_{|K}+(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)_{|K'}v_{|K'}.
\end{aligned}$$

Moreover, the treatment of the interior boundary integrals is as follows: Given an edge $\gamma\in\mathcal{E}_h^I$ shared by two adjacent elements K and K', for a fixed unit normal vector $\boldsymbol{n}_\gamma$ for each edge γ, it can be written that

$$(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)_{|K}v_{|K}+(\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)_{|K'}v_{|K'}=\boldsymbol{n}_\gamma\cdot(\boldsymbol{A}\nabla e_H)_{|K}v_{|K}-\boldsymbol{n}_\gamma\cdot(\boldsymbol{A}\nabla e_H)_{|K'}v_{|K'}.$$

By analogy with the formula below:

$$ac-bd=\frac{1}{2}(a+b)(c-d)+\frac{1}{2}(a-b)(c+d),$$

we can write the integrand as

$$\begin{aligned}
\boldsymbol{n}\cdot(\boldsymbol{A}\nabla e_H)_{|K}v_{|K}-\boldsymbol{n}\cdot(\boldsymbol{A}\nabla e_H)_{|K'}v_{|K'} &= \frac{1}{2}\left(\boldsymbol{n}\cdot(\boldsymbol{A}\nabla e_H)_{|K}+\boldsymbol{n}\cdot(\boldsymbol{A}\nabla e_H)_{|K'}\right)(v_{|K}-v_{|K'})\\
&+\frac{1}{2}\left(\boldsymbol{n}\cdot(\boldsymbol{A}\nabla e_H)_{|K}-\boldsymbol{n}\cdot(\boldsymbol{A}\nabla e_H)_{|K'}\right)(v_{|K}+v_{|K'})\\
&= \langle\boldsymbol{n}_\gamma\cdot\boldsymbol{A}\nabla e_H\rangle_\gamma[v]_\gamma+[\boldsymbol{n}_\gamma\cdot\boldsymbol{A}\nabla e_H]_\gamma\langle v\rangle_\gamma.
\end{aligned}$$

Note that

$$\sum_{\gamma\in\mathcal{E}_h^N}\int_\gamma \boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H v = \sum_{\gamma\in\mathcal{E}_h^N}\int_\gamma (g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H)v,$$

so we get

$$\begin{aligned}\sum_{K\in\mathcal{P}_h}\int_K \boldsymbol{A}\nabla e_H\cdot\nabla v &= \sum_{K\in\mathcal{P}_h}\int_K (f+\operatorname{div}(\boldsymbol{A}\nabla u_H))v + \sum_{\gamma\in\mathcal{E}_h^I}\int_\gamma \langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H\rangle_\gamma [v]_\gamma\\ &+ \sum_{\gamma\in\mathcal{E}_h^I}\int_\gamma [\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H]_\gamma\langle v\rangle_\gamma + \sum_{\gamma\in\mathcal{E}_h^D}\int_\gamma \boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H v\\ &+ \sum_{\gamma\in\mathcal{E}_h^N}\int_\gamma (g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H)v. \end{aligned}\tag{3.11}$$

From the definition of $B_h(.,.)$, we have

$$\begin{aligned}B_h(e_H,v) &= \sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla e_H,\nabla v)_K - \sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left(\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H\rangle_\gamma,[v]_\gamma\right)_\gamma\\ &-\tau\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left([e_H]_\gamma,\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma\\ &+\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left(\frac{\kappa}{|\gamma|}[e_H]_\gamma,[v]_\gamma\right)_\gamma \qquad \forall v\in X_h.\end{aligned}\tag{3.12}$$

Inserting (3.11) into (3.12) yields

$$\begin{aligned}B_h(e_H,v) &= \sum_{K\in\mathcal{P}_h}\int_K (f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\,v + \sum_{\gamma\in\mathcal{E}_h^I}\int_\gamma \langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H\rangle_\gamma\,[v]_\gamma\\ &+\sum_{\gamma\in\mathcal{E}_h^I}\int_\gamma [\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H]_\gamma\,\langle v\rangle_\gamma + \sum_{\gamma\in\mathcal{E}_h^D}\int_\gamma (\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H)\ v\\ &+\sum_{\gamma\in\mathcal{E}_h^N}\int_\gamma (g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H)\ v - \sum_{\gamma\in\mathcal{E}_I^h\cup\mathcal{E}_h^D}\left(\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H\rangle_\gamma,[v]_\gamma\right)_\gamma\\ &-\tau\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left([e_H]_\gamma,\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma + \sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left(\frac{\kappa}{|\gamma|}[e_H]_\gamma,[v]_\gamma\right)_\gamma.\end{aligned}\tag{3.13}$$

In the right hand side of (3.13), second, fourth and sixth terms cancel each other. Then, one obtains

$$\begin{aligned}B_h(e_H,v) &= \sum_{K\in\mathcal{P}_h}(f+\operatorname{div}(\boldsymbol{A}\nabla u_H),v)_K + \sum_{\gamma\in\mathcal{E}_h^I}\left([\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H]_\gamma,\ \langle v\rangle_\gamma\right)_\gamma\\ &+\sum_{\gamma\in\mathcal{E}_h^N}(g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H,v)_\gamma - \tau\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left([e_H]_\gamma,\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma\\ &+\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left(\frac{\kappa}{|\gamma|}[e_H]_\gamma,[v]_\gamma\right)_\gamma.\end{aligned}$$

Rewriting the last equality and letting

$$[\boldsymbol{n}\cdot\boldsymbol{A}\nabla e_H]_\gamma = [\boldsymbol{n}\cdot\boldsymbol{A}\nabla(u-u_H)]_\gamma = -[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma,$$

gives us following equality

$$\begin{aligned}
&\sum_{K\in\mathcal{P}_h}(f+\operatorname{div}(\boldsymbol{A}\nabla u_H),v)_K - \sum_{\gamma\in\mathcal{E}_h^I}\left([\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma,\,\langle v\rangle_\gamma\right) + \sum_{\gamma\in\mathcal{E}_h^N}(g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H,v)_\gamma\\
= \;& B_h(e_H,v)+\tau\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left([e_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma - \sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}\left(\frac{\kappa}{|\gamma|}[e_H]_\gamma,[v]_\gamma\right)_\gamma.
\end{aligned}$$

If we write $\mathcal{E}_h^I\cup\mathcal{E}_h^D$ separately and observe that $[e_H]_\gamma = e_H$, $\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma = \boldsymbol{n}\cdot\boldsymbol{A}\nabla v$, $[v]=v$ on Dirichlet boundary, we obtain:

$$\begin{aligned}
&\sum_{K\in\mathcal{P}_h}(f+\operatorname{div}(\boldsymbol{A}\nabla u_H),v)_K - \sum_{\gamma\in\mathcal{E}_h^I}\left([\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma,\,\langle v\rangle_\gamma\right) + \sum_{\gamma\in\mathcal{E}_h^N}(g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H,v)_\gamma\\
= \;& B_h(e_H,v)+\tau\sum_{\gamma\in\mathcal{E}_h^I}\left([e_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma - \sum_{\gamma\in\mathcal{E}_h^I}\left(\frac{\kappa}{|\gamma|}[e_H]_\gamma,[v]_\gamma\right)_\gamma\\
+ \;& \tau\sum_{\gamma\in\mathcal{E}_h^D}(e_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma - \sum_{\gamma\in\mathcal{E}_h^D}\left(\frac{\kappa}{|\gamma|}e_H,v\right)_\gamma \qquad \forall v\in X_h.
\end{aligned}$$

Now, in the convergence analysis, the terms containing κ make trouble. So it is important to eliminate these term. If we use test function from the subspaces $X_h\cap H_0^1(\Omega)$, then we can tackle with this trouble. Then, the last inequality becomes

$$\begin{aligned}
&\sum_{K\in\mathcal{P}_h}(f+\operatorname{div}(\boldsymbol{A}\nabla u_H),v)_K - \sum_{\gamma\in\mathcal{E}_h^I}\left([\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma,\,\langle v\rangle_\gamma\right) + \sum_{\gamma\in\mathcal{E}_h^N}(g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H,v)_\gamma\\
= \;& B_h(e_H,v)+\tau\sum_{\gamma\in\mathcal{E}_h^I}\left([e_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma + \tau\sum_{\gamma\in\mathcal{E}_h^D}(e_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \qquad \forall v\in X_h\cap H_0^1(\Omega).
\end{aligned}$$

Also we can write $B_h(e_H,v) = B_h(u_h-u_H,v)+B_h(u-u_h,v)$ and note that $B_h(u-u_h,v) = 0\ \ \forall v\in X_h$.

Note that $[e_H]_\gamma = -[u_H]_\gamma$ on $\mathcal{E}_h^I$ and $e_H = q-u_H$ on $\mathcal{E}_h^D$. This yields

$$\begin{aligned}
&\sum_{K\in\mathcal{P}_h}(f+\operatorname{div}(\boldsymbol{A}\nabla u_H),v)_K - \sum_{\gamma\in\mathcal{E}_h^I}\left([\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma,\,\langle v\rangle_\gamma\right) + \sum_{\gamma\in\mathcal{E}_h^N}(g-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H,v)_\gamma\\
= \;& B_h(u_h-u_H,v)-\tau\sum_{\gamma\in\mathcal{E}_h^I}\left([u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\right)_\gamma + \tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \qquad (3.14)
\end{aligned}$$

for all $v\in X_h\cap H_0^1(\Omega)$.

On the other hand from (2.6), one can conclude that

$$\begin{aligned} B_h(u_h-u_H,v) &= \sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla(u_h-u_H),\nabla v)_K-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_h-u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma \\ &- \tau\sum_{\gamma\in\mathcal{E}_h^D}(u_h-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma-\sum_{\gamma\in\mathcal{E}_h^I}(\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla(u_h-u_H)\rangle_\gamma,[v]_\gamma)_\gamma \\ &+ \sum_{\gamma\in\mathcal{E}_h^I}\frac{\kappa}{|\gamma|}([u_h-u_H]_\gamma,[v]_\gamma)_\gamma-\sum_{\gamma\in\mathcal{E}_h^D}(\boldsymbol{n}\cdot\boldsymbol{A}\nabla(u_h-u_H),v)_\gamma \\ &+ \sum_{\gamma\in\mathcal{E}_h^D}\frac{\kappa}{|\gamma|}(u_h-u_H,v)_\gamma \qquad \forall v\in X_h. \end{aligned} \tag{3.15}$$

Moreover, since our test function space is $X_h\cap H_0^1(\Omega)$, (3.15) becomes

$$\begin{aligned} B_h(u_h-u_H,v) &= \sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla(u_h-u_H),\nabla v)_K-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_h-u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma \\ &- \tau\sum_{\gamma\in\mathcal{E}_h^D}(u_h-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \qquad \forall v\in X_h\cap H_0^1(\Omega). \end{aligned}$$

This implies

$$\begin{aligned} &B_h(u_h-u_H,v)-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma+\tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \\ =& \sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla(u_h-u_H),\nabla v)_K-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_h-u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma \\ -& \tau\sum_{\gamma\in\mathcal{E}_h^D}(u_h-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma \\ +& \tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \qquad \forall v\in X_h\cap H_0^1(\Omega). \end{aligned}$$

In the above identity, in the right-hand side of the equality the terms u_H vanish and the rest of the terms give

$$\begin{aligned} &B_h(u_h-u_H,v)-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_H]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma+\tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_H,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \\ =& \sum_{K\in\mathcal{P}_h}(\boldsymbol{A}\nabla(u_h-u_H),\nabla v)_K-\tau\sum_{\gamma\in\mathcal{E}_h^I}\big([u_h]_\gamma,\langle\boldsymbol{n}\cdot\boldsymbol{A}\nabla v\rangle_\gamma\big)_\gamma+\tau\sum_{\gamma\in\mathcal{E}_h^D}(q-u_h,\boldsymbol{n}\cdot\boldsymbol{A}\nabla v)_\gamma \end{aligned} \tag{3.16}$$

(3.14)and (3.16) implies the key identity (3.10). ■

MARKING STRATEGY:

For some number $\theta\in(0,1)$, let $\mathcal{M}_H$ be any subset of $\mathcal{P}_H$ such that the following bulk criterion is satisfied:

$$\sum_{K\in\mathcal{M}_H}\left(\Phi_K^2+\Psi_K^2\right)\geqslant\theta\sum_{K\in\mathcal{P}_H}\left(\Phi_K^2+\Psi_K^2\right). \tag{3.17}$$

REFINEMENT STRATEGY:

A marked triangle $K \in \mathcal{P}_H$ will be cut into sixteen congruent sub-triangles.

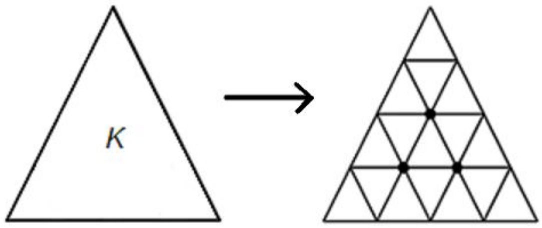


Figure 3.2: The refinement of triangle K into sixteen congruent subtriangles.

Estimation of the Conforming Part:

Theorem 3.1.5 *Let $u \in H^1(\Omega)$ such that $u = q$ on Γ_D be the solution of (2.1). Moreover, $u_H \in X_H$, $u_h \in X_h$ denote the solution of (2.7) with respect to $\mathcal{P}_H$ and $\mathcal{P}_h$, respectively. Then, sufficiently large κ, the following inequality holds*

$$\begin{aligned}
&\sum_{K\in\mathcal{M}_H} \Phi_K^2 \leqslant 288\lambda_{-1}\,\varrho^2\left(2^7 10^2 C_i^2 + 2^2 3^2 C_i^2\right) \sum_{K'\in\mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
+\quad & 288\lambda_{-1}\,\varrho^2\left(2^7 10^2 \varrho C_i^2\lambda^2 + 2^2 3^2 \varrho C_i^2 \lambda^2\right) \sum_{\gamma'\in\mathcal{E}_h^I} |\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
+\quad & 288\lambda_{-1}\,\varrho^2\left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2 \varrho C_i^2 \lambda^2\right) \sum_{\gamma'\in\mathcal{E}_h^D} |\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2 \\
+\quad & 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^9 10^2 C_i^2 + 368\varrho\left(2^7 10^2 C_i^2 + 36 C_i^2\right)\right) \sum_{K'\in\mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
+\quad & 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^{11}10^2\varrho C_i^2\lambda^2 + 368\varrho\left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2 \varrho C_i^2\lambda^2\right)\right) \sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} |\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
+\quad & 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^2 3^2\varrho + 2^6 10^2 \varrho + 368\varrho(3^2\varrho + 2^4 10^2\varrho)\right) \sum_{K\in\mathcal{P}_H} osc^2(f,K,1) \\
+\quad & 64(1+\varrho)\lambda_{-1}9\varrho\delta_{\max}(240\varrho+1) \sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0) \\
+\quad & 288\lambda_{-1}\,\varrho^2\,(3^2\varrho + 2^4 10^2\varrho) \sum_{K\in\mathcal{P}_H} osc^2(f,K,1) + 2160\lambda_{-1}\varrho^3 \sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0) \\
+\quad & \frac{6\varrho}{\pi^2}\lambda_{-1} \sum_{K\in\mathcal{P}_H} osc^2(f,K,0) + \frac{24\varrho}{\pi}\lambda_{-1} \sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0) \qquad (3.18)
\end{aligned}$$

where $\gamma \in \mathcal{E}_K$ define $\delta_{max} = \max\{\delta(\gamma) | \gamma \in \mathcal{E}_K, K \in \mathcal{P}_H\}$ where $\delta(\gamma) = \max\{\frac{|\gamma|}{|\gamma'|} \mid \gamma' \in$

$\mathcal{E}_h^I \cup \mathcal{E}_h^D, \gamma' \in \gamma\}$ *which is a finite number. Also, we can define* $\lambda_{-1} = \max\{\rho(\boldsymbol{A}_K^{-1}) | K \in \mathcal{P}_{H,0}\}$ *and* $\lambda = \max\{\rho(\boldsymbol{A}_K) | K \in \mathcal{P}_{H,0}\}$ *where* $\mathcal{P}_{H,0}$ *is initial mesh.*

The proof of (3.18) follows from the subsequent series of following lemmas.

Lemma 3.1.6 *Fix* $K \in \mathcal{M}_H$. *For* $\gamma' \in \mathcal{E}_K$, *define* $\delta(\gamma') = \max\{\frac{|\gamma'|}{|\gamma|} \mid \gamma \in \partial\mathcal{P}_H, \gamma \subset \gamma'\}$ *which is a finite number. Then the following inequality holds*

$$\begin{aligned}\Phi_K^2 &\leqslant 9\rho(\boldsymbol{A}_K^{-1})\,\varrho \sum_{\gamma' \in \mathcal{E}_K} \left(|\gamma'| \|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_{\gamma'}\|_{\gamma'}^2 + \delta(\gamma')\kappa^2|\gamma'|^{-1}\|[u_H]_{\gamma'}\|_{\gamma'}^2\right) \\ &+ \frac{6\varrho}{\pi^2}\rho(\boldsymbol{A}_K^{-1}) osc^2(f, K, 0) + \frac{24\varrho}{\pi}\rho(\boldsymbol{A}_K^{-1}) \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_H^N} osc^2(g, \gamma, 0) \end{aligned} \tag{3.19}$$

where ϱ *is defined by (2.2) and* $\rho(\boldsymbol{A}_K)$, $\rho(\boldsymbol{A}_K^{-1})$ *denote the spectral radius of the matrices* $\boldsymbol{A}_K$ *and* $\boldsymbol{A}_K^{-1}$ *respectively, defined in (2.19).*

Proof. We recall (3.2):

$$(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K \leqslant \mathfrak{C}_\sigma \sum_{\gamma' \in \mathcal{E}_K} \sum_{\substack{\gamma \in \partial\mathcal{P}_H \\ \gamma \subset \gamma'}} |\gamma| \|R_{\gamma',K}\|_\gamma^2,$$

where $R_{\gamma',K} = -[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma - \left(\frac{\kappa}{|\gamma|^2}, [u_H]_\gamma\right)_\gamma$ and $\mathfrak{C}_\sigma = \frac{3}{2}\varrho\rho(\boldsymbol{A}_K^{-1})$. Inserting those definitions in the above inequality gives

$$(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K \leqslant \frac{3}{2}\varrho\rho(\boldsymbol{A}_K^{-1}) \sum_{\gamma' \in \mathcal{E}_K} \sum_{\substack{\gamma \in \partial\mathcal{P}_H \\ \gamma \subset \gamma'}} |\gamma| \| -[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma - \left(\frac{\kappa}{|\gamma|^2}, [u_H]_\gamma\right)_\gamma \|_\gamma^2.$$

By triangle inequality, we get

$$(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K}, \boldsymbol{\sigma}_{\mathcal{T}_K})_K \leqslant 3\varrho\rho(\boldsymbol{A}_K^{-1}) \sum_{\gamma' \in \mathcal{E}_K} \sum_{\substack{\gamma \in \partial\mathcal{P}_H \\ \gamma \subset \gamma'}} |\gamma| \left(\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 + \| \left(\frac{\kappa}{|\gamma|^2}, [u_H]_\gamma\right)_\gamma \|_\gamma^2 \right) \tag{3.20}$$

and also

$$\| \left(\frac{\kappa}{|\gamma|^2}, [u_H]_\gamma\right)_\gamma \|_\gamma^2 = \int_\gamma \left(\int_\gamma \frac{\kappa}{|\gamma|^2}[u_H]_\gamma\right)^2 = \frac{\kappa^2}{|\gamma|^4}\int_\gamma \left(\int_\gamma [u_H]_\gamma\right)^2.$$

Cauchy-Schwarz's inequality implies that

$$\left(\int_\gamma [u_H]_\gamma\right)^2 \leqslant \int_\gamma 1^2 \int_\gamma [u_H]_\gamma^2 = |\gamma| \int_\gamma [u_H]_\gamma^2.$$

Thus

$$\| \left(\frac{\kappa}{|\gamma|^2}, [u_H]_\gamma\right)_\gamma \|_\gamma^2 \leqslant \frac{\kappa^2}{|\gamma|^4}|\gamma| \int_\gamma \left(\int_\gamma [u_H]_\gamma^2\right) = \frac{\kappa^2}{|\gamma|^2}\int_\gamma [u_H]_\gamma^2 = \frac{\kappa^2}{|\gamma|^2}\|[u_H]_\gamma\|_\gamma^2.$$

Moreover, fineness parameter $\delta(\gamma')$ allows us to write the following inequality,

$$3\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}\sum_{\substack{\gamma\in\partial\mathcal{P}_H\\ \gamma\subset\gamma'}}|\gamma|\frac{\kappa^2}{|\gamma|^2}\|[u_H]_\gamma\|_\gamma^2 \leqslant 3\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}\delta(\gamma')\kappa^2|\gamma'|^{-1}\|[u_H]_{\gamma'}\|_{\gamma'}^2. \quad (3.21)$$

Now, since $|\gamma| \leqslant |\gamma'|$, we have:

$$\begin{aligned} 3\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}\sum_{\substack{\gamma\in\partial\mathcal{P}_H\\ \gamma\subset\gamma'}}|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant 3\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}|\gamma'|\sum_{\substack{\gamma\in\partial\mathcal{P}_H\\ \gamma\subset\gamma'}}\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 \\ &= 3\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}|\gamma'|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_{\gamma'}\|_{\gamma'}^2. \quad (3.22) \end{aligned}$$

Using the inequalities (3.21) and (3.22) into (3.20) gives

$$(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_{\mathcal{T}_K},\boldsymbol{\sigma}_{\mathcal{T}_K})_K \leqslant 3\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}\Big(|\gamma'|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_{\gamma'}\|_{\gamma'}^2+\delta(\gamma')\kappa^2|\gamma'|^{-1}\|[u_H]_{\gamma'}\|_{\gamma'}^2\Big) \quad (3.23)$$

Definition (3.1) and inequality (3.23) imply

$$\begin{aligned} \Phi_K^2 &\leqslant 9\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}\Big(|\gamma'|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_{\gamma'}\|_{\gamma'}^2+\delta(\gamma')\kappa^2|\gamma'|^{-1}\|[u_H]_{\gamma'}\|_{\gamma'}^2\Big) \\ &+ 3\mathfrak{C}_K^2\|f-\Pi_K^{(0)}f\|_K^2+\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^N}6\mathfrak{C}_\gamma^2\|g-\Pi_\gamma^{(0)}g\|_\gamma^2. \quad (3.24) \end{aligned}$$

Now note that by Lemma A.0.2, we have $\frac{h_K^2}{|K|} \leqslant 2\varrho$ and using the definitions (2.3) and (3.3), then

$$\begin{aligned} 3\mathfrak{C}_K^2\|f-\Pi_K^{(0)}f\|_K^2 &= 3\frac{h_K^2}{\pi^2}\rho(\boldsymbol{A}_K^{-1})\|f-\Pi_K^{(0)}f\|_K^2 \\ &\leqslant \frac{6\varrho}{\pi^2}\rho(\boldsymbol{A}_K^{-1})|K|\|f-\Pi_K^{(0)}f\|_K^2 \\ &= \frac{6\varrho}{\pi^2}\rho(\boldsymbol{A}_K^{-1})osc^2(f,K,0). \quad (3.25) \end{aligned}$$

Similarly Lemma A.0.2, definitions (2.4) and (3.4) imply,

$$\begin{aligned} 6\mathfrak{C}_\gamma^2\|g-\Pi_\gamma^{(0)}g\|_\gamma^2 &= 6\frac{|\gamma|}{|K|}\frac{h_K}{\pi}\Big(\frac{h_K}{\pi}+\max_{\substack{\gamma'\in\mathcal{E}_K\\ \gamma'\neq\gamma}}|\gamma'|\Big)\rho(\boldsymbol{A}_K^{-1})\|g-\Pi_\gamma^{(0)}g\|_\gamma^2 \\ &= 6\Big(\frac{h_K^2}{|K|\pi^2}+\frac{h_K}{|K|\pi}\max_{\substack{\gamma'\in\mathcal{E}_K\\ \gamma'\neq\gamma}}|\gamma'|\Big)\rho(\boldsymbol{A}_K^{-1})|\gamma|\|g-\Pi_\gamma^{(0)}g\|_\gamma^2 \\ &\leqslant 6\Big(\frac{2\varrho}{\pi^2}+\frac{2\varrho}{\pi}\Big)\rho(\boldsymbol{A}_K^{-1})osc^2(g,\gamma,0) \\ &\leqslant \frac{24\varrho}{\pi}\rho(\boldsymbol{A}_K^{-1})osc^2(g,\gamma,0) \quad (3.26) \end{aligned}$$

Using (3.25) and (3.26) into (3.24), we get

$$\begin{aligned}\Phi_K^2 &\leqslant 9\varrho\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma'\in\mathcal{E}_K}\left(|\gamma'|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_{\gamma'}\|_{\gamma'}^2+\delta(\gamma')\kappa^2|\gamma'|^{-1}\|[u_H]_{\gamma'}\|_{\gamma'}^2\right)\\ &+ \frac{6\varrho}{\pi^2}\rho(\boldsymbol{A}_K^{-1})osc^2(f,K,0)+\frac{24\varrho}{\pi}\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^N}osc^2(g,\gamma,0).\end{aligned}$$

■

Remark 3.1.7 *An upper bound for $\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2$ is needed. There are three cases:*

1. *If γ is an interior edge of $\mathcal{P}_H$, then we find an upper bound for $\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2$.*
2. *If $\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^N$, then we find an upper bound for $\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H-\Pi_\gamma^{(0)}g\|_\gamma^2$.*
3. *If $\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^D$, then $[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma=0$. This implies we do not have any contribution from this edge.*

The next two lemmas deal with the first two cases.

