On the Investment Implications of Bankruptcy Laws Under Sequential Investment Decisions

by Ayşe Yeliz Kaçamak

Submitted to the Social Sciences Institute
in partial fulfillment of the requirements for the degree of
Master of Arts

Sabancı University
July 2011

ON THE INVESTMENT IMPLICATIONS OF BANKRUPTCY LAWS UNDER SEQUENTIAL INVESTMENT DECISIONS

APPROVED BY

Assoc. Prof. Dr. Özgür Kıbrıs ....
(Thesis Supervisor)

Assist. Prof. Dr. Mehmet Barlo ....

Assist. Prof. Dr. Şerif Aziz Şimşir ....

DATE OF APPROVAL: ....

Acknowledgements

I would like to thank my thesis advisor Özgür Kıbrıs, for his continuous support, and helpful comments throughout this thesis. Furthermore, I am grateful for him, Mehmet Barlo and İzak Atiyas, for providing many opportunities,encouraging and supporting me throughout my undergraduate and graduate education. Also, all Economics Faculty deserve my gratitude for their sincerity, and supporting me through hard times.

I am also grateful to Can Ürgün, who never stopped supporting me, for everything. Both my undergraduate and graduate years would not be same without him.

I also would like to thank my friends, Tansel Uras, Gizem Çavuşlar, Halit Erdoğan, Haluk Çitçi, Zeynel Harun Alioğulları and Rüştü Duran, for their invaluable friendship, support and help they provide during my graduate years and in the process of this thesis.

Finally my family deserves most of my gratitude, for their continuous support and love they provide throughout my life.

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Ayşe Yeliz Kaçamak
Economics, M.A. Thesis, 2011
Supervisor: Özgür Kıbrıs

Abstract

Keywords: bankruptcy, noncooperative investment game, proportional, equal awards, equal losses.

Axiomatic analysis of bankruptcy problems reveals three major principles: (*i*) proportionality (PRO), (*ii*) equal awards (EA), and (*iii*) equal losses (EL). However, most real life bankruptcy procedures implement only the proportionality principle. We construct a noncooperative investment game with sequential investment decisions to explore whether the explanation lies in the alternative implications of these principles on investment behavior. Our results are as follows. First EL always induces higher total investment than PRO which in turn induces higher total investment than EA;this is consistent with the findings of Kıbrıs and Kıbrıs (2010) who analyze an investment game with simultaneous investment decisions. Second, we observe that under both EA and EL, changing the order of moves from simultaneous to sequential effects increases total investment, independent of the identity of the first-mover. Finally, we also compare these principles in terms of social welfare induced in equilibrium and have following results. A switch from PRO to EA or EL, may decrease both egalitarian and utilitarian welfare independent of the setting used. Moreover, a transition from a simultaneous case to a sequential case may increase egalitarian welfare independent of the rule used. This transition may also increase utilitarian social welfare under EA but decrease it under EL.

SIRALI YATIRIM KARARLARI ALINAN ORTAMLARDA İFLAS KURALLARININ YATIRIMCI DAVRANIŞINA ETKİSİ

Ayşe Yeliz Kaçamak
Ekonomi Yüksek Lisans Tezi, 2011
Tez Danışmanı: Özgür Kıbrıs

Özet

Anahtar Kelimeler: iflas, işbirlikçi olmayan yatırım oyunu, orantısal, eşit kazançlar, eşit kayıplar.

İflas problemlerinin aksiyomatik analizi üç ana orensibi ortaya çıkarıyor: (*i*) orantısallık (PRO), (*ii*) eşit kazançlar (EA), ve (*iii*) eşit kayıplar (EL). Fakat gerçek dünyadaki iflas süreçlerinde sadece proportionality principle uygulanmakta. Bu uygulamadaki sebeplerin yatırım davranışından kaynaklanıp kaynaklanmadığını incelemek amacıyla işbirlikçi olmayan, ardışık kararların alındığı bir yatırım oyunu kuruyoruz. Elde ettiğiniz sonuçlar şöyle. Öncelikle EL her zaman PRO'dan PRO da EA'dan daha yüksek toplam yatırıma teşvik ediyor. Bu bulgular eşzamanlı yatırım kararlarının verildiği bir yatırım oyunu analize eden Kıbrıs ve Kıbrıs (2010)'la tutarlıdır. İkinci olarak eşzamanlı bir oyundan, sıralı bir oyuna geçildiğinde ilk hareket eden oyuncunun kimliğinden bağımsız olarak, hem EA hem EL altında toplam yatırımı artıyor. Son olarak bu prensipleri denge halinde yarattıkları sosyal refah açısından karşılaştırıyoruz, bu karşılaştırma sonucu elde ettiğimiz sonuçlar şöyle. PRO'dan hem EL'ye hem de EA'ya geçtiğimizde, egalitaryan ve utilitaryan refahta düşüş gözlemlenebilir. Ek olarak eşzamanlı kararların alındığı bir oyundan, ardışık kararların alındığı bir oyuna geçtiğimizde, hangi kuralın kullanıldığından bağımsız olarak egalitaryan refahta artış gözlemlenebilir. Bu geçiş EA altında, utilitaryan refahta da artış sağlayabilirken, EL altında, düşüşe sebep olabilir.

TABLE OF CONTENTS

1. Introduction 1

2.Model..10

3. Equilibria Under Alternative Bankruptcy Rules 13

3.1 When Big Investor Moves First.............................13

3.2 When Small Investor Moves First..........................16

4.Comparisons of Equilibrium Investment............................20

4.1 Individual Investment Decisions............................20

4.2 Total Investment...24

5.Comparisons of Equilibrium Welfare................................27

5.1 Egalitarian Social Welfare Levels..........................30

5.2 Utilitarian Social Welfare Levels...........................35

6. Conclusion...40

References..43

7. Appendix: Proofs...47

8. Figures...69

1 Introduction

Following the seminal work of O'Neill (1982), a vast literature focused on the axiomatic analysis of "bankruptcy problems". As the name suggests, a canonical example to this problem is the case of a bankrupt firm whose monetary worth is to be allocated among its creditors. Each creditor holds a *claim* on the firm and the firm's liquidation value is less than the total of the creditors' claims. The axiomatic literature provided a large variety of "bankruptcy rules" as solutions to this problem. The most central of these rules are all based on one (or more) of three central principles: (*i*) **proportionality**, (*ii*) **equal awards**, and (*iii*) **equal losses**.[1]

Bankruptcy has also been a central topic in corporate finance where researchers analyze a large number of issues related to it (*e.g.* see Hotchkiss *et al* (2008)).[2] This literature shows that, in practice almost every country uses the following rule to allocate the liquidation value of a bankrupt firm.[3] First, creditors are sorted into different priority groups (such as secured creditors or unsecured creditors). These groups are served sequentially. That is, a creditor is not awarded a share until creditors in higher priority groups are fully

[1] As their names suggest, these principles suggest that the agents' shares should be chosen, respectively, (*i*) proportional to their investments, (*ii*) so as to equate their awarded shares, (*iii*) so as to equate their losses from initial investment. There are bankruptcy rules purely based on one of these principles (such as the Proportional, Constrained Equal Awards, Constrained Egalitarian, Constrained Equal Losses rules) as well as rules that apply different principles on different types of problems (such as the Talmud rule which uses both equal awards and equal losses principles).

[2] This is not surprising considering that in US between 1999 and 2009, more than 551000 firms filed for Chapter 7 bankruptcy and more than 22.16 billion USD were allocated in these cases (see http://www.justice.gov/ust/index.htm).

