ALGEBRAIC GEOMETRIC METHODS IN STUDYING SPLINES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

NESLİHAN ÖS SİPAHİ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

DECEMBER 2013

Approval of the thesis:

ALGEBRAIC GEOMETRIC METHODS IN STUDYING SPLINES

submitted by **NESLİHAN ÖS SİPAHİ** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics** ______________

Assoc. Prof. Dr. Mohan Lal Bhupal
Supervisor, **Mathematics Department, METU** ______________

Assoc. Prof. Dr. Selma Altınok Bhupal
Co-supervisor, **Mathematics Departmans, Hacettepe University** ______________

Examining Committee Members:

Prof. Dr. Turgut Önder
Mathematics Department, METU ______________

Assoc. Prof. Dr. Mohan Lal Bhupal
Mathematics Department, METU ______________

Prof. Dr. Yıldıray Ozan
Mathematics Department, METU ______________

Assoc. Prof. Dr. Sefa Feza Arslan
Mathematics Department, MSGSU ______________

Assoc. Prof. Dr. Müfit Sezer
Mathematics Department, Bilkent University ______________

Date: ______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: NESLİHAN ÖS SİPAHİ

Signature :

ABSTRACT

ALGEBRAIC GEOMETRIC METHODS IN STUDYING SPLINES

Sipahi, Neslihan Ös
Ph.D., Department of Mathematics
Supervisor : Assoc. Prof. Dr. Mohan Lal Bhupal
Co-Supervisor : Assoc. Prof. Dr. Selma Altınok Bhupal

December 2013, 103 pages

In this thesis, our main objects of interest are piecewise polynomial functions (splines). For a polyhedral complex Δ in R^n, $C^r(\Delta)$ denotes the set of piecewise polynomial functions defined on Δ. Determining the dimension of the space of splines with polynomials having degree at most k, denoted by $C^r_k(\Delta)$, is an important problem, which has many applications. In this thesis, we first give an exposition on splines and introduce different algebraic geometric methods used to compute the dimension of splines both on polyhedral and simplicial complexes. Then we generalize the important result of Mcdonald and Schenck [23] on planar splines on a polyhedral complex. Also, by using the method in [18], we make generalizations on the dimension of the spaces of splines on simplicial complexes in dimension three. This generalizaton includes simplicial complexes having no interior points, and octahedrons with one interior point. In the latter case, we make some generalizations by considering the number of linearly independent interior planes.

Keywords: Spline, Polyhedral complex, Dimension formula, Hilbert polynomial, Homology modules

ÖZ

PARÇALI POLİNOM FONKSİYONLARINI ÇALIŞMAK İÇİN CEBİRSEL GEOMETRİK YÖNTEMLER

Sipahi, Neslihan Ös

Doktora, Matematik Bölümü

Tez Yöneticisi : Doç. Dr. Mohan Lal Bhupal

Ortak Tez Yöneticisi : Doç. Dr. Selma Altınok Bhupal

Aralık 2013 , 103 sayfa

Bu tezde odaklanacağımız temel nesneler parçalı tanımlı polinom fonksiyonlardır. Δ, $\mathbb{R}^n$'de çok yüzlü bir bölge belirtmek üzere, Δ üzerindeki düzgünlük derecesi r olan parçalı tanımlı polinom fonksiyonlar $C^r(\Delta)$ ile gösterilir. $C^r_k(\Delta)$, $C^r(\Delta)$'nın derecesi en fazla k olan polinomları içeren bir alt kümesidir ve bir vektör uzayı oluşturur ve bu vektör uzayının boyutunun hesaplanması, birçok uygulaması olan önemli bir problemdir. Bu tezde, öncelikle parçalı tanımlı polinom fonksiyonları hakkındaki çalışmaları özetleyip, hem polihedral, hem de simpleksler kompleksleri üzerindeki parçalı tanımlı polinom fonksiyonların boyut hesaplamasında kullanılan farklı cebirsel geometrik yöntemleri tanımlıyoruz. Daha sonra, Mcdonald ve Schenck'in [23], bir polihedral kompleks Δ üzerinde tanımlı düzlemsel parçalı polinom fonksiyonlarının oluşturduğu vektör uzayının boyutuna ilişkin önemli sonucunu genelleştiriyoruz. Ayrıca, [18] makalesindeki metodu kullanarak, üç boyuttaki bir simpleksler kompleksi üzerindeki parçalı polinom fonksiyonlarının boyutlarına ilişkin genelleştirmeler yapıyoruz. Bu genelleştirmeler, hiç iç noktası olmayan simpleksler komplekslerini ve bir iç noktası olan sekizyüzlüleri kapsıyor. Sekizyüzlüler durumunda, üzerinde tanımlı spline uzaylarının boyutlarını inceleyip onların lineer bağımsız iç düzlemlerinin sayısına bakarak boyutları konusunda bazı genelleştirmeler yapıyoruz.

Anahtar Kelimeler: Spline, Polihedral kompleks, Boyut formülü, Hilbert polinomu, Modül homolojileri

To my mother and father

ACKNOWLEDGMENTS

First, I would like to thank Assoc. Prof. Dr. Sefa Feza Arslan, my former advisor, for introducing me the splines and guiding me to study on this subject. Since he had to leave METU, I could not go on being his doctorate student, but we always kept studying together. He spent his priceless time for me without any expectation, except the pleasure of learning new subjects, so I want to thank him heartfeltedly. He is my role model as an academician.

I would also like to thank Assoc. Prof. Dr. Mohan Lal Bhupal, my current advisor. For three and a half years, I have been his doctorate student and he made it possible for me to continue working on the thesis I started with Assoc. Prof. Dr. Sefa Feza Arslan. He was always very polite towards me by answering all my questions and he always helped me diligently.

It was also a great honour for me to study with my co-advisor, Assoc. Prof. Dr. Selma Altınok Bhupal. She was a great advisor and an instructor, with her discipline and scientific guidance. Especially, the chapter on polyhedral simplices and results there, which we obtained together were impossible without her guidance and help. Above all she is a great mother, and with the Bhupals, I always felt as a member of their family.

I am also very thankful for the financial support of The Scientific and Technical Research Council of Turkey (TÜBİTAK).

Also, I should thank my friends, whom I have met during my doctorate education. We have handled lots of difficulties and despite the hardness of our education, we have shared so much happy memories. Without them everything would be less meaningful.

I want to express my gratitude to my parents Kamil Ös and Suzan Ös. They devoted their whole lives to me and my sisters Fatma Bilge and Nagihan Özerkmen. We always felt their love, support and trust with us. I had the most precious thing in the world, the happy childhood just by means of my family.

I am also thankful to my husband Ali Sipahi. While we were students, we've started to share a love and now, we became parents and tasted the greatest love with our daughter Mai Nehir Sipahi.

TABLE OF CONTENTS

ABSTRACT v

ÖZ vi

ACKNOWLEDGMENTS ix

TABLE OF CONTENTS x

LIST OF TABLES xiii

LIST OF FIGURES xiv

CHAPTERS

1 INTRODUCTION 1

2 THEORETICAL BACKGROUND 5

2.1 Polyhedral Complexes 5

2.2 Simplicial Complexes 7

2.3 Simplicial Homology 8

2.4 Univariate Splines 9

2.5 Multivariate Splines 11

2.5.1 Literature on multivariate splines 12

2.5.2 Main techniques for multivariate splines 14

2.5.3 Hilbert function, Hilbert series and Hilbert polynomial 16

3 ALGEBRAIC METHODS TO CALCULATE THE DIMENSION OF SPLINES 19

3.1 Matrix Method 19

3.2 Hilbert Series Method 23

3.3 Homology Method 25

3.4 The Method of Using Relations Among Fat Points, Inverse Systems and Splines 27

3.4.1 Fat Points and Inverse Systems 27

3.4.2 Piecewise Polynomial Functions on Simplicial Complexes 31

4 SPLINES WITH MIXED SMOOTHNESS DEGREE ON POLYHEDRAL COMPLEXES 37

4.1 Splines with Mixed Smoothness Degrees on Polyhedral Complexes 37

4.2 Examples 45

5 RESULTS IN DIMENSION THREE 63

5.1 Simplicial complexes with no interior point 63

5.1.1 Dimension of a general tetrahedron 67

5.2 All Possible Dimensions for an Octahedron 72

5.2.1 Octahedrons Having 3 Interior Hyperplanes 72

5.2.2 Octahedrons Having 4 Interior Hyperplanes 79

5.2.3 Octahedrons Having 5 Interior Hyperplanes 86

5.2.4 Octahedrons Having 6 interior Hyperplanes 88

5.2.5 Octahedrons Having 7 interior Hyperplanes 90

5.2.6 Octahedrons Having 8 Interior Hyperplanes 91

5.2.7 Octahedrons Having 9 Interior Hyperplanes 93

5.2.8 Octahedrons Having 10 Interior Hyperplanes . . . 94

5.2.9 Octahedrons Having 11 Interior Hyperplanes . . . 95

5.2.10 Octahedrons Having 12 Interior Hyperplanes . . . 97

6 CONCLUSION 99

REFERENCES 101

CURRICULUM VITAE 103

LIST OF TABLES

TABLES

Table 4.1 Table giving the some properties of $G_p(\Delta)$ 57

Table 4.2 Table giving the some properties of $G_p(\Delta)$ 61

LIST OF FIGURES

FIGURES

Figure 2.1 Example of a 2-dimensional polyhedral complex 6

Figure 2.2 Example of a non-polyhedral complex 6

Figure 2.3 Example of a non-pure complex 6

Figure 2.4 Example of a hereditary and non-hereditary complexes 7

Figure 2.5 Directed simplex 8

Figure 2.6 The difference between a simplicial complex and the polyhedral complex 8

Figure 2.7 Homogenization of the polyhedral complex Δ 15

Figure 3.1 2-dimensional simplicial complex with four 2-cells 20

Figure 3.2 Planar simplicial complex 34

Figure 3.3 Complicated figure 34

Figure 4.1 Polyhedral complex containing extra codimension-2 space except the interior vertices 41

Figure 4.2 Associated dual graph $G_{p_0}(\Delta)$ for the point p_0 in Fig. 4.1 42

Figure 4.3 Δ with four 2-faces, six interior 1-faces and three interior vertices . 45

Figure 4.4 Ordered form of the linear forms of the Fig. 4.1 45

Figure 4.5 $G_{p_0}(\Delta)$ 48

Figure 4.6 $G_{p_1}(\Delta)$ 49

Figure 4.7 Polyhedral complex with thirteen 2-faces and twenty interior 1-faces 51

Figure 4.8 Ordered forms of the points and the linear forms of the Fig. 4.7 . . 52

Figure 4.9 $G_{p_0}(\Delta)$ for p_0 54

Figure 4.10 Polyhedral complex resembling to a spider web 55

Figure 4.11 Ordered forms of the points and the linear forms of the Fig. 4.10 . . 56

Figure 4.12 Associated dual graph of p_0 60

Figure 5.1 Simplicial complex with n-gon base having no interior point 64

Figure 5.2 Tetrahedron with no interior point 67

Figure 5.3 Regular octahedron 72

Figure 5.4 Octahedron having four hyperplanes 79

Figure 5.5 Octahedron constructed by moving a vertex of a regular octahedron through a line 81

Figure 5.6 Octahedron constructed by moving a vertex of a regular octahedron leaving in a plane 82

Figure 5.7 Octahedron having 5 interior hyperplanes 87

Figure 5.8 Octahedron with six hyperplanes 89

Figure 5.9 Octahedron with 7 hyperplanes 90

CHAPTER 1

INTRODUCTION

Our main objects of interest in this thesis are piecewise polynomial functions, which are also called splines or finite elements. For a polyhedral complex Δ in R^n, $C^r(\Delta)$ is the set of piecewise polynomial functions (splines) defined on Δ that are continuously differentiable up to order r. For each k in $\mathbb{N}$, $C^r_k(\Delta)$ is the subset of $C^r(\Delta)$ consisting of the piecewise polynomial functions (splines) on Δ, such that on each face σ of Δ, the restriction of the piecewise polynomial function to σ has degree less than or equal to k. Determining the dimension of the vector space $C^r_k(\Delta)$ has crucial importance, especially in geometric modelling and approximation theory. Splines are used in several areas related with geometric design, graphics, and robotics supported by computers. They increase the power to control of the shape of a surface. Computing the dimension of $C^r_k(\Delta)$ involves several different branches of mathematics such as algebra and geometry.

In this thesis we utilize not only the combinatorics, but also the geometry of Δ and the algebraic properties of the functions forming the splines in order to compute the dimension of $C^r_k(\Delta)$. This is not an easy computation especially when the dimension of the space in which Δ embedded is greater than 2. We present several algebraic geometric methods to compute the dimensions of the vector spaces of splines on 2 and 3 dimensional polyhedral complexes. We make some generalizations on the dimensions of the vector spaces of splines on some special complexes.

In [11], Billera and Rose presented a homological approach in the study of splines. They showed that for a polyhedral complex Δ and fixed smoothness degree r, $C^r_k(\Delta) \simeq C^r(\hat{\Delta})_k$, and $C^r(\hat{\Delta}) = \cup_{k\geq 0} C^r(\hat{\Delta})_k$, where $\hat{\Delta}$ is defined to be the homogenization of Δ.

(Homogenization is explained in Chapter 2). Hence the dimension of the vector space of splines is given as the Hilbert function of a graded algebra. Thus $\dim C^r_k(\Delta)$ is given by a polynomial in k, $f(\Delta, r, k)$, for sufficiently large k, which is called the Hilbert polynomial. Consequently, throughout the thesis, we are interested in computing Hilbert series and Hilbert functions. Computation of Hilbert series is very important for computational commutative algebra and algebraic geometry and appear in various contexts.

In [18], Geramita and Schenck, presented a connection between fat points and the inverse systems and using this gave the free resolution of an ideal generated by the mixed powers of homogeneous linear forms. Using the latter together with the Hilbert function, they were able to compute the dimension of planar splines on simplicial complexes for any mixed smoothness degree r. In this thesis, we apply this method to splines in 3-space to obtain new results.

In [8], Alfeld and Schumaker obtain all three coefficients of the polynomial giving the dimension of the vector space of splines on a simplicial complex Δ for dimension 2. Later, in [23], Mcdonald and Schenck improve their result and determine three coefficients of the polynomial $f(\Delta, r, k)$, which gives the dimension of the vector space of splines on a d-dimensional polyhedral complex Δ with fixed smoothness degree r, for sufficiently large k. In this thesis, one of our aims is to refine the formula given by Mcdonald and Schenck to mixed smoothness degrees. We obtain all three coefficients of the polynomial $f(\Delta, \alpha, k)$ giving the dimension of the vector space of splines with mixed smoothness degrees $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{f_1^0})$ on a 2-dimensional polyhedral complex Δ, each α_i is the smoothness degree on the corresponding interior 1-face of Δ, and f_1^0 is the number of the interior 1-faces of Δ. More precisely, we prove the following theorem:

Theorem 1.0.1 *For a 2-dimensional hereditary, pure polyhedral complex Δ, we have*

$$\begin{aligned} HP(C^r(\hat{\Delta}), k) &= (f_2 - f_1^0)\binom{k+2}{2} + \sum_{i=1}^{f_1^0} \binom{k+2-\alpha_i-1}{2} + \mathrm{a}_0(N) \\ &= (f_2 - f_1^0)\binom{k+2}{2} + \sum_{i=1}^{f_1^0} \binom{k+2-\alpha_i-1}{2} + \sum_{\psi_j \in H_1(G_{\xi_i}(\Delta))} c_j, \end{aligned} \tag{1.1}$$

where $c_j = \mathrm{a}_0(HP(\hat{R}/I_{\psi_j}))$ *for* $I_{\psi_j} = (l_1^{\alpha_1+1}, l_2^{\alpha_2+1}, \cdots, l_{n_j}^{\alpha_{n_j}+1})$.

In Chapter 2, we give some preliminaries and theoretical background necessary for understanding the computations and generalizations contained in this thesis. We also give a survey of the literature on splines.

In Chapter 3, we present methods contained in the literature to compute the dimensions of splines and give a detailed review of the article of Geramita and Schenck [18], since the techniques and results will be used both in Chapters 4 and 5 to obtain new results. In this article, a formula for the dimension of planar splines for any 2-dimensional simplicial complex is given. Their method depends on constructing a special chain complex, and it transforms the computation of the dimensions of the vector spaces of splines on Δ of degree less than or equal to k to the problem of computing the Hilbert functions of ideals generated by powers of homogeneous linear forms.

In Chapter 4, by using the results of Geramita and Schenck [18] together with the results of McDonald and Schenck [23] for polyhedral complexes with fixed r smoothness, we obtain a formula defining all three coefficients of the Hilbert polynomials of the vector spaces corresponding to mixed degree splines. In this sense, we generalize the formula in [23] to the case of mixed smoothness degree.

In Chapter 5, by modifying the method developed in [18] to 3-dimensional simplicial complexes, we obtain a general formula giving the dimensions of the space of splines defined on n-gons with no interior points. Furthermore, we give a formulae for the dimensions of the spaces of splines on octahedrons by considering the number of their linearly independent interior hyperplanes.

CHAPTER 2

THEORETICAL BACKGROUND

In this chapter, we present preliminary definitions and the necessary theoretical background.

2.1 Polyhedral Complexes

Let C be a subspace of $\mathbb{R}^d$. If, for any two elements $c_1, c_2 \in C$, we have $tc_1 + (1-t)c_2 \in C$ for $t \in I = [0, 1]$, then C is called a **convex set**. The **convex hull**, Conv(S), of a set S consisting of finitely many points in $\mathbb{R}^d$ is the smallest convex set that contains S, which is given as a set by

$$\mathrm{Conv}(S)\left\{\sum_{i=1}^{|S|} \alpha_i x_i \ : \ \alpha_i \geq 0 \text{ and } \sum_{\mathrm{i}=1}^{|\mathrm{S}|} \alpha_{\mathrm{i}} = 1\right\},$$

where $|S|$ is the number of points $x_i \in S$.

The convex hull of a finite set in $\mathbb{R}^d$ is called a **polytope**. The lower dimensional boundaries of a polytope are called its faces. For example, any triangle and any line of an octahedron is a face of it.

Definition 2.1.1 ([22]) $\Delta \in \mathbb{R}^d$ *which is the finite union of polytopes is a* ***polyhedral complex****, if the faces of each elements of* Δ *are elements of* Δ*, and the intersection of any two elements of* Δ *is an element of* Δ*. 3-dimensional polyhedral complexes are called as* ***polyhedron****.*

Note that a k-dimensional element is called as a k-cell and we can think of a complex as the union its cells.

Example 2.1.2 *Fig. 2.1 is an example of a planar polyhedral complex, which has three 2-cells, seven 1-cells, five 0-cells and the empty set.*

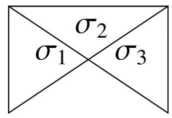

Figure 2.1: Example of a 2-dimensional polyhedral complex

In contrast to the polyhedral complex above, Fig. 2.2 is not a polyhedral complex, since the intersection of some of its elements is not an element of the complex again.

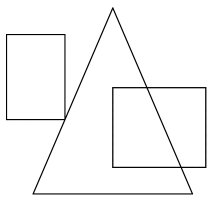

Figure 2.2: Example of a non-polyhedral complex

Definition 2.1.3 *If each maximal element of a polyhedral complex* $\Delta \subset \mathbb{R}^d$ *is d-dimensional (with respect to inclusion), then* Δ *is called* ***pure*** *d-dimensional.*

Example 2.1.4 *Here is an example of a non-pure complex, since* σ_2 *has dimension one:*

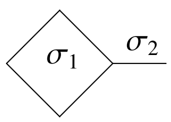

Figure 2.3: Example of a non-pure complex

Definition 2.1.5 *In a complex* Δ*, if two d-dimensional polytopes intersect along a common* $d - 1$ *dimensional face, they are called* ***adjacent****.* Δ *is called a* ***hereditary***

complex *if, for any τ in Δ (including the empty set), any d-dimensional polytopes σ and σ' in Δ that contain τ can be connected by a sequence of d-dimensional polytopes in Δ such that $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_m = \sigma'$, where each σ_i contains τ and for each i, σ_i and σ_{i+1} are adjacent.*

Example 2.1.6 *In Fig. 2.4 given below, the left one is not hereditary, since p is contained in both σ_1 and σ_2, but there is no sequence of 2-cells containing p connecting σ_1 and σ_2. On the other hand, the figure on the right is a hereditary complex.*

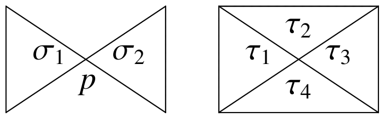

Figure 2.4: Example of a hereditary and non-hereditary complexes

2.2 Simplicial Complexes

A simplicial complex is a special polyhedral complex. Polyhedral complexes are constructed by using polytopes, while simplicial complexes are constructed as the finite union of simplices, which are special polytopes. Hence, all the definitions given for polyhedral complexes are also valid for simplicial complexes.

A d-simplex is a d-dimensional polytope, which is the convex hull of its $d+1$ vertices $v_0, v_1, \cdots, v_d$ satisfying the condition that $\{v_1 - v_0, v_2 - v_0, \cdots, v_d - v_0\}$ are linearly independent. In particular, a d-simplex is the set of points

$$C = \{a_0v_0 + a_1v_1 + \cdots + a_dv_d \ : \ a_i \geq 0, \text{ for } 0 \leq \mathrm{i} \leq \mathrm{d}, \text{ and } \sum_{\mathrm{i}=0}^{\mathrm{d}} \mathrm{a_i} = 1\}.$$

We can denote a d-simplex by $\{v_0, v_1, \cdots, v_d\}$. If we give an ordering to the vertices of the simplex we show this by using the notation $\langle v_0, v_1, \cdots, v_d\rangle$. For example, a 1-simplex is a line, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. When we write $\langle v_0, v_1, v_2\rangle$, we work on a triangle with rotation sense as in the Fig. 2.5.

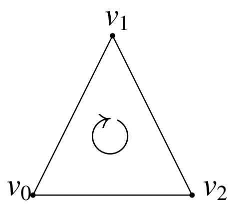

Figure 2.5: Directed simplex

Example 2.2.1 *Fig. 2.6 below demonstrates the difference between a polyhedral complex and a simplicial complex:*

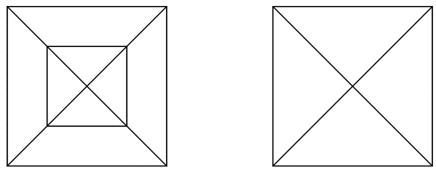

Figure 2.6: The difference between a simplicial complex and the polyhedral complex

Here the figure on the left is a polyhedral complex, but it is not a simplicial complex, since it contains some elements that are not triangles. On the other hand, the figure on the right is a simplicial complex, since each element of it is a simplex and obviously, it is a polyhedral complex.

In this thesis, we focus on both simplicial and polyhedral complexes. In fact, making generalizations about splines over simplicial complexes is much easier than that in the polyhedral complex case. Some results about splines over simplicial complexes are still open in the polyhedral case. In the next chapter, we give some methods that we use to compute the dimension of the vector spaces of splines, but we will see that some of these methods work only in the simplicial complex case.

2.3 Simplicial Homology

A chain complex is a sequence of abelian groups or modules, which are connected by homomorphisms satisfying

$$\cdots \to \mathrm{A_{d+1}} \xrightarrow{\partial_{d+1}} \mathrm{A_d} \xrightarrow{\partial_d} \cdots \to \mathrm{A_1} \xrightarrow{\partial_1} \mathrm{A_0} \to 0,$$
$$\partial_i \circ \partial_{i+1} = 0.$$

For a $(d+1)$-simplex $\sigma = \langle v_0, v_1, \cdots, v_d \rangle$, where v_i's are the vertices, we define the boundary map ∂_d as

$$\partial_n(\sigma) = \sum_{i=0}^{d} (-1)^i \sigma|_{\langle v_0, v_1, \cdots, \hat{v}_i, \cdots, v_d \rangle}.$$

Here ˆ denotes a deleted vertex.

An open simplex is a simplex with all its proper faces deleted. Let Δ_i^0 denote the set of i-dimensional interior faces. Given a complex κ of R-modules on the interior faces of Δ,

$$0 \to \oplus_{\sigma \in \Delta_d} \kappa(\sigma) \xrightarrow{\partial_d} \oplus_{\gamma \in \Delta_{d-1}^0} \kappa(\gamma) \xrightarrow{\partial_{d-1}} \cdots \to \oplus_{\beta \in \Delta_0^0} \kappa(\beta) \to 0,$$

the i^{th} homology of this complex is defined to be $H_i(\kappa) = \ker \partial_i / \operatorname{im} \partial_{i+1}$.

2.4 Univariate Splines

Referring to [22], we now give the basic definitions and theory about splines. We use the notation and theory in . First, we give the definition of a univariate spline (a spline defined over a polyhedral complex Δ on the real line). Let Δ be the interval $[a, b]$ in $\mathbb{R}$. Consider the piecewise function F on the interval $[a, b]$, defined by

$$F(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c], \\ f_2(x) & \text{if } x \in [c, b]. \end{cases} \tag{2.1}$$

where f_1 and f_2 are polynomials in $\mathbb{R}[x]$, and c satisfies $a < c < b$. In this case, F is called a spline or piecewise polynomial function. By well-definedness of F, $f_1(c) = f_2(c)$, and this makes F continuous on $[a, b]$. The trivial case $f_1 = f_2$ is not interesting. If we take different polynomials, we get extra control over the graph of the function. Since f_1 and f_2 are polynomials, they have derivatives of any orders, hence, for any $r \geq 0$, we can consider splines differentiable up to order r:

$$F^{(r)}(x) = \begin{cases} f_1^{(r)}(x) & \text{if } x \in [a, c], \\ f_2^{(r)}(x) & \text{if } x \in [c, b]. \end{cases}$$

If we want F to be a C^r function (namely, F to be differentiable up to order r) on $[a, b]$, we should have $f_1^{(k)}(c) = f_2^{(k)}(c)$ for any k, where $0 \leq k \leq r$. Algebraically, this property is equivalent to the next proposition.

Proposition 2.4.1 ([22, Proposition 3.2]) *The piecewise polynomial function F defined in Eqn. (2.1) is a C^r function on $[a, b]$ if and only if the polynomial $f_1 - f_2$ is divisible by $(x - c)^{r+1}$.*

Proof. Suppose $F \in C^r$. We need to show that $(x-c)^{r+1}$ divides $f_1 - f_2$. We prove this by induction on r. For $r = 0$ we have $f_1(c) = f_2(c)$. Hence, $f_1(x) - f_2(x) = (x - c)p(x)$ for some $p(x) \in \mathbb{R}[x]$. For the induction step, assume that $F \in C^{r-1}$, since $F \in C^{r-1}$, there exists a polynomial $h(x)$ satisfying, $f_1(x) - f_2(x) = (x - c)^r h(x)$. We take the r^{th} derivatives of both sides. Every term on the right-hand side, except the one $r!h(x)$ contains $(x - c)$. Since $f_1^{(r)}(c) - f_2^{(r)}(c) = 0$, $(x - c)$ divides $h(x)$, so $(x - c)^{r+1}$ divides $f_1(x) - f_2(x)$.

Conversely, suppose that $f_1(x) - f_2(x)$ is divisible by $(x - c)^{r+1}$, then $(f_1 - f_2)(x) = (x - c)^{r+1} p(x)$ for some $p(x) \in \mathbb{R}[x]$. When we take any t^{th} derivative of the equality with $0 \leq t \leq r$, each term in the right-hand side contains $(x - c)$, which implies that $F \in C^r$. ■

We denote the piecewise polynomial function F in Eqn. (2.1) by the ordered pair $(f_1, f_2) \in \mathbb{R}[x]^2$. In this sense, C^r splines form a subspace of $\mathbb{R}[x]^2$, under componentwise addition and scalar multiplication. The set of all splines (f_1, f_2) with the degrees of the polynomials f_1 and f_2 less than or equal to k is a finite dimensional vector space $C_k^r \subset \mathbb{R}[x]^2$. Any spline (f_1, f_2) in C_k^r can be rewritten as a sum of pairs $(f_1, f_1) + (0, f_2 - f_1)$, in which each summand still has degree less than or equal to k. Here (f_1, f_1) is obviously a C^r spline for any $r \geq 0$. But the spline $(0, f_2 - f_1)$ is not always C^r. By Prop. 2.4.1, $(0, f_2 - f_1)$ is C^r spline if and only if $(x - c)^{r+1}$ divides $f_2 - f_1$. If this spline is not trivial ($f_1 \neq f_2$), a necessary condition for $(x - c)^{r+1}$ to divide $f_2 - f_1$ is that $r + 1 \leq k$. If $r + 1 \leq k$, linear combinations of $(0, (x - c)^{r+1}), (0, (x - c)^{r+2}), \cdots, (0, (x - c)^k)$ are elements of C_k^r. This gives us the following proposition.

Proposition 2.4.2 ([22, Proposition 3.4]) *In a subdivided interval* $[a,b] = [a,c] \cup [c,b]$*, the dimension of the space* C_k^r *for one-variable spline functions is equal to*

$$\dim_{\mathbb{R}} C_k^r = \begin{cases} k+1 & \text{if } r+1 > k, \\ 2k-r+1 & \text{if } r+1 \leq k. \end{cases}$$

Note that, if $r+1 > k$, C_k^r contains only the trivial splines, so it has $(1,1)$, (x,x), $\cdots$, (x^k, x^k) as basis elements. But if $r+1 \leq k$, in addition to the trivial splines, we also have non trivial ones, and they are linear combinations of $(0,(x-c)^{r+1})$, $(0,(x-c)^{r+2})$, $\cdots$, $(0,(x-c)^k)$, which are also among the basis elements.

Proposition 2.4.3 ([25, Önerme 3.3]) *Let* $[a,b]$ *be a closed interval divided into* s *subintervals in such a way that* $[a,b] = [x_0,x_1] \cup [x_1,x_2] \cup \cdots \cup [x_{s-2},x_{s-1}] \cup [x_{s-1},x_s]$.
i-) *For an* s *tuple of polynomials* $(f_1, f_2, \cdots, f_s)$*, define a function* F *on the interval* $[a,b]$ *such that* $f_i = F|_{[x_{i-1},x_i]}$*. Then* $F \in C^r$ *if and only if for* $1 \leq i \leq s-1$, $f_{i+1} - f_i \in \langle (x-x_i)^{r+1} \rangle$.
ii-) *The dimension of the space of* C^r *splines with polynomials having at most k degree is equal to*

$$\dim C_k^r = \begin{cases} k+1 & \text{if } r+1 > k, \\ s(k-r)+r+1 & \text{if } r+1 \leq k. \end{cases}$$

Proof. For a detailed proof, see [25, Önerme 3.3]. ∎

2.5 Multivariate Splines

While working on $\mathbb{R}$, we divided intervals into subintervals that intersect at common endpoints. We apply the same idea to polyhedral complexes in $\mathbb{R}^d$ by dividing them into polytopes intersecting along common faces. Hence, we work on d-dimensional pure and hereditary polyhedral complexes (and sometimes on simplicial complexes having these properties) in $\mathbb{R}^d$. Ordering the d-cells of the complex $\sigma_1, \sigma_2, \cdots, \sigma_m$, we consider a polyhedral complex Δ as the union of the d-cells: $\Delta = \cup_{i=1}^m \sigma_i$. For a d-complex Δ and $i \leq d$, let Δ_i, Δ^0, Δ_i^0, $f_i(\Delta)$ and $f_i^0(\Delta)$ denote the set of i-dimensional faces, the set of interior faces, the set of i-dimensional interior faces, the number of i-dimensional faces and the number of i-dimensional interior faces of Δ, respectively. Note that d-faces are always considered as interior.

For a d-complex Δ and $r \in \mathbb{N}$, $C^r(\Delta)$ is the set of the functions $F : \Delta \to \mathbb{R}$ satisfying:

i) For each $\sigma \in \Delta_d$, $F|_\sigma$ is in $\mathbb{R}[x_1, \cdots, x_d]$. (In fact $F|_{\sigma_i}$ is equal to the polynomial f_i.)

ii) F is continuously differentiable up to order r on Δ.