Lemma 3.1.8 *Fix $K\in\mathcal{M}_H$ and assume K has been refined using our refinement strategy. Define $\mathcal{P}_{h,K}=\{K'\in\mathcal{P}_h,K'\subseteq K\}$ and $\mathcal{E}_{h,K}^I=\{\gamma'\in\mathcal{E}_h^I,\gamma'\subseteq K\}$. Recall $\delta(\gamma)=\max\{\frac{|\gamma|}{|\gamma'|}|\ \gamma'\in\mathcal{E}_h^I,\gamma'\subset\gamma\}$ Then, for any interior edge $\gamma\in\partial\mathcal{P}_H$ and for sufficiently small ϵ, such that $18\epsilon\varrho<1$ we have*

$$\begin{aligned}|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant \frac{4}{\epsilon(1-18\epsilon\varrho)}\left(\frac{h_K^2}{4}\|f+div(\boldsymbol{A}\nabla u_H)\|_K^2\right.\\ &+ 4C_i^2\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\ &+ 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\ &+ 16\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2.\end{aligned}$$

Proof. Let $\gamma\in\mathcal{E}_K$ is an interior edge of $\mathcal{P}_H$. Let $\tilde{v}$ be the extension of $[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma$ to K, let ℓ be the piecewise linear function which is different from zero inside the shaded region and on γ, zero elsewhere, i.e., is zero on the boundaries a, b and c, as in Figure 3.3. Also we assume that ℓ takes the value 1 at the midpoint of γ and $1/2$ at the quarter point of γ.

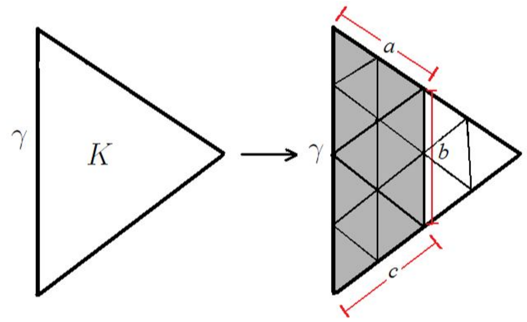


Figure 3.3: Support of ℓ

Define $v = \tilde{v}\ell$. Then $v \in X_h$. Since $u_H \in \mathbb{P}_1(K)$, for any $\gamma' \in \mathcal{E}_h^I \cap \mathring{K}$,

$$[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma'} = 0.$$

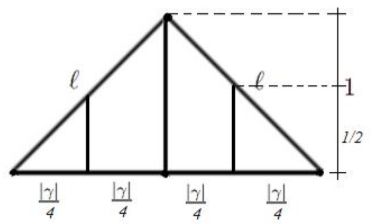
In addition, since ℓ is zero on the edges a, b and c in Figure 3.3, we have

$$\sum_{\gamma' \in \mathcal{E}_h^I} \left([\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma'},\, \langle v \rangle_{\gamma'}\right)_{\gamma'} = \left([\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma},\, \langle v \rangle_{\gamma}\right)_{\gamma}.$$

Since γ is an interior edge of $\mathcal{P}_H$, $v = 0$ on $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$. Moreover, $[v] = 0$ on $\gamma' \in \mathcal{E}_{h,K}^I$ except the case when $\gamma' \subset \gamma$, then (3.10) is rewritten as

$$\begin{aligned}\left([\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma},\, \langle v \rangle_{\gamma}\right)_{\gamma} &= (f + \operatorname{div}(\boldsymbol{A}\nabla u_H), v)_K - \sum_{K' \in \mathcal{P}_{h,K}} (\boldsymbol{A}\nabla(u_h - u_H), \nabla v)_{K'} \\ &\quad + \tau \sum_{\gamma' \in \mathcal{E}_{h,K}^I} \left([u_h]_{\gamma'}, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_{\gamma'}\right)_{\gamma'} - \tau \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} (q - u_h, \boldsymbol{n} \cdot \boldsymbol{A}\nabla v)_{\gamma'}.\end{aligned}$$

Now recall that $\ell = 1$ at the midpoint of γ, $\ell = 1/2$ at the quarter point of γ and 0 at the end points of γ,



$$\begin{aligned}\left([\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma},\, \langle v \rangle_{\gamma}\right)_{\gamma} &= \frac{1}{2}\int_{\gamma} |[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma}|^2 \ell = \frac{1}{2}|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma}|^2 \int_{\gamma} \ell \qquad (3.27) \\ &= \frac{|\gamma|}{4}|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma}|^2 = \frac{1}{4}\int_{\gamma} |[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma}|^2 = \frac{1}{4}\|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_{\gamma}\|_{\gamma}^2.\end{aligned}$$

Let ν_{max} be the height of the triangle K, as in Figure 3.4, since $0 \leqslant \ell \leqslant 1$, $\tilde{v}$ is constant, then

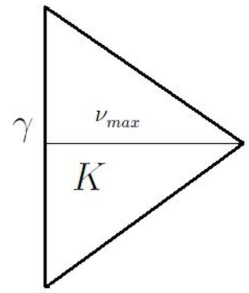


Figure 3.4: Height of the triangle K

$$\begin{aligned}
\|v\|_K^2 = \|\tilde{v}\ell\|_K^2 \leqslant \|\tilde{v}\|_K^2 &= \int_K |[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma|^2 = |[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma|^2 |K| \\
&= |[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma|^2 \left(\frac{\nu_{max}|\gamma|}{2}\right) \\
&= \frac{1}{2}\nu_{max} \int_\gamma |[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma|^2 \\
&= \frac{\nu_{max}}{2} \|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2.
\end{aligned}$$

Let ϱ be a shape regularity parameter, then by Lemma A.0.5, we have,

$$\nu_{max} \leqslant \varrho|\gamma|,$$

then

$$\|v\|_K^2 \leqslant \frac{\varrho}{2}|\gamma| \|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2. \tag{3.28}$$

Now applying Cauchy-Schwarz's inequality to right-hand side of (3.27) and using (3.27) and multiplying both sides by $|\gamma|$ yield

$$\begin{aligned}
|\gamma|\frac{1}{4}\|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant |\gamma| \|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K \, \|v\|_K + |\gamma| \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'} \, \|\nabla v\|_{K'} \\
&+ |\gamma| \sum_{\gamma' \in \mathcal{E}_{h,K}^I} \|[u_h]_{\gamma'}\|_{\gamma'} \, \|\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v\rangle_{\gamma'}\|_{\gamma'} \\
&+ |\gamma| \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} \|q - u_h\|_{\gamma'} \, \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla v\|_{\gamma'}.
\end{aligned} \tag{3.29}$$

Using the trace and inverse estimates in Chapter 2.7, the following bounds are obtained.

By (2.14), for $K' \in \mathcal{P}_{h,K}$

$$\|\nabla v\|_{K'} \leqslant C_i h_{K'}^{-1} \|v\|_{K'}.$$

By (2.15) and (2.18), for $\gamma' = \partial K_1 \cap \partial K_2$:

$$\|\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v\rangle_{\gamma'}\|_{\gamma'} \leqslant C_i \rho(\boldsymbol{A}_K) \sqrt{\varrho} \left(h_{K_1}^{-3/2} \|v\|_{K_1} + h_{K_2}^{-3/2} \|v\|_{K_2}\right).$$

By (2.16) and (2.18), for $\gamma' \in \mathcal{E}_h^D \cap \partial K$:

$$\|\boldsymbol{n} \cdot \boldsymbol{A}\nabla v\|_{\gamma'} \leqslant C_i \rho(\boldsymbol{A}_K) 2\sqrt{\varrho} h_{K'}^{-3/2} \|v\|_{K'}.$$

Then (3.29) becomes,

$$\begin{aligned} |\gamma| \frac{1}{4} \|[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 \leqslant\ & |\gamma| \|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K \|v\|_K + \sum_{K' \in \mathcal{P}_{h,K}} C_i |\gamma| h_{K'}^{-1} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'} \|v\|_{K'} \\ + & |\gamma| \sum_{\substack{\gamma' \in \mathcal{E}_{h,K}^I \\ \gamma' = \partial K_1 \cap \partial K_2}} \sqrt{\varrho} C_i \rho(\boldsymbol{A}_K) \|[u_h]_{\gamma'}\|_{\gamma'} \left(h_{K_1}^{-3/2} \|v\|_{K_1} + h_{K_2}^{-3/2} \|v\|_{K_2} \right) \\ + & |\gamma| \sum_{\substack{\gamma' \in \mathcal{E}_h^D \cap \partial K \\ \gamma' \subset \partial K'}} 2\sqrt{\varrho} C_i \rho(\boldsymbol{A}_K) \|q - u_h\|_{\gamma'} \, h_{K'}^{-3/2} \|v\|_{K'} \end{aligned}$$

If the initial mesh is one of the following three meshes, then the ratio $\frac{|\gamma|}{h_{K_2}}$ is bounded by following constants. Moreover, if we have a finer refinement than above examples on the

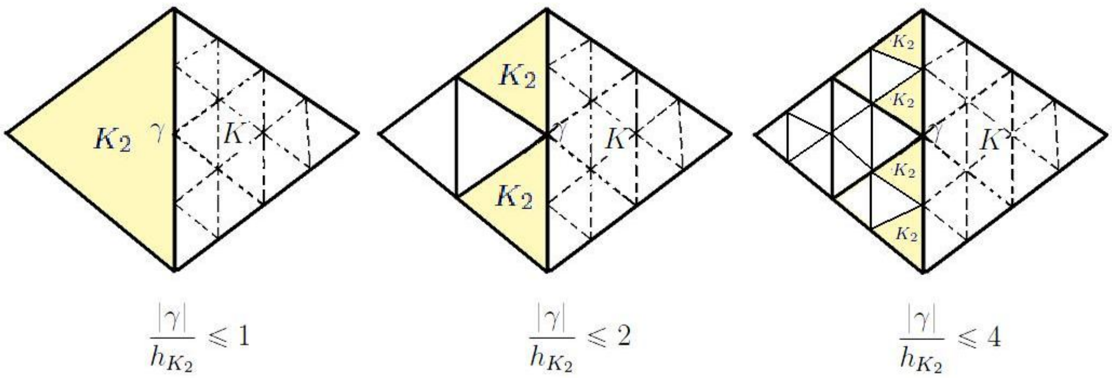


Figure 3.5: An example of initial meshes

edge γ, we can bound the ratio by $\delta(\gamma)$ which is a finite number that measures the fineness of γ with respect to γ'. We can write that $\frac{|\gamma|}{h_{K_2}} \leqslant \frac{|\gamma|}{|\gamma'|} \leqslant \delta(\gamma)$ where $\gamma' = \gamma \cap \partial K_2$.

Now in Figure 3.6 let K' be an arbitrary triangle in $\mathcal{P}_{h,K}$ and $h_{K'}$ denotes the length of the longest edge of element K'.

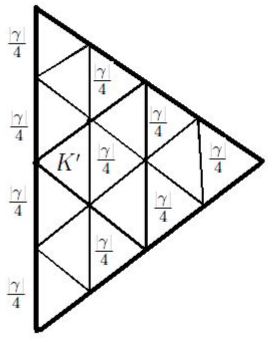


Figure 3.6: Refinement of K

Therefore we can easily write that $\frac{|\gamma|}{4} \leqslant h_{K'} \;\; \Rightarrow \;\; |\gamma| h_{K'}^{-1} \leqslant 4$. Also it is obvious that if γ' is an arbitrary edge of a triangle K' then $|\gamma'| \leqslant h_{K'}$, so by elementary computation we have $|\gamma| h_{K'}^{-3/2} = |\gamma| h_{K'}^{-1} h_{K'}^{-1/2} \leqslant 4|\gamma'|^{-1/2}$. Consequently, using those above relation between the edges, we have

$$
\begin{aligned}
|\gamma|\frac{1}{4}\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 \;\; &\leqslant \;\; h_K\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K\,\|v\|_K + \sum_{K'\in\mathcal{P}_{h,K}} 4C_i\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}\,\|v\|_{K'} \\
&+ \sum_{\substack{\gamma'\in\mathcal{E}_{h,K}^I \\ \gamma'=\partial K_1\cap\partial K_2}} 4\sqrt{\varrho}C_i\rho(\boldsymbol{A}_K)|\gamma'|^{-1/2}\|[u_h]_{\gamma'}\|_{\gamma'}\,(\|v\|_{K_1} + \|v\|_{K_2}) \\
&+ \sum_{\substack{\gamma'\in\mathcal{E}_h^D\cap\partial K \\ \gamma'\subset\partial K'}} 8\sqrt{\varrho}C_i\rho(\boldsymbol{A}_K)|\gamma'|^{-1/2}\|q - u_h\|_{\gamma'}\|v\|_{K'}
\end{aligned}
\tag{3.30}
$$

By Young's inequality for $\epsilon > 0$

$$
\begin{aligned}
|\gamma|\frac{1}{4}\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 \;\; &\leqslant \;\; \epsilon\|v\|_K^2 + \frac{h_K^2}{4\epsilon}\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ \sum_{K'\in\mathcal{P}_{h,K}} (\epsilon\|v\|_{K'}^2 + \frac{4}{\epsilon}C_i^2\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2) \\
&+ \sum_{\substack{\gamma'\in\mathcal{E}_{h,K}^I \\ \gamma'=\partial K_1\cap\partial K_2}} \left(\epsilon(\|v\|_{K_1} + \|v\|_{K_2})^2 + \frac{4}{\epsilon}\varrho C_i^2\rho(\boldsymbol{A}_K)^2|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\right) \\
&+ \sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} \left(\epsilon\|v\|_{K'}^2 + \frac{16}{\epsilon}\varrho C_i^2\rho(\boldsymbol{A}_K)^2|\gamma'|^{-1}\|q - u_h\|_{\gamma'}^2\right)
\end{aligned}
$$

Now note that $(\|v\|_{K_1} + \|v\|_{K_2})^2 \leqslant 2\|v\|_{K_1}^2 + 2\|v\|_{K_2}^2$ and also K_1 and K_2 interior element. So, for each edge of a triangle we have this sum. Since we have three edges in total we get

$6\|v\|_{K_1}^2 + 6\|v\|_{K_2}^2$. Finally we have

$$\begin{aligned}
|\gamma|\frac{1}{4}\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant 9\epsilon\|v\|_K^2 + \frac{1}{\epsilon}\left(\frac{h_K^2}{4}\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 + \sum_{K'\in\mathcal{P}_{h,K}} 4C_i^2\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\right.\\
&+ \sum_{\gamma'\in\mathcal{E}_{h,K}^I} 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ \left.\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} 16\varrho C_i^2\rho(\boldsymbol{A}_K)^2|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2\right)
\end{aligned}$$

By using (3.28) it can be written

$$\begin{aligned}
|\gamma|\frac{1}{4}\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant \frac{9\varrho}{2}\epsilon|\gamma|\;\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 + \frac{1}{\epsilon}\left(\frac{h_K^2}{4}\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2\right.\\
&+ 4C_i^2\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ \left.16\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2\right)
\end{aligned}$$

For sufficiently small ϵ, i.e. if $1-18\epsilon>0$, we arrive at

$$\begin{aligned}
|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant \frac{4}{\epsilon(1-18\epsilon\varrho)}\left(\frac{h_K^2}{4}\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2\right.\\
&+ 4C_i^2\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ 16\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2.
\end{aligned}$$

■

Corollary 3.1.9 *In Lemma 3.1.8, take $\epsilon=(36\varrho)^{-1}$. Then we have*

$$\begin{aligned}
|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant 288\varrho\left(\frac{h_K^2}{4}\|f+div(\boldsymbol{A}\nabla u_H)\|_K^2\right.\\
&+ 4C_i^2\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ \left.16\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2\right). \qquad (3.31)
\end{aligned}$$

Lemma 3.1.10 *Let $K \in \mathcal{M}_H$. Define $\mathcal{P}_{h,K} = \{K' \in \mathcal{P}_h, K' \subseteq K\}$ and $\mathcal{E}^I_{h,K} = \{\gamma' \in \mathcal{E}^I_h, \gamma' \subseteq K\}$. Then, for any $\gamma \in \partial\mathcal{P}_H \cap \mathcal{E}^N_H$ and for sufficiently small ϵ, such that $10\epsilon\varrho < 1$ we have*

$$
\begin{aligned}
|\gamma| \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi^{(0)}_\gamma g\|^2_\gamma \leqslant \; & \frac{2}{\epsilon(1-10\epsilon\varrho)} \Big(\frac{h_K^2}{4} \|f + div(\boldsymbol{A}\nabla u_H)\|^2_K \\
& + 4C_i^2 \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|^2_{K'} + 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}^I_{h,K}} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|^2_{\gamma'} \\
& + 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}^D_h \cap \partial K} |\gamma'|^{-1} \|q - u_h\|^2_{\gamma'} + 3\varrho osc^2(g, \gamma, 0) \Big).
\end{aligned}
$$

Proof. We construct a test function $v \in X_h$ as follows. Let $\tilde{v}$ be the extension of $\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi^{(0)}_\gamma g$ to K and let ℓ be the piecewise linear function which is different from zero inside the shaded region as in Figure 3.3 and on γ, zero elsewhere, i.e., it is zero on the boundaries a, b and c, in Figure 3.3. Also we assume that ℓ takes the value 1 at the midpoint of γ and $1/2$ at the quarter point of γ. Take $v = \tilde{v}\ell$, then $v \in X_h$. It can be written that,

$$
(\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi^{(0)}_\gamma g, v)_\gamma = (\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - g, v)_\gamma + (g - \Pi^{(0)}_\gamma g, v)_\gamma. \tag{3.32}
$$

Note that, the test function v have following properties:

- $v = 0$ outside of element K,
- $v = 0$ on interior and Dirichlet edges of $\mathcal{P}_H$,
- $[\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H] = 0$ on all $\gamma' \in \mathcal{E}^I_h \backslash \partial\mathcal{P}_H$.

Using this v in (3.10) allow us to write

$$
\begin{aligned}
(g - \boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H, v)_\gamma = \; & -(f + \mathrm{div}(\boldsymbol{A}\nabla u_H), v)_K + \sum_{K' \in \mathcal{P}_{h,K}} (\boldsymbol{A}\nabla(u_h - u_H), \nabla v)_{K'} \\
& - \tau \sum_{\gamma' \in \mathcal{E}^I_{h,K}} \big([u_h]_{\gamma'}, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_{\gamma'}\big)_{\gamma'} \\
& + \tau \sum_{\gamma' \in \mathcal{E}^D_h \cap \partial K} (q - u_h, \boldsymbol{n} \cdot \boldsymbol{A}\nabla v)_{\gamma'}.
\end{aligned} \tag{3.33}
$$

Equations (3.32) and (3.33) imply

$$
\begin{aligned}
(\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi^{(0)}_\gamma g, v)_\gamma = \; & (f + \mathrm{div}(\boldsymbol{A}\nabla u_H), v)_K - \sum_{K' \in \mathcal{P}_{h,K}} (\boldsymbol{A}\nabla(u_h - u_H), \nabla v)_{K'} \\
& + \tau \sum_{\gamma' \in \mathcal{E}^I_{h,K}} \big([u_h]_{\gamma'}, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_{\gamma'}\big)_{\gamma'} - \tau \sum_{\gamma' \in \mathcal{E}^D_h \cap \partial K} (q - u_h, \boldsymbol{n} \cdot \boldsymbol{A}\nabla v)_{\gamma'} \\
& + (g - \Pi^{(0)}_\gamma g, v)_\gamma.
\end{aligned} \tag{3.34}
$$

Since $\ell = 1$ at the midpoint of γ, $\ell = 1/2$ at the quarter point of γ and 0 at the end points of γ, we obtain similar to (3.27)

$$\begin{aligned}
\left(\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g,\ v\right)_\gamma &= \int_\gamma |\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 \ell = |\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 \int_\gamma \ell \\
&= \frac{|\gamma|}{2} |\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 = \frac{1}{2} \int_\gamma |\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 \\
&= \frac{1}{2} \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2. \qquad (3.35)
\end{aligned}$$

Now applying Cauchy-Schwarz's inequality to right-hand side of (3.34) and using (3.35) and multiplying both sides by $|\gamma|$ give

$$\begin{aligned}
|\gamma| \frac{1}{2} \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 &\leqslant |\gamma| \|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K \, \|v\|_K + |\gamma| \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'} \, \|\nabla v\|_{K'} \\
&+ |\gamma| \sum_{\gamma' \in \mathcal{E}_{h,K}^I} \|[u_h]_{\gamma'}\|_{\gamma'} \, \|\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla v \rangle_{\gamma'}\|_{\gamma'} \qquad (3.36) \\
&+ |\gamma| \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} \|q - u_h\|_{\gamma'} \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla v\|_{\gamma'} + |\gamma| \|g - \Pi_\gamma^{(0)} g\|_\gamma \|v\|_\gamma.
\end{aligned}$$

Using the trace and inverse inequalities (2.14)-(2.18) in the right-hand side of (3.36), we obtain,

$$\begin{aligned}
|\gamma| \frac{1}{2} \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 &\leqslant |\gamma| \|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K \, \|v\|_K \\
&+ C_i |\gamma| \sum_{K' \in \mathcal{P}_{h,K}} h_{K'}^{-1} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'} \, \|v\|_{K'} \\
&+ \sqrt{\varrho} C_i \rho(\boldsymbol{A}_K) |\gamma| \sum_{\substack{\gamma' \in \mathcal{E}_{h,K}^I \\ \gamma' = \partial K_1 \cap \partial K_2}} \|[u_h]_{\gamma'}\|_{\gamma'} \left(h_{K_1}^{-3/2} \|v\|_{K_1} + h_{K_2}^{-3/2} \|v\|_{K_2} \right) \\
&+ 2\sqrt{\varrho} C_i \rho(\boldsymbol{A}_K) |\gamma| \sum_{\substack{\gamma' \in \mathcal{E}_h^D \cap \partial K \\ \gamma' \subset \partial K'}} \|q - u_h\|_{\gamma'} h_{K'}^{-3/2} \|v\|_{K'} \\
&+ 2\sqrt{3\varrho} |\gamma| h_K^{-1/2} \|g - \Pi_\gamma^{(0)} g\|_\gamma \|v\|_K.
\end{aligned}$$

Moreover, as in the previous proof, using the relation between the edges of a triangle, we see that $|\gamma| \leqslant h_K$ and $\forall K' \in \mathcal{P}_{h,K}$ $\frac{|\gamma|}{4} \leqslant h_{K'}$ and $|\gamma| h_{K'}^{-3/2} \leqslant 4|\gamma'|^{-1/2}$ for any edge γ' of K'. So we finally have:

$$
\begin{aligned}
|\gamma|\frac{1}{2}\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 \quad &\leqslant \quad h_K\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K\,\|v\|_K + 4C_i \sum_{K'\in\mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}\,\|v\|_{K'} \\
&+ \quad 4\sqrt{\varrho}C_i\rho(\boldsymbol{A}_K) \sum_{\substack{\gamma'\in\mathcal{E}_{h,K}^I \\ \gamma'=\partial K_1\cap\partial K_2}} |\gamma'|^{-1/2}\|[u_h]_{\gamma'}\|_{\gamma'}\,(\|v\|_{K_1} + \|v\|_{K_2}) \\
&+ \quad 8\sqrt{\varrho}C_i\rho(\boldsymbol{A}_K) \sum_{\substack{\gamma'\in\mathcal{E}_h^D\cap\partial K \\ \gamma'\subset\partial K}} |\gamma'|^{-1/2}\|q - u_h\|_{\gamma'}\|v\|_{K'} \\
&+ \quad 2\sqrt{3\varrho}|\gamma|^{1/2}\|g - \Pi_\gamma^{(0)} g\|_\gamma\|v\|_K.
\end{aligned}
$$