[3] Procedures on the liquidation of the firm and its allocation among creditors exist in bankruptcy laws of every country. For examples, see Chapter 7 of the U.S. Bankruptcy Code or the Receivership code in U.K. In some countries such as Sweden or Finland, these procedures provide the only option for the resolution of bankruptcy. Bankruptcy laws of some other countries, such as U.S., also offer procedures (such as Chapter 11) for reorganization of the bankrupt firm.

reimbursed. Second, in each priority group, the shares of the creditors are determined *proportional* to their claims.[4]

In this paper, we explore why in actual bankruptcy laws, *proportionality* has been preferred over the other two principles. Our starting observation is that alternative bankruptcy rules affect the investment behavior in a country in different ways. In a way, each rule induces a different noncooperative game among the investors. Comparing the equilibria of these games, in terms of total investment or social welfare, might provide us ways of comparing alternative bankruptcy rules and thus, the principles underlying them, in a way that is not previously considered in either the axiomatic literature or the corporate finance literature on bankruptcy, both discussed at the end of this section.

This paper follows up on Kıbrıs and Kıbrıs (2010) who analyzes the above question using games where the investors move simultaneously. Their analysis is aimed to model investment situations where the investors are not informed about the others' decisions. This however, is not purely the case in many real life settings. Firms going public could be a good example for a sequential setting where big investor(s) move first, also venture capital can be a good example for a sequential setting where small investor(s) move first. To analyze such environments, in this paper we construct a game where the investment decisions are sequential. Finally, we also use the sequential model to ask mechanism-design type questions. More specifically, we compare the equilibria of the simultaneous and sequential versions of the investment game to see if total investment in the economy or equilibrium social welfare can be improved by a particular design of the order of moves.

As a representation of the proportionality principle, we use the *Proportional rule* (hereafter, **PRO**), which assigns each investor a share proportional to his investment. We then look at the "unconstrained equal awards rule" *(***EA***)*

[4] This is a very old and common practice, referred to as a *pari passu* distribution; the term meaning "proportionally, at an equal pace, without preference" (see Black's Law Dictionary, 2004).

which always chooses equal division as a representation of the equal awards principle. Thirdly as a representation of the equal losses principle we use the "unconstrained equal losses rule" *(***EL***)* which always equates the investors' losses.

For each one of these bankruptcy rules, we construct a simple game among 2 investors who sequentially choose how much money to invest in a firm. The total of these investments determine the value of the firm. The firm is a lottery which either brings a positive return or goes bankrupt. In the latter case, its liquidation value is allocated among the investors according to the prespecified bankruptcy rule. For each bankruptcy rule, we analyze the subgame perfect Nash equilibria of the corresponding investment game, under sequential setting, both the cases where big investor moves first and the cases where small investor moves first. (We capture this asymmetry in the agents by allowing them to differ in risk aversion levels, which will be explained below)[5]. We then compare these equilibria.

In our model, agents have Constant Absolute Risk Aversion preferences and are weakly ordered according to their degrees of risk aversion. (This ordering is without loss of generality since the agents are identical in other dimensions.) The agents do not face liquidity constraints and thus, their income levels are not relevant. However, as is standard in the literature, it is possible to interpret the agents' risk aversion levels as a decreasing function of their income levels. (Thus, less risk averse agents can be thought of as richer, bigger investors.) Alternatively, each agent can be taken to represent an investment fund. In this case, the income level is irrelevant. The risk-aversion parameter attached to each investment fund then represents the type of that fund. Since we do not restrict possible configurations of risk aversion, this allows us to compare the three principles in terms of how they treat different types of agents (such as big

[5] The characteristics of the rules are different, in the sense how they treat different agents,. Therefore, by changing the order of moves, we try to observe how the rules treat the agents according to their identities

versus small investors) as well as how they react to changes in the risk-aversion distribution.

Our analysis compares bankruptcy rules in terms of two criteria. Our first criterion is **total equilibrium investment** which is a simple measure of how a bankruptcy rule affects investment behavior in the economy. It is reasonable to think that a government prefers bankruptcy rules that induce higher total investment in the economy. Thus, a bankruptcy rule that induces higher total investment than *PRO* might be considered a superior alternative to it. On the other hand, it is not clear that an increase in total investment will also increase the welfare of the investors. Thus, our second criterion is **equilibrium social welfare**. Egalitarianism and utilitarianism present two competing and central notions of measuring social welfare. We therefore compare bankruptcy rules in terms of both **egalitarian** and **utilitarian** social welfare that they induce in equilibrium.

Before going any further it will be beneficial to describe the findings of Kıbrıs and Kıbrıs (2010) in more detail. Their model is similar to ours except (*i*) there is an arbitrary number of agents and (*ii*) investment decisions are taken simultaneously. Like us, they compare Nash equilibria of investment games (induced by different bankruptcy rules) in terms of (*i*) total equilibrium investment and (*ii*) equilibrium social welfare (both egalitarian and utilitarian). They compare the Nash equilibria of the investment games under PRO, EA, and EL as well as mixtures of PRO and EA and mixtures of PRO and EL. Their main results are as follows. The investment game has a unique Nash equilibrium for every parameter combination and for each bankruptcy rule. These equilibria are such that, at all parameters values (*i*) *EL* induces higher total investment than *PRO* which in turn induces higher total investment than *EA*; (*ii*) *PRO* induces higher egalitarian social welfare than both *EA* and *EL* in interior equilibria; (*iii*) *PRO* induces higher utilitarian social welfare than *EL* in interior equilibria but its relation to *EA* depends on the parameter values (however, a numerical analysis shows that on a large part of the parameter space, PRO induces higher utilitarian social welfare than EA). Thus, in the confines of their simple model, PRO outperforms EA in almost every criterion. Also, switching from PRO to EL increases total investment but decreases both egalitarian and utilitarian social welfare. The rest of the related literature will be given at the end of the introduction section.

A summary of our main results is as follows. Independent of the order of moves, the investment game has a unique subgame perfect Nash equilibrium for every parameter combination and for each bankruptcy rule. These equilibria exhibit the following properties. In terms of individual investment levels (i) under EA, compared to the simultaneous-move game, both agents invest more under the sequential setting, independent of the identity of the first mover; (ii) under EL, compared to the simultaneous-move game, both agents invest more if they are the first mover, and they invest less if they are the second mover (and this finding is independent of their risk aversion levels); (iii) under PRO, the order of moves has no effect on the individual investment decisions. In terms of total investment, (iv) EL does better than PRO which in turn does better than EA; (v) independent of the order of moves, both EL and EA perform better under sequential setting than under simultaneous setting; (vi) under PRO, order of moves has no effect on total investment.

To compare social welfare induced by rules we mostly conduct numerical analysis. This analysis show that in terms of social welfare (vii) PRO induces higher egalitarian social welfare than EA which in turn induces higher egalitarian social welfare than EL in interior equilibria independent of the setting ; $(viii)$ Under EA when big investor is the first mover utilitarian social welfare is maximized. Both PRO and EA induce higher utilitarian social welfare than EL in interior equilibria independent of the setting.(ix)When EA (EL) is considered a transition from a simultaneous case to a sequential case may increase(decrease) egalitarian and utilitarian social welfare independent of the order of moves.

PRO is advantageous to the other rules also in the sense that only under PRO do the investors have dominant strategies (which are strictly dominant). Thus, for planning purposes, the government has a stronger prediction on investor behavior under PRO.

Finally, potential heterogeneity of the agents' risk attitudes plays an important role in our analysis. Bankruptcy rules are very different in terms of the incentives that they provide for big versus small investors. The equal losses principle offers relatively better protection to the bigger (i.e. less risk averse) investors. The equal awards principle does the opposite. The proportionality principle strikes a compromise by offering the same proportion of their investment to every agent. We also observe that under different rules an agent reacts very differently to changes in the others' risk attitudes: under EA (EL) his investment decreases (increases) as the other agents get more risk averse; under PRO, however, his investment remains constant. This once again makes the equilibrium prediction under PRO more reliable since under PRO, the agents, in determining their investment strategies, need not be informed about the risk-aversion (or income) profile of the other investors. A detailed summary of our findings as well as their possible policy implications is presented in *Section 6*.

