BELL-LIKE INEQUALITIES IN QUANTUM INFORMATION THEORY

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ABSTRACT

BELL-LIKE INEQUALITIES IN QUANTUM INFORMATION THEORY

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Bell and Bell-like inequalities are essential tools for identifying and inspecting quantum entanglement, a key phenomenon in quantum information theory. As the development of Quantum Information Theory progresses more and more Bell-like inequalities are formulated. An investigation of these and further inequalities will be given in their correspondences to Bell's and Kochen-Specker theorems. Also Hardy's test will be studied and particular applications of it for bipartite and tripartite systems are going to be probed.

Keywords: Bell's Theorem, Bell Inequalities, CHSH, Locality, Hidden Variables, EPR, Leggett-Garg, Kochen-Specker, KCBS, GHZ, Hardy's Test

ÖZ

KUANTUM BİLGİ TEORİSİNDE BELL-TÜRÜ EŞİTSİZLİKLER

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Bell ve Bell-türü eşitsizlikler kuantum bilgi teorisinin kilit noktalarından birisi olan kuantum dolanıklığı incelemek ve belirlemek için temel araçlardır. Kuantum Bilgi Teorisi ilerledikçe ortaya çok sayıda çeşitli Bell-türü eşitsizlikler de sürülmektedir. Bu tür ve ötesindeki eşitsizlikler, Bell ve Kochen-Specker teoremleri ile olan bağlantıları içerisinde ele alınacaktır. Ek olarak Hardy testi çalışılacak, iki ve üç parçacıklı sistemler için özel uygulamaları incelenecektir.

Anahtar Kelimeler: Bell Teoremi, Bell Eşitsizlikleri, CHSH, Yerellik, Saklı Değişkenler, EPR, Leggett-Garg, Kochen-Specker, KCBS, GHZ, Hardy Testi

To J. S. Bell, who has started it all

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EPR	Einstein-Podolsky-Rosen
LGI	Leggett-Garg Inequality
KCBS	Klyachko-Can-Binicioğlu-Shumovsky
MABK	Mermin-Ardehali-Belinski-Klyshko
CHSH	Clauser-Horne-Shimony-Holt
BCHS	Bell-Clauser-Horne-Shimony
BKS	Bell-Kochen-Specker
KS	Kochen-Specker
GHZ	Greenberger-Horne-Zeilinger
NIM	Non-Invasive Measurement
MR	Macro-Realism
LHV	Local Hidden Variables
QM	Quantum Mechanics
LC	Local Causality
NLHV	Non-local Hidden Variables
SQUID	Superconducting Quantum Interference Device
Sec.	Section
Eqn.	Equation
Fig.	Figure

CHAPTER 1

INTRODUCTION

There are many beginnings in the field of physics. EPR paper[1] and the concept of entanglement were the initial push for the Quantum Information Theory and Bell's 1964 paper on the EPR 'paradox'[2] was the cornerstone on which an entire literature is built upon. Bell's insight on the subject, his ability to successfully transform mostly philosophical debates into formal assumptions within a context of logical structure was not an underrated one. This approach, which later assumed the name Bell's Theorem, allowed physicists to form such statements that their negation would demonstrate the falsification of classical ideas such as locality, non-contextuality and so on, while bringing forth the requirement for a revision of the most basic physical intuitions of the modern era.

Main logic behind Bell's theorem can be best understood as *reductio ad absurdum* in the form of a "no-go theorem". One formulates a statement using explicit assumptions; demonstrating the falsehood of this statement, which leads to a physical impossibility, opens up these assumptions to debate. In Bell's case it was a contradiction between quantum predictions which can be verified by experimental data and a theory of hidden variables using the requirements introduced by Einstein as the *a priori* assumptions derived from a definition known as "criterion of reality"[1]. Bell constructed an inequality, which now known as Bell's inequality, where its bound on expectation values denoted the limits on a certain bipartite system and he demonstrated that quantum predictions violate these limits. This violation of Bell's inequality is considered to be the first experimentally testable formulation of EPR argument, hence some even consider this as the opening shot of 'experimental metaphysics'[3].

After the development of Bell's theorem a huge amount of literature piled up rather quickly and many new forms of inequalities and some theorems focused on similar problems which Bell's theorem was addressing. In the following chapters of this study the most common of these inequalities will be investigated, their similarities and differences are going to be emphasized to establish their relations to Bell's theorem and the lineage of many inequalities. In addition to these, some theorems addressing the problem of micro/macro-reality divide will also be a focus of investigation. Their resemblances and dissimilarities with respect to the Bell's theorem are going to be studied.

Chapter 2 consists several key points on which the following literature has built upon. The problem of completeness of quantum theory asserted by Einstein, Podolsky and Rosen in 1935[1] and immediate responses to this paper will be the starting point of this study. Famous Bohr-Einstein debates following the initial papers and how two different formulations of physical reality have formed and sustained themselves will be the next. The concept of hidden variables, one particular and very strong use of it by Bohm and its differences with the quantum theory at hand will also be investigated in chapter 2.

Following the same line of thought in chapter 3 the development of Bell's theorem and Bell's thinking is going to be focused on. Inequalities such as BCHS, CHSH and MABK will be investigated to see the common patterns between them and to identify the key points of inequalities as Bell inequality.

In chapter 4 other types of inequalities and theorems regarding the problem of realism will be studied. These can be summarized as the Leggett-Garg inequalities[4], which are sometimes referred to as temporal Bell inequalities, and the Kochen-Specker theorem[5] dealing with the concept of contextuality.

In chapter 5, Bell's theorem without the use of inequalities is going to be studied. Firstly the GHZ experiment[6] and how Bell's theorem, which was understood through an inequality, can be formulated without inequalities will be investigated. After that a more recent development of this type, without inequalities, called Hardy's test[7] will be the matter of importance. Several different uses of Hardy's test on generic and spin states are going to be presented and some contemporary studies and experiments based on it will be mentioned.

Finally in chapter 6 and 7 all of the subjects above are going to be discussed with respect to their relevance to the problems of macro/micro - realism, locality, contextuality and their resemblances in themselves. Following this, the study will be concluded with closing statements and remarks. In the appendix some derivations of useful constructions will be shown explicitly.

CHAPTER 2

EPR PROBLEM AND HIDDEN VARIABLES

In this chapter the problem introduced by Einstein-Podolsky-Rosen and its consequences are going to be discussed gradually in several sections. Section 2.1 will be an introduction to the original paper, its key ideas and some immediate and recognized responses to it will be included here. In section 2.2 the publicized debates developed between Einstein and Bohr will be brought forth and how these debates nourished the construction of several important concepts in the field is going to be investigated. Section 2.3 will focus on a special branch of alternative theories to quantum theory itself divulged around the idea of adding hidden variables to the wave function. Finally in section 2.4 the formal background that led to the construction of Bell's theorem will be introduced briefly and only in the context of their relevance to the EPR problem and the rest of this study.

2.1 EPR Paper and Responses

Einstein-Podolsky-Rosen's 1935 dated paper named "Can Quantum-Mechanical Description of Reality Be Considered Complete?"[1] can be considered as an epoch moment for the problem of locality in quantum mechanics. It has over 5000 citations standing alone in `http://journals.aps.org/`, some of it from 2015. After 80 years, although its mathematical representations are of little use to anyone, this paper hasn't lost its relevance to the contemporary arguments of physicists and the problem of locality (or rather non-locality) stands as an unsolved problem in physics.

In this section the conceptual developments that arose from the EPR paper and the immediate responses will be of focus and the mathematics of this paper will mostly be omitted since it was highly theoretical and 'not even rigorously true'[8]. However ideas and assumptions sprouting from this original paper has encapsulated the debates over many areas from completeness of quantum theory to the nature of physical reality itself. To this extent it would only be appropriate to look at what Einstein-Podolsky-Rosen introduced as a condition of completeness;

"*every element of the physical reality must have a counterpart in the physical theory*"[1]

They continue by disserting that elements of the physical reality cannot be determined by considerations other than of experimental approaches and measurements and denote that a comprehensive definition of reality is unnecessary in their case. However a criterion of reality is established in the following form;

"*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*" [1]

After some dealing with the position and momentum of a system they reach the conclusion that;

"*when the momentum of a particle is known, its coordinate has no physical reality*"[1]

Moreover they claim that either (1) 'for two physical quantities with non-commuting operators there is no simultaneous reality' or (2) 'that the quantum mechanical description of reality given by the wave function is not complete'. To further the argument they examine two separated systems which have interacted between t=0 and t = T and then ceased to interact. Since they no longer interact the authors make the assumption that "no real change can take place in the second system in consequence of anything that may be done to the first system"[1]. Hence using this argument and assigning two wavefunctions to the same reality (by expanding it in two different bases) they reach to a point where starting from the argument that the wave function gives a complete description of reality indicates that two physical quantities with non-commuting operators can have simultaneous reality. So the negation of (2) leads to the negation of (1), then it is concluded as the initial assumption that wave function gives complete description of reality is not holding true[1].

This famous paper has two closing statements which are, although spectacularly wrong, leads to two of the still on-going debates in the field of foundations of quantum mechanics. First one is; "This makes the reality of P and Q[which are observables corresponding to the second system] depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this."[1] This is the point in history where the concept of quantum entanglement is first expressed (though still not named) in the literature. This counter-intuitive idea was developed by the authors to refute quantum theory in the sense that it is not 'reasonable', which later turned out to be one of the foundational cornerstones of the theory itself. The ingeniousness of this paper led to this groundbreaking discovery which allowed the problem of locality to be formalized.

The second statement was that; "While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible"[1]. This is also a very important statement which led a considerable amount of physicists to search for a theory that gives a 'more' complete description of the physical reality than the wave function approach of quantum theory itself. The scope of this (still continuing) search for this 'new' theory is more extensively investigated in section 2.3.

With these closing statements and above mentioned assumptions, criterion and descriptions the EPR paper was a very controversial and hence a fruitful piece of study. Responses to it was immediate. Niels Bohr published an article titled [9] same as EPR's, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" which has the following abstract;

"It is shown that a certain 'criterion of physical reality' formulated in a recent article with the above title by A. Einstein, B. Podolsky and N. Rosen contains an essential ambiguity when it is applied to quantum phenomena. In this connection a viewpoint termed 'complementarity' is explained from which quantum-mechanical description of physical phenomena would seem to fulfill, within its scope, all rational demands of completeness.” [9]

Bohr’s paper develops around several topics ranging from time measurements in quantum theory to transformation theorem of quantum mechanics, from commutation relationships to experimental arrangements and so on. However the repeating idea in this paper is that it is impossible to control the interactions between the measuring device and the measured object, hence the notion of "without in any way disturbing a system" is not an applicable condition when dealing with quantum phenomenon. The problem of locality introduced by EPR is basically seen as a mishandling of quantum concepts and reference frames.

Another and again very important response came from Schrödinger named "Discussion of Probability Relations Between Separated Systems"[10] coining the term ‘entanglement’ and explaining at length how the situation proposed by EPR is in fact accurate and measurements on one of the entangled states may help predict the future measurements on the accompanying entangled state.

Also Schrödinger published an important and a very famous paper titled "Die gegenwärtige Situation in der Quantenmechanik"[11] which is translated as "The Present Situation in Quantum Mechanics" where he explains key developments in quantum theory, its differences from classical mechanics an so forth. He makes reference to the EPR paper and problems introduced by this paper as well and answers them.

Although there are countless number of responses and references to the EPR paper, Wendell H. Furry’s 1936 article named "Note on the Quantum Mechanical Theory of Measurement"[12] should also be noted. It relates von Neumann’s theory of measurement and the EPR problem through using the notion of ‘reduction of wave packets’. An emphasis on the relation of a system and the means used to observe it is made as a concluding remark here and following Bohr’s lead the issue of measuring devices is highlighted again. Also as an interesting note the closing statement underlines the problem of the distinction between subject and object, which is a recurring focus in investigations concerning reality in the quantum ‘world’.

2.2 Einstein-Bohr debates

The EPR paper introduced and highlighted some important problems within the reach of physics but it is also a metaphysical piece. Its opening statement is "In a complete theory there is an element corresponding to each element of reality."[1], which emphasized a philosophical divide in the field of physics. In this divide champions of the sides came forth and argued their cases against each other and in front of the entire physics community of those times. The most famous of these debates were the series of letters and articles which in their totality called as the Bohr-Einstein debates. Although it was named Bohr-Einstein debates other towering figures in the history of physics such as Max Born, Werner Heisenberg and Erwin Schrödinger played important roles. Born corresponded with Einstein continuously through the period now known as the Quantum Revolution and especially Schrödinger took on himself to answer Einstein's assertions with regard to the incomplete or problematic structure of quantum mechanics.

Standing points of these two giants can be best emphasized through their own words. Einstein believed in a way of conducting physics such that it is a matter of truth seeking, finding the right tools to understand what is actually out there in the physical world. He explains his grasp on the dealings of physics as; "…the concepts of physics relate to a real outside world… It is further characteristic of these physical objects that they are thought of as arranged in a space-time continuum. An essential aspect of this arrangement of things in physics is that they lay claim, at a certain time, to an existence independent of one another, provided these objects 'are situated in different parts of space'." [13]

However Bohr did not believe in this one-to-one correspondence, through his readings of Kant and similar philosophers he developed an understanding of physics that is more concerned with explanation and prediction, while being skeptical about a world outside that can be exactly encapsulated by a physical theory. In his own words; "There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature."[14]

Though Einstein's and Bohr's points corresponded to two of the extremes of their times there was a vast majority standing in between. Physicists such as Heisenberg agreed Bohr's ideas but hesitated to generalize them into the physics of everyday life, which is now referred to as macro-reality. A brief note on his views can be said as; "This again emphasizes a subjective element in the description of atomic events, since the measuring device has been constructed by the observer, and we have to remember that what we observe is not nature in itself but nature exposed to our method of questioning. Our scientific work in physics consists in asking questions about nature in the language that we possess and trying to get an answer from experiment by the means that are at our disposal.“ [15]

The separation between levels of physical reality into macro and micro later played an important role in theory developing as well as in philosophy of physics. Physicists following the lead of classical physicists and later updated versions by Einstein argues that there is a consistent reality that both govern the stars and the atomic particles and it has an in-itself existence. Others take a standing in the center and assert that reality is scale dependent and macro-reality has different rules than micro-reality, former has definiteness and sharp precision while the latter has a statistically un-deterministic nature. Finally the extremes in Bohr's line of thought argues that the consistency seen in the macro level is again an illusion and can only be attributed to the high approximation powers of standing theories. There are other, more contemporary developments in each of these line of thoughts, which will be discussed in 6.1.

Different beliefs yield different theories and various ways of problematization. These points taken as micro-realism, macro-realism and realism as a whole led to a variety of different roads in the context of physical theories. Physicists such as John S. Bell devoted their lives to constitute a theory that is consistent with realism [16, 17]. While others tried to establish the triumph of Bohr's beliefs and introduced methods to refute the consistent structure of the macroscopic realism[4]. And a few has even proposed drastic changes in the current understanding of basic concepts such as time[18] or introduced new concepts such as universal background noise[19, 20] in their search for a new theory.

The ideas and theories are always in flux, new approaches are being developed and old ones are being rigorously tested everyday. However what is constant in physics is and always have been the correspondence of theory and results of experiments. This is probably best explained by Feynman's following quote from the book *The Character of Physical Law*; “It does not make any difference how beautiful your guess is. It does not make any difference how smart you are, who made the guess, or what his name is – if it disagrees with experiment it is wrong. That is all there is to it.” [21]

2.3 Hidden Variables

In this section a general approach to construct an alternative theory to quantum mechanics will be of the focus. This search can be summarized as the hidden variable program and its purpose can best be explained through the following lines from Bell's article named "On the Problem of Hidden Variables in Quantum Mechanics"[22]; "The question at issue is whether the quantum mechanical states can be regarded as ensembles of states further specified by additional variables, such that given values of these variables together with the state vector determine precisely the results of individual measurements."

There are several known [18, 23, 24] and in some respect still on-going versions of this program. Aharonov's approach uses a two-vector formalism that allow future events to effect the outcomes of present measurements. Nelson's way is a derivation of quantum mechanics from classical mechanics through random variables and stochastic systems. Finally, Bohmian mechanics involves allowing a definite position function with nonlocal properties. All these have in common is that their aim is to replicate the predictions of quantum mechanics but eliminate the discrepancies of it (such as wave function collapse) through introducing ‘hidden’ variables. Hidden here is used in the sense that this new variable cannot be set during the preparation period of state at hand.

As there are theories making use of the hidden variables there are strong impossibility proofs that prohibits the addition of hidden variables to quantum theory in order to obtain same predictions. The common weakness of these proofs is that they use certain assumptions and proofs with respect to hidden variables but the generalization process of them to cover the entire class of hidden variable theories is problematic. As stated in "A Refutation of the Proof by Jauch and Piron that Hidden Variables Can be Excluded in Quantum Mechanics" Bohm himself declares that this kind of proofs, such as von Neumann's, exclude only a certain very restricted class of hidden variable theories[25].

Simplified and shortened versions of some of these impossibility proofs of hidden variables can be found in Bell's article[22] for [26, 27, 28] and their restrictions with respect to Bohm's pilot-wave theory [24] can again be found in Bell's another article named "On the Impossible Pilot Wave"[29].

The notation used in the following section will be a faithful representation of Bohm's original demonstration and will not be followed in the rest of this study since rest will use mostly Dirac's bra-ket notation formalism.

2.4 Bohm's spin arrangement

Bohm investigates the EPR problem in his book 'Quantum Theory' between pages 611–623 as a finishing argument for the 22nd chapter titled 'Quantum Theory of the Measurement Process'.[24] In the 16th section 'The Hypothetical Experiment of Einstein, Rosen and Podolsky' of this chapter Bohm introduces a 'somewhat modified' version of the experiment in a conceptually equivalent form, but considerably easier to treat mathematically. This modified version is the spin arrangement of the EPR problem that is generally known and it is sometimes even referred to as the EPR-Bohm correlations. It is the version that has been most widely used in the literature following that era.

In this version a molecule containing two atoms in a state in which the total spin is zero with spin of each atom being $\frac{1}{2}$ is supposed. The classical counterpart of this supposition takes place as to assume two particles with spin angular momentum zero, hence if the first particle has a certain angular-momentum vector than the second particle will have an angular-momentum vector that is of the same magnitude but opposite direction. So that by measuring one particle the experimenter can obtain information about the other one. Since taking a measurement of the first particle is consistent with EPR's requirement of 'without in any way disturbing' for the second particle this new version proposed by Bohm is consistent with the original situation proposed by EPR.

Bohm starts from the most general state consisting two spin-half particles;

$$\begin{aligned} \Psi_a &= u_+(1)u_+(2) \quad \Psi_b = u_-(1)u_-(2) \\ \Psi_c &= u_+(1)u_-(2) \quad \Psi_d = u_-(1)u_+(2) \end{aligned} \tag{2.1}$$

are the spins aligned in $\hat{z}$-direction, where $\pm$ signs correspond to $\pm\frac{\hbar}{2}$. Following the line of argument above to obtain a total spin zero state with particle spins corresponding to opposite values with each other Ψ_c and Ψ_d are required in such a way with $J_z = 0$ and $J = 0$,

$$\Psi_0 = \frac{\Psi_c - \Psi_d}{\sqrt{2}} \tag{2.2}$$

Now describing the process of measurement of σ_x with eigenfunctions v_+, v_- result is similarly $J = 0$ and $J_x = 0$,

$$\Psi_0 = \frac{v_+(1)v_-(2) - v_-(1)v_+(2)}{\sqrt{2}} \tag{2.3}$$

After measurement the state becomes

$$\Psi = \frac{v_+(1)v_-(2)e^{i\alpha_1} + v_-(1)v_+(2)e^{i\alpha_2}}{\sqrt{2}} \tag{2.4}$$

where Bohm takes α_1 and α_2 as uncontrollable phase factors. Hence it is seen from here as well that the value of σ_x for each particle is also correlated.

As an addition to these, Bohm states another important point corresponding to this problem. He defines a mean value of any function $g(o_2)$ of the spin variables of the second particle alone and looks to the average of this function after a measurement on the first particle. Average of this function is the same as what is obtained without a measurement of the spin variables of the first particle. He concludes that the behaviour of the two spins, however, are correlated despite the fact that each behaves in a way that does not depend on what actually happens to the other after interaction has ceased.[24]

Bohm's brilliant redefinition of the problem at hand allowed it to become much more open to mathematical interpretation. Bell himself was much affected by Bohm's ideas and vocally advertised de Broglie-Bohm theory in more than one occasion[30, 29].

For more of the correspondence between these two great minds one can look at Sheldon Goldstein's publication for the 50th year anniversary of Bell's theorem, of a talk he gave on Bell's view on Bohm at a memorial conference for him, titled "Bell on Bohm"[31]. This modified version of the EPR argument allowed Bell to formulate his inequality and theorem.

CHAPTER 3

BELL'S THEOREM AND BELL INEQUALITIES

In the previous chapter the EPR paper and the whirlwind of debates, arguments and theorizing that followed it were discussed. One of the main topics in these was the problem of locality and how to address it. Although approaches such as de Broglie – Bohm Theory and other hidden variable theories are somewhat inclusive of the problem, they still can not demonstrate it clearly enough to directly respond to Einstein's point of concern. The main focus of this chapter is the one that did not end the discussions but allowed everyone to speak about the problem at hand through mathematical, logical and most importantly physical aspects, rather than solely philosophical or vaguely formal manners.

In this chapter the renowned 1964 dated paper of Bell[2] and further progress in that line will be investigated. Bell's theorem and generation of Bell inequalities are going to be shown in the context of Bell's own and consecutive papers regarding his work. Further developments and other inequalities formulated from Bell's approach such as BCHS and CHSH inequalities consisting of bipartite spin half states and a product of a set of papers which together called as MABK inequality consisting N-partite spin half particles will also be of importance. There will also be pieces from more contemporary discussions regarding Bell's theorem and Bell type of inequalities. In addition to those, how this line of thought effected the current understanding of physical reality among academia of physics will be examined, though not exhaustively.

3.1 "On the Einstein-Podolsky-Rosen Paradox" paper

This single article is probably one of the most cited and well-read papers regarding the notions of quantum entanglement, non-locality and hidden variable theories, also it is considered to be one of the cornerstones of Quantum Information Theory. It is published by J. S. Bell in 1964 and consist the method now known as the Bell's theorem. This theorem in fact is not a well established and clearly stated theorem but it is a general understanding or an approach, a way of stating certain assumptions in a formal manner.

The opening statements of this work are the following lines,

"The paradox of Einstein, Podolsky and Rosen was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality."[2]

In his 2014 dated paper titled "The Two Theorems of John Bell "[32] for a special issue celebrating the 50th anniversary of Bell's 1964 paper, H. M. Wiseman makes a point that although there is a single Bell's theorem there are more than one interpretation of it and this mainly depend on how Bell himself in time developed his own approach and it didn't reach to as many physicists as his original paper did. In the above first lines of the 1964 paper, Bell argues that the EPR argument was two-fold, it depended on quantum theory being either not causal or not local. However his later views, presented in 1975 at the Sixth GIFT Seminar, Jaca and reproduced in 1976 titled "The theory of local beables"[16] indicates a particular and singular argument introduced by EPR and that is 'local causality'. (Further discussions on 'Local Causality' will be conducted in Sec. 6.2)

The second chapter of the latter publication is titled 'Local Causality'. The single page explanation can be summarized as; if an event B does not belong to the backward light cone of an event A, the outcome of the event A can not depend on B in any way. Assume the event A depends on some set of variables denoted as Λ. If B does not lie in the backward light cone of A then the following relation should be satisfied, meaning the outcome obtained from A does not depend on B;

$$(A|\Lambda, B) = (A|\Lambda) \tag{3.1}$$

During the following chapters in that paper Bell again shows, by using the simplified 1974 version of CHSH inequality[33], that quantum mechanics is not a locally causal theory. However, he also explains that, in chapter 7, relativistic quantum mechanics is locally causal in the human sense. The previous statement emphasizes that although there is discrepancy in formalism there are no such problems in experiment, hence it does not allow faster than light transmission of messages. This is again stressed by Bell in his 1981 instructive study titled "Bertlmann's socks and the nature of reality"[34] that this is not seen by EPR as a real action at a distance but rather a 'spooky action at a distance' indicating it having a correspondence only in wave function formalism.