By Young's inequality for $\epsilon > 0$

$$
\begin{aligned}
|\gamma|\frac{1}{2}\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 \quad &\leqslant \quad \epsilon\|v\|_K^2 + \frac{h_K^2}{4\epsilon}\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ \quad \sum_{K'\in\mathcal{P}_{h,K}} (\epsilon\|v\|_{K'}^2 + \frac{4}{\epsilon}C_i^2\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2) \\
&+ \quad \sum_{\gamma'\in\mathcal{E}_{h,K}^I} \left(\epsilon(\|v\|_{K_1} + \|v\|_{K_2})^2 + \frac{4}{\epsilon}\varrho C_i^2\rho(\boldsymbol{A}_K)^2|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\right) \\
&+ \quad \sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} (\epsilon\|v\|_{K'}^2 + \frac{16}{\epsilon}\varrho C_i^2\rho(\boldsymbol{A}_K)^2|\gamma'|^{-1}\|q - u_h\|_{\gamma'}^2) \\
&+ \quad \epsilon\|v\|_K^2 + \frac{3}{\epsilon}\varrho|\gamma|\|g - \Pi_\gamma^{(0)} g\|_\gamma^2.
\end{aligned}
$$

Note that $(\|v\|_{K_1}^2 + \|v\|_{K_2})^2 \leqslant 2\|v\|_{K_1}^2 + 2\|v\|_{K_2})^2$ and for each edge of a triangle we have this sum. Totally, we have three edges for each triangle, we have $6\|v\|_{K_1}^2 + 6\|v\|_{K_2})^2$. Then we get

$$
\begin{aligned}
|\gamma|\frac{1}{2}\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 \quad \leqslant \quad & 10\epsilon\|v\|_K^2 + \frac{1}{\epsilon}\Bigg(\frac{h_K^2}{4}\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
& + 4C_i^2 \sum_{K'\in\mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
& + 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_{h,K}^I} |\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
& + 16\varrho C_i^2\rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} |\gamma'|^{-1}\|q - u_h\|_{\gamma'}^2 \\
& + 3\varrho|\gamma|\|g - \Pi_\gamma^{(0)} g\|_\gamma^2\Bigg)
\end{aligned}
$$

Similarly as in (3.28), using the height v_{max} defined in Figure 3.4 and using Lemma A.0.5, we get following upper bound for the function v in L_2-norm:

$$\begin{aligned}
\|v\|_K^2 = \|\tilde{v}\ell\|_K^2 \leqslant \|\tilde{v}\|_K^2 &= \int_K |\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 = |\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 |K| \\
&= |\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 \left(\frac{v_{max}|\gamma|}{2}\right) \\
&= \frac{1}{2} v_{max} \int_\gamma |\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g|^2 \\
&= \frac{v_{max}}{2} \|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 \\
&\leqslant \frac{\varrho}{2} |\gamma| \|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2. \qquad (3.37)
\end{aligned}$$

(3.37) yield

$$\begin{aligned}
|\gamma|\frac{1}{2}\|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 &\leqslant \frac{10\varrho}{2}\epsilon|\gamma|\ \|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 + \frac{1}{\epsilon}\Bigg(\frac{h_K^2}{4}\|f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ 4C_i^2 \sum_{K'\in\mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} |\gamma'|^{-1}\|q - u_h\|_{\gamma'}^2 + 3\varrho|\gamma|\|g - \Pi_\gamma^{(0)} g\|_\gamma^2\Bigg)
\end{aligned}$$

We now require an assumption on ϵ. If ϵ satisfies $1 - 10\epsilon > 0$ we obtain:

For sufficiently small ϵ and by (2.4), we arrive at

$$\begin{aligned}
|\gamma|\|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 \leqslant\ & \frac{2}{\epsilon(1 - 10\epsilon\varrho)}\Big(\frac{h_K^2}{4}\|f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ 4C_i^2 \sum_{K'\in\mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} |\gamma'|^{-1}\|q - u_h\|_{\gamma'}^2 + 3\varrho\, osc^2(g,\gamma,0)\Big).
\end{aligned}$$

■

Corollary 3.1.11 *In Lemma 3.1.10, take* $\epsilon = (20\varrho)^{-1}$*. Then*

$$\begin{aligned}
|\gamma|\|\boldsymbol{n}\cdot \boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)} g\|_\gamma^2 \leqslant\ & 80\varrho\Big(\frac{h_K^2}{4}\|f + div(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ 4C_i^2 \sum_{K'\in\mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K} |\gamma'|^{-1}\|q - u_h\|_{\gamma'}^2 + 3\varrho osc^2(g,\gamma,0)\Big). \qquad (3.38)
\end{aligned}$$

Lemma 3.1.12 *Let $K \in \mathcal{M}_H$ and $\mathcal{P}_{h,K} = \{K' \in \mathcal{P}_h, K' \subseteq K\}$ and $\mathcal{E}^I_{h,K} = \{\gamma \in \mathcal{E}^I_h, \gamma \subseteq K\}$. For sufficiently small ϵ, such that $20\epsilon < 1$ we have*

$$\begin{aligned}
h_K^2\|f + div(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{640C_i^2}{\epsilon(9-180\epsilon)} \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&+ \frac{640\varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)} \sum_{\gamma \in \mathcal{E}^I_{h,K}} |\gamma|^{-1}\|[u_h]_\gamma\|_\gamma^2 \\
&+ \frac{2560\varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)} \sum_{\gamma \in \mathcal{E}^D_h \cap \partial K} |\gamma|^{-1}\|q - u_h\|_\gamma \\
&+ (4\varrho + \frac{80\varrho}{\epsilon(9-180\epsilon)})\, osc^2(f, K, 1).
\end{aligned}$$

Proof. For $K \in \mathcal{M}_H$ consider the following refinement denoted by $\mathcal{P}_{h,K}$:

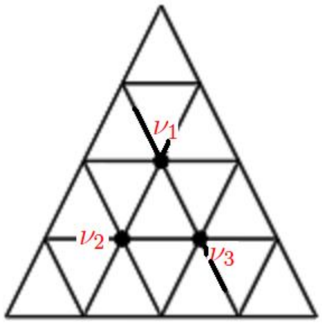


Figure 3.7: Refinement of K

We introduce the finite dimensional space

$$S_K = \{v \in C^0(K),\ v_{|K'} \in \mathbb{P}_1(K')\ \forall K' \in \mathcal{P}_{h,K},\ v = 0 \text{ on } \partial K\}.$$

If we extend functions in S_K by 0 outside of K, then it can be said that S_K is a subset of X_h. Also, it is easily seen that a function in S_K is uniquely determined by its values at the nodes v_1, v_2, v_3 shown in Figure 3.7. Thus $dim(S_K) = 3 = dim(\mathbb{P}_1(K))$. Furthermore, a basis $\{\phi_1, \phi_2, \phi_3\}$ for S_K can be constructed by "gluing" together Lagrangian-type functions corresponding to the individual triangles in $\mathcal{P}_{h,K}$.

Now letting $\{\psi_1, \psi_2, \psi_3\}$ be the usual Lagrangian basis for $\mathbb{P}_1(K)$, we form the "Gramian" matrix G given by $Gij = (\phi_j, \psi_i)_K$, $i, j = 1, 2, 3$. We next show that G is nonsingular.

Showing that G is nonsingular: Let v_1, v_2, v_3 denote the three nodes shown in Figure 3.7. Let $\mathbf{v}^2 = (v_1^2, v_2^2)$, $\mathbf{v}^3 = (v_1^3, v_2^3)$ be the vectors starting from v_1 and terminating at v_2 and v_3, respectively. Let ϕ_1, ϕ_2, ϕ_3 be the basis functions corresponding to the nodes v_1, v_2, v_3 and denote their supports by S^1, S^2, S^3. Clearly,

$$\begin{aligned} \phi_2(x,y) &= \phi_1(x - v_1^2, y - v_2^2) \qquad \forall (x,y) \in S^2 \text{ and} \\ \phi_3(x,y) &= \phi_1(x - v_1^3, y - v_2^3) \qquad \forall (x,y) \in S^3. \end{aligned}$$

Suppose there exists $\psi(x,y) = ax+by+c \in \mathbb{P}_1(K)$ such that $(\phi_j, \psi) = 0$, $j = 1,2,3$. Showing that $a = b = c = 0$, implies the linear independence of the rows of G.

$$\begin{aligned} 0 = (\psi, \phi_2)_K &= \int_{S^2} \psi(x,y)\, \phi_2(x,y) dxdy = \int_{S^2} \psi(x,y)\, \phi_1(x - v_1^2, y - v_2^2) dxdy \\ &= \int_{S^1} \psi(x + v_1^2, y + v_2^2)\, \phi_1(x,y) dxdy \\ &= \int_{S^1} \psi(x,y)\, \phi_1(x,y) dxdy + (a v_1^2 + b v_2^2) \int_{S^1} \phi_1(x,y) dxdy. \end{aligned}$$

Now $\int_{S^1} \psi(x,y)\, \phi_1(x,y) dxdy = (\psi, \phi_1) = 0$. On the other hand, ϕ_1 is nonnegative and nonzero; thus we conclude from the above that $a v_1^2 + b v_2^2 = 0$. In a similar way, we obtain $a v_1^3 + b v_2^3 = 0$. Since the vectors $\mathbf{v}^2, \mathbf{v}^3$ are linearly independent, it follows that $a = b = 0$. Now that this has been shown, the fact that $c = 0$ readily follows from $(\psi, \phi_1)_K = 0$.

<u>*Showing norm equivalence:*</u> Let $T_K : \mathbb{P}_1(K) \to S_K$ denotes the operator given by $(T_K v, \chi)_K = (v, \chi)_K \quad \forall \chi \in S_K$. Then $\|T_K \cdot\|_K$ is a norm equivalent to $\|\cdot\|_K$ on $\mathbb{P}_1(K)$ with constants that are independent of h_K because of Lemma A.0.8.

In Lemma A.0.8, it is proved that $\|T_K \cdot\|_K$ is a norm and equivalent to $\|\cdot\|_K$ such that

$$\forall v \in \mathbb{P}_1(K) \quad \frac{3}{2\sqrt{10}} \|v\|_K \leqslant \|T_K v\|_K \leqslant \|v\|_K. \tag{3.39}$$

We also remark that

$$\|T_K v\|_K^2 = (T_K v, T_K v)_K = (v, T_K v)_K. \tag{3.40}$$

Now we can start the estimation of $h_K^2 \|f + \operatorname{div}(\boldsymbol{A} \nabla u_H)\|_K^2$.

To estimate $f + \operatorname{div}(\boldsymbol{A}\nabla u_H)$ we take $v = T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))$ that is extended by 0 outside of K.

Note that since our test function $v = 0$ on ∂K, we do not have any contribution from boundary terms in (3.10). Also $u_H \in \mathbb{P}_1(K)$, then for any $\gamma \in \mathcal{E}_h^I \cap \mathring{K}$,

$$[\boldsymbol{n} \cdot \boldsymbol{A} \nabla u_H]_\gamma = 0.$$

In addition since $v \in C^0(K)$, for any $\gamma \in \mathcal{E}_h^I \cap \mathring{K}$, $[v]_\gamma = 0$. Therefore, by (3.40) and the key identity (3.10) we get,

$$\begin{aligned}
\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_K^2 &= (\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H), T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_K \\
&= (f + \operatorname{div}(\boldsymbol{A}\nabla u_H), T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_K \\
&+ (\Pi_K^{(1)} f - f, T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_K \\
&= \sum_{K' \in \mathcal{P}_{h,K}} (\boldsymbol{A}\nabla(u_h - u_H), \nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_{K'} \\
&- \tau \sum_{\gamma \in \mathcal{E}_{h,K}^I} ([u_h]_\gamma, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\rangle_\gamma)_\gamma \\
&+ \tau \sum_{\gamma \in \mathcal{E}_h^D \cap \partial K} (q - u_h, \boldsymbol{n} \cdot \boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_\gamma \\
&+ (\Pi_K^{(1)} f - f, T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_K.
\end{aligned}$$

Using (3.39), it can be written that

$$\begin{aligned}
\frac{9}{40}\|\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \sum_{K' \in \mathcal{P}_{h,K}} (\boldsymbol{A}\nabla(u_h - u_H), \nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_{K'} \\
&- \tau \sum_{\gamma \in \mathcal{E}_{h,K}^I} ([u_h]_\gamma, \langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\rangle_\gamma)_\gamma \\
&+ \tau \sum_{\gamma \in \mathcal{E}_h^D \cap \partial K} (q - u_h, \boldsymbol{n} \cdot \boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_\gamma \\
&+ (\Pi_K^{(1)} f - f, T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_K. \qquad (3.41)
\end{aligned}$$

To complete the proof, each component of (3.41) will be handled separately. Before engaging the proof we immediately notice that $\forall K' \in \mathcal{P}_{h,K},\ h_K = 4h_{K'}$ and $\forall \gamma \in \mathcal{E}_{h,K}^I,\ |\gamma| \leqslant h_{K'} = \frac{h_K}{4} \Rightarrow h_{K'}^{-3/2} = 8h_K^{-3/2}$ where $\gamma \in \partial K'$. Now using Cauchy-Schwarz's, (2.14), Young's inequalities for any $\epsilon > 0$ and (3.39) yield

$$\begin{aligned}
\textstyle\sum_{K' \in \mathcal{P}_{h,K}} & (\boldsymbol{A}\nabla(u_h - u_H), \nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)))_{K'} \\
\leqslant & \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'} \|\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K'} \\
\leqslant & \sum_{K' \in \mathcal{P}_{h,K}} C_i h_{K'}^{-1} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'} \|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K'} \\
\leqslant & \frac{\epsilon}{2}\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_K^2 + \frac{C_i^2}{2\epsilon} \sum_{K' \in \mathcal{P}_{h,K}} h_{K'}^{-2} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\epsilon}{2}\|\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 + \frac{C_i^2}{2\epsilon}\sum_{K'\in\mathcal{P}_{h,K}} h_{K'}^{-2}\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&= \frac{\epsilon}{2}\|\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 + \frac{16C_i^2}{2\epsilon}h_K^{-2}\sum_{K'\in\mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2. \qquad (3.42)
\end{aligned}
$$

Similarly using Cauchy-Schwarz's, (2.15), (2.18), and Young's inequalities for any $\epsilon > 0$, respectively

$$
\begin{aligned}
&\textstyle\sum_{\gamma\in\mathcal{E}_{h,K}^I} ([u_h]_\gamma, \langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\rangle_\gamma)_\gamma \\
&\leqslant \sum_{\gamma\in\mathcal{E}_{h,K}^I} \|[u_h]_\gamma\|_\gamma \|\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\rangle_\gamma\|_\gamma \\
&\leqslant \sum_{\substack{\gamma\in\mathcal{E}_{h,K}^I\\ \gamma=\partial K_1\cap\partial K_2}} \|[u_h]_\gamma\|_\gamma \sqrt{\varrho}\Big(h_{K_1}^{-1/2}\|\boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1} \\
&\qquad\qquad + h_{K_2}^{-1/2}\|\boldsymbol{A}\nabla T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}\Big) \\
&\leqslant \sum_{\substack{\gamma\in\mathcal{E}_{h,K}^I\\ \gamma=\partial K_1\cap\partial K_2}} \|[u_h]_\gamma\|_\gamma C_i\rho(\boldsymbol{A}_K)\sqrt{\varrho}\Big(h_{K_1}^{-3/2}\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1} \\
&\qquad\qquad + h_{K_2}^{-3/2}\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}\Big) \\
&= \sum_{\substack{\gamma\in\mathcal{E}_{h,K}^I\\ \gamma=\partial K_1\cap\partial K_2}} \|[u_h]_\gamma\|_\gamma C_i\rho(\boldsymbol{A}_K)\sqrt{\varrho}\Big(8h_{K}^{-3/2}\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1} \\
&\qquad\qquad + 8h_{K}^{-3/2}\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}\Big) \\
&= \sum_{\substack{\gamma\in\mathcal{E}_{h,K}^I\\ \gamma=\partial K_1\cap\partial K_2}} \Bigg(\frac{64\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}h_K^{-3}\|[u_h]_\gamma\|_\gamma^2 + \frac{\epsilon}{2}\Big(\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1} \\
&\qquad\qquad + \|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}\Big)^2\Bigg).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\Big(\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1} + \|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}\Big)^2 \\
&\leqslant 2\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1}^2 + 2\|T_K(\Pi_K^{(1)} f + \operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}^2.
\end{aligned}
$$

Since γ is an interior edges, for each triangle we get three times above sum.

So, we have $6\left(\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_1}^2+\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}^2\right)$. Then using (3.39),

$$
\begin{aligned}
&\sum_{\gamma\in\mathcal{E}_{h,K}^I}([u_h]_\gamma,\langle \boldsymbol{n}\cdot\boldsymbol{A}\nabla T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\rangle_\gamma)_\gamma\\
\leqslant\;& 3\epsilon\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_K^2+\frac{64\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}\sum_{\gamma\in\mathcal{E}_{h,K}^I}h_K^{-3}\|[u_h]_\gamma\|_\gamma^2\\
\leqslant\;& 3\epsilon\|\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2+\frac{64\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}\sum_{\gamma\in\mathcal{E}_{h,K}^I}h_K^{-3}\|[u_h]_\gamma\|_\gamma^2. \qquad (3.43)
\end{aligned}
$$

In (3.43) consider the edge $\gamma\in\mathcal{E}_{h,K}^I$ where $\gamma=\partial K_1\cap\partial K_2$. If this $\gamma\in\partial K$ then one of the triangles, write K_2, will be out of the element K. Then from the definition of $v=T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))$, we have that $v=0$ on ∂K and out of the element K. This implies $\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K_2}=0$ in that case.

To estimate the third component of (3.41), use Cauchy-Schwarz's, (2.16), (2.18), Young's inequalities for any $\epsilon>0$ and (3.39), respectively

$$
\begin{aligned}
&\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K}(q-u_h,\boldsymbol{n}\cdot\boldsymbol{A}\nabla T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H)))_\gamma\\
\leqslant\;&\sum_{\substack{\gamma\in\mathcal{E}_h^D\cap\partial K\\ \gamma\in\partial K'}}\|q-u_h\|_\gamma\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_\gamma\\
\leqslant\;&\sum_{\substack{\gamma\in\mathcal{E}_h^D\cap\partial K\\ \gamma\in\partial K'}}\|q-u_h\|_\gamma 2\sqrt{\varrho}h_{K'}^{-1/2}\|\boldsymbol{A}\nabla T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K'}\\
\leqslant\;&\sum_{\substack{\gamma\in\mathcal{E}_h^D\cap\partial K\\ \gamma\in\partial K'}}\|q-u_h\|_\gamma 2\sqrt{\varrho}C_i\rho(\boldsymbol{A}_K)h_{K'}^{-3/2}\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K'}\\
=\;&\sum_{\substack{\gamma\in\mathcal{E}_h^D\cap\partial K\\ \gamma\in\partial K'}}\|q-u_h\|_\gamma 16\sqrt{\varrho}C_i\rho(\boldsymbol{A}_K)h_K^{-3/2}\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K'}\\
\leqslant\;&\sum_{\substack{\gamma\in\mathcal{E}_h^D\cap\partial K\\ \gamma\in\partial K'}}\left(\frac{\epsilon}{2}\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_{K'}^2+\frac{256\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}h_K^{-3}\|q-u_h\|_\gamma^2\right)\\
=\;&\frac{\epsilon}{2}\|T_K(\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H))\|_K^2+\frac{256\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K}h_K^{-3}\|q-u_h\|_\gamma^2\\
\leqslant\;&\frac{\epsilon}{2}\|\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2+\frac{256\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K}h_K^{-3}\|q-u_h\|_\gamma^2. \qquad (3.44)
\end{aligned}
$$

Finally for the term $(\Pi_K^{(1)} f - f, T_K(\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)))_K$, use Cauchy-Schwarz's, (3.39) and Young's inequalities for any $\epsilon > 0$, respectively

$$\begin{aligned}
(\Pi_K^{(1)} f - f, T_K(\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)))_K &\leqslant \|\Pi_K^{(1)} f - f\|_K \|T_K(\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H))\|_K \\
&\leqslant \|\Pi_K^{(1)} f - f\|_K \|\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K \qquad (3.45) \\
&\leqslant \frac{\epsilon}{2}\|\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K^2 + \frac{1}{2\epsilon}\|\Pi_K^{(1)} f - f\|_K^2.
\end{aligned}$$

Now using (3.42), (3.43), (3.44) and (3.45) in (3.41),

$$\begin{aligned}
\frac{9}{40}\|\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{9\epsilon}{2}\|\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K^2 + \frac{16C_i^2}{2\epsilon}\sum_{K'\in\mathcal{P}_{h,K}} h_K^{-2}\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&+ \frac{64\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}\sum_{\gamma\in\mathcal{E}_{h,K}^I} h_K^{-3}\|[u_h]_\gamma\|_\gamma^2 \\
&+ \frac{256\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{2\epsilon}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K} h_K^{-3}\|q - u_h\|_\gamma^2 \\
&+ \frac{1}{2\epsilon}\|\Pi_K^{(1)} f - f\|_K^2. \qquad (3.46)
\end{aligned}$$

Multiply both sides of (3.46) by h_K^2, Lemma A.0.2 and the definition (2.3) give

$$h_K^2\|\Pi_K^{(1)} f - f\|_K^2 \leqslant 2\varrho|K|\|\Pi_K^{(1)} f - f\|_K^2 = 2\varrho\, osc^2(f, K, 1).$$

Also, using the relation $|\gamma| \leqslant \frac{h_K}{4}$ for all $\gamma \in \mathcal{E}_{h,K}^I$ for a small $\epsilon > 0$, we get

$$\begin{aligned}
h_K^2\|\Pi_K^{(1)} f + \mathrm{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{320C_i^2}{\epsilon(9-180\epsilon)}\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&+ \frac{1280\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)}\sum_{\gamma\in\mathcal{E}_{h,K}^I} h_K^{-1}\|[u_h]_\gamma\|_\gamma^2 \\
&+ \frac{5120\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)\epsilon)}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K} h_K^{-1}\|q - u_h\|_\gamma^2 \\
&+ \frac{20}{\epsilon(9-180\epsilon)} h_K^2\|\Pi_K^{(1)} f - f\|_K^2 \\
&\leqslant \frac{320C_i^2}{\epsilon(9-180\epsilon)}\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&+ \frac{320\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)}\sum_{\gamma\in\mathcal{E}_{h,K}^I} |\gamma|^{-1}\|[u_h]_\gamma\|_\gamma^2 \\
&+ \frac{1280\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K} |\gamma|^{-1}\|q - u_h\|_\gamma^2 \\
&+ \frac{40\varrho}{\epsilon(9-180\epsilon)} osc^2(f, K, 1) \qquad (3.47)
\end{aligned}$$

Triangle inequality, Lemma (A.0.2) and definition (2.3) imply

$$\begin{aligned}
h_K^2\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant 2h_K^2\|\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2+2h_K^2\|f-\Pi_K^{(1)}f\|_K^2\\
&\leqslant 2h_K^2\|\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2+4\varrho|K|\|f-\Pi_K^{(1)}f\|_K^2\\
&\leqslant 2h_K^2\|\Pi_K^{(1)}f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2+4\varrho\, osc^2(f,K,1) \qquad (3.48)
\end{aligned}$$

Inequalities (3.47) and (3.48) imply that

$$\begin{aligned}
h_K^2\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{640C_i^2}{\epsilon(9-180\epsilon)}\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+\frac{640\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)}\sum_{\gamma\in\mathcal{E}_{h,K}^I}|\gamma|^{-1}\|[u_h]_\gamma\|_\gamma^2\\
&+\frac{2560\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{\epsilon(9-180\epsilon)}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K}|\gamma|^{-1}\|q-u_h\|_\gamma\\
&+(4\varrho+\frac{80\varrho}{\epsilon(9-180\epsilon)})\, osc^2(f,K,1)
\end{aligned}$$

Corollary 3.1.13 *In Lemma 3.1.8, pick $\epsilon=\frac{1}{40}$, then we have*

■

$$\begin{aligned}
h_K^2\|f+div(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{2^9 10^2 C_i^2}{9}\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+\frac{2^9 10^2\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{9}\sum_{\gamma\in\mathcal{E}_{h,K}^I}|\gamma|^{-1}\|[u_h]_\gamma\|_\gamma^2\\
&+\frac{2^{11}10^2\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{9}\sum_{\gamma\in\mathcal{E}_h^D\cap\partial K}|\gamma|^{-1}\|q-u_h\|_\gamma\\
&+(4\varrho+\frac{2^6 10^2\varrho}{9})\, osc^2(f,K,1) \qquad (3.49)
\end{aligned}$$