$C^r_k(\Delta)$ is a subset of $C^r(\Delta)$. It consists of elements $F \in C^r(\Delta)$ satisfying $F|_\sigma$ is a polynomial of degree at most k for any $\sigma \in \Delta_d$. F is defined to have r smoothness at a point p if for each $\sigma \in \Delta_d$ containing p, $F|_\sigma$ has the same value up to order r at p.

Adjacent d-cells σ_i and σ_j intersect along a $(d-1)$-cell, since we work on hereditary complexes. The intersection is denoted as τ_{ij}, which is contained in an affine hyperplane $\mathrm{V}(l_{ij})$, where $l_{ij} \in \mathbb{R}[x_1, x_2, \cdots, x_d]$ is a polynomial of degree one.

2.5.1 Literature on multivariate splines

In [2], Courant gave the idea of using continuous splines in approximation theory. Influenced by Courant's idea, Strang introduced the problem of finding the dimension of the spaces of splines in [3], [4]. Strang made the following conjecture: For a generically embedded planar 2-manifold Δ,

$$\dim C^1_m(\Delta) = \binom{m+2}{2} f_2 - (2m+1) f_1^0 + 3 f_0^0, \tag{2.2}$$

where f_2 is the number of triangles in Δ, f_1^0 and f_0^0 the number of interior edges and interior vertices, respectively. In [5], Morgan and Scott showed that for $m \geq 5$ and for any embedding of Δ, the dimension of $C^1_m(\Delta)$ is equal to the right hand side of Eqn. (2.2) plus the number of rectangles triangulated by crossing diagonals. They also gave an explicit basis for $C^1_m(\Delta)$ for $m \geq 5$. In [6], they showed that their solution was not true for $m = 2$ by an example. The Morgan-Scott formula was shown to be a lower bound for all $m \geq 2$ by Schumaker, see [7]. Also, he gave a lower bound for the dimension of $C^r_m(\Delta)$ for $m \geq r + 1$. By improving Morgan and Scott's result, Alfeld and Schumaker showed that Schumaker's lower bound gives exactly the dimension of C^r_m for $m \geq 4r + 1$ for all r [8].

In [10], Billera considered the problem in a homological way, and by using the homological approach on the triangulated manifolds Δ in $\mathbb{R}^2$. He gave lower bounds on the dimension of $C^r_m(\Delta)$ for all r. He proved Strang's Conjecture in the affirmative in the case $r = 1$ over a triangulated manifold in $\mathbb{R}^2$. In [11], Billera and Rose gave computational techniques to find bases for the spaces $C^r(\Delta)$ by using the Gröbner basis methods. Defining $C^r(\Delta)$ as the kernel of a map between free modules, they gave an exact sequence of graded modules. In [12], Billera and Rose focused on the freeness of $C^r(\Delta)$ depending on Δ, r, and d as an R-module, because in the case of freeness of $C^r(\Delta)$, the dimension is independent of the embedding of Δ in $\mathbb{R}^d$ just on its combinatorics. When $d = 2$, they proved that $C^r(\Delta)$ is free if and only if Δ is a manifold with boundary. They also showed that the module $C^r(\hat{\Delta})$ is free if and only if $C^r(star(\sigma))$ is free for all faces σ of Δ (This has great importance, because $\hat{\Delta}$ is the star of the origin.).

In [13], Schenck and Stillman showed that for dimension 2, $C^r(\hat{\Delta})$ is free if and only if $H_1(R/J)$ vanishes. They also gave a non-freeness result for 2-dimensional complexes having interior edges, such that edges which do not reach the boundary. In this case, there exists an r_0 satisfying that $C^r(\hat{\Delta})$ is non-free for all $r > r_0$.

In [14], Schenk and Stillman showed that the dimensions of the splines given by Billera and Rose were the same with the bounds on dimensions given by Alfeld and Schumaker. They extended the Alfeld and Schumaker's bounds for all degrees in $d = 2$ case.

In [16], Schenck defined a complex, top homology module of which was isomorphic to $C^r(\hat{\Delta})$ and he determined bounds on the dimension of the homology modules. He showed that for all $i < d$, $\dim_{\mathbb{R}} H_i(R/J) \leq i - 1$.

Geramita and Schenck, in [18], searched the relation between the ideals of fat points and splines on a d-dimensional simplicial complex Δ embedded in $\mathbb{R}^d$. Moreover, by using this relation, they derived a formula, which gave the number of planar splines for sufficiently high degree.

In [17], Dalbec and Schenck focused on Rose conjecture saying that for a fixed simplicial complex Δ, the freeness of $C^r(\hat{\Delta})$ implies the freeness of $C^{r-1}(\hat{\Delta})$. They showed that the conjecture is true for $d = 2$, but they gave a counter example in $d = 3$. Dalbec and Schenck's results are still not known to be true, when $\hat{\Delta}$ is a polyhedral complex.

In [21], Rose considered the spline modules on polyhedral complexes as the syzygy module of its dual graph with edges weighted by powers of linear forms. By using some techniques without changing the isomorphism class of the syzygy module, Rose took the dual graph into pieces and by these pieces, she calculated the homological dimension of the Hilbert series of the module.

McDonald and Schenck, in [23], extended the formula of Alfeld and Schumaker, giving the dimension of the splines on a simplicial complex in $\mathbb{R}^2$ by a polynomial $f(\Delta, r, k)$, depending on the complex Δ, smoothness r and degree k (for sufficiently large k). They gave the first three coefficients of the polynomial $f(\Delta, r, k)$ for polyhedral complexes of any dimension. In $d = 2$ case, they give the dimension of the splines.

2.5.2 Main techniques for multivariate splines

The theorem given for univariate splines is generalized to multivariate spline case as follows in [22]:

Proposition 2.5.1 ([22, Proposition 3.7]) *Let Δ be a pure and hereditary complex that contains m d-cells σ_i and $F \in C^r(\Delta)$ such that $f_i = F|_{\sigma_i} \in \mathbb{R}[x_1, \cdots, x_d]$ for $1 \leq i \leq m$. Then for each adjacent pair σ_i, σ_j in Δ, $f_i - f_j \in \langle l_{ij}^{r+1} \rangle$. Conversely, any m-tuple of polynomials $(f_1, \cdots, f_m)$ satisfying $f_i - f_j \in \langle l_{ij}^{r+1} \rangle$ for each adjacent pair σ_i, σ_j of d-cells in Δ defines an element $F \in C^r(\Delta)$ by setting $F|_{\sigma_i} = f_i$.*

Proof. For a detailed proof, see [25, Önerme 3.6]. ■

From now on, we consider $F \in C^r(\Delta)$ as an m-tuple of R-polynomials such as $F = (f_1, f_2, \cdots, f_m)$, where $f_i = F|_{\sigma_i}$ and $R = \mathbb{R}[x_1, x_2, \cdots, x_d]$. By this representation $C^r(\Delta)$ can be regarded as an R-module under the pointwise addition and multiplication.

For $h \in R$, $(f_1, f_2, \cdots, f_m)$ and $(g_1, g_2, \cdots, g_m) \in C^r(\Delta)$,

$(f_1, f_2, \cdots, f_m) + (g_1, g_2, \cdots, g_m) = (f_1 + g_1, f_2 + g_2, \cdots, f_m + g_m)$ and
$h.(f_1, f_2, \cdots, f_m) = (h.f_1, h.f_2, \cdots, h.f_m)$.

We embed Δ into $\mathbb{R}^{d+1}$ by sending any element $(a_1, a_2, \cdots, a_d)$ in $\mathbb{R}^d$ to $(a_1, a_2, \cdots, a_d, 1)$ in $\mathbb{R}^{d+1}$. In other words, we embed Δ to the hyperplane $x_{d+1} = 1$, and construct the convex hull of each σ in Δ_d with the origin $p = (0, 0, \cdots, 0)$ in $\mathbb{R}^{d+1}$. Hence, we get $\hat{\Delta} = \Delta.p$, join of Δ with p.

We can give an example demonstrating the homogenization of a polyhedral complex Δ:

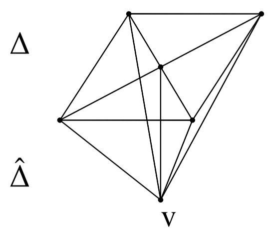

Figure 2.7: Homogenization of the polyhedral complex Δ

Example 2.5.2 *Here Δ is a simplicial complex, which is a quadrilateral splited into 4 triangles by its diagonals. We embed $\Delta \in \mathbb{R}^2$ into $\mathbb{R}^3$. In this process, any point (a, b) in Δ converts to $(a, b, 1)$ in $\mathbb{R}^3$ and by joining these points with v, we derived $\hat{\Delta}$ which is a pentahedron.*

In this way $C^r(\hat{\Delta})$ becomes an $\hat{R}$-module, where $\hat{R} = R[x_{d+1}]$. And any function on Δ can be carried to a cone over $\hat{\Delta}$, by taking the homogenization of itself. We define the homogenization ${}^hf$ of $f(x_1, x_2, \cdots, x_d) \in R$ as follows: ${}^hf = x_{d+1}^{\partial f} f(\frac{x_1}{x_{d+1}}, \frac{x_2}{x_{d+1}}, \cdots, \frac{x_d}{x_{d+1}})$. Here ∂f defines the total degree of f. We define the set of C^r-splines over $\hat{\Delta}$ as $C^r(\hat{\Delta})$. In [11], Billera and Rose showed that if $F = (f_1, f_2, \cdots, f_m) \in C^r(\Delta)$ then ${}^hF \in C^r(\hat{\Delta})$, and it is defined as
${}^hF ={}^h (f_1, f_2, \cdots, f_m) = (x_{d+1}^{\partial F - \partial f_1}({}^hf_1), x_{d+1}^{\partial F - \partial f_2}({}^hf_2), \cdots, x_{d+1}^{\partial F - \partial f_m}({}^hf_m))$, where ∂F means the maximum of the ∂f_i's.

Billera and Rose in [11] proved $C^r(\hat{\Delta})$ is a finitely generated graded module over $\hat{R}$. Moreover, they have shown that there is a vector space isomorphism between $C^r_k(\Delta)$ and $C^r(\hat{\Delta})_k$, which is the set of functions defined on $\hat{\Delta}$ having degree exactly k. Hence, we can consider the graded module $C^r(\hat{\Delta})$ as the union of $C^r(\hat{\Delta})_k$'s. Namely,

$C^r(\hat{\Delta}) = \cup_{k\geq 0} C^r(\hat{\Delta})_k$. Instead of calculating $\dim_{\mathbb{R}} C^r_k(\Delta)$, we will generally be interested in finding the dimension of $C^r(\hat{\Delta})_k$. This leads us to Hilbert function and series computations. Hence, let us recall the basic definitions and theorems about Hilbert functions and series.

2.5.3 Hilbert function, Hilbert series and Hilbert polynomial

A finitely generated graded $k[x_1, \cdots, x_d]$-module S can be written as $\bigoplus_{i\geq 0} S_i$ such that $S_0 = k$, where k is a field and S_i denotes the i^{th} degree graded part of S. The Hilbert function of S, $HF(S,k) : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$HF : i \mapsto \dim_k S_i$$

mapping i to the dimension of the k-vector space S_i.

Hilbert series of S, $HS(S,t)$, is defined to be the generating function

$$HS(S,t) = \sum_{i\geq 0} HF(S,i)t^i.$$

Theorem 2.5.3 ([24]) *Hilbert Series of any graded ring S can be given in rational forms as:*

1-)$HS(S,t) = \dfrac{Q(t)}{(1-t)^d}$, *where* $Q(t) \in \mathbb{Z}[t]$ *and* $d = \dim k[x_1, \cdots, x_d]$.

2-)$HS(S,t) = \dfrac{G(t)}{(1-t)^s}$, *where* $G(t) \in \mathbb{Z}[t]$ *is not divisible by* $(1-t)$ *and* $s = \dim S$.

For i sufficiently large, the Hilbert function of S is equal to a polynomial with rational coefficients having degree $d-1$. This polynomial is called as Hilbert polynomial $HP(S,t)$.

Now, since we have seen above that $C^r(\hat{\Delta})$ is a finitely generated graded $\hat{R}$-module, the dimension of its graded pieces can be determined by using the Hilbert series:

$$
\begin{aligned}
HS(C^r(\hat{\Delta}), t) &= \sum_{k\geq 0} HF(C^r(\hat{\Delta}), k)t^k, \\
&= \sum_{k\geq 0} \dim_{\mathbb{R}} C^r(\hat{\Delta})_k t^k, \\
&= \sum_{k\geq 0} \dim_{\mathbb{R}} C^r_k(\Delta) t^k.
\end{aligned}
$$

CHAPTER 3

ALGEBRAIC METHODS TO CALCULATE THE DIMENSION OF SPLINES

3.1 Matrix Method

In this section, we first present the main technique, which can be used both for polyhedral and simplicial complexes. As we have mentioned before, we work with pure and hereditary complexes. Let $F \in C^r(\Delta)$ be a spline, where Δ is a d-dimensional complex. Two adjacent d-cells σ_i and σ_j of Δ intersect along a $(d-1)$-dimensional interior face τ_{ij}, which is contained in the hyperplane l_{ij}. This can be written algebraically as follows:

$$f_i - f_j + g_{ij} l_{ij}^{r+1} = 0, \text{ where } \mathrm{g_{ij}} \in \mathbb{R}[\mathrm{x}_1, \cdots, \mathrm{x_d}],$$

and here we denote $F|_{\sigma_i}$ and $F|_{\sigma_j}$ by f_i and f_j.

Example 3.1.1 *We apply these ideas to compute the splines over the simplicial complex Δ given in the Fig. 3.1. Δ is a 2-dimensional simplicial complex with four 2-cells. We label these cells as $\sigma_1, ..., \sigma_4$ in a clockwise fashion. An element F in $C^r(\Delta)$ is represented as (f_1, f_2, f_3, f_4), where $f_i = F|_{\sigma_i}$ for $1 \le i \le 4$. Here σ_1 and σ_2 intersect along a line segment contained in the $y = 0$ plane. Hence we can express all the intersections as below:*

$$\begin{aligned} f_1 - f_2 + g_1 y^{r+1} &= 0, \\ f_2 - f_3 + g_2 (x-y)^{r+1} &= 0, \\ f_3 - f_4 + g_3 y^{r+1} &= 0, \\ f_1 - f_4 + g_4 x^{r+1} &= 0. \end{aligned} \tag{3.1}$$

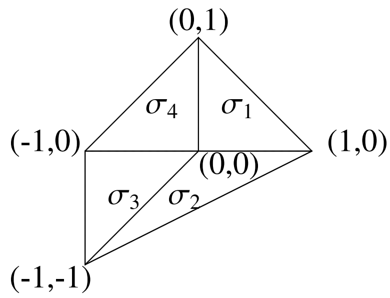

Figure 3.1: 2-dimensional simplicial complex with four 2-cells

where $g_i(x, y) \in \mathbb{R}[x, y]$. We can rewrite these equations in a matrix form as follows.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & (y)^{r+1} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & (x-y)^{r+1} & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & (y)^{r+1} & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & (x)^{r+1} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (3.2)$$

Here the elements of $C^r(\Delta)$ are given as (f_1, f_2, f_3, f_4). If we consider the map $\mathbb{R}[x, y]^8 \to \mathbb{R}[x, y]^4$ given by the matrix,

$$M(\Delta, r) = \begin{pmatrix} 1 & -1 & 0 & 0 & (y)^{r+1} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & (x-y)^{r+1} & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & (y)^{r+1} & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & (x)^{r+1} \end{pmatrix},$$

the first four components of the elements of the kernel of this map give the elements of $C^r(\Delta)$.

We can now generalize these arguments. Suppose Δ is a d-dimensional complex in $\mathbb{R}[x_1, \cdots, x_d]$ having m d-dimensional cells, then by labeling these cells as σ_1, σ_2,

$\cdots$, σ_m, a spline $F \in C^r(\Delta)$ can be given as $(f_1, f_2, \cdots, f_m)$. The d-cells intersect along $(d-1)$-cells, which we denote by τ_{ij} and are contained in hyperplanes l_{ij}. We fix an order and rename these hyperplanes as $l_1, l_2, \cdots, l_e$. Then we get the $M(\Delta, r)$,the $e \times (m+e)$ matrix, which is represented by:

$$M(\Delta, r) = (\partial(\Delta)|D). \tag{3.3}$$

Here the $e \times m$ matrix $\partial(\Delta)$ part of M contains only 1, -1 and 0 as components. In its s^{th} row, we check the τ_s, if it is $\sigma_i \cap \sigma_j$, and if $i < j$, then it takes 1 in its $\partial(\Delta)_{si}$ component and -1 in its $\partial(\Delta)_{sj}$ component and all the other entries in this row are zero. The D part of M is an $e \times e$ diagonal matrix containing l_i^{r+1} in its diagonals:

$$\begin{pmatrix} l_1^{r+1} & 0 & \cdots & 0 \\ 0 & l_2^{r+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_e^{r+1} \end{pmatrix}.$$

This construction leads to the following results given in [22, Proposition 3.10].

Proposition 3.1.2 *[22, Proposition 3.10] For a pure and hereditary complex Δ in $\mathbb{R}^d$, let $M(\Delta, r)$ be defined as in Eqn. 3.3:*

(i) *An m-tuple $(f_1, f_2, \cdots, f_m)$ is in $C^r(\Delta)$, if and only if there exists $(g_1, g_2, \cdots, g_e)$ satisfying $(f_1, f_2, \cdots, f_m, g_1, g_2, \cdots, g_e)$ is in the kernel of the map*

$$\mathbb{R}[x_1, x_2, \cdots, x_d]^{m+e} \to \mathbb{R}[x_1, x_2, \cdots, x_d]^e,$$

defined by the matrix $M(\Delta, r)$.

(ii) *$C^r(\Delta)$ is a module over $\mathbb{R}[x_1, x_2, \cdots, x_d]$. If we construct the module of syzygies on the columns of $M(\Delta, r)$, then the projection homomorphism*

$$\mathbb{R}[x_1, x_2, \cdots, x_d]^{m+e} \to \mathbb{R}[x_1, x_2, \cdots, x_d]^m,$$

of the syzygy module onto the first m components gives $C^r(\Delta)$.
(iii) *$C^r_k(\Delta)$ is a finite dimensional vector subspace of $C^r(\Delta)$.*

Proof. See [22, p.395]. ■

Since $C^r(\Delta)$ has a module structure over $\mathbb{R}[x_1, x_2, \cdots, x_d]$, we can use Gröbner basis to determine the kernel of $M(\Delta, r)$, and we can calculate the dimensions of $C^r_k(\Delta)$, for any r, if $C^r(\Delta)$ is free (we will explain the reason of this in the homology method section). Consider the Ex. 3.1.1. By using CoCoA, we find a Gröbner basis for $\ker(M(\Delta, 1))$ as below:

```
Use R::=QQ[x,y];
M:=Module([1,0,0,1],[-1,1,0,0],[0,-1,1,0],[0,0,-1,-1],[y^2,0,0,0],
[0,(x-y)^2,0,0],[0,0,y^2,0],[0,0,0,x^2]);
SyzOfGens(M);
```

Hence, the kernel of the map is given as:

$$\text{Module}([[1,1,1,1,0,0,0,0],[0,y^2,y^2,0,1,0,-1,0],[0,-xy^2+1/2y^3,-1/2x^2y,-1/2x^2y,-x+1/2y,-1/2y,0,-1/2y],[0,1/2xy^2,-1/2x^3+x^2y,-1/2x^3+x^2y,1/2x,-1/2x,0,-1/2x+y]]),$$

In [12, Theorem 3.5], Billera and Rose have shown that for a 2-complex Δ, $C^r(\Delta)$ is free if and only if Δ is a manifold with boundary. In Ex. 3.1.1, Δ is a manifold with boundary and hence, $C^1(\Delta)$ is a free module, so $h_1 = (1,1,1,1), h_2 = (0,0,y,^2,y^2), h_3 = (0,-xy^2+1/2y^3,-1/2x^2y,-1/2x^2y)$ and $h_4 = (0,1/2xy^2,-1/2x^3+x^2y,-1/2x^3+x^2y)$ construct a basis for $C^1(\Delta)$. Any element of $C^1(\Delta)$ can be written in the form:

$$t_1(1,1,1,1)+t_2(0,0,y^2,y^2)+t_3(0,-xy^2+1/2y^3,-1/2x^2y,-1/2x^2y)+$$

$$t_4(0,1/2xy^2,-1/2x^3+x^2y,-1/2x^3+x^2y),$$

where $t_i \in \mathbb{R}[x,y]$ for all $1 \leq i \leq 4$.

We can now compute the dimension of $C^1_k(\Delta)$ for each k. For $k < 2$, $C^1_k(\Delta)$ contains only the splines $t_1(1,1,1,1)$. Hence, for $k = 0$, the only generator is (1,1,1,1) and $\dim C^1_0 = 1$. For $k = 1$, the generators are $\langle(1,1,1,1),(x,x,x,x),(y,y,y,y)\rangle$ and $\dim C^1_1 = 3$. For $k \geq 2$, by counting the monomials, which give degree k multiplied with the generators, we get the following general dimension formula for $C^1_k(\Delta)$:

$$\dim C^1_k(\Delta) = \begin{cases} 1 & \text{if } k = 0, \\ 3 & \text{if } k = 1, \\ \binom{k+2}{2} + \binom{k+2-2}{2} + 2\binom{k+2-3}{2} & \text{if } k \geq 2. \end{cases}$$

Hence if we make the calculations we get $\dim_{\mathbb{R}} C^1_k = 2k^2 - 2k + 3$ for $k \geq 1$.

Note once again that, this method is applicable, only in the case $C^r(\Delta)$ is free.

3.2 Hilbert Series Method

If we don't need to find a generating set for $C^r(\Delta)$, we can determine the dimension of the spline space by a computation of a Hilbert series. To do this, we consider the homogenized polyhedral complex $\hat{\Delta}$, which has been presented in the previous chapter. Hence, $C^r(\hat{\Delta})$ becomes a graded $\mathbb{R}$-algebra and by using the isomorphism $C^r_k(\Delta) \cong C^r(\hat{\Delta})_k$ given by Billera and Rose in [11], we can determine $\dim_{\mathbb{R}} C^r_k(\Delta)$ for each k. In the previous section, we have defined $M(\Delta, r)$. Recalling that, $M(\Delta, r) = (\partial(\Delta)|D)$, if we take the homogenizations of the entries of this matrix, only D, the $e \times e$ part changes, because the $\partial(\Delta)$ part is homogeneous. The l_{ij}'s, which correspond to entries of the diagonal of D, may not be homogeneous, and we denote their homogenizations as L_{ij}. Hence the matrix defining the map from $\hat{R}^{m+e}$ to $\hat{R}^e$ is denoted by $M(\hat{\Delta}, r)$. $\hat{\Delta}$ is still hereditary by the construction of homogenization (since we have a cone of d-dimensional Δ in $d+1$-space with the top point v =origin). Then [11, Proposition 4.3] implies that $C^r(\hat{\Delta}) \cong \ker(M(\hat{\Delta}, r))$, since $\hat{\Delta}$ is hereditary. Also we have the following exact sequence of $\hat{R}$-modules:

$$0 \to \ker(M(\hat{\Delta}, r)) \to \hat{R}^m \oplus \hat{R}(-r-1)^e \to \operatorname{im} M(\hat{\Delta}, r) \to 0,$$

where $\hat{R} = \mathbb{R}[x_1, x_2, \cdots, x_{d+1}]$.

We use the additivity of Hilbert series, and we have the equality:

$$HS(\ker(M(\hat{\Delta}, r)), t) = HS(\hat{R}^m \oplus \hat{R}(-r-1)^e, t) - HS(\operatorname{im} M(\hat{\Delta}, r), t).$$

Thus, we obtain:

$$HS(C^r(\hat{\Delta}), t) = HS(\hat{R}^m \oplus \hat{R}(-r-1)^e, t) - HS(\operatorname{im} M(\hat{\Delta}, r), t).$$

Note that $HS(\hat{R}, t) = \dfrac{1}{(1-t)^{d+1}}$ and $HS(\hat{R}^m, t) = \dfrac{m}{(1-t)^{d+1}}$. For the shifted graded module $\hat{R}(-r-1)$, we have $\hat{R}(-r-1)_k = \hat{R}_{k-r-1}$ for $k \in \mathbb{N}$ satisfying $k - r - 1 \geq 0$, so $HS(\hat{R}(-r-1), t) = t^{r+1} HS(\hat{R}, t) = \dfrac{t^{r+1}}{(1-t)^{d+1}}$. Then we can express the Hilbert series of $\hat{R}^m \oplus \hat{R}(-r-1)^e$ as $HS(\hat{R}^m \oplus \hat{R}(-r-1)^e, t) = \dfrac{m + et^{r+1}}{(1-t)^{d+1}}$. We also need to determine the Hilbert series $HS(\operatorname{im} M(\hat{\Delta}, r))$. Since the columns of $M(\hat{\Delta}, r)$ construct a basis for $\operatorname{im} M(\hat{\Delta}, r)$, we can compute the Hilbert series of this module by using Buchberger algorithm. By using CoCoA, we compute the Hilbert series of the Ex. 3.1.1.

```
Use R :: = QQ [x,y,z];
M : = [[1,0,0,1],[-1,1,0,0],[0,-1,1,0],[0,0,-1,-1],[y^2,0,0,0],
[0,(x-y)^2,0,0],[0,0,y^2,0],[0,0,0,x^2]];
N : = Module (M);
I : = LT(N);
Hilbert(I);
H(0) = 3
H(t) = 2t^2 + 6t + 1   for t >= 1.
```

Then the Hilbert series of $\operatorname{im} M(\hat{\Delta}, 1)$ is the infinite sum:

$$\begin{aligned} HS(\operatorname{im} M(\hat{\Delta}, 1), t) &= 3 + (2^2 + 6. + 1)t + (8 + 12 + 1)t^2 + \cdots \\ &\quad + (2n^2 + 6n + 1)t^n + \cdots \\ &= 3 + 9t + 21t^2 + \cdots + (2n^2 + 6n + 1)t^n + \cdots \end{aligned} \tag{3.4}$$

We can now compute the Hilbert series of $C^1(\hat{\Delta})$:

$$\begin{aligned} HS(C^1(\hat{\Delta}), t) &= HS(\hat{R}^4 \oplus \hat{R}(-1-1)^4, t) - HS(\operatorname{im} M(\hat{\Delta}, 1), t) \\ &= \frac{4 + 4t^2}{(1-t)^3} - (3 + 9t + 21t^2 + \cdots + (2n^2 + 6n + 1)t^n + \cdots). \end{aligned} \tag{3.5}$$

To compute the Hilbert series of $C^1(\hat{\Delta})$, we need to compute $\dfrac{4 + 4t^2}{(1-t)^3}$. Since $\dfrac{1}{(1-t)} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$, we have

$$\begin{aligned} \frac{4 + 4t^2}{(1-t)^3} &= (4 + 4t^2)(1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + \cdots) \\ &= 4 + 12t + 28t^2 + 52t^3 + 84t^4 + 124t^5 + \cdots \end{aligned}$$

Then the result is

$$\begin{aligned}HS(C^1(\hat{\Delta}), t) &= (4 + 12t + 28t^2 + 52t^3 + 84t^4 + 124t^5 + \cdots) - (3 + 9t + 21t^2 \\ &\quad + 37t^3 + 57t^4 + 81t^5 \cdots + (2n^2 + 6n + 1)t^n + \cdots) \\ &= 1 + 3t + 7t^2 + 15t^3 + 27t^4 + 43t^5 + \cdots\end{aligned}$$

In the above infinite sum, each coefficient of t^k gives the $\dim_{\mathbb{R}} C^r(\hat{\Delta})_k$. By using Eqn. (3.5), we can also compute the Hilbert function. For the ring $\hat{R} = \mathbb{R}[x_1, x_2, \cdots, x_{d+1}]$, the Hilbert function is $HF(\hat{R}) = \binom{k+d}{d}$ and $HF(\hat{R}(-i)) = \binom{k+d-i}{d}$. Hence,

$$\begin{aligned}HF(C^1(\hat{\Delta}), k) &= HF(\hat{R}^4 \oplus \hat{R}(-1-1)^4, k) - HF(\operatorname{im} M(\hat{\Delta}, 1), k) \\ &= 4\binom{k+2}{2} + 4\binom{k+2-1-1}{2} - (2k^2 + 6k + 1),\ k \geq 1 \\ &= 4\binom{k+2}{2} + 4\binom{k}{2} - (2k^2 + 6k + 1),\ k \geq 1 \\ &= 2k^2 - 2k + 3,\ k \geq 1.\end{aligned}$$

3.3 Homology Method

In [10], Billera has introduced the homological view to focus on the splines. Recall that for a simplicial complex Δ, $\sigma \in \Delta^0$ denotes the set of interior faces of Δ. Billera defined L_σ to be the homogeneous ideal of $\hat{\sigma}$, whose generators are homogeneous linear polynomials. Then for a fixed $r \geq 0$, he has defined a complex $\mathfrak{J}$ of ideals on Δ:

$$\mathfrak{J} := R/\hat{I}_\sigma^{r+1}.$$

Hence $\mathfrak{J}$ is a subcomplex of the constant complex R on Δ^0 satisfying that $R(\sigma) = R$ for any $\sigma \in \Delta^0$. Then the quotient of $\mathcal{R}$ by $\mathfrak{J}$ gives the short exact sequence:

$$0 \to \mathfrak{J} \to \mathcal{R} \to \mathcal{R}/\mathfrak{J} \to 0,$$

which gives a long exact sequence in homology,

$$\cdots \to H_i(\mathfrak{J}) \to H_i(\mathcal{R}) \to H_i(\mathcal{R}/\mathfrak{J}) \to H_{i-1}(\mathfrak{J}) \to H_{i-1}(\mathcal{R}) \to H_{i-1}(\mathcal{R}/\mathfrak{J}) \to \cdots$$

Billera showed that the spline module is isomorphic to the top homology module $H_d(\mathcal{R}/\mathfrak{J})$. Thus, he was able to compute the dimension of the splines by the computation of the dimension of a homology module.

Later, Schenck in [16] has described a complex $\mathcal{J}$ of ideals on interior faces of Δ:

$$
\begin{aligned}
&\mathcal{J}(\sigma) = 0 && \text{for } \sigma \in \Delta_d,\\
&\mathcal{J}(\tau) = \hat{I}_\tau^{r+1} && \text{for } \tau \in \Delta_{d-1}^0,\\
&\mathcal{J}(\xi) = \sum_{\xi\in\tau} \hat{I}_\tau^{r+1} && \text{for } \xi \in \Delta_{d-2}^0, \tau \in \Delta_{d-1}^0,\\
&\quad\vdots && \quad\vdots\\
&\mathcal{J}(\nu) = \sum_{\nu\in\tau} \hat{I}_\tau^{r+1} && \text{for } \nu \in \Delta_0^0, \tau \in \Delta_{d-1}^0.
\end{aligned}
$$

Then, there is a short exact sequence

$$0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0.$$

This sequence corresponds to a long exact homology sequence:

$$\cdots \to H_i(\mathcal{J}) \to H_i(\mathcal{R}) \to H_i(\mathcal{R}/\mathcal{J}) \to H_{i-1}(\mathcal{J}) \to H_{i-1}(\mathcal{R}) \to H_{i-1}(\mathcal{R}/\mathcal{J}) \to \cdots$$

We have $H_d(\mathcal{R}/\mathfrak{J}) = H_d(\mathcal{R}/\mathcal{J})$, since on d and $d-1$-interior faces $\mathfrak{J}$ and $\mathcal{J}$ are equal.

In [16], Schenck has proved that for all $i < d$, the dimension of $H_i(\mathcal{R}/\mathcal{J})$ is $\leq i-1$. Then in $d = 3$ case, we have $H_1(\mathcal{R}/\mathcal{J}) = 0$ and $H_2(\mathcal{R}/\mathcal{J})$ is at most one.