It is noted by Wiseman[32] that Bell's early terminology is not well adjusted to the specifications he was making. He used separability and locality interchangeably in more than one paper, he introduced strong arguments such as realism and determinism without clearly identifying the concepts at hand and so on. However in his later writings this attitude has slowly disappeared and much more complete works of Bell can be found in his post-1964 years. Wiseman attributes the shroud of mystery concerning Bell's theorem mostly to the misunderstanding of his previous works by some hasty researches in the physics community. All these said, Bell continued using the word locality implying local causality in his later writings as well hence in this study the word locality will be preserved and used to imply local causality argument introduced by Bell[16].

Continuing with Bell's 1964 paper it seems rather important to construct what was later to be shown in contradiction. Similar to EPR's description of elements of reality and completeness of a theory, Bell here introduces a 'requirement of locality'. He argues that "the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past"[2] is what creates the essential difficulty in EPR argument to quantum theory. Now in here another note should be included that the EPR argument does not exactly correspond or require this kind of locality in their paper. The term broadly defined in the above quote as ‘operations’ is in fact has a much more limited use in EPR paper, denoting not operations in general but the measurement outcomes provided by these.

Now following the demonstration of [2] in a faithful manner, since quantum theory does not allow pre-determination and the argument is that this should not be the case for a complete theory, Bell introduces a λ that may be a single variable, a set of variables or even a set of functions discrete or continuous, to address this issue. The result A of measuring $\sigma_1.\hat{a}$ is then determined by $\hat{a}$ and λ together, and the result B of measuring $\sigma_2.\hat{b}$ in the same instance is determined by $\hat{b}$ and λ, where $\hat{a}$ and $\hat{b}$ are directions. So now the following can be argued,

$$A(a,\lambda) = \pm 1 \quad B(b,\lambda) = \pm 1 \tag{3.2}$$

To get the expectation value of the product of the two components at hand the probability distribution $\rho(\lambda)$ is to be used in the context:

$$P(a,b) = \int d\lambda \rho(\lambda) A(a,\lambda) B(b,\lambda) \tag{3.3}$$

This should correpond to the quantum mechanical expectation value of a singlet state:

$$\langle \sigma_1.\hat{a}\sigma_2.\hat{b} \rangle = -\hat{a}.\hat{b} \tag{3.4}$$

Now for normalized probability distribution ρ there is

$$\int d\lambda \rho(\lambda) = 1$$

and because of eqn. 3.2, P in eqn. 3.3 cannot be less than -1 and it can be -1 at $\hat{a} = \hat{b}$ only if

$$A(a,\lambda) = -B(a,\lambda)$$

Using this eqn. 3.3 can be rewritten in the form

$$P(a,b) = -\int d\lambda \rho(\lambda) A(a,\lambda) A(b,\lambda)$$

And for another unit vector $\hat{c}$ there is

$$P(a,b) - P(a,c) = -\int d\lambda \rho(\lambda) [A(a,\lambda)A(b,\lambda) - A(a,\lambda)A(c,\lambda)]$$

after re-arranging these

$$P(a,b) - P(a,c) = \int d\lambda \rho(\lambda) A(a,\lambda) A(b,\lambda) [A(b,\lambda)A(c,\lambda) - 1]$$

Here note that $A(i, \lambda)$ has possible outcomes ± 1 for all i's, hence $[A(i, \lambda)]^2 = 1$ holds for $i = a, b, ...$ Now through the conditions defined in eqn. 3.2 it becomes

$$|P(a, b) - P(a, c)| \leq \int d\lambda \rho(\lambda)[1 - A(b, \lambda)A(c, \lambda)]$$

The second term on the right is in fact $P(b, c)$ and the whole thing can be written as

$$|P(a, b) - P(a, c)| \leq 1 + P(b, c) \tag{3.5}$$

This is the first of well-known Bell inequalities. For P not constant the left hand side is in order |b - c| for small |b - c|, hence P(b,c) cannot be stationary at -1 the minimum value which is for $\hat{b} = \hat{c}$, and it cannot be equal to the quantum mechanical expectation value expressed in equation 3.4.

In addition to this the formal proof of the quantum mechanical correlation 3.4 not being approximated arbitrarily closely by the form 3.3 can be shown in the following lines. This proof would not take into account the failure of the approximation at isolated points. Instead of equations 3.3 and 3.4 consider these functions

$$\overline{P(a, b)} \quad \text{and} \quad \overline{-\hat{a}.\hat{b}}$$

where the bar represent, independent averaging of $P(a', b')$ and $-\hat{a}'.\hat{b}'$ over vectors $\hat{a}'$ and $\hat{b}'$ within small angles of $\hat{a}$ and $\hat{b}$. For all $\hat{a}$ and $\hat{b}$ assume that the difference is bounded by ϵ:

$$|\overline{P(a, b)} + \hat{a}.\hat{b}| \leq \epsilon \tag{3.6}$$

Now it will be shown that ϵ cannot be made arbitrarily small for a small finite δ of the following form. Suppose that for all $\hat{a}$ and $\hat{b}$;

$$|\overline{\hat{a}.\hat{b}} - \hat{a}.\hat{b}| \leq \delta$$

Then from eqn. 3.6 it follows

$$|\overline{P(a, b)} + \hat{a}.\hat{b}| \leq \epsilon + \delta \tag{3.7}$$

and from eqn. 3.3

$$\overline{P(a, b)} = \int d\lambda \rho(\lambda) \bar{A}(a, \lambda) \bar{B}(b, \lambda) \tag{3.8}$$

where the averages are bounded as given below;

$$|\bar{A}(a, \lambda)| \leq 1 \quad \text{and} \quad |\bar{B}(b, \lambda)| \leq 1 \tag{3.9}$$

So that from equations 3.7 and 3.8 together with $\hat{a} = \hat{b}$

$$\int d\lambda\rho(\lambda)[\bar{A}(b,\lambda)\bar{B}(b,\lambda) + 1] \leq \epsilon + \delta \tag{3.10}$$

Now see that from eqn. 3.8

$$\begin{aligned} \bar{P}(a,b) - \bar{P}(a,c) = &\int d\lambda\rho(\lambda)\bar{A}(a,\lambda)\bar{B}(b,\lambda)[\bar{A}(b,\lambda)\bar{B}(c,\lambda) + 1] \\ &- \int d\lambda\rho(\lambda)\bar{A}(a,\lambda)\bar{B}(c,\lambda)[\bar{A}(b,\lambda)\bar{B}(b,\lambda) + 1] \end{aligned} \tag{3.11}$$

Using 3.11 and 3.9 together

$$\begin{aligned} |\bar{P}(a,b) - \bar{P}(a,c)| = &\int d\lambda\rho(\lambda)[\bar{A}(b,\lambda)\bar{B}(c,\lambda) + 1] \\ &+ \int d\lambda\rho(\lambda)[\bar{A}(b,\lambda)\bar{B}(b,\lambda) + 1] \end{aligned} \tag{3.12}$$

And together with 3.8 and 3.10 eqn 3.12 becomes of the form

$$|\bar{P}(a,b) - \bar{P}(a,c)| \leq 1 + \bar{P}(b,c) + \epsilon + \delta$$

Finally using 3.7

$$|\hat{a}.\hat{c} - \hat{a}.\hat{b}| - 2(\epsilon + \delta) \leq 1 - \hat{b}.\hat{c} + 2(\epsilon + \delta)$$

or in a more orderly manner

$$|\hat{a}.\hat{c} - \hat{a}.\hat{b}| + \hat{b}.\hat{c} - 1 \leq 4(\epsilon + \delta) \tag{3.13}$$

Now take for example $\hat{a}.\hat{c} = 0$ and $\hat{a}.\hat{b} = \hat{b}.\hat{c} = \frac{1}{\sqrt{2}}$. Then

$$4(\epsilon + \delta) \geq \sqrt{2} - 1$$

So that for small finite δ, the value of ϵ cannot be taken as arbitrarily small. Hence it is shown that the quantum mechanical value cannot be represented, either accurately or arbitrarily closely, in the form of the eqn. 3.3.

Furthermore Bell demonstrates that if A could depend on $\hat{b}$ and B could depend on $\hat{a}$ there would be no difficulty in reproducing the quantum mechanical correlation 3.5. However this would mean in a system like that, for given values of the hidden variables, the results of measurements with one magnet depend on the setting of the distant magnet. As a conclusion he wrote;

"In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote."[2]

3.2 BCHS and CHSH Inequalities

3.2.1 BCHS or Bell-Clauser-Horne-Shimony inequalities

There are many articles using inequalities of the CHSH or Bell-CHSH or BCHS types while referring to the same basic set of articles, mainly [2, 35, 36, 37]. However since the original paper of CHSH uses correlation functions and defining spaces over λ there was a lot of room for simplification hence in the following years what is commonly referred to as CHSH inequality has changed. This new and simplified version of it depended upon expectation values and average values in the classical sense, while another form of derivation sprouting from the first set of initial papers began to form rapidly after Arthur Fine's paper titled "Hidden Variables, Joint Probability, and the Bell Inequalities"[38]. In this section of the study main focus will be on what is now commonly called as Fine's theorem and its connection to the papers [35] [36]. This theorems connection to the Bell's inequality and CHSH inequality will be explained through following the demonstration of de Muynck's 1986 article [39].

In Arthur Fine's 1982 article titled "Hidden Variables, Joint Probability, and the Bell Inequalities"[38] a series of statements has been established and their equivalence is argued. These statements, taken from the abstract of his paper, are the following:

(1) There is a deterministic hidden-variables model for the experiment.

(2) There is a factorizable, stochastic model.

(3) There is one joint distribution for all observables of the experiment, returning the experimental probabilities.

(4) There are well-defined, compatible joint distributions for all pairs and triples of commuting and non-commuting observables.

(5) The Bell inequalities hold.

The logical connections between all these statements are not as solid as Fine wished them to be, however for demonstrative purposes equivalence between them are clear and instructive enough. Following the arguments of Clauser-Horne in [33] Fine describes two space-time regions R_1 and R_2 that are space-like separated. There are non-commuting observables A, A' for R_1 and B, B' for R_2 with values ± 1. Probabilities of the experiment are the observed distributions for A, A', B, B' and four additional compatible pairs AB, AB', A'B and A'B'. Another definition is, for $A = +1$ complement of it is denoted by $\bar{A} = -1$ and P(S) is a function defined as the probability of enclosed observable S taking the value +1. So that P($\bar{A}$) denotes A's complement $\bar{A}$ taking the value +1 hence P($\bar{A}$) corresponds to the probability of A taking value -1.

Similar to Bell's hidden variable notation, a normalized probability density function $\rho(\lambda)$ is defined on Λ which the set of hidden variables required for the 'complete' state specifications. Now another notation is introduced by Fine such that $A(\lambda), A'(\lambda), B(\lambda), B'(\lambda)$ each defined on Λ with values ± 1 satisfy the integration relations

$$P(S) = \int \tilde{S}(\lambda)\rho(\lambda)d\lambda \tag{3.14}$$

and

$$P(ST) = \int \tilde{S}(\lambda)\tilde{T}(\lambda)\rho(\lambda)d\lambda \tag{3.15}$$

where $\tilde{S}$ is a special notation identifying the relation $\tilde{S}(\lambda) = 1 \rightarrow S(\lambda) = 1$ and $\tilde{S}(\lambda) = 0 \rightarrow S(\lambda) = -1$. See that this holds since $S(\lambda) = -1$ gives P(S) = 0 due to P(..) giving the probability of S taking the value +1. And for $\tilde{S}(\lambda) = 1$ the integral $\int \tilde{S}(\lambda)\rho(\lambda)d\lambda$ gives 1 which indicates P(S) is 1, so it is consistent with S(λ) taking the value +1. Also see that if instead of P(S) its complement P($\bar{S}$) is used then taking $\tilde{S}(\lambda) \rightarrow 1 - \tilde{S}(\lambda)$ and re-arranging as

$$P(\bar{S}) = \int [1 - \tilde{S}(\lambda)]\rho(\lambda)d\lambda \tag{3.16}$$

will provide the same distribution relation. For $\tilde{S}(\lambda) = 1$, P($\bar{S}$) will be zero meaning that P(S) = 1 and this is consistent with $\tilde{S}(\lambda) \rightarrow S(\lambda) = 1$ relation. Inverse of it and similar conventions for eqn. 3.15 with compatible pairs of observables also hold.

Now that it is possible to argue that the existence of a deterministic hidden-variables model is strictly equivalent to the existence of a joint probability distribution function P(A A' B B') for the four observables of the experiment[38]. To look for this

$$P(AA'BB') = \int \tilde{A}(\lambda)\tilde{A}'(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \tag{3.17}$$

and see that

$$P(AA'BB') + P(A\bar{A}'BB') + P(AA'B\bar{B}') + P(A\bar{A}'B\bar{B}') = P(AB) \tag{3.18}$$

For explicit demonstration of this relation refer to the Appendix section B.1.

If $P(AA'BB')$ is given then a simple way to define a deterministic hidden-variables model is of the following sort. Let Λ consist of all sixteen quadruples of the form $\lambda =< a_1, a_2, a_3, a_4 >$ where $a_i = \pm 1$ and introduce

$$A(\lambda) = a_1 \quad A'(\lambda) = a_2 \quad B(\lambda) = a_3 \quad B'(\lambda) = a_4 \tag{3.19}$$

Then define $\rho(\lambda)$ as $\rho(a_1, a_2, a_3, a_4) = P(AA'BB')$ where $S_i = S$ is used if $a_i = 1$ and $S_i = \bar{S}$ is used if $a_i = -1$. Thus the idea of deterministic hidden variables is just the idea of a suitable joint probability function[38].

Now to show that the existence of this in its own accord is a contradiction with quantum theory Fine introduces a proposition that for triples A, B, B' and A', B, B' with probability distributions $P(ABB')$ and $P(A'BB')$ it is possible to construct P(A A' B B') and P(B B') which is a well-defined joint probability distribution of two non-commuting observables which is a violation of quantum mechanical requirements.

Setting $P(AA'BB') = \frac{P(ABB')P(A'BB')}{P(BB')}$ with $P(AA'BB') \to 0$ for $P(BB') = 0$ can be shown and justified as

$$\begin{aligned} P(ABB') = P(AA'BB') + P(A\bar{A}'BB') &\leq P(A'B) + P(\bar{A}'B') \\ &= P(A'B') + P(B') - P(A'B) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} P(\bar{A}BB') = P(\bar{A}A'BB') + P(A\bar{A}'BB') &\leq P(A'B') + P(\bar{A}'B) \\ &= P(A'B') + P(B) - P(A'B) \end{aligned} \tag{3.21}$$

For explicit calculations again refer to Appendix B.1. Then it can easily be seen that

$$0 \leq P(A\bar{B}\bar{B}') = P(A) - P(AB) - P(A'B) + P(AB') + P(ABB') \tag{3.22}$$

together with

$$0 \leq P(\bar{A}\bar{B}\bar{B}') = 1 - P(B) - P(A) - P(B') + P(AB) + P(AB') + P(\bar{A}BB') \quad (3.23)$$

Using eqn. 3.20 for $P(ABB')$ in eqn. 3.22 and eqn. 3.21 for $P(\bar{A}BB')$ in eqn. 3.23 gives

$$-1 \leq P(AB) + P(AB') + P(A'B') - P(A'B) - P(A) - P(B') \leq 0 \quad (3.24)$$

By changing A to A', B to B' and other variations, seven more inequalities of this sort can be obtained. Fine refers these eight inequalities collectively as Bell/CH inequalities[38] or from later references they are called BCHS inequalities. It is showed by these that every deterministic hidden variable theory restricts the probabilities of the experiment so as to satisfy these eight inequalities.

Now following de Muynck[39] take the observables A, A', B, B' only as A_i consistent with eqn. 3.19. Expectation value of a product A_iA_j is given by

$$\langle A_iA_j \rangle = P_{ij}(+,+) + P_{ij}(-,-) - P_{ij}(+,-) - P_{ij}(-,+) \quad (3.25)$$

where $P_{ij}(+,+)$ stands for $P(A_i = +1, A_j = +1)$ and similarly for the others. The inequality in eqn. 3.24 can be re-written in the form

$$-1 \leq P(A_1, A_2) + P(A_1, A_3) + P(A_3, A_4) - P(A_2, A_4) - P(A_1) - P(A_3) \leq 0 \quad (3.26)$$

By re-naming and shortening the middle term in the following manner

$$\begin{aligned} Q(A_1, A_2, A_3, A_4) &= P(A_1, A_2) + P(A_1, A_3) \\ &+P(A_3, A_4) - P(A_2, A_4) - P(A_1) - P(A_3) \end{aligned} \quad (3.27)$$

It can be argued that

$$\begin{aligned} \langle A_1A_2 \rangle + \langle A_1A_3 \rangle + \langle A_3A_4 \rangle - \langle A_2A_4 \rangle = Q(+,+,+,+) + Q(-,-,-,-) \\ -Q(+,-,-,+) - Q(-,+,+,-) \end{aligned} \quad (3.28)$$

for explicit demonstration see Appendix section B.2. From 3.26 it follows as

$$Q(+,+,+,+) \geq -1, \qquad Q(-,-,-,-) \geq -1,$$

$$-Q(+,-,-,+) \geq 0, \qquad -Q(-,+,+,-) \geq 0,$$

Giving the following inequality

$$(\langle A_1A_2\rangle + \langle A_1A_3\rangle + \langle A_3A_4\rangle - \langle A_2A_4\rangle) \geq -2 \tag{3.29}$$

Just by changing A_2 with A_3, it can also be argued that

$$(\langle A_1A_2\rangle + \langle A_1A_3\rangle + \langle A_2A_4\rangle - \langle A_3A_4\rangle) \geq -2 \tag{3.30}$$

And from these equations 3.29 and 3.30 together the following can be obtained

$$|\,\langle A_2A_4\rangle - \langle A_3A_4\rangle\,| \geq 2 + \langle A_1A_2\rangle + \langle A_1A_3\rangle \tag{3.31}$$

In the above relation taking $\langle A_1A_2\rangle = -1$ will give Bell's inequality of 3.5.

In conclusion to this section, it can be said that inequalities such as BCHS, CHSH and their kind are generally noted as Bell inequalities since they are all equivalent to each other and can be shown to satisfy Bell's original inequality. However, as Bell acknowledges in the conclusion part of his work [16] the 1969 paper of CHSH [35] is a very suitable prototype for any Bell inequality to be derived from. Different inequalities may be required for different experimental configurations but they all should be, in principle, obtainable from the CHSH inequality.

3.2.2 Violation of CHSH inequality

In the previous section it is shown that any Bell inequality for dichotomic observables can be derived from one another. In this section a straightforward violation of CHSH inequality will be demonstrated and since each Bell inequality can be derived from one another, violation of one serves the purpose of violation of Bell inequalities as a whole. Experimental violations of this sort has been reported, starting from Aspect experiments in 1981 [37].

A simple spin singlet state corresponding to a bipartite system with particles at points A and B can be defined as

$$|\Psi\rangle_{AB} = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{3.32}$$

Operators concerning spin-half states are explicitly derived in appendix sec. A.1, hence the following arguments with regard to the state at hand can be said

$$\sigma_{Ax}\,|0\rangle_A = |1\rangle_A \qquad \sigma_{Ax}\,|1\rangle_A = |0\rangle_A$$

$$\sigma_{Ay} |0\rangle_A = i |1\rangle_A \qquad \sigma_{Ay} |1\rangle_A = -i |0\rangle_A$$

$$\sigma_{Az} |0\rangle_A = |0\rangle_A \qquad \sigma_{Az} |1\rangle_A = |1\rangle_A$$

Similarly all these relations apply to B as well. By defining an overall σ_i operation on the system as $\sigma_{Ai} + \sigma_{Bi}$ where $i = x, y, z$, the below equations are applicable

$$\sigma_x |\Psi\rangle_{AB} = (\sigma_{Ax} + \sigma_{Bx}) |\Psi\rangle_{AB} = \sigma_{Ax} |\Psi\rangle_{AB} + \sigma_{Bx} |\Psi\rangle_{AB}$$

$$\sigma_y |\Psi\rangle_{AB} = (\sigma_{Ay} + \sigma_{By}) |\Psi\rangle_{AB} = \sigma_{Ay} |\Psi\rangle_{AB} + \sigma_{By} |\Psi\rangle_{AB}$$

$$\sigma_z |\Psi\rangle_{AB} = (\sigma_{Az} + \sigma_{Bz}) |\Psi\rangle_{AB} = \sigma_{Az} |\Psi\rangle_{AB} + \sigma_{Bz} |\Psi\rangle_{AB}$$

See that $\sigma_{Ai} |\Psi\rangle_{AB}$ leaves the state of the particle B unchanged and same goes for operations on B leaving A unchanged, even though the system in its entirety is affected. By using this and the equations above it can easily be argued that the relations below hold

$$\sigma_x |\Psi\rangle_{AB} = 0 \qquad \sigma_y |\Psi\rangle_{AB} = 0 \qquad \sigma_z |\Psi\rangle_{AB} = 0$$

From the definition of $\hat{S_n}$ that gives σ_n, again it is easy to see that

$$\sigma_n |\Psi\rangle_{AB} = (\sigma_{An} + \sigma_{Bn}) |\Psi\rangle_{AB} = 0$$

which indicates that the system at hand is rotationally invariant and it is a s = 0 state. Now for simplicity leaving out $\hbar$ and constants, one can define $\sigma_{An} = \pm 1$ and the corresponding $\sigma_{Bn} = \mp 1$ to satisfy $\sigma_n = \sigma_{An} + \sigma_{Bn} = 0$ for $\sigma_n |\Psi\rangle = 0$ therefore these give $\sigma_{An}\sigma_{Bn} = -1$. Also this can be investigated by looking at the following relations,

$$\sigma_{Ax}\sigma_{Bx} |\Psi\rangle = \sigma_{Ax}[\frac{|00\rangle - |11\rangle}{\sqrt{2}}] = \frac{|10\rangle - |01\rangle}{\sqrt{2}} = - |\Psi\rangle$$

$$\sigma_{Ay}\sigma_{By} |\Psi\rangle = \sigma_{Ay}[\frac{-i |00\rangle - i |11\rangle}{\sqrt{2}}] = \frac{-i^2 |10\rangle + i^2 |01\rangle}{\sqrt{2}} = - |\Psi\rangle$$

$$\sigma_{Ax}\sigma_{Bx} |\Psi\rangle = \sigma_{Az}[\frac{- |01\rangle - |10\rangle}{\sqrt{2}}] = \frac{- |01\rangle + |10\rangle}{\sqrt{2}} = - |\Psi\rangle$$

From these relations $\sigma_{An}\sigma_{Bn} |\Psi\rangle = - |\Psi\rangle$ can be derived too. Hence it gives

$$\langle\Psi|\sigma_{An}\sigma_{Bn}|\Psi\rangle = -1 \tag{3.33}$$

Now check for $\langle\Psi|\sigma_{An}\sigma_{Bm}|\Psi\rangle$ where $\hat{n} \neq \hat{m}$. First of all expand σ_{Bm} and σ_{An} as,

$$\sigma_{Bm} = m_x\sigma_{Bx} + m_y\sigma_{By} + m_z\sigma_{Bz} \quad \text{and} \quad \sigma_{An} = n_x\sigma_{Ax} + n_y\sigma_{Ay} + n_z\sigma_{Az}$$

Applying σ_{Bm} on $|\Psi\rangle$ will result,

$$\sigma_{Bm}|\Psi\rangle = m_x[\frac{|00\rangle - |11\rangle}{\sqrt{2}}] + m_y[\frac{-i|00\rangle - i|11\rangle}{\sqrt{2}}] + m_z[\frac{-|01\rangle - |10\rangle}{\sqrt{2}}]$$

On this applying σ_{An} will yield,

$$\sigma_{An}\sigma_{Bm}|\Psi\rangle = n_x\sigma_{Ax}\sigma_{Bm}|\Psi\rangle + n_y\sigma_{Ay}\sigma_{Bm}|\Psi\rangle + n_z\sigma_{Az}\sigma_{Bm}|\Psi\rangle$$