Lemma 3.1.14 *Fix $K\in\mathcal{M}_H$. For $\gamma\in\mathcal{E}_K$, the following inequality holds*

$$\begin{aligned}
&\kappa^2\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\\
\leqslant\;& 64(1+\varrho)\left(\frac{2^9 10^2 C_i^2}{9}+368\varrho\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
+\;& 64(1+\varrho)\left(\frac{2^{11}10^2\varrho C_i^2\lambda^2}{9}+368\varrho\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
+\;& 64(1+\varrho)\left(4\varrho+\frac{2^6 10^2\varrho}{9}+368\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\right)\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)\\
+\;& 64(1+\varrho)(240\varrho+1)\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0). \qquad (3.50)
\end{aligned}$$

Proof. By [59] (3.20) we have

$$
\begin{aligned}
&\kappa^2 \sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 \qquad (3.51)\\
\leqslant\;& 64(1+\varrho)\left(\sum_{K\in\mathcal{P}_H} h_K^2\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 + \sum_{\gamma\in\mathcal{E}_H^I}|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 + \sum_{\gamma\in\mathcal{E}_H^N}|\gamma|\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H - g\|_\gamma^2\right)
\end{aligned}
$$

Now from previous chapter remember that by (3.49) we have

$$
\begin{aligned}
h_K^2\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{2^9 10^2 C_i^2}{9}\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ \frac{2^9 10^2\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{9}\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ \frac{2^{11} 10^2\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{9}\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}\\
&+ (4\varrho+\frac{2^6 10^2\varrho}{9})\; osc^2(f,K,1)
\end{aligned}
$$

From this inequality we deduce that

$$
\begin{aligned}
\sum_{K\in\mathcal{P}_H} h_K^2\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 &\leqslant \frac{2^9 10^2 C_i^2}{9}\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ \frac{2^{11} 10^2\varrho C_i^2\lambda^2}{9}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ (4\varrho+\frac{2^6 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H} osc^2(f,K,1) \qquad (3.52)
\end{aligned}
$$

and by (3.31)

$$
\begin{aligned}
|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant 288\varrho\Bigg(\frac{h_K^2}{4}\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2\\
&+ 4C_i^2\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ 4\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ 16\varrho C_i^2\rho(\boldsymbol{A}_K)^2\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2\Bigg). \qquad (3.53)
\end{aligned}
$$

For all $\gamma \in \mathcal{E}_H^I$, 3.53 becomes

$$\begin{aligned}
\sum_{\gamma\in\mathcal{E}_H^I} |\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant 288\varrho\sum_{K\in\mathcal{P}_H}\frac{h_K^2}{4}\|f+\operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ 288\varrho 4C_i^2\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
&+ 288\varrho 16\varrho C_i^2\lambda^2\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&\leqslant 288\varrho\frac{2^7 10^2 C_i^2}{9}\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
&+ 288\varrho\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 288\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H} osc^2(f,K,1) \\
&+ 288\varrho 4C_i^2\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
&+ 288\varrho 16\varrho C_i^2\lambda^2\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&= 288\varrho\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
&+ 288\varrho\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 288\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H} osc^2(f,K,1) \qquad (3.54)
\end{aligned}$$

we deduce following inequality from (3.54)

$$\begin{aligned}
\sum_{\gamma\in\mathcal{E}_H^I} |\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 &\leqslant 288\varrho\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\
&+ 288\varrho\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 288\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H} osc^2(f,K,1) \qquad (3.55)
\end{aligned}$$

Now note that by triangle inequality

$$|\gamma|\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H - g\|_\gamma^2 \leqslant |\gamma|\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H - \Pi_\gamma^{(0)}g\|_\gamma^2 + |\gamma|\|g-\Pi_\gamma^{(0)}g\|_\gamma^2$$

By using (3.38) and (2.4) it can be written that

$$
\begin{aligned}
|\gamma| \|\boldsymbol{n} \cdot \boldsymbol{A} \nabla u_H - g\|_\gamma^2 &\leqslant 80\varrho\Big(\frac{h_K^2}{4}\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ 4C_i^2 \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} |\gamma'|^{-1} \|q - u_h\|_{\gamma'}^2 + 3\varrho osc^2(g, \gamma, 0)\Big) + osc^2(g, \gamma, 0)
\end{aligned}
$$

Summing for all $\gamma \in \mathcal{E}_H^N$ then

$$
\begin{aligned}
\sum_{\gamma \in \mathcal{E}_h^N} |\gamma| \|\boldsymbol{n} \cdot \boldsymbol{A}\nabla u_H - g\|_\gamma^2 &\leqslant 80\varrho\Big(\sum_{K \in \mathcal{P}_H} \frac{h_K^2}{4} \|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2 \\
&+ 4C_i^2 \sum_{K' \in \mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + 16\varrho C_i^2 \lambda^2 \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2\Big) \\
&+ (240\varrho + 1) \sum_{\gamma \in \mathcal{E}_H^N} osc^2(g, \gamma, 0) \\
&\leqslant 80\varrho\Big(\frac{2^7 10^2 C_i^2}{9} \sum_{K' \in \mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&+ \frac{2^9 10^2 \varrho C_i^2 \lambda^2}{9} \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ (\varrho + \frac{2^4 10^2 \varrho}{9}) \sum_{K \in \mathcal{P}_H} osc^2(f, K, 1) \\
&+ 4C_i^2 \sum_{K' \in \mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + 16\varrho C_i^2 \lambda^2 \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2\Big) \\
&+ (240\varrho + 1) \sum_{\gamma \in \mathcal{E}_H^N} osc^2(g, \gamma, 0) \\
&= 80\varrho\left(\frac{2^7 10^2 C_i^2}{9} + 4C_i^2\right) \sum_{K' \in \mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
&+ 80\varrho\left(\frac{2^9 10^2 \varrho C_i^2 \lambda^2}{9} + 16\varrho C_i^2 \lambda^2\right) \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ 80\varrho(\varrho + \frac{2^4 10^2 \varrho}{9}) \sum_{K \in \mathcal{P}_H} osc^2(f, K, 1) \\
&+ (240\varrho + 1) \sum_{\gamma \in \mathcal{E}_H^N} osc^2(g, \gamma, 0)
\end{aligned}
$$

Then we get

$$\sum_{\gamma\in\mathcal{E}_H^N}|\gamma|\|\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H-g\|_\gamma^2 \leqslant 80\varrho\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2$$
$$+\ 80\varrho\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2$$
$$+\ 80\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)$$
$$+\ (240\varrho+1)\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0) \tag{3.56}$$

Using (3.52), (3.55) and (3.56) in (3.51) we get

$$\kappa^2\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2$$
$$\leqslant 64(1+\varrho)\Bigg(\frac{2^9 10^2 C_i^2}{9}\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2$$
$$+\ \frac{2^{11}10^2\varrho C_i^2\lambda^2}{9}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2$$
$$+\ (4\varrho+\frac{2^6 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)$$
$$+\ 288\varrho\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2$$
$$+\ 288\varrho\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2$$
$$+\ 288\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)$$
$$+\ 80\varrho\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2$$
$$+\ 80\varrho\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2$$
$$+\ 80\varrho(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)$$
$$+\ (240\varrho+1)osc^2(g,\gamma,0)\Bigg)$$

$$
\begin{aligned}
= \quad & 64(1+\varrho)\left(\frac{2^9 10^2 C_i^2}{9} + 368\varrho\left(\frac{2^7 10^2 C_i^2}{9} + 4C_i^2\right)\right) \sum_{K' \in \mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
+ \quad & 64(1+\varrho)\left(\frac{2^{11} 10^2 \varrho C_i^2 \lambda^2}{9} + 368\varrho\left(\frac{2^9 10^2 \varrho C_i^2 \lambda^2}{9} + 16\varrho C_i^2 \lambda^2\right)\right) \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
+ \quad & 64(1+\varrho)\left(4\varrho + \frac{2^6 10^2 \varrho}{9} + 368\varrho(\varrho + \frac{2^4 10^2 \varrho}{9})\right) \sum_{K \in \mathcal{P}_H} osc^2(f, K, 1) \\
+ \quad & 64(1+\varrho)(240\varrho + 1) \sum_{\gamma \in \mathcal{E}_H^N} osc^2(g, \gamma, 0)
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
& \kappa^2 \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\
\leqslant \quad & 64(1+\varrho)\left(\frac{2^9 10^2 C_i^2}{9} + 368\varrho\left(\frac{2^7 10^2 C_i^2}{9} + 4C_i^2\right)\right) \sum_{K' \in \mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
+ \quad & 64(1+\varrho)\left(\frac{2^{11} 10^2 \varrho C_i^2 \lambda^2}{9} + 368\varrho\left(\frac{2^9 10^2 \varrho C_i^2 \lambda^2}{9} + 16\varrho C_i^2 \lambda^2\right)\right) \sum_{\gamma' \in \mathcal{E}_h^I \cup \mathcal{E}_h^D} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
+ \quad & 64(1+\varrho)\left(4\varrho + \frac{2^6 10^2 \varrho}{9} + 368\varrho(\varrho + \frac{2^4 10^2 \varrho}{9})\right) \sum_{K \in \mathcal{P}_H} osc^2(f, K, 1) \\
+ \quad & 64(1+\varrho)(240\varrho + 1) \sum_{\gamma \in \mathcal{E}_H^N} osc^2(g, \gamma, 0).
\end{aligned}
$$

■

Now note that in (3.31) we have the term $\frac{h_K^2}{4}\|f + \operatorname{div}(\boldsymbol{A}\nabla u_H)\|_K^2$. By using (3.49) we get following bound for (3.31).

$$
\begin{aligned}
|\gamma|\|[\boldsymbol{n}\cdot\boldsymbol{A}\nabla u_H]_\gamma\|_\gamma^2 \quad \leqslant \quad & 288\varrho\Bigg(\frac{2^7 10^2 C_i^2}{9} \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\
+ \quad & \frac{2^7 10^2 \varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{9} \sum_{\gamma' \in \mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
+ \quad & \frac{2^9 10^2 \varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{9} \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} |\gamma'|^{-1} \|q - u_h\|_{\gamma'} \\
+ \quad & (\varrho + \frac{2^4 10^2 \varrho}{9})\, osc^2(f, K, 1) \\
+ \quad & 4C_i^2 \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ \quad 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} |\gamma'|^{-1} \|q - u_h\|_{\gamma'}^2 \Bigg)
\end{aligned}
$$

Also if $\gamma \in \mathcal{E}_K \cap \mathcal{E}_H^N$, we have a contributions from Neumann boundary, too. When we look at (3.38), we see that (3.31) has greater upper bound than (3.38). Also both of the inequalities include same terms except the data oscillation term. So we have just a data oscillation from Neumann boundary. Then for all $\gamma \in \partial \mathcal{P}_H$ we have following bound for (3.31) ,

$$
\begin{aligned}
|\gamma| \|[\boldsymbol{n} \cdot \boldsymbol{A} \nabla u_H]_\gamma\|_\gamma^2 \quad &\leqslant \quad 288\varrho \Bigg(\frac{2^7 10^2 C_i^2}{9} \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A} \nabla (u_h - u_H)\|_{K'}^2 \\
&+ \quad \frac{2^7 10^2 \varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{9} \sum_{\gamma' \in \mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ \quad \frac{2^9 10^2 \varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{9} \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} |\gamma'|^{-1} \|q - u_h\|_{\gamma'} \\
&+ \quad (\varrho + \frac{2^4 10^2 \varrho}{9})\, osc^2(f, K, 1) \\
&+ \quad 4C_i^2 \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A} \nabla (u_h - u_H)\|_{K'}^2 \\
&+ \quad 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ \quad 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} |\gamma'|^{-1} \|q - u_h\|_{\gamma'}^2 \Bigg) \\
&+ \quad 240\varrho^2 osc^2(g, \gamma, 0)
\end{aligned}
$$

If we rearrange the last inequality we get,

$$
\begin{aligned}
|\gamma| \|[\boldsymbol{n} \cdot \boldsymbol{A} \nabla u_H]_\gamma\|_\gamma^2 \quad &\leqslant \quad 288\varrho \Bigg(\left(\frac{2^7 10^2 C_i^2}{9} + 4C_i^2 \right) \sum_{K' \in \mathcal{P}_{h,K}} \|\boldsymbol{A} \nabla (u_h - u_H)\|_{K'}^2 \\
&+ \quad \left(\frac{2^7 10^2 \varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{9} + 4\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \right) \sum_{\gamma' \in \mathcal{E}_{h,K}^I} |\gamma'|^{-1} \|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+ \quad \left(\frac{2^9 10^2 \varrho C_i^2 \rho(\boldsymbol{A}_K)^2}{9} + 16\varrho C_i^2 \rho(\boldsymbol{A}_K)^2 \right) \sum_{\gamma' \in \mathcal{E}_h^D \cap \partial K} |\gamma'|^{-1} \|q - u_h\|_{\gamma'} \\
&+ \quad (\varrho + \frac{2^4 10^2 \varrho}{9})\, osc^2(f, K, 1) \Bigg) + 240\varrho^2 osc^2(g, \gamma, 0)
\end{aligned}
\tag{3.57}
$$

Using (3.57) into (3.19) give ,

$$\begin{aligned}
\Phi_K^2 \quad &\leqslant \quad 9\rho(\boldsymbol{A}_K^{-1})\,\varrho\,288\varrho\Bigg(\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_{h,K}}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ \quad \left(\frac{2^7 10^2\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{9}+4\varrho C_i^2\rho(\boldsymbol{A}_K)^2\right)\sum_{\gamma'\in\mathcal{E}_{h,K}^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ \quad \left(\frac{2^9 10^2\varrho C_i^2\rho(\boldsymbol{A}_K)^2}{9}+16\varrho C_i^2\rho(\boldsymbol{A}_K)^2\right)\sum_{\gamma'\in\mathcal{E}_h^D\cap\partial K}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}\\
&+ \quad (\varrho+\frac{2^4 10^2\varrho}{9})\,osc^2(f,K,1)\Bigg)+9\rho(\boldsymbol{A}_K^{-1})\,\varrho\,240\varrho^2\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^N}osc^2(g,\gamma,0)\\
&+ \quad 9\rho(\boldsymbol{A}_K^{-1})\varrho\delta(\gamma)\kappa^2\sum_{\gamma\in\mathcal{E}_K}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\\
&+ \quad \frac{6\varrho}{\pi^2}\rho(\boldsymbol{A}_K^{-1})osc^2(f,K,0)+\frac{24\varrho}{\pi}\rho(\boldsymbol{A}_K^{-1})\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^N}osc^2(g,\gamma,0) \qquad (3.58)
\end{aligned}$$

Above inequality implies that for a marking triangle K, we have given upper bound. For all marking triangle K, one can reach the following upper bound. Before doing this, let us introduce some definitions.

Let $\lambda_{-1}=\max\,\{\rho(\boldsymbol{A}_K^{-1})|K\in\mathcal{P}_{H,0}\}$ and let $\lambda=\max\,\{\rho(\boldsymbol{A}_K)|K\in\mathcal{P}_{H,0}\}$ where $\mathcal{P}_{H,0}$ is initial mesh. Let $\delta_{\max}=\max\{\delta(\gamma)\ |\gamma\in\mathcal{E}_K, K\in\mathcal{P}_H\}$. Then, from (3.58) we can conclude that,

$$\begin{aligned}
\sum_{K\in\mathcal{M}_H}\Phi_K^2 \quad &\leqslant \quad 2592\lambda_{-1}\,\varrho^2\left(\left(\frac{2^7 10^2 C_i^2}{9}+4C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\right.\\
&+ \quad \left(\frac{2^7 10^2\varrho C_i^2\lambda^2}{9}+4\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ \quad \left.\left(\frac{2^9 10^2\varrho C_i^2\lambda^2}{9}+16\varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^D}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}\right)\\
&+ \quad 9\lambda_{-1}\varrho\delta_{\max}\kappa^2\sum_{\gamma\in\mathcal{E}_H^I}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2+9\lambda_{-1}\varrho\delta_{\max}\kappa^2\sum_{\gamma\in\mathcal{E}_H^D}|\gamma|^{-1}\|q-u_H\|_\gamma^2\\
&+ \quad 2592\lambda_{-1}\,\varrho^2\,(\varrho+\frac{2^4 10^2\varrho}{9})\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)+2160\lambda_{-1}\varrho^3\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0)\\
&+ \quad \frac{6\varrho}{\pi^2}\lambda_{-1}\sum_{K\in\mathcal{P}_H}osc^2(f,K,0)+\frac{24\varrho}{\pi}\lambda_{-1}\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0)
\end{aligned}$$

Consequently, rearranging the above inequality and doing some elementary computation, we arrive the inequality (3.18).

$$\begin{aligned}
\sum_{K\in\mathcal{M}_H} \Phi_K^2 \;\leqslant\;& 288\lambda_{-1}\,\varrho^2\left(2^7 10^2 C_i^2 + 2^2 3^2 C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+\; 288\lambda_{-1}\,\varrho^2\left(2^7 10^2 \varrho C_i^2\lambda^2 + 2^2 3^2 \varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+\; 288\lambda_{-1}\,\varrho^2\left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2 \varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^D}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2\\
&+\; 9\lambda_{-1}\varrho\delta_{\max}\kappa^2\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\\
&+\; 288\lambda_{-1}\,\varrho^2\,(3^2\varrho + 2^4 10^2\varrho)\sum_{K\in\mathcal{P}_H} osc^2(f,K,1) + 2160\lambda_{-1}\varrho^3\sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0)\\
&+\; \frac{6\varrho}{\pi^2}\lambda_{-1}\sum_{K\in\mathcal{P}_H} osc^2(f,K,0) + \frac{24\varrho}{\pi}\lambda_{-1}\sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0) \qquad (3.59)
\end{aligned}$$

In the above inequality since we have a term which is dependent on the penalty parameter we should also bound this term with the difference between fine mesh and coarse mesh for the guarantied error reduction.

Plugging the inequality (3.50) into (3.59) allow us to write

$$\begin{aligned}
&\sum_{K\in\mathcal{M}_H} \Phi_K^2\\
\leqslant\;& 288\lambda_{-1}\,\varrho^2\left(2^7 10^2 C_i^2 + 2^2 3^2 C_i^2\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
+\;& 288\lambda_{-1}\,\varrho^2\left(2^7 10^2 \varrho C_i^2\lambda^2 + 2^2 3^2 \varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
+\;& 288\lambda_{-1}\,\varrho^2\left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2 \varrho C_i^2\lambda^2\right)\sum_{\gamma'\in\mathcal{E}_h^D}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2\\
+\;& 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^9 10^2 C_i^2 + 368\varrho\left(2^7 10^2 C_i^2 + 36 C_i^2\right)\right)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
+\;& 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^{11} 10^2 \varrho C_i^2\lambda^2 + 368\varrho\left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2 \varrho C_i^2\lambda^2\right)\right)\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
+\;& 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^2 3^2\varrho + 2^6 10^2\varrho + 368\varrho(3^2\varrho + 2^4 10^2\varrho)\right)\sum_{K\in\mathcal{P}_H} osc^2(f,K,1)\\
+\;& 64(1+\varrho)\lambda_{-1}9\varrho\delta_{\max}(240\varrho+1)\sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0)\\
+\;& 288\lambda_{-1}\,\varrho^2\,(3^2\varrho + 2^4 10^2\varrho)\sum_{K\in\mathcal{P}_H} osc^2(f,K,1) + 2160\lambda_{-1}\varrho^3\sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0)\\
+\;& \frac{6\varrho}{\pi^2}\lambda_{-1}\sum_{K\in\mathcal{P}_H} osc^2(f,K,0) + \frac{24\varrho}{\pi}\lambda_{-1}\sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0)
\end{aligned}$$

Estimation of the Nonconforming Part:

Theorem 3.1.15 *Let $u \in H^1(\Omega)$ such that $u = q$ on Γ_D and $u_H \in X_H$ be the solution of (2.1) and its DG approximations (2.7) with respect to $\mathcal{P}_H$. Then, there holds*

$$\sum_{K\in\mathcal{M}_H} \Psi_K^2 \leqslant 4C_i^2\lambda \sum_{\gamma\in\mathcal{E}_H^I} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 + 4C_i^2\lambda \sum_{\gamma\in\mathcal{E}_H^D} |\gamma|^{-1}\|q-u_H\|_\gamma^2 + (4C^2+6)\sum_{\gamma\in\mathcal{E}_H^D} osc^2(q,\gamma) \tag{3.60}$$

where for a given initial mesh $\mathcal{P}_{H,0}$, we define $\lambda = \max\{\rho(\boldsymbol{A}_K)|K \in \mathcal{P}_{H,0}\}$ and C is an constant which is independent of the size of the elements in the mesh $\mathcal{P}_H$.

Proof. Estimator for the nonconforming part given as following

$$\Psi_K = |||u_H - \mathcal{S}(u_H)|||_K + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^D} \inf_{\substack{v\in H_\gamma^1(K)\\ v_{|\gamma}=q-q_{I,H}}} |||v|||_K$$

where

$$\mathcal{S}(u_H)(x_m) = \begin{cases} q_{I,H}(x_m) & \text{if } m \in \mathcal{V}_D(\mathcal{T}_K) \\ u_{H|K}(x_m) & \text{if } m \in \mathcal{V}(\mathcal{T}_K)\setminus\mathcal{V}_{\partial K}(\mathcal{T}_K) \\ \frac{1}{\#\Omega_m}\sum_{K'\in\Omega_m} u_{H|K'}(x_m) & \text{if } m \in \mathcal{V}\cap\mathcal{V}_{\partial K}(\mathcal{T}_K)\setminus\mathcal{V}_D(\mathcal{T}_K) \end{cases}$$

In [4], it was proved that

$$\inf_{\substack{v\in H_\gamma^1(K)\\ v_{|\gamma}=q-q_I}} |||v|||_K \leqslant Cosc(q,\gamma).$$

Then it can be written that,

$$\Psi_K^2 \leqslant 2|||u_H - \mathcal{S}(u_H)|||_K^2 + 4C^2 \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_H^D} osc^2(q,\gamma)\Big). \tag{3.61}$$

and by [4], we have

$$|||u_H - \mathcal{S}(u_H)|||_K^2 \leqslant \frac{C_i^2\rho(\boldsymbol{A}_K)}{6} \sum_{m\in\mathcal{V}_{\partial K}(\mathcal{T}_K)\cap\mathcal{V}} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2 \tag{3.62}$$

Proof of (3.62): For every element $K \in \mathcal{P}_H$, $A_K = A_{|K} \in \mathbb{R}^{2\times 2}$, there exist positive constant $\rho(\boldsymbol{A}_K)$ satisfying

$$|||v|||_K = (\boldsymbol{A}\nabla v, \nabla v)_K \leqslant \|A\|_K\|\nabla v\|_K^2 \leqslant \rho(\boldsymbol{A}_K)\|\nabla v\|_K^2 \qquad \forall v \in H^1(K). \tag{3.63}$$

Using (3.63) and applying the Markov inequality (2.14) give

$$|||u_H - \mathcal{S}(u_H)|||_K^2 \leqslant \rho(\boldsymbol{A}_K)\|\nabla(u_H - \mathcal{S}(u_H))\|_K^2 \leqslant C_i^2\rho(\boldsymbol{A}_K)h_K^{-2}\|u_H - \mathcal{S}(u_H)\|_K^2$$

and evaluating this integral, using the following quadrature rule based on edge midpoints which is exact for quadratic functions, see Lemma A.0.7,

$$\int_K f = \frac{|K|}{3}(f(m_{\gamma_1}) + f(m_{\gamma_1}) + f(m_{\gamma_3}))$$

where m_{γ_i} $i = 1, 2, 3$ refer to midpoints of the edges γ_i of the triangle K. Applying this quadrature rule, we have

$$\|u_H - \mathcal{S}(u_H)\|_K^2 = \frac{1}{3}|K| \sum_{\gamma \in \mathcal{E}_K} |u_H(m_\gamma) - \mathcal{S}(u_H)(m_\gamma)|^2.$$

The restriction of $(u_{H|K} - \mathcal{S}(u_H))$ to an edge $\gamma \subset \partial K$ is a linear function of arc length, which means that the value at the midpoint m_γ is the average value of the values at the endpoints of the edge, and therefore,

$$|u_H(m_\gamma) - \mathcal{S}(u_H)(m_\gamma)| = \frac{1}{2} \sum_{m \in \mathcal{V}(\gamma)} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|.$$

where $\mathcal{V}(\gamma)$ index the vertices at the endpoints of γ. This implies

$$|u_H(m_\gamma) - \mathcal{S}(u_H)(m_\gamma)|^2 \leqslant \frac{1}{2} \sum_{m \in \mathcal{V}(\gamma)} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2.$$

Then

$$\sum_{\gamma \in \mathcal{E}_K} |u_H(m_\gamma) - \mathcal{S}(u_H)(m_\gamma)|^2 \leqslant \sum_{m \in \mathcal{V}(K)} u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2$$

where $\mathcal{V}(K)$ index the set of the vertices of element K. Therefore,

$$|||u_H - \mathcal{S}(u_H)|||_K^2 \leqslant \frac{C_i^2\rho(\boldsymbol{A}_K)}{3} h_K^{-2}|K| \sum_{m \in \mathcal{V}(K)} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2 \tag{3.64}$$

and by elementary computation it can be shown that $h_K^{-2}|K|$ is bounded by $1/2$ such that for an arbitrary triangle K. Indeed: $|K| = \frac{h_K a}{2} \leqslant \frac{h_K^2}{2}.$