The paper is organized as follows. In *Section 2*, we present the model. In *Section 3*, we calculate and analyze the subgame perfect Nash equilibrium induced by each rule. In *Section 4*, we compare bankruptcy rules in terms of individual investments and the total investment they induce in equilibrium. In *Section 5*, we then compare them in terms of egalitarian and utilitarian social welfare. We summarize our findings and conclude in *Section 6*. *Appendix* contains the proofs. Finally, *Figures* contains graphs used throughout the text..

Related Literature.

The axiomatic literature on bankruptcy and taxation problems contains many studies that analyze the properties of alternative bankruptcy rules. For example, Dagan (1996), Schummer and Thomson (1997), Herrero and Villar (2002), and Yeh (2001) analyze properties of CEA. Yeh (2001a), Herrero and Villar (2002), and Herrero (2003) analyze properties of CEL. Aumann and Maschler (1985) and the following literature discuss properties of a Talmudic rule. O'Neill (1982), Moulin (1985a,b), Young (1988), Chun (1988a), de Frutos (1999), Ching and Kakkar (2000), Chambers and Thomson (2002), and Ju, Miyagawa, and Sakai (2007) analyze properties of PRO. Thomson (2003 and 2008) presents a detailed review of the extensive axiomatic literature.

The corporate finance literature also contains a large number of papers that study bankruptcy (e.g. see Bebchuck (1988), Aghion, Hart, and Moore (1992), Atiyas (1995), Hart (1999), Stiglitz (2001)). However, most of these papers study reorganization procedures such as *Chapter 11* in the US. There are some papers that discuss liquidation procedures (and some, such as Baird (1986) argue that they are superior to reorganization procedures). For example, Bris, Welch, and Zhu (2006) use a comprehensive data set from the US to compare liquidation and reorganization procedures in terms of costs and efficiency. Stromberg (2000) uses Swedish data to evaluate liquidation procedures. Also, Hotchkiss *et al* (2008) summarize bankruptcy laws in different countries and as part of it, describe liquidation procedures (as these constitute the only resolution to bankruptcy in some countries). Finally, there are studies that discuss the implications of priority classes on investor behavior. However, all of these studies take the existing proportional allocation rule (*i.e.* PRO) as a given, nonchanging constant and does not discuss its merits in relation to alternative rules.

There are previous papers that employ game theoretical tools to analyze bankruptcy problems. Aumann and Maschler (1985), Curiel, Maschler, and Tijs, (1987), and Dagan and Volij (1993) relate bankruptcy rules to cooperative game theoretical solutions. Chun (1989) presents a noncooperative game that implements an egalitarian surplus sharing rule. Dagan, Serrano, and Volij (1997) present a noncooperative game that implements a large family of consistent bankruptcy rules by employing the rule's two-person version in the design. Chang and Hu (2008) carry out a similar analysis for a class of "f-just" rules. Herrero (2003) implements the CEA and CEL bankruptcy rules. Garcia-Jurado, Gonzalez-Diaz, and Villar (2006) present noncooperative games for a large class of "acceptable" rules. Finally, Eraslan and Yılmaz (2007) and the literature cited therein analyze negotiation games that arise during reorganization of the bankrupt firm. None of these papers however focus on investment implications of these bankruptcy rules.

This thesis is closely related to Kıbrıs and Kıbrıs (2010), previously described, and Karagözoğlu (2010) who also designs a noncooperative game and analyzes investment implications of a class of rules that include PRO, CEA, and CEL. Aside from the fact that Karagözoğlu considers the constrained rules CEA and CEL, the main differences are as follows. In Karagözoğlu (2010) model, (i) there are two types of agents (high income and low income) who (ii) simultaneously choose either zero or full investment of their income, and (iii) the agents are risk neutral. Due to these differences, our results are quite different. In Karagözoğlu (2010), PRO maximizes total investment whereas in our setting, the maximizer of total investment is the EL (as seen in Section 4).[6] On the other hand, both studies find PRO to induce higher total investment than EA and CEA, respectively. Also, Karagözoğlu (2010) does not carry out a welfare analysis but additionally analyzes a class of rules

[6]According to Kıbrıs and Kıbrıs, this difference is due to two reasons. First, Karagözoğlu uses binary strategies and this limits the sensitivity of equilibrium total investment to the problem's parameters. Thus, when in binary strategies the two rules induce equal investment, it might be that EL exceeds PRO when we take into account how much the agents do invest. The second reason is the difference between EL and CEL.They show in *Appendix B* that CEL induces more types of equilibria than EL and in some of them, PRO induces more total investment than CEL.

that includes the Talmud rule (the TAL family by Moreno-Ternero and Villar, 2006) and discusses the case of two firms.

2 Model

Let $N = \{1, 2\}$ be the set of agents. Each $i \in N$ has the following **Constant Absolute Risk Aversion (CARA) utility function** $u_i : \mathbb{R}_+ \to \mathbb{R}$ on money: $u_i(x) = -e^{-a_i x}$. Assume that, each $i \in N$ is risk averse, that is, $a_i > 0$. Also assume that $a_1 \leq a_2$.

Each agent i **invests** $s_i \in \mathbb{R}_+$ units of wealth on a risky company. The company has value $\sum_N s_i$ after investments. With **success probability** $p \in (0, 1)$, this value brings a **return** $r \in (0, 1]$ and becomes $(1 + r) \sum_N s_i$. With the remaining probability $(1-p)$, the company goes bankrupt and its value becomes $\beta \sum_N s_i$ where $\beta \in (0, 1)$ is the **fraction that survives bankruptcy**.

In case of bankruptcy, the value of the firm is allocated among the agents according to a prespecified bankruptcy rule. Formally, a **bankruptcy problem** is a vector of claims (*i.e.* investments) $s = (s_1, .., s_n) \in \mathbb{R}^n_+$ and an endowment $E \in \mathbb{R}_+$ satisfying $\sum_N s_i \geq E$. In our model, the bankrupt firm retains β fraction of its capital.[7] Thus $E = \beta \sum_N s_i$ is a function of s. As a result, the vector s (together with β) is sufficient to fully describe the bankruptcy problem at hand. Thus in our setting, the class of all bankruptcy problems is simply $\mathbb{R}^3_+$.

A **bankruptcy rule** F assigns each $s \in \mathbb{R}^2_+$ to an allocation $x \in \mathbb{R}^2$ satisfying $\sum_N x_i = \beta \sum_N s_i$. In this paper, we will focus on the following bankruptcy rules. The **Proportional Rule (PRO)** is defined as follows: for each $i \in N$, $PRO_i(s) = \beta s_i$. The **Equal Awards rule (EA)** is defined as $EA_i(s) = \frac{\beta}{2} \sum_N s_i$.The **Equal Losses rule (EL)** is defined as $EL_i(s) = s_i - \frac{1-\beta}{2} \sum_N s_j$.

For each bankruptcy rule F, we analyze two **sequential investment games** it induces over the agents. The two games only differ in terms of which agent moves first. Thus, throughout the paper we will refer to these games as

[7] This assumption is supported by empirical evidence from Bris, Welch, and Zhu (2006) who note that the firm scale is fairly unrelated to percent value changes in bankruptcy.

F_{12} and F_{21}, showing the order of moves. Finally since we will compare the sequential setting with the simultaneous setting analyzed in Kıbrıs and Kıbrıs (2010), we will use F_{sim} to denote the game under F in a simultaneous setting.