As we have mentioned before, one method for determining the dimension of a spline was to compute the Hilbert series and Schenck has shown that it depends on the modules $\oplus_{\beta\in\Delta_i^0}\mathcal{R}/\mathcal{J}(\beta)$ for $i = d, d-1, d-2$.

In [13] and [14], Schenck and Stillman worked on the $d = 2$ case and they have shown that the module $C^r(\hat{\Delta})$ is free if and only if $H_1(\mathcal{R}/\mathcal{J})$ vanishes, and $C^r(\hat{\Delta})$ can be free only, if Δ is a topological disk. Note that if Δ is a d-ball, $H_i(\mathcal{R}) = 0$, for $i \neq d$. By the long exact sequence, $C^r(\hat{\Delta}) \simeq R \oplus H_{d-1}(\mathcal{J})$ and $H_i(\mathcal{R}/\mathcal{J}) \simeq H_{i-1}(\mathcal{J})$ for all $i \leq d-1$. Hence, as a conclusion, $H_i(\mathcal{J})$ has dimension less than or equal to i, for $i < d-1$. So, we can restrict the study of spline modules to the study of $H_{d-1}(\mathcal{J})$, which is much easier.

In [16], Schenck shows that, for a complex Δ satisfying $H_i(\mathcal{R}) = 0$ for all $i < d$, then $C^r(\hat{\Delta})$ is free if and only if $H_i(\mathcal{J}) = 0$ for all $i < d - 1$. If we know that the module $C^r(\hat{\Delta})$ is free, then we can determine $C^r(\hat{\Delta})$ just by the computation of the Hilbert series of the $\mathcal{R}/\mathcal{J}(\sigma)$, $\sigma \in \Delta_i^0$.

In this section, we have shown that the dimension of splines is known, if the homology modules can be computed, but this is not always easy. Hence, in the next section, we give another method to find $\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k$ by using the Euler characteristic equation.

3.4 The Method of Using Relations Among Fat Points, Inverse Systems and Splines

In this section, we give the method given in the article [18] of Geramita and Schenck. In this article, by showing the connection between fat points and the inverse systems, they give the free resolution of an ideal generated by the mixed powers of homogeneous linear forms. By using this, they compute the dimension of 2-dimensional splines. We will also observe that this method does not work in higher dimensions.

3.4.1 Fat Points and Inverse Systems

Let $\mathbb{X} = \{P_1, \cdots, P_s\}$ be a finite set of points such that $P_i = [p_{i0} : p_{i1} : \cdots : p_{id}] \in \mathbb{P}^d$, $I(P_i) = \wp_i \subseteq R = k[x_0, \cdots, x_d]$, and $L_{P_i} = \sum_{j=0}^{d} p_{i_j} y_j$. The ideal $I = \cap_{i=1}^{m} \wp_i^{\alpha_i}$, $\alpha_i \geq 1$ is defined to be a fat points ideal. For $S = k[y_0, \cdots, y_d]$, R acts on S by partial differentiation, such that $x_j y_i = \partial(y_i)/\partial(y_j)$. Since I is a submodule of R, it also acts on S, so we can determine the elements of S annihilated by I, and define as $I^{-1} = \{f \in S | I.f = 0\}$, in other words $ann_S(I)$. The connection between I^{-1} and fat points ideals can be understood by the following theorem (Note that the notation presented here will be used throughout the section):

Theorem 3.4.1 *[[19]] Let $I = \wp_1^{n_1+1} \cap \wp_2^{n_2+1} \cdots \cap \wp_s^{n_s+1}$ be an ideal of fat points defined* *as above:*

$$(I^{-1})_t = \begin{cases} S_t & \text{if } t \le \max\{n_i\}, \\ L_{P_1}^{t-n_1} S_{n_1} + \cdots + L_{P_s}^{t-n_s} S_{n_s} & \text{if } t \ge \max\{n_i + 1\}. \end{cases}$$

and

$$\dim_k (I^{-1})_t = \dim_k (R/I, t) = HF(R/I, t).$$

Proof. See [20], p.22. ■

Here $L_{P_1}^{t-n_1} S_{n_1} + \cdots + L_{P_s}^{t-n_s} S_{n_s}$ represents the t^{th} graded part of the ideal generated by $L_{P_1}^{t-n_1}, \cdots, L_{P_s}^{t-n_s}$ in S.

By this theorem, we obtain a relation between fat points and the ideals generated by powers of homogeneous linear forms.

Corollary 3.4.2 ([18, Corollary 2.3]) *For any pairwise linearly independent homogeneous linear forms $L_1, \cdots, L_s$ in $S = k[y_0, y_1]$, where $0 < \alpha_1 \le \cdots \le \alpha_s$ in $\mathbb{Z}$, let J be the ideal generated by the elements $L_1^{\alpha_1}, \cdots, L_s^{\alpha_s}$. Then for each integer t, the dimension of the vector space J_t is given by the equality:*

$$\dim_k J_t = \min\{t + 1, \sum_{i=1}^{s} \max\{t - \alpha_i + 1, 0\}\}.$$

Proof.. Since $\alpha_1 \le \cdots \le \alpha_s$, there exists an integer q, which is the largest one, satisfying the inequality $t - \alpha_i + 1 > 0$, then we can arrange the summation as:

$$\min\{t + 1, \sum_{i=1}^{s} \max\{t - \alpha_i + 1, 0\}\} = \min\{t + 1, \sum_{i=1}^{q} (t - \alpha_i + 1)\}.$$

The integers larger than q, satisfy $t - \alpha_i + 1 < 0$ then $t - \alpha_i \le 0$, hence $S_{t-\alpha_i} = 0$. Since $J_t = L_1^{\alpha_1} S_{t-\alpha_1} + \cdots + L_s^{\alpha_s} S_{t-\alpha_s}$, it turns to, $J_t = L_1^{\alpha_1} S_{t-\alpha_1} + \cdots + L_q^{\alpha_q} S_{t-\alpha_q}$. By Theo. 3.4.1 if we choose $n_i = t - \alpha_i$ for $I = \wp_1^{t-\alpha_1+1} \cap \wp_2^{t-\alpha_2+1} \cdots \cap \wp_q^{t-\alpha_q+1}$, we have

$$(I^{-1})_t = \begin{cases} S_t & \text{if } t \le \max\{t - \alpha_i\}, \\ L_1^{\alpha_1} S_{t-\alpha_1} + \cdots + L_q^{\alpha_q} S_{t-\alpha_q} & \text{if } t \ge \max\{t - \alpha_i + 1\}. \end{cases}$$

Since t is always greater than $t - \alpha_i$, we obtain for $(I^{-1})_t = J_t$ and by using Theo. 3.4.1, $\dim_k J_t = \dim_k(R/I, t) = \dim_k(R_t/I_t) = t + 1 - \dim_k I_t$.

Her, I is a principal ideal generated by a linear form F having degree $\sum_{i=1}^{q}(t-\alpha_i+1)$. Hence,

$$\dim_k I_t = \begin{cases} 0 & \text{if } t < \sum_{i=1}^{q}(t-\alpha_i+1), \\ \dim_k S(-\sum_{i=1}^{q}(t-\alpha_i+1))_t & \text{if } t \geq \sum_{i=1}^{q}(t-\alpha_i+1). \end{cases}$$

The case $t < \sum_{i=1}^{q}(t-\alpha_i+1)$ is equivalent to saying that $t+1 \leq \sum_{i=1}^{q}(t-\alpha_i+1)$, which gives $\dim_k J_t = t+1$. The other case $t \geq \sum_{i=1}^{q}(t-\alpha_i+1)$ is equivalent to $t+1 > \sum_{i=1}^{q}(t-\alpha_i+1)$ and $\dim_k J_t = \sum_{i=1}^{q}(t-\alpha_i+1)$. When we combine these two results, we reach the conclusion:

$$\dim_k J_t = \min\{t+1, \sum_{i=1}^{q}(t-\alpha_i+1)\}.$$

■

By this corollary, we obtain the conclusion that the dimension of the ideals generated by the pairwise linearly independent linear forms do not depend on the linear forms, but on their powers. We can also use this corollary in deciding the minimal generator set of an ideal generated by powers of homogeneous linear forms in $k[y_0, y_1]$.

Example 3.4.3 *Let $L_1, L_2, L_3, L_4, L_5 \in k[y_0, y_1]$ be pairwise linearly independent homogeneous linear forms. For $J = (L_1^5, L_2^7, L_3^8, L_4^8, L_5^9)$, $\dim_k J_9 = \min\{9+1, (9-5+1)+(9-7+1)+(9-8+1)+(9-8+1)+(9-9+1)\} = 10$. If we construct an ideal $J' = (L_1^5, L_2^7, L_3^8, L_4^8)$, then $\dim_k(J')_9 = \min\{9+1, (9-5+1)+(9-7+1)+(9-8+1)+(9-8+1))\} = 10$. This implies that $L_5^9 \in J'$, hence $J = J'$. In the same way, we can show that $L_2^7 \notin (L_1^5)$, $L_3^8 \notin (L_1^5, L_2^7)$ and $L_4^8 \notin (L_1^5, L_2^7, L_3^8)$. Hence, the minimal generator set of J is $J' = (L_1^5, L_2^7, L_3^8, L_4^8)$.*

Corollary 3.4.4 *Let $J = (L_1^{\alpha_1}, \cdots, L_s^{\alpha_s})$, where $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, then for $m \geq 2$,*

$$L_{m+1}^{\alpha_{m+1}} \notin (L_1^{\alpha_1}, \cdots, L_m^{\alpha_m}) \Leftrightarrow \alpha_{m+1} \leq \frac{\sum_{i=1}^{m}\alpha_i - m}{m-1}.$$

Proof. Let $J_i = (L_1^{\alpha_1}, \cdots, L_i^{\alpha_i})$, then $L_{m+1}^{\alpha_{m+1}} \notin J_m$ if and only if $(J_m)_{\alpha_{m+1}} \neq (J_{m+1})_{\alpha_{m+1}}$. By Cor. 3.4.2,

$$\dim_k(J_m)_{\alpha_{m+1}} = \min\{\alpha_{m+1}+1, \textstyle\sum_{i=1}^{m}(\alpha_{m+1}-\alpha_i+1)\},$$
$$\dim_k(J_{m+1})_{\alpha_{m+1}} = \min\{\alpha_{m+1}+1, \textstyle\sum_{i=1}^{m+1}(\alpha_{m+1}-\alpha_i+1)\}.$$

Then we see that $(J_m)_{\alpha_{m+1}} \neq (J_{m+1})_{\alpha_{m+1}}$ if and only if $\alpha_{m+1} + 1 > \sum_{i=1}^{m}(\alpha_{m+1} - \alpha_i + 1)$, which implies that $\alpha_{m+1}(m-1) < \sum_{i=1}^{m} \alpha_i - m + 1$ and that gives, $\alpha_{m+1}(m-1) \leq \sum_{i=1}^{m} \alpha_i - m$. Hence, $\alpha_{m+1} \leq \dfrac{\sum_{i+1}^{m} \alpha_i - m}{m-1}$. ■

From now on, when we write an ideal $J = (L_1^{\alpha_1}, \cdots, L_t^{\alpha_t})$, we suppose that it has a minimal set of generators. For the inequality $\sum_{i=1}^{t}(r - \alpha_i + 1) > r$, Ω is defined to be the least integer r satisfying the inequality. Hence,

$$\Omega = \left\lfloor \frac{\sum_{i=1}^{t} \alpha_i - t}{t-1} \right\rfloor + 1.$$

Theorem 3.4.5 ([18]) *Let $J = (L_1^{\alpha_1}, \cdots, L_t^{\alpha_t})$. Then,*

$$H(S/J, i) = \begin{cases} i+1 & \text{if } 0 \leq i < \alpha_1, \\ (i+1) - \sum_{\{j|\alpha_j \leq i\}}(i - \alpha_j + 1) & \text{if } \alpha_1 \leq i < \Omega, \\ 0 & \text{if } i \geq \Omega. \end{cases}$$

Proof. [25, Teorem 3.4] ■

Theorem 3.4.6 ([18]) *For a minimally generated ideal $J = (L_1^{\alpha_1}, \cdots, L_t^{\alpha_t})$, so that $\Omega = \left\lfloor \dfrac{\sum_{i=1}^{t} \alpha_i - t}{t-1} \right\rfloor + 1$, J has a resolution:*

$$0 \to S(-\Omega)^{N_\Omega} \oplus S(-\Omega - 1)^{N_{\Omega+1}} \to \oplus_{i=1}^{t} S(-\alpha_i) \to J \to 0,$$

where $N_\Omega = \sum_{i=1}^{t}(\Omega - \alpha_i + 1) - (\Omega + 1)$ and $N_{\Omega+1} = \sum_{i=1}^{t} \alpha_i + (1-t)\Omega$.

Proof. [25, Teorem 3.5] ■

Example 3.4.7 *Recall Ex. 3.4.3, where we have shown that J is minimally generated by $(L_1^5, L_2^7, L_3^8, L_4^8)$. Then $\Omega = \left\lfloor \dfrac{5+7+8+8-4}{4-1} \right\rfloor + 1 = 9$, $N_\Omega = (9-5+1) + (9-7+1) + (9-8+1) + (9-8+1) - (9+1) = 2$, and $N_{\Omega+1} = 5+7+8+8+(1-4)9 = 1$. Hence, J has the resolution:*

$$0 \to S(-9)^2 \oplus S(-10) \to S(-5) \oplus S(-7) \oplus S(-8)^2 \to J \to 0.$$

3.4.2 Piecewise Polynomial Functions on Simplicial Complexes

In this section, we use the notation in the multivariable splines section. We denote by $\mathcal{R}$ the chain complex defined by $\mathcal{R}_i = R^{f_i^0}$, where $R = \mathbb{R}[x_1, \cdots, x_d, x_{d+1}]$. Here, ∂_i denotes the ordinary simplicial boundary map:

$$\mathcal{R} : \cdots \to \oplus_{\alpha\in\Delta^0_{i+1}} R (= R^{f^0_{i+1}}) \xrightarrow{\partial_{i+1}} \oplus_{\beta\in\Delta^0_i} R \xrightarrow{\partial_i} \oplus_{\gamma\in\Delta^0_{i-1}} R \xrightarrow{\partial_{i-1}} \cdots$$

Here the homology of the chain complex $\mathcal{R}$ is the simplicial relative homology with coefficients in R. For any interior face γ, $\mathcal{J}(\gamma)$ is defined to be the ideal generated by mixed powers of linear forms, which define hyperplanes incident to $\hat{\gamma}$, i.e., $\mathcal{J}(\gamma) = \sum_{\gamma\subseteq\tau_i\in\Delta^0_{d-1}} L_{\tau_i}^{\alpha_i+1}$. If we constrict the chain complex $\mathcal{R}$ to the ideals $\mathcal{J}$ we get the following chain complex:

$$\mathcal{J} : \cdots \to \oplus_{\alpha\in\Delta^0_{i+1}} \mathcal{J}(\alpha) \xrightarrow{\partial_{i+1}} \oplus_{\beta\in\Delta^0_i} \mathcal{J}(\beta) \xrightarrow{\partial_i} \oplus_{\gamma\in\Delta^0_{i-1}} \mathcal{J}(\gamma) \xrightarrow{\partial_{i-1}} \cdots$$

Hence, we can define the quotient of $\mathcal{R}$ by $\mathcal{J}$, and we get a chain complex $\mathcal{R}/\mathcal{J}$:

$$\mathcal{R}/\mathcal{J} : \cdots \to \oplus_{\alpha\in\Delta^0_{i+1}} \mathcal{R}/\mathcal{J}(\alpha) \xrightarrow{\partial_{i+1}} \oplus_{\beta\in\Delta^0_i} \mathcal{R}/\mathcal{J}(\beta) \xrightarrow{\partial_i} \oplus_{\gamma\in\Delta^0_{i-1}} \mathcal{R}/\mathcal{J}(\gamma) \xrightarrow{\partial_{i-1}} \cdots$$

The top homology of this complex $H_d(\mathcal{R}/\mathcal{J})$ is equal to the module $C^\alpha(\hat{\Delta})$. This equality has been shown for equal α_i's in [16]. But, it has also been verified for mixed α_i's. By using Euler characteristic equation, we can understand $C^\alpha(\hat{\Delta})$, if we understand modules and the lower homology modules of the complex. Here understanding the modules in the chain complex is equivalent to understanding the ideals generated by the powers of linear forms, in other words, fat points ideals.

In [16], by using localization techniques, it has been shown that, for fixed α_i's, $H_i(\mathcal{R}/\mathcal{J})$ has dimension at most $i - 1$ as an R-module. These techniques work also for the case, when α_i's are mixed.

We now consider the planar case, Δ embedded in $\mathbb{R}^2$. Then, $H_1(\mathcal{R}/\mathcal{J})$ is a zero dimensional $\mathbb{R}[x, y, z]$ module, so vanishes in sufficiently high degrees. The short exact sequence of complexes:

$$0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0,$$

gives a long exact sequence of homology modules:

$$\cdots \to H_i(\mathcal{J}) \to H_i(\mathcal{R}) \to H_i(\mathcal{R}/\mathcal{J}) \to H_{i-1}(\mathcal{J}) \to \cdots$$

Since every interior vertex is connected to at least one outer vertex via an edge, then $\langle v_b, v_i \rangle$ is meaningful for an inner vertex v_i and an outer vertex v_b which are connected to each other. And $\partial_1(\langle v_b, v_i \rangle) = \langle v_i \rangle - \langle v_b \rangle = \langle v_i \rangle \in \operatorname{im} \partial_1$, thus $\langle v_i \rangle + \operatorname{im} \partial_1 = \operatorname{im} \partial_1$. Which means that under the module of $\operatorname{im} \partial_1$ every vertex is equivalent to a vertex on the boundary. Hence, $H_0(\mathcal{R}) = 0$. Then by the long exact sequence, $H_0(\mathcal{R}/\mathcal{J})$ is also equal to zero.

Theorem 3.4.8 *With the notation given above, for sufficiently large k,*

$$\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k = \dim_{\mathbb{R}} \sum_{i=0}^{2} (-1)^i \oplus_{\beta \in \Delta^0_{2-i}} \mathcal{R}/\mathcal{J}(\beta)_k. \tag{3.6}$$

Proof. By applying Euler characteristic equation to the complex:

$$\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J}),$$

we get

$$\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k = \dim \sum_{i=0}^{2} (-1)^i \oplus_{\beta \in \Delta^0_{2-i}} \mathcal{R}/\mathcal{J}(\beta)_k + \dim_{\mathbb{R}} \sum_{i=0}^{1} (-1)^i H_{1-i}(\mathcal{R}/\mathcal{J})_k.$$

As mentioned before, we know that $H_0(\mathcal{R}/\mathcal{J}) = 0$ and for sufficiently large k, $H_1(\mathcal{R}/\mathcal{J}) = 0$. Since $H_2(\mathcal{R}/\mathcal{J})$ is equal to the spline module $C^{\alpha}(\hat{\Delta})$, we obtain the conclusion. ∎

Since we know,

$$\dim_{\mathbb{R}} \oplus_{\sigma \in \Delta^0_2} \mathcal{R}_k = f^0_2 \binom{k+2}{2},$$

and

$$\dim_{\mathbb{R}} \oplus_{\tau \in \Delta^0_1} \mathcal{R}/\mathcal{J}(\tau)_k = \sum_{i=1}^{f^0_1} \left[\binom{k+2}{2} - \binom{k+2-\alpha_i-1}{2} \right],$$

we need to find

$$\dim_{\mathbb{R}} \oplus_{\gamma \in \Delta^0_0} \mathcal{R}/\mathcal{J}(\gamma)_k.$$

In fact, we have computed this in the previous subsection. Here, we can translate γ_i to the origin, so $\mathcal{J}(\gamma_i)$ contains linear forms in variables x and y. Thus,

$$\mathcal{R}/\mathcal{J}(\gamma_i) \simeq \mathbb{R}[z] \otimes_{\mathbb{R}} \mathbb{R}[x,y]/\mathcal{J}(\gamma_i).$$

By naming β_j as α_{j+1}, let $\beta^i = (\beta_1, ..., \beta_t)$ be the exponent vector of $\mathcal{J}(\gamma_i)$ with minimal generator set $L_1^{\beta_1}, \cdots, L_t^{\beta_t}$. Then $\Omega^i = \left\lfloor \dfrac{\sum_{j=1}^t \beta_j - t}{t-1} \right\rfloor + 1$. Hence, by the above isomorphism and Theo. 4.1.11, $\mathcal{R}/\mathcal{J}(\gamma_i)$ has the following free resolution:

$$0 \to \mathcal{R}(-\Omega^i)^{N_{\Omega^i}} \oplus \mathcal{R}(-\Omega^i - 1)^{N_{\Omega^i+1}} \to \oplus_{j=1}^t R(-\beta_j) \to \mathcal{R} \to \mathcal{R}/\mathcal{J}(\gamma_i) \to 0,$$

where $N_{\Omega^i} = \sum_{j=1}^t (\Omega^i - \beta_j + 1) - (\Omega^i + 1)$ and $N_{\Omega^i+1} = \sum_{j=1}^t \beta_j + (1-t)\Omega^i$.

By the constriction of the resolution to the degree k and using the additivity of the Hilbert polynomial we obtain the equality:

$$\dim_{\mathbb{R}} \oplus_{\gamma_i \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma_i)_k = \binom{k+2}{2} - \sum_{\beta_j \in \beta^i} \binom{k+2-\beta_j}{2} + N_{\Omega^i} \binom{k - \Omega^i + 2}{2} + N_{\Omega^i+1} \binom{k - \Omega^i - 1 + 2}{2}.$$

Theorem 3.4.9 *For a simplicial complex Δ embedded in $\mathbb{R}^2$, we have*

$$\begin{aligned} \dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= (f_2^0 - f_1^0 + f_0^0)\binom{k+2}{2} + \sum_{i=1}^{f_1^0} \binom{k+2-\alpha_i-1}{2} \\ &\quad - \sum_{i=1}^{f_0^0} \left[\sum_{\beta_j \in \beta^i} \binom{k+2-\beta_j}{2} - N_{\Omega^i}\binom{k+2-\Omega^i}{2} - N_{\Omega^i+1}\binom{k+2-\Omega^i-1}{2} \right], \end{aligned}$$

for $k >> 0$.

(3.7)

Proof. The Hilbert polynomial of each graded module can be determined and then by applying Theo. 3.4.8, the conclusion can be drawn. ■

Example 3.4.10 *Let's apply this result to Ex. 3.1.1. Δ has points at $(1,0)$, $(0,1)$, $(-1,0)$, $(-1,-1)$ and $\alpha = (1,1,1,1)$ and it has only one interior vertex γ (the origin) and $\mathcal{J}(\gamma) = \langle y^2, x^2, (y+x-2z)^2 \rangle$, which is the minimal generating set for $\mathcal{J}(\gamma)$. Then $\Omega = \lfloor (2+2+2-3)/(3-1) \rfloor + 1 = 2$ and $N_2 = (2-2+1).3 - (2+1) = 0$, $N_{2+1} = 2+2+2+(1-3).2 = 2$. By Theo. 3.4.9,*

$$\begin{aligned} \dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= (4-4+1)\binom{k+2}{2} + 4.\binom{k+2-1-1}{2} \\ &\quad - \left[3.\binom{k+2-2}{2} - 0.\binom{k+2-2}{2} - 2.\binom{k+2-2-1}{2} \right] \\ &= \binom{k+2}{2} + \binom{k}{2} + 2.\binom{k-1}{2}. \end{aligned}$$

for $k >> 0$.

Example 3.4.11 *Let Δ be a planar simplicial complex with vertices at $(0,0)$, $(-1,2)$, $(-1,-2)$, $(-3,0)$, $(5,0)$ and $\alpha = (1,2,2,3)$.*

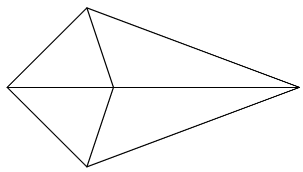

Figure 3.2: Planar simplicial complex

In this example, Δ has only one interior vertex γ (the origin) and $\mathcal{J}(\gamma) = \langle (2x+y)^2, y^3, (2x-y)^3, y^4 \rangle$, which has the minimal generating set $\langle (2x+y)^2, y^3, (2x-y)^3 \rangle$. Then, $\Omega = \lfloor (2+3+3-3)/(3-1) \rfloor + 1 = 3$ and $N_3 = (3-2+1)+2.(3-3+1)-(3+1) = 0$, $N_{3+1} = 2+3+3+(1-3).3 = 2$. By Theo. 3.4.9,

$$\begin{aligned}\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= (4-4+1)\binom{k+2}{2} + \binom{k+2-1-1}{2} + 2.\binom{k+2-2-1}{2} \\ &\quad - \left[\binom{k+2-2}{2} + 2.\binom{k+2-3}{2} - 0.\binom{k+2-3}{2} - 2.\binom{k+2-3-1}{2}\right] \\ &= \binom{k+2}{2} + 2.\binom{k-2}{2} \text{for } k >> 0.\end{aligned}$$

Example 3.4.12 *In this example, we have a more complicated figure:*

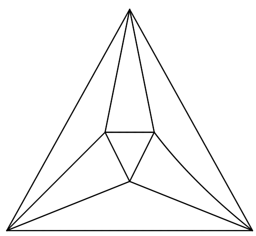

Figure 3.3: Complicated figure

On the three edges of the interior vertices, we take $\alpha_i = 2$ and on the six edges connecting the interior vertices to the boundary vertices, we have $\alpha_i = 3$. Hence, for each of the interior vertices $\mathcal{J}(\gamma) = (L_1^3, L_2^3, L_3^4, L_4^4)$ where each L_i are distinct linear forms. Then, $\mathcal{J}(\gamma)$ is minimally generated by (L_1^3, L_2^3, L_3^4). Then,
$\Omega = \lfloor (3+3+4+4-4)/(4-1) \rfloor + 1 = 4$ *and*
$N_4 = (4-3+1)+(4-3+1)+(4-4+1)-(4+1) = 0$, $N_{4+1} = 3+3+4+(1-3).4 = 2$.
By Theo. 3.4.9,

$$\begin{aligned}
\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= (7-9+3)\binom{k+2}{2} + 3\binom{k+2-2-1}{2} + 6\binom{k+2-3-1}{2} \\
&\quad - 3\left[2\binom{k+2-3}{2} + \binom{k+2-4}{2} - 0\binom{k+2-4}{2} - 2\binom{k+2-4-1}{2}\right] \\
&= \binom{k+2}{2} - 3\binom{k-1}{2} + 3\binom{k-2}{2} + 6\binom{k-3}{2} \text{for } k >> 0.
\end{aligned}$$

If we try to apply the method given by Geramita and Schenck in dimension 3, we have two important problems to consider: one is the Hilbert function of the corresponding fat points, which is much harder to compute than the case $\mathbb{P}^1$, and the second is the second homology of the corresponding chain complex, which is not always zero. In Chapter 5, we will use this method to obtain some new results.

CHAPTER 4

SPLINES WITH MIXED SMOOTHNESS DEGREE ON POLYHEDRAL COMPLEXES[1]

4.1 Splines with Mixed Smoothness Degrees on Polyhedral Complexes

In [11], Billera and Rose give a homological approaches to splines. They show that for a polyhedral complex Δ and fixed smoothness degree r, $C^r_k(\Delta) \simeq C^r(\hat{\Delta})_k$, and $C^r(\hat{\Delta}) = \cup C^r(\hat{\Delta})_k$. Hence the dimension of the vector space of splines becomes the Hilbert function of a graded algebra. This means that $\dim C^r_k(\Delta)$ is given by a polynomial in k, $f(\Delta, r, k)$, for sufficiently large k, called the Hilbert polynomial. In [23], McDonald and Schenck give a method to determine the first three coefficients of the polynomial $f(\Delta, r, k)$. In case of dimension 2, their method gives the exact polynomial. In this chapter we will extend their method by the techniques given in [18] for the splines having different smoothness degree $\alpha = \{\alpha_1, \ldots, \alpha_{f^0_{d-1}}\}$ on 2-dimensional polyhedral complex Δ.

In [11], for a d-dimensional polyhedral complex Δ, Billera and Rose give the exact sequence

$$0 \to C^r(\hat{\Delta}) \to \hat{R}^{f^d} \oplus \hat{R}^{f^0_{d-1}}(-r-1) \xrightarrow{\phi} \hat{R}^{f^0_{d-1}} \to N \to 0,$$

where

$$\phi = \left(\begin{array}{c|ccc} & l^{r+1}_{\tau_1} & & \\ \partial_d & & \ddots & \\ & & & l^{r+1}_{f^0_{d-1}} \end{array}\right).$$

and N is the cokernel of ϕ and $\hat{R} = \mathbb{R}[x, y, z]$. By applying Hilbert polynomial $HP()$

[1] This work has been done under the supervision of co-advisor Selma Altınok; see [1].

to the sequence we get

$$HP(C^r(\hat{\Delta}), k) = f_d HP(\hat{R}, k) + f_{d-1}^0 HP(\hat{R}(-r-1), k) - f_{d-1}^0 HP(\hat{R}, k) + HP(N), k).$$

In [11], Billera and Rose show that N is supported on primes of codimension at least 2. Hence calculating the k_{d-2} coefficient of the Hilbert polynomial of $C^r(\hat{\Delta})$ is equivalent to calculating the k_{d-2} coefficient of the Hilbert polynomial of N since the contribution of the other terms of the k_{d-2} coefficient of the Hilbert polynomial of $C^r(\hat{\Delta})$ can be calculated combinatorially. Moreover Billera and Rose's result implies that N has a Hilbert polynomial of degree $d-2$. Because of that in $d = 2$ case $HP(N, k)$ becomes a constant, (we will use $a_{d-2}(M)$ to denote the k_{d-2} coefficient of any graded module M), hence we represent it as $a_0(N)$.

In the exact sequence N is the cokernel of the map ϕ, so

$$N \simeq (\bigoplus_{\tau \in \Delta_{d-1}^0} R/l_\tau^{r+1})/\partial_d.$$

Using localization techniques, Mcdonald and Schenck show that any codimension-2 associated primes of N are linear. This result produces a list of candidates for codimension-2 linear primes. As a result of this, they describe the submodule of N supported in dimension $(d-2)$. The only contribution to the k^{d-2} coefficient of the Hilbert polynomial of N comes from the elements of that submodule. The work of Mcdonald and Schenck is done for the fixed smoothness degree r, that is, all d-faces intersect along $(d-1)$- interior faces with the same smoothness degree r.

In [23] Mcdonald and Schenck states the following equality;

$$\begin{aligned}
HP(C^r(\hat{\Delta}), k) &= (f_2 - f_1^0)\binom{k+2}{2} + f_1^0\binom{k+2-r-1}{2} + a_0(N) \\
&= \frac{f_2}{2}k^2 + 3\frac{f_2 - 2(r+1)f_1^0}{2}k + f_2 + (\binom{r}{2} - 1)f_1^0 + \sum_{\psi_j \in H_1(G_{\xi_i}(\Delta))} c_j,
\end{aligned} \tag{4.1}$$

where $c_j = a_0(HP(\hat{R}/I_{\psi_j}))$.

Here a_0 gives the constant term of the Hilbert polynomial of R/I_{ψ_j}, where $I_{\psi_j} = (l_1^{r+1}, l_2^{r+1}, \ldots, l_{n_j}^{r+1})$.

In [23, Lemma 3.13], for a codimension-2 minimally generated ideal

$$I_\psi = (l_1^{r+1}, l_2^{r+1}, \ldots, l_{n_j}^{r+1}),$$

$\hat{R}/I_\psi$ has the free resolution:

$$0 \to \hat{R}(-r-1-\alpha(\psi))^{s_1(\psi)} \oplus \hat{R}(-r-2-\alpha(\psi))^{s_2(\psi)} \to \hat{R}(-r-1)^n \to \hat{R} \to \hat{R}/I,$$

where $\alpha(\psi) = \left\lfloor \frac{r+1}{n-1} \right\rfloor$, $s_1(\psi) = (n-1)\alpha(\psi) + n - r - 2$, and $s_2(\psi) = r + 1 - (n-1)\alpha(\psi)$.