Then (3.64) becomes,

$$\begin{aligned}
|||u_H - \mathcal{S}(u_H)|||_K^2 &\leqslant \frac{C_i^2\rho(\boldsymbol{A}_K)}{6} \sum_{m \in \mathcal{V}(K)} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2 \\
&\leqslant \frac{C_i^2\rho(\boldsymbol{A}_K)}{6} \sum_{m \in \mathcal{V}(\mathcal{T}_K)} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2. \\
&= \frac{C_i^2\rho(\boldsymbol{A}_K)}{6} \sum_{m \in \mathcal{V}_{\partial K}(\mathcal{T}_K)} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2. \\
&\leqslant \frac{C_i^2\rho(\boldsymbol{A}_K)}{6} \sum_{m \in \mathcal{V}_{\partial K}(\mathcal{T}_K) \cap \mathcal{V}} |u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2 \qquad (3.65)
\end{aligned}$$

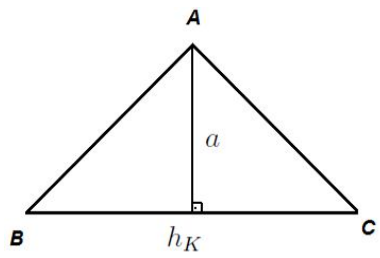


Figure 3.8: Relation between area and edge

Now, one needs to find an upper bound for $|u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2$. **Case (i):** First consider the case when $m \notin \mathcal{V}_D(\mathcal{T}_K)$, i.e., x_m is not a vertex lying on the Dirichlet boundary. Upon observing that

$$\sum_{K' \in \Omega_m} \frac{1}{\#\Omega_m} = 1,$$

we can obtain following equality after inserting the definition of $\mathcal{S}(u_H)(x_m)$

$$u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m) = \begin{cases} 0 & \text{if } m \in \mathcal{V}(\mathcal{T}_K) \setminus \mathcal{V}_{\partial K}(\mathcal{T}_K) \\ \frac{1}{\#\Omega_m} \sum_{K' \in \Omega_m \setminus K} (u_{H|K}(x_m) - u_{H|K'}(x_m)) & \text{if } m \in \mathcal{V} \cap \mathcal{V}_{\partial K}(\mathcal{T}_K) \setminus \mathcal{V}_D(\mathcal{T}_K) \end{cases}.$$

Note that we removed K from the set Ω_m, because if $K \in \Omega_m$, then $\mathcal{S}(u_H)(x_m) = u_{H|K}(x_m)$ so this implies $u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m) = 0$. Then,

$$u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m) = \frac{1}{\#\Omega_m} \sum_{K' \in \Omega_m \setminus K} (u_{H|K}(x_m) - u_{H|K'}(x_m)).$$

The above equality implies that

$$|u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)| \leqslant \frac{1}{\#\Omega_m} \sum_{K' \in \Omega_m \setminus K} |\left(u_{H|K}(x_m) - u_{H|K'}(x_m)\right)|.$$

Now, if $x_m \not\subset \overline{\Gamma_D}$ is a point of the closure of one element $K \in \mathcal{P}_H$ only then last inequality implies $|u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)| = 0$. Therefore, let x_m be a common point of the closure of more than one element in $\mathcal{P}_H$. We shall first bound the contribution $|u_{H|K}(x_m) - u_{H|K'}(x_m)|$ from elements $K, K' \in \Omega_m$ whose boundaries both contain an edge $\gamma \in \partial\mathcal{P}_H$. In this case by definition of jump and trace inverse estimates which is defined in (2.13) we can write that

$$\begin{aligned} |u_{H|K}(x_m) - u_{H|K'}(x_m)| = |[u_H]_\gamma(x_m)| &\leqslant \|[u_H]_\gamma\|_{L_\infty(\gamma)} \\ &\leqslant 2|\gamma|^{-1/2}\|[u_H]_\gamma\|_\gamma. \end{aligned}$$

Then

$$|u_{H|K}(x_m) - u_{H|K'}(x_m)| \leqslant 2|\gamma|^{-1/2}\|[u_H]_\gamma\|_\gamma.$$

This relation is valid for pairs of elements sharing a common edge γ. If the closure of elements K and K' consists of only the common point x_m then we can then write $|u_{H|K}(x_m)-u_{H|K'}(x_m)|$ as a telescoping sum of the jumps in u_H across neighboring edges, from which we can use the previous inequality to obtain

$$|u_{H|K}(x_m) - u_{H|K'}(x_m)| \leqslant 2 \sum_{\substack{\gamma\in\mathcal{E}_H^I \\ x_m\subset\overline{\gamma}}} |\gamma|^{-1/2}\|[u_H]_\gamma\|_\gamma$$

Case(ii): If $x_m \in \mathcal{V}_D(\mathcal{T}_K)$. Let x_m be an endpoint of an edge $\gamma \in \mathcal{E}_K \cap \mathcal{E}_H^D$ for an element $K' \in \Omega_m$ where it is allowed for the possibility that $K' = K$ and observe that

$$|u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)| \leqslant |u_{H|K}(x_m) - u_{H|K'}(x_m)| + |u_{H|K'}(x_m) - q_I(x_m)|$$

The first term on the right-hand side of the above inequality can be bounded using previous step and we can also write that

$$\begin{aligned} |u_{H|K'}(x_m) - q_I(x_m)| &\leqslant \|u_H - q_I\|_{L_\infty(\gamma)} \\ &\leqslant 2|\gamma|^{-1/2}\|u_H - q_I\|_\gamma \\ &\leqslant 2|\gamma|^{-1/2}\left(\|u_H - q\|_\gamma + \|q - q_I\|_\gamma\right) \end{aligned} \tag{3.66}$$

to bound second term, In [77] by Corollary 3.15, it can be written that

$$\|q - q_I\|_\gamma \leqslant \frac{1}{\sqrt{2}}|\gamma|^{1/2} osc(q,\gamma).$$

For $m \in \mathcal{V}_{\partial K}(\mathcal{T}_K) \cap \mathcal{V}$ for $K \in \Omega_m$, we can write

$$|u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)| \leqslant 2 \sum_{\substack{\gamma\in\mathcal{E}_H^I: \\ x_m\subset\overline{\gamma}}} |\gamma|^{-1/2}\|[u_H]\|_\gamma + 2 \sum_{\substack{\gamma\in\mathcal{E}_H^D: \\ x_m\subset\overline{\gamma}}} |\gamma|^{-1/2}\|u_H - q\|_\gamma + \sqrt{2} \sum_{\substack{\gamma\in\mathcal{E}_H^D: \\ x_m\subset\overline{\gamma}}} osc(q,\gamma).$$

This implies,

$$|u_{H|K}(x_m) - \mathcal{S}(u_H)(x_m)|^2 \leqslant 12 \sum_{\substack{\gamma\in\mathcal{E}_H^I: \\ x_m\subset\overline{\gamma}}} |\gamma|^{-1}\|[u_H]\|_\gamma^2 + 12 \sum_{\substack{\gamma\in\mathcal{E}_H^D: \\ x_m\subset\overline{\gamma}}} |\gamma|^{-1}\|u_H - q\|_\gamma^2 + 6 \sum_{\substack{\gamma\in\mathcal{E}_H^D: \\ x_m\subset\overline{\gamma}}} osc^2(q,\gamma) \tag{3.67}$$

Combining the estimates (3.61), (3.62) and (3.67) gives,

$$\begin{aligned} \Psi_K^2 &\leqslant 4C_i^2\rho(\boldsymbol{A}_K) \sum_{\gamma\in\tilde{\mathcal{E}}_K} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 + 4C_i^2\rho(\boldsymbol{A}_K) \sum_{\gamma\in\tilde{\mathcal{E}}_K\cap\mathcal{E}_H^D} |\gamma|^{-1}\|u_H - q\|_\gamma^2 \\ &+ (4C^2 + 6) \sum_{\gamma\in\tilde{\mathcal{E}}_K\cap\mathcal{E}_H^D} osc^2(q,\gamma), \end{aligned}$$

$\tilde{\mathcal{E}}_K$ denote the set of edges having any vertex of K as an endpoint such that

$$\tilde{\mathcal{E}}_K = \{\gamma \in \partial\mathcal{P}_H : \overline{\gamma} \cap \overline{K}\}.$$

Define $\lambda = \max\{\rho(\boldsymbol{A}_K) | K \in \mathcal{P}_{H,0}\}$ where $\mathcal{P}_{H,0}$ is a given initial mesh. Then, for all $K \in \mathcal{M}_H$, we get (3.60)

$$\begin{aligned} \sum_{K\in\mathcal{M}_H} \Psi_K^2 &\leqslant 4C_i^2\lambda \sum_{\gamma\in\mathcal{E}_H^I} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 + 4C_i^2\lambda \sum_{\gamma\in\mathcal{E}_H^D} |\gamma|^{-1}\|q-u_H\|_\gamma^2 \\ &+ (4C^2+6) \sum_{\gamma\in\tilde{\mathcal{E}}_K\cap\mathcal{E}_H^D} osc^2(q,\gamma) \end{aligned}$$

■

By collecting the estimates (3.18) and (3.60) we have,

$$\begin{aligned} &\sum_{K\in\mathcal{M}_H} (\Phi_K^2 + \Psi_K^2) \\ \leqslant\ & 288\lambda_{-1}\,\varrho^2 \left(2^7 10^2 C_i^2 + 2^2 3^2 C_i^2\right) \sum_{K'\in\mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 \\ +\ & 288\lambda_{-1}\,\varrho^2 \left(2^7 10^2 \varrho C_i^2\lambda^2 + 2^2 3^2 \varrho C_i^2 \lambda^2\right) \sum_{\gamma'\in\mathcal{E}_h^I} |\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\ +\ & 288\lambda_{-1}\,\varrho^2 \left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2 \varrho C_i^2 \lambda^2\right) \sum_{\gamma'\in\mathcal{E}_h^D} |\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2 \\ +\ & 4C_i^2\lambda \sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 \\ +\ & 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^9 10^2 C_i^2 + 368\varrho\left(2^7 10^2 C_i^2 + 36 C_i^2\right)\right) \sum_{K'\in\mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2 \\ +\ & 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^{11}10^2\varrho C_i^2\lambda^2 + 368\varrho\left(2^9 10^2 \varrho C_i^2\lambda^2 + 2^4 3^2\varrho C_i^2\lambda^2\right)\right) \sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} |\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\ +\ & 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^2 3^2\varrho + 2^6 10^2\varrho + 368\varrho(3^2\varrho + 2^4 10^2\varrho)\right) \sum_{K\in\mathcal{P}_H} osc^2(f,K,1) \\ +\ & 64(1+\varrho)\lambda_{-1}9\varrho\delta_{\max}(240\varrho+1) \sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0) \\ +\ & 288\lambda_{-1}\,\varrho^2\,(3^2\varrho + 2^4 10^2\varrho) \sum_{K\in\mathcal{P}_H} osc^2(f,K,1) + \frac{6\varrho}{\pi^2}\lambda_{-1} \sum_{K\in\mathcal{P}_H} osc^2(f,K,0) \\ +\ & \left(\frac{24\varrho}{\pi}\lambda_{-1} + 2160\lambda_{-1}\varrho^3\right) \sum_{\gamma\in\mathcal{E}_H^N} osc^2(g,\gamma,0) \\ +\ & (4C^2+6) \sum_{\gamma\in\mathcal{E}_H^D} osc^2(q,\gamma) \end{aligned}$$

For simplicity let us write last inequality by following way

$$\begin{aligned}
\sum_{K\in\mathcal{M}_H}(\Phi_K^2+\Psi_K^2) &\leqslant A_1\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2+A_2\sum_{\gamma'\in\mathcal{E}_h^I}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ A_3\sum_{\gamma'\in\mathcal{E}_h^D}|\gamma'|^{-1}\|q-u_h\|_{\gamma'}^2+A_4\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\\
&+ A_5\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2+A_6\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ A_7\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)+A_8\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0)\\
&+ A_9\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)+A_{10}\sum_{K\in\mathcal{P}_H}osc^2(f,K,0)\\
&+ A_{11}\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0)+A_{12}\sum_{\gamma\in\mathcal{E}_H^D}osc^2(q,\gamma)\\
&\leqslant (A_1+A_5)\sum_{K'\in\mathcal{P}_h}\|\boldsymbol{A}\nabla(u_h-u_H)\|_{K'}^2\\
&+ (A_3+A_6)\sum_{\gamma'\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\\
&+ A_4\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\\
&+ (A_7+A_9)\sum_{K\in\mathcal{P}_H}osc^2(f,K,1)\\
&+ (A_8+A_{11})\sum_{\gamma\in\mathcal{E}_H^N}osc^2(g,\gamma,0)\\
&+ A_{10}\sum_{K\in\mathcal{P}_H}osc^2(f,K,0)+A_{12}\sum_{\gamma\in\mathcal{E}_H^D}osc^2(q,\gamma)
\end{aligned}\tag{3.68}$$

where

$$\begin{aligned}
A_1 &= 288\lambda_{-1}\varrho^2\left(2^7 10^2 C_i^2+2^2 3^2 C_i^2\right)\\
A_2 &= 288\lambda_{-1}\varrho^2\left(2^7 10^2\varrho C_i^2\lambda^2+2^2 3^2\varrho C_i^2\lambda^2\right)\\
A_3 &= 288\lambda_{-1}\varrho^2\left(2^9 10^2\varrho C_i^2\lambda^2+2^4 3^2\varrho C_i^2\lambda^2\right)\\
A_4 &= 4C_i^2\lambda\\
A_5 &= 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^9 10^2 C_i^2+368\varrho\left(2^7 10^2 C_i^2+36C_i^2\right)\right)\\
A_6 &= 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^{11}10^2\varrho C_i^2\lambda^2+368\varrho\left(2^9 10^2\varrho C_i^2\lambda^2+2^4 3^2\varrho C_i^2\lambda^2\right)\right)\\
A_7 &= 64(1+\varrho)\lambda_{-1}\varrho\delta_{\max}\left(2^2 3^2\varrho+2^6 10^2\varrho+368\varrho(3^2\varrho+2^4 10^2\varrho)\right)\\
A_8 &= 64(1+\varrho)\lambda_{-1}9\varrho\delta_{\max}(240\varrho+1)\\
A_9 &= 288\lambda_{-1}\varrho^2\,(3^2\varrho+2^4 10^2\varrho)
\end{aligned}$$

$$
\begin{aligned}
A_{10} &= \frac{6\varrho}{\pi^2}\lambda_{-1} \\
A_{11} &= \left(\frac{24\varrho}{\pi}\lambda_{-1} + 2160\lambda_{-1}\varrho^3\right) \\
A_{12} &= (4C^2 + 6)
\end{aligned}
$$

3.2 Proof of Error Reduction :

Lemma 3.2.1 (Karakashian, Pascal,2007) *Following inequality holds for $\mathcal{P}_H$ for sufficently large κ*

$$B_H(e_H, e_H) \geqslant \frac{1}{2}\sum_{K\in\mathcal{P}_H} \|\!|\nabla e_H\|\!|_K^2 + \left(\frac{\kappa^2}{4C_1} - \kappa - \frac{224\varrho}{3} - \frac{\lambda\varrho}{4}\right) \sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 \quad (3.69)$$

where $C_1 = 64(1+\varrho)\left(\frac{2^2 10^2 C_i^2\lambda}{9^2} + \frac{11^2 C_i^2\lambda + 40C_i^4\lambda + 9^2\lambda}{72}\right)$ *and* $\lambda = \max\{\,\rho(\boldsymbol{A}_K)|\, K \in \mathcal{P}_{H,0}\}$.

Proof. Proof is given in Lemma A.0.9. ■

Before engaging the proof, let

$$osc^2 := \sum_{K\in\mathcal{P}_H} osc^2(f, K, 0) + osc^2(f, K, 1) + \sum_{K\in\mathcal{E}_H^N} osc^2(g, \gamma, 0) + \sum_{K\in\mathcal{E}_H^D} osc^2(q, \gamma).$$

By coercivity of bilinear form in X_h we have

$$B_h(u_h - u_H) \geqslant \frac{1}{2}\sum_{K\in\mathcal{P}_h} \|A\nabla(u_h - u_H)\|_K^2 + \frac{1}{2}\sum_{\gamma\in\mathcal{E}_h^I\cup\mathcal{E}_H^D} |\gamma|^{-1}\|[u_h - u_H]\|_\gamma^2. \quad (3.70)$$

Proof can be found in [74].

Also by (3.68), it can be written that

$$
\begin{aligned}
\sum_{K\in\mathcal{M}_H} (\Phi_K^2 + \Psi_K^2) \;&\leqslant\; \underbrace{(A_1 + A_5)}_{=\tilde{A_1}} \sum_{K'\in\mathcal{P}_h} \|\boldsymbol{A}\nabla(u_h - u_H)\|_{K'}^2 + \underbrace{(A_3 + A_6)}_{=\tilde{A_2}} \sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} |\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2 \\
&+\; A_4 \sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} |\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2 + \underbrace{(A_7 + A_9)}_{=\tilde{A_3}}\, osc^2\,. \quad (3.71)
\end{aligned}
$$

Now by (3.9) we have

$$B_H(e_H, e_H) + \kappa(\delta_{\max} - 1) \sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D} |\gamma|^{-1}\|[u_H]\|_\gamma^2 \geqslant B_h(e_H, e_H).$$

Using (3.8), (3.70), (3.71) and marking strategy (3.17), respectively we get

$$\begin{aligned}
B_H(e_H,e_H) &+ \kappa(\delta_{\max}-1)\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]\|_\gamma^2 \geqslant B_h(e_H,e_H)\\
&= B_h(e_h,e_h)+B_h(u_h-u_H,u_h-u_H)\\
&\geqslant B_h(e_h,e_h)+\frac{1}{2}\sum_{K\in\mathcal{P}_h}\|A\nabla(u_h-u_H)\|_K^2\\
&\geqslant B_h(e_h,e_h)+\frac{1}{2\tilde{A}_1}\sum_{K\in\mathcal{M}_H}(\Phi_K^2+\Psi_K^2)-\frac{1}{2}\left(\frac{\tilde{A}_2}{\tilde{A}_1}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\right.\\
&+\left.\frac{A_4}{\tilde{A}_1}\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2+\frac{\tilde{A}_3}{\tilde{A}_1}\,osc^2\right)\\
&\geqslant B_h(e_h,e_h)+\frac{\theta}{2\tilde{A}_1}\sum_{K\in\mathcal{P}_H}(\Phi_K^2+\Psi_K^2)-\frac{1}{2}\left(\frac{\tilde{A}_2}{\tilde{A}_1}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2\right.\\
&+\left.\frac{A_4}{\tilde{A}_1}\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2+\frac{\tilde{A}_3}{\tilde{A}_1}\,osc^2\right) \qquad (3.72)
\end{aligned}$$

By [59] we have,

$$\kappa\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\leqslant\frac{C_1}{\kappa}\sum_{K\in\mathcal{P}_H}|\!|\!|e_H|\!|\!|_K^2$$

where $C_1=64(1+\varrho)\left(\frac{2^2 10^2 C_i^2\lambda}{9^2}+\frac{11^2C_i^2\lambda+40C_i^4\lambda+9^2\lambda}{72}\right)$. Also by [69] we have

$$\sum_{K\in\mathcal{P}_H}|\!|\!|e_H|\!|\!|_K^2\leqslant\sum_{K\in\mathcal{P}_H}\left(\Phi_K^2+\Psi_K^2\right).$$

So we can write following inequality

$$\kappa\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2\leqslant\frac{C_1}{\kappa}\sum_{K\in\mathcal{P}_H}\left(\Phi_K^2+\Psi_K^2\right). \qquad (3.73)$$

Then inequality (3.72) becomes

$$\begin{aligned}
B_H(e_H,e_H) &+ \frac{C_1(\delta_{\max}-1)}{\kappa}\sum_{K\in\mathcal{P}_H}\left(\Phi_K^2+\Psi_K^2\right)\geqslant B_h(e_h,e_h)+\frac{\theta}{2\tilde{A}_1}\sum_{K\in\mathcal{P}_H}(\Phi_K^2+\Psi_K^2)\\
&-\frac{1}{2}\left(\frac{\tilde{A}_2}{\tilde{A}_1}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2+\frac{A_4}{\tilde{A}_1}\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2+\frac{\tilde{A}_3}{\tilde{A}_1}\,osc^2\right)
\end{aligned}$$

Then for $\kappa>\frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\theta}$ we have following inequality

$$\begin{aligned}
B_H(e_H,e_H) &\geqslant B_h(e_h,e_h)+\left(\frac{\theta}{2\tilde{A}_1}-\frac{C_1(\delta_{\max}-1)}{\kappa}\right)\sum_{K\in\mathcal{P}_H}(\Phi_K^2+\Psi_K^2) \qquad (3.74)\\
&-\frac{1}{2}\left(\frac{\tilde{A}_2}{\tilde{A}_1}\sum_{\gamma'\in\mathcal{E}_h^I\cup\mathcal{E}_h^D}|\gamma'|^{-1}\|[u_h]_{\gamma'}\|_{\gamma'}^2+\frac{A_4}{\tilde{A}_1}\sum_{\gamma\in\mathcal{E}_H^I\cup\mathcal{E}_H^D}|\gamma|^{-1}\|[u_H]_\gamma\|_\gamma^2+\frac{\tilde{A}_3}{\tilde{A}_1}\,osc^2\right)
\end{aligned}$$

By (3.69) (this result holds for a generic mesh), it can be written that

$$\left(\frac{\kappa^2}{4C_1} - \kappa - \frac{224\varrho}{3} - \frac{\lambda\varrho}{4}\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \leqslant B_H(e_H, e_H) \tag{3.75}$$

and

$$\left(\frac{\kappa^2}{4C_1} - \kappa - \frac{224\varrho}{3} - \frac{\lambda\varrho}{4}\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_h]_\gamma \|_\gamma^2 \leqslant B_h(e_h, e_h). \tag{3.76}$$

For simplicity let us define $\left(\frac{\kappa^2}{4C_1} - \kappa - \frac{224\varrho}{3} - \frac{\lambda\varrho}{4}\right) := \alpha$

Using (3.75) and (3.76) into (3.74) we get

$$\begin{aligned} B_H(e_H, e_H) &\geqslant B_h(e_h, e_h) + \left(\frac{\theta}{2\tilde{A}_1} - \frac{C_1(\delta_{\max} - 1)}{\kappa}\right) \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2) \\ &- \frac{1}{2}\left(\frac{\tilde{A}_2}{\tilde{A}_1\alpha} B_h(e_h, e_h) + \frac{A_4}{\tilde{A}_1\alpha} B_H(e_H, e_H) + \frac{\tilde{A}_3}{\tilde{A}_1}\, osc^2\right) \end{aligned}$$

If we rearrange last inequality

$$\begin{aligned} &\left(1 + \frac{A_4}{2\tilde{A}_1\alpha}\right) B_H(e_H, e_H) + \frac{\tilde{A}_3}{2\tilde{A}_1}\, osc^2 \\ &\geqslant \left(1 - \frac{\tilde{A}_2}{2\tilde{A}_1\alpha}\right) B_h(e_h, e_h) + \left(\frac{\theta}{2\tilde{A}_1} - \frac{C_1(\delta_{\max} - 1)}{\kappa}\right) \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2) \end{aligned} \tag{3.77}$$

Now, by (A.13), we have

$$B_H(e_H, e_H) \leqslant 2 \sum_{K \in \mathcal{P}_H} \|\!|\!| e_H |\!|\!|_K^2 + \left(\frac{56\lambda\varrho}{3} + \lambda\varrho + 3\kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2$$

Using (3.73) in the last inequality and recalling

$$\sum_{K \in \mathcal{P}_H} |\!|\!| e_H |\!|\!|_K^2 \leqslant \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2)$$

one gets,

$$\begin{aligned} B_H(e_H, e_H) &\leqslant 2 \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2) + \left(\frac{56\lambda\varrho}{3} + \lambda\varrho + 3\kappa\right) \frac{C_1}{\kappa^2} \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2) \\ &= \left(2 + \left(\frac{56\lambda\varrho}{3} + \lambda\varrho + 3\kappa\right) \frac{C_1}{\kappa^2}\right) \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2) \\ &= \beta \sum_{K \in \mathcal{P}_H} (\Phi_K^2 + \Psi_K^2) \end{aligned} \tag{3.78}$$

where $\beta = \left(2 + \left(\frac{56\lambda\varrho}{3} + \lambda\varrho + 3\kappa\right) \frac{C_1}{\kappa^2}\right)$. Using the last result into (3.77) we get,

$$\begin{aligned} &\left(1 + \frac{A_4}{2\tilde{A}_1\alpha}\right) B_H(e_H, e_H) + \frac{\tilde{A}_3}{2\tilde{A}_1}\, osc^2 \\ \geqslant\; &\left(1 - \frac{\tilde{A}_2}{2\tilde{A}_1\alpha}\right) B_h(e_h, e_h) + \frac{1}{\beta}\left(\frac{\theta}{2\tilde{A}_1} - \frac{C_1(\delta_{\max} - 1)}{\kappa}\right) B_H(e_H, e_H). \end{aligned}$$