Without loss of generality, we will define Game F_{12} where agent 1 moves first and agent 2 moves second. Game F_{21}, where agent 2 moves first, is defined similarly. In Game F_{12}, agent 1's **strategy set** is $S_1 = \mathbb{R}_+$ from which he chooses an investment level s_1. Agent 2's strategy set is the set of functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, or formally $S_2 = \mathbb{R}_+^{\mathbb{R}_+}$. Let $S = \prod_N S_i$. However with an abuse of notation we will refer to S_2 as the set of actions (values) induced by the strategies (functions) in the whole strategy set given s_1.

A strategy profile $s \in S$ corresponds for agent i to the lottery that brings the net return $(1+r)s_i - s_i = rs_i$ with probability p and the net return $F_i(s) - s_i$ with the remaining probability $(1-p)$. Note that $F_i(s) - s_i \leq 0$. The interpretation is that the agent initially borrows s_i at an interest rate normalized to 0. If the investment is successful, he receives $(1+r)\, s_i$, pays back s_i, and is left with his profit rs_i. In case of bankruptcy, he only receives back $F_i(s)$ and has to pay back s_i, so his net profit becomes $F_i(s) - s_i$. The same lottery is obtained from an environment where each agent i allocates his monetary endowment between a riskless asset (whose return is normalized to 0) and the risky company. In this second interpretation, assume that the agent does not have a liquidity constraint. That is, he is allowed to invest more than his endowment. This assumption only serves to rid us from (the rather unrealistic) boundary cases where some agents spend all their monetary endowment on the risky firm. Alternatively, one can impose a liquidity constraint but focus on equilibria which are in the interior of the strategy spaces.

Agent i's expected payoff from a strategy profile $s \in S$ is thus

$$\begin{aligned} U_i^F(s) &= pu_i(rs_i) + (1-p)u_i(F_i(s) - s_i). \\ &= -pe^{-a_i r s_i} - (1-p)e^{-a_i F_i(s_i, s_{-i}) + a_i s_i} \end{aligned} \tag{1}$$

Let $U^F = \left(U_1^F, U_2^F\right)$. Let $H = \{\emptyset\} \cup (\mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_+)$ be the history set of the investment game where $\emptyset$ denotes the unique starting history. Moreover let $Z = \mathbb{R}_+ \times \mathbb{R}_+$ denotes the set of terminal nodes hence $H/Z = \{\emptyset\} \cup (\mathbb{R}_+)$ denotes the set of non-terminal (decision) nodes. As mentioned before we are going to look at two games; (i) in Game F_{12}, the big investor moves first and the small investor moves after observing the big investor's action, (ii) in Game F_{21}, the small investor moves first and, observing his action the big investor makes his decision. Therefore, the player functions $P_{12} : H/Z \to N$ and $P_{21} : H/Z \to N$ will be defined as follows: $P_{12}(h) = \begin{cases} 1 & if \quad h = \emptyset, \\ 2 & if \quad o/w \end{cases}$ when the big investor is the first mover and will be defined as $P_{21}(h) = \begin{cases} 2 & if \quad h = \emptyset, \\ 1 & if \quad o/w \end{cases}$ when the small investor moves first.

The **sequential investment game induced by the bankruptcy rule F and with the order of moves** ij is then defined as $G^{F_{ij}} = \langle S, H, P_{ij}, U^F \rangle$. Let $\epsilon(G^{F_{ij}})$ denote the set of subgame perfect Nash **equilibria** of G^F.

3 Equilibria Under Alternative Bankruptcy Rules

We start by analyzing the subgame perfect Nash equilibria of each game. This section serves as a preliminary for our comparisons of individual and total investment (in Section 4) and welfare (in Section 5).

3.1 When Big Investor Moves First

Throughout the first part of this section we assume that agent 1 moves first. He is called the big investor because he has relatively lower risk aversion level.

Proportional Rule (PRO):

The following proposition shows that under PRO, the investment game has a strictly dominant strategy equilibrium. Moreover, this strictly dominant strategy equilibrium is independent of the order of moves.

Proposition 1 *If* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \leq 0$, *the investment game under the rule* PRO *has a unique dominant strategy equilibrium* $(0, ..., 0)$. *Otherwise, the game has a strictly dominant strategy equilibrium* s^* *in which each agent* i *chooses a positive investment level* s_i^* *given by*

$$s_i^* = \frac{1}{a_i\,(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right).$$

There is no other Nash equilibria.

Note that, if $pr > (1-p)(1-\beta)$, all agents choose a positive investment level at the dominant strategy equilibrium. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of success, p, with the loss incurred on unit investment in case of failure, $(1-\beta)$, weighted by the probability of failure, $(1-p)$. Investing in the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure.

Equilibrium investment levels are ordered as $s_1^* \geq s_2^*$. Also, s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

Equal Awards (EA):

The following proposition determines the form of the subgame perfect Nash equilibria under EA where the big agent is the first mover.

Proposition 2 *At the unique subgame perfect Nash equilibrium of* Game under EA_{12} *the equilibrium investment actions are as follows:*
(i) if

$$\frac{2pr}{(1-p)(2-\beta)} \geq 1$$

or

$$\frac{2pr}{(1-p)(2-\beta)} < 1$$

and

$$\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{a_1(2+2r-\beta)}{\beta a_2}} + \left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) > 0$$

then

$$\begin{aligned} s_1^* &= \frac{\left(a_2(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_2a_1(r+1)(r-\beta+1)} \\ s_2^* &= \frac{\left(a_2\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)} \end{aligned}$$

(ii) otherwise

$$
\begin{aligned}
s_1^* &= 0 \\
s_2^* &= 0
\end{aligned}
$$

Equilibrium investment levels are ordered as $s_1^* \geq s_2^*$. Also, s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

Equal Losses (EL):

The following proposition shows that the subgame perfect Nash equilibrium under EL when the big investor is the first mover, is of the form $s_1^* \geq s_2^*$.

Proposition 3 *At the unique subgame perfect Nash equilibrium of* Game under EL_{12} *the equilibrium investment actions are as follows:*
(i) if

$$\frac{2pr}{(1-\beta)(1-p)} > 1$$

and

$$\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) < \left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\dfrac{a_1(1+2r-\beta)}{(1-\beta)a_2}}$$

then

$$s_1^* = \frac{\left(a_2(2r+(1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) - a_1(1-\beta)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 r a_1 (r+1-\beta)\, 2}$$

$$s_2^* = \frac{\left(-a_2(1-\beta)\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + \ln\frac{2pr}{(1-p)(1-\beta)}(2r+(1-\beta))\, a_1\right)}{a_2 r a_1 (r+1-\beta)\, 2}$$

(ii) if

$$\frac{2pr}{(1-\beta)(1-p)} > 1$$

and

$$(2r + (1-\beta))\, a_1 < (1-\beta)\, a_2$$

then

$$s_1^* = \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{(2r+1-\beta)a_1}$$
$$s_2^* = 0$$

(iii) *otherwise*

$$\begin{aligned} s_1^* &= 0 \\ s_2^* &= 0 \end{aligned}$$

Equilibrium investment levels are ordered as $s_1^* \geq s_2^*$. Also s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

3.2 When Small Investor Moves First

In the second part of this section we assume that agent 2 moves first. He is called the small investor because he has relatively higher risk aversion level.

Proportional Rule (PRO**):**

As noted in the previous subsection, the investment game under PRO has a unique dominant strategy equilibrium independent of the order of moves.

Equal Awards (EA**):**

The following proposition determines the form of the unique subgame perfect Nash equilibrium under EA where the small agent is the first mover.