Here we are interested in a free resolution of $\hat{R}/I_\psi$ for a codimension-2 minimally generated ideal $I_{\psi_j} = (l_1^{\alpha_1+1}, l_2^{\alpha_2+1}, \ldots, l_{n_j}^{\alpha_{n_j}+1})$.

Our aim is to refine Eqn. (4.1) for different smoothness degrees of polynomials.

Theorem 4.1.1 *For a 2-dimensional hereditary, pure polyhedral complex Δ, we have*

$$\begin{aligned} HP(C^r(\hat{\Delta}), k) &= (f_2 - f_1^0)\binom{k+2}{2} + \sum_{i=1}^{f_1^0} \binom{k+2-\alpha_i-1}{2} + a_0(N) \\ &= (f_2 - f_1^0)\binom{k+2}{2} + \sum_{i=1}^{f_1^0} \binom{k+2-\alpha_i-1}{2} + \sum_{\psi_j \in H_1(G_{\xi_i}(\Delta))} c_j, \end{aligned} \tag{4.2}$$

where $c_j = a_0(HP(\hat{R}/I_{\psi_j}))$ for $I_{\psi_j} = (l_1^{\alpha_1+1}, l_2^{\alpha_2+1}, \ldots, l_{n_j}^{\alpha_{n_j}+1})$ will be given explicitly.

From now on we suppose that the smoothness degree is mixed, that is, $\alpha = (\alpha_1, \ldots, \alpha_{f_{d-1}^0})$,where the α_i are not necessarily all equal. Then the exact sequence changes into the following one:

$$0 \to C^r(\hat{\Delta}) \to \hat{R}^{f^d} \oplus \bigoplus_{i=0}^{f_{d-1}^0} \hat{R}(-\alpha_i - 1) \xrightarrow{\phi} \hat{R}^{f_{d-1}^0} \to N \to 0,$$

where

$$\phi = \left(\begin{array}{c|ccc} & l_{\tau_1}^{\alpha_1+1} & & \\ \partial_d & & \ddots & \\ & & & l_{\tau_{f_{d-1}^0}}^{\alpha_{f_{d-1}^0}+1} \end{array} \right).$$

The cokernel N of ϕ is isomorphic to

$$N \simeq \Big(\bigoplus_{\tau_i \in \Delta_{d-1}^0} R/l_{\tau_i}^{r_i+1}\Big)/\partial_d.$$

We can start to convert the theory constructed in [23] for a mixed smoothness degree.

Lemma 4.1.2 *If J is a codimension-2 associated prime ideal of N, then J contains a linear form $\langle l_\tau \rangle$, where $\tau \in \Delta^0_{d-1}$.*

Proof.([23, Lemma 3.1]) Suppose that J contains no l_τ, then l_τ becomes invertible in R_J, and this implies that $N_J = 0$. Since J is an associated prime of N, it is not possible. ■

Lemma 4.1.3 (Version of [23, Lemma 3.2] for a mixed order) *For a codimension-2 linear space ξ, if $\sigma \in \Delta_d$ has at most one facet whose linear span contains ξ, then in the localization $N_{I(\xi)}$, every generator of N corresponding to a facet of σ goes to zero*

Proof. Suppose that l_{τ_1} is the linear span of the facet of σ that contains ξ, then all the other facets of σ become invertible in $R_{I(\xi)}$. The generators of N corresponding to σ are as follows:

$$\langle 1 + l_{\tau_1}^{\alpha_1+1}, l_{\tau_2}^{\alpha_2+1}, \ldots, l_{\tau_j}^{\alpha_j+1} \rangle + \operatorname{im} \partial_d,$$
$$\langle l_{\tau_1}^{\alpha_1+1}, 1 + l_{\tau_2}^{\alpha_2+1}, \ldots, l_{\tau_j}^{\alpha_j+1} \rangle + \operatorname{im} \partial_d,$$
$$\vdots$$
$$\langle l_{\tau_1}^{\alpha_1+1}, l_{\tau_2}^{\alpha_2+1}, \ldots, 1 + l_{\tau_j}^{\alpha_j+1} \rangle + \operatorname{im} \partial_d.$$

Here it is clear that $R - I(\xi)$ annihilates the generators of N except the first one, since it contains all l_{τ_i}, for $i \geq 2$. But if we multiply the first component of N with the element $f_1 - f_2$ of $R - I(\xi)$ then it falls into $\operatorname{im} \partial_d$. So $R - I(\xi)$ annihilates N, i.e., in the localization $N_{I(\xi)}$ every generators of N corresponding to a facet of σ goes to zero. ■

Theorem 4.1.4 *If J is a codimension-2 associated prime ideal of N then it has the form $\langle l_{\tau_i}, l_{\tau_j} \rangle$ for $\tau_i, \tau_j \in \Delta^0_{d-1}$.*

Proof. Suppose that there is only one l_τ in J, then by Lem. 4.1.2 only one hyperplane which is the linear span of $\tau \in \Delta^0_{d-1}$ contains $V(J)$. Then, except for l_τ, all the linear forms l_{τ_i} becomes invertible in R_J, by Lem. 4.1.3 N_J vanishes. This contradicts the fact that J is an associated prime. ■

To determine the codimension-2 associated primes on a polyhedral complex we need more geometric knowledge than in case of a simplicial complex. In the simplicial case, the codimension-2 associated primes are the vertices of the complex. In light of the above information we can determine the candidates of the codimension-2 associated primes of N. To decide exactly which codimension-2 associated primes exist we will give a kind of dual graph definition which is related to both the combinatorics and the geometry of a complex Δ.

Definition 4.1.5 ([23, Definition 3.5]) *For Δ a d-dimensional polyhedral complex and ξ a codimension-2 linear subspace, $G_\xi(\Delta)$ is defined to be the graph whose vertices correspond to $\sigma \in \Delta_d$ which has a $(d-1)$-dimensional face whose linear span contains ξ. Two vertices of $G_\xi(\Delta)$ are connected if and only if the corresponding d-faces intersect along a $(d-1)$-face whose linear span contains ξ.*

Illustrate the definition, I will give here the example given in [23] by Mcdonald and Schenck.

Example 4.1.6 *Let Δ be the polyhedral complex drawn as in the Fig. 4.1.*

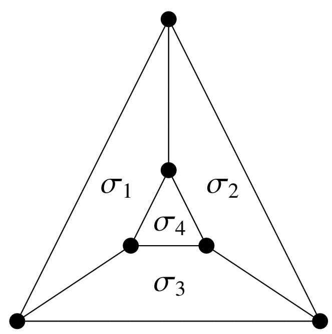

Figure 4.1: Polyhedral complex containing extra codimension-2 space except the interior vertices

In this example, for every interior vertex v, $G_v(\Delta)$ is a triangle. Apart from the interior vertices, there is another codimension-2 space which is the intersection point of the edges connecting the interior vertices with the boundary vertices. If we name this point as p_0, then $G_{p_0}(\Delta)$ is also a triangle and depicted as in the Fig. 4.2

In Fig. 4.1, if we move the top point of the outer triangle we get another polyhedral

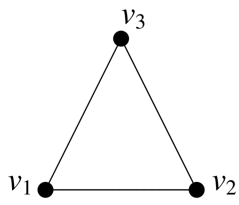

Figure 4.2: Associated dual graph $G_{p_0}(\Delta)$ for the point p_0 in Fig. 4.1

complex Δ' having the same combinatorial properties as Δ. But in the new figure there is no additional codimension-2 space as above.

Corollary 4.1.7 ([23, Corollary 3.8]) *$G_\xi(\Delta)$ is homotopic to a disjoint union of circles and segments*

Proof. This corollary is a consequence of [23, Lemma 3.7]. ■

By this stage, we get all the candidates of the codimension-2 associated primes of N, but we are not sure which of them exactly exist. Mcdonald and Schenck give two important results [23, Theorem 3.9, Proposition 3.10] which we now restate for mixed smoothness degree but we will use the same technique with them in their proof.

Theorem 4.1.8 ([23], Theorem 3.9) *Let Δ be a polyhedral complex and ξ a codimension-2 linear prime, then*

$$N_{I(\xi)} \simeq \bigoplus_{\psi \in H_1(G_\xi(\Delta))} (R/I_\psi)_{I(\xi)},$$

where $\psi \in H_1(G_\xi(\Delta))$ denotes the set of components of $G_\xi(\Delta)$ that are homotopic to a circle, and

$$I_\psi = \left\langle l_{\tau_i}^{\alpha_i+1} \mid \tau_i \in \Delta_{d-1}^0 \textit{corresponds to an edge of } \psi \right\rangle.$$

Proof. By Cor. 4.1.7, $G_\xi(\Delta)$ is formed from circles and segments but the generators of N lying on segments in the localization $N_{I(\xi)}$ go to zero by Lem. 4.1.3. Then for any $\sigma \in \Delta$ corresponding to a vertex having valence two, there are elements $\tau_i, \tau_j \in \Delta_{d-1}^0$ such that in the localization $R_{I(\xi)}$, l_{τ_i}, l_{τ_j} are not invertible but all the other linear forms of facets of σ become units. If we eliminate the column corresponding to σ in ∂_d with the unit entries of the columns of $D_{I(\xi)}$ we get a column having nonzero entries only in the row corresponding to τ_i, τ_j. Repeating these steps we see that the cycle corresponds to an essential submodule of $N_{I(\xi)}$, with the generator quotiented by the $(\alpha_k + 1)$ powers of the forms corresponding to the edges of the cycle. ■

Proposition 4.1.9 ([23, Proposition 3.10]) *Let $\mathcal{P}$ be the set of all codimension 2 associated primes of N. Then there is a graded exact sequence*

$$0 \longrightarrow K \longrightarrow N \longrightarrow \bigoplus_{\substack{\psi \in H_1(G_\xi(\Delta)) \\ Q \in \mathcal{P}}} \hat{R}/I_\psi \longrightarrow C \longrightarrow 0,$$

where K and C are supported in codimension at least 3. ■

Corollary 4.1.10 *Let $\mathcal{P}$ be the set of all codimension-2 associated primes of N. Then*

$$\begin{aligned} a_{d-2}(C^\alpha(\hat{\Delta})) =& a_{d-2}(\hat{R}^{f_d - f^0_{d-1}}) + \sum_{i=1}^{f^0_{d-1}} a_{d-2}(\hat{R}(-\alpha_i - 1)) \\ &+ \sum_{Q \in \mathcal{P}} \sum_{\psi_j \in H_1(G_{V(Q)}(\Delta))} a_{d-2}(\hat{R}/I_{\psi_j}). \end{aligned}$$

■

We take I_ψ as the ideal minimally generated by $\left\langle l_1^{\alpha_1+1}, l_2^{\alpha_2+1}, \ldots, l_n^{\alpha_n+1} \right\rangle$ with different exponents. In [18], Geramita and Schenck give the following lemma:

Lemma 4.1.11 *Any minimally generated ideal $I_\psi = \langle l_1^{\alpha_1+1}, l_2^{\alpha_2+1}, \ldots, l_n^{\alpha_n+1} \rangle \subset \mathbb{R}[x_0, x_1, x_2] = \hat{R}$, $\hat{R}/I_\psi$ has a resolution as follows:*

$$0 \to \hat{R}(-\Omega)^{N_\Omega} \oplus \hat{R}(-\Omega - 1)^{N_{\Omega+1}} \to \oplus_{i=1}^n \hat{R}(-\alpha_i - 1) \to \hat{R} \to \hat{R}/I_\psi \to 0,$$

where $\Omega = \left\lfloor \frac{\sum_{i=1}^n (\alpha_i+1) - n}{n-1} \right\rfloor + 1$, $N_\Omega = \sum_{i=1}^n (\Omega - (\alpha_i + 1) + 1) - (\Omega + 1)$ *and* $N_{\Omega+1} = \sum_{i=1}^n (\alpha_i + 1) + (1 - n)\Omega$.

It follows that the Hilbert polynomial of $\hat{R}/I_\psi$ is

$$\begin{aligned} HP(\hat{R}/I_\psi, k) = \binom{k+2}{2} - \sum_{i=1}^n \binom{k + 2 - \alpha_i - 1}{2} + \\ N_\Omega \binom{k + 2 - \Omega}{2} + N_{\Omega+1} \binom{k - \Omega - 1 + 2}{2}. \quad (4.3) \end{aligned}$$

Hence, it gives that:

$$a_1(\hat{R}/I_\psi, k) = a_2(\hat{R}/I_\psi, k) = 0,$$
$$a_0(\hat{R}/I_\psi, k) = 1 - \sum_i^n \binom{\alpha_i}{2} + N_\Omega \binom{\Omega - 1}{2} + N_{\Omega+1} \binom{\Omega}{2}.$$

Proof of Theo. 4.1.1. We have

$$HP(C^\alpha(\hat{\Delta}), k) = (f_2 - f_1^0)\binom{k+2}{2} + \sum_{j=1}^{f_1^0} \binom{k + 2 - \alpha_j - 1}{2} + a_0(N),$$

where $a_0(N) = HP(N, k)$. We need to determine $a_0(N)$. By Prop. 4.1.9, we obtain

$$a_0(N) = \sum_{Q \in \mathcal{P}} \sum_{\psi_j \in H_1(G_{V(Q)}(\Delta))} a_0(\hat{R}/I_{\psi_j}).$$

Hence the theorem follows. ■

As a result of this, we obtain

Corollary 4.1.12 *Under the same assumption of Theo. 4.1.1,*

$$\begin{aligned} HP(C^r(\hat{\Delta}), k) =& \frac{f_2}{2}k^2 + \frac{1}{2}(3f_2 - 2\sum_{i=1}^{f_1^0}(\alpha_i + 1))k + f_2 + \sum_{i=1}^{f_1^0}(\binom{\alpha_i}{2} - 1) \\ &+ \sum_{Q \in \mathcal{P}} \sum_{\psi_j \in H_1(G_{V(Q)}(\Delta))} 1 - \sum_i^{n_j} \binom{\alpha_i}{2} + N_{\Omega_j}\binom{\Omega_j - 1}{2} + N_{\Omega_j+1}\binom{\Omega_j}{2}. \end{aligned}$$

In the planer simplicial case, the codimension 2 associated primes J corresponding to a codimension 2 space ξ are vertices of a simplex. We are interested in ξ such that $H_1(G_\xi(\Delta)) \neq 0$, which are exactly the interior vertices. For each interior vertex there is only one corresponding cycle in the dual graph $G_\xi(\Delta)$. Hence we obtain a formula for planer mixed splines on a simplicial complex as follows.

Corollary 4.1.13 *For mixed splines on a simplicial planer complex Δ,*

$$\begin{aligned} HP(C^\alpha(\hat{\Delta}), k) =& (f_2 - f_1^0 + f_0^0)\binom{k+2}{2} + \sum_{i=1}^{f_1^0}\binom{k + 2 - \alpha_i - 1}{2} \\ &- \sum_{j=1}^{f_0^0}\left[\sum_{i=1}^{n_j}\binom{k + 2 - \alpha_i - 1}{2} - N_{\Omega_j}\binom{k + 2 - \Omega_j}{2} - N_{\Omega_{j+1}}\binom{k - \Omega_j - 1 + 2}{2}\right], \end{aligned} \tag{4.4}$$

for k sufficiently large.

This equality coincide with the formula given by Geramita and Schenk in [18].

4.2 Examples

Example 4.2.1 *In this example we will apply Theo. 4.1.1 to the complex Δ with four 2-faces, six interior 1-faces and three interior vertices as given in the Fig. 4.1. We choose $p_1 = (0, 2)$, $p_2 = (2, -1)$, $p_3 = -(2, -1)$, $p_4 = (0, 4)$, $p_5 = (4, -2)$,*

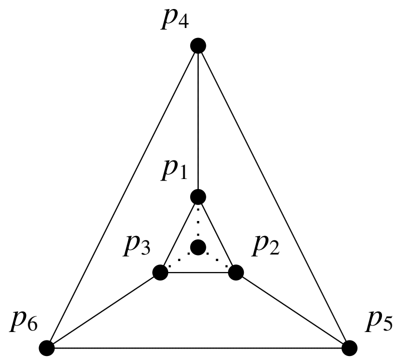

Figure 4.3: Δ with four 2-faces, six interior 1-faces and three interior vertices

$p_6 = (-4, -2)$. In the Fig. 4.3, p_0 is the intersection of the lines connecting the interior vertices with the boundary vertices, that is, the intersection point of the dotted lines. Here l_i implies the homogenized form of the linear space l_i. Then we have

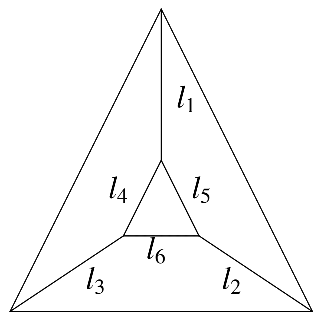

Figure 4.4: Ordered form of the linear forms of the Fig. 4.1

$$l_1 : x = 0, \qquad l_2 : x + 2y = 0, \qquad l_3 : x - 2y = 0,$$
$$l_4 : 2y - 3x - 4z = 0, \qquad l_5 : 2y + 3x - 4z = 0, \qquad l_6 : y + z = 0.$$

2-faces intersect along the interior 1-faces with smoothness degree α_i. Here α is given as

$$\begin{aligned}\alpha &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)\\ &= (1, 2, 1, 1, 1, 2).\end{aligned}$$

Let $F = (f_1, f_2, f_3, f_4) \in \mathbb{R}[x, y, z]^4$ satisfying $F|_{\hat{\sigma}_i} = f_i$. Then $F \in C^r(\hat{\Delta})$ if and only if it satisfies the algebraic property given below;

$$f_1 - f_2 + g_1 x^2 = 0, \qquad f_1 - f_4 + g_4(2y - 3x - 4z)^2 = 0,$$
$$f_2 - f_3 + g_2(x + 2y)^3 = 0, \qquad f_2 - f_4 + g_5(2y + 3x - 4z)^2 = 0,$$
$$f_1 - f_3 + g_3(x - 2y)^2 = 0, \qquad f_3 - f_4 + g_6(y + z)^3 = 0.$$

If we rewrite these equations in a matrix form we get ϕ as follows

$$\phi.(f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4, g_5, g_6)^T = 0,$$

where

We can write down the graded exact sequence:

$$0 \to C^r(\hat{\Delta}) \to \hat{R}^4 \oplus \hat{R}(-2)^4 \oplus \hat{R}(-3)^2 \xrightarrow{\phi} \hat{R}^6 \to N \to 0.$$

By using the additivity of Hilbert polynomial on exact sequences we get

$$\begin{aligned}HP(C^r(\hat{\Delta})) &= -2\binom{k+2}{2} + 4\binom{k}{2} + 2\binom{k-1}{2} + HP(N, k),\\ &= 2k^2 - 8k + HP(N, k).\end{aligned} \tag{4.5}$$

To find the elements of N contributing to the k^0 coefficient of the Hilbert polynomial, we define the codimension-2 prime ideals associated to N. By [23] possible candidates of the codimension-2 prime ideals associated to N can be determined as

$$\phi = \begin{pmatrix} 1 & -1 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & (x+2y)^3 & & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & (x-2y)^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & (2y-3x-4z)^2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & (2y+3x-4z)^2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & (y+z)^3 \end{pmatrix}.$$

follows:

$$Q_1 : \langle l_1, l_2\rangle = \langle x, 2y + x\rangle, \qquad Q_9 : \langle l_2, l_6\rangle = \langle 2y + x, y + z\rangle,$$

$$Q_2 : \langle l_1, l_3\rangle = \langle x, 2y - x\rangle, \qquad Q_{10} : \langle l_3, l_4\rangle = \langle 2y - x, 2y - 3x - 4z\rangle,$$

$$Q_3 : \langle l_1, l_4\rangle = \langle x, 2y - 3x - 4z\rangle, \qquad Q_{11} : \langle l_3, l_5\rangle = \langle 2y - x, 2y + 3x - 4z\rangle,$$

$$Q_4 : \langle l_1, l_5\rangle = \langle x, 2y + 3x - 4z\rangle, \qquad Q_7 : \langle l_2, l_4\rangle = \langle 2y + x, 2y - 3x - 4z\rangle,$$

$$Q_5 : \langle l_1, l_6\rangle = \langle x, y + z\rangle, \qquad Q_8 : \langle l_2, l_5\rangle = \langle 2y + x, 2y + 3x - 4z\rangle,$$

$$Q_6 : \langle l_2, l_3\rangle = \langle 2y + x, 2y - x\rangle, \qquad Q_{14} : \langle l_4, l_6\rangle = \langle 2y - 3x - 4z, y + z\rangle,$$

$$Q_{12} : \langle l_3, l_6\rangle = \langle 2y - x, y + z\rangle, \qquad Q_{15} : \langle l_5, l_6\rangle = \langle 2y + 3x - 4z, y + z\rangle,$$

$$Q_{13} : \ \langle l_4, l_5\rangle = \langle 2y - 3x - 4z, 2y + 3x - 4z\rangle.$$

Now we need to decide which codimension-2 associated primes of N has non-zero $H_1(G_{V(Q)}(\Delta))$. By [23, Corollary 3.11], it is shown that

$$a_0(C^r(\hat{\Delta})) = a_0(\hat{R}^{f_2 - f_1^0}) + \sum_{Q\in\wp} \sum_{\psi_j \in H_1(G_{V(Q)}(\Delta))} a_0(\hat{R}/I_{\psi_j}).$$

Here $\psi \in H_1(G_{V(Q)}(\Delta)$ means that ψ is a component of $G_{V(Q)}(\Delta)$ homotopic to S^1, and $I_\psi = \langle l_\tau^{\alpha_i+1} | \tau \in \hat{\Delta}_{d-1}^0$ corresponds to an edge of $\psi\rangle$.

For $Q_1 = \langle x, 2y + x\rangle \in \wp$, $V(Q_1) = V(Q_2) = V(Q_6) = p_0$ and then G_{p_0} is as follows: This figure is homotopic to S^1. For the ideal $I_{\psi_0} = \langle x^2, (x + 2y)^3, (x - 2y)^3\rangle$, it is the

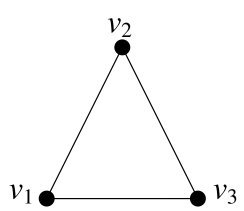

Figure 4.5: $G_{p_0}(\Delta)$

minimally generated ideal $\langle x^2, (x - 2y)^2\rangle$. Hence $\hat{R}/I_{\psi_0}$ has the following resolution by Lem. 4.1.11:

$$0 \to \hat{R}(-4) \to \hat{R}(-2)^2 \to \hat{R} \to \hat{R}/I_{\psi_0} \to 0,$$

where $\Omega_{p_0} = 3$, $N_{\Omega_{p_0}} = 0$ *and* $N_{\Omega_{p_0}+1} = 1$. *Then the constant term of the Hilbert polynomial of* $\hat{R}/I_{\psi_{p_0}}$ *is equal to*

$$\begin{aligned} HP(\hat{R}/I_{\psi_0}, k) &= \binom{k+2}{2} - 2\binom{k}{2} + \binom{k-2}{2} \\ &= \frac{1}{2}[(k+2).(k+1) - 2.k.(k-1) + (k-2).(k-3)] \\ &= 4. \end{aligned}$$

Similarly, for $Q_3 = \langle x, 2y-3x-4\rangle \in \wp, V(Q_3) = p_1 = V(Q_4) = V(Q_{13})$ *and then* G_{p_1} *is as follows: Again, it is homotopic to* S^1. *The ideal* $I_{\psi_1} = \langle x^2, (2y+3x-4z)^2, (2y-3x-$

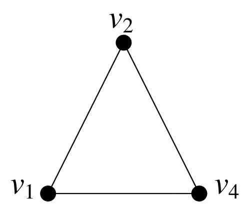

Figure 4.6: $G_{p_1}(\Delta)$

$4z)^2\rangle$ *is itself a minimally generated ideal. Hence,* $\hat{R}/I_{\psi_1}$ *has the following resolution by Lem. 4.1.11:*

$$0 \to \hat{R}(-2)^0 \oplus \hat{R}(-3)^2 \to \hat{R}(-2)^3 \to \hat{R} \to \hat{R}/I_{\psi_1} \to 0,$$

where $\Omega_{p_1} = 2$, $N_{\Omega_{p_1}} = 0$ *and* $N_{\Omega_{p_1}+1} = 2$. *Then the constant term of the Hilbert polynomial of* $\hat{R}/I_{\psi_{p_1}}$ *is:*

$$\begin{aligned} HP(\hat{R}/I_{\psi_1}, k) &= \binom{k+2}{2} - 3\binom{k}{2} + 2\binom{k-1}{2} \\ &= \frac{1}{2}[(k+2)(k+1) - 3k(k-1) + 2(k-1)(k-2)] \\ &= 3. \end{aligned}$$

For $Q_8 = \langle x+2y, 2y+3x-4\rangle \in \wp, V(Q_8) = p_2 = V(Q_9) = V(Q_{15})$, *and then* G_{p_2} *becomes a triangle means that it is homotopic to* S^1. *The ideal* $I_{\psi_2} = \langle (x+2y)^3, (y+z)^3, (2y+3x-4z)^2\rangle$ *is minimally generated ideal. Then* $\hat{R}/I_{\psi_2}$ *has the following resolution by Lem. 4.1.11:*

$$0 \to \hat{R}(-4)^2 \to \hat{R}(-2) \oplus \hat{R}(-3)^2 \to \hat{R} \to \hat{R}/I_{\psi_2} \to 0,$$

where $\Omega_{p_2} = 3$*,* $N_{\Omega_{p_2}} = 0$ *and* $N_{\Omega_{p_2}+1} = 2$*. Then the constant term of the Hilbert polynomial of* $\hat{R}/I_{\psi_{p_2}}$ *is equal to*

$$\begin{aligned} HP(\hat{R}/I_{\psi_2}, k) &= \binom{k+2}{2} - \binom{k}{2} - 2\binom{k-1}{2} + 2\binom{k-2}{2}, \\ &= \frac{1}{2}[(k+2).(k+1) - k.(k-1) - 2.(k-1).(k-2) + 2.(k-2).(k-3)], \\ &= 5. \end{aligned}$$

For $Q_{10} = \langle x - 2y, 2y - 3x - 4\rangle \in \wp, V(Q_{10}) = p_3 = V(Q_{12}) = V(Q_{14})$*, and then* G_{p_3} *is homotopic to* S^1*. The ideal* $I_{\psi_3} = \langle (2y-3x-4z)^2, (y+z)^3, (x-2y)^2\rangle$ *is minimally generated ideal* $\langle (x-2y)^2, (2y-3x-4z)^2\rangle$*. Then the constant term of the Hilbert polynomial of* $\hat{R}/I_{\psi_{p_3}}$ *is equal to* $\hat{R}/I_{\psi_{p_0}}$*, hence it has* $HP(\hat{R}/I_{\psi_{p_3}}, k) = 4$*. Then by the Eqn. (4.5)* $HP(C^r(\hat{\Delta}), k)$ *is equivalent to:*

$$\begin{aligned} HP(C^r(\hat{\Delta}), k) &= -2\binom{k+2}{2}^2 + 4\binom{k}{2} + 2\binom{k-1}{2} + HP(N, k), \\ &= 2k^2 - 8k + 4 + 3 + 5 + 4, \\ &= 2k^2 - 8k + 16. \end{aligned}$$

To check whether the result is true we use the exact sequence

$$0 \to \ker(\mathrm{M}(\hat{\Delta}, \alpha)) \to \hat{R}^4 \oplus \hat{R}(-2)^4 \oplus \hat{R}(-3)^2 \to \mathrm{im}(\mathrm{M}(\hat{\Delta}, \alpha)) \to 0,$$

and calculate $HP(\ker(\mathrm{M}(\hat{\Delta}, \alpha)), k)$ *since it is equivalent to* $HP(C^r(\hat{\Delta}), k)$*. Firstly we calculate* $HP(\mathrm{im}(\mathrm{M}(\hat{\Delta}, \alpha)), k)$ *by using CoCoA as follows:*

```
Use R ::= QQ[x,y,z];
M:=[[1,0,1,1,0,0],[-1,1,0,0,1,0],[0,-1,-1,0,0,1],[0,0,0,1,1,1],
[x^2,0,0,0,0,0],[0,(x+2y)^3,0,0,0,0],[0,0,(x-2y)^2,0,0,0],
[0,0,0,(2y-3x-4z)^2,0,0],[0,0,0,0,(2y+3x-4z)^2,0],[0,0,0,0,0,
(y+z)^3]];
N:=Module(M);
I:=LT(N);
Hilbert(I);
H(0) = 3
H(1) = 9
H(2) = 22
H(t) = 3t^2 + 9t - 10   for t >= 3
```

Then $HP(\mathrm{im}(\mathrm{M}(\hat{\Delta}, \alpha)), k) = 3k^2 + 9k - 10$ *for* $k \geq 3$. *By the above exact sequence we have*

$$\begin{aligned} HP(\ker(\mathrm{M}(\hat{\Delta}, \alpha)), k) &= 4\binom{k+2}{2} + 4\binom{k}{2} + 2\binom{k-1}{2} - HP(\mathrm{im}(\mathrm{M}(\hat{\Delta}, \alpha)), k) \\ &= 2(k^2 + 3k + 2) + 2(k^2 - k) + (k^2 - 3k + 2) - (3k^2 + 9k - 10) \\ &= 2k^2 - 8k + 16. \end{aligned}$$

As we see two results coincide.