Investigating each part in its own,

$$n_x\sigma_{Ax}\sigma_{Bm}|\Psi\rangle = n_x m_x[\frac{|10\rangle - |01\rangle}{\sqrt{2}}] - n_x m_y[i\frac{|10\rangle + |01\rangle}{\sqrt{2}}] - n_x m_z[\frac{|00\rangle + |11\rangle}{\sqrt{2}}]$$

$$n_y\sigma_{Ay}\sigma_{Bm}|\Psi\rangle = n_y m_x[i\frac{|10\rangle + |01\rangle}{\sqrt{2}}] + n_y m_y[\frac{|10\rangle - |01\rangle}{\sqrt{2}}] + n_y m_z[i\frac{|00\rangle - |11\rangle}{\sqrt{2}}]$$

$$n_z\sigma_{Az}\sigma_{Bm}|\Psi\rangle = n_z m_x[\frac{|00\rangle + |11\rangle}{\sqrt{2}}] + n_z m_y[i\frac{-|00\rangle + |11\rangle}{\sqrt{2}}] + n_x m_z[\frac{-|01\rangle + |10\rangle}{\sqrt{2}}]$$

Following the previous line of argument it can be said that,

$$\langle\Psi|\sigma_{An}\sigma_{Bm}|\Psi\rangle = \langle\Psi|n_x\sigma_{Ax}\sigma_{Bm} + n_y\sigma_{Ay}\sigma_{Bm} + n_z\sigma_{Az}\sigma_{Bm}|\Psi\rangle$$

which can be separated as,

$$\langle\Psi|n_x\sigma_{Ax}\sigma_{Bm}|\Psi\rangle + \langle\Psi|n_y\sigma_{Ay}\sigma_{Bm}|\Psi\rangle + \langle\Psi|n_z\sigma_{Az}\sigma_{Bm}|\Psi\rangle$$

By taking the hermitian conjugate of eqn. 3.32 the following is reached,

$$_{AB}\langle\Psi| = \frac{\langle 01| - \langle 10|}{\sqrt{2}}$$

Hence using the braket the following results can be acquired,

$$\begin{aligned}\langle\Psi|n_x\sigma_{Ax}\sigma_{Bm}|\Psi\rangle = {} & \frac{n_x m_x}{2}(-1) + \frac{n_x m_y}{2}(-i) + \frac{n_x m_z}{2}(0) \\ & + \frac{n_x m_x}{2}(-1) + \frac{n_x m_y}{2}(i) + \frac{n_x m_z}{2}(0)\end{aligned}$$

That is

$$\langle\Psi|n_x\sigma_{Ax}\sigma_{Bm}|\Psi\rangle = -n_x m_x$$

Again using the bra-ket notation for the other two in the following manner

$$\begin{aligned}\langle\Psi|n_y\sigma_{Ay}\sigma_{Bm}|\Psi\rangle = {} & \frac{n_y m_x}{2}(i) + \frac{n_y m_y}{2}(i^2) + \frac{n_y m_z}{2}(0) \\ & + \frac{n_y m_x}{2}(-i) + \frac{n_y m_y}{2}(i^2) + \frac{n_y m_z}{2}(0)\end{aligned}$$

Giving

$$\langle\Psi|n_y\sigma_{Ay}\sigma_{Bm}|\Psi\rangle = -n_y m_y$$

And finally

$$\begin{aligned}\langle\Psi|n_z\sigma_{Az}\sigma_{Bm}|\Psi\rangle = &\frac{n_z m_x}{2}(0) + \frac{n_z m_y}{2}(0) + \frac{n_z m_z}{2}(-1)\\ &+\frac{n_z m_x}{2}(0) + \frac{n_z m_y}{2}(0) + \frac{n_z m_z}{2}(-1)\end{aligned}$$

Resulting in

$$\langle\Psi|n_z\sigma_{Az}\sigma_{Bm}|\Psi\rangle = -n_z m_z$$

Using these together it can be argued that

$$\langle\Psi|\sigma_{An}\sigma_{Bm}|\Psi\rangle = -n_x m_x - n_y m_y - n_z m_z = -\hat{n}.\hat{m} \tag{3.34}$$

for any given $\hat{n}$ and $\hat{m}$. See that this is in accord with eqn. 3.33 since $\hat{n}.\hat{n}$ gives 1. Now define $S_{An} = \pm 1$ and the rest, where the difference between σ_n and S_n being $S_n \equiv S(\lambda, \hat{n})$ while $\sigma_n \equiv \sigma(\bar{n})$ Following the most basic form of CHSH inequality (usually derived from and attributed to [35, 36]), which argues the statement below

$$-2 \leq \langle (S_{An} + S_{Am})S_{B\ell} + (S_{An} - S_{Am})S_{Bp}\rangle \leq 2 \tag{3.35}$$

where A and B are two particles and $\hat{n}, \hat{\ell}, \hat{m}$ and $\hat{p}$ are arbitrarily chosen directions of measurement. For a certain set of directions(Fig. 3.1) it can be shown that the inequality 3.34 is violated.

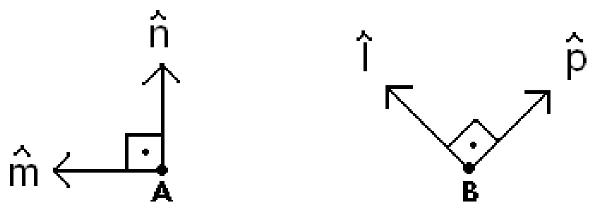

Figure 3.1: Directions of measurements for maximal violation

For the following set of measurements,

$$\langle\sigma_{An}\sigma_{B\ell}\rangle = -\hat{n}.\hat{\ell} = \frac{1}{\sqrt{2}} \qquad \langle\sigma_{Am}\sigma_{B\ell}\rangle = -\hat{m}.\hat{\ell} = \frac{1}{\sqrt{2}}$$

$$\langle\sigma_{An}\sigma_{Bp}\rangle = -\hat{n}.\hat{p} = \frac{1}{\sqrt{2}} \qquad \langle\sigma_{Am}\sigma_{Bp}\rangle = -\hat{m}.\hat{p} = -\frac{1}{\sqrt{2}}$$

With $\hat{n} \perp \hat{m}$ and $\hat{\ell} \perp \hat{p}$ while $\theta = \frac{\pi}{4}$ and $\phi = \frac{3\pi}{4}$ where θ is the angle between $\hat{m}$ and $\hat{\ell}$ while ϕ is the angle between $\hat{m}$ and $\hat{p}$. Since the system is rotationally invariant, the only dependence is to the angles between the directions and not to the directions themselves. Using these directions, the maximum value of $2\sqrt{2}$ allowed by QM can be obtained.

3.3 MABK Inequalities

In this section Mermin-Ardehali-Belinski-Klyshko inequalities [40, 41, 42], which are usually summarized as MABK, will be of the focus. Since there is a vast and mostly contemporary literature on the possible experimental realizations of this type of inequalities, with additions and refutations from many sources, the main point of this section is not to study these exhaustively but to explain their relevance to the Bell's theorem. Through this context, a brief derivation and explanation of the correspondence between them is going to be investigated and their categorical understanding as the general class of multipartite Bell inequalities will be examined.

Mermin's Inequality

Following the GHZ[6] era, Mermin, whom was a prominent figure already in Bell's theorem related literature, published an article titled "Extreme Quantum Entanglement in a Superposition of Macroscopically Distinct States"[40]. The term 'macroscopically distinct' here is a particularly important term to be noticed. More on this term and the concept of macro-realism will be discussed in Sec 4.1. Furthermore, the GHZ approach, which is a use of Bell's theorem without inequalities, will be the focus of Sec 5.1.

In Mermin's article a Bell-type inequality is derived for a state of n spin-1/2 particles which are in a superposition state much like the GHZ state for n = 3 case. This is the first noticeable and realizable generalization of Bell's inequality to n-partite systems in the literature, hence it is an important development in the field. It is a rather straightforward demonstration, using distribution and correlation functions instead of expectation values to address the cases where the measurements are imperfect and the extreme values are failed to be attained. Main difference from the GHZ approach is the emphasis on realizable observations with imperfections.

The demonstration starts with the definition of the basic state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[|00..0\rangle + i\,|11..1\rangle] \tag{3.36}$$

with $|0\rangle$ corresponding to upward spin in $\hat{z}$-direction and $|1\rangle$ corresponds to downward spin in $\hat{z}$-direction. And the operator $\hat{A}$ is defined in the following way;

$$\hat{A} = \frac{1}{2i}[\prod_{j=1}^{n}(\sigma_x{}^j + i\sigma_y{}^j) - \prod_{j=1}^{n}(\sigma_x{}^j - i\sigma_y{}^j)] \tag{3.37}$$

where $|\Phi\rangle$ is an eigenstate to it with eigenvalues 2^{n-1}. Expanding this product will provide

$$\begin{aligned} 2^{n-1} &= \langle\Phi|\sigma_y{}^1\sigma_x{}^2...\sigma_x{}^n|\Phi\rangle + \langle\Phi|\sigma_x{}^1\sigma_y{}^2...\sigma_x{}^n|\Phi\rangle + .. + \langle\Phi|\sigma_x{}^1\sigma_x{}^2...\sigma_y{}^n|\Phi\rangle \\ &- \langle\Phi|\sigma_y{}^1\sigma_y{}^2\sigma_y{}^3\sigma_x{}^4...\sigma_x{}^n|\Phi\rangle - ... + \langle\Phi|\sigma_y{}^1\sigma_y{}^2\sigma_y{}^3\sigma_y{}^4\sigma_y{}^5...\sigma_x{}^n|\Phi\rangle + ... \end{aligned} \tag{3.38}$$

The above expression represents a set of expectation values for experiments to be conducted. This form of the expansion is due to the ‘i’ factor coming from the Pauli matrix of σ_y. Since there is a $\frac{1}{2i}$ in front of the expansion the sign is of the order i^{m-1}, where m is the number of σ_y’s in that term of the expansion which is always odd due to the cancellation of even σ_y numbered terms. The total number of the terms in 3.38 is given as

$$\sum_{j\ odd}\binom{n}{j} = 2^{n-1}$$

This is obtained through the following steps;

$$\left.\begin{aligned} (1+1)^n &= \sum_j \binom{n}{j} \\ (1-1)^n &= \sum_j \binom{n}{j}(-1)^j \end{aligned}\right\} \rightarrow \sum_{j\ odd}\binom{n}{j} = \frac{1}{2}[(1+1)^n - (1-1)^n] = 2^{n-1}$$

Since each term must lie between -1 and +1 to obtain the equality of 2^{n-1} in 3.38 each term has to have its extreme values. This gives that $|\Phi\rangle$ must be an eigenstate of the operators appearing in every term. Mermin shows that this idea alone is a refutation of EPR argument due to the commutation relations of σ_x and σ_y elements in these terms.

This refutation takes place in his earlier article titled "Quantum Mysteries Revisited" [43] where he applies a pedagogical and non-technical method he previously used on EPR problem to the GHZ experiment.

As the paper at hand does not deal with extreme values but the resulting distribution functions of imperfect measurements, Mermin introduces correlation functions $E_{p_1...p_n}$ where $\langle\Phi|{\sigma_{p_1}}^1...{\sigma_{p_n}}^n|\Phi\rangle$ fails to attain the extreme values ±1. Now it is inquired that whether the measured distribution functions $P_{p_1...p_n}(m_1, m_2, ...m_n)$ (with each s_j either being x or y and each m_j being + in $\hat{z}$-direction or - in $\hat{z}$-direction) that describe the outcomes of the 2^{n-1} different kinds of experiments that must be performed on n particles in the state $|\Phi\rangle$ to obtain the correlation functions yielded in 3.38 can be all represented in the conditionally independent form as follows

$$P_{p_1...p_n}(m_1, m_2, ...m_n) = \int d\lambda\rho(\lambda)[{p^1}_{p_1}(m_1, \lambda)...{p^n}_{p_n}(m_n, \lambda)] \tag{3.39}$$

Mermin takes this form of representation as a hallmark of a local theory that accounts for the correlations entirely in terms of information jointly available to the particles when they left their common source[40]. The set of parameters λ is common to all n particles, with distribution $\rho(\lambda)$, is subject only to the requirement that the outcome of any one detector for given λ does not depending on the choice of component to be measured at any of the other detectors, or in other words that correlation functions which can be represented in this form obeys local causality introduced by Bell[16].

If a representation 3.39 is accepted, then the mean of a product of x or y components for the spins of all particles can be given in the following manner

$$E_{p_1...p_n} = \int d\lambda\rho(\lambda){E_{p_1}}^1(\lambda)...{E_{p_n}}^n(\lambda) \tag{3.40}$$

where the correlation function is given as

$${E_s}^j(\lambda) = {p_p}^j(+, \lambda) - {p_p}^j(-, \lambda) \tag{3.41}$$

Now, the theoretical value of the linear combinations of experimentally measured correlation functions from the expansion 3.38 of 3.37 is given as

$$F = \int d\lambda\rho(\lambda)\frac{1}{2i}[\prod_{j=1}^{n}({E_x}^j + i{E_y}^j) - \prod_{j=1}^{n}({E_x}^j - i{E_y}^j)] \tag{3.42}$$

The quantum mechanical expectation value of $\hat{A}$ corresponding to this is

$$F = \langle\Phi|\hat{A}|\Phi\rangle = 2^{n-1} \tag{3.43}$$

However in 3.42 a much more strict bound can be formulated. Each of the 2n quantities ${E_x}^j, {E_y}^j$ appearing in 3.42 is constrained by 3.41 to the region in between -1 and +1. Since the integrand of 3.42 is linear in each ${E_s}^j$ (if the other 2n-1 is held fixed), it assumes its extreme values only at the boundaries of each of their domains. It is therefore bounded everywhere by the largest of the extreme values it takes at where the points ${E_x}^j$ and ${E_y}^j$ is independently taken to be -1 or +1. Since 3.42 is equivalent to

$$F = Im[\int d\lambda\rho(\lambda)\prod_{j=1}^{n}({E_x}^j + i{E_y}^j) \tag{3.44}$$

at the extreme points F can be taken as just the imaginary part of an average of the product $\prod_{j=1}^{n}({E_x}^j + i{E_y}^j)$ with complex numbers each with $\sqrt{2}$ magnitude and phases of the form $\pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$. For even n this product can lie along the imaginary axis with value $\sqrt{2^n}$ and for odd n it is at $\frac{\pi}{4}$ degrees to the imaginary axis and its Im(..) can only attain the maximum value of $\sqrt{2^{n-1}}$. Hence if F can be represented in 3.42 then

$$F \rightarrow \begin{cases} F \leq 2^{\frac{n}{2}}, n \quad \text{even} \\ F \leq 2^{\frac{n-1}{2}}, n \quad \text{odd} \end{cases} \tag{3.45}$$

relations should hold. However when $n \geq 3$ these bounds are violated. In the appendix section B.3.1 an explicit demonstration for n = 2 and n = 3 particle systems are studied. It is shown in [44] that the maximal violation of Mermin's inequality can only be obtained for GHZ states and the states obtained from them by local unitary transformations, hence other maximally entangled states such as W states does not maximally violate this inequality.

The important point of Mermin's inequality is that its violation increases exponentially with number of particles. While n increases linearly, violation of the bounds increases in the order of $2^{\frac{n}{2}}$, hence for macroscopic systems it gets significant values.

In Ardehali's addition[41] to Mermin's original paper, a lemma of the following sort is introduced:

Lemma: If u, u', v and v' are random variables having probability distribution function P(u, v, u', v'), then the following elementary relation always holds[41]:

$$\sum P(u,u',v,v')[u(v-v')+u'(v+v')] \leq 2max\{|u|,|u'|\}max\{|v|,|v'|\} \quad (3.46)$$

instead of random variables here using the correlation functions from 3.41 will yield the same results with 3.45.

As a final point to this section, in Belinski and Klyshko's extensive study of 1993 named "Interference of light and Bell's theorem" [42] the fifth section is titled as "Bell Theorem For N Observers". They refer to an extensive literature on the subject mainly between 1989 and 1992 such as GHZ[6], Mermin[40] and other studies of that period [45, 41, 46], highlighting the increasing interest on the subject. Authors argue that for Mermin's inequality at n=2 and n=3 gives familiar results with previously studied subjects however as N $\rightarrow \infty$ a new quantum effect can appear and they study this through introduction of a hypothetical photon interference experiment. Mathematical results derived from their work is much similar to Mermin's original derivation, however since there is a physical setup introduced the notation and corresponding elements such as the phase or visibility of interference requires additional assumptions to the problem at hand.

Combined together these demonstrations are called as MABK inequalities. Term is generally used for non-GHZ tripartite and N > 3 cases. Important point of these type of Bell inequalities is generally taken to be as their potential to take the experimental verifications of no-local-hidden variable theory type of tests to a macroscopic level.

CHAPTER 4

OTHER TYPES OF INEQUALITIES

Till this chapter the development of Bell's theorem and Bell-type inequalities on the arguments introduced by EPR, and later developed by Bohm, has been the focus of the study. Generalization of Bell-type inequalities to further cases by Mermin and other parties has been investigated, several derivations re-stated and the core issues of EPR problem such as locality, realism and completeness has been introduced. As mentioned earlier, Bell's theorem is considered by some as the opening shot of 'experimental metaphysics' [3], however as expected it does not stand alone. In this chapter two other approaches to the presumably metaphysical problems through physical experimentation will be investigated. The problem of 'macrorealism' vs 'microrealism' in section 4.1 and the issue of contextuality in section 4.2.

Problematization of these subjects were already considered long before John Bell, in the early days of quantum revolution. Schrödinger's cat [11] is a strong motivation for Leggett's pursuit of his 'program' [47] and the roots of Kochen-Specker theorem lies in de Broglie's pilot wave and von Neumann's refutation of it. However the importance of these two approaches does not lie only within the problems that they consider but also the insights they provide to tackle them. Kochen and Specker were mathematicians hence their insights on the subject of contextuality did allow new outlooks on the structure of Hilbert space on which quantum mechanics is built upon. And Anthony Leggett's background on both experimental and philosophical matters of physics allowed him to be in a quite unique position to both assert highly philosophical (hence strongly logical) and up-to-date experimentally aware propositions to the subject at hand.

4.1 Leggett-Garg Inequality

The focus of this section will be what is generally referred to as Leggett-Garg Inequalities(or LGIs). These types of inequalities were developed by Anupam Garg and Anthony J. Leggett in 1985 to distinguish theories supporting macrorealism(MR) from quantum mechanics(QM) [4]. In this section key points of the Leggett-Garg's original paper will be presented, together with their response to some immediate critism. Later a somewhat comprehensive and contemporary understanding of what this type of inequalities do correspond to is going to be investigated through passages and highlights from Leggett's own reviews[48, 49], comments from his own summer lecture series at the Institute for Quantum Computing, University of Waterloo[47] and several other review articles [50, 51]. The LGIs are closely related to highly experimental objects such as SQUIDs, fullerenes, magnetic biomolecules, quantum-optical systems and so on, although these subjects will not be investigated a brief literature review on the experimental realization of LGIs can be found at the end of this section.

4.1.1 "Is the flux there when nobody looks?" paper

The original paper by Anthony J. Leggett and Anupam Garg dated 1985, where the LGI experiment is introduced has the title "Quantum Mechanics versus Macroscopic Realism: Is the Flux There when Nobody Looks?" [4]. In this paper an idealized 'macroscopic quantum coherence' experiment is introduced with the conjunction of two general assumptions which are,

(S1) Macroscopic Realism: A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states.

(S2) Noninvasive measurability at the macroscopic level: It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics.[4]

From later references, the assumption of macroscopic realism simply states that a macroscopic object which has available to it two or more macroscopically distinct states is at 'almost all' times in one of these states. The 'almost all' statement here is important due to the fact that there are transition periods between these macroscopically distinct states, however it is also noted in [4] that effects caused by these transitions are smaller than the relevant ones by several orders of magnitude hence can be ignored for any and all experimental purposes.[49].

For (S2), which is referred to as NIM(non-invasive measurability), Leggett denotes that this assumption is not a quantum mechanical assumption in itself, hence the violation of any LGI can be understood as the falsification of NIM rather than MR. However, he adds that any proposition of a MR theory without the assumption or the possibility of NIMs seems unlikely to him [47].

The original paper argues that for an isolated SQUID, QM predicts that if the flux is initially in one well, it will oscillate back and forth with some frequency ω_0. Below figure can be considered to investigate the situation at hand,

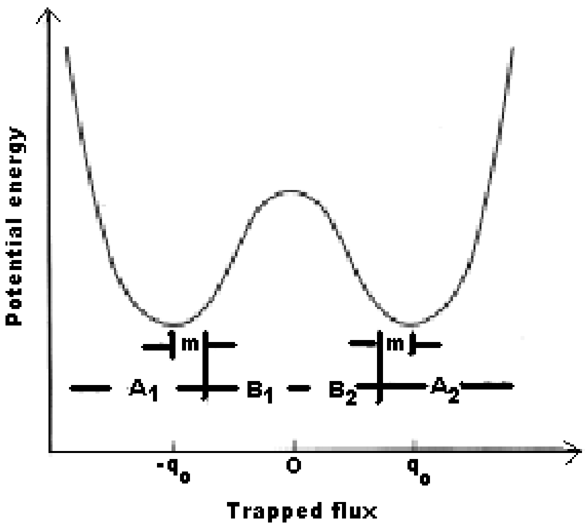

Figure 4.1: SQUID oscillation

As shown in Fig. 4.1 it is possible to divide the values of the trapped flux into four regions A_1, B_1, B_2 and A_2. Define a quantity Q which equals to +1 if the system is observed to be in region A_2 and -1 if observed in A_1, for now the probabilities of finding the system in B_1 or B_2 are ignored since they are minuscule. It follows from (S1) that for a system prepared in time t_0,

(i) joint probability densities $\rho(Q_1, Q_2), \rho(Q_1, Q_2, Q_3)$ and so on, can be defined for Q to have values Q_i at times t_i (for $t_0 < t_1 < t_2 ...$)

(ii) it is possible to argue correlation functions of the form $K_{ij} = \langle Q_i Q_j \rangle$

These joint probability densities should be consistent hence

$$\sum_{Q_2=\pm 1} \rho(Q_1, Q_2, Q_3) = \rho(Q_1, Q_3) \tag{4.1}$$

From just this, considering times t_i corresponding to polarizer settings, Bell-type inequalities such as

$$1 + K_{12} + K_{23} + K_{13} \geq 0 \tag{4.2}$$

or

$$|K_{12} + K_{23}| + |K_{14} - K_{24}| \leq 2 \tag{4.3}$$

can be constructed. Since $Q_i = \pm 1$, $K_{ij} = \langle Q_i Q_j \rangle$ will provide $|K_{ij}| \leq 1$. Assume $Q_1 = +1$, then there are four possible time development scenarios of the form;

$$\begin{aligned}
Q_1 &= +1 \quad Q_2 = +1 \quad Q_3 = +1 \\
Q_1 &= +1 \quad Q_2 = -1 \quad Q_3 = +1 \\
Q_1 &= +1 \quad Q_2 = -1 \quad Q_3 = -1 \\
Q_1 &= +1 \quad Q_2 = +1 \quad Q_3 = -1
\end{aligned} \tag{4.4}$$

giving the K_{ij} values below.

$$\begin{aligned}
K_{12} &= +1 \quad K_{23} = +1 \quad K_{13} = +1 \\
K_{12} &= -1 \quad K_{23} = -1 \quad K_{13} = +1 \\
K_{12} &= -1 \quad K_{23} = +1 \quad K_{13} = -1 \\
K_{12} &= +1 \quad K_{23} = -1 \quad K_{13} = -1
\end{aligned} \tag{4.5}$$

It is clear to see that these values do satisfy the inequality constructed in eqn. 4.2. Similar scenarios can be considered for $Q_1 = -1$ case as well. Following the same line of thought a fourth Q value as Q_4 can be defined and through similar steps the inequality in eqn. 4.3 can be obtained.

Violation of these inequalities will falsify the assumptions (S1) and (S2) which led to their construction. However in this form alone LGI has many more assumptions than just (S1) and (S2) and the original paper continues for several pages just to explain the relationships between dissipation patterns and tunneling frequency for underdamped or heavily damped conditions.