Then we have

$$\begin{aligned}
&\left(1+\frac{A_4}{2\tilde{A}_1\alpha}-\frac{1}{\beta}\left(\frac{\theta}{2\tilde{A}_1}-\frac{C_1(\delta_{\max}-1)}{\kappa}\right)\right)B_H(e_H,e_H)+\frac{\tilde{A}_3}{2\tilde{A}_1}\,osc^2 \\
\geqslant\;&\left(1-\frac{\tilde{A}_2}{2\tilde{A}_1\alpha}\right)B_h(e_h,e_h). \qquad (3.79)
\end{aligned}$$

Recall that we have an assumption on κ such that $\kappa > \frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\theta}$. Ultimate goal of the thesis was that

$$B_h(e_h,e_h)\leqslant \eta B_H(e_H,e_H)$$

where $0<\eta<1$. Then following inequality should be satisfied

$$1+\frac{A_4}{2\tilde{A}_1\alpha}-\frac{1}{\beta}\left(\frac{\theta}{2\tilde{A}_1}-\frac{C_1(\delta_{\max}-1)}{\kappa}\right)<1-\frac{\tilde{A}_2}{2\tilde{A}_1\alpha}$$

If we rearrange above inequality we get

$$\frac{A_4}{2\tilde{A}_1\alpha}+\frac{\tilde{A}_2}{2\tilde{A}_1\alpha}+\frac{1}{\beta}\frac{C_1(\delta_{\max}-1)}{\kappa}<\frac{1}{\beta}\frac{\theta}{2\tilde{A}_1}$$

Multiply each side $2\tilde{A}_1$

$$\frac{A_4}{\alpha}+\frac{\tilde{A}_2}{\alpha}+\frac{1}{\beta}\frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\kappa}<\frac{\theta}{\beta}.$$

Now multiply with $\frac{\beta}{\theta}$ and α each side

$$\frac{\beta}{\theta}A_4+\frac{\beta}{\theta}\tilde{A}_2+\frac{\alpha}{\kappa}\frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\theta}<\alpha$$

and take all term right hand side

$$0<\alpha-\frac{\alpha}{\kappa}\frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\theta}-\frac{\beta}{\theta}\tilde{A}_2-\frac{\beta}{\theta}A_4.$$

Recall that

$$\alpha=\frac{\kappa^2}{4C_1}-\kappa-\frac{224\varrho}{3}-\frac{\lambda\varrho}{4}.$$

Then it can be written that

$$\alpha=\kappa^2\left(\frac{1}{4C_1}-\frac{1}{\kappa}-\frac{224\varrho}{3\kappa^2}-\frac{\lambda\varrho}{4\kappa^2}\right).$$

Since we have an assumption on κ such that $\kappa>\frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\theta}$, then it can be easily seen that $0<\frac{1}{4C_1}-\frac{1}{\kappa}-\frac{224\varrho}{3\kappa^2}-\frac{\lambda\varrho}{4\kappa^2}<1$, define this quantity as $\epsilon:=\frac{1}{4C_1}-\frac{1}{\kappa}-\frac{224\varrho}{3\kappa^2}-\frac{\lambda\varrho}{4\kappa^2}$. So we have $\alpha=\epsilon\kappa^2$ where $0<\epsilon<1$. Then we have

$$0<\epsilon\kappa^2-\epsilon\kappa\frac{2\tilde{A}_1C_1(\delta_{\max}-1)}{\theta}-\frac{\beta}{\theta}\tilde{A}_2-\frac{\beta}{\theta}A_4.$$

Divide both side with ϵ,

$$0 < \kappa^2 - \kappa \frac{2\tilde{A_1} C_1 (\delta_{\max} - 1)}{\theta} - \frac{\beta}{\theta\epsilon}\tilde{A_2} - \frac{\beta}{\theta\epsilon} A_4.$$

For simplicity let us define $0 < \frac{2\tilde{A_1} C_1 (\delta_{\max} - 1)}{\theta} := \xi$ and $0 < \frac{\beta}{\theta\epsilon}\tilde{A_2} + \frac{\beta}{\theta\epsilon} A_4 := \mu$, we get

$$0 < \kappa^2 - \kappa\xi - \mu.$$

By solving above second order equation we get

$$\kappa > \frac{\xi + \sqrt{\xi^2 + 4\mu}}{2}$$

Choosing $\kappa > \frac{\xi + \sqrt{\xi^2 + 4\mu}}{2}$, one can guaranteed convergence in adaptive strategy.

CHAPTER 4

ON the CONSTANTS in INVERSE INEQUALITIES in L_2

In this chapter we shall present the exact constants in univariate and multivariate Markov inequality in L_2-norm. Using orthogonal polynomials, we reduce the problem to a simple eigenvalue problem and we establish bounds with known constants for Markov inequalities on an arbitrary 1-simplex, 2-simplex and 3-simplex. In [83], a similar technique is used to obtain the constants in the *hp*-trace inequalities. In this thesis, we obtain the same constant with [77] for a linear polynomial u and smaller constant for higher order polynomials in 1D.

The remainder of this chapter is organized as follows. In Section 4.1, the Markov inequality is proved for one dimensional domain and an explicit constant is found in L_2 norm. Section 4.2 gives us a Markov inequality constant for polynomial in two variables on triangular domain, while in Section 4.3, we give the result for a tetrahedron. Finally, in Section 4.4 we provide a brief conclusion.

4.1 One Dimensional Domain

In this section, we state and prove Markov inequality for a polynomial u total degree N on a finite interval and we give a closed form for the constants up to polynomial degree 4. For the case $5 \leqslant N \leqslant 10$, we give numerical values for this constant.

Theorem 4.1.1 *(Markov Inequality on a finite interval) For an interval $K = [a, b]$ and for a polynomial u of total degree $N = 1, 2, 3, 4$ the following result holds*

$$\|u'\|_{L_2(K)} \leqslant \frac{2\sqrt{C_N}}{b-a}\|u\|_{L_2(K)}$$

where $C_1 = 3$, $C_2 = 15$, $C_3 = \frac{45+\sqrt{1605}}{2}$ and $C_4 = \frac{105+3\sqrt{805}}{2}$.

Proof. Consider the reference interval $\hat{K} = [-1, 1]$ and associate L_2- orthonormal polynomial, the classical Legendre polynomial. The reference interval $\hat{K} = [-1, 1]$ is mapped to interval $K = [a, b]$ by following transformation, which send -1 to a, 1 to b :

$$x = \frac{(b-a)r}{2} + \frac{b+a}{2}$$

where $r \in [-1, 1]$ and $x \in [a, b]$. By chain rule and using a scaling argument

$$\begin{aligned}
\|u'\|_{L_2(K)} &= \left\| \frac{du}{dr}\frac{dr}{dx} \right\|_{L_2(K)} \\
&= \left| \frac{dr}{dx} \right| \left\| \frac{du}{dr} \right\|_{L_2(K)} \\
&= \frac{2}{|b-a|} \left| \frac{b-a}{2} \right|^{1/2} \left\| \frac{du}{dr} \right\|_{L_2(\hat{K})} \\
&\leqslant \frac{2}{|b-a|} \sqrt{C_N} \left| \frac{b-a}{2} \right|^{1/2} \|u\|_{L_2(\hat{K})} \\
&= \frac{2}{|b-a|} \sqrt{C_N} \|u\|_{L_2(K)}
\end{aligned}$$

where C_N can be determined for a given polynomial order N by solving the following eigenvalue problem for the maximum eigenvalue,

$$\left(\frac{d\phi_n}{dr}, \frac{d\phi_m}{dr} \right)_{\hat{K}} u_m = \lambda (\phi_n, \phi_m)_{\hat{K}} u_m$$

Here $\{\phi_n\}_{n=1}^{n=N+1}$ is an orthonormal basis of the reference interval $\hat{K} = [-1, 1]$. Einstein summation is assumed for repeated indices. The L_2 inner product on $\hat{K}$ is denoted by $(\cdot, \cdot)_{\hat{K}}$. Defining $\mathbf{S}_{nm} = \left(\frac{d\phi_n}{dr}, \frac{d\phi_m}{dr} \right)_{\hat{K}}$, $\mathbf{M}_{nm} = (\phi_n, \phi_m)_{\hat{K}}$ and using the orthonormality of the basis give us $\mathbf{M} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix. Then, the above problem reduces to a classical eigenvalue problem

$$\mathbf{S}_{nm} u_m = \lambda u_m.$$

Let C_N be the maximum eigenvalue λ, then we can write:

$$\left\| \frac{du}{dr} \right\|^2_{L_2(\hat{K})} \leqslant C_N \|u\|^2_{L_2(\hat{K})}.$$

For $N = 1$ with orthonormal basis function on the reference interval,

$$\begin{aligned} \phi_0 &= \tfrac{\sqrt{2}}{2}, \\ \phi_1 &= \tfrac{\sqrt{6}}{2} r, \end{aligned} \qquad \text{and} \qquad \mathbf{S} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix},$$

thus $C_1 = 3$.

Note that basis functions are hierarchical. Then, for $N = 2$ with orthonormal basis function on the reference interval,

$$
\begin{aligned}
\phi_0 &= \frac{\sqrt{2}}{2}, \\
\phi_1 &= \frac{\sqrt{6}}{2} r, \\
\phi_2 &= \frac{\sqrt{10}}{4}(3r^2 - 1),
\end{aligned}
$$

and

$$
\mathbf{S} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 15 \end{array} \right],
$$

and in this case $C_2 = 15$. We note that the 2×2 submatrix of $\mathbf{S}$ is the matrix that we get in the case of $N = 1$.

For $N = 3$, orthonormal basis functions on the reference interval are given,

$$
\begin{aligned}
\phi_0 &= \frac{\sqrt{2}}{2}, \\
\phi_1 &= \frac{\sqrt{6}}{2} r, \\
\phi_2 &= \frac{\sqrt{10}}{4}(3r^2 - 1), \\
\phi_3 &= \frac{\sqrt{14}}{4}(5r^3 - 3r),
\end{aligned}
$$

and

$$
\mathbf{S} = \left[\begin{array}{cc|c|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & \sqrt{21} \\ \hline 0 & 0 & 15 & 0 \\ \hline 0 & \sqrt{21} & 0 & 42 \end{array} \right],
$$

and in this case $C_3 = \frac{45+\sqrt{1605}}{2}$.

Similarly for $N = 4$ with orthonormal basis function on the reference interval,

$$\begin{aligned}
\phi_0 &= \frac{\sqrt{2}}{2}, \\
\phi_1 &= \frac{\sqrt{6}}{2} r, \\
\phi_2 &= \frac{\sqrt{10}}{4}(3r^2 - 1), \\
\phi_3 &= \frac{\sqrt{14}}{4}(5r^3 - 3r), \\
\phi_4 &= \frac{3\sqrt{2}}{16}(35r^4 - 30r^2 + 3),
\end{aligned}$$

and

$$\mathbf{S} = \left[\begin{array}{cc|c|c|c}
0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & \sqrt{21} & 0 \\
\cline{1-2}
0 & 0 & 15 & 0 & 9\sqrt{5} \\
\cline{1-3}
0 & \sqrt{21} & 0 & 42 & 0 \\
\cline{1-4}
0 & 0 & 9\sqrt{5} & 0 & 0
\end{array}\right],$$

and in this case $C_4 = \frac{105+3\sqrt{805}}{2}$.

A closed form bound on the eigenvalues for higher order is not obvious. However, it may be possible to use Gerschgörin's theorem to localize these eigenvalues.

Numerical values for C_N were computed as shown in Table 4.1. We remark that for higher order polynomials, the largest eigenvalues are obtained by using Matlab. We compare C_N constants with the constants of [77], which are given with the formula $N^2(N+1)(N+1/2)$. The results show that our constant is consistent and smaller than the constant of [77]. Additionally, the eigenvalues clearly scale asymptotically as $N^2(N+1)(N+1/2)$, where $\beta = \frac{C_{N_1}}{C_{N_2}} / \frac{N_1^2(N_1+1)(N_1+1/2)}{N_2^2(N_2+1)(N_2+1/2)}$. ■

N	C_N	$N^2(N+1)(N+1/2)$	$\frac{C_N}{N^2(N+1)(N+1/2)}$	β
1	3.0000	3	1.0000	2.0000
2	15.0000	30	0.5000	1.4813
3	42.5312	126	0.3375	1.2783
4	95.0588	360	0.2641	1.1793
5	184.7262	825	0.2239	1.1245
6	326.1508	1638	0.1991	1.0914
7	536.3742	2940	0.1824	1.0699
8	834.8615	4896	0.1705	1.0552
9	1243.5042	7695	0.1616	1.0447
10	1786.6229	11550	0.1547	

Table 4.1: Experimentally determined constants in the discrete Markov inequality on an interval

4.2 Two Dimensional Domain

In this section, we discuss the Markov inequality for a 2-simplex and we give a closed form for the constant for polynomial degree 1 and 2. For $3 \leqslant N \leqslant 10$, numerical values are given.

Theorem 4.2.1 *(Markov Inequality for a planar triangle) For a planar triangle K, let $|\partial K|$ be the perimeter length of K and $|K|$ be the area of triangle K. Then, for a polynomial u of total degree $N = 1, 2, 3, 4$ the following result holds*

$$\|\nabla \boldsymbol{u}\|_{L_2(K)} \leqslant \sqrt{C_N}\frac{|\partial K|}{|K|}\|\boldsymbol{u}\|_{L_2(K)}$$

where $C_1 = 6$, $C_2 = \frac{45}{2}$, $C_3 = 56.8879$, $C_4 = 119.8047$.

Proof. Let $\hat{K}$ be the right angle reference triangle with

$$\hat{K} = \{(r,s)| -1 \leqslant r, s \leqslant 1; r+s \leqslant 0\}.$$

The reference triangle $\hat{K}$ is mapped to triangle K by following transformation, which send $(-1,-1)$ to (x_1, y_1), $(1,-1)$ to (x_2, y_2), $(-1,1)$ to (x_3, y_3) : $(r,s) \in \hat{K} \mapsto (x,y) \in K$,

$$\begin{aligned} x &= r\frac{(x_2-x_1)}{2} + s\frac{(x_3-x_1)}{2} + \frac{(x_2+x_3)}{2} \\ y &= r\frac{(y_2-y_1)}{2} + s\frac{(y_3-y_1)}{2} + \frac{(y_2+y_3)}{2}\ . \end{aligned} \tag{4.1}$$

In vector form, the transformation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{(x_2-x_1)}{2} & \frac{(x_3-x_1)}{2} \\ \frac{(y_2-y_1)}{2} & \frac{(y_3-y_1)}{2} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} + \begin{bmatrix} \frac{(x_2+x_3)}{2} \\ \frac{(y_2+y_3)}{2} \end{bmatrix}$$

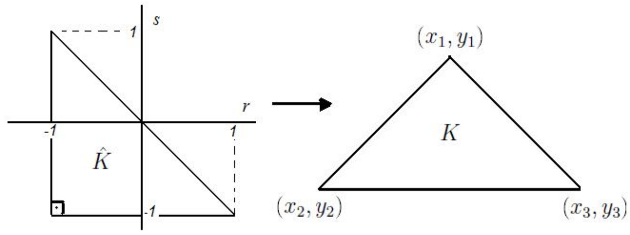


Figure 4.1: Mapping from the reference triangle $\hat{K}$ to the physical triangle K.

or $z = J\hat{z} + z_1$.

The standard formula for a change of variables in a multiple integral gives

$$\int_K f(x,y)dxdy = |\det(J)| \int_{\hat{K}} g(r,s)drds$$

where g is defined by

$$g(r,s) = f\left(r\frac{(x_2 - x_1)}{2} + s\frac{(x_3 - x_1)}{2} + \frac{(x_2 + x_3)}{2}, r\frac{(y_2 - y_1)}{2} + s\frac{(y_3 - y_1)}{2} + \frac{(y_2 + y_3)}{2}\right).$$

The Jacobian factor is constant:

$$|\det(J)| = \frac{1}{4}|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)|$$

Moreover

$$\frac{|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)|}{2} = |K|,$$

then

$$|\det(J)| = \frac{|K|}{2}.$$

Chain rule, triangle inequality and standard scaling argument give,

$$\begin{aligned}
\|\nabla u\|_{L_2(K)} &= \|\nabla r\frac{\partial u}{\partial r} + \nabla s\frac{\partial u}{\partial s}\|_{L_2(K)} \\
&\leqslant |\nabla r| \left\|\frac{\partial u}{\partial r}\right\|_{L_2(K)} + |\nabla s| \left\|\frac{\partial u}{\partial s}\right\|_{L_2(K)} \\
&= |\nabla r| \left(|\det(J)|^{1/2} \left\|\frac{\partial u}{\partial r}\right\|_{L_2(\hat{K})}\right) + |\nabla s| \left(|\det(J)|^{1/2} \left\|\frac{\partial u}{\partial s}\right\|_{L_2(\hat{K})}\right) \\
&\leqslant (|\nabla r| + |\nabla s|) \sqrt{C_N}|\det(J)|^{1/2}\|u\|_{L_2(\hat{K})} \\
&= (|\nabla r| + |\nabla s|) \sqrt{C_N}\|u\|_{L_2(K)} \qquad (4.2)
\end{aligned}$$

where $|\nabla r| = \sqrt{(\nabla r)^T(\nabla r)}$ and same for $|\nabla s|$.

Similar to the one dimensional case, utilizing Einstein summation for repeated indices and using the orthonormality of the basis give us the following eigenvalue problem

$$\mathbf{S}_{nm}u_m = \lambda u_m$$

where $\mathbf{S}_{nm} := \left(\frac{\partial \phi_n}{\partial r}, \frac{\partial \phi_m}{\partial r}\right)_{\hat{K}}$, and $\{\phi_n\}_{n=1}^{n=(N+1)(N+2)/2}$ is an orthonormal basis of the reference triangle $\hat{K}$.

Before giving the proof, we first compute $|\nabla r| + |\nabla s|$ in (4.2):

$$|\nabla r| + |\nabla s| = (r_x^2 + r_y^2)^{1/2} + (s_x^2 + s_y^2)^{1/2}$$

by (4.1) we have

$$J = \frac{\partial(x,y)}{\partial(r,s)} = \begin{bmatrix} x_r & x_s \\ y_r & y_s \end{bmatrix} = \begin{bmatrix} \frac{(x_2-x_1)}{2} & \frac{(x_3-x_1)}{2} \\ \frac{(y_2-y_1)}{2} & \frac{(y_3-y_1)}{2} \end{bmatrix}$$

Then

$$J^{-1} = \frac{\partial(r,s)}{\partial(x,y)} = \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} = \frac{1}{\det(J)} \begin{bmatrix} \frac{(y_3-y_1)}{2} & -\frac{(x_3-x_1)}{2} \\ -\frac{(y_2-y_1)}{2} & \frac{(x_2-x_1)}{2} \end{bmatrix} \tag{4.3}$$

Therefore (4.2) and (4.3) allow us to say,

$$\begin{aligned} (r_x^2 + r_y^2)^{1/2} + (s_x^2 + s_y^2)^{1/2} &= \frac{\left[(y_3-y_1)^2 + (x_3-x_1)^2\right]^{1/2} + \left[(y_2-y_1)^2 + (x_2-x_1)^2\right]^{1/2}}{2|\det(J)|} \\ &= \frac{\operatorname{dist}((x_1,y_1),(x_3,y_3)) + \operatorname{dist}((x_1,y_1),(x_2,y_2))}{|K|} \\ &\leqslant \frac{|\partial K|}{|K|} \end{aligned}$$

where $\operatorname{dist}(\cdot,\cdot)$ denotes the distance between two points. We conclude that

$$|\nabla r| + |\nabla s| \leqslant \frac{|\partial K|}{|K|}.$$

We note in particular that as

$$\frac{|\partial K|}{|K|} \approx h_K^{-1}$$

where h_K is the longest edge of the element K and also h_K^{-1} is common constant in Markov inequality in literature [35, 56].

Now, defining C_N as the maximum eigenvalue λ allows us to state:

$$\left\|\frac{\partial u}{\partial r}\right\|_{L_2(\hat{K})}^2 \leqslant C_N \|u\|_{L_2(\hat{K})}^2 ,$$

and by symmetry, the same constant applies for the norm of the s-derivative of u.

For $N = 1$ with orthonormal basis functions on the reference triangle $\hat{K}$ and the matrix $\mathbf{S}$ is given by:

$$\begin{aligned} \phi_{0,0} &= \tfrac{\sqrt{2}}{2}, \\ \phi_{0,1} &= \tfrac{1+3s}{2}, \\ \phi_{1,0} &= \tfrac{\sqrt{3}(1+2r+s)}{2} \end{aligned} \qquad \text{and } \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

thus $C_1 = 6$.

Since basis functions are hierarchical, the first three basis functions are same with the $N = 1$ case. Then, for $N = 2$, the orthonormal basis is:

$$\begin{aligned} \phi_{0,0} &= \frac{\sqrt{2}}{2}, \\ \phi_{0,1} &= \frac{1+3s}{2}, \\ \phi_{1,0} &= \frac{\sqrt{3}(1+2r+s)}{2}, \\ \phi_{0,2} &= \frac{\sqrt{6}(2s+5s^2-1)}{4}, \\ \phi_{1,1} &= \frac{3\sqrt{2}(3+5s)(1+2r+s)}{8}, \\ \phi_{2,0} &= \frac{\sqrt{30}(1+6r+4s+6r^2+6rs+6s^2)}{8}, \end{aligned}$$

and stiffness matrix is given by,

$$\mathbf{S} = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 2\sqrt{6} & 0 \\ \hline 0 & 0 & 0 & 2\sqrt{6} & \frac{33}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{45}{2} \end{array}\right],$$

and in this case $C_2 = \frac{45}{2}$.

For $N = 3$

$$\mathbf{S} = \left[\begin{array}{cccccc|c|cc|cc} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&6&2\sqrt{6}&3\sqrt{2}&0&0&\sqrt{42} \\ 0&0&0&0&2\sqrt{6}&\frac{33}{2}&10\sqrt{3}&0&0&0 \\ 0&0&0&0&3\sqrt{2}&10\sqrt{3}&36&0&0&0 \\ 0&0&0&0&0&0&0&\frac{45}{2}&6\sqrt{3}&0 \\ 0&0&0&0&0&0&0&6\sqrt{3}&44&0 \\ 0&0&0&0&\sqrt{42}&0&0&0&0&56 \end{array}\right],$$

and in this case $C_3 = 56.8879$.

By similar argument, for $N = 4$ we get the numeric value for $C_4 = 119.8047$ and the eigenvalues scale asymptotically as N^4 as in one dimensional case. Table 4.2 presents numerical values for C_N and compares the constants experimentally determined. ■

N	C_N	$\frac{C_N}{N^4}$
1	6.0000	6.0000
2	22.5000	1.4063
3	56.8879	0.7023
4	119.8047	0.4680
5	224.1195	0.3586
6	385.2210	0.2972
7	620.8674	0.2586
8	951.2557	0.2322
9	1399.0115	0.2132
10	1989.1818	0.1989

Table 4.2: Experimentally determined constants in the discrete Markov inequality on a triangle

4.3 Three Dimensional Domain

In this section, we consider a tetrahedron and we find the closed form for the constant up to polynomial degree 3.

Theorem 4.3.1 *(Markov Inequality for a tetrahedron) For a tetrahedral element K, let $|\partial K|$ denote the surface area of the K and $|K|$ denote the volume of the K, then for a polynomial u of total degree $N = 1, 2, 3, 4$ the following result holds*

$$\|\nabla v\|_{L_2(K)} \leqslant 2\sqrt{C_N \frac{|\partial K|}{|K|}}\|v\|_{L_2(K)}$$

where $C_1 = 10$, $C_2 = \frac{63}{2}$, $C_3 = 42 + 12\sqrt{7}$, $C_4 = 148.4089$.