Example 4.2.2 *In this example, we work on the polyhedral complex* Δ *given in the Fig. 4.7:* Δ *has thirteen 2-faces and twenty interior 1-faces. We choose,*

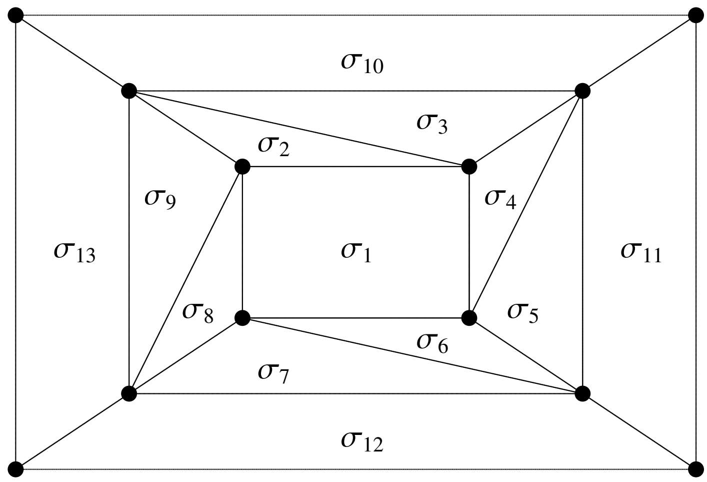

Figure 4.7: Polyhedral complex with thirteen 2-faces and twenty interior 1-faces

$p_1 = (-2, 1)$, $p_2 = (2, 1)$, $p_3 = (2, -1)$, $p_4 = (-2, -1)$, $p_5 = (-4, 2)$,
$p_6 = (4, 2)$, $p_7 = (4, -2)$, $p_8 = (-4, -2)$, $p_9 = (-8, 4)$, $p_{10} = (8, 4)$,
$p_{11} = (8, -4)$, $p_{12} = (-8, -4)$,

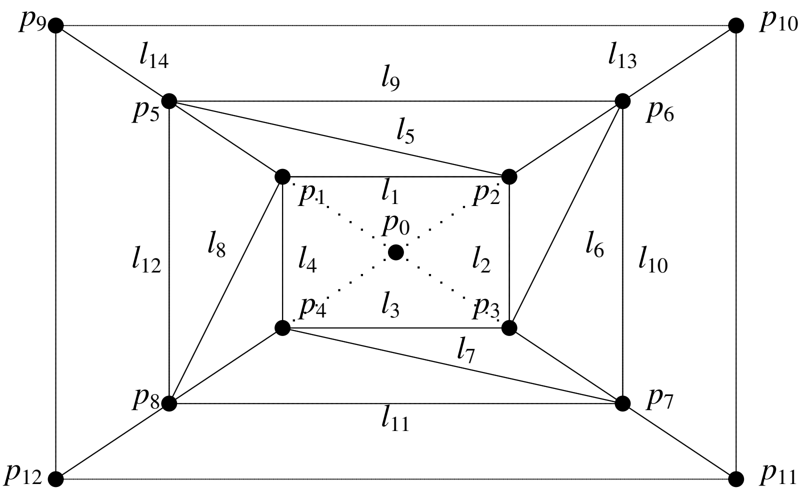

Figure 4.8: Ordered forms of the points and the linear forms of the Fig. 4.7

and smoothness degree

$$\begin{aligned} r &= (r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}) \\ &= (1, 1, 2, 1, 2, 2, 3, 2, 2, 1, 3, 2, 2, 1, 3, 2, 3, 2, 1, 2). \end{aligned}$$

And $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}) \in \mathbb{R}[x, y]^{13}$ *gives an element of* $C^r(\hat{\Delta})$

if and only if

$$
\begin{aligned}
f_2 - f_3 + g_1(x+6y-8z)^2 &= 0, & f_5 - f_{11} + g_{11}(x-4z)^4 &= 0,\\
f_3 - f_4 + g_2(x-2y)^2 &= 0, & f_{11} - f_{12} + g_{12}(x+2y)^3 &= 0,\\
f_4 - f_5 + g_3(3x-2y-8z)^3 &= 0, & f_7 - f_{12} + g_{13}(y+2z)^3 &= 0,\\
f_5 - f_6 + g_4(x+2y)^2 &= 0, & f_{12} - f_{13} + g_{14}(x-2y)^2 &= 0,\\
f_6 - f_7 + g_5(x+6y+8z)^3 &= 0, & f_9 - f_{13} + g_{15}(x+4z)^4 &= 0,\\
f_7 - f_8 + g_6(x-2y)^3 &= 0, & f_{10} - f_{13} + g_{16}(x+2y)^3 &= 0,\\
f_8 - f_9 + g_7(3x-2y+8z)^4 &= 0, & f_1 - f_2 + g_{17}(y-z)^4 &= 0,\\
f_2 - f_9 + g_8(x+2y)^3 &= 0, & f_1 - f_4 + g_{18}(x-2z)^3 &= 0,\\
f_3 - f_{10} + g_9(y-2z)^3 &= 0, & f_1 - f_6 + g_{19}(y+z)^2 &= 0,\\
f_{10} - f_{11} + g_{10}(x-2y)^2 &= 0, & f_1 - f_8 + g_{20}(x+2z)^3 &= 0.
\end{aligned}
$$

Hence the matrix form ϕ satisfies:

$$\phi(f_1, f_2, \ldots, f_{13}, g_1, g_2, \ldots, g_{20})^T = 0,$$

where

$$\phi = \left(\begin{array}{c|ccc} & l_{\tau_1}^{\alpha_1+1} & & \\ \partial_d & & \ddots & \\ & & & l_{\tau_{20}}^{\alpha_{20}+1} \end{array}\right).$$

So, we have the graded exact sequence:

$$0 \to C^r(\hat{\Delta}) \to \hat{R}^{13} \oplus \hat{R}(-2)^6 \oplus \hat{R}(-3)^{10} \oplus \hat{R}(-4)^4 \xrightarrow{\phi} \hat{R}^{20} \to N \to 0.$$

By using the additivity of Hilbert polynomial on exact sequences we get

$$
\begin{aligned}
HP(C^r(\hat{\Delta}, k)) &= -7\binom{k+2}{2} + 6\binom{k}{2} + 10\binom{k-1}{2} + 4\binom{k-2}{2} + HP(N, k)\\
&= \frac{13}{2}k^2 - \frac{77}{2}k + 15 + HP(N, k).
\end{aligned}
\tag{4.6}
$$

To find the elements of N contributing to the k^0 coefficient of the Hilbert polynomial of $C^r(\hat{\Delta})$, we find the codimension-2 prime ideals associated to N satisfying that $H_1(G_{V(Q)}(\Delta)) \neq 0$. At the points p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8 in Δ together with the point p_0 which is not contained in Δ, $H_1(G_{V(Q)})$ is homotopic to a circle. Hence by the Theo. 4.1.1 we get

$$
\begin{aligned}
HP(N, k) =& a_0(\hat{R}/I_{\psi_0}) + a_0(\hat{R}/I_{\psi_1}) + a_0(\hat{R}/I_{\psi_2}) + a_0(\hat{R}/I_{\psi_3}) + a_0(\hat{R}/I_{\psi_4}) + a_0(\hat{R}/I_{\psi_5})\\
&+ a_0(\hat{R}/I_{\psi_6}) + a_0(\hat{R}/I_{\psi_7}) + a_0(\hat{R}/I_{\psi_8}).
\end{aligned}
$$

For example, for p_0, $G_{p_0}(\Delta)$ is as follows: $G_{p_0}(\Delta)$ consists of a tetrahedron and four

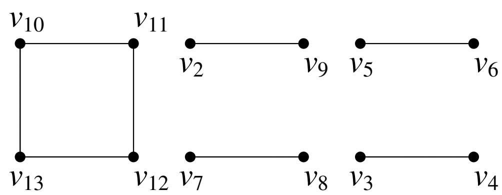

Figure 4.9: $G_{p_0}(\Delta)$ for p_0

line segments. In case of the tetrahedron component, $G_{p_0}(\Delta)$ is homotopic to a circle. So $I_{\psi_{p_0}} = \langle (x-2y)^2, (x+2y)^3 \rangle$, and it is minimally generated. Then it has the following resolution:

$$0 \to \hat{R}(-4)^0 \oplus \hat{R}(-5) \to \hat{R}(-2) \oplus \hat{R}(-3) \to \hat{R} \to \hat{R}/I_{\psi_0} \to 0,$$

where $\Omega_{p_0} = 4$, $N_{\Omega_{p_0}} = 0$ and $N_{\Omega_{p_0}+1} = 1$. Then the constant term of the Hilbert polynomial of $\hat{R}/I_{\psi_{p_0}}$ is equal to

$$\begin{aligned} HP(\hat{R}/I_{\psi_{p_0}}, k) =& \binom{k+2}{2} - \binom{k}{2} - \binom{k-1}{2} + \binom{k-3}{2} \\ =& \frac{1}{2}[(k+2).(k+1) - k.(k-1) - (k-1).(k-2) + (k-3).(k-4)] \\ =& 6. \end{aligned}$$

We find the result in a similar way at the other points, and we obtain the table: When we replace all the calculations in Eqn. (4.6) we find $HP(C^r(\hat{\Delta}), k)$ as follows:

$$\begin{aligned} HP(C^r(\hat{\Delta}), k) &= \frac{13}{2}k^2 - \frac{77}{2}k + 15 + HP(N, k) \\ &= \frac{13}{2}k^2 - \frac{77}{2}k + 15 + 6 + 8 + 4 + 4 + 5 + 5 + 5 + 5 + 6 \\ &= \frac{13}{2}k^2 - \frac{77}{2}k + 63. \end{aligned}$$

for sufficiently large k.

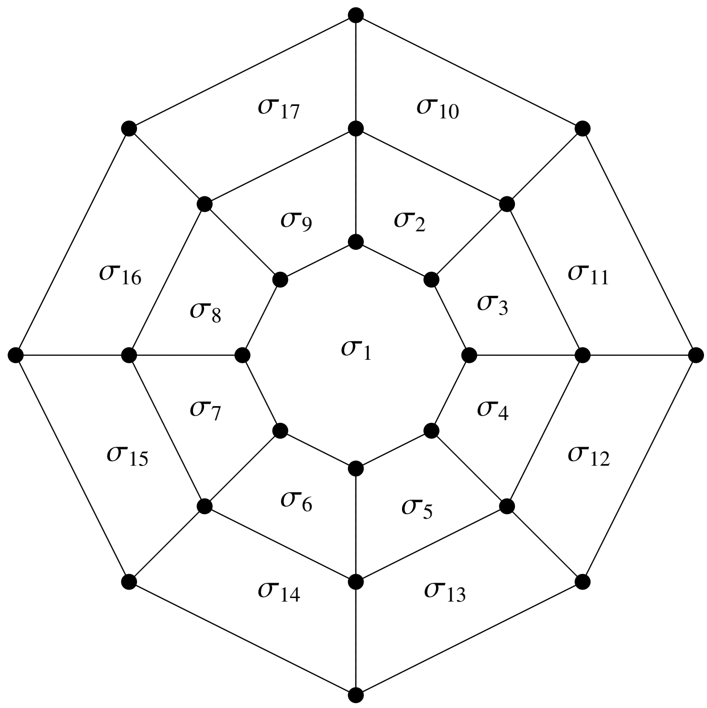

Figure 4.10: Polyhedral complex resembling to a spider web

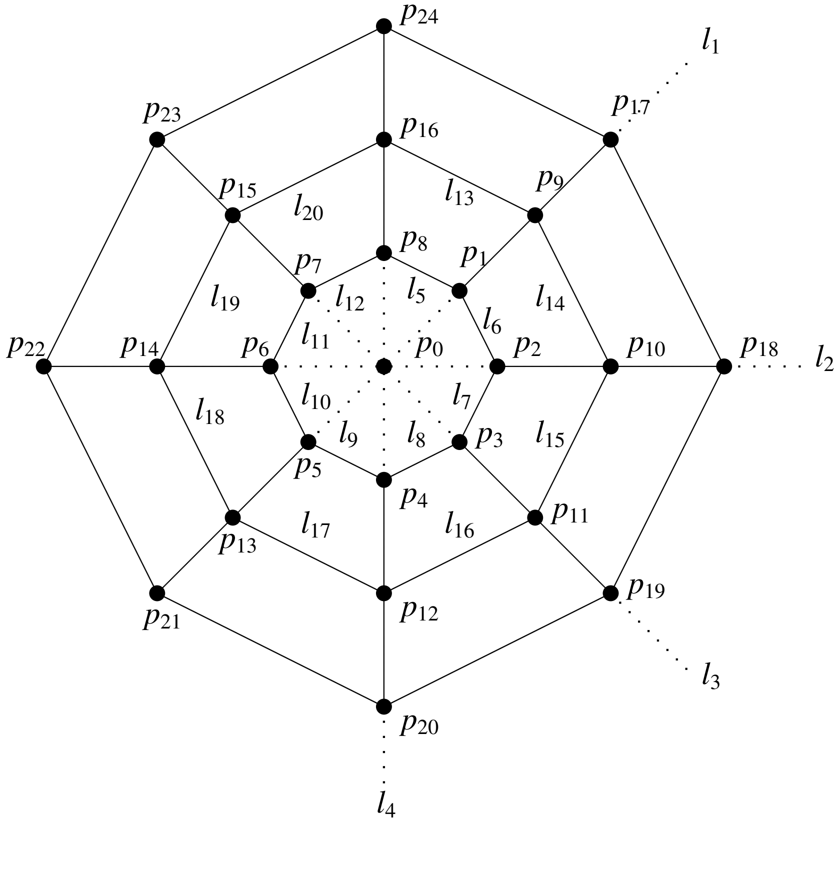

Figure 4.11: Ordered forms of the points and the linear forms of the Fig. 4.10

Table 4.1: Table giving the some properties of $G_p(\Delta)$

$p \in \Delta_0^0$	$G_p(\Delta)$	I_ψ m.g.ideal	$HP(R/I_\psi, k)$
p_0	a tetrahedron and 4 line segments	$\langle l_{14}^2, l_{13}^3 \rangle$	6
p_1	a tetrahedron and 3 line segments	$\langle l_4^3, l_{13}^3, l_3^4 \rangle$	8
p_2	a tetrahedron and 3 line segments	$\langle l_5^2, l_{14}^2 \rangle$	4
p_3	a tetrahedron and 3 line segments	$\langle l_3^2, l_4^2 \rangle$	4
p_4	a tetrahedron and 3 line segments	$\langle l_1^2, l_7^3, l_{14}^3 \rangle$	5
p_5	a pentagon and 2 line segments	$\langle l_5^2, l_{13}^3, l_9^3 \rangle$	5
p_6	a pentagon and 2 line segments	$\langle l_{14}^2, l_6^3, l_9^3 \rangle$	5
p_7	a pentagon and 2 line segments	$\langle l_{13}^2, l_7^3, l_{11}^3 \rangle$	5
p_8	a pentagon and 2 line segments	$\langle l_2^2, l_{11}^3 \rangle$	6

Example 4.2.3 *In this example, we work on the polyhedral complex Δ given in the Fig. 4.10: Δ has seventeen 2-faces, and thirty two interior 1-faces. We choose*

$$\begin{aligned}
&p_1 = (3,3), && p_2 = (4,0), && p_3 = (3,-3), && p_4 = (0,-4), && p_5 = (-3,-3),\\
&p_6 = (-4,0), && p_7 = (-3,3), && p_8 = (0,4), && p_9 = (6,6), && p_{10} = (8,0),\\
&p_{11} = (6,-6), && p_{12} = (0,-8), && p_{13} = (-6,-6), && p_{14} = (-8,0), && p_{15} = (-6,6),\\
&p_{16} = (0,8), && p_{17} = (9,9), && p_{18} = (12,0), && p_{19} = (9,-9), && p_{20} = (0,-12),\\
&p_{21} = (-9,-9), && p_{22} = (-12,0), && p_{23} = (-9,9), && p_{24} = (0,12),
\end{aligned}$$

and smoothness degree

$$\begin{aligned}
r =&(r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21},\\
&r_{22}, r_{23}, r_{24}, r_{25}, r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32})\\
=&(3,3,3,2,3,3,2,2,3,3,2,2,2,3,2,3,3,2,2,3,2,3,3,2,2,2,3,3,3,3,2,2).
\end{aligned}$$

Then $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}) \in \mathbb{R}[x,y]^{17}$ gives an *element of* $C^r(\hat{\Delta})$ *if and only if*

$$\begin{aligned}
f_1 - f_2 + g_1.(x+3y-12z)^4 &= 0, & f_2 - f_{10} + g_{17}.(x+3y-24z)^4 &= 0,\\
f_1 - f_3 + g_2.(3x+y-12z)^4 &= 0, & f_3 - f_{11} + g_{18}.(3x+y-24z)^3 &= 0,\\
f_1 - f_4 + g_3.(3x-y-12z)^4 &= 0, & f_4 - f_{12} + g_{19}.(3x-y-24z)^3 &= 0,\\
f_1 - f_5 + g_4.(x-3y-12z)^3 &= 0, & f_5 - f_{13} + g_{20}.(x-3y-24z)^4 &= 0,\\
f_1 - f_6 + g_5.(x+3y+12z)^4 &= 0, & f_6 - f_{14} + g_{21}.(x+3y+24z)^3 &= 0,\\
f_1 - f_7 + g_6.(3x+y+12z)^4 &= 0, & f_7 - f_{15} + g_{22}.(3x+y+24z)^4 &= 0,\\
f_1 - f_8 + g_7.(3x-y+12z)^3 &= 0, & f_8 - f_{16} + g_{23}.(3x-y+24z)^4 &= 0,\\
f_1 - f_9 + g_8.(x-3y+12z)^3 &= 0, & f_9 - f_{17} + g_{24}.(x-3y+24z)^3 &= 0,\\
f_2 - f_3 + g_9.(x-y)^4 &= 0, & f_{10} - f_{11} + g_{25}.(x-y)^3 &= 0,\\
f_3 - f_4 + g_{10}.y^4 &= 0, & f_{11} - f_{12} + g_{26}.y^3 &= 0,\\
f_4 - f_5 + g_{11}.(x+y)^3 &= 0, & f_{12} - f_{13} + g_{27}.(x+y)^4 &= 0,\\
f_5 - f_6 + g_{12}.x^3 &= 0, & f_{13} - f_{14} + g_{28}.x^4 &= 0,\\
f_6 - f_7 + g_{13}.(x-y)^3 &= 0, & f_{14} - f_{15} + g_{29}.(x-y)^4 &= 0,\\
f_7 - f_8 + g_{14}.y^4 &= 0, & f_{15} - f_{16} + g_{30}.y^4 &= 0,\\
f_8 - f_9 + g_{15}.(x+y)^3 &= 0, & f_{16} - f_{17} + g_{31}.(x+y)^3 &= 0,\\
f_2 - f_9 + g_{16}.x^4 &= 0, & f_{10} - f_{17} + g_{32}.x^3 &= 0.
\end{aligned}$$

Hence the matrix form ϕ *satisfies:*

$$\phi(f_1, f_2, \ldots, f_{17}, g_1, g_2, \ldots, g_{32})^T = 0,$$

where

$$\phi = \left(\begin{array}{c|ccc} & l_{\tau_1}^{\alpha_1+1} & & \\ \partial_d & & \ddots & \\ & & & l_{\tau_{32}}^{\alpha_{32}+1} \end{array}\right).$$

Then we have the graded exact sequence:

$$0 \to C^r(\hat{\Delta}) \to \hat{R}^{17} \oplus \hat{R}(-3)^{15} \oplus \hat{R}(-4)^{17} \xrightarrow{\phi} \hat{R}^{32} \to N \to 0.$$

By using the additivity of Hilbert polynomial on exact sequences we get

$$\begin{aligned}
HP(C^r(\hat{\Delta})) &= -15\binom{k+2}{2} + 15\binom{k-1}{2} + 17\binom{k-2}{2} + HP(N,k)\\
&= \frac{17}{2}k^2 - \frac{175}{2}k + 51 + HP(N,k).
\end{aligned} \tag{4.7}$$

To find the elements of N contributing to the k^0 coefficient of the Hilbert polynomial, we determine the possible candidates for codimension-2 prime ideals associated to N:

$$
\begin{array}{lll}
l_1 : x - y = 0, & l_8 : x - 3y - 12z = 0, & l_{15} : 3x - y - 24z = 0, \\
l_2 : y = 0, & l_9 : x + 3y + 12z = 0, & l_{16} : x - 3y - 24z = 0, \\
l_3 : x + y = 0, & l_{10} : 3x + y + 12z = 0, & l_{17} : x + 3y + 24z = 0, \\
l_4 : x = 0, & l_{11} : 3x - y + 12z = 0, & l_{18} : 3x + y + 24z = 0, \\
l_5 : x + 3y - 12z = 0, & l_{12} : x - 3y + 12z = 0, & l_{19} : 3x - y + 24z = 0, \\
l_6 : 3x + y - 12z = 0, & l_{13} : x + 3y - 24z = 0, & l_{20} : x - 3y + 24z = 0, \\
l_7 : 3x - y - 12z = 0, & l_{14} : 3x + y - 24z = 0. &
\end{array}
$$

To find the elements of N contributing to the k^0 coefficient of the Hilbert polynomial of $C^r(\hat{\Delta})$, we find the codimension-2 prime ideals associated to N satisfying that $H_1(G_{V(Q)}(\Delta)) \neq 0$. At the points p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16} in Δ together with the point p_0 which is not contained in Δ, $H_1(G_{V(Q)})$ is homotopic to a circle. Hence by Theo. 4.1.1 we get

$$
\begin{aligned}
HP(N,k) =& a_0(\hat{R}/I_{\psi_0}) + a_0(\hat{R}/I_{\psi_1}) + a_0(\hat{R}/I_{\psi_2}) + a_0(\hat{R}/I_{\psi_3}) + a_0(\hat{R}/I_{\psi_4}) + a_0(\hat{R}/I_{\psi_5}) \\
&+ a_0(\hat{R}/I_{\psi_6}) + a_0(\hat{R}/I_{\psi_7}) + a_0(\hat{R}/I_{\psi_8}) + a_0(\hat{R}/I_{\psi_9}) + a_0(\hat{R}/I_{\psi_{10}}) + a_0(\hat{R}/I_{\psi_{11}}) \\
&+ a_0(\hat{R}/I_{\psi_{12}}) + a_0(\hat{R}/I_{\psi_{13}}) + a_0(\hat{R}/I_{\psi_{14}}) + a_0(\hat{R}/I_{\psi_{15}}) + a_0(\hat{R}/I_{\psi_{16}}).
\end{aligned}
$$

For example, for p_0, $G_{p_0}(\Delta)$ is as follows:
Both figures are homotopic to cycles. For the left one we get the ideal,

$$I_{\psi_{0,1}} = \langle (x-y)^4, y^4, (x+y)^3, x^3, (x-y)^3, y^4, (x+y)^3, x^4 \rangle.$$

It is minimally generated by $\langle x^3, (x-y)^3, (x+y)^3 \rangle$. Then $\hat{R}/I_{\psi_{0,1}}$ has the following resolution:

$$0 \to \hat{R}(-4) \oplus \hat{R}(-5) \to \hat{R}(-3)^3 \to \hat{R} \to \hat{R}/I_{\psi_{0,1}} \to 0,$$

where $\Omega_{p_{0,1}} = 4$, $N_{\Omega_{p_{0,1}}} = 1$ and $N_{\Omega_{p_{0,1}}+1} = 1$. The Hilbert polynomial of $\hat{R}/I_{\psi_{p_{0,1}}}$ is constant:

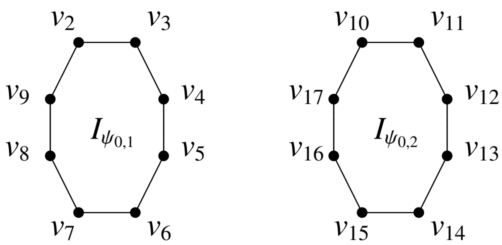

Figure 4.12: Associated dual graph of p_0

$$
\begin{aligned}
HP(\hat{R}/I_{\psi_{0,1}}, k) = &\binom{k+2}{2} - 3\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-3}{2} \\
=&\frac{1}{2}[(k+2)(k+1) - 3(k-1)(k-2) + (k-2)(k-3)+ \\
&2(k-3)(k-4)] \\
=&7.
\end{aligned}
$$

For the figure on the right we have the ideal,

$$I_{\psi_{0,2}} = \langle (x-y)^3, y^3, (x+y)^4, x^4, (x-y)^4, y^4, (x+y)^3, x^3 \rangle.$$

It is minimally generated by $\langle x^3, y^3, (x-y)^3, (x+y)^3 \rangle$. Then $\hat{R}/I_{\psi_{0,2}}$ has the following resolution:

$$0 \to \hat{R}(-4)^3 \to \hat{R}(-3)^4 \to \hat{R} \to \hat{R}/I_{\psi_{0,2}} \to 0,$$

where $\Omega_{p_{0,2}} = 3$, $N_{\Omega_{p_{0,2}}} = 0$ and $N_{\Omega_{p_{0,2}}+1} = 3$. The Hilbert polynomial of $\hat{R}/I_{\psi_{p_{0,2}}}$ is constant:

$$
\begin{aligned}
HP(\hat{R}/I_{\psi_{0,2}}, k) = &\binom{k+2}{2} - 4\binom{k-1}{2} + 3\binom{k-2}{2}, \\
= &\frac{1}{2}[(k+2).(k+1) - 4.(k-1).(k-2) + 3.(k-2).(k-3)], \\
= &6.
\end{aligned}
$$

By applying the same method to the other points, we obtain:
When we replace all the calculations in Eqn. (4.7) we find $HP(C^r(\hat{\Delta}), k)$ as follows:

Table 4.2: Table giving the some properties of $G_p(\Delta)$

$p \in \Delta_0^0$	$G_p(\Delta)$	I_ψ m.g.ideal	$HP(R/I_\psi, k)$
p_0	2 octagons	$\langle l_4^3, l_1^3, l_3^3\rangle$and$\langle l_1^3, l_2^3, l_3^3, l_4^3\rangle$	7 and 6
p_1	a triangle and 3 line segments	$\langle l_1^4, l_5^4, l_6^4\rangle$	12
p_2	a triangle and 3 line segments	$\langle l_2^4, l_6^4, l_7^4\rangle$	12
p_3	a triangle and 3 line segments	$\langle l_3^3, l_8^3, l_7^3\rangle$	8
p_4	a triangle and 3 line segments	$\langle l_4^3, l_8^3, l_9^4\rangle$	8
p_5	a triangle and 3 line segments	$\langle l_1^3, l_9^4, l_{10}^4\rangle$	10
p_6	a triangle and 3 line segments	$\langle l_{11}^3, l_2^4, l_{10}^4\rangle$	10
p_7	a triangle and 3 line segments	$\langle l_3^3, l_{11}^3, l_{12}^3\rangle$	7
p_8	a triangle and 3 line segments	$\langle l_{12}^3, l_4^4, l_5^4\rangle$	10
p_9	a quadrilateral and 2 line segments	$\langle l_1^3, l_{14}^4, l_{13}^4\rangle$	8
p_{10}	a quadrilateral and 2 line segments	$\langle l_2^3, l_{14}^3, l_{15}^3\rangle$	7
p_{11}	a quadrilateral and 2 line segments	$\langle l_3^3, l_{15}^3, l_{16}^4\rangle$	8
p_{12}	a quadrilateral and 2 line segments	$\langle l_4^3, l_{17}^3, l_{16}^4\rangle$	8
p_{13}	a quadrilateral and 2 line segments	$\langle l_1^3, l_{17}^3, l_{18}^4\rangle$	8
p_{14}	a quadrilateral and 2 line segments	$\langle l_2^4, l_{18}^4, l_{19}^4\rangle$	12
p_{15}	a quadrilateral and 2 line segments	$\langle l_3^3, l_{20}^3, l_{19}^4\rangle$	8
p_{16}	a quadrilateral and 2 line segments	$\langle l_4^3, l_{20}^3, l_{13}^4\rangle$	8

$$
\begin{aligned}
HP(C^r(\hat{\Delta}), k) &= \frac{17}{2}k^2 - \frac{175}{2}k + 51 + HP(N, k) \\
&= \frac{17}{2}k^2 - \frac{175}{2}k + 51 + 7 + 6 + 12 + 12 + 8 + 8 + 10 + 10 + 7 + 10 + 8 \\
&\quad + 7 + 8 + 8 + 8 + 12 + 8 + 8 \\
&= \frac{17}{2}k^2 - \frac{175}{2}k + 208,
\end{aligned}
$$

for sufficiently large k.

CHAPTER 5

RESULTS IN DIMENSION THREE

In this Chapter, we apply the method for planar splines given by Geramita and Schenck [18] to dimension 3. While applying the method, we have two important problems to consider: one of them is the computation of the Hilbert function of the corresponding fat points, which is much harder than the case $\mathbb{P}^1$, corresponding to the planar splines. The other problem is the second homology of the corresponding chain complex, which is not always zero.

First, we develop a formula for simplicial complexes with no interior points.

5.1 Simplicial complexes with no interior point

In this section, we first show that the dimension of the vector space of a spline defined on an n-gon base, Δ, in $\mathbb{R}^3$ having no interior point is just related with the geometry of Δ. In other words, it has $H_2(R/J) = 0$. Suppose Δ is the convex hull of the vertices $v_0, v_1, v_2, v_3, ..., v_{n+1}$ having no interior point with an n-gon base designed as in the below picture, where $n \in \mathbb{N}$. Δ is a pure, hereditary polyhedral complex with n interior 3-cells, n interior 2-cells, 1 interior 1-cell and 0 interior 0-cell. We can

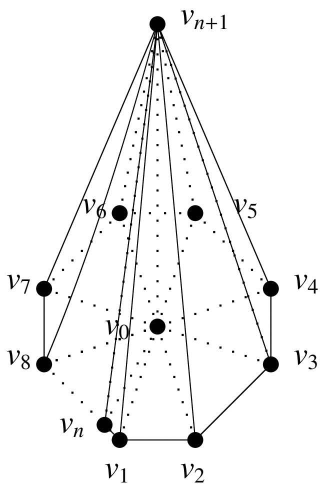

Figure 5.1: Simplicial complex with n-gon base having no interior point

number the 3-cells as $\sigma_1, \sigma_2, ..., \sigma_n$ in the following way.

$$\begin{aligned}
\sigma_1 &= \mathrm{Conv}(\{v_0, v_1, v_2, v_{n+1}\}),\\
\sigma_2 &= \mathrm{Conv}(\{v_0, v_2, v_3, v_{n+1}\}),\\
\sigma_3 &= \mathrm{Conv}(\{v_0, v_3, v_4, v_{n+1}\}),\\
&\vdots\\
\sigma_n &= \mathrm{Conv}(\{v_0, v_n, v_1, v_{n+1}\}).
\end{aligned}$$

We embed Δ into the hyperplane $x_4 = 1 \subseteq \mathbb{R}^4$, thus we can form $\hat{\Delta}$ over Δ with the vertex (v) at the origin.

The homology of the chain complex $\mathcal{R}$ is the simplicial relative homology with coefficients in $R = \mathbb{R}[x, y, z, w]$.

$$\mathcal{R} : 0 \to R^n \xrightarrow{\partial_3} \oplus_{\alpha\in\Delta_2^0} R \xrightarrow{\partial_2} \oplus_{\alpha\in\Delta_1^0} R \xrightarrow{\partial_1} 0.$$

The interior faces of the simplicial complex are:

$$\Delta_2^0 = \{\langle v_0, v_1, v_{n+1}\rangle, \langle v_0, v_2, v_{n+1}\rangle, ..., \langle v_0, v_n, v_{n+1}\rangle\},$$

$$\Delta_1^0 = \{\langle v_{n+1}, v_0\rangle\},$$

$$\Delta_0^0 = \{\}.$$

As we have defined in Chapter 3, for any interior face γ the ideal $\mathcal{J}(\gamma) = \sum_{\gamma\subseteq\tau_i\in\Delta_2^0} L_{\tau_i}^{\alpha_i}$. For $\gamma \in \Delta_2^0$, $\mathcal{J}(\gamma)$'s are given as:

$$\begin{aligned}
\mathcal{J}(\langle v_0, v_1, v_{n+1}\rangle) =& \langle p_1^2\rangle,\\
\mathcal{J}(\langle v_0, v_2, v_{n+1}\rangle) =& \langle p_2^2\rangle,\\
\mathcal{J}(\langle v_0, v_3, v_{n+1}\rangle) =& \langle p_3^2\rangle,\\
&\vdots\\
\mathcal{J}(\langle v_0, v_n, v_{n+1}\rangle) =& \langle p_n^2\rangle.
\end{aligned}$$

and for $\gamma \in \Delta_1^0$, $\mathcal{J}(\langle v_{n+1}, v_0\rangle) = \langle p_1^2, p_2^2, ..., p_n^2\rangle$. Here, p_i's denote the linear polynomials corresponding to the planes.