To overcome the problem of NIM, Leggett-Garg introduces the idea of a ideal negative result experiment or ideal negative measurement. They do not argue that this is the only possible non-invasive measurement type but it is a type of NIM, which has its counter-arguments standing as well [52, 47]. The ideal negative measurement scheme follows the line of thought that constructing a device which only interacts with the system if Q has +1 value and not interacting with Q = -1, or vice versa, can be considered as a non-invasive measurement. Most of the critism on this is based upon the fact that the state collapse occurs in both cases, however Leggett answers that this (S2) is not a quantum mechanical assumption.

One of the criticism to this paper came in 1988[52] directly arguing the possibility of NIM and the macroscopic nature of a SQUID. Peres noted that the amount of ΔX lost during the measurement must be gained as ΔP due to the uncertainty principle, hence the 'arbitrarily small perturbation on its subsequent dynamics' part of (S2) does not hold. In their response [53] authors formulate a measure ζ of 'invasiveness' and show that it can be made of the order 10^{-7}, which is far too small to spoil the projected experiment [53].

4.1.2 Temporal Bell Inequalities

LGIs are sometimes also referred to as temporal Bell inequalities as well, since original Bell inequalities use space-like separated regions and LGIs use temporally separated regions. In [47], using this approach, a much more clear and comprehensive version of the above derivation can be found. First of all, it is important to note that after 2008 [49] a third assumption is added to (S1) and (S2) which reads as,

(S3) Induction: Properties of an ensemble are determined exclusively by initial conditions and not what will happen in a later time

Leggett introduces this assumption while noting that further developments in physics may come from debating over the concept of 'arrow of time', however for any MR theory that can be formulated in current or contemporary physics (S3) is a reasonable and a 'should be' assumption to be made.

Take a two-state system which can be considered as the generalized version of the one argued in [4]. Divide this ensemble to three sub-ensembles which consists of,

(E1) The quantity Q is measured at t_1 and t_2

(E2) The quantity Q is measured at t_2 and t_3

(E3) The quantity Q is measured at t_1 and t_3

for $t_1 < t_2 < t_3$. For the experimental average of these sub-ensembles take;

$$\langle Q(t_1)Q(t_2)\rangle_{exp} = \langle Q_1 Q_2\rangle_{exp}$$

$$\langle Q(t_2)Q(t_3)\rangle_{exp} = \langle Q_2 Q_3\rangle_{exp}$$

$$\langle Q(t_1)Q(t_3)\rangle_{exp} = \langle Q_1 Q_3\rangle_{exp}$$

respectively for (E1), (E2) and (E3). And define the correlation function K_{exp} as

$$K_{exp} = \langle Q_1 Q_2\rangle_{exp} + \langle Q_2 Q_3\rangle_{exp} - \langle Q_1 Q_3\rangle_{exp} = K_{exp}(t_1, t_2, t_3)$$

Now assume a simple two state Hamiltonian[47, 49] of the form:

$$\hat{H} = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}$$

which will provide

$$\langle Q(t_i)Q(t_j)\rangle = \cos\Omega(t_i - t_j) \tag{4.6}$$

It should be noted here that for the equation above, general state before time t_i is rather irrelevant. Now the question asked by Leggett is that ‘What does MR predict?’. Answer to this question can be given in several steps. First of all, by (S1) it can be argued that at any given time, quantities Q_1, Q_2, Q_3 simultaneously exist (whether measured or not) and take values ± 1. Building on that and remembering the arguments which led to eqn. 4.2 the following can be constructed by using simple algebra.

$$Q_1 Q_2 + Q_2 Q_3 - Q_1 Q_3 \leq 1 \tag{4.7}$$

Hence for a single ensemble

$$\langle Q_1 Q_2\rangle_{ens} + \langle Q_2 Q_3\rangle_{ens} - \langle Q_1 Q_3\rangle_{ens} \leq 1 \tag{4.8}$$

By using above eqn.s 4.7 and 4.8 if measurements are non-invasive then $\langle Q_i Q_j \rangle_{ens} = \langle Q_i Q_j \rangle_{exp}$. Thus any theory of MR class predicts the Bell-like inequality of the form:

$$K_{MR} \leq 1 \tag{4.9}$$

This is known as 'Temporal Bell Inequality' [47] and is violated by QM for appropriate choices of time. For example, taking the time intervals $t_3 - t_2 = t_2 - t_1 = \frac{\pi}{3\Omega}$ gives $K_{QM} = \frac{3}{2}$, which clearly violates the above given inequality.

Here Leggett highlights some *a priori* objections to the program and their counter arguments. First objection, he notes, is that for 'Macroscopic' objects the action S is of the scale $S \gg \hbar$ hence the predictions of QM will reduce to those of CM(Classical Mechanics). His counter argument to this objection is to use such systems which have microscopic energy levels but have a reasonably macroscopic variable embedded in it. He gives SQUIDs as a notable example [47, 49, 48].

Second objection to the program he notes as 'this program is totally ridiculous since decoherence will kill you stone dead' [47]. He says that this objection mainly depends on pre-2000 literature where the effect of decoherence was over-estimated. Although macroscopic objects do have closely spaced energy levels most of these states are irrelevant to the question of superposition, for example in SQUIDs there exist many degrees of freedom but most of them are irrelevant to the interference effect that would have been observed with experiments.

The final objection he highlights is asymmetry of the program. He means by this that if such an experiment supports QM it can clearly be argued that MR should let go, however if such an experiment supports MR than an argument can always be made that writing a Hamiltonian or a Lagrangian for macroscopic objects is a much more complicated process than what has been anticipated [47]. In this matter he answers with the following line of logic.

If theory T predicts experimental consequence E, and that experiment finds E, this *does not* establish that T is correct.

However, if theory T predicts experimental consequence E, and for that experiment 'not-E' is found, this does establish the falsity of T. It is also noted by Leggett[49] that concepts such as realism cannot be brought into questioning alone, hence only a set of propositions which uses the argument of realism can be refuted but never the concept realism by itself.

Before finishing up this section, another standard form[51] of the LGIs should be mentioned. For the known ± 1 valued Q_is the following form can easily be formulated.

$$-1 \leq \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle + \langle Q_1 Q_3 \rangle \leq 3 \tag{4.10}$$

The upper bound 3 cannot be exceeded since 4.6, however the lower bound -1 can be violated through appropriate choice of $t_i - t_j$ sets.

Further Remarks on Leggett-Garg Inequalities

Leggett-Garg Inequalities, in contrast to Bell inequalities, are still a contemporary subject in progress. There are papers, as recent as March 2015[54], developing the current form of the inequalities. Using the guidelines of [50] there can be found 13 experimental ideas developed by different teams just in between 2010 and 2014, the context of these tries will be mentioned shortly. References with respect to used physical objects and measurement methods can be found in the table[50] below.

Recent developments (after 2000) in condensed matter theory and advancing experimental capabilities can be considered to be the most important factors of the increasing studies in LGIs. But also interest from fields even like biophysics, to explore the possible quantum effects in bio-molecular processes (such as photosynthesis [55, 56, 57] and bio-systems in general [58]) acts as a propellant for the physicists to push towards the frontier between macro and micro systems with tenacity.

Table 4.1: Experimental violations of LGI

Physical object	Measurement method	Reference
superconducting qubits	CWM	[59]
	W/SW	[60]
nitrogen-vacany centre	STAT	[61]
	W	[62]
NMR(nuclear magnetic resonance)	P	[63, 64]
	INM	[65]
photons	W/SW	[66, 67, 68]
	P	[69]
$Nd^{3+} : YVO_4$ crystal	STAT	[70]
phosphorus impurties in silicon	INM	[71]

The abbreviations used for the measurement methods can be identified as:

P: Projective; INM: ideal negative measurement; STAT: stationarity

CWM: Continuous weak measurement; W/SW: weak/semi-weak measurement

Most of the experiments given in [50] test variations of LGIs, however the authors note that these systems can hardly be considered as macroscopic hence the violations are not unexpected.

It is noted by Leggett[72] that the only actual experiment which claims to have tested the Leggett-Garg inequality at the macroscopic level is Palacios-Laloy et al., Nature Physics 6,442 (2010)[59]. However he adds that this is problematic because of their 'weak measurement' technique. 'Weak measurement' can be used for purposes of establishing that everything is consistent with QM, but refuting macrorealism through measurements with this technique is open to criticism since the analysis far too heavily depends on QM itself. He notes that most of the experiments done at the microscopic level such as [66] also shares the same fate and distinguishes [71] as the only one to his knowledge that those not suffer from this defect. But there are attempts at devising macroscopic analog of that experiment which is hoped to eliminate the problems encountered in previous experiments.

4.2 Kochen-Specker Theorem and KCBS Inequality

In this section, the arguments of Simon B. Kochen and Ernst Specker[5] and a contemporary inequality derived from it [73] will be of the focus. This approach is widely referred to as Kochen-Specker Theorem and deals with the concept of contextuality. In the literature there are two different naming of this approach as BKS theorem or KS theorem corresponding to Bell-Kochen-Specker or Kochen-Specker alone, there are two possible reasons to consider this situation. The first one, which can be attributed to the use of BKS Theorem, is that both Bell's article "On the Problem of Hidden Variables in Quantum Mechanics"[22] and Kochen-Specker's study "The Problem of Hidden Variables in Quantum Mechanics"[5] uses Gleason's original study [27] and focus on the problem of hidden variables in quantum mechanics. BKS Theorem indicates a general method which consist of placing certain constraints on a possible theory of hidden variables, then demonstrating that these constraints are not applicable in reproducing the predictions of quantum mechanics and ruling out a class of hidden variable theories bound to the constraints introduced. In this sense, both Bell's theorem and Kochen-Specker's theorem are specific examples of BKS theorem. However it should be noted that, with respect to this use of the term, BKS theorem in itself does not specify the constraints (local causality in Bell's case and contextuality in Kochen-Specker case).

The second reason is basically the timing and language of Kochen-Specker. Although it is submitted in 1966 there is no reference or use of either Bell's papers or the argument of locality in their 29 pages long study. They exclusively focus on the possibility of embedding a partial algebra that is in accord with the predictions of quantum mechanical observables into a commutative algebra. After proving the possibility of this, authors demonstrate that although this algebra can be embedded, there will be contradictions due to the non-commutative operators in quantum theory [5]. In this context, Kochen-Specker theorem and Bell's theorem are considered to be complements of each other rather than being equivalent theorems demonstrating the same thing.

Although their approach is ingenious and original, the KS theorem was much less popular than Bell's theorem most probably due to its highly complicated and intricate geometrical structure[74] until Peres's refinement in 1990[75]. In their original study Kochen-Specker expressed the non-existence condition by using 117 vectors, which represented difficulty for any experimental applications. After Peres's paper studies along that line accelerated and finally with KCBS inequality strong experimental applications opened up. In the following parts of this section Mermin's 1990 dated article "Simple unified form for the major no-hidden-variables theorems"[74] and Cabello's 1996 dated article "Bell-Kochen-Specker theorem: A proof with 18 vectors"[76] will be used as guidelines for demonstrating KS theorem, also Peres's articles [75, 77] will be of use. For the KCBS inequality original article [73] and Cabello's 2013 'explanation' [78] is going to be examined.

4.2.1 A simple proof of Kochen-Specker theorem

The Kochen-Specker theorem indicates that for operators A, B, C with [A, B] = [A, C] = 0 and [B, C] $\neq 0$, the outcome of a measurement of A cannot be independent of whether A is measured alone, or together with B or C [22]. As their study relies on Gleason's, this theorem is an extended use of the concept Quantum contextuality, which is shown by Gleason to exists only in dimensions greater than two [27]. Contextuality provides that the measurement result of a quantum observable depends on the measurement context, that is whether the observable is measured by itself or with other commuting observables. These other measurements can be simultaneous or previous to the measurement of the relevant observable.

The original proof of Kochen-Specker uses three dimensional state space, which can be associated with the spin states of a spin-1 particle. Peres's argument instead uses four dimensional space of two spin-half particles[75]. Considering the following six operators ${\sigma_x}^1{\sigma_y}^2, {\sigma_y}^1{\sigma_x}^2, {\sigma_x}^1, {\sigma_x}^2, {\sigma_y}^1, {\sigma_y}^2$, assigning simultaneous values to these must fail if the system is in spin singlet state given in eqn. 3.32 that is,

$$|\Psi\rangle_{AB} = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{4.11}$$

Now define a function $f({\sigma_i}^j)$, for $i = x, y$ and $j = 1, 2$, which corresponds to these simultaneously assigned values. Kochen-Specker argues and defines that [5] for commeasurable observables A_1, A_2 the following holds;

$$f(A_1A_2) = f(A_1)f(A_2) \tag{4.12}$$

In this case for the spin singlet state relations below should be satisfied,

$$f({\sigma_x}^2) = -f({\sigma_x}^1) \quad \text{and} \quad f({\sigma_y}^2) = -f({\sigma_y}^1)$$

together with

$$f({\sigma_x}^1{\sigma_y}^2) = f({\sigma_x}^1)f({\sigma_y}^2) \quad \text{and} \quad f({\sigma_y}^1{\sigma_x}^2) = f({\sigma_y}^1)f({\sigma_x}^2)$$

Hence $f({\sigma_x}^1{\sigma_y}^2)f({\sigma_y}^1{\sigma_x}^2)$ can be re-written as $f({\sigma_x}^1)f({\sigma_y}^2)f({\sigma_y}^1)f({\sigma_x}^2)$ but is $f({\sigma_x}^2) = -f({\sigma_x}^1)$ and $f({\sigma_y}^2) = -f({\sigma_y}^1)$ are given so together they basically give $f({\sigma_x}^1{\sigma_y}^2)f({\sigma_y}^1{\sigma_x}^2) = 1$. But ${\sigma_x}^1{\sigma_y}^2$ and ${\sigma_y}^1{\sigma_x}^2$ commute with each other and their product is ${\sigma_z}^1{\sigma_z}^2$, which requires $f({\sigma_x}^1{\sigma_y}^2)f({\sigma_y}^1{\sigma_x}^2) = f({\sigma_z}^1{\sigma_z}^2) = 1$ however for a spin singlet state $f({\sigma_z}^1{\sigma_z}^2) = -1$ is known. Therefore simultaneous values cannot be assigned for a spin singlet state. (For detailed demonstration of above relations see Appendix sec. B.4)

The only downside of Peres's argument is that it applies to a particular state, rather than the original demonstration of Kochen-Specker, which is state-independent. Mermin proposes an advancement [74] to Peres's argument by adding three more operators to the six operators at hand, namely ${\sigma_x}^1{\sigma_x}^2, {\sigma_y}^1{\sigma_y}^2, {\sigma_z}^1{\sigma_z}^2$. For mutually commuting operators Mermin defines six constraints of the following types;

$$\begin{aligned}
f({\sigma_x}^1{\sigma_x}^2)f({\sigma_x}^1)f({\sigma_x}^2) &= 1 \\
f({\sigma_y}^1{\sigma_y}^2)f({\sigma_y}^1)f({\sigma_y}^2) &= 1 \\
f({\sigma_x}^1{\sigma_y}^2)f({\sigma_x}^1)f({\sigma_y}^2) &= 1 \\
f({\sigma_y}^1{\sigma_x}^2)f({\sigma_y}^1)f({\sigma_x}^2) &= 1 \\
f({\sigma_x}^1{\sigma_y}^2)f({\sigma_y}^1{\sigma_x}^2)f({\sigma_z}^1{\sigma_z}^2) &= 1 \\
f({\sigma_x}^1{\sigma_x}^2)f({\sigma_y}^1{\sigma_y}^2)f({\sigma_z}^1{\sigma_z}^2) &= -1
\end{aligned} \tag{4.13}$$

Multiplying all of the equations given in 4.13 the left hand side will give +1 since every term appears twice and $(\pm 1)^2 = +1$. However the right hand side will clearly result in -1, hence a contradiction is arrived at, independent of any particular state.

Further remarks on Kochen-Specker theorem

In a proof with 18 vectors by Cabello et al.[76] the general characteristics of a non-contextual hidden-variables (NCHV) theory are defined as follows

(1) In an individual system each projection operator (defined as $|u\rangle\langle u|$) has a unique answer in the form of yes or no (1 or 0) which is independent of which of the other observables are considered to being observed simultaneously (non-contextuality condition)

(2) For each set of one dimensional projection operators, in an n-dimensional Hilbert space, whose sum gives the identity operator there is only one operator with answer 1 while there are n-1 operators which give the answer 0.

According to these, if the answer to a projection operation over a given vector is one, than the rest of the orthogonal projection operators should result in zero. Using this and a notation of the form $f(u_i)$ denoting the answer to the $|u_i\rangle\langle u_i|$ operation, authors introduced a straightforward proof using nine sets of orthogonal four-dimensional vectors of the following types [76]:

$$\begin{aligned}
f(0,0,0,1)+f(0,0,1,0)+f(1,1,0,0)+f(1,-1,0,0)&=1\\
f(0,0,0,1)+f(0,1,0,0)+f(1,0,1,0)+f(1,0,-1,0)&=1\\
f(1,-1,1,-1)+f(1,-1,-1,1)+f(1,1,0,0)+f(0,0,1,1)&=1\\
f(1,-1,1,-1)+f(1,1,1,1)+f(1,0,-1,0)+f(0,1,0,-1)&=1\\
f(0,0,1,0)+f(0,1,0,0)+f(1,0,0,1)+f(1,0,0,-1)&=1\\
f(1,1,-1,1)+f(1,1,1,1)+f(1,0,0,-1)+f(0,1,-1,0)&=1\\
f(1,1,-1,1)+f(1,1,1,-1)+f(1,-1,0,0)+f(0,0,1,1)&=1\\
f(1,1,-1,1)+f(-1,1,1,1)+f(1,0,1,0)+f(0,1,0,-1)&=1\\
f(1,1,1,-1)+f(-1,1,1,1)+f(1,0,0,1)+f(0,1,-1,0)&=1
\end{aligned} \tag{4.14}$$

These vectors can be considered as pure spin states of a bipartite spin-half system. There are 18 different vectors used in this setup and 12 of those are separable and can be written in the form of $(u_1,u_2)^1 \otimes (u_3,u_4)^2$ where u_i corresponds to the eigenvalues of operators ${\sigma_x}^j$ and ${\sigma_z}^j$ applied on different particles.

Six remaining vectors are entangled and cannot be written in a separated manner. These vectors can be represented in terms of their connections with the observables ${\sigma_z}^1 \otimes {\sigma_z}^2$, ${\sigma_z}^1 \otimes {\sigma_x}^2$, ${\sigma_x}^1 \otimes {\sigma_z}^2$ and ${\sigma_x}^1 \otimes {\sigma_x}^2$. As an example, f(1, -1, 1,1) which is an eigenvector of ${\sigma_z}^1 \otimes {\sigma_x}^2$ and ${\sigma_x}^1 \otimes {\sigma_z}^2$ with values ± 1 can be associated with the question, does the operations ${\sigma_z}^1 \otimes {\sigma_x}^2$ and ${\sigma_x}^1 \otimes {\sigma_z}^2$ have well-defined outcomes with respect to a hidden variable with values -1 and +1? If the answer is yes, f(1,-1,1,1) = 1, if no, it is 0.

Considering these, it is clear to see that eqn. 4.14 cannot provide simultaneous answers to all assigned value questions since adding all nine equations yield an even number on the left hand side while giving 9 on the right hand side, which is a clear contradiction. This proof of Cabello et al. [76] is only one of the many attempts in the era to refine the argument of Peres's on the Kochen-Specker theorem. Another important point is that the above given proof is a state-independent form and for a specific state (a spin singlet state is given as an example [76]) the number of vectors required for a proof is significantly reduces. This asserts Mermin's analysis on his paper [74] that the high complexity in Kochen and Specker's original demonstration is most probably due to their success of demonstrating their proof without using a specific state.

4.2.2 KCBS Inequality

In 2008 Klyachko-Can-Binicioğlu-Shumovsky(KCBS) published an article titled "Simple Test for Hidden Variables in Spin-1 Systems" [73] discussing the non-classical nature of most spin-1 states, its physical implications, possible underlying cause and experimental verification through a Bell type inequality. They use a similar line of argument with Kochen-Specker [5] and assume a hidden distribution of variables which is compatible with the distributions of commuting observables in accord with quantum mechanics. Consider a non-negative function of the following form,

$$\sum_{A_j \text{commute}} f_j(a_j) \geq 0 \tag{4.15}$$

while assuming a hidden distribution over all variables a_i compatible with distributions of other commuting observables a_j where $i \in j$. Then taking the expectation value of this distribution leads to a Bell-type inequality.

$$\sum_{A_j \text{commute}} \langle\Psi|f_j(A_j)|\Psi\rangle \geq 0 \tag{4.16}$$

where A_j are commuting observables and $f(A_j)$ represents the hidden distribution. Their insight was to connect the marginal problem, which deals with the problem of existence of a joint probability distribution of random variables compatible with given partial distributions, with a geometrical proof. This proof is that the marginal problem[79] corresponds to the existence of a body in $\mathbb{R}^n$ of a non-negative density with some given projections onto coordinate subspaces [73]. Using this together with the constraints they obtained from CHSH [35] inequality by Fine's theorem[38], for a spin-1 system with a cyclic quintuplet of unit vectors of the form $\ell_i \perp \ell_{i+1}$ with $i = 1, .., 5$ with the use of an appropriate software they obtained a geometrical object which is now known as the KCBS pentagram[73].

Now consider the spin projection operators S_{ℓ_i}. For successive indices the following holds,

$$[{S_{\ell_i}}^2, {S_{\ell_{i+1}}}^2] = 0 \tag{4.17}$$

due to orthogonality of $\vec{\ell_i}$ and $\vec{\ell}_{i+1}$. Instead of squares of spin projection operators, a new observable of the form deemed more appropriate and formulated as,

$$A_i = 2{S_{\ell_i}}^2 - 1$$

which has eigenvalues ± 1. Using this the following inequality is obtained similar to the form of CHSH inequality,

$$\langle A_1 A_2 \rangle + \langle A_2 A_3 \rangle + \langle A_3 A_4 \rangle + \langle A_4 A_5 \rangle + \langle A_5 A_1 \rangle \geq -3 \tag{4.18}$$

and by using the relation below

$$A_i A_{i+1} = 2{S_{\ell_i}}^2 + 2{S_{\ell_{i+1}}}^2 - 3 \tag{4.19}$$

this later is recast into the following form,

$$\langle {S_{\ell_1}}^2 \rangle_\Psi + \langle {S_{\ell_2}}^2 \rangle_\Psi + \langle {S_{\ell_3}}^2 \rangle_\Psi + \langle {S_{\ell_4}}^2 \rangle_\Psi + \langle {S_{\ell_5}}^2 \rangle_\Psi \geq 3 \tag{4.20}$$

which is called the pentagram inequality. The relation in eqn. 4.19 is obtained through using this relation,

$$A_{\ell_i} = I - 2 \left| \ell_i \right\rangle \left\langle \ell_i \right| = 2{S_{\ell_i}}^2 - I \tag{4.21}$$

where $|\ell\rangle$ corresponds to $|0\rangle_\ell$ in Hilbert space representation of a spin-1 particle.

Hence the pentagram inequality can be written in the form[73];

$$\sum_{k=1}^{5} | \langle \ell_k | \Psi \rangle |^2 \leq 2 \tag{4.22}$$

The neutrally polarized spin-1 state gives $|\langle \ell_k|\Psi\rangle|^2 = \cos^2(\ell_k\Psi) = \frac{1}{\sqrt{5}}$ which violates the above inequality.

KCBS inequality reduces the number of involved spin projection operators from 31 to 5 [73] for the contextuality tests, however it is state dependent. The authors argue that there cannot be a test of non-contextual hidden variables (NCHV) for three dimensional quantum systems with less then 5 observables and that other possible tests involving 5 observables can all be reduced to the pentagram inequality.

After the publication of their paper, experimental verifications of KCBS inequality has started being tested. A brief summary of the successful experimental implementations can be found below:

Table 4.2: Experimental violations of KCBS Inequality

Physical object	Year	Reference
photons	2009	[80]
	2011	[81]
	2012	[82]
	2013	[83]
	2014	[84, 85]
	2015	[86]
$^{171}Yb^{+}$ ions	2013	[87, 88]

CHAPTER 5

BELL'S THEOREM WITHOUT INEQUALITIES

In this chapter an alternative approach to the Bell's theorem will be presented. Bell's theorem is usually considered as constructing inequalities using local hidden variables to show contradictions in order to demonstrate the falsification of LHV. There are several examples and alternative approaches to the EPR problem, however this chapter will consist of a method introduced by Greenberger, Horne and Zeilinger around 1989-1990[6] which is widely known as the GHZ experiment. In section 5.1 the history, development and indications of solely the GHZ experiment will be investigated. Another method known as Hardy's Test, which uses the GHZ logic but deviates from it in a profound manner, will be of the focus in section 5.2.