Proposition 4 *At the unique subgame perfect Nash equilibrium of* Game under EA_{21} *the equilibrium investment actions are as follows:*

(*i*) *if*

$$\frac{2pr}{(1-\beta)(1-p)} \geq 1$$

or

$$\frac{2pr}{(1-p)(2-\beta)} < 1$$

and

$$\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{a_2(2+2r-\beta)}{\beta a_1}} + \left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) > 0$$

then

$$\begin{aligned} s_1^* &= \frac{\left(a_1\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)} \\ s_2^* &= \frac{\left(a_1(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)} \end{aligned}$$

(*ii*) *otherwise*

$$\begin{aligned} s_1^* &= 0 \\ s_2^* &= 0 \end{aligned}$$

Equilibrium investment levels do not have a clear ordering. However, s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

Equal Losses (EL):

The following proposition shows that the subgame perfect Nash equilibrium under EL when the small investor is the first mover.

Proposition 5 *At the unique subgame perfect Nash equilibrium of* Game under EL_{21} *the equilibrium investment actions are as follows:*
(i) if

$$\frac{2pr}{(1-\beta)(1-p)} > 1$$

and

$$\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(2r+(1-\beta))} > \left(\frac{2pr}{(1-p)(1-\beta)}\right)^{a_2(1-\beta)}$$

and

$$\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{a_2(1+2r-\beta)} > \left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(1-\beta)}$$

then

$$s_1^* = \frac{\left(a_2(2r+(1-\beta))\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) - a_1\ln\left(\frac{p(2r+(1-\beta))}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_2 r a_1 (r+1-\beta)\,2}$$

$$s_2^* = \frac{\left(a_1(2r+(1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) - a_2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_1 r a_2 (r+1-\beta)\,2}$$

(ii) if

$$\frac{2pr}{(1-\beta)(1-p)} > 1$$

and

$$\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{a_2(1+2r-\beta)} < \left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(1-\beta)}$$

then

$$s_1^* = 0$$

$$s_2^* = \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{(2r+1-\beta)a_2}$$

(iii) if

$$\frac{2pr}{(1-\beta)(1-p)} > 1$$

and

$$\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(2r+(1-\beta))} < a_2 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{(1-\beta)a_2}$$

then

$$s_1^* = \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{(2r+1-\beta)a_1}$$

$$s_2^* = 0$$

(iv) otherwise

$$\begin{aligned} s_1^* &= 0 \\ s_2^* &= 0 \end{aligned}$$

Equilibrium investment levels do not have a clear ordering. However, s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

4 Comparisons of Equilibrium Investment

4.1 Individual Investment Decisions

In this section, we compare bankruptcy rules in terms of the individual investment levels that they induce in equilibrium. We also look at the effect of the order of moves on investment behaviour. In order to see the effects of transition from a simultaneous setting to a sequential setting, let us first check the big investor's investment levels in a numerical example where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_2 = 10$; *Figures 1* and *2* respectively demonstrate s_1^* under EA and EL as a function of a_1. And then let us also check the small investor's investment levels in a numerical example where $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$; *Figures 3* and *4* respectively demonstrate s_2^* under EA and EL as a function of a_2.

As can be observed from the graphs no matter which rule we use the sequential case induces higher investment for the first mover when compared to the simultaneous setting. As can be seen in *Figure 1*, under EA, the investment levels of the (big) investor 1 are ordered as,

$$s_1^{EA,12} > s_1^{EA,21} > s_1^{EA,sim}.$$

Therefore, one can claim that the big investor increases his investment under EA with sequential moves whether he is the first mover or not. Under EA_{12} big investor is the first mover, therefore, it is reasonable for him to make higher investment than the investment he makes under EA_{sim}. Moreover the investment game under EA is supermodular; in other words if an agent increases his investment, the other agent's best response is to increase his investment as well. By the above logic we expect the small investor to increase his investment when he is the first mover, that is, in the case of EA_{21}. Observing this, due to supermodularity, the big investor will increase his investment as well. This ordering is not specific to this numerical example, in fact we have the following proposition.

Proposition 6 *In the interior subgame perfect Nash equilibrium of investment games under EA the big investor's investment decisions s_1^* have the following order*

$$s_1^{EA,12} > s_1^{EA,21} > s_1^{EA,sim}.$$

In *Figure 2*,under EL, s_1^* is ordered as,

$$s_1^{EL,12} > s_1^{EL,sim} > s_1^{EA,21}$$

Which leads to the idea that big investor increases his investment under EL with sequential moves when he is the first mover however he decreases his investment when he is the second mover.

The reason why big investor increases his investment when he is the first mover, under EL_{12} is the same as EA_{12}. However unlike EA_{21} he decreases his investment when he is the second mover. This is mainly because, investment game under EL is submodular. Under EL_{21} small investor will increase his investment decision since he is the first mover however due to submodularity, big investor's best response is to decrease his investment decision. These give us the following proposition.

Proposition 7 *In the interior subgame perfect Nash equilibrium of investment games under EL the big investor's investment decisions s_1^* have the following order.*

$$s_1^{EL,12} > s_1^{EL,sim} > s_1^{EL,21}$$

Figure 3 looks at the small investor, the more risk averse agent 2, under EA and shows that his equilibrium choice s_2^* increases under both EA_{12} and EA_{21} compared to the simultaneous case. However, unlike the big investor, his investment decision under EA does not have clear ordering for EA_{12} and EA_{21} since these two curves intersects at a certain risk aversion level. The intuition behind this intersection is as follows. Both investors increase their investment levels when they are the first movers, however, the amount of this increase is not same across the agents due to the difference in their risk aversion levels. As expected big investor, who has lower risk aversion level, increases his investment under EA_{12} more than the increase in small investor's investment decision under EA_{21}. When small investor's risk aversion level reaches a certain degree, the increase in his investment as a best response to the big agent under EA_{12} exceeds the increase in his investment under EA_{21}. These results are summarized in the following proposition.

Proposition 8 *In terms of small investor's investment decision s_2^*. In the interior subgame perfect Nash equilibrium of investment games under EA, simultaneous setting and sequential setting where big investor moves first have the following order:*

$$s_2^{EA,12} > s_2^{EA,sim}$$

simultaneous setting and sequential setting where small investor moves first have the following order:

$$s_2^{EA,21} > s_2^{EA,sim}$$

sequential setting where big investor moves first and sequential setting where small investor moves first do not have a clear ordering.

Figure 4 depicts the small investor's behavior under EL. Due to this figure small investor's investment decision ordering is as,

$$s_2^{EL,21} > s_2^{EL,sim} > s_2^{EL,12}$$

Although, at first glance this ordering seems to be different than the ordering of the big investor's investment decisions, the interpretation is same. The small investor makes the highest investment decision under EL_{21}, when he is the first mover. However, again due to submodularity he decreases his investment when he is the second mover, under EL_{12}, compared to EL_{sim}. Again this is not specific to this certain numerical example.

Proposition 9 *In the interior subgame perfect Nash equilibrium of investment games under EL the small investor's investment decisions* s_2^* *have the following order.*

$$s_2^{EL,21} > s_2^{EL,sim} > s_2^{EL,12}$$

In summary both agents increase their investment decisions under EA when they move sequentially, independent of the identity of the first mover, compared to the case they move simultaneously. Under EL both agents increase their investment decisions in a sequential setting if they are the first movers, but they decrease their investment when they are the second movers.

Now let us check individual investment levels in a numerical example for two investors where $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$ and for which *Figures 5* and *6* respectively demonstrate s_1^* and s_2^* as a function of a_2 for the three extreme rules under three settings: PRO, $EA_{12}, EA_{21}, EA_{sim}$, and $EL_{12}, EL_{sim}, EL_{21}$

As can be seen in *Figure 5*, in terms of s_1^*, the three rules are ordered as $EL > PRO > EA$ independent of the setting. Also, s_1^* is independent of a_2 under PRO but it is increasing (decreasing) in a_2 under EL (EA). This demonstrates a general phenomenon. The bigger investor, that is, the relatively less risk averse agent 1, faces very different incentives under the three rules. In case of bankruptcy, he is protected best by EL and worst by EA whereas his share under PRO is independent of the other agents. This reflects on his investment choices.