If we take the quotient of $\mathcal{R}$ by $\mathcal{J}$, we obtain the following chain complex:

$$\mathcal{R}/\mathcal{J} : 0 \to R^n \xrightarrow{\partial_3} R/\langle p_1^2\rangle \oplus R/\langle p_2^2\rangle \oplus ... \oplus R/\langle p_n^2\rangle \xrightarrow{\partial_2} R/\langle p_1^2, p_2^2, ..., p_n^2\rangle \xrightarrow{\partial_1} 0.$$

Note that, ∂_3 acts on each σ as:

$$\begin{aligned}
\partial_3(\langle v_0, v_1, v_2, v_{n+1}\rangle) =& \langle v_1, v_2, v_{n+1}\rangle - \langle v_0, v_2, v_{n+1}\rangle + \langle v_0, v_1, v_{n+1}\rangle -\\
& \langle v_0, v_1, v_2\rangle,\\
\partial_3(\langle v_0, v_2, v_3, v_{n+1}\rangle) =& \langle v_2, v_3, v_{n+1}\rangle - \langle v_0, v_3, v_{n+1}\rangle + \langle v_0, v_2, v_{n+1}\rangle -\\
& \langle v_0, v_2, v_3\rangle,\\
&\vdots\\
\partial_3(\langle v_0, v_n, v_1, v_{n+1}\rangle) =& \langle v_n, v_1, v_{n+1}\rangle - \langle v_0, v_1, v_{n+1}\rangle + \langle v_0, v_n, v_{n+1}\rangle -\\
& \langle v_0, v_n, v_1\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\partial_3(g_1, g_2, ..., g_n) =& (g_1 - g_n)\langle v_0, v_1, v_{n+1}\rangle + (-g_1 + g_2)\langle v_0, v_2, v_{n+1}\rangle\\
& + (-g_2 + g_3)\langle v_0, v_3, v_{n+1}\rangle + ... + (-g_{n-1} + g_n)\langle v_0, v_1, v_{n+1}\rangle\\
=& ((g_1 - g_n), (-g_1 + g_2), (-g_2 + g_3), ..., (-g_{n-1} + g_n)).
\end{aligned}$$

Also, note that ∂_2 acts on Δ_2^0 as:

$$\begin{aligned}
\partial_2(\langle v_0, v_1, v_{n+1}\rangle) =& \langle v_1, v_{n+1}\rangle - \langle v_0, v_{n+1}\rangle + \langle v_0, v_1\rangle,\\
\partial_2(\langle v_0, v_2, v_{n+1}\rangle) =& \langle v_2, v_{n+1}\rangle - \langle v_0, v_{n+1}\rangle + \langle v_0, v_2\rangle,\\
\partial_2(\langle v_0, v_3, v_{n+1}\rangle) =& \langle v_3, v_{n+1}\rangle - \langle v_0, v_{n+1}\rangle + \langle v_0, v_3\rangle,\\
&\vdots\\
\partial_2(\langle v_0, v_n, v_{n+1}\rangle) =& \langle v_n, v_{n+1}\rangle - \langle v_0, v_{n+1}\rangle + \langle v_0, v_n\rangle.
\end{aligned}$$

Thus, we have:

$$\partial_2(g_1, g_2, ..., g_n) = (g_1 + g_2 + ... + g_n)\langle v_{n+1}, v_0\rangle.$$

Now, it is easy to check that $\mathcal{R}/\mathcal{J}$ is a complex:

$$\begin{aligned}
\partial_2(\partial_3) &= \partial_2((g_1 - g_n), (-g_1 + g_2), ..., (-g_{n-1} + g_n))\\
&= g_1 - g_n - g_1 + g_2 - ... - g_{n-1} + g_n\\
&= 0.
\end{aligned}$$

$\partial_1(\partial_2) = 0$, since $\partial_1 = 0$.

By applying the Euler characteristic equation (Eqn. (3.4.8)) to the complex $\mathcal{R}/\mathcal{J}$, we obtain the formula
$\dim_{\mathbb{R}} H_3(\mathcal{R}/\mathcal{J})_k = \dim \sum_{i=0}^{3}(-1)^i \oplus_{\beta\in\Delta^0_{3-i}} \mathcal{R}/\mathcal{J}(\beta)_k + \dim_{\mathbb{R}} \sum_{i=0}^{2}(-1)^i H_{2-i}(\mathcal{R}/\mathcal{J})_k$, in which the top homology $\dim_{\mathbb{R}} H_3(\mathcal{R}/\mathcal{J})_k$ is precisely the module $\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k$.

To compute $\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k$, we need to calculate $\dim_{\mathbb{R}}(H_2(\mathcal{R}/\mathcal{J}))_k$ and $\dim_{\mathbb{R}}(H_1(\mathcal{R}/\mathcal{J}))_k$. But for the complexes constructed above, $H_1(\mathcal{R}/\mathcal{J})_k = 0$, since $\ker\partial_0 = 0$. We claim that $\dim_{\mathbb{R}}(H_2(\mathcal{R}/\mathcal{J}))_k$ is also 0 for all k. Since $H_2(\mathcal{R}/\mathcal{J}) = H_1(\mathcal{J}) = \ker\partial_1/im\partial_2$, we want to show that the map ∂_2 is onto. To check the surjectivity of ∂_2, we take any $f \in \oplus_{\lambda\in\Delta^0_2}\mathcal{J}(\lambda) = \langle p_1^2, p_2^2, ..., p_n^2\rangle$ such that $f = h_1p_1^2 + h_2p_2^2 + ... + h_np_n^2$. We should check that there is a $g = (g_1, g_2, ..., g_n) \in \oplus_{\lambda\in\Delta^0_3}\mathcal{J}(\lambda)$ satisfying $\partial_2(g) = f$. If we choose $g_1 = -h_1$, $g_2 = -h_2$,..., $g_n = -h_n$, then $g = (g_1, g_2, ..., g_n)$ satisfies this condition. Thus ∂_2 is onto, which implies that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) = 0$, and hence $\dim_{\mathbb{R}}(H_2(\mathcal{R}/\mathcal{J})) = 0$.

As a result, for the simplicial complexes not containing any interior point, we obtain the formula 5.1:

$$\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k = \dim \sum_{i=0}^{d}(-1)^i \oplus_{\beta\in\Delta^0_{d-i}} \mathcal{R}/\mathcal{J}(\beta)_k. \tag{5.1}$$

For these type of simplical complexes, the planes defined by $p_1, ..., p_n$ form a pencil of planes with the intersection line containing $\langle v_{n+1}, v_0\rangle$. The plane defined by p_i is a linear combination of the planes defined by p_j and p_k, if p_i, p_j and p_k define distinct planes. Hence, we have 2 situations:

a) $R/\langle p_1^2, p_2^2, ..., p_n^2\rangle = R/\langle p_i^2, p_j^2\rangle$ case, resulting with the formula:

$$\begin{aligned}\dim C_k^\alpha(\hat{\Delta}) &= n\binom{k+3}{3} - n(\binom{k+3}{3} - \binom{k+1}{3}) + 4k \\ &= n\binom{k+1}{3} + 4k \qquad \forall\, k \geq 0.\end{aligned}$$

b) $R/\langle p_1^2, p_2^2, ..., p_n^2\rangle = R/\langle p_i^2, p_j^2, p_i p_j\rangle$ case, resulting with the formula:

$$\begin{aligned}\dim C_k^\alpha(\hat{\Delta}) &= n\binom{k+3}{3} - n(\binom{k+3}{3} - \binom{k+1}{3}) + 3k + 1 \\ &= n\binom{k+1}{3} + 1 + 3k \qquad \forall\, k \geq 0.\end{aligned}$$

Remark 5.1.1 *Note that, all the arguments are still valid, if the points $v_0, v_1, ..., v_n$ do not lie on the same plane.*

5.1.1 Dimension of a general tetrahedron

We can now give the detailed computations for the dimension of $C_k^\alpha(\Delta)$, when Δ is a tetrahedron with no interior point. Let Δ be a tetrahedron having vertices at the

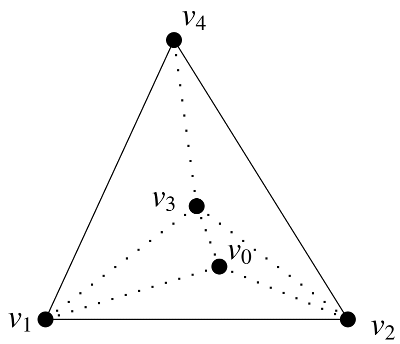

Figure 5.2: Tetrahedron with no interior point

points $v_0 = (a_0, b_0, c_0)$, $v_1 = (a_1, b_1, c_1)$,

$v_2 = (a_2, b_2, c_2)$, $v_3 = (a_3, b_3, c_3)$, $v_4 = (a_4, b_4, c_4)$. Hence, Δ, Conv$(\{v_1, v_2, v_3, v_4\})$, is a pure and a hereditary polyhedral complex with three 3-cells, three interior 2-cells, one interior 1-cell and zero interior 0-cell. Let's number the 3-cells as $\sigma_1, \sigma_2, \sigma_3$ in the following way:

$$\sigma_1 = \text{Conv}(\{v_0, v_1, v_2, v_4\}), \sigma_2 = \text{Conv}(\{v_0, v_2, v_3, v_4\}), \sigma_3 = \text{Conv}(\{v_0, v_3, v_1, v_4\}).$$

In the figure, there is no interior point hence $\oplus_{\gamma \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma) = 0$. By embedding Δ into the hyperplane $x_4 = 1 \subseteq \mathbb{R}^4$, we can form $\hat{\Delta}$ over Δ with the vertex $v = (0, 0, 0, 0)$. In that case, the chain complex becomes:

$$\mathcal{R}/\mathcal{J} : 0 \to \mathcal{R}^3 \xrightarrow{\partial_3} \oplus_{\alpha \in \Delta_2^0} \mathcal{R}/\mathcal{J}(\alpha) \xrightarrow{\partial_2} \oplus_{\beta \in \Delta_1^0} \mathcal{R}/\mathcal{J}(\beta) \xrightarrow{\partial_1} 0.$$

For the interior faces, we have:

$$\mathcal{J}(\gamma) = \{L_{\tau_i}^{\alpha_{i+1}} : \gamma \subset \tau_i \subseteq \Delta_2^0\},$$

$\mathcal{J}(\langle v_0, v_1, v_4 \rangle) = [(b_0c_1 - b_0c_4 - b_1c_0 + b_1c_4 + b_4c_0 - b_4c_1)x + (-a_0c_1 + a_0c_4 + a_1c_0 - a_1c_4 - a_4c_0 + a_4c_1)y + (a_0b_1 - a_0b_4 - a_1b_0 + a_1b_4 + a_4b_0 - a_4b_1)z + (-a_0b_1c_4 + a_0b_4c_1 + a_1b_0c_4 - a_1b_4c_0 - a_4b_0c_1 + a_4b_1c_0)]^2 (= p_1^2)$,

$\mathcal{J}(\langle v_0, v_2, v_4 \rangle) = [(b_0c_2 - b_0c_4 - b_2c_0 + b_2c_4 + b_4c_0 - b_4c_2)x + (-a_0c_2 + a_0c_4 + a_2c_0 - a_2c_4 - a_4c_0 + a_4c_2)y + (a_0b_2 - a_0b_4 - a_2b_0 + a_2b_4 + a_4b_0 - a_4b_2)z + (-a_0b_2c_4 + a_0b_4c_2 + a_2b_0c_4 - a_2b_4c_0 - a_4b_0c_2 + a_4b_2c_0)]^2 (= p_2^2)$,

$\mathcal{J}(\langle v_0, v_3, v_4 \rangle) = [(b_0c_3 - b_0c_4 - b_3c_0 + b_3c_4 + b_4c_0 - b_4c_3)x + (-a_0c_3 + a_0c_4 + a_3c_0 - a_3c_4 - a_4c_0 + a_4c_3)y + (a_0b_3 - a_0b_4 - a_3b_0 + a_3b_4 + a_4b_0 - a_4b_3)z + (-a_0b_3c_4 + a_0b_4c_3 + a_3b_0c_4 - a_3b_4c_0 - a_4b_0c_3 + a_4b_3c_0)]^2 (= p_3^2)$,

$\mathcal{J}(\langle v_4, v_0 \rangle) = \langle p_1^2, p_2^2, p_3^2 \rangle$.

In this case, $\mathcal{R}/\mathcal{J}$ is the following complex:

$\mathcal{R}/\mathcal{J} : 0 \to \mathcal{R}^3 \xrightarrow{\partial_3} \mathcal{R}/\langle p_1^2 \rangle \oplus \mathcal{R}/\langle p_2^2 \rangle \oplus \mathcal{R}/\langle p_3^2 \rangle \xrightarrow{\partial_2} \mathcal{R}/\langle p_1^2, p_2^2, p_3^2 \rangle \xrightarrow{\partial_1} 0.$

Hence, we have the formula:

$$\begin{aligned}
\dim C^{\alpha}(\hat{\Delta})_k &= \dim \sum_{i=0}^{3} (-1)^{i+1} \sum_{\beta \in \Delta_i^0} (\mathcal{R}/\mathcal{J}(\beta))_k \\
&= \dim \sum_{k \in \Delta_3^0} \mathcal{R}_k - \dim \sum_{\beta \in \Delta_2^0} (\mathcal{R}/\mathcal{J}(\beta))_k + \dim \sum_{\beta \in \Delta_1^0} (\mathcal{R}/\mathcal{J}(\beta))_k \\
&= 3\binom{k+3}{3} - 3(\binom{k+3}{3} - \binom{k+1}{3}) + \dim(\mathcal{R}/\langle p_1^2, p_2^2, p_3^2 \rangle)_k.
\end{aligned}$$

We need to calculate $\dim(\mathcal{R}/\langle p_1^2, p_2^2, p_3^2 \rangle)_k$. We write the defining equations of the planes,

$p_1^2 = [(b_0c_1 - b_0c_4 - b_1c_0 + b_1c_4 + b_4c_0 - b_4c_1)x + (-a_0c_1 + a_0c_4 + a_1c_0 - a_1c_4 - a_4c_0 + a_4c_1)y + (a_0b_1 - a_0b_4 - a_1b_0 + a_1b_4 + a_4b_0 - a_4b_1)z + (-a_0b_1c_4 + a_0b_4c_1 + a_1b_0c_4 - a_1b_4c_0 - a_4b_0c_1 + a_4b_1c_0)]^2$,

$p_2^2 = [(b_0c_2 - b_0c_4 - b_2c_0 + b_2c_4 + b_4c_0 - b_4c_2)x + (-a_0c_2 + a_0c_4 + a_2c_0 - a_2c_4 - a_4c_0 + a_4c_2)y + (a_0b_2 - a_0b_4 - a_2b_0 + a_2b_4 + a_4b_0 - a_4b_2)z + (-a_0b_2c_4 + a_0b_4c_2 + a_2b_0c_4 - a_2b_4c_0 - a_4b_0c_2 + a_4b_2c_0)]^2$,

$p_3^2 = [(b_0c_3 - b_0c_4 - b_3c_0 + b_3c_4 + b_4c_0 - b_4c_3)x + (-a_0c_3 + a_0c_4 + a_3c_0 - a_3c_4 - a_4c_0 + a_4c_3)y + (a_0b_3 - a_0b_4 - a_3b_0 + a_3b_4 + a_4b_0 - a_4b_3)z + (-a_0b_3c_4 + a_0b_4c_3 + a_3b_0c_4 - a_3b_4c_0 - a_4b_0c_3 + a_4b_3c_0)]^2$.

in the matrix form:

$$A = \begin{pmatrix} b_0(c_1 - c_4) + b_1(c_4 - c_0) + b_4(c_0 - c_1) & a_0(c_4 - c_1) + a_1(c_0 - c_4) + a_4(c_1 - c_0) & a_0(b_1 - b_4) + a_1(b_4 - b_0) + a_4(b_0 - b_1) & a_0(b_1c_4 - b_4c_1) + a_1(b_4c_0 - b_0c_4) + a_4(b_0c_1 - b_1c_0) \\ b_0(c_2 - c_4) + b_2(c_4 - c_0) + b_4(c_0 - c_2) & a_0(c_4 - c_2) + a_2(c_0 - c_4) + a_4(c_2 - c_0) & a_0(b_2 - b_4) + a_2(b_4 - b_0) + a_4(b_0 - b_2) & a_0(b_2c_4 - b_4c_2) + a_2(b_4c_0 - b_0c_4) + a_4(b_0c_2 - b_2c_0) \\ b_0(c_3 - c_4) + b_3(c_4 - c_0) + b_4(c_0 - c_3) & a_0(c_4 - c_3) + a_3(c_0 - c_4) + a_4(c_3 - c_0) & a_0(b_3 - b_4) + a_3(b_4 - b_0) + a_4(b_0 - b_3) & a_0(b_3c_4 - b_4c_3) + a_3(b_4c_0 - b_0c_4) + a_4(b_0c_3 - b_3c_0) \end{pmatrix}.$$

By using Singular, we can calculate the row reduced echelon form of the matrix A:

$$\text{rowred(A)} = \begin{pmatrix} 1 & 0 & (-a_0 + a_4)/(c_0 - c_4) & (-a_0c_4 + a_4c_0)/(c_0 - c_4) \\ 0 & 1 & (-b_0 + b_4)/(c_0 - c_4) & (-b_0c_4 + b_4c_0)/(c_0 - c_4) \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, we should consider the case $c_0 = c_4$. We can write c_0 instead of c_4 in A, and it gives the following row reduced form of A:

$$\text{rowred(A)} = \begin{pmatrix} 1 & (-a_0 + a_4)/(b_0 - b_4) & 0 & (-a_0b_4 + a_4b_0)/(b_0 - b_4) \\ 0 & 0 & 1 & c_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, we have one more case $c_0 = c_4$ and $b_0 = b_4$, and by replacing c_4 by c_0 and b_4 by b_0 in A, we get the following row reduced form:

$$\text{rowred(A)} = \begin{pmatrix} 0 & 1 & 0 & b_0 \\ 0 & 0 & 1 & c_0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These three cases show that p_3 can be written by a linear combination of, p_1 and p_2 which means $\dim(\mathcal{R}/\langle p_1^2, p_2^2, p_3^2\rangle)_k = \dim(\mathcal{R}/\langle p_1^2, p_2^2, p_1p_2\rangle)_k$.

Hence, we have the formula (The second case explained in the previous section):

$$\begin{aligned} \dim C^\alpha(\hat{\Delta})_k &= 3\binom{k+3}{3} - 3(\binom{k+3}{3} - \binom{k+1}{3}) + 3k + 1 \\ &= 3\binom{k+1}{3} + 1 + 3k \qquad \forall\, k \geq 0. \end{aligned}$$

Remark 5.1.2 *Again, note that, all the arguments are valid, if the points v_0, v_1, v_2, v_3 do not lie on the same plane.*

5.2 All Possible Dimensions for an Octahedron

In this section, we examine all possible situations for the dimension of $C_k^\alpha(\Delta)$, when Δ is an octahedron with one interior point. The dimension depends on the number of interior two faces contained in the plane defined by L_{ij}. This plane contains the interior 2-cell σ_{ij}, which is the intersection of the adjacent 2-cells σ_i and σ_j. Hence, an octahedron with one interior point can have from 3 to 12 different interior hyperplanes. Here, we consider each case and try to give a formula for $\dim_{\mathbb{R}} C_k^\alpha(\Delta)$, when the smoothness degree is 1, such that $r = 1$. In our calculations, we take the interior vertex of the octahedron v_0 as $(0, 0, 0)$, since by a change of coordinates, it is always possible to translate a point to the origin.

5.2.1 Octahedrons Having 3 Interior Hyperplanes

If we consider a regular octahedron, then the octahedron has three interior hyperplanes. For such a construction $H_1(\mathcal{J})$ is always zero, [16]. Thus, we can calculate the dimension of $C_k^\alpha(\Delta)$ just by combinatorial calculations. Below we work on a specific octahedron having three interior hyperplanes, but we will show that any octahedron having three interior hyperplanes have the same dimension for $C_k^\alpha(\Delta)$.

Here, we work on an octahedron having vertices at $v_0 = (0, 0, 0), v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (-1, 0, 0), v_4 = (0, -1, 0), v_5 = (0, 0, 1), v_6 = (0, 0, -1)$, given as: Δ has

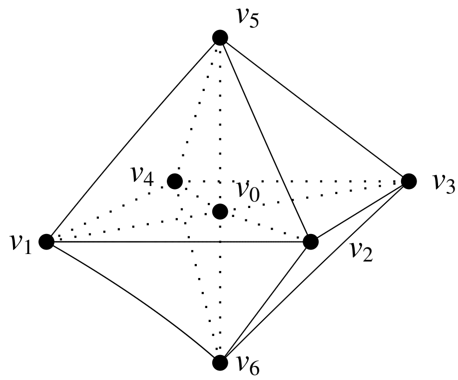

Figure 5.3: Regular octahedron

eight 3-cells which we name as:

$$
\begin{aligned}
\sigma_1 &= \mathrm{Conv}(\{v_0, v_1, v_2, v_5\}), & \sigma_5 &= \mathrm{Conv}(\{v_6, v_0, v_1, v_2\}),\\
\sigma_2 &= \mathrm{Conv}(\{v_0, v_2, v_3, v_5\}), & \sigma_6 &= \mathrm{Conv}(\{v_6, v_0, v_2, v_3\}),\\
\sigma_3 &= \mathrm{Conv}(\{v_0, v_3, v_4, v_5\}), & \sigma_7 &= \mathrm{Conv}(\{v_6, v_0, v_3, v_4\}),\\
\sigma_4 &= \mathrm{Conv}(\{v_0, v_4, v_1, v_5\}), & \sigma_8 &= \mathrm{Conv}(\{v_6, v_0, v_4, v_1\}).
\end{aligned}
$$

Δ_2^0 is the set of the interior two faces of Δ, and it has 12 elements.

$$
\begin{aligned}
\Delta_2^0 =&\{\langle v_0, v_1, v_2\rangle, \langle v_0, v_2, v_3\rangle, \langle v_0, v_3, v_4\rangle, \langle v_0, v_4, v_1\rangle, \langle v_0, v_2, v_5\rangle,\\
&\langle v_0, v_3, v_5\rangle, \langle v_0, v_4, v_5\rangle, \langle v_0, v_1, v_5\rangle, \langle v_6, v_0, v_2\rangle, \langle v_6, v_0, v_3\rangle,\\
&\langle v_6, v_0, v_4\rangle, \langle v_6, v_0, v_1\rangle\}.
\end{aligned}
$$

The set of the interior one faces, Δ_1^0, contains 6 elements:

$\Delta_1^0 = \{\langle v_0, v_1\rangle, \langle v_0, v_2\rangle, \langle v_0, v_3\rangle, \langle v_0, v_4\rangle, \langle v_5, v_0\rangle, \langle v_6, v_0\rangle\}$.

Δ_0^0, which is the set of the interior vertices has only one element, such that $\Delta_0^0 = \{\langle v_0\rangle\}$.

Let, ∂_i be the relative simplicial boundary map, $\mathcal{R}$ is the chain complex defined by $\mathcal{R}_i = R^{f_i^0}$. So the homology of $\mathcal{R}$ is the relative simplicial homology, with coefficients in R.

$$
\begin{aligned}
\mathcal{R} : 0 \to\ & \langle v_0, v_1, v_2, v_5\rangle \oplus \langle v_0, v_2, v_3, v_5\rangle \oplus \langle v_0, v_3, v_4, v_5\rangle \oplus \langle v_0, v_4, v_1, v_5\rangle\\
&\oplus \langle v_6, v_0, v_1, v_2\rangle \oplus \langle v_6, v_0, v_2, v_3\rangle \oplus \langle v_6, v_0, v_3, v_4\rangle \oplus \langle v_6, v_0, v_4, v_1\rangle\\
&\xrightarrow{\partial_3} \langle v_0, v_1, v_2\rangle \oplus \langle v_0, v_2, v_3\rangle \oplus \langle v_0, v_3, v_4\rangle \oplus \langle v_0, v_4, v_1\rangle\\
&\oplus \langle v_0, v_2, v_5\rangle \oplus \langle v_0, v_3, v_5\rangle \oplus \langle v_0, v_4, v_5\rangle \oplus \langle v_0, v_1, v_5\rangle \oplus \langle v_6, v_0, v_2\rangle\\
&\oplus \langle v_6, v_0, v_3\rangle \oplus \langle v_6, v_0, v_4\rangle \oplus \langle v_6, v_0, v_1\rangle \xrightarrow{\partial_2} \langle v_0, v_1\rangle \oplus \langle v_0, v_2\rangle\\
&\oplus \langle v_0, v_3\rangle \oplus \langle v_0, v_4\rangle \oplus \langle v_5, v_0\rangle \oplus \langle v_6, v_0\rangle \xrightarrow{\partial_1} \langle v_0\rangle \xrightarrow{\partial_0} 0.
\end{aligned}
$$

∂_3 acts on the three faces of Δ as in the following way:

$$\begin{aligned}
\partial_3(\langle v_0, v_1, v_2, v_5\rangle) =& \langle v_1, v_2, v_5\rangle - \langle v_0, v_2, v_5\rangle + \langle v_0, v_1, v_5\rangle - \\
& \langle v_0, v_1, v_2\rangle, \\
\partial_3(\langle v_0, v_2, v_3, v_5\rangle) =& \langle v_2, v_3, v_5\rangle - \langle v_0, v_3, v_5\rangle + \langle v_0, v_2, v_5\rangle - \\
& \langle v_0, v_2, v_3\rangle, \\
\partial_3(\langle v_0, v_3, v_4, v_5\rangle) =& \langle v_3, v_4, v_5\rangle - \langle v_0, v_4, v_5\rangle + \langle v_0, v_3, v_5\rangle - \\
& \langle v_0, v_3, v_4\rangle, \\
\partial_3(\langle v_0, v_4, v_1, v_5\rangle) =& \langle v_4, v_1, v_5\rangle - \langle v_0, v_1, v_5\rangle + \langle v_0, v_4, v_5\rangle - \\
& \langle v_0, v_4, v_1\rangle, \\
\partial_3(\langle v_6, v_0, v_1, v_2\rangle) =& \langle v_0, v_1, v_2\rangle - \langle v_6, v_1, v_2\rangle + \langle v_6, v_0, v_2\rangle - \\
& \langle v_6, v_0, v_1\rangle, \\
\partial_3(\langle v_6, v_0, v_2, v_3\rangle) =& \langle v_0, v_2, v_3\rangle - \langle v_6, v_2, v_3\rangle + \langle v_6, v_0, v_3\rangle - \\
& \langle v_6, v_0, v_2\rangle, \\
\partial_3(\langle v_6, v_0, v_3, v_4\rangle) =& \langle v_0, v_3, v_4\rangle - \langle v_6, v_3, v_4\rangle + \langle v_6, v_0, v_4\rangle - \\
& \langle v_6, v_0, v_3\rangle, \\
\partial_3(\langle v_6, v_0, v_4, v_1\rangle) =& \langle v_0, v_4, v_1\rangle - \langle v_6, v_4, v_1\rangle + \langle v_6, v_0, v_1\rangle - \\
& \langle v_6, v_0, v_4\rangle.
\end{aligned}$$

Hence, the image of an element $(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8)$ under the map ∂_3 is:

$$\begin{aligned}
\partial_3(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8) =& (g_5 - g_1)\langle v_0, v_1, v_2\rangle + (g_6 - g_2)\langle v_0, v_2, v_3\rangle \\
& + (g_7 - g_3)\langle v_0, v_3, v_4\rangle + (g_8 - g_4)\langle v_0, v_4, v_1\rangle \\
& + (g_2 - g_1)\langle v_0, v_2, v_5\rangle + (g_3 - g_2)\langle v_0, v_3, v_5\rangle \\
& + (g_4 - g_3)\langle v_0, v_4, v_5\rangle + (g_1 - g_4)\langle v_0, v_1, v_5\rangle \\
& + (g_5 - g_6)\langle v_6, v_0, v_2\rangle + (g_6 - g_7)\langle v_6, v_0, v_3\rangle \\
& + (g_7 - g_8)\langle v_6, v_0, v_4\rangle + (g_8 - g_5)\langle v_6, v_0, v_1\rangle,
\end{aligned}$$

$$\begin{aligned}
\partial_3(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8) =& ((g_5 - g_1), (g_6 - g_2), (g_7 - g_3), (g_8 - g_4), (g_2 - g_1), (g_3 - g_2), \\
& (g_4 - g_3), (g_1 - g_4), (g_5 - g_6), (g_6 - g_7), (g_7 - g_8), (g_8 - g_5)).
\end{aligned}$$

The actions of ∂_2 on the interior two faces are given below:

$$\partial_2(\langle v_0, v_1, v_2\rangle) = \langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle,$$

$$
\begin{aligned}
\partial_2(\langle v_0, v_2, v_3\rangle) &= \langle v_2, v_3\rangle - \langle v_0, v_3\rangle + \langle v_0, v_2\rangle,\\
\partial_2(\langle v_0, v_3, v_4\rangle) &= \langle v_3, v_4\rangle - \langle v_0, v_4\rangle + \langle v_0, v_3\rangle,\\
\partial_2(\langle v_0, v_4, v_1\rangle) &= \langle v_4, v_1\rangle - \langle v_0, v_1\rangle + \langle v_0, v_4\rangle,\\
\partial_2(\langle v_0, v_2, v_5\rangle) &= \langle v_2, v_5\rangle - \langle v_0, v_5\rangle + \langle v_0, v_2\rangle,\\
\partial_2(\langle v_0, v_3, v_5\rangle) &= \langle v_3, v_5\rangle - \langle v_0, v_5\rangle + \langle v_0, v_3\rangle,\\
\partial_2(\langle v_0, v_4, v_5\rangle) &= \langle v_4, v_5\rangle - \langle v_0, v_5\rangle + \langle v_0, v_4\rangle,\\
\partial_2(\langle v_0, v_1, v_5\rangle) &= \langle v_1, v_5\rangle - \langle v_0, v_5\rangle + \langle v_0, v_1\rangle,\\
\partial_2(\langle v_6, v_0, v_2\rangle) &= \langle v_0, v_2\rangle - \langle v_6, v_2\rangle + \langle v_6, v_0\rangle,\\
\partial_2(\langle v_6, v_0, v_3\rangle) &= \langle v_0, v_3\rangle - \langle v_6, v_3\rangle + \langle v_6, v_0\rangle,\\
\partial_2(\langle v_6, v_0, v_4\rangle) &= \langle v_0, v_4\rangle - \langle v_6, v_4\rangle + \langle v_6, v_0\rangle,\\
\partial_2(\langle v_6, v_0, v_1\rangle) &= \langle v_0, v_1\rangle - \langle v_6, v_1\rangle + \langle v_6, v_0\rangle.
\end{aligned}
$$

So the image of an element $(h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{12})$ under the map ∂_2 is:

$$
\begin{aligned}
\partial_2(h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{12}) &= (h_1 - h_4 + h_8 + h_{12})\langle v_0, v_1\rangle\\
&+ (-h_1 + h_2 + h_5 + h_9)\langle v_0, v_2\rangle + (-h_2 + h_3 + h_6 + h_{10})\langle v_0, v_3\rangle\\
&+ (-h_3 + h_4 + h_7 + h_{11})\langle v_0, v_4\rangle + (h_5 + h_6 + h_7 + h_8)\langle v_5, v_0\rangle\\
&+ (h_9 + h_{10} + h_{11} + h_{12})\langle v_6, v_0\rangle.
\end{aligned}
$$

The action of ∂_1 is:

$$
\partial_1(c_1, c_2, c_3, c_4, c_5, c_6) = (-c_1 - c_2 - c_3 - c_4 + c_5 + c_6)\langle v_0\rangle.
$$

Now, we are able to show that $\mathcal{R}$ is a complex.

$$
\begin{aligned}
\partial_2(\partial_3) = \partial_2(&(g_5 - g_1), (g_6 - g_2), (g_7 - g_3), (g_8 - g_4), (g_2 - g_1), (g_3 - g_2), (g_4 - g_3),\\
&(g_1 - g_4), (g_5 - g_6), (g_6 - g_7), (g_7 - g_8), (g_8 - g_5)),\\
=&((g_5 - g_1) - (g_8 - g_4) + (g_1 - g_4) + (g_8 - g_5), -(g_5 - g_1) + (g_6 - g_2)+\\
&(g_2 - g_1) + (g_5 - g_6), -(g_6 - g_2) + (g_7 - g_3) + (g_3 - g_2) + (g_6 - g_7),\\
&-(g_7 - g_3) + (g_8 - g_4) + (g_4 - g_3) + (g_7 - g_8), (g_2 - g_1) + (g_3 - g_2)+\\
&(g_4 - g_3) + (g_1 - g_4), (g_5 - g_6) + (g_6 - g_7) + (g_7 - g_8) + (g_8 - g_5)),\\
=&(0, 0, 0, 0, 0, 0).
\end{aligned}
$$

$$\begin{aligned}\partial_1(\partial_2) = \partial_1((&h_1 - h_4 + h_8 + h_{12}), (-h_1 + h_2 + h_5 + h_9), (-h_2 + h_3 + h_6 + h_{10}),\\ &(-h_3 + h_4 + h_7 + h_{11}), (h_5 + h_6 + h_7 + h_8), (h_9 + h_{10} + h_{11} + h_{12})),\\ &= -(h_1 - h_4 + h_8 + h_{12}) - (-h_1 + h_2 + h_5 + h_9) - (-h_2 + h_3 + h_6 + h_{10})\\ &\quad -(-h_3 + h_4 + h_7 + h_{11}) + (h_5 + h_6 + h_7 + h_8) + (h_9 + h_{10} + h_{11} + h_{12}) = 0.\end{aligned}$$

Hence, $\mathcal{R}$ is a complex. We define for any interior face γ the ideal $\mathcal{J}(\gamma) = \sum_{\gamma\subseteq\tau_i\in\Delta_2^0} L_{\tau_i}^{\alpha_i}$. $\mathcal{J}(\gamma)$ is an ideal generated by the powers of homogeneous linear forms. In our examples, because of the choice of v_0 all the linear forms will be homogeneous.

$$\begin{aligned}&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = z^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = x^2,\\ &\mathcal{J}(\langle v_0, v_2, v_3\rangle) = z^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = y^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2,\\ &\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2,\\ &\mathcal{J}(\langle v_0, v_4, v_1\rangle) = z^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = y^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = y^2,\end{aligned}$$

$$\begin{aligned}&\mathcal{J}(\langle v_0, v_1\rangle) = \langle y^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_2\rangle) = \langle x^2, z^2\rangle,\\ &\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2\rangle,\\ &\mathcal{J}(\langle v_5, v_0\rangle) = \langle x^2, y^2\rangle, && \mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2\rangle,\end{aligned}$$

$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2\rangle.$$

Taking $\hat{\Delta}$ instead of Δ makes calculations easier, but it makes no change in the linear forms, since they are all homogeneous. So, we will be working in $R = \mathbb{R}[x, y, z, w]$. ∂_i also gives us a map on the quotient of $\mathcal{R}$ by $\mathcal{J}$. There is a short exact sequence of complexes:

$$0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0.$$

This sequence gives us the long exact sequence of homology modules:

$$\begin{aligned}0 \to H_2(\mathcal{J}) \to H_2(\mathcal{R}) \to &H_2(\mathcal{R}/\mathcal{J}) \to H_1(\mathcal{J}) \to H_1(\mathcal{R}) \to\\ &H_1(\mathcal{R}/\mathcal{J}) \to H_0(\mathcal{J}) \to H_0(\mathcal{R}) \to H_0(\mathcal{R}/\mathcal{J}) \to 0.\end{aligned}$$

Here, R is a manifold. Hence, $H_i(\mathcal{R}) = 0 \quad \forall i < d$. By [16, Theorem 4.10], if Δ is a complex such that $H_i(\mathcal{R}) = 0 \quad \forall i < d$, then $C^r(\Delta)$ is free if and only if $H_i(\mathcal{J}) = 0 \ \ \forall i < d - 1$. Since $H_i(\mathcal{R}) = 0 \ \ \forall i < d$, we get the following equalities:

$$H_2(\mathcal{R}/\mathcal{J}) = H_1(\mathcal{J}), \qquad H_1(\mathcal{R}/\mathcal{J}) = H_0(\mathcal{J}).$$

Working with $H_i(\mathcal{J})$ makes our calculations easier, hence instead of finding $H_i(\mathcal{R}/\mathcal{J})$, we will be interested in $H_i(\mathcal{J})$.