Proof. Let $\hat{K}$ be standard tetrahedron with

$$\hat{K} = \{(r, s, t)| - 1 \leqslant r, s, t \leqslant 1; r + s + t \leqslant -1\}.$$

The standard tetrahedron $\hat{K}$ is mapped to the physical tetrahedron K by an affine mapping, which sends $(-1, -1, -1)$ to (x_1, y_1, z_1), $(1, -1, -1)$ to (x_2, y_2, z_2), $(-1, 1, -1)$ to (x_3, y_3, z_3), $(-1, -1, 1)$ to (x_4, y_4, z_4) : $(r, s, t) \in \hat{K} \mapsto (x, y, z) \in K$,

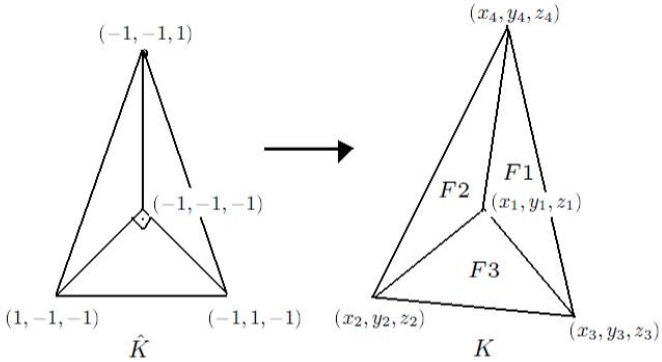


Figure 4.2: Mapping from the reference tetrahedron $\hat{K}$ to the physical tetrahedron K where $F1$, $F2$ and $F3$ denote faces of the physical tetrahedron K.

$$\begin{aligned} x &= r\frac{(x_2 - x_1)}{2} + s\frac{(x_3 - x_1)}{2} + t\frac{(x_4 - x_1)}{2} + \frac{(x_2 + x_3 + x_4 - x_1)}{2} \\ y &= r\frac{(y_2 - y_1)}{2} + s\frac{(y_3 - y_1)}{2} + t\frac{(y_4 - y_1)}{2} + \frac{(y_2 + y_3 + y_4 - y_1)}{2} \\ z &= r\frac{(z_2 - z_1)}{2} + s\frac{(z_3 - z_1)}{2} + t\frac{(z_4 - z_1)}{2} + \frac{(z_2 + z_3 + z_4 - z_1)}{2}. \end{aligned} \tag{4.4}$$

In vector form, the transformation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{(x_2-x_1)}{2} & \frac{(x_3-x_1)}{2} & \frac{(x_4-x_1)}{2} \\ \frac{(y_2-y_1)}{2} & \frac{(y_3-y_1)}{2} & \frac{(y_4-y_1)}{2} \\ \frac{(z_2-z_1)}{2} & \frac{(z_3-z_1)}{2} & \frac{(z_4-z_1)}{2} \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} + \begin{bmatrix} \frac{(x_2+x_3+x_4-x_1)}{2} \\ \frac{(y_2+y_3+y_4-y_1)}{2} \\ \frac{(z_2+z_3+z_4-z_1)}{2} \end{bmatrix}$$

or $k = J\hat{k} + k_1$.

The Jacobian factor is constant:

$$\begin{aligned}|\det(J)| &= \frac{1}{8}|(x_2-x_1)(y_3-y_1)(z_4-z_1)+(y_2-y_1)(z_3-x_1)(x_4-x_1)\\ &\quad +(z_2-z_1)(x_3-x_1)(y_4-y_1)-(x_4-x_1)(y_3-y_1)(z_2-z_1)\\ &\quad -(y_4-y_1)(z_3-z_1)(x_2-x_1)-(z_4-z_1)(x_3-x_1)(y_2-y_1)|\end{aligned}$$

Moreover,

$$\begin{aligned}&\frac{1}{6}|(x_2-x_1)(y_3-y_1)(z_4-z_1)+(y_2-y_1)(z_3-x_1)(x_4-x_1)+(z_2-z_1)(x_3-x_1)(y_4-y_1)\\ &-(x_4-x_1)(y_3-y_1)(z_2-z_1)-(y_4-y_1)(z_3-z_1)(x_2-x_1)-(z_4-z_1)(x_3-x_1)(y_2-y_1)|\\ &=|K|\end{aligned}$$

where $|K|$ is the volume of the tetrahedron K, then

$$|\det(J)| = \frac{3}{4}|K|. \tag{4.5}$$

Chain rule, the triangle inequality and standard scaling argument give

$$\begin{aligned}\|\nabla u\|_{L_2(K)} &= \left\|\nabla r\frac{\partial u}{\partial r}+\nabla s\frac{\partial u}{\partial s}+\nabla t\frac{\partial u}{\partial t}\right\|_{L_2(K)}\\ &\leqslant |\nabla r|\left\|\frac{\partial u}{\partial r}\right\|_{L_2(K)}+|\nabla s|\left\|\frac{\partial u}{\partial s}\right\|_{L_2(K)}+|\nabla t|\left\|\frac{\partial u}{\partial t}\right\|_{L_2(K)}\\ &= |\nabla r|\left(|\det(J)|^{1/2}\left\|\frac{\partial u}{\partial r}\right\|_{L_2(\hat{K})}\right)+|\nabla s|\left(|\det(J)|^{1/2}\left\|\frac{\partial u}{\partial s}\right\|_{L_2(\hat{K})}\right)\\ &+ |\nabla t|\left(|\det(J)|^{1/2}\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\hat{K})}\right)\\ &\leqslant (|\nabla r|+|\nabla s|+|\nabla t|)\sqrt{C_N}|\det(J)|^{1/2}\|u\|_{L_2(\hat{K})}\\ &= (|\nabla r|+|\nabla s|+|\nabla t|)\sqrt{C_N}\|u\|_{L_2(K)}.\end{aligned}$$

The constant C_N can be determined by solving the following classical eigenvalue problem for the maximum eigenvalue.

$$\underbrace{\left(\frac{\partial\phi_n}{\partial r},\frac{\partial\phi_m}{\partial r}\right)_{\hat{K}}}_{\mathbf{S}_{nm}} u_m = \lambda\underbrace{(\phi_n,\phi_m)_{\hat{K}}}_{\mathbf{M}_{nm}=I} u_m\ ,$$

where $\{\phi_n\}_{n=1}^{n=(N+1)(N+2)(N+3)/6}$ is an orthonormal basis defined on the reference tetrahedron $\hat{K}$.

First of all let us compute $|\nabla r| + |\nabla s| + |\nabla t|$:

$$|\nabla r| + |\nabla s| + |\nabla t| \quad = \quad (r_x^2 + r_y^2 + r_z^2)^{1/2} + (s_x^2 + s_y^2 + s_z^2)^{1/2} + (t_x^2 + t_y^2 + t_z^2)^{1/2}$$

by (4.4) we have

$$J = \frac{\partial(x,y,z)}{\partial(r,s,t)} = \begin{bmatrix} x_r & x_s & x_t \\ y_r & y_s & y_t \\ z_r & z_s & z_t \end{bmatrix} = \begin{bmatrix} \frac{(x_2-x_1)}{2} & \frac{(x_3-x_1)}{2} & \frac{(x_4-x_1)}{2} \\ \frac{(y_2-y_1)}{2} & \frac{(y_3-y_1)}{2} & \frac{(y_4-y_1)}{2} \\ \frac{(z_2-z_1)}{2} & \frac{(z_3-z_1)}{2} & \frac{(z_4-z_1)}{2} \end{bmatrix}$$

Then, inverse of the Jacobian matrix J can be defined following way

$$|J^{-1}| = \left|\frac{\partial(r,s,t)}{\partial(x,y,z)}\right| = \begin{bmatrix} |r_x| & |r_y| & |r_z| \\ |s_x| & |s_y| & |s_z| \\ |t_x| & |t_y| & |t_z| \end{bmatrix} = \frac{1}{|\det(J)|} \begin{bmatrix} \frac{|\partial F1|}{2} & \frac{|\partial F2|}{2} & \frac{|\partial F3|}{2} \\ \frac{|\partial F1|}{2} & \frac{|\partial F2|}{2} & \frac{|\partial F3|}{2} \\ \frac{|\partial F1|}{2} & \frac{|\partial F2|}{2} & \frac{|\partial F3|}{2} \end{bmatrix} \tag{4.6}$$

where $|\partial F1|$, $|\partial F2|$ and $|\partial F3|$ denote the area of the faces $F1$, $F2$, $F3$ of the element K, respectively.

Therefore (4.5) and (4.6) allow us to write,

$$\begin{aligned} |\nabla r| + |\nabla s| + |\nabla t| \quad &= \quad 3\frac{\left[|\partial F1|^2 + |\partial F2|^2 + |\partial F3|^2\right]^{1/2}}{2|\det(J)|} \\ &\leqslant \quad 2\frac{|\partial K|}{|K|} \end{aligned}$$

We conclude that

$$|\nabla r| + |\nabla s| + |\nabla t| \leqslant 2\frac{|\partial K|}{|K|}$$

Note that

$$\frac{|\partial K|}{|K|} \approx h_K^{-1}$$

where h_K is the longest edge of the element K and it is a common constant in inverse inequalities [35, 56].

Let us define C_N as the maximum eigenvalue λ allows us to state:

$$\left\|\frac{\partial u}{\partial r}\right\|_{L_2(\hat{K})}^2 \leqslant C_N \|u\|_{L_2(\hat{K})}^2 ,$$

and by symmetry the same constant applies for the norm of the partial derivative of u with respect to s and t.

For $N = 1$ with orthonormal basis functions on the reference tetrahedron $\hat{K}$:

$$\begin{aligned}
\phi_{0,0,0} &= \frac{\sqrt{3}}{2}, \\
\phi_{1,0,0} &= \frac{\sqrt{30}}{4}(2 + 2r + s + t), \\
\phi_{0,1,0} &= \frac{\sqrt{10}}{4}(2 + 3r + s) \\
\phi_{0,0,1} &= \frac{\sqrt{5}}{2}(1 + 2t)
\end{aligned}$$

and

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix},$$

thus $C_1 = 10$.

For $N = 2, 3, 4$ using same argument we get $C_2 = \frac{63}{2}$, $C_3 = 42 + 12\sqrt{7}$, $C_4 = 148.4089$, respectively.

Numerical values for C_N are computed in Table 4.3. The eigenvalues clearly scale asymptotically as N^4. ■

N	C_N	$\frac{C_N}{N^4}$
1	10.0000	10.0000
2	31.5000	1.9688
3	73.7490	0.9105
4	148.4089	0.5797
5	269.5513	0.4313
6	452.0694	0.3488
7	717.7792	0.2990
8	1085.8205	0.2651
9	1587.8353	0.2420
10	2245.8720	0.2246

Table 4.3: Experimentally determined constants in the discrete Markov inequality

4.4 Conclusion

As a conclusion, we have obtained some explicit expressions for the Markov inequality constants on 1-simplex, 2-simplex and 3-simplex in L_2-norm. Since we choose the max eigenvalue of the system without losing any data, we could say that our estimates are also sharp.

From this work, computable constants in some inverse inequalities are obtained. One can effectively utilize these results to give guaranteed computable upper bounds of a priori and posteriori error estimation of finite element solutions.

CHAPTER 5

CONCLUSIONS and FUTURE WORK

In the introduction, I expressed the hope that the work in this thesis could be a "first step" towards a fully computable convergence analysis. In this final chapter, I will conclude by describing the progress made towards this goal in terms of my development for convergence analysis. I will also suggest some future research directions that could provide the next steps along higher order elements.

5.1 Conclusions

The aim of this thesis has been to express a fully computable convergence analysis for the first order symmetric DG finite element approximations. In Chapter 2, an introduction and summary of the method for the given model problem are established. Moreover, some useful result and inequalities are given. In Chapter 3, the convergence of an adaptive Interior Penalty Discontinuous Galerkin method (IPDG) is studied for a 2D model second order elliptic boundary value problem. Based on a residual type a posteriori error estimator, it is proved that after each refinement step of the adaptive scheme, we achieve a guaranteed reduction of the global discretization error in the broken energy seminorm associated with the IPDG method. In contrast to recent work on convergence of adaptive IPDG methods [23, 52, 59], the convergence analysis is to free of unknown constants. The main ingredients of the proof of the error reduction property are the reliability and discrete local efficiency of the estimator, a special marking strategy which is called Dofler method that takes care of a proper selection of edges and elements for refinement, and a Galerkin orthogonality property with respect to the energy inner product.

In Chapter 4, the explicit bounds for the finite element inverse inequality is derived. This was accomplished by using orthonormal polynomials on the 1D, 2D and 3D simplex and realizing that a special ordering makes the associated matrices hierarchical. The results are sharp with respect to the geometry of the elements, with respect to the polynomial order of the finite element space, and with respect to the physical dimension of the element.

5.2 Suggestions for Future Work

There are some possible research direction to improve this work. First of all, higher order DG finite element can be used to get convergence result. Moreover, a relatively large amount of new nodes must be created by refining each marked element. It was the deficiency of the refinement strategy. By using new refinement technique, it is possible to decrease number of the degrees of freedom. By this new refinement strategy, a contraction property of the adaptive discontinuous Galerkin finite element method can be proved without further assumptions on refine. Also, there is no study on the convergence of non-symmetric interior penalty DG (NIPG) method for elliptic problems. Although, in numerical example NIPG convergence faster, in theoretically it is not easy to prove convergence of the method. One of the reason is that;

Galerkin orthogonality is essential to show error reduction property. However, for NIPG method Galerkin orthogonality does not hold. That is one of the difficulty for this problem. It might be possible to use Quasi-Galerkin orthogonality. So, for future work it would be interesting study on the convergence of NIPG method.

For the inverse estimates, results was given just for simplices. This work can be extendable for quadrilaterals elements .

5.3 Summary

In summary, a fully computable convergence result is one of the main contribution of this thesis. This work is the first in the literature. A posteriori error estimator which is introduced by [7] is actually is an estimator not indicator. This is also first to use this type of residual error estimator for convergence analysis.

Another additive of this thesis in the finite element analysis is inverse inequality constant. This is also the most comprehensive work for this inequlaity.

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APPENDIX A

SOME USEFUL INEQUALITIES

Lemma A.0.1 *Let ABC is a triangle with the edge a, b, and c. Let the radius of the incircle triangle be r. If A denote the area of the triangle ABC, then*

$$r = \frac{2A}{a+b+c}$$

Proof. The center of the incircle of a triangle is located at the intersection of the angle bisectors of the triangle. Given the side lengths of the triangle, it is possible to determine the radius of the circle. Use the fact that the sum of the areas of the smaller triangles is equal to the area of the larger triangle to obtain an expression for the radius.Denote A as a area of a triangle,

$$\begin{aligned} \frac{1}{2}a \cdot r + \frac{1}{2}b \cdot r + \frac{1}{2}c \cdot r &= A \\ \frac{1}{2}r \cdot (a+b+c) &= A \\ r &= \frac{2A}{a+b+c} \end{aligned} \tag{A.1}$$

■

Lemma A.0.2 *Let K is an arbitrary triangle with the edges named by γ', γ, and h_K where h_K denotes the longest edge of an triangle K and let $|K|$ denote the area of the triangle K and ϱ denotes the shape regularity parameter. Then, the ratio between h_K^2 and K is given,*

$$\frac{h_K^2}{|K|} \leqslant 2\varrho$$

Proof. By shape regularity property of the triangle K, we have $\varrho \geqslant \frac{h_K}{2r}$ where $r = \frac{2|K|}{|\gamma|+|\gamma'|+h_K}$, then

$$\begin{aligned} \varrho \geqslant \frac{h_K}{2r} &= \frac{h_K}{2\frac{2|K|}{|\gamma|+|\gamma'|+h_K}} \\ &= \frac{h_K}{4|K|}(|\gamma| + |\gamma'| + h_K) \end{aligned}$$

From the triangle inequality, the sum of the lengths of any two sides of a triangle always exceeds the length of the third side, i.e., $|\gamma| + |\gamma'| \geqslant h_K$ then

$$\begin{aligned} \frac{h_K}{4|K|}(|\gamma| + |\gamma'| + h_K) &\geqslant \frac{h_K}{4|K|}(2h_K) \\ &= \frac{h_K^2}{2|K|}. \end{aligned}$$

So, we get

$$\varrho \geqslant \frac{h_K^2}{2|K|} \Rightarrow 2\varrho \geqslant \frac{h_K^2}{|K|}.$$

Actually, it can be concluded that

$$2\varrho \geqslant \frac{h_K|\gamma|}{|K|}$$

and

$$2\varrho \geqslant \frac{h_K|\gamma'|}{|K|}.$$

■

Lemma A.0.3 *Let K is an arbitrary triangle with the edges named by γ', γ, and h_K where h_K denotes the longest edge of an triangle K and ϱ is a shape regularity parameter. Then, the ratio between the edges γ and γ' is,*

$$\frac{|\gamma|}{|\gamma'|} \leqslant \varrho.$$

Proof. *Case 1*: Assume that ABC is an acute triangle whose angles are all acute (i.e. less than 90°). if h_K is a largest edge of a triangle then $\frac{\pi}{3} \leqslant \theta < \frac{\pi}{2}$. By shape regularity we have

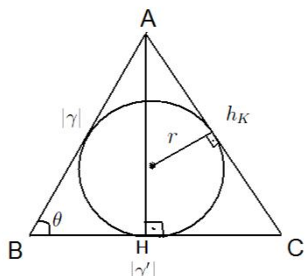

Figure A.1: Acute Triangle

$\varrho \geqslant \frac{h_K}{2r}$ where $r = \frac{2|K|}{|\gamma|+|\gamma'|+h_K}$ and by triangle inequality $|\gamma'| + h_K \geqslant |\gamma|$ and also $\frac{h_K}{|AH|} \geqslant 1$

then

$$
\begin{aligned}
\varrho \geqslant \frac{h_K}{2r} &= \frac{h_K}{2\frac{2|K|}{|\gamma|+|\gamma'|+h_K}} \\
&= \frac{h_K}{4|K|}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{4(|\gamma'||AH|)/2}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{2(|\gamma'||AH|)}(|\gamma| + |\gamma'| + h_K) \\
&\geqslant \frac{h_K}{2(|\gamma'||AH|)}(2|\gamma|) \\
&\geqslant \frac{|\gamma|}{|\gamma'|}. \qquad \text{(A.2)}
\end{aligned}
$$

Case 2: Assume that ABC is a right triangle (i.e. 90°). If h_K is a largest edge of a triangle then $\theta = \frac{\pi}{2}$. By shape regularity we have $\varrho \geqslant \frac{h_K}{2r}$ where $r = \frac{2|K|}{|\gamma|+|\gamma'|+h_K}$ and by triangle inequality

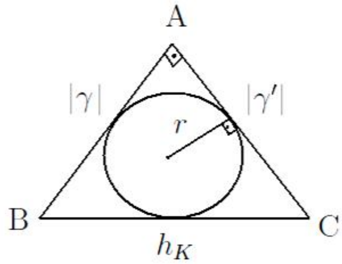

Figure A.2: Right Triangle

$|\gamma'| + h_K \geqslant |\gamma|$ and also $\frac{h_K}{|\gamma|} \geqslant 1$ then

$$
\begin{aligned}
\varrho \geqslant \frac{h_K}{2r} &= \frac{h_K}{2\frac{2|K|}{|\gamma|+|\gamma'|+h_K}} \\
&= \frac{h_K}{4|K|}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{4(|\gamma||\gamma'|)/2}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{2(|\gamma||\gamma'|)}(|\gamma| + |\gamma'| + h_K) \\
&\geqslant \frac{h_K}{2(|\gamma||\gamma'|)}(2|\gamma|) \\
&\geqslant \frac{|\gamma|}{|\gamma'|} \qquad \text{(A.3)}
\end{aligned}
$$

Case 3: Assume that ABC is an obtuse triangle whose angles are all obtuse (i.e. greater than 90°). If h_K is a largest edge of a triangle then $\frac{\pi}{2} < \theta < \pi$.

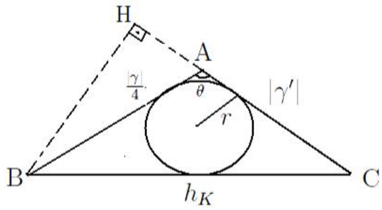

Figure A.3: Obtuse Triangle

By shape regularity we have $\varrho \geqslant \frac{h_K}{2r}$ where $r = \frac{2|K|}{|\gamma|+|\gamma'|+h_K}$ and by triangle inequality $|\gamma'| + h_K \geqslant |\gamma|$ and also $\frac{h_K}{|BH|} \geqslant 1$ then

$$\begin{aligned}
\varrho \geqslant \frac{h_K}{2r} &= \frac{h_K}{2\frac{2|K|}{|\gamma|+|\gamma'|+h_K}} \\
&= \frac{h_K}{4|K|}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{4(|\gamma'||BH|)/2}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{2(|\gamma'||BH|)}(|\gamma| + |\gamma'| + h_K) \\
&\geqslant \frac{h_K}{2(|\gamma'||BH|)}(2|\gamma|) \\
&\geqslant \frac{|\gamma|}{|\gamma'|}.
\end{aligned}$$

■

Lemma A.0.4 *Let K is an arbitrary triangle with the edges named by γ', γ, and h_K where h_K denotes the longest edge of an triangle K and ϱ is a shape regularity parameter. Then, the ratio between the edges h_K and γ is,*

$$\frac{h_K}{|\gamma|} \leqslant \varrho.$$

Proof. We want to find a relation between the longest edge and other edge of a triangle,

By shape regularity we have $\varrho \geqslant \frac{h_K}{2r}$ where $r = \frac{2|K|}{|\gamma|+|\gamma'|+h_K}$ and by triangle inequality

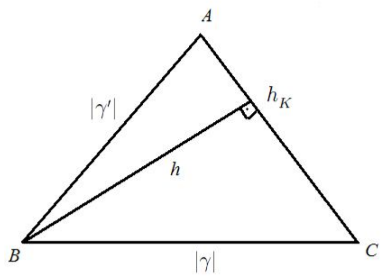

Figure A.4: A triangle with the longest edge h_K

$|\gamma| + |\gamma'| \geqslant |h_K|$ and also $|\gamma| \geqslant h$ then

$$\begin{aligned}
\varrho \geqslant \frac{h_K}{2r} &= \frac{h_K}{2\frac{2|K|}{|\gamma|+|\gamma'|+h_K}} \\
&= \frac{h_K}{4|K|}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{4(h_K h)/2}(|\gamma| + |\gamma'| + h_K) \\
&= \frac{h_K}{2(h_K h)}(|\gamma| + |\gamma'| + h_K) \\
&\geqslant \frac{h_K}{2(h_K h)}(2h_K) \\
&= \frac{h_K}{h} \\
&\geqslant \frac{h_K}{|\gamma|}.
\end{aligned}$$

■

Lemma A.0.5 *Let K be an arbitrary triangle with the edges h_K, $|\gamma|$, $|\gamma'|$ and h_K denote the longest edge of the triangle K. Let h be the height which belongs the edge γ and, ϱ be shape regularity parameter. Then*

$$\frac{h}{|\gamma|} \leqslant \varrho$$

Proof. K is the triangle with the edges h_K, $|\gamma|$, $|\gamma'|$.