Figure 6 looks at the smaller investor, the more risk averse agent 2 and shows that under all three rules, his equilibrium choice s_2^* is decreasing in

a_2. Also, the three rules do not have a fixed order in terms of s_2^*. For low risk aversion levels (*i.e.* when agent 2 is not too different than agent 1), the ordering of the three rules in terms of s_2^* is $EL > PRO > EA$, same as s_1^*. But it is reversed for high risk aversion levels. For this case, agent 2 is protected best under EA and worst under EL and this reflects to his equilibrium investment choices under them. It is also interesting to note that, for risk aversion levels in between the two extremes, it is PRO that induces the highest investment level s_2^* on agent 2.

4.2 Total Investment

Looking at individual investment levels, one does not observe a clear ordering of the three rules. However, in terms of total investment, we obtain a strong result. The following three theorems establish that, in terms of total investment, the rules analyzed in the previous section are ordered as

$$EL_{12} > EL_{21} > EL_{sim} > PRO > EA_{12} > EA_{21} > EA_{sim}.$$

Theorem 1 *In the sequential investment game where the big investor moves first, EL induces a higher equilibrium total investment than PRO, which in turn induces a higher equilibrium total investment than EA.*

Theorem 2 *In the sequential investment game where the small investor moves first, EL induces a higher equilibrium total investment than PRO, which in turn induces a higher equilibrium total investment than EA.*

Theorem 3 *In terms of total investment, both EA and EL performs better under sequential setting than the simultaneous setting.*

Corollary 10 *By above three theorems in terms total investment we have the* *following ordering of rules*

$$EL_{12} > EL_{21} > EL_{sim} > PRO > EA_{12} > EA_{21} > EA_{sim}.$$

In the simultaneous case EL_{sim} induces higher investment than EA_{sim}. Moreover under sequential setting, although investment games under EA are supermodular and those under EL are submodular games, EL still induces higher investment independent of the first mover's identity. This is mainly because; when the big investor is the first mover, he increases his investment under EL_{12} more than he increases under EA_{12} as a best response to the small investor, because he is better protected under EL. And although, small investor decreases his investment under EL_{12}, the significant increase in the big agent's investment decision, causes EL_{12} to induce higher total investment level than EA_{12}.

In the case where small investor moves first ;under EL_{21}, as a best response to small investor, big investor decreases his investment decision. However, since he is better protected under EL, this decrease is not a drastic one. So the increase in small investor's investment level is more than enough to compensate this loss, resulting EL_{21} to induce higher investment than EL_{sim}. On the other hand although both agents increase their investment levels under EA_{21}, the total increase is still is not enough to exceed even EL_{sim}. Therefore by above logic we have $EL_{21} > EA_{21}$.

It is interesting to note that, even when all agents are identical in terms of risk aversion, the ordering of the rules in terms of total investment is as above. Particularly, EL still induces more total investment than the other rules. This means that these rules not only differ in terms of how they treat big versus small investors (as discussed at the beginning of this section), but they also differ in terms of the investment incentives that they provide in a symmetric game where all agents are identical in terms of risk aversion. Moreover, even when the risk aversion levels are equal sequential case still induces higher total investment . This can be observed in *Figures 5* and *6* by choosing $a_2 = a_1 = 3$.

5 Comparisons of Equilibrium Welfare

In this section, we look at the individual and social welfare levels induced by the Nash equilibria under the PRO, EA, and EL rules with aforementioned three settings. We compare these three rules under three settings in terms of both egalitarian and utilitarian social welfare.[8]

In *Figure 7*, we fix $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_2 = 10$ and demonstrate individual welfare level of agent 1 as a function of a_1. As noted above, an agent's welfare under PRO is independent of his risk aversion level. Thus, it remains constant at -0.6. The individual welfare under both EA and EL depends on a_1 independent of the setting. At relatively smaller risk aversion levels agent 1 receives highest payoff under EL rules. However, as he gets more risk averse his payoff decreases under EL but increases under EA. This is because of the characteristics of these two rules. Big (small) agent is more protected under $EL(EA)$ than $EA(EL)$. This effect is obvious when two agents differ in terms of risk aversion drastically. However, as the difference in risk aversion levels gets smaller, the difference in payoffs agent 1 receives from these two rules also gets smaller.

In *Figure 8* we can see individual welfare level of agent 2 as a function of a_2 where $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_1 = 3$ is fixed. Similarly like agent 1, agent 2's welfare under PRO is independent of a_2 and remains constant at -0.6. His welfare under both EA and EL depends on a_2 independent of the setting. By the same logic explained above, when a_2 is close to a_1. Agent 2's welfare from EA and EL gets closer independent of the setting. As the difference gets larger the difference in agent 2's welfare level under EA and EL rules gets larger as well. This is again because of the characteristics of these two rules.

[8] Note that this is a partial analysis since we only look at the investment game. Therefore our welfare analysis include only the players' utilities. However incorporating our investment game into a general equilibrium model will give us a complete analysis, i which we can analyze sociatal welfare.

Figure 9 tries to capture the effect of transition from a simultaneous to a sequential setting and changing order of moves under EA rules where $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_1 = 3$. The individual welfare levels of agent 2 is ordered as

$$EA_{12} > EA_{21} > EA_{sim}$$

Whereas, the individual welfare levels of agent 1 is ordered as

$$EA_{21} > EA_{12} > EA_{sim}$$

According to above inequalities we can say that transition from a simultaneous setting to sequential setting increases both individual welfares and therefore, total welfare under EA independent of the identity of the first mover.

Moreover, as expected most of the time the small investor gets higher payoff under EA rules. However, this is violated when we change the setting into sequential and the second agent is the first mover. This is mainly because; when risk aversion levels are close enough to each other, the mover effect dominates the risk aversion effect. Therefore, under these conditions agent 2 may invest more than the agent 1, which in turn results with EA rule favoring agent 1 rather than agent 2.

Figure 10 demonstrates the effect of transition from a simultaneous to a sequential setting and changing order of moves under EL rules where $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_1 = 3$. The individual welfare levels of agent 1 are ordered as

$$EL_{12} > EL_{sim} > EL_{21}$$

Whereas, agent 2's welfare levels are ordered as

$$EL_{21} > EL_{sim} > EL_{12}$$

According to above inequalities, when compared to the simultaneous case, an agent's welfare increases when he is the first mover under a sequential setting, and it decreases when he is the second mover under a sequential setting.
Under EL rules the investor who makes higher investment is favored. Under simultaneous and sequential settings where agent 1 is the first mover, agent 1 always makes a higher investment. However, when we reverse the order of moves, again the risk aversion effect is dominated by the mover effect, and when risk aversions are close to each other agent 2 may make higher investment than agent 1. Therefore, in the picture agent 1 always gets higher utility than agent 2 under a simultaneous setting and a sequential where agent 1 is the first mover. However, agent 2's utility may exceed agent 1's under a sequential setting where agent 2 is the first mover.

At the symmetric case (when $a_1 = a_2 = 3$) agents 1 and 2 receive identical welfare levels under simultaneous setting independent of the rule. However, when we transcede to the sequential setting agent's welfare levels are not equal.