For the case $i = 0$, $H_0(\mathcal{J}) = \ker \partial_0 / \operatorname{im} \partial_1$. Here, our question is that 'is ∂_1 onto?' So we should check that $\forall f = f_1 x^2 + f_2 y^2 + f_3 z^2 \in \langle x^2, y^2, z^2 \rangle$, is there any

$$\begin{aligned} g = (g_1, g_2, g_3, g_4, g_5, g_6) \in & \langle y^2, z^2 \rangle \oplus \langle x^2, z^2 \rangle \oplus \langle y^2, z^2 \rangle \oplus \\ & \langle x^2, z^2 \rangle \oplus \langle x^2, y^2 \rangle \oplus \langle x^2, y^2 \rangle, \end{aligned}$$

satisfying

$$\partial_1(g_1, g_2, g_3, g_4, g_5, g_6) = f_1 x^2 + f_2 y^2 + f_3 z^2,$$

which implies that

$$-g_1 - g_2 - g_3 - g_4 + g_5 + g_6 = f_1 x^2 + f_2 y^2 + f_3 z^2,$$

where

$$\begin{aligned}
g_1 &\in \langle y^2, z^2 \rangle && \Rightarrow g_1 = a_1 y^2 + b_1 z^2, \\
g_2 &\in \langle x^2, z^2 \rangle && \Rightarrow g_2 = a_2 x^2 + b_2 z^2, \\
g_3 &\in \langle y^2, z^2 \rangle && \Rightarrow g_3 = a_3 y^2 + b_3 z^2, \\
g_4 &\in \langle x^2, z^2 \rangle && \Rightarrow g_4 = a_4 x^2 + b_4 z^2, \\
g_5 &\in \langle x^2, y^2 \rangle && \Rightarrow g_5 = a_5 x^2 + b_5 y^2, \\
g_6 &\in \langle x^2, y^2 \rangle && \Rightarrow g_6 = a_6 x^2 + b_6 y^2.
\end{aligned}$$

If we choose

$$g_3 = g_4 = g_5 = g_6 = 0,$$

and

$$a_1 = -f_2, a_2 = -f_1, b_1 = -f_3, b_2 = c_1 = c_2 = 0,$$

then the equality is satisfied. Thus,

$$\begin{aligned} \partial_1(g_1, g_2, g_3, g_4, g_5, g_6) &= -g_1 - g_2 \\ &= f_1 x^2 + f_2 y^2 + f_3 z^2. \end{aligned}$$

This result implies that ∂_1 is onto. So, $H_0(\mathcal{J}) = 0$.

By the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$, we obtain the following equality, since $H_3(\mathcal{R}/\mathcal{J})$ is the homogeneous spline module $C^\alpha(\hat{\Delta})$.

$$\begin{aligned}
\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k &= \dim_{\mathbb{R}} \sum_{i=0}^{3} (-1)^{i+1} \sum_{\beta\in\Delta_i^0} (\mathcal{R}/\mathcal{J}(\beta))_k + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\
&= \dim_{\mathbb{R}} \sum_{k\in\Delta_3^0} \mathcal{R}_k - \dim_{\mathbb{R}} \sum_{\beta\in\Delta_2^0} (\mathcal{R}/\mathcal{J}(\beta))_k + \dim_{\mathbb{R}} \sum_{\beta\in\Delta_1^0} (\mathcal{R}/\mathcal{J}(\beta))_k \\
&\quad - \dim_{\mathbb{R}} \sum_{\beta\in\Delta_0^0} (\mathcal{R}/\mathcal{J}(\beta))_k + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\
&= 8\binom{k+3}{3} - 12 \dim_{\mathbb{R}} \mathcal{R}/\langle x^2\rangle + 2\dim_{\mathbb{R}}(\mathcal{R}/\langle y^2, z^2\rangle)_k \\
&\quad + 2\dim_{\mathbb{R}}(\mathcal{R}/\langle x^2, z^2\rangle)_k + \dim_{\mathbb{R}}(\mathcal{R}/\langle x^2, y^2\rangle)_k \\
&\quad - \dim_{\mathbb{R}}(\mathcal{R}/\langle x^2, y^2, z^2\rangle)_k + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k.
\end{aligned}$$

But by [16] we know $H_1(\mathcal{J}) = H_2(\mathcal{R}/\mathcal{J})$ and it is zero. Hence, the dimension of $C^r(\hat{\Delta})_k$ depends only on combinatorial calculations. When we compute these dimensions, we get the following results:

$$\dim_{\mathbb{R}}(\mathcal{R}/\langle x^2\rangle)_k = (k+1)^2,$$

$$\dim_{\mathbb{R}}(\mathcal{R}/\langle y^2, z^2\rangle)_k = \begin{cases} 1 & \text{if } k = 0, \\ 4k & \text{if } k \geq 1. \end{cases}$$

$$\dim_{\mathbb{R}}(\mathcal{R}/\langle x^2, z^2\rangle)_k = \begin{cases} 1 & \text{if } k = 0, \\ 4k & \text{if } k \geq 1. \end{cases}$$

$$\dim_{\mathbb{R}}(\mathcal{R}/\langle x^2, y^2\rangle)_k = \begin{cases} 1 & \text{if } k = 0, \\ 4k & \text{if } k \geq 1. \end{cases}$$

$$\dim_{\mathbb{R}}(\mathcal{R}/\langle x^2, y^2, z^2\rangle)_k = \begin{cases} 1 & \text{if } k = 0, \\ 4 & \text{if } k = 1, \\ 7 & \text{if } k = 2, \\ 8 & \text{if } k \geq 3. \end{cases}$$

Replacing these in the equality, we reach the conclusion:

$$\begin{aligned}
\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 6.4k - 8 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k, \\
&= 8\binom{k+3}{3} - 12k^2 - 20 \ \ \forall \ \ k \geq 3.
\end{aligned}$$

In fact, this formula is valid for any octahedron having 3 interior two faces, since during the calculations, we need only to know $\sum_{\beta\in\Delta_1^0}\dim_{\mathbb{R}}(\mathcal{R}/\mathcal{J}(\beta))_k$ and $\dim_{\mathbb{R}}(\mathcal{R}/\mathcal{J}(v_0))_k$, but for the first dimension any interior line can only be contained in two different hyperplanes and this always gives the same result with ours. For the last calculation, we should consider the hyperplanes containing v_0 and by our configuration, it is contained in 3 hyperplanes that intersect at a point, v_0, which implies that we can't rewrite any of them by the linear combinations of the others (If it was possible, we would be able to put all the vertices in a space having dimension 2, which would never give an octahedron). Hence the dimension is uniquely determined for a regular octahedron with one interior point.

5.2.2 Octahedrons Having 4 Interior Hyperplanes

We can construct an octahedron with four hyperplanes by moving any boundary vertex of a regular octahedron on one of the four boundary edges of the chosen vertex. To be able to make a generalization, we consider the case that we obtain by moving one of the vertices of the figure above. Hence, $\Delta = \text{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$, where $v_0 = (0,0,0), v_1 = (1/2,-1/2,0), v_2 = (0,1,0), v_3 = (-1,0,0), v_4 = (0,-1,0), v_5 = (0,0,1), v_6 = (0,0,-1)$, in which we move vertex v_1 on the v_1v_4 edge.

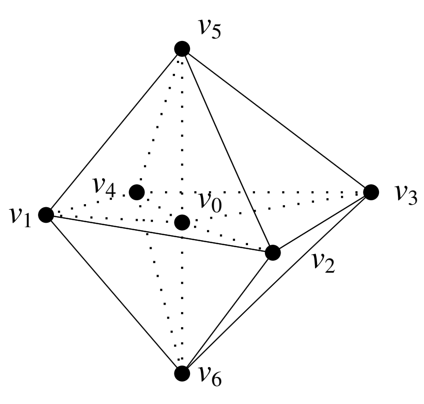

Figure 5.4: Octahedron having four hyperplanes

For any interior face γ of the simplex, we determine the ideal $\mathcal{J}(\gamma)$:

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = z^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = x^2,\\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = z^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = y^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2,\\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2,\\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = z^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (x+y)^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = (x+y)^2,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1\rangle) = \langle (x+y)^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_2\rangle) = \langle x^2, z^2\rangle,\\
&\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2\rangle,\\
&\mathcal{J}(\langle v_5, v_0\rangle) = \langle x^2, y^2, (x+y)^2\rangle, && \mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2, (x+y)^2\rangle,
\end{aligned}$$

$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2, (x+y)^2\rangle.$$

We can show that ∂_1 is onto. So, $H_0(\mathcal{J}) = 0$. By $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$, we get the following equality:

$$\begin{aligned}
\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k &= 8.\binom{k+3}{3} - 12.(k+1)^2 + 4.4k + 2.(3k+1) - 6 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k\\
&= 8\binom{k+3}{3} - 12k^2 - 2k - 16 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \;\; \forall \;\; k \geq 2.
\end{aligned}$$

In this nongeneric configuration, we know $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J}) = H_1(\mathcal{J}) = 0$ for all r by [16]. So the dimension of our simplex, $\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k$ is computable just by combinatorial calculations and is equal to:

$$\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 2k - 16 \;\; \forall \;\; k \geq 2.$$

In fact, we can generalize this to any point over the v_4v_1 line. First of all, let's consider the regular octahedron with hyperplanes $\{x, y, z\}$. Hence moving v_1 to any point along the line v_4v_1, we have $v_1^{'} = (1,0,0) + t[(0,-1,0) - (1,0,0)]$, where $t \in [0,1]$, thus $v_1^{'} = (1-t, -t, 0)$. We need to find the ideal $\mathcal{J}(\gamma)$ for any interior face γ of the simplex:

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1^{'}, v_2\rangle) = z^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = x^2,\\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = z^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = y^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2,\\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2,\\
&\mathcal{J}(\langle v_0, v_4, v_1^{'}\rangle) = z^2,
\end{aligned}$$

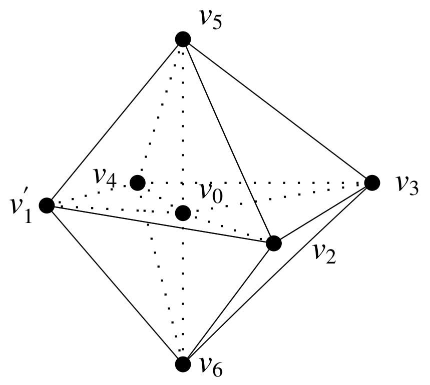

Figure 5.5: Octahedron constructed by moving a vertex of a regular octahedron through a line

$$\mathcal{J}(\langle v_0, v_1', v_5\rangle) = (xt - y(t-1))^2,$$
$$\mathcal{J}(\langle v_6, v_0, v_1'\rangle) = (xt - y(t-1))^2.$$

$$\mathcal{J}(\langle v_0, v_1'\rangle) = \langle (xt - y(t-1))^2, z^2\rangle, \qquad \mathcal{J}(\langle v_0, v_2\rangle) = \langle x^2, z^2\rangle,$$
$$\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2\rangle, \qquad \mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2\rangle,$$

$$\mathcal{J}(\langle v_5, v_0\rangle) = \langle x^2, y^2, (xt - y(t-1))^2\rangle,$$
$$\mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2, (xt - y(t-1))^2\rangle,$$
$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2, (xt - y(t-1))^2\rangle.$$

∂_1 is onto and so, $H_0(\mathcal{J}) = 0$.

By $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$, we get the result:

$$\begin{aligned}\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k &= 8.\binom{k+3}{3} - 12.(k+1)^2 + 4.4k + 2.(3k+1) - 6 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\ &= 8\binom{k+3}{3} - 12k^2 - 2k - 16 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k \geq 2.\end{aligned}$$

Since $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J}) = 0$ for all r, the dimension of our simplex, $\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k$ is computable just by combinatorial calculations and is equal to:

$$\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 2k - 16 \ \ \forall \ \ k \geq 2.$$

Here, the result is independent of the boundary vertex we've chosen to move, since when we move any boundary vertex through one of the incident edges of its hyperplanes (for a better comprehension, one can rename the vertices and call the moved vertex as v_1), we get a hyperplane which can be written as a linear combination of the other two and when $r = 1$, this gives the opportunity to decide the dimension in the same way as in our example.

We can generalize the above cases. Let's take the regular octahedron having three interior hyperplanes $\{x, y, z\}$ and move its vertex v_2 to a new point, remaining still in the same plane with $\mathrm{Conv}(\{v_0, v_2, v_3, v_4\})$ (When we take a point in the regular octahedron, it is contained exactly in two hyperplanes. Here, by moving, we mean to leave it in one of the hyperplanes and move it out of the other). Thus, I choose $\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$, where $v_0 = (0, 0, 0)$, $v_1 = (1, 0, 0)$, $v_2 = (2, 3, 0)$, $v_3 = (-1, 0, 0)$, $v_4 = (0, -1, 0)$, $v_5 = (0, 0, 1)$, $v_6 = (0, 0, -1)$. The calculation of the

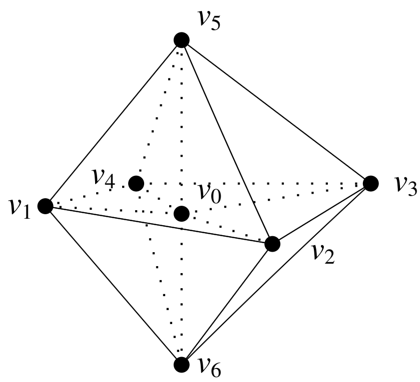

Figure 5.6: Octahedron constructed by moving a vertex of a regular octahedron leaving in a plane

ideal $\mathcal{J}(\gamma)$ for any interior face γ is as follows:

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = z^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = (3x - 2y)^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = y^2 \\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = z^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = (3x - 2y)^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2, \\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = x^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2, \\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = z^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = y^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = y^2,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1\rangle) = \langle y^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_2\rangle) = \langle (3x - 2y)^2, z^2\rangle, \\
&\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2\rangle, \\
&\mathcal{J}(\langle v_5, v_0\rangle) = \langle x^2, y^2, (3x - 2y)^2\rangle, && \mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2, (3x - 2y)^2\rangle,
\end{aligned}$$

$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2, (3x - 2y)^2\rangle.$$

∂_1 is onto and so, $H_0(\mathcal{J}) = 0$. By $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$, we get the following equality:

$$\begin{aligned}\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12.(k+1)^2 + 4.4k + 2.(3k+1) - 6 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\ &= 8\binom{k+3}{3} - 12k^2 - 2k - 16 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k \geq 2.\end{aligned}$$

In this nongeneric octahedron, we don't know whether $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is 0, so we use an algebraic calculation to compute $\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k$.

Interior faces are given as:

$$\begin{array}{lll}
\tau_1 \cap \tau_2 \subset V(3x-2y), & \sigma_1 \cap \sigma_2 \subset V(3x-2y), & \sigma_1 \cap \tau_1 \subset V(z), \\
\tau_2 \cap \tau_3 \subset V(y), & \sigma_2 \cap \sigma_3 \subset V(y), & \sigma_2 \cap \tau_2 \subset V(z), \\
\tau_3 \cap \tau_4 \subset V(x), & \sigma_3 \cap \sigma_4 \subset V(x), & \sigma_3 \cap \tau_3 \subset V(z), \\
\tau_1 \cap \tau_4 \subset V(y), & \sigma_1 \cap \sigma_4 \subset V(y), & \sigma_4 \cap \tau_4 \subset V(z).
\end{array}$$

An element $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \in \mathbb{R}[x, y, z, w]^8$ gives an element of $C^r(\hat{\Delta})$ if and only if

$$\begin{array}{rrr}
f_1 - f_2 + (3x-2y)^2 f_9 = 0, & f_5 - f_6 + (3x-2y)^2 f_{13} = 0, & f_1 - f_5 + z^2 f_{17} = 0, \\
f_2 - f_3 + y^2 f_{10} = 0, & f_6 - f_7 + y^2 f_{14} = 0, & f_2 - f_6 + z^2 f_{18} = 0, \\
f_3 - f_4 + x^2 f_{11} = 0, & f_7 - f_8 + x^2 f_{15} = 0, & f_3 - f_7 + z^2 f_{19} = 0, \\
f_1 - f_4 + y^2 f_{12} = 0, & f_5 - f_8 + y^2 f_{16} = 0, & f_4 - f_8 + z^2 f_{20} = 0.
\end{array}$$

for some $f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{19}, f_{20} \in \mathbb{R}[x, y, z, w]$. These equations can be rewritten again in the vector matrix form.

$$M(\hat{\Delta}, r) = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & (3x-2y)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & (3x-2y)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^2
\end{pmatrix}.$$

Thus, the elements of $C^r(\hat{\Delta})$ are projections onto the first eight components of elements of the kernel of the map $\mathbb{R}[x, y, z, w]^{20} \to \mathbb{R}[x, y, z, w]^8$ defined by the matrix $M(\hat{\Delta}, r)$. The graded $\mathbb{R}[x, y, z, w]$-modules construct the following exact sequence (Let $\hat{R}$ be $\mathbb{R}[x, y, z, w]$):

$$0 \to \ker M(\hat{\Delta}, 1) \to \hat{R}^8 \oplus \hat{R}(-2)^{12} \to \operatorname{im} M(\hat{\Delta}, 1) \to 0.$$

Since the Hilbert series of M module and the LT(M) are the same, to calculate the Hilbert series of $\operatorname{im} M(\hat{\Delta})$, we apply the Buchberger algorithm, and we can calculate the Hilbert series of the M module generated by the columns of the $M(\hat{\Delta})$. By using the CoCoA program, we calculate the Hilbert series of $\operatorname{im} M(\hat{\Delta})$:

```
Use R :: = QQ [x,y,z,w];
M : = [[1,0,0,1,0,0,0,0,1,0,0,0,],[-1,1,0,0,0,0,0,0,0,1,0,0],
[0,-1,1,0,0,0,0,0,0,0,1,0],[0,0,-1,-1,0,0,0,0,0,0,0,1],
[0,0,0,0,1,0,0,1,-1,0,0,0],[0,0,0,0,-1,1,0,0,0,-1,0,0],
[0,0,0,0,0,-1,1,0,0,0,-1,0],[0,0,0,0,0,0,-1,-1,0,0,0,-1],
[(3x-2y)^2,0,0,0,0,0,0,0,0,0,0,0],[0,y^2,0,0,0,0,0,0,0,0,0,0],
[0,0,x^2,0,0,0,0,0,0,0,0,0],[0,0,0,y^2,0,0,0,0,0,0,0,0],
[0,0,0,0,(3x-2y)^2,0,0,0,0,0,0,0],[0,0,0,0,0,y^2,0,0,0,0,0,0],
[0,0,0,0,0,0,x^2,0,0,0,0,0],[0,0,0,0,0,0,0,y^2,0,0,0,0],
[0,0,0,0,0,0,0,0,z^2,0,0,0],[0,0,0,0,0,0,0,0,0,z^2,0,0],
[0,0,0,0,0,0,0,0,0,0,z^2,0],[0,0,0,0,0,0,0,0,0,0,0,z^2] ];
N : = Module (M);
I : = LT(N);
Hilbert(I);
H(0)=7
H(1)=28
H(t)=2t^3+12t^2+16 for  t>=2
```

The Hilbert series's of the graded module $\ker M(\hat{\Delta})$ is the following

$$H(\ker M(\hat{\Delta}, r), u) = H(\hat{R}^m \oplus \hat{R}(-r-1)^e, u) - H(\operatorname{im} M(\hat{\Delta}, r), u).$$

Here we need to calculate $H(\hat{R}^m \oplus \hat{R}(-r-1)^e, u)$ which is equal to:

$$\begin{aligned} H(\hat{R}^8 \oplus \hat{R}(-2)^{12}, u) &= 8/(1-u)^{n+1} + 12u^2/(1-u)^{n+1}, \quad n = 3 \\ &= 8 + 12u^2/(1-u)^4. \end{aligned}$$

Thus

$$\begin{aligned} H(\ker M(\hat{\Delta}, 1), u) =& 8 + 12u^2/(1-u)^4 \; - (7 + 28u + \cdots + (2t^3 + 12t^2 + 1)u^t + ...) \\ =& 1 + 4u + 12u^2 + 30u^3 + 64u^4 + 122u^5 + 212u^6 + ... \end{aligned}$$

Here, $H(\ker M(\hat{\Delta}, 1), u) = \dim_{\mathbb{R}} C_u^r(\hat{\Delta})$. When we compare the two results, we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) = 0$, i.e., the dimension of Δ depends only on combinatorial calculations.

Moving v_2 by leaving it in the same plane with the $\mathrm{Conv}(\{v_0, v_3, v_4, v_1\})$ and preserving the convexity of the octahedron always give the same result with this example. Because, under all circumstances, we get an hyperplane which is a linear combination of the others two ($p_1 = ap_2 + bp_3$, where $a, b \in \mathbb{R}$). In fact, in the last method the important thing is the relation among the columns (or rows) of the matrix, but changing a and b in the equation $(ax + by)^2$ does not result with a change. (And also we can choose any vertex to move, not specifically v_2). Hence for such kind of configurations, we know that

$$\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k = 8.\binom{k+3}{3} - 12k^2 - 2k - 16.$$

5.2.3 Octahedrons Having 5 Interior Hyperplanes

The way of getting an octahedron having five hyperplanes is moving any boundary vertex of a regular octahedron, in such a way that the vertex no more stays in both of the hyperplanes containing it. Hence we can consider the example moving one of the vertices of the first figure.

In this example we consider the case $\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$, where $v_0 = (0,0,0), v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (-1,0,0), v_4 = (0,-1,0), v_5 = (1,1,1), v_6 = (0,0,-1)$ in which we move vertex v_5 not through any incident edges of it. For any interior face γ the ideal $\mathcal{J}(\gamma)$ is as below.

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = z^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = (x-z)^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2, \\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = z^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = (y-z)^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2, \\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = (x-z)^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = y^2, \\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = z^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (y-z)^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = x^2,
\end{aligned}$$

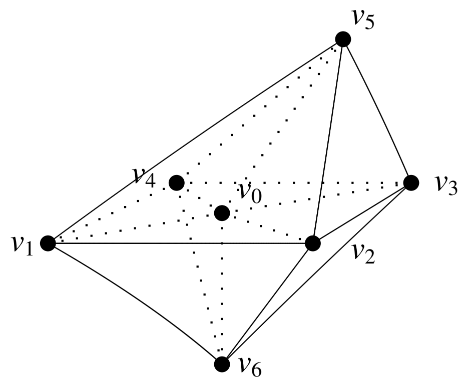

Figure 5.7: Octahedron having 5 interior hyperplanes

$$\mathcal{J}(\langle v_0, v_1\rangle) = \langle y^2, z^2, (y-z)^2\rangle, \qquad \mathcal{J}(\langle v_0, v_2\rangle) = \langle x^2, z^2, (x-z)^2\rangle,$$
$$\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2, (y-z)^2\rangle, \qquad \mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2, (x-z)^2\rangle,$$
$$\mathcal{J}(\langle v_5, v_0\rangle) = \langle (x-z)^2, (y-z)^2\rangle, \qquad \mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2\rangle,$$

$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2, (x-z)^2, (y-z)^2\rangle.$$

Since we are working on a manifold we know that $H_i(\mathcal{R}) = 0 \ \ \forall i < d$. And we can show that ∂_1 is onto. So, $H_0(\mathcal{J}) = 0$.
By the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ we get the equality

$$\begin{aligned}\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 4(3k+1) + 2.4k - 5 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\ &= 8\binom{k+3}{3} - 12k^2 - 4k - 13 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k > 1.\end{aligned}$$

Here we don't know what $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is, and to calculate it we use the matrix method.

By using the CoCoA program we calculate the $\dim_{\mathbb{R}} C^1(\hat{\Delta})_u$ as the following
$\dim_{\mathbb{R}} C^1(\hat{\Delta})_u = 1 + 4u + 11u^2 + 28u^3 + 61u^4 + 117u^5 + 205u^6 + 333u^7 + ...$
When we compare two results we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) \neq 0$. And we reach the following conclusion

$$\dim_{\mathbb{R}}(H_1(\mathcal{J}))_0 = 0, \ \ \dim_{\mathbb{R}}(H_1(\mathcal{J}))_1 = 0,$$
$$\dim_{\mathbb{R}}(H_1(\mathcal{J}))_2 = 0, \ \ \dim_{\mathbb{R}}(H_1(\mathcal{J}))_3 = 1,$$
$$\dim_{\mathbb{R}}(H_1(\mathcal{J}))_t = 2 \text{ for all } t \geq 4.$$

We can generalize this special example. In the example three planes intersect along a line three times, which means that one of the planes can be written as a linear combination of the others two. Hence the dimension of $\mathcal{R}/\langle p_1^2, p_2^2, (a_1.p_1 + a_2.p_2)^2\rangle$ in $\mathbb{R}$ is equal to $3k + 1$ (since it is isomorphic to $(\mathcal{R}/\langle x^2, y^2, xy\rangle)$), where $a_1, a_2 \in \mathbb{R}$. For the ideal $\mathcal{J}(\langle v_0\rangle)$ the generators can be identified as $\{p_1^2, p_2^2, p_3^2, p_4 = (a.p_1 + b.p_3)^2, p_5 = (c.p_2 + d.p_3)^2\}$ where $a, b, c, d \in \mathbb{R}$, by using the property of three planes intersecting along a line, which has the same dimension with the ideal $\langle x^2, y^2, z^2, xz, yz\rangle$. For the combinatoric part knowing these dimensions are enough but what can be said about the $dim_{\mathbb{R}}(H_1(\mathcal{J}))_k$. In fact, in the second part of the calculation we used a special matrix identifying our simplex. For any octahedron having five interior faces we can rename the vertices to make the moved vertex v_5, after that we will have $M(\hat{\Delta}, r)$ similar to the previous one, it is obvious that p_4 and p_5 will be different but they will still be the linear combinations of the others as given in the example. Hence the relation among the rows of the matrix $M(\hat{\Delta}, r)$ will be the same as the previous example. Hence for this case we have

$$dim_{\mathbb{R}} C^r(\hat{\Delta})_k = 8\binom{k+3}{3} - 12(k+1)^2 + 4(3k+1) + 2.4k - 5 + 2 \text{for all k} \geq 4,$$

$$dim_{\mathbb{R}} C^r(\hat{\Delta})_k = 8\binom{k+3}{3} - 12(k+1)^2 + 4(3k+1) + 2.4k - 3 \text{for all k} \geq 4.$$

5.2.4 Octahedrons Having 6 interior Hyperplanes

In a regular octahedron any boundary vertex is contained in two inner hyperplanes, and any two of the vertices share at least one of these hyperplanes. To obtain an octahedron having 6 hyperplanes we move two vertices of the octahedron that contained in three inner hyperplanes, through one of their four boundary edges in such a way that they don't stay in any of their comman hyperplanes after the movement (Here I won't consider the case that vertices and v_0 lie on the same line, since their movement considered in the case 'octahedrons having 4 interior hyperplanes').
We consider the example for $\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$, where $v_0 = (0, 0, 0)$, $v_1 = (1/2, -1/2, 0)$, $v_2 = (0, 1/2, -1/2)$, $v_3 = (-1, 0, 0)$, $v_4 = (0, -1, 0)$, $v_5 = (0, 0, 1)$, $v_6 = (0, 0, -1)$ in which we move vertex v_1 through the v_1v_4 edge, and v_2 through the v_2v_6 edge. We will find the ideal $\mathcal{J}(\gamma)$ for any interior face γ.

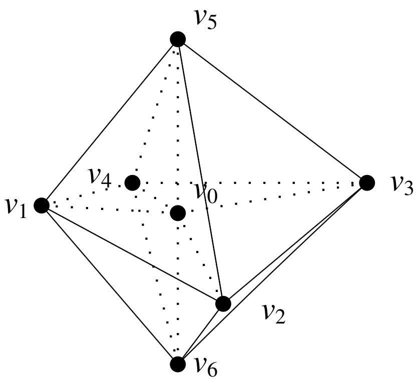

Figure 5.8: Octahedron with six hyperplanes

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = (x+y+z)^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = x^2,\\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = (y+z)^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = y^2,\\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = x^2,\\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = z^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (x+y)^2,\\
&\mathcal{J}(\langle v_6, v_0, v_1\rangle) = (x+y)^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = x^2,\\
&\mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2 && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2\rangle, && \mathcal{J}(\langle v_0, v_2\rangle) = \langle x^2, (y+z)^2, (x+y+z)^2\rangle,\\
&\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2, (y+z)^2\rangle, && \mathcal{J}(\langle v_0, v_1\rangle) = \langle (x+y+z)^2, (x+y)^2, z^2\rangle,\\
&\mathcal{J}(\langle v_5, v_0\rangle) = \langle x^2, y^2, (x+y)^2\rangle, && \mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2, (x+y)^2\rangle,
\end{aligned}$$

$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2, (x+y)^2, (y+z)^2, (x+y+z)^2\rangle.$$

We show that ∂_1 is onto. So, $H_0(\mathcal{J}) = 0$.