5.1 GHZ Experiment

The original introduction of GHZ dates to a workshop held on October 21 and 22, 1988 at George Mason University titled "Bell's Theorem, Quantum Theory and Conceptions of the Universe" and the citation in general referred to as is the original publication of GHZ argument is a book titled the same with the workshop edited by Menas Kafatos in 1989 [6] and the relevant section of it has the title "Going Beyond Bell's Theorem". A more well-known version of the argument is published in 1990 with Shimony joining the authors for an extensive and self-contained review which has the title "Bell's theorem without inequalities"[89].

In this section both the original 1989 and the extended 1990 version of the demonstrations will be revisited and their relevance to the Bell's theorem is going to be highlighted by comparing some of the arguments with the derivation of Bell's from section 3.1. Although it is a rather straightforward demonstration the importance of the original idea lies on the emphasis that for bipartite systems Bell inequalities hold for perfect correlations, however for N-partite systems with $N \geq 3$ they do not. Just by using Bell's original assumptions on the EPR problem it is possible to show that contradiction arises between a local hidden variable(LHV) theory and quantum mechanical predictions.

A similar line of argument can be developed for GHZ states using the Kochen-Specker theorem instead of Bell's theorem and the argument of contextuality, that is assigning simultaneous values to non-commuting observables will result in contradictory outcomes. The applicability of both theorems to the GHZ states provides additional insight and opens up the possibility of a unified no-go theorem for hidden variable theories[74].

5.1.1 Going Beyond Bell's Theorem

Remember that in the singlet state eqn. 3.4

$$\langle(\sigma_1.\hat{a})(\sigma_2.\hat{b})\rangle = -\hat{a}.\hat{b} \tag{5.1}$$

which can be re-written in the form

$$P(a,b) = \langle(\sigma_1.\hat{a})(\sigma_2.\hat{b})\rangle = -\hat{a}.\hat{b} = -cos(a.b) \tag{5.2}$$

and leading to the famous Bell inequality of the form 3.5

$$|P(a,b) - P(a,c)| \leq 1 + P(b,c) \tag{5.3}$$

GHZ notes that for the case $\hat{a} = \pm\hat{b}$ this inequality gives no information at all. They call this the super-classical case[6]. The question they asked was that is it always possible to find a classical model for the super-classical cases and the answer was no. However this could not be demonstrated through bipartite spin-singlet state used in deriving Bell's theorem so they had to consider a more complex consisting four spin-half particles in a state that is similar to Bohm's.

$$|\Phi\rangle = \frac{|0011\rangle - |1100\rangle}{\sqrt{2}} \tag{5.4}$$

Through a similar reasoning with eqn. 5.2 for the quantum mechanical expectation value for the spins in four given directions can be denoted in the following manner

$$P(n_1, n_2, n_3, n_4) = -cos(\alpha + \beta - \gamma - \sigma) \tag{5.5}$$

where each these n_i is taken to be in the xy-plane at angles $\alpha, \beta, \gamma, \sigma$ respectively. If any of the two angles are fixed the other two behaves similarly with the bipartite case, so Bell's inequality will hold. Now for the case

$$\alpha + \beta + \gamma + \sigma = 0, \pi$$

the cosine term will result in either +1 or -1, hence measuring the three of the spins along these given directions will allow the experimenter to predict the fourth with 100 percent certainty, which is exactly the super-classical case. After introducing the λ element and proper parametrization it becomes

$$A(\alpha, \lambda)B(\beta, \lambda)C(\gamma, \lambda)D(\sigma, \lambda) = +1 \quad \text{or} \quad -1 \tag{5.6}$$

for $\alpha + \beta + \gamma + \sigma = 0$ and $\alpha + \beta + \gamma + \sigma = \pi$ cases. However this condition cannot be satisfied. Keeping two of the parameters constant and varying the other two continuously is possible hence the only logical result is A = B = C = D = constant, but this is impossible since the product sometimes equals to +1 and sometimes -1. This holds for any value of λ hence no integration is required either. The authors note that making definite predictions in the EPR sense using a classical, deterministic, local theory which reproduces quantum theory in general is impossible, however to show this one must go beyond the Bell's theorem[6].

5.1.2 Bell's theorem without inequalities

In 1990 a much more comprehensive study on the subject is published by GHSZ (Greenberger, Horne, Shimony and Zeilinger)[89]. It has several sections in which the clear links between previous study[6] and EPR program(as authors call it) is established, an extensive version of the original derivation is done and the more commonly known version of GHZ states are introduced.

Remember the initial GHZ state 5.4 and the quantum mechanical expectation value correspondence as 5.5. Now for four arbitrary directions such as $\hat{n}_1, \hat{n}_2, \hat{n}_3, \hat{n}_4$ corresponding to particles $i = 1, 2, 3, 4$, the expectation value can be written as

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle = \langle \Phi | \sigma_1 \sigma_2 \sigma_3 \sigma_4 | \Phi \rangle \tag{5.7}$$

where σ_i represents $\hat{n}_i.\vec{\sigma}$ on the ith particle. Since 5.4 is given, its hermitian conjugate is its bra version in the bra-ket notation hence

$$\langle \Phi | = \frac{\langle 0011 | - \langle 1100 |}{\sqrt{2}} \tag{5.8}$$

So that the eqn. 5.7 can be written in the explicit form as

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle = \frac{[\langle 0011 | - \langle 1100 |] \sigma_1 \sigma_2 \sigma_3 \sigma_4 [| 0011 \rangle - | 1100 \rangle]}{2}$$

which becomes

$$\begin{aligned} \langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle = \frac{1}{2} [& \langle 0011 | \sigma_1 \sigma_2 \sigma_3 \sigma_4 | 0011 \rangle - \langle 1100 | \sigma_1 \sigma_2 \sigma_3 \sigma_4 | 0011 \rangle \\ & + \langle 1100 | \sigma_1 \sigma_2 \sigma_3 \sigma_4 | 1100 \rangle - \langle 0011 | \sigma_1 \sigma_2 \sigma_3 \sigma_4 | 1100 \rangle] \end{aligned} \tag{5.9}$$

After some calculations (which can be found in appendix section B.5.1) this can be represented as

$$\begin{aligned} \langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle = & \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \cos(\theta_4) \\ - \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) & \cos(\phi_1 + \phi_2 - \phi_3 - \phi_4) \end{aligned} \tag{5.10}$$

Setting $\hat{n}_i$ to be on xy-plane will reduce this equation into

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle = - \cos(\phi_1 + \phi_2 - \phi_3 - \phi_4) \tag{5.11}$$

Now for the perfect correlation cases where $\phi_1 + \phi_2 - \phi_3 - \phi_4 = 0, \pi$ values of $P(\hat{n}_1, \hat{n}_2, \hat{n}_3, \hat{n}_4)$ will be ∓ 1. Analogous to eqn. 5.6 these can be stated as

$$\phi_1 + \phi_2 - \phi_3 - \phi_4 = 0 \rightarrow A(\phi_1, \lambda) B(\phi_2, \lambda) C(\phi_3, \lambda) D(\phi_4, \lambda) = -1 \tag{5.12}$$

and

$$\phi_1 + \phi_2 - \phi_3 - \phi_4 = \pi \rightarrow A(\phi_1, \lambda) B(\phi_2, \lambda) C(\phi_3, \lambda) D(\phi_4, \lambda) = +1 \tag{5.13}$$

Consider the following implications of these restrictions

$$A(0, \lambda) B(0, \lambda) C(0, \lambda) D(0, \lambda) = -1 \tag{5.14}$$

$$A(\phi,\lambda)B(0,\lambda)C(\phi,\lambda)D(0,\lambda) = -1 \tag{5.15}$$

$$A(\phi,\lambda)B(0,\lambda)C(0,\lambda)D(\phi,\lambda) = -1 \tag{5.16}$$

$$A(2\phi,\lambda)B(0,\lambda)C(\phi,\lambda)D(\phi,\lambda) = -1 \tag{5.17}$$

Using these together certain other implications can be obtained as well. For example eqn.s 5.14 and 5.15 will result in

$$A(\phi,\lambda)C(\phi,\lambda) = A(0,\lambda)C(0,\lambda) \tag{5.18}$$

Eqns. 5.14 and 5.16 will give

$$A(\phi,\lambda)D(\phi,\lambda) = A(0,\lambda)D(0,\lambda) \tag{5.19}$$

and using these two together (5.18 and 5.19) yields

$$\frac{C(\phi,\lambda)}{D(\phi,\lambda)} = \frac{C(0,\lambda)}{D(0,\lambda)} \tag{5.20}$$

which can be rewritten as

$$C(\phi,\lambda)D(\phi,\lambda) = C(0,\lambda)D(0,\lambda) \tag{5.21}$$

since $D(\phi,\lambda)$ is ± 1 and is equal to its inverse, same applies to $D(0,\lambda)$ as well. Combining the equations 5.17 and 5.21 gives

$$A(2\phi,\lambda)B(0,\lambda)C(0,\lambda)D(0,\lambda) = -1 \tag{5.22}$$

and comparing this with eqn. 5.14 results in

$$A(2\phi,\lambda) = A(0,\lambda) = constant \quad \forall \quad \phi \tag{5.23}$$

(a much simpler version of the above demonstration can be found in appendix section B.5.2). This result alone is physically troublesome since $A(\phi,\lambda)$ represents an intrinsic spin quantity of the first particle and should change sign as $\phi \to \phi + \pi$. Now to show the mathematical contradiction just consider the eqn. 5.13 which allows

$$A(\phi+\pi,\lambda)B(0,\lambda)C(\phi,\lambda)D(0,\lambda) = +1 \tag{5.24}$$

Also considering eqn. 5.15 the following relation holds

$$A(\phi + \pi, \lambda) = -A(\phi, \lambda) \tag{5.25}$$

however this result clearly contradicts with eqn. 5.23. Also see that since $B(\phi, \lambda)$ has been set to $B(0, \lambda)$ for the entire demonstration this method can clearly be applied to three spin-half particles case as well.

5.1.3 Kochen-Specker theorem applied to GHZ state

A brief but instructive version of the GHZ proof is given by Mermin[74] using Kocher-Specker theorem[5], advanced by Peres's argument[75]. An eight dimensional space consisting of three spin half particles is considered such as;

$$|\Psi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \tag{5.26}$$

Ten operators ${\sigma_x}^1$, ${\sigma_y}^1$, ${\sigma_x}^2$, ${\sigma_y}^2$, ${\sigma_x}^3$, ${\sigma_y}^3$, ${\sigma_x}^1{\sigma_y}^2{\sigma_y}^3$, ${\sigma_y}^1{\sigma_x}^2{\sigma_y}^3$, ${\sigma_y}^1{\sigma_y}^2{\sigma_x}^3$ and ${\sigma_x}^1{\sigma_x}^2{\sigma_x}^3$ are taken and simultaneous values are assigned to each. Using the line of argument from 4.12, five constraints on the commuting subsets of these operators can be identified as:

$$\begin{aligned}
f({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)f({\sigma_x}^1)f({\sigma_y}^2)f({\sigma_y}^3) &= 1 \\
f({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)f({\sigma_y}^1)f({\sigma_x}^2)f({\sigma_y}^3) &= 1 \\
f({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3)f({\sigma_y}^1)f({\sigma_y}^2)f({\sigma_x}^3) &= 1 \\
f({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3)f({\sigma_x}^1)f({\sigma_x}^2)f({\sigma_x}^3) &= 1 \\
f({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3)f({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)f({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)f({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3) &= -1
\end{aligned} \tag{5.27}$$

See that relations of the form $f({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)f({\sigma_x}^1)f({\sigma_y}^2)f({\sigma_y}^3)$ is in fact of the form $[f({\sigma_x}^1)f({\sigma_y}^2)f({\sigma_y}^3)]^2 = 1$ from 4.12 hence the first four of the above equations are self-evident. For the fifth one consider the operations

$$\begin{aligned}
{\sigma_x}^1{\sigma_x}^2{\sigma_x}^3\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right) &= \frac{|000\rangle + |111\rangle}{\sqrt{2}} \\
{\sigma_x}^1{\sigma_y}^2{\sigma_y}^3\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right) &= -\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)
\end{aligned} \tag{5.28}$$

and so on, each ${\sigma_y}^j$ will bringforth an i and two ${\sigma_y}^j$s will cause a minus sign. As a result the fifth relation will become (+1)(-1)(-1)(-1) = -1.

Now, by multiplying all of the relations given in 5.27 the left hand side will result in +1 since each term appears twice, but the right hand side will give -1. This is a clear contradiction, giving that for an entangled state such as GHZ the context of measurements affect the outcomes, hence assigning simultaneous values using hidden variable argument results in contradictory terms. Although Mermin derived the above relation for GHZ states, it is in fact state independent, for detailed demonstration see appendix section B.5.3.

5.2 Hardy's Test

In 1992 theoretical physicist Lucien Hardy has introduced an approach to test the validity of local realistic theories without using inequalities in an article named 'Quantum Mechanics, Local Realistic Theories, and Lorentz-Invariant Realistic Theories' [7]. He explains the main difference between his approach and the GHZ experiment as his version being "only applies to the 1/16th of the experiments whereas the GHZ result applies to every experiment" [7]. The physical system used is analogous to a spin-1/2 one but it is a generalized physical quantity with two eigenvalues. Still his thinking provides an alternative method to the GHZ and is a relatively straightforward demonstration using two particles compared to GHZ setup which requires three particles at least [6].

The original paper in 1992 consists of a setup with an electron and a positron. Using two sets of beam splitters which leads the particles to different paths where a possibility of intersection and a probability of annihilation is introduced, using the hidden variable argument of realistic theories he shows a clear contradiction between the results and the assumption of locality. However in the second paper [90] physical setup is omitted and a more formal version of the argument is presented in the form where no physical process, such as the annihilation of electron and the positron, is mentioned and the contradiction is presented only by choosing the appropriate constants for initial states. Also in the second paper [90] this test is applied to spin-1/2. Results show that the contradiction applies to every entangled state except the maximally entangled one. This point will be highlighted in the discussion part of this work at hand and a comparison with Bell inequalities will be investigated.

In the following sections both the derivations of first[7] and the second[90]papers are going to be presented. An addition by S. Goldstein will be shown[91]. A basic demonstration for a bipartite spin-half state will be given in section 5.3.1 and for a tripartite spin-half state in 5.3.2. There are further generalizations in literature such as the generalization to two spin-s particles in [92], generalization to N spin half particles in [93] and a generalization attempt to N spin-s particles by [94]. More recent developments such as a review by Ahoronov "Revisiting Hardy's Paradox: Counterfactual Statements, Real Measurements, Entanglement and Weak Values" [95], an implementation using joint weak measurement on a photon pair [96] and an experimental try of this implementation by a Japanese team [97] are also relevant study subjects for Hardy's test.

5.2.1 Derivation of Hardy's Paradox by using e^+ e^- pair

In this gedanken experiment Hardy proposed using two Mach-Zehnder-type interferometers $MZ^{\pm}$, one for positrons (MZ^+) and one for electrons (MZ^-). The paths of these particles are arranged in a manner that is shown in Fig. 5.1 at one point they may interact at a point P.

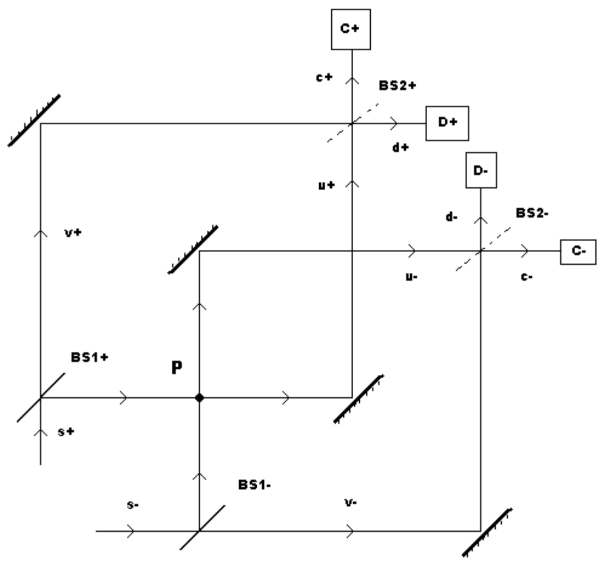

Figure 5.1: Hardy's setup

Outcome of this interaction is given as:

$$|u^+u^-\rangle \rightarrow |\gamma\rangle \tag{5.29}$$

where $|u^{\pm}\rangle$ corresponds to the states of positron and electron along that path and $|\gamma\rangle$ is the state of the radiation produced on annihilation. It is this interaction that allows the possibility of obtaining $|d^{+}d^{-}\rangle$ final state where both the electron and the positron arrive at the corresponding detectors $D^{\pm}$. Later, during the derivation of Hardy's second paper it will become clear that this setup is only one of the many possible setups where arranging the constants of given states in such manners where contradictory results appear if they are accepted as physical realities rather than distributions over probabilities.

To explicitly show the derivation, consider that an electron and a positron are set in the point of origin for the paths noted in the above figure 5.1. Given that the initial state is $|s^{\pm}\rangle$ where $|s^{+}\rangle$ represents the initial state of the positron and $|s^{-}\rangle$ representing the initial state of the electron, the first beam splitters (denoted in the Fig. 5.1 as BS1+ and BS1-) act on the state in the following manner:

$$|s^{+}\rangle \rightarrow \frac{1}{\sqrt{2}}(i\,|u^{+}\rangle + |v^{+}\rangle) \qquad \text{and} \qquad |s^{-}\rangle \rightarrow \frac{1}{\sqrt{2}}(i\,|u^{-}\rangle + |v^{-}\rangle) \tag{5.30}$$

The operation of $BS2^{+}$ is given as:

$$|u^{+}\rangle \rightarrow \frac{1}{\sqrt{2}}(|c^{+}\rangle + i\,|d^{+}\rangle) \qquad \text{and} \qquad |v^{+}\rangle \rightarrow \frac{1}{\sqrt{2}}(i\,|c^{+}\rangle + |d^{+}\rangle) \tag{5.31}$$

And the operation of $BS2^{-}$ is given as:

$$|u^{-}\rangle \rightarrow \frac{1}{\sqrt{2}}(|c^{-}\rangle + i\,|d^{-}\rangle) \qquad \text{and} \qquad |v^{-}\rangle \rightarrow \frac{1}{\sqrt{2}}(i\,|c^{-}\rangle + |d^{-}\rangle) \tag{5.32}$$

Hence by just considering these relations it is easy to argue that in the absence of $BS2^{\pm}$:

$$|u^{\pm}\rangle \rightarrow |c^{\pm}\rangle \qquad \text{and} \qquad |v^{\pm}\rangle \rightarrow |d^{\pm}\rangle \tag{5.33}$$

For a given initial state of the form $|s^{+}s^{-}\rangle$ after passing through BS1$\pm$ respectively, the state will evolve to the following form:

$$|s^{+}s^{-}\rangle \rightarrow \frac{1}{2}(i\,|u^{+}\rangle+|v^{+}\rangle)(i\,|u^{-}\rangle+|v^{-}\rangle) \rightarrow \frac{1}{2}(-\,|\gamma\rangle+i\,|u^{+}v^{-}\rangle+i\,|v^{+}u^{-}\rangle+|v^{+}v^{-}\rangle) \tag{5.34}$$

Note that this is due to eqn. 5.29 where interaction of e^{-} and e^{+} at point P leads to the annihilation of these two particles and cause a radiation which is denoted by $|\gamma\rangle$ as a state.

In the absence of both BS2$^+$ and BS2$^-$ the final state will be:

$$\frac{1}{2}(-|\gamma\rangle + i\,|c^+d^-\rangle + i\,|d^+c^-\rangle + |d^+d^-\rangle) \tag{5.35}$$

If BS2$^+$ is kept in its place while BS2$^-$ is removed, the final state would be:

$$\frac{1}{2\sqrt{2}}(-\sqrt{2}\,|\gamma\rangle - |c^+c^-\rangle + 2i\,|c^+d^-\rangle + i\,|d^+c^-\rangle) \tag{5.36}$$

Similarly, if BS2$^-$ is kept in its place while BS2$^+$ is removed, the final state would be:

$$\frac{1}{2\sqrt{2}}(-\sqrt{2}\,|\gamma\rangle - |c^+c^-\rangle + i\,|c^+d^-\rangle + 2i\,|d^+c^-\rangle) \tag{5.37}$$

And finally if both BS2$^+$ and BS2$^-$ are kept in their places:

$$\frac{1}{4}(-2\,|\gamma\rangle - 3\,|c^+c^-\rangle + i\,|c^+d^-\rangle + i\,|d^+c^-\rangle + |d^+d^-\rangle) \tag{5.38}$$

The argument of realism is introduced at this point by assuming that the state of the pair of particles is already established prior to the measurement via the hidden variable λ. Certain sets of arrangements are used in the original paper where the absence or existence of second beam splitters(BS2$^\pm$) create different scenarios. These different situations are denoted in the paper with C$^\pm$(∞,λ) noting the outcome of C$^\pm$ detectors where BS2$^\pm$ are absent and C$^\pm$(0,λ) noting where BS2$^\pm$ are present. Same goes for D$^\pm$ as well.

The important point here is that since now realism is introduced, outcomes for the positron and electron are supposed to be independent of each other. For example the value of D$^\pm$(0,λ) is to be independent of whether BS2- is in place or not. From eqn. 5.35 the following argument can be made:

$$C^+(\infty,\lambda)C^-(\infty,\lambda) = 0 \tag{5.39}$$

for every experiment since by the arrangement of paths and interaction at point P it is guaranteed that no $|u^+u^-\rangle$ final state can emerge. Similarly from eqn. 5.36:

$$D^+(\infty,\lambda) = 1 \quad \rightarrow \quad C^-(\infty,\lambda) = 1 \tag{5.40}$$

this is due to the term $|d^+c^-\rangle$ where it is the only term that contains $|d^+\rangle$ that is when the positron is detected by the detector D$^+$. And from eqn. 5.37:

$$D^-(\infty, \lambda) = 1 \quad \rightarrow \quad C^+(\infty, \lambda) = 1 \tag{5.41}$$

again this argument is based upon the fact that only $|c^+d^-\rangle$ term contains $|d^-\rangle$ which gives out a detection by the detector D$^-$. Finally from eqn. 5.38:

$$D^+(\infty, \lambda)D^-(\infty, \lambda) = 1 \tag{5.42}$$

this holds for $\frac{1}{16}$ of the experiments.

Now the contradiction is clear to see. For the cases where eqn. 5.42 is satisfied, eqn. 5.41 and 5.40 should also be satisfied. Hence 5.40 and 5.41 together gives that:

$$C^+(\infty, \lambda)C^-(\infty, \lambda) = 1 \tag{5.43}$$

This statement is the opposite with the one made in eqn. 5.39, hence providing a contradiction between local realism and quantum mechanics. The point where locality comes into importance is that if the hidden variable λ were not a local one than statements like D$^+$(0,λ) or C$^-$(∞,λ) would be impossible to claim. Rather than that, only joint measurement results should have been claimed, preventing any contradiction to arise. However the one at hand is not an all out contradiction since 5.43 only occurs for $\frac{1}{16}$th of experiments while 5.39 is valid for all of the experiments. Hardy notes that this is the main dissimilarity between his approach and GHZ result, since GHZ applies to every experiment.[7]

5.2.2 Derivation of Hardy's Paradox via two leveled generic states

The second paper of Hardy on this matter is named "Nonlocality for Two Particles without Inequalities for Almost All Entangled States" [90]. He states that the purpose of this paper was to show that the proof of non-locality he introduced can be run for any entangled state except the maximally entangled ones, this is in comparison to his previous derivation where a particular physical setup and arrangement of states were required.