Finally *Figure 11* depicts total welfare levels induced by aforementioned seven rules as a function of a_2, where $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_1 = 3$. As it can be seen from the picture EL rules induce lower welfare levels than PRO. This is mainly because although the big investor's utility is usually maximized under EL, the small agent usually suffer drastically. This effect reflects to the total welfare comparison. When we compare PRO and EA rules, we can see that for small risk aversion levels where agent 1 and agent 2 are close to each other in terms of risk aversion, PRO induces higher total welfare. However, when agent 2 gets more risk averse, his utility increases, and this increase is greater than the decrease in agent 1 utility, therefore, EA rules exceed PRO in terms of total welfare after a critical risk aversion level.
If we compare EA rules in terms of setting the order is proportional to the order of agent 2's individual welfare levels, that is,

$$EA_{12} > EA_{21} > EA_{sim}$$

Therefore, we can conclude that according to our picture transceding to a sequential setting may increase total welfare.
However, the order of EL in terms of setting according to the total welfare they induce is not clear. For small risk aversion levels of agent 2 simultaneous case induces higher total welfare, on the other hand as a_2 gets higher EL_{21} gets closer to simultaneous case. Due to *Figure 11*, transceding from simultaneous setting to a sequential may decrease total welfare.

Lemma 1 [9]*Assume* $a_1 \leq a_2$. *Then,*
(i) $U_1^{PRO}(\epsilon\left(G^{PRO}\right)) = U_2^{PRO}(\epsilon\left(G^{PRO}\right))$,
(ii) $U_1^{EA}\left(\epsilon\left(G_{sim}^{EA}\right)\right) \leq U_2^{EA}\left(\epsilon\left(G_{sim}^{EA}\right)\right)$, *equality achieved if* $a_1 = a_2$
(iii) $U_1^{EL}\left(\epsilon\left(G_{sim}^{EL}\right)\right) \geq U_2^{EL}\left(\epsilon\left(G_{sim}^{EL}\right)\right)$, *equality achieved if* $a_1 = a_2$
(iv) $U_1^{EL_{12}}\left(\epsilon\left(G_{12}^{EL}\right)\right) \geq U_2^{EL}\left(\epsilon\left(G_{12}^{EL}\right)\right)$, *equality is never achieved.*

5.1 Egalitarian Social Welfare Levels

The **egalitarian social welfare level induced by a rule F** is the minimum utility an agent obtains at the Nash equilibrium of the investment game induced by F:

$$EG^F\left(p, r, \beta, a_1, a_2\right) = \min\left\{U_1^F(\epsilon\left(G^F\right)), U_2^F(\epsilon\left(G^F\right))\right\}.$$

We next make a numerical comparison of the Egalitarian social welfare levels induced by $EL_{12}, EL_{21}, EL_{sim}, PRO, EA_{12}, EA_{21}, EA_{sim}$ for interior equilibria (where both agents choose positive investment levels).[10]

[9] Items *(i)* *(ii)* and *(iii)* are proved in Kıbrıs and Kıbrıs (2010)
[10] We are aware of the fact that, by doing such a simulation we treat each point in the

By 1 we know that $EG^{PRO}\left(p,r,\beta,a_1,a_2\right)=U_1^{PRO}(\epsilon\left(G^{PRO}\right))=U_2^{PRO}(\epsilon\left(G^{PRO}\right))$,

$EG^{EAsim}\left(p,r,\beta,a_1,a_2\right)=U_1^{EAsim}\left(\epsilon\left(G_{sim}^{EA}\right)\right)$, $EG^{ELsim}\left(p,r,\beta,a_1,a_2\right)=U_2^{ELsim}\left(\epsilon\left(G_{sim}^{EL}\right)\right)$,

$EG^{EL_{12}}\left(p,r,\beta,a_1,a_2\right)=U_2^{EL}\left(\epsilon\left(G_{12}^{EL}\right)\right)$ .While comparing these rules we will use the corresponding Egalitarian social welfare functions, however, for the rules that have no clear ordering across agents in terms of utilities we will use $\min\left\{U_1^F(\epsilon\left(G^F\right)),U_2^F(\epsilon\left(G^F\right))\right\}$ as our function.

5.1.1 PRO vs. EL rules

PRO vs. EL_{12}

This case corresponds to 1497729 parameter combinations. Among them, PRO induce a higher social welfare every time. The two rules never induce the same welfare level.

PRO vs. EL_{21}

This case corresponds to 1290567 parameter combinations. Among them, PRO induce a higher social welfare every time. The two rules never induce the same welfare level.

PRO vs. EL_{sim}

Due to Kıbrıs and Kıbrıs we know that PRO induces higher Egalitarian social welfare than EL_{sim} independent of the parameters used.

5.1.2 PRO vs. EA rules

PRO vs. EA_{12}

This case corresponds to 5443671 parameter combinations. Among them, PRO induce a higher social welfare every time. The two rules never induce the same welfare level.

PRO vs. EA_{21}

parameter space as equally important. However there may be some points that may be more to our interest. Nevertheless, since it is not possible to do such an analysis we assume each point has the same measure.

This case corresponds to 3569074 parameter combinations. Among them, PRO induce a higher social welfare every time. The two rules never induce the same welfare level.

PRO vs. EA_{sim}

Due to Kıbrıs and Kıbrıs we know that PRO induces higher Egalitarian social welfare than EA_{sim} independent of the parameters used.

5.1.3 EA rules

EA_{sim} vs. EA_{12}

This case corresponds to 3337553 parameter combinations. Among them, EA_{12} induce a higher social welfare 1297509 times (38 percent) and EA_{sim} induce a higher social welfare 2040044 times (62 percent). The two rules never induce the same welfare level.

EA_{sim} vs. EA_{21}

This case corresponds to 3337553 parameter combinations. Among them, EA_{21}induce a higher social welfare every time. The two rules never induce the same welfare level.

EA_{12} vs. EA_{21}

This case corresponds to 3569074 parameter combinations. Among them, EA_{12} induce a higher social welfare 1151577 times (32 percent) and EA_{sim} induce a higher social welfare 2417479 times (68 percent). The two rules never induce the same welfare level.

5.1.4 EL rules

EL_{sim} vs. EL_{12}

This case corresponds to 448875 parameter combinations. Among them, EL_{12} induce a higher social welfare 293143 times (65 percent) and EL_{sim} induce a higher social welfare 155732 times (35 percent). The two rules never induce the same welfare level.

EL_{sim} vs. EL_{21}

This case corresponds to 320264 parameter combinations. Among them, EL_{21} induce a higher social welfare 235984 times (73 percent) and EL_{sim} induce a higher social welfare 84280 times (27 percent). The two rules never induce the same welfare level.

EL_{12} vs. EL_{21}

This case corresponds to 1206143 parameter combinations. Among them, EL_{12} induce a higher social welfare 457178times (37 percent) and EL_{21} induce a higher social welfare 748965 times (63 percent). The two rules never induce the same welfare level.

5.1.5 EA rules vs. EL rules

EL_{12} vs. EA_{12}

This case corresponds to 1386471 parameter combinations. Among them, EA_{12} induce a higher social welfare 849592 times (61 percent). The two rules never induce the same welfare level and EL_{12} induce a higher social welfare 536879 times (38 percent).

EL_{21} vs. EA_{21}

This case corresponds to 1185588 parameter combinations. Among them, EL_{21} induce a higher social welfare 274660times (24 percent) and EA_{21} induce a higher social welfare 910928 times (76 percent). The two rules never induce the same welfare level.

EL_{12} vs. EA_{21}

This case corresponds to 134365 parameter combinations. Among them, EA_{21} induce a higher social welfare 946180times (70 percent) and EL_{12} induce a higher social welfare 397425 times (70 percent). The two rules never induce the same welfare level.

EA_{sim} vs. EL_{12}

This case corresponds to 1326168 parameter combinations. Among them,

EA_{sim} induce a higher social welfare 831789 times (62 percent) and EL_{12} induce a higher social welfare 494379 times (38 percent). The two rules never induce the same welfare level.

EL_{sim} vs. EA_{12}

This case corresponds to 1513392 parameter combinations. Among them, EL_{sim} induce a higher social welfare658214 times (44 percent) and EA_{12} induce a higher social welfare 831789 times (56 percent). The two rules never induce the same welfare level.