By $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ we get the following equality.

$$\begin{aligned}
\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= 8.\binom{k+3}{3} - 12.(k+1)^2 + 5(3k+1) + 4k - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k\\
&= 8\binom{k+3}{3} - 12k^2 - 5k - 11 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k \geq 1.
\end{aligned}$$

Schenck in [16] obtains this case and says that $\dim_{\mathbb{R}} H_1(\mathcal{J})_k = 0$. By the movement, 5 interior lines contained in three hyperplanes such that one of them is the linear combination of the others hence the combinatoric equality is unique.

$$\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 5k - 11 \ \ \forall \ \ k \geq 1.$$

5.2.5 Octahedrons Having 7 interior Hyperplanes

We give a sample of an octahedron with seven hyperplanes. In this example, one boundary vertex of the regular octahedron was removed not leaving it in any inner hyperplanes incident to itself, and another vertex which share an edge with the first one was removed through one of the four boundary lines incident to it. We consider the case $\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$, where $v_0 = (0,0,0), v_1 = (1,0,0), v_2 = (-1/2, 1/2, 0), v_3 = (-1,0,0), v_4 = (0,-1,0), v_5 = (1,1,1), v_6 = (0,0,-1)$. For any

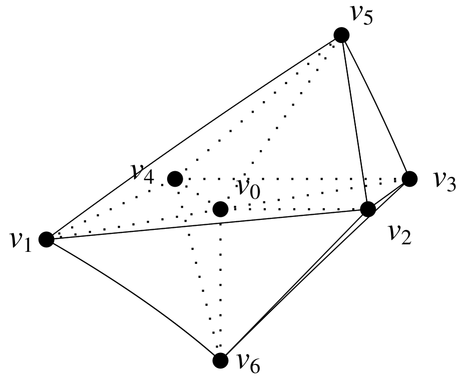

Figure 5.9: Octahedron with 7 hyperplanes

interior face γ the ideal $\mathcal{J}(\gamma)$ is decided below.

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = z^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = y^2, && \mathcal{J}(\langle v_6, v_0, v_2\rangle) = (x+y)^2,\\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = z^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = x^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = (x+y-2z)^2,\\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = z^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = (x-z)^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = (y-z)^2,\\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = z^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (y-z)^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = x^2,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1\rangle) = \langle y^2, z^2, (y-z)^2\rangle, && \mathcal{J}(\langle v_0, v_2\rangle) = \langle (x+y)^2, z^2, (x+y-2z)^2\rangle,\\
&\mathcal{J}(\langle v_0, v_3\rangle) = \langle y^2, z^2, (y-z)^2\rangle, && \mathcal{J}(\langle v_0, v_4\rangle) = \langle x^2, z^2, (x-z)^2\rangle,\\
&\mathcal{J}(\langle v_6, v_0\rangle) = \langle x^2, y^2, (x+y)^2\rangle, && \mathcal{J}(\langle v_5, v_0\rangle) = \langle (x-z)^2, (y-z)^2, (x+y-2z)^2\rangle,
\end{aligned}$$

$$\mathcal{J}(\langle v_0\rangle) = \langle x^2, y^2, z^2, (x-z)^2, (y-z)^2, (x+y)^2, (x+y-2z)^2\rangle.$$

$H_i(\mathcal{R}) = 0 \ \ \forall i < d$, and ∂_1 is onto. So, $H_0(\mathcal{J}) = 0$.

By the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ we get the following equality.

$$\begin{aligned}\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 6(3k+1) - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\ &= 8\binom{k+3}{3} - 12k^2 - 6k - 10 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k > 0.\end{aligned}$$

Here we don't know what $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is. And we use the matrix method to calculate the $\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k$. By using CoCoA we find that

$$\dim_{\mathbb{R}} C^{r}(\hat{\Delta})_u = 1 + 4u + 11u^2 + 26u^3 + 56u^4 + 110u^5 + 196u^6 + ...$$

When we compare two results we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) \neq 0$. And we reach the following conculusion

$$\dim_{\mathbb{R}}(H_1(\mathcal{J})_k = \begin{cases} 0 & \text{if } k = 0, 1, \\ 1 & \text{if } k = 2, \\ 2 & \text{if } k \geq 3. \end{cases}$$

5.2.6 Octahedrons Having 8 Interior Hyperplanes

In this subsection we consider a Δ with eight hyperplanes in a specialized sample. $\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$ where $v_0 = (0,0,0), v_1 = (2,-2,-1), v_2 = (1,3,-2), v_3 = (-2,1,1), v_4 = (-1,-2,2), v_5 = (0,0,3), v_6 = (0,0,-3)$. For any interior face γ the ideal $\mathcal{J}(\gamma)$ is decided below.

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = (7x + 3y + 8z)^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = (3x - y)^2, \\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = (5x + 3y + 7z)^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = (x + 2y)^2, \\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = (4x + 3y + 5z)^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = (2x - y)^2, \\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = (2x + y + 2z)^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (x + y)^2, \\
&\mathcal{J}(\langle v_6, v_0, v_2\rangle) = (3x - y)^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = (x + 2y)^2, \\
&\mathcal{J}(\langle v_6, v_0, v_4\rangle) = (2x - y)^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = (x + y)^2,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{J}(\langle v_0, v_1\rangle) =& \langle (7x+3y+8z)^2, (2x+y+2z)^2, (x+y)^2\rangle,\\
\mathcal{J}(\langle v_0, v_2\rangle) =& \langle (7x+3y+8z)^2, (5x+3y+7z)^2, (3x-y)^2\rangle,\\
\mathcal{J}(\langle v_0, v_3\rangle) =& \langle (5x+3y+7z)^2, (4x+3y+5z)^2, (x+2y)^2\rangle,\\
\mathcal{J}(\langle v_0, v_4\rangle) =& \langle (4x+3y+5z)^2, (2x+y+2z)^2, (2x-y)^2\rangle,\\
\mathcal{J}(\langle v_5, v_0\rangle) =& \langle (3x-y)^2, (x+2y)^2, (2x-y)^2, (x+y)^2\rangle,\\
\mathcal{J}(\langle v_6, v_0\rangle) =& \langle (3x-y)^2, (x+2y)^2, (2x-y)^2, (x+y)^2\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}(\langle v_0\rangle) = \langle & (7x+3y+8z)^2, (5x+3y+7z)^2, (4x+3y+5z)^2, (2x+y+2z)^2.\\
& (3x-y)^2, (x+2y)^2, (2x-y)^2, (x+y)^2\rangle
\end{aligned}
$$

We know that $H_i(\mathcal{R}) = 0 \;\; \forall i < d$. and we can show that $H_0(\mathcal{J}) = \ker \partial_0 / \operatorname{im} \partial_1 = 0$.

By using the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ we get that

$$
\begin{aligned}
\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 6(3k+1) - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k\\
&= 8\binom{k+3}{3} - 12k^2 - 6k - 10 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \;\; \forall \;\; k > 0
\end{aligned}
$$

Here we don't know what $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is. Here we use the matrix method to calculate the $\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k$, and we find that

$$
\dim_{\mathbb{R}} C^r_u(\hat{\Delta}) = 1 + 4u + 11u^2 + 26u^3 + 56u^4 + 110u^5 + 196u^6 + \ldots
$$

When we compare two results we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) \neq 0$. And we reach the following conclusion

$$
\dim_{\mathbb{R}}(H_1(\mathcal{J}))_k = \begin{cases} 0 & \text{if } k = 0, 1,\\ 1 & \text{if } k = 2,\\ 2 & \text{if } k \geq 3. \end{cases}
$$

By the examples given in the latter two subsections, we see that dimension of an octahedron having seven and eight hyperplanes depend not on combinatoric only, but also on the dimension of $H_2(\mathcal{R}/\mathcal{J})$. And most interestingly, even they have different constructions, they have the same result. Another important situation need to be mentioned here is that any octahedron having more than six hyperplanes will have the same combinatoric part for $r = 1$. Since each line have to be contained at least three hyperplanes in each cases and there are six interior lines that brings the $6.(3k+1)$ into the calculation. Hence calculating dimension of $C^1_k(\hat{\Delta})$ is depend on calculation of the dimension of the $H_2(\mathcal{R}/\mathcal{J})$.

5.2.7 Octahedrons Having 9 Interior Hyperplanes

We will consider a Δ having nine hyperplanes by the following sample.
$\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$ where $v_0 = (0,0,0)$, $v_1 = (3/2, 3/2, 0)$, $v_2 = (0,1,-1)$, $v_3 = (-3,0,0)$, $v_4 = (0,-3,0)$, $v_5 = (-2,0,2)$, $v_6 = (0,0,-3)$. For any interior face γ the ideal $\mathcal{J}(\gamma)$ is decided below.

$$
\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = (x-y-z)^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = (x+y+z)^2,\\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = (y+z)^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = (y)^2,\\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = (z)^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = (x+z)^2,\\
&\mathcal{J}(\langle v_6, v_0, v_2\rangle) = (x)^2, && \mathcal{J}(\langle v_6, v_0, v_3\rangle) = (y)^2,\\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = (z)^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (x-y+z)^2,\\
&\mathcal{J}(\langle v_6, v_0, v_4\rangle) = (x)^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = (x-y)^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}(\langle v_0, v_1\rangle) &= \langle (x-y-z)^2, z^2, (x-y+z)^2, (x-y)^2\rangle,\\
\mathcal{J}(\langle v_0, v_2\rangle) &= \langle (x-y-z)^2, (y+z)^2, (x+y+z)^2, x^2\rangle,\\
\mathcal{J}(\langle v_0, v_3\rangle) &= \langle (y+z)^2, z^2, y^2\rangle,\\
\mathcal{J}(\langle v_0, v_4\rangle) &= \langle z^2, (x+z)^2, x^2\rangle,\\
\mathcal{J}(\langle v_5, v_0\rangle) &= \langle (x+y+z)^2, y^2, (x+z)^2, (x-y+z)^2\rangle,\\
\mathcal{J}(\langle v_6, v_0\rangle) &= \langle x^2, y^2, (x-y)^2\rangle,
\end{aligned}
$$

$$\mathcal{J}(\langle v_0\rangle) = \langle (x-y-z)^2, (y+z)^2, z^2, (x+y+z)^2, y^2, (x+z)^2, (x-y+z)^2, x^2, (x-y)^2\rangle.$$

We know that $H_i(\mathcal{R}) = 0 \;\; \forall i < d$. and also we can show that $H_0(\mathcal{J}) = \ker\partial_0/\operatorname{im}\partial_1 = 0$.

By the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ get the following equality.

$$\begin{aligned}\dim_{\mathbb{R}} C^1(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 6(3k+1) - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\ &= 8\binom{k+3}{3} - 12k^2 - 6k - 10 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k, \quad \forall \ k > 0.\end{aligned}$$

Here we don't know what $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is. Thus we apply the matrix method to calculate the $\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k$.

By using the CoCoA program we calculate the Hilbert series's of the graded module $\dim_{\mathbb{R}} C^r(\hat{\Delta})_k$.

$$\dim_{\mathbb{R}} C^r(\hat{\Delta})_k = 1 + 4u + 10u^2 + 24u^3 + 54u^4 + 108u^5 + 194u^6 + ...$$

When we compare two results we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) = 0$. And the dimension of $C^1(\hat{\Delta})$ is computed by combinatoric relations.

$$\dim_{\mathbb{R}} C^1(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 6k - 10.$$

5.2.8 Octahedrons Having 10 Interior Hyperplanes

In this part we will consider Δ with ten hyperplanes in a special example.
$\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$ where $v_0 = (0,0,0)$, $v_1 = (-4,-3,-1)$, $v_2 = (1,-3,1)$, $v_3 = (2,1,2)$, $v_4 = (-1,1,-1)$, $v_5 = (0,-1,3)$, $v_6 = (-1,-1,-2)$. For any interior face γ the ideal $\mathcal{J}(\gamma)$ is decided below.

$$\begin{aligned}
&\mathcal{J}(\langle v_0, v_1, v_2\rangle) = (2x - y - 5z)^2, && \mathcal{J}(\langle v_0, v_2, v_5\rangle) = (8x + 3y + z)^2,\\
&\mathcal{J}(\langle v_0, v_2, v_3\rangle) = (x - z)^2, && \mathcal{J}(\langle v_0, v_3, v_5\rangle) = (5x - 6y - 2z)^2,\\
&\mathcal{J}(\langle v_0, v_3, v_4\rangle) = (x - z)^2, && \mathcal{J}(\langle v_0, v_4, v_5\rangle) = (2x + 3y + z)^2,\\
&\mathcal{J}(\langle v_0, v_4, v_1\rangle) = (4x - 3y - 7z)^2, && \mathcal{J}(\langle v_0, v_1, v_5\rangle) = (5x - 6y - 2z)^2,\\
&\mathcal{J}(\langle v_6, v_0, v_2\rangle) = (7x + y - 4z)^2, && \mathcal{J}(\langle v_6, v_0, v_4\rangle) = (3x + y - 2z)^2,\\
&\mathcal{J}(\langle v_6, v_0, v_3\rangle) = (2y - z)^2, && \mathcal{J}(\langle v_6, v_0, v_1\rangle) = (5x - 7y + z)^2,
\end{aligned}$$

$$\mathcal{J}(\langle v_0, v_1\rangle) = \langle (2x-y-5z)^2, (4x-3y-7z)^2, (5x-6y-2z)^2, (5x-7y+z)^2\rangle,$$
$$\mathcal{J}(\langle v_0, v_2\rangle) = \langle (2x-y-5z)^2, (x-z)^2, (8x+3y+z)^2, (7x+y-4z)^2\rangle,$$
$$\mathcal{J}(\langle v_0, v_3\rangle) = \langle (x-z)^2, (5x-6y-2z)^2, (2y-z)^2\rangle,$$
$$\mathcal{J}(\langle v_0, v_4\rangle) = \langle (x-z)^2, (4x-3y-7z)^2, (2x+3y+z)^2, (3x+y-2z)^2\rangle,$$
$$\mathcal{J}(\langle v_5, v_0\rangle) = \langle 7(8x+3y+z)^2, y^2, (5x-6y-2z)^2, (2x+3y+z)^2\rangle,$$
$$\mathcal{J}(\langle v_6, v_0\rangle) = \langle (7x+y-4z)^2, (2y-z)^2, (3x+y-2z)^2, (5x-7y+z)^2\rangle,$$

$$\mathcal{J}(\langle v_0\rangle) = \langle (2x-y-5z)^2, (x-z)^2, (4x-3y-7z)^2, (8x+3y+z)^2, (5x-6y-2z)^2,$$
$$(2x+3y+z)^2, (5x-7y+z)^2, (7x+y-4z)^2, (2y-z)^2, (3x+y-2z)^2\rangle.$$

We know that $H_i(\mathcal{R}) = 0 \ \ \forall i < d$, and we show that $H_0(\mathcal{J}) = \ker \partial_0 / \operatorname{im} \partial_1 = 0$.

By the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ gives the following equality

$$\begin{aligned}\dim_{\mathbb{R}} C^1(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 6(3k+1) - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \\ &= 8\binom{k+3}{3} - 12k^2 - 6k - 10 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k > 0.\end{aligned}$$

Here we don't know what $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is. Thus we use the matrix method to calculate it.

By using the CoCoA program we calculate the Hilbert series's of the graded module $\dim_{\mathbb{R}} C^r_u(\hat{\Delta})$ as follows.

$$\dim_{\mathbb{R}} C^r_u(\hat{\Delta}) = 1 + 4u + 10u^2 + 24u^3 + 54u^4 + 108u^5 + 194u^6 + ...$$

When we compare two results we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) = 0$. And The dimension of $C^1(\hat{\Delta})$ is computed by combinatoric relations.

$$\dim_{\mathbb{R}} C^1(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 6k - 10.$$

5.2.9 Octahedrons Having 11 Interior Hyperplanes

In this section we will consider an example with Δ having eleven hyperplanes.
$\Delta = \text{Conv}(v_0, v_1, v_2, v_3, v_4, v_5, v_6)$ where $v_0 = (0,0,0), v_1 = (2,0,0), v_2 = (1,2,-1),$

$v_3 = (-2,-1,0), v_4 = (0,-2,0), v_5 = (2,3,2), v_6 = (1,0,-2)$. For any interior face γ the ideal $\mathcal{J}(\gamma)$ is decided below.

$$\begin{aligned}
&\mathcal{J}(\langle v_0,v_1,v_2\rangle) = (y+2z)^2, && \mathcal{J}(\langle v_0,v_2,v_5\rangle) = (7x-4y-z)^2,\\
&\mathcal{J}(\langle v_0,v_2,v_3\rangle) = (x-2y-3z)^2, && \mathcal{J}(\langle v_0,v_3,v_5\rangle) = (x-2y+2z)^2,\\
&\mathcal{J}(\langle v_0,v_3,v_4\rangle) = (z)^2, && \mathcal{J}(\langle v_0,v_4,v_5\rangle) = (x-z)^2,\\
&\mathcal{J}(\langle v_0,v_4,v_1\rangle) = (z)^2, && \mathcal{J}(\langle v_0,v_1,v_5\rangle) = (2y-3z)^2,\\
&\mathcal{J}(\langle v_6,v_0,v_2\rangle) = (4x-y+2z)^2, && \mathcal{J}(\langle v_6,v_0,v_4\rangle) = (2x+z)^2,\\
&\mathcal{J}(\langle v_6,v_0,v_3\rangle) = (2x-4y+z)^2, && \mathcal{J}(\langle v_6,v_0,v_1\rangle) = (y)^2,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{J}(\langle v_0,v_1\rangle) = \langle (y+2z)^2,(z)^2,(2y-3z)^2,(y)^2\rangle,\\
&\mathcal{J}(\langle v_0,v_2\rangle) = \langle (y+2z)^2,(x-2y-3z)^2,(7x-4y-z)^2,(4x-y+2z)^2\rangle,\\
&\mathcal{J}(\langle v_0,v_3\rangle) = \langle (x-2y-3z)^2,(z)^2,(x-2y+2z)^2,(2x-4y+z)^2\rangle,\\
&\mathcal{J}(\langle v_0,v_4\rangle) = \langle (z)^2,(x-z)^2,(2x+z)^2\rangle,\\
&\mathcal{J}(\langle v_5,v_0\rangle) = \langle (7x-4y-z)^2,(x-2y+2z)^2,(x-z)^2,(2y-3z)^2\rangle,\\
&\mathcal{J}(\langle v_6,v_0\rangle) = \langle (4x-y+2z)^2,(2x-4y+z)^2,(2x+z)^2,(y)^2\rangle,
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}(\langle v_0\rangle) = \langle &(y+2z)^2,(x-2y-3z)^2,z^2,(7x-4y-z)^2,(x-2y+2z)^2,(x-z)^2,\\
&(2y-3z)^2,(y)^2,(4x-y+2z)^2,(2x-4y+z)^2,(2x+z)^2\rangle.
\end{aligned}$$

We know that $H_i(\mathcal{R}) = 0 \ \ \forall i < d$, and we can show that $H_0(\mathcal{J}) = \ker\partial_0/\operatorname{im}\partial_1 = 0$.

By applying the Euler characteristic equation, $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ we get the following equality.

$$\begin{aligned}
\dim_{\mathbb{R}} C^1(\hat{\Delta})_k &= 8\binom{k+3}{3} - 12(k+1)^2 + 6(3k+1) - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k\\
&= 8\binom{k+3}{3} - 12k^2 - 6k - 10 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \ \ \forall \ \ k > 0.
\end{aligned}$$

Here we don't know what $\dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})$ is. Thus we use the matrix method to calculate the $\dim_{\mathbb{R}} C^{\alpha}(\hat{\Delta})_k$.

By using the CoCoA program we calculate the Hilbert series of the graded module $\dim_{\mathbb{R}} C^r(\hat{\Delta})_u$ as following

$$\dim_{\mathbb{R}} C^r(\hat{\Delta})_u = 1 + 4u + 10u^2 + 24u^3 + 54u^4 + 108u^5 + 194u^6 + \ldots$$

When we compare two results we see that $\dim_{\mathbb{R}}(H_1(\mathcal{J})) = 0$. So the dimension of $C^1(\hat{\Delta})$ is computed by combinatoric relations.

$$\dim_{\mathbb{R}} C^1(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 6k - 10.$$

5.2.10 Octahedrons Having 12 Interior Hyperplanes

An octahedron can have at most twelve hyperplanes, in this part we will consider this nongeneric case. In [15], Schenck remarks that in nongeneric octahedrons $\dim_{\mathbb{R}} H_1(J) = 0$ when $r = 1$. Thus dimension of that octahedron can be find by the combinatorial calculations. Now we will calculate dimension of a nongeneric octahedron and extend it to the general nongeneric octahedrons. We consider the example for
$\Delta = \mathrm{Conv}(\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\})$, where $v_1 = (3,1,1)$, $v_2 = (5,-3,2)$, $v_3 = (-3,-2,-1)$, $v_4 = (-2,2,0)$, $v_5 = (1,1,3)$, $v_6 = (0,1,3)$ and $v_0 = (0,0,0)$.

We find for any interior face γ of the simplex the ideal $\mathcal{J}(\gamma)$.

$$\begin{aligned}
\mathcal{J}(\langle v_0, v_1, v_2\rangle) &= (5x - y - 14z)^2, & \mathcal{J}(\langle v_0, v_2, v_5\rangle) &= (11x + 13y - 8z)^2,\\
\mathcal{J}(\langle v_0, v_2, v_3\rangle) &= (7x - y - 19z)^2, & \mathcal{J}(\langle v_0, v_3, v_5\rangle) &= (5x - 8y + z)^2,\\
\mathcal{J}(\langle v_0, v_3, v_4\rangle) &= (x + y - 5z)^2, & \mathcal{J}(\langle v_0, v_4, v_5\rangle) &= (3x + 3y - 2z)^2,\\
\mathcal{J}(\langle v_0, v_4, v_1\rangle) &= (x + y - 4z)^2, & \mathcal{J}(\langle v_0, v_1, v_5\rangle) &= (x - 4y + z)^2,\\
\mathcal{J}(\langle v_6, v_0, v_2\rangle) &= (11x + 15y - 5z)^2, & \mathcal{J}(\langle v_6, v_0, v_3\rangle) &= (5x - 9y + 3z)^2,\\
\mathcal{J}(\langle v_6, v_0, v_4\rangle) &= (3x + 3y - z)^2, & \mathcal{J}(\langle v_6, v_0, v_1\rangle) &= (2x - 9y + 3z)^2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}(\langle v_0, v_1\rangle) &= \langle (5x - y - 14z)^2, (x + y - 4z)^2, (x - 4y + z)^2, (2x - 9y + 3z)^2\rangle,\\
\mathcal{J}(\langle v_0, v_2\rangle) &= \langle (5x - y - 14z)^2, (7x - y - 19z)^2, (11x + 13y - 8z)^2,\\
&\qquad (11x + 15y - 5z)^2\rangle,\\
\mathcal{J}(\langle v_0, v_3\rangle) &= \langle (7x - y - 19z)^2, (x + y - 5z)^2, (5x - 8y + z)^2, (5x - 9y + 3z)^2\rangle,\\
\mathcal{J}(\langle v_0, v_4\rangle) &= \langle (x + y - 5z)^2, (x + y - 4z)^2, (3x + 3y - 2z)^2, (3x + 3y - z)^2\rangle,\\
\mathcal{J}(\langle v_5, v_0\rangle) &= \langle (11x + 13y - 8z)^2, (5x - 8y + z)^2, (3x + 3y - 2z)^2, (x - 4y + z)^2\rangle,\\
\mathcal{J}(\langle v_6, v_0\rangle) &= \langle (11x + 15y - 5z)^2, (5x - 9y + 3z)^2, (3x + 3y - z)^2, (2x - 9y + 3z)^2\rangle,
\end{aligned}$$

$$\begin{aligned}\mathcal{J}(\langle v_0\rangle) = \langle &(5x-y-14z)^2, (7x-y-19z)^2, (x+y-5z)^2, (x+y-4z)^2,\\ &(11x+13y-8z)^2, (5x-8y+z)^2, (3x+3y-2z)^2, (x-4y+z)^2,\\ &(11x+15y-5z)^2, (5x-9y+3z)^2, (3x+3y-z)^2, (2x-9y+3z)^2\rangle.\end{aligned}$$

We can show that ∂_1 is onto. So, $H_0(\mathcal{J}) = 0$.

By $\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J})$ we write the following equality.

$$\begin{aligned}\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k &= 8.\binom{k+3}{3} - 12.(k+1)^2 + 6(3k+1) - 4 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k\\ &= 8\binom{k+3}{3} - 12k^2 - 6k - 10 + \dim_{\mathbb{R}} H_2(\mathcal{R}/\mathcal{J})_k \;\; \forall \;\; k \geq 1.\end{aligned}$$

Since $\dim_{\mathbb{R}} H_1(\mathcal{J})_k = 0$, we can find the dimension of the octahedron just by combinatorial calculations. Here the combination is unique for any nongeneric case since any interior line is contained in 4 hyperplanes and two of the hyperplanes generate the other two hence $\dim_{\mathbb{R}}(\mathcal{R}/\mathcal{J}\langle v_i, v_j\rangle)_k = 3k+1$. Also v_0 is contained in 12 hyperplanes which are intersecting through 6 lines hence $\dim_{\mathbb{R}}(\mathcal{R}/\mathcal{J}\langle v_0\rangle)_k = 4$ for $k \geq 1$. Thus,

$$\dim_{\mathbb{R}} C^\alpha(\hat{\Delta})_k = 8\binom{k+3}{3} - 12k^2 - 6k - 10 \;\; \forall \;\; k \geq 1.$$

In cases, after octahedrons having six hyperplanes case, in our calculations, we get the same combinatorial part. But, in the examples having seven and eight hyperplanes, the combinatorial part was different for the dimension of $C^\alpha(\hat{\Delta})$, but for the rest they were equal.

CHAPTER 6

CONCLUSION

In this thesis, we studied splines on simplicial and polyhedral complexes. After presenting some results of Geramita and Schenck given in [18] in two dimension case, we tried to extend their results to three dimension. To do this, we needed to extra homology that need to be calculated, which was not easy at all. We considered several methods to calculate the dimension of splines, and by comparing their results, we tried to obtain a way of controlling the dimension of the homology module ($H_2(R / J)$). We proved that on pure and hereditary simplicial complexes, if the dimension of the second homology module is zero, then the result depends only on the geometry of the complexes, and could be calculated by some combinatorial calculations. That made it possible to make some generalizations. We have given a formula for the dimension of the splines on n-gons having no interior points. Also, by using results given in [16], we give explicit formulas for the dimensions of the splines on octahedrons with one interior point. The formulas depend on the number of interior 2-faces.

Also in this thesis, one of our aims was to generalize a result of Mcdonald and Schenck [23]. Their result gives the coefficients of the polynomial showing the dimension of a spline space defined on a 2-dimensional polyhedral complex with the fixed smoothness degree r. By connecting these ideas, with the work of [18], we gave a formula for the coefficients of the Hilbert polynomial of a spline for the mixed smoothness degree case, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{f_1^0})$.

In dimension 3, still very little is known for splines on a simplicial complex. Hence there are many open problems in dimension 3, which we plan to attack in the future.

In this thesis, we have seen that different branches of mathematics are unexpectedly connected, in our study it was among the geometry, algebra and combinatorics, so that by interaction between seemingly distant branches may yield with new results.

REFERENCES

[1] S. Altınok, N. Sipahi, *Piecewise Polynomials with Different Smoothness Degree on Polyhedral Complexes*, Submitted.

[2] R. Courant, *Variational methods for the solution of problems of equilibrium and vibration*, Bull. Amer. Math. Soc. **49** (1943), 1-2.

[3] G. Strang, *Piecewise Polynomials and the Finite Element Method*, Bull. Amer. Math. Soc. **79** (1973), 1128-1137.

[4] - , *The dimension of piecewise polynomial spaces and one-sided approximation*, Proc. Conf. Numerical Solution of differential Equations (Dundee 1973), lecture Notes in Math., vol.**365** Springer-Verlag, New York, 1974, pp. 144-152.

[5] J. Morgan and R. Scott, *A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$*, Math Comp. **29** (1975), 736-740.

[6] - , *The dimension of the space of C^1 piecewise polynomials*, unpublished manuscript, 1975.

[7] L. L. Schumaker, *On the dimension of spaces of piecewise polynomials in two variables, Multivariate Approximation Theory* (W. Schemp and K. Zeller, eds.), Birkhauser, Basel, 1979, pp. 396-412

[8] P. Alfeld and L. L. Schumaker, *The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r + 1$*, Const. Approx. **3** (1987), 189-197.

[9] L. J. Billera, *The Algebra of Continuous Piecewise Polynomials*, Advances in Mathematics **76** 170-183, (1989).

[10] L. J. Billera, *Homology of Smooth Splines: Generic Triangulations and a Conjecture of Strang*, Transactions of the American Mathematical Society, Volume **310** Number 1, November (1988).

[11] L. J. Billera, L. L. Rose, *A dimension series for multivariate splines*. Discrete Comput. Geom. **6** (1991), 107-128.

[12] L. J. Billera, L. L. Rose, *Modules of piecewise polynomials and their freeness*, Math. Z. **209** (1992), 485-497.

[13] H. Schenck and M. Stillman, *Local cohomology of bivariate splines*, J. Pure Appl. Algebra 118-119 (1997), 535-548.

[14] H. Schenck and M. Stillman, *A family of ideals of Minimal regularity and the Hilbert series of* $C^r(\hat{\Delta})$, Advances in Applied Mathematics **19** (1997), 169-182.

[15] D. Eisenbud, *Commutative Algebra with a View towards Algebraic Geometry*, Springer-Verlag, New York, 1995.

[16] H. Schenck, *A Spectral Sequence for Splines*, Advences in Applied Mathematics **19** (1997), 183-199.

[17] J. P. Dalbec, H. Schenck, *On a conjecture of Rose*, Journal of Pure and Applied Algebra **165** (2001), 151-154.

[18] A. V. Geramita, H. Schenck, *Fat Points, Inverse Systems, and Piecewise Polynomial Functions*, Journal of Algebra **204** (1998), 116-128.

[19] J. Ensalem and A. Iarrobino, *Inverse system of a symbolic power I*, J. Algebra **174** (1995), 1080-1090.

[20] A. V. Geramita, *Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties, and parameter spaces for Gorenstein ideals*,in Queen's Paper in Pure and Applied Mathematics, Vol. **102** pp. 1-114, Queen's Univ., Kingston, Ontario, 1996.

[21] L. Rose, *Graphs, syzygies, and multivariate splines*, Discrete Comp. Geom. **32** (2004), 623-637.

[22] Cox, D. and Little, J. and O'Shea, D., *Using Algebraic Geometry*, Springer, Springer-Verlag, New York, (2005), 389,390.

[23] T. McDonald, H. Schenck, *Piecewise Polynomials on Polyhedral Complexes*, Adv. in Appl. Math.**42** (2009), no. 1, 82-93.

[24] M. F. Atiyah, I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, London, (1969).

[25] S. Pehlivan, *Parçalı Polinom Fonksiyonları ve Uygulamaları*, MSc. Thesis, Adnan Menderes University, Aydın, 2008.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Sipahi (Ös), Neslihan

Nationality: Turkish (TC)

Date and Place of Birth: 03.04.1982, İzmir

Marital Status: Married

Phone: 05365297147

Fax: -

EDUCATION

Degree	Institution	Year of Graduation
M.S.	DEU	2005
High School	Çimentaş Anatolian High School	2000

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
1	İstanbul Aydın University	Research Assistant

PUBLICATIONS

F. Arslan, N. Sipahi, N. Şahin, Monomial curve families supporting Rossi's conjecture, Journal of Symbolic Computation, **55** (2013), 10-18.