In his derivation Hardy devises the generic state in a certain manner so that the outcome of $|u_1 u_2\rangle$ vanishes. This was known to be arbitrary and any other state can be set to vanish by choosing different constants. This has been shown by S. Goldstein while he generalized this approach of Hardy to suit a more complete formal structure [91].

As a start, choosing appropriate basis states $|\pm_i\rangle$ for particle i with i = 1,2 is required. These states do not correspond to any particular physical property, they could be associated to any appropriate quantity with respect to the experimental setup at hand. An entangled state with two particles can be written in the following form by Schmidt decomposition:

$$|\Psi\rangle = \alpha\,|{+}{+}\rangle - \beta\,|{-}{-}\rangle \tag{5.44}$$

where α and β are two real constants obeying normalization relation and $|{+}{+}\rangle$ and $|{-}{-}\rangle$ corresponding to the states where both particles are in $|+\rangle$ state and both particles are in $|-\rangle$ state respectively. As mentioned earlier the minus sign in between is chosen for later convenience. Now introduce another set of basis states, $|u_i\rangle$ and $|v_i\rangle$ which relates to the original basis vectors in the following manner:

$$|+\rangle_i = c_1\,|u_i\rangle + ic_2{}^*\,|v_i\rangle \tag{5.45}$$

$$|-\rangle_i = ic_2\,|u_i\rangle + c_1{}^*\,|v_i\rangle \tag{5.46}$$

see that $\langle +|-\rangle_i = ic_1{}^*c_2 - ic_2c_1{}^* = 0$ so they hold. Also for the inverse relations:

$$|u_i\rangle = c_1{}^*\,|+\rangle_i - ic_2{}^*\,|-\rangle_i \tag{5.47}$$

$$|v_i\rangle = -ic_2\,|+\rangle_i + c_1\,|-\rangle_i \tag{5.48}$$

again c_1 and c_2 satisfies the normalization relation and the new states are orthogonal to each other due to the orthogonality of the old basis states. Substituting eqn.s 5.45 and 5.46 into eqn. 5.44 gives:

$$\begin{aligned}|\Psi\rangle = (\alpha c_1{}^2 + \beta c_2{}^2)\,|u_1u_2\rangle + i(\alpha c_2{}^*c_1 - \beta c_2c_1{}^*)\,|u_1v_2\rangle +\\ i(\alpha c_2{}^*c_1 - \beta c_2c_1{}^*)\,|v_1u_2\rangle - [\alpha(c_2{}^*)^2 + \beta(c_1{}^*)^2]\,|v_1v_2\rangle\end{aligned} \tag{5.49}$$

Call $\alpha {c_1}^2 + \beta {c_2}^2 = 0$ thus giving:

$$\frac{{c_2}^2}{\alpha} = -\frac{{c_1}^2}{\beta} = 0$$

or equivalently taking the positive square roots and rearranging:

$$c_2 = k\sqrt{\alpha} \qquad \text{and} \qquad c_1 = ik\sqrt{\beta} \tag{5.50}$$

See that the negative square root solution can be obtained by just putting $\sqrt{\beta} \to -\sqrt{\beta}$ at any stage. The constant k here can be assumed to be real by choosing the phases of c_1 and c_2 appropriately. Hence using the normalization property of c_1 and c_2 together with eqn. 5.50 the following k can be attributed:

$$k^2 = \frac{1}{|\alpha| + |\beta|} \tag{5.51}$$

Substituting eqn. 5.50 into 5.49 and using 5.51 together with ignoring overall phase will give that:

$$|\Psi\rangle = \sqrt{\alpha\beta}\,|u_1 v_2\rangle + \sqrt{\alpha\beta}\,|v_1 u_2\rangle + (|\alpha| - |\beta|)\,|v_1 v_2\rangle \tag{5.52}$$

which can be rewritten in the following form

$$\begin{aligned} |\Psi\rangle = &\left(\frac{\sqrt{\alpha\beta}}{|\alpha| - |\beta|}\,|u_1\rangle + \sqrt{|\alpha| - |\beta|}\,|v_1\rangle\right) \\ \otimes &\left(\frac{\sqrt{\alpha\beta}}{|\alpha| - |\beta|}\,|u_2\rangle + \sqrt{|\alpha| - |\beta|}\,|v_2\rangle\right) - \frac{\alpha\beta}{|\alpha| - |\beta|}\,|u_1 u_2\rangle \end{aligned} \tag{5.53}$$

Now a third set of basis vectors are to be introduced in order to reach the desired point:

$$|w_i\rangle = c_3\,|u_i\rangle + c_4\,|v_i\rangle \tag{5.54}$$

$$|x_i\rangle = -{c_4}^*\,|u_i\rangle + {c_3}^*\,|v_i\rangle \tag{5.55}$$

with the following inverse relations

$$|u_i\rangle = {c_3}^*\,|w_i\rangle - c_4\,|x_i\rangle \tag{5.56}$$

$$|v_i\rangle = {c_4}^* |w_i\rangle + c_3 |x_i\rangle \tag{5.57}$$

where

$$c_3 = \frac{\sqrt{\alpha\beta}}{\sqrt{1-|\alpha\beta|}} \quad \text{and} \quad c_4 = \frac{|\alpha|-|\beta|}{\sqrt{1-|\alpha\beta|}}$$

Normalization of c_3 and c_4 follows from the normalization of α and β. Using 5.54 in 5.53:

$$|\Psi\rangle = N(|w_1 w_2\rangle - {c_3}^2 |u_1 u_2\rangle) \tag{5.58}$$

with

$$N = \frac{1-|\alpha\beta|}{|\alpha|-|\beta|}$$

is obtained. Hence using eqn.s 5.54 and 5.56 in 5.58 the $|\Psi\rangle$ state of the two particles can be written in the following four equivalent forms:

$$|\Psi\rangle = N(c_3 c_4 |u_1 v_2\rangle + c_3 c_4 |v_1 u_2\rangle + {c_4}^2 |v_1 v_2\rangle) \tag{5.59}$$

$$|\Psi\rangle = N(|w_1\rangle \otimes (c_3 |u_2\rangle + c_4 |v_2\rangle) - {c_3}^2 ({c_3}^* |w_1\rangle - c_4 |x_1\rangle) \otimes |u_2\rangle) \tag{5.60}$$

$$|\Psi\rangle = N((c_3 |u_1\rangle + c_4 |v_1\rangle) \otimes |w_2\rangle - {c_3}^2 |u_2\rangle \otimes ({c_3}^* |w_2\rangle - c_4 |x_2\rangle)) \tag{5.61}$$

$$|\Psi\rangle = N(|w_1 w_2\rangle - {c_3}^2 ({c_3}^* |w_1\rangle - c_4 |x_1\rangle) \otimes ({c_3}^* |w_2\rangle - c_4 |x_2\rangle)) \tag{5.62}$$

Now consider the observables U_i and X_i with the corresponding operators,

$$U_i = |u_i\rangle \langle u_i| \quad \text{and} \quad X_i = |x_i\rangle \langle x_i|$$

These physical quantities each can take values 0 and 1 with respect to the eigenvalues of corresponding operators. Also see that these operators in general do not commute so it is not possible to measure both of them on the same particle at the same time. By using 5.59 it can shown that when U_1 and U_2 are measured

$$U_1 U_2 = 0 \tag{5.63}$$

is obtained due to the absence of $|u_1u_2\rangle$ term in the state. Similar to the first derivations logic, from 5.60 it is shown that

$$\text{if} \quad X_1 = 1 \quad \text{then} \quad U_2 = 1 \tag{5.64}$$

this is due to only the term $|x_1u_2\rangle$ containing $|x_1\rangle$. Continuing the same line of argument, it is obtained from 5.61 that

$$\text{if} \quad X_2 = 1 \quad \text{then} \quad U_1 = 1 \tag{5.65}$$

Finally from 5.62 if X_1 and X_2 are measured respectively for the first and second particles then there is a chance that $X_1 = 1$ and $X_2 = 1$ are obtained with $|Nc_3{}^2c_4{}^2|^2$ probability.

$$X_1X_2 = 1 \tag{5.66}$$

Now again introducing the notion of realism and assuming that there exist some hidden variables λ which describe the state of each individual pair of particles a contradiction between local realism and quantum mechanics can be shown. Again similarly to the first derivation the argument of locality comes into existence due to the discrimination of U_1, X_1 independent of U_2, X_2 and so on. This separation of particles in the local realist theories, which is the underlying idea behind the assumption that systems can be investigated locally without considering any effect coming from outside of their local area which is exactly the assumption that is shown to be contradicting with quantum mechanical predictions.

To make it simpler one can always translate this into a logical chain in the following form,

$$X_1 = 1 \rightarrow U_2 = 1 \rightarrow U_1 = 0 \quad \text{and} \quad X_1 = 1 \rightarrow X_2 = 1 \rightarrow U_1 = 1 \tag{5.67}$$

which gives that for the same system with the same initial condition $(X_1 = 1)$ one can argue that by using hidden variables $U_1 = 1$ and $U_1 = 0$ are both attainable at the same time.

S. Goldstein's one page long paper named "Nonlocality without Inequalities for Almost All Entangled States for Two Particles" came just a few months later and got published in 1994. His argument was to generalize the Hardy's results in the sense that it allows one of the four observables at hand to become almost arbitrary [91].

Let $|u_i\rangle$ and $|v_i\rangle$ be bases for particles i where i = 1, 2, and a state of the form

$$|\Psi\rangle = a\,|v_1 v_2\rangle + b\,|u_1 v_2\rangle + c\,|v_1 u_2\rangle$$

with non-zero a, b, c. Important point here is that the term $|u_1 u_2\rangle$ is missing which basically corresponds to the annihilation process defined in the first paper by Hardy. Similarly with the previous papers the following operators are defined

$$U_i = |u_i\rangle\langle u_i| \quad \text{and} \quad W_i = |w_i\rangle\langle w_i|$$

And (i) $U_1 U_2 = 0$ together with (ii) $U_1 = 0$ indicating $W_2 = 1$ and (iii) $U_2 = 0$ implying $W_1 = 1$ are established for the following $|w_i\rangle$'s

$$|w_1\rangle = \frac{a\,|v_1\rangle + b\,|u_1\rangle}{\sqrt{|a|^2 + |b|^2}} \quad \text{and} \quad |w_2\rangle = \frac{a\,|v_2\rangle + c\,|u_2\rangle}{\sqrt{|a|^2 + |c|^2}}$$

This gives (iv) $W_1 = W_2 = 0$ with non-vanishing probability since a, b, c are not zero. It can be seen that when local realism is assumed (i) – (iii) gives $W_1 W_2 = 1$ but (iv) $W_1 W_2 = 0$ occurs as well hence a contradiction.

Goldstein furthers his argument to demonstrate that not just for spin-1/2 bases but for any basis $|u_1\rangle$, $|v_1\rangle$ for the first particle the general state $|\Psi\rangle$ assumes the form

$$|\Psi\rangle = |u_1\rangle \otimes |{f_2}^u\rangle + |v_1\rangle \otimes |{f_2}^v\rangle \tag{5.68}$$

in which, for almost all states $|\Psi\rangle$, $|{f_2}^u\rangle \neq 0$ and $|{f_2}^v\rangle$ is neither proportional to nor orthogonal to $|{f_2}^u\rangle$, that is $|{f_2}^v\rangle = a\,|{f_2}^u\rangle + |{f_2}^{u^\perp}\rangle$ where $a \neq 0$ and $\langle {f_2}^u | {f_2}^{u^\perp}\rangle = 0$. The importance of this statement is that this construction fails when $|{f_2}^u\rangle$ is proportional to $|{f_2}^v\rangle$ which means that $|\Psi\rangle$ is a product state. Also Goldstein notes that $\langle {f_2}^u | {f_2}^u\rangle = \langle {f_2}^v | {f_2}^v\rangle$ indicates that $|\Psi\rangle$ is maximally entangled.

So that for particles associated with higher than two dimensions in Hilbert space the following decomposition can be applied

$$|\Psi\rangle = \Sigma c_i\,|{e_1}^i\rangle \otimes |{f_2}^i\rangle$$

where e forms partially a basis for the first particles and same applies to f for the second particle. So that for any entangled state that is not maximally entangled $c_i \neq c_j$ hence the previous argument following the eqn. 5.68 can be applied.

Goldstein concludes that this shows generally a "proof of nonlocality" can be reached from every entangled state which is not maximally entangled. He stresses that the importance of this approach mainly relies on it making no reference to probability or probabilistic inequalities.

5.3 Applying Hardy's Test

In this section of the study some key points of Hardy's test will be investigated through several applications. Firstly, a bipartite spin-1/2 system is going to be studied to show an easy demonstration of Hardy's test by using the arguments of logical realism similar to Cabello's case [98]. Secondly, the same line of argument will be followed with the generic tripartite spin-1/2 state case. The main point of this second exercise will be to show that for tripartite states single logical chain is enough to demonstrate Hardy's desired result.

5.3.1 Hardy's Test for two spin-half particles

Consider a generic separable state of the following form

$$|\Psi\rangle = c_1 |00\rangle + c_2 |01\rangle + c_3 |10\rangle + c_4 |11\rangle$$

with the general normalization relation $|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = 1$ and $|0\rangle$ representing spin-up in the $\hat{z}$ direction and $|1\rangle$ representing spin-down in $\hat{z}$ direction.

Now take $c_1 = 0$ to make this an entangled state with the following properties

$$|\Psi\rangle = c_2 |01\rangle + c_3 |10\rangle + c_4 |11\rangle \tag{5.69}$$

where

$$|c_2|^2 + |c_3|^2 + |c_4|^2 = 1$$

And define two operations of the form

$$\hat{U}_i = |U_i\rangle \langle U_i| \quad \hat{D}_i = |D_i\rangle \langle D_i|$$

Operation $\hat{U}_i$ deals with the outcome of i'th particles σ_z measurement and provides the results 0 or 1 depending on whether the i'th particle is spin-up in $\hat{z}$ direction, in which case it gives 1, or whether it's not, in which case it gives 0. Similarly $\hat{D}_i$ deals with the outcomes of i'th particles σ_x measurement and so on.

To satisfy Hardy's test there are four basic requirements.

$$U_1 U_2 = 0 \tag{5.70}$$

$$D_1 = 1 \rightarrow U_2 = 1 \tag{5.71}$$

$$D_2 = 1 \rightarrow U_1 = 1 \tag{5.72}$$

$$D_1 D_2 = 1 \quad \text{with} \quad prob. \neq 0 \tag{5.73}$$

To satisfy 5.70 the eqn. 5.69 is enough. Noting that $|+\rangle$, $|-\rangle$ represents spin-up/down in $\hat{x}$, for 5.71 a state of the following manner should be constructed from 5.69 by changing the basis and choosing appropriate constants;

$$|\Psi\rangle = (...) \, |+\rangle_1 \otimes |0\rangle_2 + (...) \, |-\rangle_1 \otimes |...\rangle_2 \tag{5.74}$$

where the contents of (...) parts are irrelevant for the requirement at hand. A state of this form can easily be constructed like expanding the eigenkets of the first particle in $|+\rangle$, $|-\rangle$ basis ;

$$|\Psi\rangle = \left(\frac{|+\rangle_1 + |-\rangle_1}{\sqrt{2}}\right) \otimes c_2 \, |1\rangle_2 + \left(\frac{|+\rangle_1 - |-\rangle_1}{\sqrt{2}}\right) \otimes c_3 \, |0\rangle_2 + \left(\frac{|+\rangle_1 - |-\rangle_1}{\sqrt{2}}\right) \otimes c_4 \, |1\rangle_2$$

this is

$$|\psi\rangle = \frac{1}{\sqrt{2}} [c_2(|{+1}\rangle + |{-1}\rangle) + c_3(|{+0}\rangle - |{-0}\rangle) + c_4(|{+1}\rangle - |{-1}\rangle)]$$

For this to resemble eqn. 5.74 the condition $c_2 = -c_4$ is a necessary one. After choosing them in that manner it becomes

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [c_3(|{+0}\rangle - |{-0}\rangle) - 2c_4 \, |{-1}\rangle] \tag{5.75}$$

It is clear that this equation satisfies the requirement 5.71. And for 5.72 to hold a state of the following kind is needed

$$|\Psi\rangle = (...) \, |0\rangle_1 \otimes |+\rangle_2 + (...) \, |...\rangle_1 \otimes |-\rangle_2 \tag{5.76}$$

Hence expanding the state of the second particle in $|+\rangle, |-\rangle$ basis while keeping the first particle in $|0\rangle, |1\rangle$ basis should be done. That gives the following state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[-c_4 |0\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2) + c_3 |1\rangle_1 \otimes (|+\rangle_2 + |-\rangle_2) + c_4 |1\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2)]$$

after simplifying this eqn.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[c_4(|0-\rangle - |0+\rangle + |1+\rangle - |1-\rangle) + c_3(|1+\rangle + |1-\rangle)]$$

For this eqn. to satisfy the relation 5.75, $c_3 = -c_4$ is a necessary condition. By $|c_2|^2 + |c_3|^2 + |c_4|^2 = 1$ relation and using necessary conditions the constants of the state can be written as

$$c_1 = 0 \quad , \quad c_2 = \frac{1}{\sqrt{3}} \quad , \quad c_3 = \frac{1}{\sqrt{3}} \quad , \quad c_4 = -\frac{1}{\sqrt{3}}$$

Hence the state is

$$|\Psi\rangle = \frac{|01\rangle + |10\rangle - |11\rangle}{\sqrt{3}}$$

And it finally becomes

$$|\Psi\rangle = \frac{|0+\rangle + 2\,|1-\rangle - |0-\rangle}{\sqrt{6}}$$

This equation satisfies the relation 5.72. Finally expanding both particles in $|0\rangle, |1\rangle$ will give 5.73. Following the steps below

$$|\Psi\rangle = \frac{1}{2\sqrt{3}}[(|+\rangle_1 + |-\rangle_1) \otimes (|+\rangle_2 - |-\rangle_2) + (|+\rangle_1 - |-\rangle_1) \otimes (|+\rangle_2 + |-\rangle_2) - (|+\rangle_1 - |-\rangle_1) \otimes (|+\rangle_2 - |-\rangle_2)] \tag{5.77}$$

after further simplifications

$$|\Psi\rangle = \frac{1}{2\sqrt{3}}[|++\rangle + |-+\rangle - |--\rangle - |+-\rangle + |++\rangle + |+-\rangle - |-+\rangle - |--\rangle - |++\rangle + |+-\rangle + |-+\rangle - |--\rangle] \tag{5.78}$$

This will give the following eqn. which satisfies 5.73 in the $\frac{1}{12}$th of the experiments.

$$|\Psi\rangle = \frac{|++\rangle - 3\,|--\rangle + |+-\rangle + |-+\rangle}{2\sqrt{3}} \tag{5.79}$$

The argument of logical realism allows one to propose that if $D_1 = 1 \rightarrow U_2 = 1$ and $D_2 = 1 \rightarrow U_1 = 1$ holds for all of the experiments, together with a certain probability of obtaining $D_1 D_2 = 1$ it can be deduced that $U_1 U_2 = 1$ is applicable for some of the cases. However it is known that $U_1 U_2 = 0$ for all of the cases, hence a contradiction arises.

5.3.2 Hardy's Test for three spin-half particles

Similar to the two spin-half case above consider a generic tripartite spin-half state of the following form

$$|\Psi\rangle = c_1 |000\rangle + c_2 |001\rangle + c_3 |010\rangle + c_4 |011\rangle + c_5 |100\rangle + c_6 |101\rangle + c_7 |110\rangle + c_8 |111\rangle$$

Taking c_1 to zero will give out an entangled state which is of the type that satisfies Hardy's test.

$$|\Psi\rangle = c_2 |001\rangle + c_3 |010\rangle + c_4 |011\rangle + c_5 |100\rangle + c_6 |101\rangle + c_7 |110\rangle + c_8 |111\rangle \quad (5.80)$$

Now consider the logical chain that is required to show a contradiction in the following manner

$$|00\rangle_{12} \rightarrow |1\rangle_3 \rightarrow |+\rangle_2 \rightarrow |1\rangle_1 \quad (5.81)$$

The first two steps of this chain, $|00\rangle_{12} \rightarrow |1\rangle_3$ and $|1\rangle_3 \rightarrow |+\rangle_2$, have probability of 1 while the last step $|+\rangle_2 \rightarrow |1\rangle_1$ has a probability of occurrence that is greater than zero but less than one. To demonstrate these steps two additional forms of 5.80 is required and their general forms can be written in the following manners. For the $|1\rangle_3 \rightarrow |+\rangle_2$ step

$$|\Psi\rangle = (...)_1 \otimes |+\rangle_2 \otimes |1\rangle_3 + (...)_1 \otimes (...)_2 \otimes |0\rangle_3 \quad (5.82)$$

and for the $|+\rangle_2 \rightarrow |1\rangle_1$ step

$$|\Psi\rangle = \alpha |1\rangle_1 \otimes |+\rangle_2 \otimes |...\rangle_3 + (...)_1 \otimes |-\rangle_2 \otimes |...\rangle_3 \quad (5.83)$$

with non-vanishing α are required. So that starting from eqn. 5.80 and expanding the second particles state in the $|+\rangle$, $|-\rangle$ basis;

$$\begin{aligned} |\Psi\rangle = &\frac{c_2[|0\rangle_1 \otimes (|+\rangle_2 + |-\rangle_2) \otimes |1\rangle_3]}{\sqrt{2}} + \frac{c_3[|0\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2) \otimes |0\rangle_3]}{\sqrt{2}} \\ &+ \frac{c_4[|0\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2) \otimes |1\rangle_3]}{\sqrt{2}} + \frac{c_5[|1\rangle_1 \otimes (|+\rangle_2 + |-\rangle_2) \otimes |0\rangle_3]}{\sqrt{2}} \\ &+ \frac{c_6[|1\rangle_1 \otimes (|+\rangle_2 + |-\rangle_2) \otimes |1\rangle_3]}{\sqrt{2}} + \frac{c_7[|1\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2) \otimes |0\rangle_3]}{\sqrt{2}} \\ &+ \frac{c_8[|1\rangle_1 \otimes (|+\rangle_2 - |-\rangle_2) \otimes |1\rangle_3]}{\sqrt{2}} \end{aligned} \quad (5.84)$$

after tidying this up

$$|\Psi\rangle = \frac{c_2}{\sqrt{2}}[|0+1\rangle + |0-1\rangle] + \frac{c_3}{\sqrt{2}}[|0+0\rangle - |0-0\rangle] + \frac{c_4}{\sqrt{2}}[|0+1\rangle - |0-1\rangle]$$
$$+\frac{c_5}{\sqrt{2}}[|1+0\rangle + |1-0\rangle] + \frac{c_6}{\sqrt{2}}[|1+1\rangle + |1-1\rangle] + \frac{c_7}{\sqrt{2}}[|1+0\rangle - |1-0\rangle]$$
$$+\frac{c_8}{\sqrt{2}}[|1+1\rangle - |1-1\rangle] \tag{5.85}$$

For this eqn. to resemble 5.82 the terms containing $|-\rangle_2$ should cancel out each other and terms containing $|+\rangle_2$ should remain intact. That is obtained through enforcing the necessary conditions

$$c_2 = c_4 \quad , \quad c_6 = c_8 \quad \text{and} \quad c_2 \neq -c_6$$

This new form of the state is as follows

$$|\Psi\rangle = [\frac{2c_2\,|0+\rangle_{12}}{\sqrt{2}} + \frac{2c_6\,|1+\rangle_{12}}{\sqrt{2}}] \otimes |1\rangle_3 + (...)_{12} \otimes |0\rangle_3 \tag{5.86}$$

Hence $|1\rangle_3 \rightarrow |+\rangle_2$ can clearly be seen from this expression. And following the same line of logic to reach 5.83 the second part of the above equation can be written explicitly as

$$|\Psi\rangle = [\frac{c_3}{\sqrt{2}}(|0+\rangle_{12} - |0-\rangle_{12}) + \frac{c_5}{\sqrt{2}}(|1+\rangle_{12} + |1-\rangle_{12})$$
$$+\frac{c_7}{\sqrt{2}}(|1+\rangle_{12} - |1-\rangle_{12})] \otimes |0\rangle_3 + (...)_{12} \otimes |1\rangle_3 \tag{5.87}$$

Hence with the probability $|\alpha|^2 = \frac{|c_7|^2+|c_5|^2}{2} + 2|c_6|^2$ the step $|+\rangle_2 \rightarrow |1\rangle_1$ holds. So in summary

$$|00\rangle_{12} \rightarrow (...) \rightarrow |1\rangle_1$$

can occur with non-vanishing probability. Meaning that constructing a local realist structure yields a contradiction since measurement results on distant entangled particles affect the measurement outcomes of other particles.