By shape regularity we have $\frac{h_K}{2r} \leqslant \varrho$ where r is the radius of a circle which is inscribed in the triangle K. By triangle inequality, $h_K + |\gamma| \geqslant |\gamma|$ and $\frac{h_K}{h} \geqslant 1$

$$\begin{aligned}
\varrho \geqslant \frac{h_K}{2r} = \frac{h_K(|\gamma| + |\gamma'| + h_K)}{4|K|} &= \frac{h_K(|\gamma| + |\gamma'| + h_K)}{2|\gamma|h} \\
&\geqslant \frac{h_K(2|\gamma'|)}{2|\gamma|h} \\
&\geqslant \frac{|\gamma'|}{|\gamma|}.
\end{aligned} \tag{A.4}$$

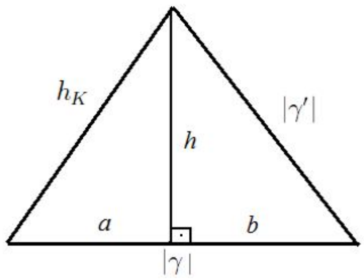

Figure A.5: Triangle with the height h

Now it can be written that $h \leqslant |\gamma'|$ and $h \leqslant h_K$, then we have

$$2h \leqslant |\gamma'| + h_K.$$

Using Lemma A.0.4 and inequality (A.4) give us

$$2h \leqslant |\gamma'| + h_K \leqslant \varrho|\gamma| + \varrho|\gamma| = 2\varrho|\gamma| \Rightarrow \frac{h}{|\gamma|} \leqslant \varrho.$$

■

Lemma A.0.6 *Let K be an arbitrary triangle. For a linear polynomial u, the following multivariate Markov inequality hold on the simplex*

$$\|\nabla v\|_K \leqslant 4\sqrt{6}\varrho h_K^{-1}\|v\|_K$$

Proof. From Chapter 4, one can obtained that if u is a linear polynomial, following Markov inequality holds in 2D-simplex

$$\|\nabla v\|_K \leqslant \frac{2h_K}{|K|}\sqrt{C_N}\|v\|_K$$

where $C_N = 6$. Lemma A.0.2 yield that,

$$\frac{2h_K}{|K|} \leqslant 4\varrho h_K^{-1}$$

Then $\forall u \in \mathbb{P}_1(K)$, we can say that

$$\|\nabla u\|_K \leqslant 4\sqrt{6}\varrho h_K^{-1}\|u\|_K$$

■

Lemma A.0.7 *Let K be an arbitrary triangle with vertices* (x_1, y_1), (x_2, y_2), (x_3, y_3). *Then,* $\int_K f$ *is estimated by the following rule*

$$\int_K f = \frac{|K|}{3}(f(\overline{x}_1, \overline{y}_1) + f(\overline{x}_2, \overline{y}_2) + f(\overline{x}_3, \overline{y}_3))$$

where

$$\begin{aligned}(\overline{x}_1,\overline{y}_1) &= \frac{1}{2}(x_1,y_1)+\frac{1}{2}(x_2,y_2)\\ (\overline{x}_2,\overline{y}_2) &= \frac{1}{2}(x_2,y_2)+\frac{1}{2}(x_3,y_3)\\ (\overline{x}_3,\overline{y}_3) &= \frac{1}{2}(x_3,y_3)+\frac{1}{2}(x_1,y_1).\end{aligned}$$

Proof. The proof can be found in [45]. ■

Lemma A.0.8 *Let K be an arbitrary triangle in the mesh $\mathcal{P}_H$ and $T_K : \mathbb{P}_1(K) \mapsto S_K$ denote the operator given by*

$(T_K v),\chi)_K = (v,\chi)_K \ \forall \chi \in S_K.$

Then $\|T_K \cdot \|_K$ is a norm equivalent to $\| \cdot \|_K$ on $\mathbb{P}_1(K)$ such that

$$\forall v \in \mathbb{P}_K : \qquad \frac{3}{2\sqrt{10}}\|v\|_K \leqslant \|T_K v\|_K \leqslant \|v\|_K$$

Proof. Let $T_K : \mathbb{P}_1(K) \to S_K$ denote the operator given by $(T_K v,\chi)_K = (v,\chi)_K$ for all $\chi \in S_K$. Then $\|T_K \cdot\|_K$ is a norm equivalent to $\|\cdot\|_K$ on $\mathbb{P}_1(K)$ with constants that are independent of h_K. To show this, we first show $\|T_K \cdot\|_K$ is a norm.

- Assume $T_K v = 0$ for some $v \in \mathbb{P}_1(K)$. It then follows that $(v,\phi) = 0$ for all $\phi \in S_K$. Since G is nonsingular, it follows that $v = 0$.
- $\|\alpha T_K v\| = |\alpha|\|T_K v\|$ for all $\alpha \in \mathbb{R}$
- $\|T_K(v+w)\| = \|T_K v + T_K w\| \leqslant \|T_K v\| + \|T_K w\|$

The equivalence of the norms is a consequence of finite dimensionality.

$$\|T_K v\|_K^2 = (T_K v, T_K v)_K = (v, T_K v)_K \leqslant \left(\int_K |v|^2\right)^{1/2}\left(\int_K |T_K v|^2\right)^{1/2} = \|v\|_K\|T_K v\|_K.$$

Thus,

$$\|T_K v\|_K \leqslant \|v\|_K. \tag{A.5}$$

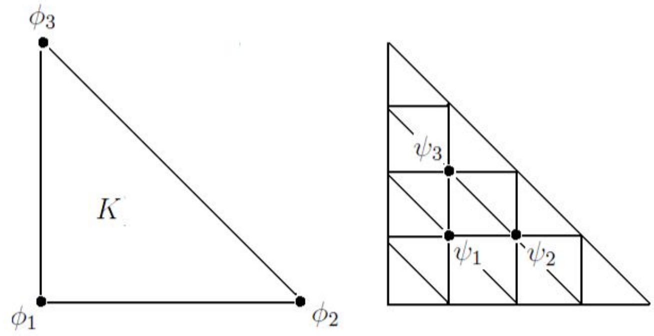

Figure A.6: Usual Lagrangian Basis function of a triangle K

$$T\phi_K = \sum_{i=1}^{3} \beta_j^k \psi_j$$

For the other side, let K be a reference element and determine $T\phi_k \in S_K$ where $1 \leqslant k \leqslant 3$:

$$(T\phi_K, \psi_i)_K = \int_K T\phi_K \psi_i = \int_K \phi_K \psi_K \Leftrightarrow \sum_{i=1}^{3} \beta_j^k \int_K \psi_j \psi_i = \int_K \phi_K \psi_i$$

Let $\beta^k = \begin{pmatrix} \beta_1^k \\ \beta_2^k \\ \beta_3^k \end{pmatrix}$ and $M_\psi = \left(\int_K \psi_i \psi_j\right)$ and $p^k = \begin{pmatrix} \int_K \phi_k \psi_1 \\ \int_K \phi_k \psi_2 \\ \int_K \phi_k \psi_3 \end{pmatrix}$

then $M\beta^k = p^k$ where $1 \leqslant k \leqslant 3$.

$P = \begin{bmatrix} p^1 & p^2 & p^3 \end{bmatrix}_{3\text{x}3}$ matrix and let $\beta = \begin{bmatrix} \beta^1 & \beta^2 & \beta^3 \end{bmatrix}_{3\text{x}3}$, then we need to solve the matrix system:

$$M_\psi \beta = P \text{ or } \beta = M_\psi^{-1} P.$$

After solving the above system we can compute the matrices,

$$X = \left(\int_K T\phi_i T\phi_j\right)_{1 \leqslant i,j \leqslant 3} \quad \text{and}$$

$$M_\phi = \left(\int_K \phi_i \phi_j\right)_{1 \leqslant i,j \leqslant 3}.$$

Therefore, the solution of the eigenvalue problem is

$$X\vec{v} = \lambda M_\phi \vec{v} \Leftrightarrow M_\phi^{-1} X\vec{v} = \lambda \vec{v},$$

select $\lambda_{\min} = \min\{\sqrt{\lambda}\}$ and check that the eigenvectors form a basis in $\mathbb{R}^3$

Claim: $\lambda_{\min} \|w\|_K \leqslant \|Tw\|_K \qquad \forall w \in \mathbb{P}_1(K)$.

Proof of Claim : Pick $w \in \mathbb{P}_1(K)$, write $w = \sum_{i=1}^{3} \alpha_i \phi_i$, let $\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$$\|w\|_K^2 = \int_K ww = \sum_{i,j} \alpha_i \alpha_j \int_K \phi_i \phi_j = \vec{\alpha}^T M_\phi \vec{\alpha}$$

where $\vec{\alpha}^T$ denotes transpose of $\vec{\alpha}$.

$$\|Tw\|_K^2 = \int_K (Tw)(Tw), \text{ but } Tw = \sum_{i=1}^{3} \alpha_i T\phi_i$$

$$\text{so } \|Tw\|_K = \sum_{i,j} \alpha_i \alpha_j \int_K T\phi_i T\phi_j = \vec{\alpha}^T X \vec{\alpha}$$

$$\text{so } C\|w\|_K \leqslant \|Tw\|_K \Leftrightarrow C^2 \vec{\alpha}^T M_\phi \vec{\alpha} \leqslant \vec{\alpha}^T X \vec{\alpha}.$$

Let λ be such that $\lambda M_\phi \vec{v} = X\vec{v}$. If $\vec{v} = (\vec{v_1}, \vec{v_2}, \vec{v_3})$ forms a basis then this implies

$$\lambda M_\phi \vec{\alpha} = X\vec{\alpha} \Rightarrow \lambda \vec{\alpha}^T M_\phi \vec{\alpha} = \vec{\alpha}^T X \vec{\alpha}$$

$$\lambda_{\min} = \min\{\sqrt{\lambda}\} \Rightarrow \lambda_{\min}^2 \vec{\alpha}^T M_\phi \vec{\alpha} \leqslant \vec{\alpha}^T X \vec{\alpha}$$

which is $\lambda_{\min}^2 = C^2 \Leftrightarrow \lambda = C$ ■

Lemma A.0.9 *[Karakashian, Pascal,2007] The following inequality holds for $\mathcal{P}_H$ for sufficently large κ*

$$B_H(e_H, e_H) \geqslant \frac{1}{2} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \left(\frac{\kappa^2}{4C_1} - \kappa - \frac{224\varrho}{3} - \frac{\lambda\varrho}{4} \right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2$$

where $C_1 = 64(1+\varrho)\left(\frac{2^2 10^2 C_i^2 \lambda}{9^2} + \frac{11^2 C_i^2 \lambda + 40 C_i^4 \lambda + 9^2 \lambda}{72}\right)$ *and* $\lambda = \max\{\rho(\boldsymbol{A}_K) | K \in \mathcal{P}_{H,0}\}$.

Proof. By (2.6) we have

$$B_H(e_H, e_H) = \sum_{K \in \mathcal{P}_H} \|e_H\|_K^2 - (1+\tau) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma\right)_\gamma + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \frac{\kappa}{|\gamma|} \|[e_H]_\gamma\|_\gamma^2 \tag{A.6}$$

We remark that if $\tau = -1$ we have

$$B_H(e_H, e_H) = \sum_{K \in \mathcal{P}_H} \|e_H\|_K^2 + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \frac{\kappa}{|\gamma|} \|[e_H]_\gamma\|_\gamma^2$$

Since $\|[e_H]_\gamma\|_\gamma = \|[u_H]_\gamma\|_\gamma$ for all $\gamma \in \mathcal{E}_H^I$ and $\|[e_H]_\gamma\|_\gamma = \|q - u_H\|_\gamma$ for all $\gamma \in \mathcal{E}_H^D$, we get

$$B_H(e_H, e_H) = \sum_{K \in \mathcal{P}_H} \|\!|e_H|\!\|_K^2 + \kappa \sum_{\gamma \in \mathcal{E}_H^I} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 + \kappa \sum_{\gamma \in \mathcal{E}_H^D} |\gamma|^{-1} \|q - u_H\|_\gamma^2.$$

If $\tau = 1$ or $\tau = 0$, we obtain an upper bound of the term $\sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma$.

We construct a function $\tilde{v} \in X_H \cap H^1(\Omega)$ satisfying $\tilde{v}_{|\Gamma_D} = q$ such that by [59], for any $u_H \in X_H$ we have

$$\sum_{K \in \mathcal{P}_H} \|\nabla(u_H - \tilde{v})\|_K^2 \leqslant \frac{8\varrho}{3} \left(\sum_{\gamma \in \mathcal{E}_H^I} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 + \sum_{\gamma \in \mathcal{E}_H^D} |\gamma|^{-1} \|q - u_H\|_\gamma^2 \right) \tag{A.7}$$

Moreover,

$$\sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma = \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [\tilde{v} - u_H]_\gamma \right)_\gamma \tag{A.8}$$

By Galerkin orthogonality we have,

$$\begin{aligned} 0 = B_H(e_H, \tilde{v} - u_H) = & \sum_{K \in \mathcal{P}_H} (\boldsymbol{A} \nabla e_H, \nabla(\tilde{v} - u_H))_K - \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [\tilde{v} - u_H]_\gamma \right)_\gamma \\ & - \tau \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla(\tilde{v} - u_H) \rangle_\gamma, [e_H]_\gamma \right)_\gamma + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[e_H]_\gamma\|_\gamma^2 \end{aligned} \tag{A.9}$$

By (A.8) and (A.9), it can be written that

$$\begin{aligned} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma = & \sum_{K \in \mathcal{P}_H} (\boldsymbol{A} \nabla e_H, \nabla(\tilde{v} - u_H))_K \\ & - \tau \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla(\tilde{v} - u_H) \rangle_\gamma, [u_H]_\gamma \right)_\gamma \\ & + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2. \end{aligned}$$

By Cauchy-Schwarz's inequality we get

$$\begin{aligned} \left| \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma \right| \leqslant & \sum_{K \in \mathcal{P}_H} \|\boldsymbol{A} \nabla e_H\|_K \|\nabla(\tilde{v} - u_H)\|_K \\ & + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \|\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla(\tilde{v} - u_H) \rangle_\gamma\|_\gamma \|[u_H]_\gamma\|_\gamma \\ & + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2. \end{aligned} \tag{A.10}$$

By (2.15) for $\gamma = \partial K_1 \cap \partial K_2$, we have

$$\begin{aligned}
\|\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla(\tilde{v} - u_H)\rangle_\gamma\|_\gamma &\leqslant \sqrt{\varrho} h_{K_1}^{-1/2} \|\boldsymbol{A}\nabla(\tilde{v} - u_H)\|_{K_1} + \sqrt{\varrho} h_{K_2}^{-1/2} \|\boldsymbol{A}\nabla(\tilde{v} - u_H)\|_{K_2} \\
&\leqslant \sqrt{\varrho} |\gamma|^{-1/2} \left(\|\boldsymbol{A}\nabla(\tilde{v} - u_H)\|_{K_1} + \|\boldsymbol{A}\nabla(\tilde{v} - u_H)\|_{K_2} \right).
\end{aligned}$$

Using last inequality into (A.10) we have

$$\begin{aligned}
\left| \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma \right| &\leqslant \sum_{K \in \mathcal{P}_H} \|\boldsymbol{A}\nabla e_H\|_K \|\nabla(\tilde{v} - u_H)\|_K \\
&+ \sum_{\substack{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D \\ \gamma = \partial K_1 \cap \partial K_2}} \sqrt{\varrho} |\gamma|^{-1/2} \left(\|\boldsymbol{A}\nabla(\tilde{v} - u_H)\|_{K_1} + \|\boldsymbol{A}\nabla(\tilde{v} - u_H)\|_{K_2} \right) \|[u_H]_\gamma\|_\gamma \\
&+ \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\
&\leqslant \sum_{K \in \mathcal{P}_H} \rho(\boldsymbol{A}_K)^{1/2} \|\!|e_H|\!\|_K \|\nabla(\tilde{v} - u_H)\|_K \\
&+ \sum_{\substack{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D \\ \gamma = \partial K_1 \cap \partial K_2}} \rho(\boldsymbol{A}_K) \sqrt{\varrho} |\gamma|^{-1/2} \left(\|\nabla(\tilde{v} - u_H)\|_{K_1} + \|\nabla(\tilde{v} - u_H)\|_{K_2} \right) \|[u_H]_\gamma\|_\gamma \\
&+ \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2.
\end{aligned}$$

the use of Young's inequality implies

$$\begin{aligned}
\left| \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A}\nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma \right| &\leqslant \frac{\epsilon}{2} \sum_{K \in \mathcal{P}_H} \rho(\boldsymbol{A}_K) \|\!|e_H|\!\|_K^2 + \frac{1}{2\epsilon} \sum_{K \in \mathcal{P}_H} \|\nabla(\tilde{v} - u_H)\|_K^2 \\
&+ \frac{\epsilon}{2} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \rho(\boldsymbol{A}_K)^2 \varrho |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 + \frac{3}{\epsilon} \sum_{K \in \mathcal{P}_H} \|\nabla(\tilde{v} - u_H)\|_K^2 \\
&+ \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\
&\leqslant \frac{\epsilon \lambda}{2} \sum_{K \in \mathcal{P}_H} \|\!|e_H|\!\|_K^2 + \frac{1}{2\epsilon} \sum_{K \in \mathcal{P}_H} \|\nabla(\tilde{v} - u_H)\|_K^2 \\
&+ \frac{\epsilon \lambda^2}{2} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \varrho |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 + \frac{3}{\epsilon} \sum_{K \in \mathcal{P}_H} \|\nabla(\tilde{v} - u_H)\|_K^2 \\
&+ \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2.
\end{aligned}$$

Now we choose $\tilde{v}$ as in (A.7),

$$\begin{aligned}
\left| \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \left(\langle \boldsymbol{n} \cdot \boldsymbol{A} \nabla e_H \rangle_\gamma, [e_H]_\gamma \right)_\gamma \right| &\leqslant \frac{\epsilon\lambda}{2} \sum_{K \in \mathcal{P}_H} |\!|\!| e_H |\!|\!|_K^2 + \frac{4\varrho}{3\epsilon} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \\
&+ \frac{\epsilon\lambda^2\varrho}{2} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 + \frac{8\varrho}{\epsilon} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \\
&+ \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \\
&= \frac{\epsilon\lambda}{2} \sum_{K \in \mathcal{P}_H} |\!|\!| e_H |\!|\!|_K^2 \\
&+ \left(\frac{28\varrho}{3\epsilon} + \frac{\epsilon\lambda^2\varrho}{2} + \kappa \right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2
\end{aligned}$$

Using this in (A.6) we obtain

$$\begin{aligned}
B_H(e_H, e_H) \geqslant \sum_{K \in \mathcal{P}_H} |\!|\!| e_H |\!|\!|_K^2 &- (1+\tau) \frac{\epsilon\lambda}{2} \sum_{K \in \mathcal{P}_H} |\!|\!| \nabla e_H |\!|\!|_K^2 \\
- (1+\tau) \left(\frac{28\varrho}{3\epsilon} + \frac{\epsilon\lambda^2\varrho}{2} + \kappa \right) &\sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 + \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} \kappa |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2
\end{aligned}$$

if $\tau = 0$ we have

$$\begin{aligned}
B_H(e_H, e_H) &\geqslant (1 - \frac{\epsilon\lambda}{2}) \sum_{K \in \mathcal{P}_H} |\!|\!| \nabla e_H |\!|\!|_K^2 \\
&- \left(\frac{28\varrho}{3\epsilon} + \frac{\epsilon\lambda^2\varrho}{2} \right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2
\end{aligned}$$

Choose $\epsilon = \frac{1}{2\lambda}$ then one gets

$$\begin{aligned}
B_H(e_H, e_H) &\geqslant \frac{3}{4} \sum_{K \in \mathcal{P}_H} |\!|\!| \nabla e_H |\!|\!|_K^2 \\
&- \left(\frac{56\varrho}{3} + \frac{\lambda\varrho}{4} \right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \\
&= \frac{1}{2} \sum_{K \in \mathcal{P}_H} |\!|\!| \nabla e_H |\!|\!|_K^2 + \frac{1}{4} \sum_{K \in \mathcal{P}_H} |\!|\!| \nabla e_H |\!|\!|_K^2 \\
&- \left(\frac{56\varrho}{3} + \frac{\lambda\varrho}{4} \right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \qquad \text{(A.11)}
\end{aligned}$$

By [59],

$$\kappa^2 \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \| [u_H]_\gamma \|_\gamma^2 \leqslant C_1 \sum_{K \in \mathcal{P}_H} |\!|\!| e_H |\!|\!|_K^2$$

where $C_1 = 64(1+\varrho)\left(\frac{2^2 10^2 C_i^2 \lambda}{9^2} + \frac{11^2 C_i^2 \lambda + 40 C_i^4 \lambda + 9^2 \lambda}{72}\right)$ Using this inequality in (A.11) we get,

$$\begin{aligned} B_H(e_H, e_H) \geqslant & \frac{1}{2} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \frac{\kappa^2}{4C_1} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\ & - \left(\frac{56\varrho}{3} + \frac{\lambda\varrho}{4}\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\ = & \frac{1}{2} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \left(\frac{\kappa^2}{4C_1} - \frac{56\varrho}{3} - \frac{\lambda\varrho}{4}\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \end{aligned}$$

if $\tau = 1$ we have

$$\begin{aligned} B_H(e_H, e_H) \geqslant & (1 - \epsilon\lambda) \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 \\ & - \left(\frac{56\varrho}{3\epsilon} + \epsilon\lambda^2\varrho + \kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \end{aligned}$$

Choose $\epsilon = \frac{1}{4\lambda}$ then one gets

$$\begin{aligned} B_H(e_H, e_H) \geqslant & \frac{3}{4} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 \\ & - \left(\frac{224\varrho}{3} + \frac{\lambda\varrho}{4} + \kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\ = & \frac{1}{2} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \frac{1}{4} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 \\ & - \left(\frac{224\varrho}{3} + \frac{\lambda\varrho}{4} + \kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \end{aligned} \tag{A.12}$$

By [59],

$$\kappa^2 \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \leqslant C_1 \sum_{K \in \mathcal{P}_H} \|e_H\|_K^2$$

where $C_1 = 64(1+\varrho)\left(\frac{2^2 10^2 C_i^2 \lambda}{9^2} + \frac{11^2 C_i^2 \lambda + 40 C_i^4 \lambda + 9^2 \lambda}{72}\right)$.

Using this inequality in (A.12) we get the desired result,

$$\begin{aligned} B_H(e_H, e_H) \geqslant & \frac{1}{2} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \frac{\kappa^2}{4C_1} \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\ & - \left(\frac{224\varrho}{3} + \frac{\lambda\varrho}{4} + \kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\ = & \frac{1}{2} \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \left(\frac{\kappa^2}{4C_1} - \kappa - \frac{224\varrho}{3} - \frac{\lambda\varrho}{4}\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \end{aligned}$$

From that proof, we can also deduce that

$$
\begin{aligned}
B_H(e_H, e_H) \leqslant & \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + (1+\tau)\frac{\epsilon\lambda}{2} \sum_{K \in \mathcal{P}_H} \|e_H\|_K^2 \\
& + (1+\tau)\left(\frac{28\varrho}{3\epsilon} + \frac{\epsilon\lambda^2\varrho}{2} + \kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2 \\
& + \kappa \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2.
\end{aligned}
$$

Pick $\tau = 1$

$$
\begin{aligned}
B_H(e_H, e_H) \leqslant & \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 + \epsilon\lambda \sum_{K \in \mathcal{P}_H} \|e_H\|_K^2 \\
& + \left(\frac{56\varrho}{3\epsilon} + \epsilon\lambda^2\varrho + 3\kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2
\end{aligned}
$$

Choose $\epsilon = \frac{1}{\lambda}$, then we get

$$
\begin{aligned}
B_H(e_H, e_H) \leqslant & \; 2 \sum_{K \in \mathcal{P}_H} \|\nabla e_H\|_K^2 \\
& + \left(\frac{56\lambda\varrho}{3} + \lambda\varrho + 3\kappa\right) \sum_{\gamma \in \mathcal{E}_H^I \cup \mathcal{E}_H^D} |\gamma|^{-1} \|[u_H]_\gamma\|_\gamma^2.
\end{aligned}
\tag{A.13}
$$

■

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Awards and Scholarships

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3. MATH 219 Introduction to Differential Equations

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Publications

1. Ozisik, S., and Riviere, B. and Warburton, T. "On the constants in inverse inequalities in L2". *submitted, also technical report TR10-19.* Submitted 2012.

2. Ozisik, S., and Riviere, B. "Fully Computable Convergence Analysis of Discontinuous Galerkin Finite Element Approximations with an arbitrary number of levels of hanging nodes". *Submitted 2012.*

3. Kaya Merdan, S., and Ozisik, S., and Riviere, B. "A fully computable Posteriori Error Estimation of Oseen Problem" *in preparation.*

Presentations

1. On The Constant in Inverse Estimates in L_2. On The Constant in Inverse Estimates in L_2. Finite Element Rodeo 2011 at Texas A and M University, TX, 25-26 February 2011.

2. Fully Computable Convergence Analysis of DGMs with an arbitrary number of levels of hanging nodes. IMA Special Event: Finite Element Circus Featuring a Scientific Celebration of Falk, Pasciak, and Wahlbin, Minnesota, November 5-6, 2010.

3. Fully Computable Convergence Analysis of DGMs with an arbitrary number of levels of hanging nodes. VIGRE Seminar Scienti

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4. Discontinuous Galerkin Methods and A Posteriori Error Estimation, Graduate Seminar, Rice Univesity, November, 2009.

5. A Posteriori Error Estimates for a Discontinuous Galerkin Approximation of Second-Order Elliptic Problems, Numerical Analysis Seminar, Rice University, September, 2009.

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1. International Conference on Applied Analysis and Algebra, Istanbul, Turkey, 2011

2. Finite Element Rodeo 2011, College Station, TX. February 25-26, 2011.

3. IMA Special Event: Finite Element Circus Featuring a Scientific Celebration of Falk, Workshops and Conferences Pasciak, and Wahlbin, Minneapolis, MN. November 5-6, 2010

4. IMA Workshop: Numerical Solutions of Partial Differential Equations: Novel Discretization Techniques, Minneapolis, MN. November 1-5, 2010

5. 2010 Joint Mathematics Meetings, San Francisco, CA. January 13-16, 2010

6. 2009 CBMS Conference on Adaptive Finite Element Methods for Partial Differential Equations, Texas A and M University, College Station,TX. May 18-22, 2009

7. Workshop on Differential Equations and Applications , Yeditepe University, Istanbul, TURKEY. June 18-23, 2009

8. Workshop in Memory of Professor Hayri Korezlioglu, Middle East Technical University (METU), Ankara, TURKEY. April 25-26,2008.

9. 6th ISAAC(International Society for Analysis, its Applications and Computation), Middle East Technical University (METU), Ankara, TURKEY. August 13-18, 2007

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