An important point to this derivation is that even when $c_6 = c_8 = 0$, which leaves no $|1\rangle_1 \otimes (..)_2 \otimes |1\rangle_3$ terms in the initial state, $|\alpha|^2$ is non-vanishing for $c_7 \neq -c_5$ and $c_7, c_5 \neq 0$. Under these conditions a similar version of the bipartite can be written.

Assume ${U_1}^1 = 1$ corresponds to the first particle having spin $|1\rangle_1$ and ${U_1}^3 = 1$ corresponds to the third particle having spin $|1\rangle_3$. So that when $c_6 = c_8 = 0$ it can be argued that $U_1{}^3 U_1{}^1 = 1$ is an impossibility since there is no corresponding $|1\rangle_1 \otimes (..)_2 \otimes |1\rangle_3$ term in the initial state. Now in the same line of argument assume $D_+{}^2 = 1$ corresponds to the second particle having spin $|+\rangle_2$. With the right arrangement it has been shown that in eqn. 5.86 $U_1{}^3 D_+{}^2 = 1$ always hold. However it has also been shown through eqn. 5.87 that $U_1{}^1 D_+{}^2 = 1$ holds with non-vanishing probability. Now assuming a hidden variable theory where these outcomes note real, definite, deterministic physical quantities with the help of a set of variables such as λ there will be a contradiction of the form

$$U_1{}^3(\lambda) U_1{}^1(\lambda) \neq 1$$

$$U_1{}^3(\lambda) D_+{}^2(\lambda) = 1$$

$$U_1{}^1(\lambda) D_+{}^2(\lambda) = 1$$

The above equation shows that Hardy's test in a single logical chain can be reached in tripartite spin-half states. That is $|u\rangle_i \rightarrow (...) \rightarrow |u^\perp\rangle_i$ with a probability for $i = 1, 2, 3$ and $|u\rangle$ corresponds to a spin state, can be shown, unless the state at hand is maximally entangled. However for a bipartite spin-half system $|u\rangle_i \rightarrow (...) \rightarrow |u^\perp\rangle_i$ with a probability, where $i = 1, 2$ is inapplicable.

CHAPTER 6

DISCUSSIONS

In the previous chapters of this study theorems, inequalities and demonstrations on some of the essential problems in foundations of quantum mechanics has been presented. This chapter's focus will be on the interrelation of those presented subjects and their relevance to the essential problems such as locality, contextuality and realism. In those lines, some basic groundwork by Bell[34], Leggett[99] and Wiseman[32] is going to be introduced to denote the historical development of these ideas. Furthermore into this chapter each inequality will be briefly revisited, their significance, important qualities and possible practical uses are going to be discussed.

6.1 Notes on Physical Reality

The problem of physical reality predates the formal branch of physics, which is generally attributed to the era following Copernican Revolution and formation of Newtonian mechanics. There are countless pages from even the early days of writing itself which address to this, still open, problem of reality. However, in the domain of formal physics the problem of reality, or more generally referred to as realism, is taken at hand from a phenomenological perspective rather than an ontological one. In this respect, realism in physics generally considers questions such as whether a hidden variable theory violating Lorentz invariance can hold rather than questions like 'what is an object?'.

In his 1981 dated paper "Bertlmann's socks and the nature of reality"[34] Bell, after discussing the EPR problem and locality, stressed upon four possible positions that might be taken on the issue of 'nature of reality', although he admits there might be other possibilities.

First of all, he consider Einstein's argument with the elements of reality and the criterion of reality, that is reinforcing the wavefunction of QM with hidden variables. Through Bell inequalities and Bell's theorem this possibility seems not a suitable approach to be taken.

Secondly, he argue another hidden variable case but in which choosing parameters a and b of distant locations as free variables are forbidden. For this case the free will of experimenter or the chance of true randomness is non-existent.

Thirdly, accepting non-locality into the framework and faster than light causal influences would solve the problem at hand. An unobservable 'aether'-like solution would be the cheapest one, Bell states, however the role of Lorentz invariance in such a theory would be quite problematic.

Finally, Bell argues that Bohr's stance was that there is no reality below a 'macroscopic level' and what the theory does is only to depict statements of prediction, it does not directly represent a coherent - sensible underlying structure of reality.

These stances of course should be taken with respect to the EPR problem and locality. In Bell's article there is no explicit mention of *contextuality*, however it can be inferred from his example with pairing socks according to their colours or lengths but not both[34].

Leggett, in his summer courses, addresses to the same problem within the view of physical theories. He classifies certain approaches with respect to their stances to or against realism at different levels. Dividing the physical reality into two of the forms micro- and macro- he asserts the notion (which was attributed to Bohr just above[34]) that such a divide is existent in the physics community[99].

A table of correspondence, similar to the one below, is given by Leggett himself[99]. It denotes the stance point of some mainstream approaches taken by physicists, acquired knowingly or unknowingly along their careers. It should be noted that approaches on this table does not cover all the different stances along the community.

Table 6.1: Leggett's table of Theory-Realism correspondence

Approach	Micro-realism	Macro-realism
Everett-Wheeler 'Many-Worlds', mentalistic, Schulman's 'Calvinistic' interpretation	yes	yes
Copenhagen interpretation, Quantum-Information theoretic approach	no	no
'orthodox' decoherence view	yes	no
(neo-)Bohmian	yes	?

Leggett uses mentalistic approach to summarize all the outlooks which argues that nothing actually happens until the human consciousness is involved. Although not being a widespread formal assumption attributed to a certain theory, this is a popular view amongst public and it effects the arguments on the foundations of quantum mechanics even in physics community. Effects of these views can also be seen in efforts like past or delayed choice experiments[100].

He uses Calvinistic in the sense that the idea of free will, where the experimenters choice of measurement parameters is free or not-constrained, is not accepted in some interpretations. It is argued that for a final given state of the universe just by applying the Schrödinger's equation backwards an initial state of the universe, although being horribly entangled, can be found[99].

The approach Leggett calls as neo-Bohmian mainly consist of updated versions of Bohmian mechanics or de Broglie-Bohm pilot wave theory. The stance of this theory with respect to macro-realism is basically unclear since the formation of pilot wave theory deals with the problem of locality by creating a non-local hidden variable(NLHV) theory. It is also a deterministic theory hence there is no measurement problem in the formal level, however the wave function collapse is a phenomenological outcome of the theory. For further remarks on the subject Arthur Fine and Sheldon Goldstein's book titled "Bohmian Mechanics and Quantum Theory: An Appraisal"[101] can be consulted.

In Wiseman's 2014 dated article[32] he generalizes the views of physicists with respect to Bell's theorem into two camps, operationalists and realists. He argues that the quantum information theoretic approach can be understood the stance of operationalist camp. The dual definition of Bell's theorem in which it generates an either-or outcome with hidden variables and locality, and that hidden variables (or realism as it is generally referred to as) should be ruled out since locality should hold. However, in addition to this mainstream stance of this camp, Wiseman states that there are other stances and gives Quantum Bayesianism (or QBism as they call themselves) as an example.

The QBist interpretation differs from the Copenhagen interpretation in pursuit of achieving a more Bohr-like interpretation of quantum mechanics. QBism takes, explicitly, the subjective view of probability, which is similar to the ones of statisticians and economists approaches [102]. In this sense assigning probability is taken as a subjective action hence disallowing even the probability of constructing a joint probability distribution to space-like separated events. This, they claim, solves the problem of non-locality in quantum mechanics[102]. Although an operationalist approach, QBism contains strong metaphysical claims such as "Inductive inference is nothing more than a broadly shared personal judgment based on habit."[102], which can be taken as a clear sign that realists are not the only ones having dealings outside the domain of physics.

Realist camp identified by Wiseman also has mainstream ideas and extremes. Wiseman places Bell into this group through two of his convictions, first 'that correlations need to be explained' and second 'that nature should have a unified description, in which antrhopocentric notions such as detector settings should play no fundamental role'[32]. Bell himself, on [103] argues that accepting 'no signaling faster than light' as the fundamental causal structure of theoretical physics is hard for him to accept. He claims that notions such as 'no signaling...' are vague and they shouldn't be. Also, in that same article Bell explains (in section 10 titled 'Quantum mechanics cannot be embedded in a locally causal theory'), the formal violation of local causality by QM, however he adds that this is only in the formal sense, a spooky-action-at-a-distance and not a 'real' one [103]. Still Bell and many other physicists, following the lead of Einstein, align themselves in the side that require their physical theory to be consistent both in formal and 'real' sense. Problems such as spooky-action-at-a-distance should be addressed, not ignored.

As an extreme point in the realist camp the demand for an objective realist theory can be given[104]. This kind of a demand, mainly coming from philosophers of physics, requires the formation of new, clearly defined, applicable notions (similar to Bell's wish[103]) on which the theory of physical reality can be constructed. This new objectively defined reality may be contextual, non-local and so on, but should be complete. In summary, they wish to follow Einstein's lead but not his understanding of what physical reality is.

6.2 Local Causality

As stated before, Bell uses the term local causality (LC)[16] to differentiate what he identifies from the EPR argument from what is commonly understood when the word locality is used. Others (such as Mermin) consistently calls the same concept as Einstein locality. Many papers dedicated to explaining what is local causality or Einstein locality (separately identifying the same concept with two different names) can be found in `http://arxiv.org/` and other sources.

A brief and superfluous description of the differences between the common notion of locality and LC can be explained by the help of figure 6.1. First of all, locality in the most general sense corresponds to that events are local. Directions chosen on a measurement device for a particle A should not play any role on distribution of particle B's experimental outcomes and vice versa.

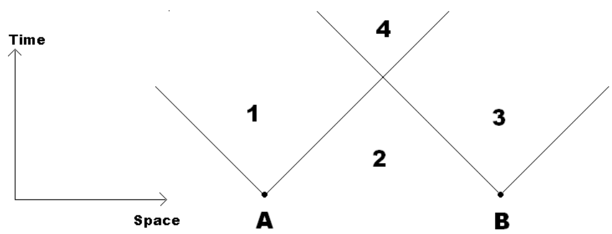

Figure 6.1: 2-D representation of light cones of two events

Now, consider the figure 6.1 above with regions R_1, R_2, R_3 and R_4 denoted in the figure as $1, 2, 3, 4$ and events A-B. Locality argument would require that outcomes of experiments done in space-like separated places, corresponding to events A and B, cannot depend on each other. This is a very general and intuitive approach. However for LC the issue at hand is more definite and precise.

Through Einstein's definition of space-time continuum, for region R_2 neither the event A nor B can play any role as a parameter effecting the probability distributions of event occurrences in that region. Similarly, for region R_1 only the event A and for region R_3 only the event B can play any role, assuming no other events took place. Only events occurring in region R_4 can depend on the occurred events A and B, for the other regions the corresponding events has not yet occurred in their frame hence cannot have any effect on those regions.

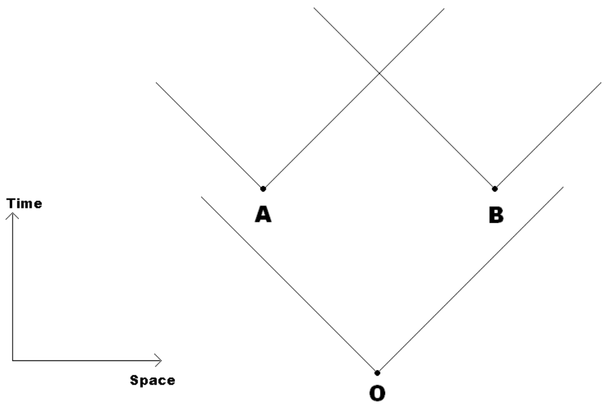

Figure 6.2: 2-D representation of light cones of three events

Again now consider the above figure 6.2. Assuming a previous event O, where both A and B could be effected from can be introduced. For the EPR problem a hidden variable approach would assume such an event O, usually the moment of interaction between two particles. However, if event O is an event where an entangled pair of particles are generated and the events A and B correspond to two space-like separated experiments done on these particles respectively, a situation arises. Bell has demonstrated that the outcome of event A does depend on the event B, and outcome of event B does depend on the event A. Through this demonstration Bell has shown that any hidden variable theory that reproduces the predictions of quantum mechanics should allow faster than light communication, and if not should satisfy an inequality, which quantum mechanics violates.

6.3 Demonstrations with and without Inequalities

In this study many demonstrations with or without inequalities have been established and investigated in their own respect. Through this section they will be revisited briefly and connections between them, together with a more general view of the entire subject, are going to be discussed.

Bell and MABK Inequalities

Bell's theorem and the initial Bell inequality eqn. 6.1 forms a groundwork on which rest of the literature has built upon.

$$|P(a,b) - P(a,c)| \leq 1 + P(b,c) \tag{6.1}$$

Two more commonly known Bell inequalities are the one that is constructed through Fine's theorem, which is called as the BCHS inequality in this study (eqn. 6.2) and the CHSH inequality 6.3.

$$-1 \leq P(AB) + P(AB') + P(A'B') - P(A'B) - P(A) - P(B') \leq 0 \tag{6.2}$$

$$-2 \leq P(AB) + P(A'B') + P(A'B) - P(AB') \leq 2 \tag{6.3}$$

Violation of these kind of inequalities have been established with experiments starting with Aspect experiments in 1981 [37].

MABK inequalities correspond to the violation of Bell-like inequalities in n-partite systems. Mermin[40] has demonstrated that for increased particle number n, the inequality (which reduces to CHSH inequality for bipartite systems) is violated with exponentially increasing values.

Legget-Garg Inequality

A general form of the LGIs can be written as eqn. 6.4 below.

$$-1 \leq \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle + \langle Q_1 Q_3 \rangle \leq 3 \tag{6.4}$$

Leggett-Garg inequality does in fact encompasses a wider discussion than a ruling out the possibility of constructing a certain class of hidden variable theories. Assumptions used to construct this inequality, macrorealism(MR), non-invasive measurement(NIM) and induction are all important points of discussion in the contemporary arguments on the nature of physical reality. For example, ideas such as no-signaling in time[54] has been proposed, which highlights that not even statements such as 'future events cannot influence past events' are not unanimously accepted among physics community [105].

KCBS Inequality

The KCBS inequality is formulated to experimentally demonstrate that assigning simultaneous values to outcomes of non-realized experiments yield contradictory results, which is the argument of Kochen-Specker theorem. It assumes the form in below eq. 6.5, called as the pentagram inequality by the authors since on a geometrical surface it creates a pentagram like five sharp edged object.

$$\langle {S_{\ell_1}}^2 \rangle_\Psi + \langle {S_{\ell_2}}^2 \rangle_\Psi + \langle {S_{\ell_3}}^2 \rangle_\Psi + \langle {S_{\ell_4}}^2 \rangle_\Psi + \langle {S_{\ell_5}}^2 \rangle_\Psi \geq 3 \tag{6.5}$$

This inequality not only just reduces the number of involved spin projection operators for state dependent contextuality tests to 5 observables, but also authors also argue that there can be no other tests for three dimensional quantum systems involving less than 5 observables. And that other tests involving 5 observables for such systems can be reduced to the inequality given above.

GHZ experiment

"Going Beyond Bell's Theorem"[6] marks an important point not just due to the advancement of theory but also for attracting attention to the GHZ state. Using a particular state to demonstrate a contradiction, without applying the method of constructing inequalities, has played an important role for the following literature of the era. Figures such as Peres [75], Mermin [74], Cabello [76] and many others applied this approach to Kochen-Specker theorem, and other approaches like Hardy's [7, 90] took inspiration from GHZ.

Other than its inspirational and instructive purposes GHZ approach also established a clear method where a single experiment can demonstrate the falsification of a certain class of hidden-variable theories, where in Bell's approach the contradiction was reached through statistical analysis. It has been shown later that, GHZ state actually goes beyond Bell's theorem also in the sense it gives consistent contradictions with the use of Kochen-Specker theorem as well [74].

Hardy's Test

Hardy's test is a particularly interesting example in the entirety of these demonstrations with and without inequalities. In the first article [7], the physical setup proposed by Hardy and the dependence of the final state to the existence or absence of the beam splitters is a direct use of locality argument. However, the second demonstration of Hardy months later [90] uses almost exactly the same mathematics and arguments, but without a physical setup, which leads to a contradiction that does not require the use of locality argument anywhere. On the contrary, the second and more generalized version of Hardy's test is actually more consistent and applicable to Kochen-Specker theorem, and direct correspondence between them can easily be shown [76].

Following the above line of argument, it can be argued that approaches like Hardy's test cannot be located solely under either Bell's theorem or Kochen-Specker theorem. The more general use of Bell-Kochen-Specker(BKS) theorem has much use in this case, which can be taken as an umbrella term for approaches that use contradictions to discard certain classes of hidden-variable theories. Hardy's test demonstrates a contradiction between hidden-variable theories and predictions of quantum mechanics, and to demonstrate this contradiction one can use either the locality argument or contextuality, both will suffice for the purpose of demonstrating this contradiction.

Unification of Theorems and Inequalities

Before finishing up the discussions on this study, it would only be appropriate to mention the efforts of unifying these theorems and inequalities. There are inequalities such as BCHS, CHSH and MABK that uses specifically Bell's theorem, space-like separated regions, local causality and local hidden variable theory. There are demonstrations of Peres, Mermin and the KCBS inequality that exclusively uses Kochen-Specker theorem, impossibility of assigning simultaneous values to the outcomes of non-realized experiments and contextuality. And there are those like GHZ and Hardy's test that can be constructed using both arguments. Finally, there are the likes of Leggett-Garg inequalities, which deals with the same problem (problem of hidden variables) however cannot be simply located under neither Bell's nor Kochen-Specker theorems.

Using a generalized term like Bell-Kochen-Specker theorem does seem to work in an informal level however it does not produce an actual statement. The EPR problem uses locality very strongly and it is understandable to argue that what EPR problem proposes is replacing quantum mechanics (or generalizing it) to a local hidden variable theory. However, Kochen and Specker also addresses to the EPR problem while introducing their theorem and locality does not play any role there. Furthermore, in assumptions that lead to the construction of LGIs where time-like separated regions are of the essence, the EPR problem still plays an important role although it has no argument concerning that type of regions. Hence just using the EPR problem or Bell's original article or Kochen-Specker's won't address the issue in its totality.

Eventhough there have been difficulties in establishing clear links between these ideas in the past, lately an accelerated effort can be seen. Approaches such as Mermin's 'Unified form' of no hidden variable theorems[74] left its place to some concepts such as Gleason's no-disturbance property[106] and monogamy relations between inequalities[107, 108] which gives more solid mathematical ground to built upon. In his 2012 dated article Cabello [78] discusses how noncontextual hidden variable(NCHV) theories and local hidden variable(LHV) theories using CHSH and KCBS inequalities can be generalized to bounds set by using only no-signalling and exclusivity (which denotes that two events are exclusive if they cannot be simultaneously true).

CHAPTER 7

CONCLUSION

In this study, the outlooks on the EPR problem and entailing hidden variable problems are investigated through formulations with respect to Bell's, Kochen-Specker and generalizations of these theorems. Several methods with and without inequalities established, their relevance to the problems at hand and to each other studied. Among these methods CHSH, MABK, LGI and KCBS inequalities are highlighted especially in their roles as archetypes for their relevant subjects. In addition to those, GHZ experiment and Hardy's test has been studied to demonstrate the without inequality approaches on the problems at hand.

A review of historical development of the problems starting from the EPR article, developing through Bohr-Einstein debates and general discussions amongst the scholars of the relevant era is taken as the focus of chapter 2. Especially the formulation of de Broglie - Bohm theory as an example to the possibility of non-local hidden variable theories plays an important role in the historical process of these problems. The concepts derived from this theory has still relevance to the contemporary discussions since Bohmian mechanics can be formulated to give the same predictions with quantum mechanics.

As the study advanced, in chapter 3 Bell's theorem had become the focus. Bell's approach to the EPR problem and his proposal to tackle with problem of completeness of QM with using the argument of locality was studied. In this context the development of the idea of local causality (LC), which is also violated formally by quantum mechanics causing a spooky-action-at-a-distance problem was highlighted. Two different branches of inequalities, mainly called in this study as BCHS and CHSH, were investigated and it is shown that both inequalities and other inequalities derived from those two are equivalent. They all can be derived from Bell's original inequality eqn. 3.5 and experimental violation of any inequality which can be called as Bell inequality due to its equivalency with Bell's original, is enough to demonstrate the violation of all Bell inequalities. Also a generalization to N-partite systems by Mermin and others has been shown.

Other types of inequalities, which can not be directly connected to Bell's original inequality was the subject of chapter 4. These other inequalities, namely Leggett-Garg and KCBS, differ from Bell inequalities through their basic assumptions. LGIs deal with the problem of macrorealism and violation of an LGI actually won't give any information on the subject of locality. Similarly, KCBS inequality uses contextuality, simultaneous value assignments to the outcomes of non-realized experiments. Violation of KCBS inequality would result in demonstrating that this assumption is wrong and this simultaneous value assignment process is an invalid one. The problem of contextuality was first formally established by Kochen and Specker using a method now being referred as Kochen-Specker theorem.

In chapter 5, which is the last body chapter of this study, Bell's theorem without inequalities were examined. Although these are referred to as Bell's theorem without inequalities they can actually be used with Kochen-Specker theorem as well. When these 'without inequalities' approaches are used with Kochen-Specker theorem, they demonstrate the negation of non-contextuality assumption and rule out non-contextual hidden variable(NCHV) theories. It is argued that a more general theorem, containing both Bell's and Kochen-Specker theorems can be thought to apply to these without inequality approaches. This theorem, called BKS, is already in use amongst the physics community however its use varies from author to author and its uses are not as clear as Bell's or Kochen-Specker theorems.

In the discussion chapter 6, subjects were re-visited and further arguments on them has been made. In section 6.1 titled 'Notes on Physical Reality', mainly four different approaches on which theories today can be located under is stressed upon. Arguments made by authors such as Bell, Leggett and Wiseman on these approaches and how these approaches deal with certain problems was also highlighted.

As a conclusion to this study several comments should be made. First of all, although some of these discussions on locality, contextuality or realism is taken to be ‘metaphysical’, their physical implications are almost as strong as the uncertainty principle. Principles such as no-signaling faster than light, no-signaling in time or exclusivity brings forth these arguments into experimental plane. Study of Bell inequalities has important uses with respect to entangled systems and studies on LGIs may deliver the exact mechanism of quantum decoherence, which would result in generation of macroscopic quantum phenomena at will.

Secondly, efforts in unifying these theorems and inequalities might reveal not only the lower bounds where quantum mechanics reduces to classical mechanics, but also the upper bounds allowed by QM, which are not always consistent with the upper bounds derived from the principles at hand. In this respect, such efforts may enforce or diminish the position of valued principles of contemporary physics and open up the path for discovering new principles of nature.

Finally, the circle of thought experiments leading actual experiments leading to further thought experiments can be seen in the development of this subject, from Einstein to Bell to Aspect to Hardy and many more. Although it is impossible to disprove general concepts such as locality or realism through experimentation, it is possible to harden the lines between what is physical and what is metaphysical. The problem of hidden variables is a clear example where through theorems of Bell’s and Kochen-Specker certain classes of this theory, LHV and NCHV respectively, are canceled out. Whether this subject can be developed to make distinctions between relativistic and non-relativistic quantum mechanics is also an open question which requires further studies.

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APPENDIX A

USEFUL TOOLS FOR QUANTUM SYSTEMS

A.1 Construction of S_n operator for spin-1/2 systems

As previously discussed, the matrix representations of Pauli matrices can be written in the following form for spin-1/2 systems:

$$S_n = \frac{\hbar}{2}\sigma_n \tag{A.1}$$

where $\hat{n}$ is an arbitrary direction. For the usual $\hat{x}, \hat{y}, \hat{z}$ directions Pauli matrices take the forms;

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.2}$$

By using these three matrices in;

$$\vec{\sigma} = \sigma_x \hat{\imath} + \sigma_y \hat{\jmath} + \sigma_z \hat{k} \tag{A.3}$$

Hence using eqn. A.2 and implementing the Pauli matrices for $\hat{x}, \hat{y}, \hat{z}$ into eqn. A.3 one reaches the σ_n operator for spin-1/2 systems through the following steps;

$$\hat{n} = \sin(\theta)\cos(\phi)\hat{\imath} + \sin(\theta)\sin(\phi)\hat{\jmath} + \cos(\theta)\hat{k}$$

and multiplying this with the $\vec{\sigma}$ given above

$$\vec{\sigma}.\vec{n} = \frac{\hbar}{2}(\sin(\theta)\cos(\phi)\sigma_x + \sin(\theta)\sin(\phi)\sigma_y + \cos(\theta)\sigma_z)$$

Now the above statement can be considered as the S_n operator. Dividing this with $\frac{\hbar}{2}$ to satisfy (1.1.1) and explicitly expanding the Pauli matrices together with the sine and cosine parameters will provide the following form;

$$\sigma_n = \begin{pmatrix} \cos(\theta) & \sin(\theta)\cos(\phi) - i\sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) + i\sin(\theta)\sin(\phi) & -\cos(\theta) \end{pmatrix}$$

Since when taken into $\sin(\theta)$ parenthesis the off-diagonal entries of this matrix given $e^{-i\phi}$ and $e^{i\phi}$ then the final form of the σ_n operator can be written as;

$$\sigma_n = \begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix} \tag{A.4}$$

APPENDIX B

EXPLICIT CALCULATIONS

B.1 Fine's Theorem

The equations 3.15, 3.16 and 3.17 are of the following forms

$$P(ST) = \int \tilde{S}(\lambda)\tilde{T}(\lambda)\rho(\lambda)d\lambda \tag{B.1}$$

and

$$P(\bar{S}) = \int [1 - \tilde{S}(\lambda)]\rho(\lambda)d\lambda \tag{B.2}$$

and

$$P(AA'BB') = \int \tilde{A}(\lambda)\tilde{A}'(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \tag{B.3}$$

Using these relations the equation 3.18 can be obtained as

$$P(AA'BB') + P(A\bar{A}'BB') + P(AA'B\bar{B}') + P(A\bar{A}'B\bar{B}') = P(AB) \tag{B.4}$$

First term is already expanded so starting from the second term

$$P(A\bar{A}'BB') = \int \tilde{A}(\lambda)[1 - \tilde{A}'(\lambda)]\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda$$

which can be rewritten as

$$P(A\bar{A}'BB') = P(ABB') - P(AA'BB')$$

Third term is

$$P(AA'B\bar{B}') = \int \tilde{A}(\lambda)\tilde{A}'(\lambda)[1 - \tilde{B}(\lambda)]\tilde{B}'(\lambda)\rho(\lambda)d\lambda$$

which correponds to

$$P(AA'B\bar{B}') = P(AA'B) - P(AA'BB')$$

And the fourth term

$$P(A\bar{A}'B\bar{B}') = \int \tilde{A}(\lambda)[1 - \tilde{A}'(\lambda)]\tilde{B}(\lambda)[1 - \tilde{B}'(\lambda)]\rho(\lambda)d\lambda$$

can be read as

$$P(A\bar{A}'B\bar{B}') = P(AB) - P(AA'B) - P(AB'B) + P(AA'BB')$$

Hence adding all these together as

$$\begin{aligned} P(AA'BB') + P(A\bar{A}'BB') + P(AA'B\bar{B}') + P(A\bar{A}'B\bar{B}') = \\ P(AA'BB') + P(ABB') - P(AA'BB') + P(AA'B) - P(AA'BB') \\ +P(AB) - P(AA'B) - P(AB'B) + P(AA'BB') = P(AB) \end{aligned} \tag{B.5}$$

gives this relation which is identical to 3.18.

Now for 3.21 and 3.22 the following can be shown explicitly

$$P(ABB') = P(AA'BB') + P(A\bar{A}'BB') \leq P(A'B') + P(B') - P(A'B)$$

and

$$P(\bar{A}BB') = P(\bar{A}A'BB') + P(A\bar{A}'BB') \leq P(A'B') + P(B) - P(A'B)$$

For the first eqn. expand P(A B B') as

$$P(ABB') = \int \tilde{A}(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda$$

it is clear to see that this can be rewritten as

$$\begin{aligned} P(ABB') = \int \tilde{A}(\lambda)\tilde{A}'(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \\ + \int \tilde{A}(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \\ - \int \tilde{A}(\lambda)\tilde{A}'(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \end{aligned} \tag{B.6}$$

and the last two terms can be re-arranged as

$$\begin{aligned} P(ABB') = \int \tilde{A}(\lambda)\tilde{A}'(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \\ + \int \tilde{A}(\lambda)[1 - \tilde{A}'(\lambda)]\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda \end{aligned} \tag{B.7}$$

which gives the following by using eqn. 3.16 on the last term

$$P(ABB') = P(AA'BB') + P(A\bar{A}'BB')$$

Now expanding the terms on the right-hand side of the inequality P(A' B') + P(B') - P(A'B) as

$$P(A'B') = \int \tilde{A}'(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda$$

$$P(B') = \int \tilde{B}'(\lambda)\rho(\lambda)d\lambda$$

and

$$-P(A'B) = -\int \tilde{A}'(\lambda)\tilde{B}(\lambda)\rho(\lambda)d\lambda$$

Adding these will give

$$P(A'B') + P(B') - P(A'B) = \int [\tilde{A}'(\lambda)\tilde{B}'(\lambda) + \tilde{B}'(\lambda) - \tilde{A}'(\lambda)\tilde{B}(\lambda)]\rho(\lambda)d\lambda$$

which is always equal or greater than

$$P(ABB') = \int \tilde{A}(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)\rho(\lambda)d\lambda$$

since any configuration of $\tilde{A}'(\lambda)\tilde{B}'(\lambda) + \tilde{B}'(\lambda) - \tilde{A}'(\lambda)\tilde{B}(\lambda)$ is necessarily greater than or equal to the corresponding configuration of $\tilde{A}(\lambda)\tilde{B}(\lambda)\tilde{B}'(\lambda)$. Same argument can be made for eqn. 3.22 as well.

B.2 Muynck's Demonstration

In section 3.2.1 equation 3.28 is as follows

$$\begin{aligned} \langle A_1A_2\rangle + \langle A_1A_3\rangle + \langle A_3A_4\rangle - \langle A_2A_4\rangle &= Q(+,+,+,+) + Q(-,-,-,-) \\ &\quad -Q(+,-,-,+) - Q(-,+,+,-) \end{aligned} \tag{B.8}$$

Left-hand side terms can be expanded in the form

$$\langle A_1A_2\rangle = P_{12}(+,+) + P_{12}(-,-) - P_{12}(+,-) - P_{12}(-,+)$$

$$\langle A_1A_3\rangle = P_{13}(+,+) + P_{13}(-,-) - P_{13}(+,-) - P_{13}(-,+)$$

$$\langle A_3A_4\rangle = P_{34}(+,+) + P_{34}(-,-) - P_{34}(+,-) - P_{34}(-,+)$$

$$-\langle A_2A_4\rangle = -P_{24}(+,+) - P_{24}(-,-) + P_{24}(+,-) + P_{24}(-,+)$$

So in total form

$$\begin{aligned}\langle A_1A_2\rangle + \langle A_1A_3\rangle + \langle A_3A_4\rangle - \langle A_2A_4\rangle = \\ P_{12}(+,+) + P_{12}(-,-) - P_{12}(+,-) - P_{12}(-,+) \\ +P_{13}(+,+) + P_{13}(-,-) - P_{13}(+,-) - P_{13}(-,+) \\ +P_{34}(+,+) + P_{34}(-,-) - P_{34}(+,-) - P_{34}(-,+) \\ -P_{24}(+,+) - P_{24}(-,-) + P_{24}(+,-) + P_{24}(-,+)\end{aligned} \tag{B.9}$$

And now expand the right-hand side terms similarly

$$Q(+,+,+,+) = P_{12}(+,+) + P_{13}(+,+) + P_{34}(+,+) - P_{24}(+,+) - P_1(+) - P_3(+)$$

$$Q(-,-,-,-) = P_{12}(-,-) + P_{13}(-,-) + P_{34}(-,-) - P_{24}(-,-) - P_1(-) - P_3(-)$$

$$-Q(+,-,-,+) = -P_{12}(+,-) - P_{13}(+,-) - P_{34}(-,+) + P_{24}(-,+) + P_1(+) + P_3(-)$$

$$-Q(-,+,+,-) = -P_{12}(-,+) - P_{13}(-,+) - P_{34}(+,-) + P_{24}(+,-) + P_1(-) + P_3(+)$$

Adding all these together will give

$$\begin{aligned}Q(+,+,+,+) + Q(-,-,-,-) - Q(+,-,-,+) - Q(-,+,+,-) = \\ P_{12}(+,+) + P_{13}(+,+) + P_{34}(+,+) - P_{24}(+,+) - P_1(+) - P_3(+) \\ +P_{12}(-,-) + P_{13}(-,-) + P_{34}(-,-) - P_{24}(-,-) - P_1(-) - P_3(-) \\ -P_{12}(+,-) - P_{13}(+,-) - P_{34}(-,+) + P_{24}(-,+) + P_1(+) + P_3(-) \\ -P_{12}(-,+) - P_{13}(-,+) - P_{34}(+,-) + P_{24}(+,-) + P_1(-) + P_3(+)\end{aligned} \tag{B.10}$$

Terms corresponding to the probability distributions of single observables will cancel out each other and the final form will be

$$\begin{aligned}Q(+,+,+,+) + Q(-,-,-,-) - Q(+,-,-,+) - Q(-,+,+,-) = \\ P_{12}(+,+) + P_{13}(+,+) + P_{34}(+,+) - P_{24}(+,+) \\ +P_{12}(-,-) + P_{13}(-,-) + P_{34}(-,-) - P_{24}(-,-) \\ -P_{12}(+,-) - P_{13}(+,-) - P_{34}(-,+) + P_{24}(-,+) \\ -P_{12}(-,+) - P_{13}(-,+) - P_{34}(+,-) + P_{24}(+,-)\end{aligned} \tag{B.11}$$

It is clear to see that this is identical to the form obtained in B.9 so that B.8 holds.

B.3 MABK

B.3.1 Mermin's inequality for n=2 and n=3

From section 3.3 eqn. 3.36 reads as;

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[|00..0\rangle + i\,|11..1\rangle] \tag{B.12}$$

and operator used in Mermin's demonstration is 3.37

$$\hat{A} = \frac{1}{2i}[\prod_{j=1}^{n}(\sigma_x{}^j + i\sigma_y{}^j) - \prod_{j=1}^{n}(\sigma_x{}^j - i\sigma_y{}^j)] \tag{B.13}$$

For bipartite states (n=2) Mermin's inequality hold and this can be shown in the following manner. The state $|\Phi\rangle$ for two particles can be written as;

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[|00\rangle + i\,|11\rangle] \tag{B.14}$$

with $|0\rangle$ corresponding to upward spin in $\hat{z}$-direction and $|1\rangle$ corresponds to downward spin in $\hat{z}$-direction. In addition to the state, for n=2 the operator $\hat{A}$ is

$$\hat{A} = \frac{1}{2i}[(\sigma_x{}^1 + i\sigma_y{}^1)(\sigma_x{}^2 + i\sigma_y{}^2) - (\sigma_x{}^1 - i\sigma_y{}^1)(\sigma_x{}^2 - i\sigma_y{}^2)] \tag{B.15}$$

Expanding this will result

$$\hat{A} = \frac{1}{2i}(\sigma_x{}^1\sigma_x{}^2 + i\sigma_x{}^1\sigma_y{}^2 + i\sigma_y{}^1\sigma_x{}^2 - \sigma_y{}^1\sigma_y{}^2 - \sigma_x{}^1\sigma_x{}^2 + i\sigma_x{}^1\sigma_y{}^2 + i\sigma_y{}^1\sigma_x{}^2 + \sigma_y{}^1\sigma_y{}^2)$$

and after simplifications

$$\hat{A} = \sigma_x{}^1\sigma_y{}^2 + \sigma_y{}^1\sigma_x{}^2$$

Together with the known Pauli matrices from eqn. A.2. Now note that

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.16}$$

Hence for a bipartite two-leveled system there is a 4x1 matrix corresponding to each ket-vector. For the state at hand those are

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{B.17}$$

And similarly for the Pauli matrices of the operator

$$\sigma_x{}^1\sigma_y{}^2 = \sigma_x{}^1 \otimes \sigma_y{}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{B.18}$$

together with

$$\sigma_y{}^1\sigma_x{}^2 = \sigma_y{}^1 \otimes \sigma_x{}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{B.19}$$

Hence applying this operator on the state at hand will provide the following result

$$\hat{A}\,|\Phi\rangle = \frac{1}{\sqrt{2}}[\begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} + i\begin{pmatrix} -i \\ 0 \\ 0 \\ 0 \end{pmatrix} + i\begin{pmatrix} -i \\ 0 \\ 0 \\ 0 \end{pmatrix}] = 2\,|\Phi\rangle \tag{B.20}$$

From eqn. 3.45 for n = 2 it can be seen that $F \leq 2$ and above calculations verify that for n = 2 this inequality holds, which was already established by Mermin[40]. However for n = 3, it does not. Now again using the relations given in equations 3.36 and 3.37 with n=3 the state and the operator can be written in the forms

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[|000\rangle + i\,|111\rangle] \tag{B.21}$$

and

$$\hat{A} = \frac{1}{2i}[(\sigma_x{}^1 + i\sigma_y{}^1)(\sigma_x{}^2 + i\sigma_y{}^2)(\sigma_x{}^3 + i\sigma_y{}^3) - (\sigma_x{}^1 - i\sigma_y{}^1)(\sigma_x{}^2 - i\sigma_y{}^2)(\sigma_x{}^3 - i\sigma_y{}^3)] \tag{B.22}$$

Following the same procedures with B.15 this operator can be written in the form

$$\hat{A} = \sigma_x{}^1\sigma_x{}^2\sigma_y{}^3 + \sigma_x{}^1\sigma_y{}^2\sigma_x{}^3 + \sigma_y{}^1\sigma_x{}^2\sigma_x{}^3 - \sigma_y{}^1\sigma_y{}^2\sigma_y{}^3 \tag{B.23}$$

which can be written in the sum of the following forms

$$\sigma_x{}^1\sigma_x{}^2\sigma_y{}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_x{}^1\sigma_y{}^2\sigma_x{}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_y{}^1\sigma_x{}^2\sigma_x{}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_y{}^1\sigma_y{}^2\sigma_y{}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Putting these in B.23 will give the following $\hat{A}$ in the matrix representation formalism

$$\hat{A} = 4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence together with

$$|000\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The operation $\hat{A}$ on the ket $|\Phi\rangle$ will give

$$\hat{A}\,|\Phi\rangle = 4\,|\Phi\rangle$$

So it is obvious that $\langle\hat{A}\rangle$ gives 4 while from 3.45 the bound is 2^1 for n=3, hence the quantum mechanical expectation value exceeds the boundary limit of the local hidden variable (λ) theory at hand. The Mermin inequality is violated with exponentially increasing values[40] for systems with $n \geq 3$.

B.4 A simple proof of Kochen-Specker theorem

A spin singlet state is given as

$$|\Psi\rangle_{12} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Define ${\sigma_x}^2$ as $I_1 \otimes {\sigma_x}^2$ where I_1 identifies the identity operator $I|\Psi\rangle = |\Psi\rangle$ for all $|\Psi\rangle$. Now see that

$$(I_1 \otimes {\sigma_x}^2)|\Psi\rangle_{12} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

and similarly for taking ${\sigma_x}^1$ as ${\sigma_x}^1 \otimes I_2$

$$({\sigma_x}^1 \otimes I_2)|\Psi\rangle_{12} = -\frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

Hence $f({\sigma_x}^2) = -f({\sigma_x}^1)$ holds. Now also see that

$$(I_1 \otimes {\sigma_y}^2)|\Psi\rangle_{12} = -i\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and

$$({\sigma_y}^1 \otimes I_2)|\Psi\rangle_{12} = i\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

giving $f({\sigma_y}^2) = -f({\sigma_y}^1)$. For ${\sigma_x}^1{\sigma_y}^2{\sigma_y}^1{\sigma_x}^2 = {\sigma_z}^1{\sigma_z}^2$ see the following matrix relations:

$${\sigma_x}^1{\sigma_y}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

and

$${\sigma_y}^1{\sigma_x}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

hence their product ${\sigma_x}^1{\sigma_y}^2{\sigma_y}^1{\sigma_x}^2$ is

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is equal to ${\sigma_z}^1{\sigma_z}^2$.

B.5 GHZ

B.5.1

Equation 5.11 is of the following form

$$\begin{aligned}\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle = \frac{1}{2}[&\langle 0011|\sigma_1\sigma_2\sigma_3\sigma_4|0011\rangle - \langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|0011\rangle \\ &+ \langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle - \langle 0011|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle]\end{aligned} \tag{B.24}$$

which consists the four elements $\langle 0011|\sigma_1\sigma_2\sigma_3\sigma_4|0011\rangle$, $\langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|0011\rangle$, $\langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle$ and $\langle 0011|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle$. Noting that the sigma operator in an arbitrary direction is given as A.4 and $\langle 0|$ and $\langle 1|$ are given as unit vectors in up and down directions along $\hat{z}$ the following relations said to hold

$$\langle 0|\sigma_n|0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos(\theta)$$

$$\langle 0|\sigma_n|1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-i\phi}\sin(\theta)$$

$$\langle 1|\sigma_n|0\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{i\phi}\sin(\theta)$$

$$\langle 1|\sigma_n|1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\cos(\theta)$$

and also noting that

$$\langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle = \langle 1|\sigma_1|1\rangle_1 \langle 1|\sigma_2|1\rangle_2 \langle 0|\sigma_3|0\rangle_3 \langle 0|\sigma_4|0\rangle_4$$

which can be generalized to other elements as well, the explicit form of these are

$$\langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle = \cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_4)$$

$$\langle 0011|\sigma_1\sigma_2\sigma_3\sigma_4|0011\rangle = \cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_4)$$

$$\langle 1100|\sigma_1\sigma_2\sigma_3\sigma_4|0011\rangle = e^{i\phi_1}\sin(\theta_1)e^{i\phi_2}\sin(\theta_2)e^{-i\phi_3}\sin(\theta_3)e^{-i\phi_4}\sin(\theta_4)$$

$$\langle 0011|\sigma_1\sigma_2\sigma_3\sigma_4|1100\rangle = e^{-i\phi_1}\sin(\theta_1)e^{-i\phi_2}\sin(\theta_2)e^{i\phi_3}\sin(\theta_3)e^{i\phi_4}\sin(\theta_4)$$

Adding these to obtain eqn. B.24 will result in

$$\begin{aligned}\langle \sigma_1\sigma_2\sigma_3\sigma_4\rangle &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_4)\\ -\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\sin(\theta_4)&\cos(\phi_1+\phi_2-\phi_3-\phi_4)\end{aligned} \tag{B.25}$$

B.5.2

Multiplying the equations 5.15, 5.16 and 5.17 will give

$$A(2\phi,\lambda)(A(\phi,\lambda))^2(B(0,\lambda))^2B(0,\lambda)(C(\phi,\lambda))^2C(0,\lambda)(D(\phi,\lambda))^2D(0,\lambda) = -1 \tag{B.26}$$

since $N(\phi,\lambda) = \pm 1$ for $N = A, B, C, D$ it can easily be argued that $(A(\phi,\lambda))^2 = (B(0,\lambda))^2 = (C(\phi,\lambda))^2 = 1$ hence giving the simplifed equation

$$A(2\phi,\lambda)B(0,\lambda)C(0,\lambda)D(0,\lambda) = -1 \tag{B.27}$$

and comparing this to eqn. 5.14 which is

$$A(0,\lambda)B(0,\lambda)C(0,\lambda)D(0,\lambda) = -1 \tag{B.28}$$

will give

$$A(2\phi,\lambda) = A(0,\lambda)$$

for all ϕ

B.5.3

The relation 4.12 states that

$$f(A_1A_2A_3) = f(A_1)f(A_2)f(A_3)$$

for three commuting operators A_1, A_2, A_3. Hence the equation

$$f({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3)f({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)f({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)f({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3)$$

can be written in the form

$$f(({\sigma_x}^1\otimes{\sigma_x}^2\otimes{\sigma_x}^3)({\sigma_x}^1\otimes{\sigma_y}^2\otimes{\sigma_y}^3)({\sigma_y}^1\otimes{\sigma_x}^2\otimes{\sigma_y}^3)({\sigma_y}^1\otimes{\sigma_y}^2\otimes{\sigma_x}^3))$$

For this to be written the only condition is that the set of operators ${\sigma_x}^1{\sigma_x}^2{\sigma_x}^3$, ${\sigma_x}^1{\sigma_y}^2{\sigma_y}^3$, ${\sigma_y}^1{\sigma_x}^2{\sigma_y}^3$ and ${\sigma_y}^1{\sigma_y}^2{\sigma_x}^3$ should be commuting. The commutation relationships are

$$\begin{aligned}
[{\sigma_x}^1{\sigma_x}^2{\sigma_x}^3, {\sigma_x}^1{\sigma_y}^2{\sigma_y}^3] &= ({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3)({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3) - ({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3) \\
[{\sigma_x}^1{\sigma_x}^2{\sigma_x}^3, {\sigma_y}^1{\sigma_x}^2{\sigma_y}^3] &= ({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3)({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3) - ({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3) \\
[{\sigma_x}^1{\sigma_x}^2{\sigma_x}^3, {\sigma_y}^1{\sigma_y}^2{\sigma_x}^3] &= ({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3)({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3) - ({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3)({\sigma_x}^1{\sigma_x}^2{\sigma_x}^3) \\
[{\sigma_x}^1{\sigma_y}^2{\sigma_y}^3, {\sigma_y}^1{\sigma_x}^2{\sigma_y}^3] &= ({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3) - ({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3) \\
[{\sigma_x}^1{\sigma_y}^2{\sigma_y}^3, {\sigma_y}^1{\sigma_y}^2{\sigma_x}^3] &= ({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3)({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3) - ({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3)({\sigma_x}^1{\sigma_y}^2{\sigma_y}^3) \\
[{\sigma_y}^1{\sigma_x}^2{\sigma_y}^3, {\sigma_y}^1{\sigma_y}^2{\sigma_x}^3] &= ({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3) - ({\sigma_y}^1{\sigma_y}^2{\sigma_x}^3)({\sigma_y}^1{\sigma_x}^2{\sigma_y}^3)
\end{aligned} \tag{B.29}$$

See that

$${\sigma_x}^1 \otimes {\sigma_x}^2 \otimes {\sigma_x}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$${\sigma_x}^1 \otimes {\sigma_y}^2 \otimes {\sigma_y}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_y{}^1 \otimes \sigma_x{}^2 \otimes \sigma_y{}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\sigma_y{}^1 \otimes \sigma_y{}^2 \otimes \sigma_x{}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using these the commutation relationships of B.29 can be shown all to be zero, hence the set of operators ($\sigma_x{}^1\sigma_x{}^2\sigma_x{}^3$, $\sigma_x{}^1\sigma_y{}^2\sigma_y{}^3$, $\sigma_y{}^1\sigma_x{}^2\sigma_y{}^3$, $\sigma_y{}^1\sigma_y{}^2\sigma_x{}^3$) is commuting. Allowing

$$f((\sigma_x{}^1 \otimes \sigma_x{}^2 \otimes \sigma_x{}^3)(\sigma_x{}^1 \otimes \sigma_y{}^2 \otimes \sigma_y{}^3)(\sigma_y{}^1 \otimes \sigma_x{}^2 \otimes \sigma_y{}^3)(\sigma_y{}^1 \otimes \sigma_y{}^2 \otimes \sigma_x{}^3))$$

to be written. Multiplying the matrix forms of operators will give the result

$$f(-I) = -f(I) = -1$$

the minus of the identity operator. Since the rest of the equations in 5.27 are already self-evident and state independent due to the fact that $f(\sigma_i{}^j)^2 = 1$ independent of i's or j's, with the above relation established the entire form of 5.27 can shown to be state independent.