EA_{sim} vs. EL_{21}

This case corresponds to 1172243 parameter combinations. Among them, EA_{sim} induce a higher social welfare 823507 times (70 percent) and EL_{21} induce a higher social welfare 348736 times (29 percent). The two rules never induce the same welfare level.

EA_{sim} vs. EL_{sim}

This case corresponds to 1294986 parameter combinations. Among them, EA_{sim} induce a higher social welfare 949377 times (73 percent) and EL_{sim} induce a higher social welfare 345609 times (27 percent). The two rules never induce the same welfare level.

EL_{21} vs. EA_{12}

This case corresponds to 1192701 parameter combinations. Among them, EA_{12} induce a higher social welfare 775930times (65 percent) and EL_{21} induce a higher social welfare 416772 times (35 percent). The two rules never induce the same welfare level.

EL_{sim} vs. EA_{21}

This case corresponds to 755209 parameter combinations. Among them, EA_{21} induce a higher social welfare 501131 times (66 percent) and EA_{21} induce a higher social welfare 254078 times (34 percent). The two rules never induce the same welfare level.

According to this numerical analysis, in terms of Egalitarian welfare induced by the rules, they exhibit following ordering

$$PRO > EA_{21} > EA_{sim} > EA_{12} > EL_{21} > EL_{12} > EL_{sim}$$

Transition from simultaneous case to a sequential case increases Egalitarian social welfare induced under EL rules independent of the order of moves whereas may decreases under EA_{12} and increase under EA_{21}.

5.2 Utilitarian Social Welfare Levels

The **utilitarian social welfare level induced by a rule F** is the total utility the two agents obtain at the Nash equilibrium of the investment game induced by F:

$$UT^F\left(p, r, \beta, a_1, a_2\right) = U_1^F(\epsilon\left(G^F\right)) + U_2^F(\epsilon\left(G^F\right)).$$

We next compare numerically the Utilitarian social welfare levels induced by EL_{12}, EL_{21}, EL_{sim}, PRO, EA_{12}, EA_{21}, EA_{sim} for interior equilibria (where both agents choose positive investment levels)[11]

5.2.1 PRO vs. EL rules

PRO vs. EL_{12}

This case corresponds to 2278616 parameter combinations. Among them, PRO induce a higher social welfare every time. The two rules never induce the same welfare level.

PRO vs. EL_{21}

[11] We are aware of the fact that, by doing such a simulation we treat each point in the parameter space as equally important. However there may be some points that may be more to our interest. Nevertheless, since it is not possible to do such an analysis we assume each point has the same measure.

This case corresponds to 2407596 parameter combinations. Among them, PRO induce a higher social welfare every time. The two rules never induce the same welfare level.

PRO vs. EL_{sim}

Due to Kıbrıs and Kıbrıs we know that PRO induces higher Utilitarian social welfare than EL_{sim} independent of the parameters used.

5.2.2 PRO vs. EA rules

PRO vs. EA_{12}

This case corresponds to 5524804 parameter combinations. Among them, EA_{12} induce a higher social welfare 2489872 times (45 percent) and PRO induce a higher social welfare 3034932 times (55 percent). The two rules never induce the same welfare level.

PRO vs. EA_{21}

This case corresponds to 5404342 parameter combinations. Among them, PRO induce a higher social welfare 2950961 times (55 percent) and EA_{21} induce a higher social welfare 2453381 times (45 percent). The two rules never induce the same welfare level.

PRO vs. EA_{sim}

This case corresponds to 2726424 parameter combinations. Among them, PRO induce a higher social welfare 1659219 times (61 percent) and EA_{sim} induce a higher social welfare 1067205 times (39 percent). The two rules never induce the same welfare level.

5.2.3 EA rules

EA_{sim} vs. EA_{12}

This case corresponds to 525336 parameter combinations. Among them, EA_{12} induce a higher social welfare every time. The two rules never induce the same welfare level.

EA_{sim} vs. EA_{21}

This case corresponds to 525336 parameter combinations. Among them, EA_{21} induce a higher social welfare every time. The two rules never induce the same welfare level.

EA_{12} vs. EA_{21}

This case corresponds to 5404342 parameter combinations. Among them, EA_{12} induce a higher social welfare every time. The two rules never induce the same welfare level.

5.2.4 EL rules

EL_{sim} vs. EL_{12}

This case corresponds to 2099043 parameter combinations. Among them, EL_{sim} induce a higher social welfare 1494900 times (71 percent) and EL_{12} induce a higher social welfare 604143 times (29 percent). The two rules never induce the same welfare level.

EL_{sim} vs. EL_{21}

This case corresponds to 2800745 parameter combinations. Among them, EL_{sim} induce a higher social welfare 1656734 times (60 percent) and EL_{21} induce a higher social welfare 1144011 times (40 percent). The two rules never induce the same welfare level.

EL_{12} vs. EL_{21}

This case corresponds to 2278616 parameter combinations. Among them, EL_{21} induce a higher social welfare every time. The two rules never induce the same welfare level.

5.2.5 EA rules vs. EL rules

EL_{12} vs. EA_{12}

This case corresponds to 1500943 parameter combinations. Among them, EA_{12} induce a higher social welfare 1494900 times (96 percent) and EL_{12}

induce a higher social welfare 6043 times (4 percent). The two rules never induce the same welfare level.

EL_{21} vs. EA_{21}

This case corresponds to 2784548 parameter combinations. Among them, EA_{21} induce a higher social welfare 1645550 times (59 percent) and EL_{21} induce a higher social welfare 1138998 times (41 percent). The two rules never induce the same welfare level.

EL_{12} vs. EA_{21}

This case corresponds to 2069680 parameter combinations. Among them, EA_{21} induce a higher social welfare 1485381 times (72 percent) and EL_{12} induce a higher social welfare 584299 times (28 percent). The two rules never induce the same welfare level.

EA_{sim} vs. EL_{12}

This case corresponds to 2019852 parameter combinations. Among them, EA_{sim} induce a higher social welfare 1479352 times (73 percent) and EL_{12} induce a higher social welfare 284166 times (38 percent). The two rules never induce the same welfare level.

EL_{sim} vs. EA_{12}

This case corresponds to 4352013 parameter combinations. Among them, EA_{12} induce a higher social welfare every time. The two rules never induce the same welfare level.

EA_{sim} vs. EL_{21}

This case corresponds to 2712991 parameter combinations. Among them, EA_{sim} induce a higher social welfare 1639000 times (60 percent) and EL_{21} induce a higher social welfare 1073991 times (40 percent). The two rules never induce the same welfare level.

EA_{sim} vs. EL_{sim}

This case corresponds to 1294986 parameter combinations. Among them, EA_{sim} induce a higher social welfare 851736 times (66 percent) and EL_{sim}

induce a higher social welfare 540500 times (34 percent). The two rules never induce the same welfare level.

EL_{21} vs. EA_{12}

This case corresponds to 2823713 parameter combinations. Among them, EA_{12} induce a higher social welfare 1656734 times (58 percent) and EL_{21} induce a higher social welfare 1166979 times (42 percent). The two rules never induce the same welfare level.

EL_{sim} vs. EA_{21}

This case corresponds to 4274562 parameter combinations. Among them, EA_{21} induce a higher social welfare every time. The two rules never induce the same welfare level.

6 Conclusion

Our analysis compares the proportionality, equal awards, and equal losses principles under three settings (simultaneous, sequential where big investor moves first, sequential where small investor moves first) in terms of two criteria (total investment and social welfare induced in equilibrium) that were not considered in a sequential setting before. Our findings are as follows:
(i) In terms of total investment, there is a constant ranking of these rules which is independent of the parameter values considered:

Total Investment: $EL_{12} > EL_{21} > EL_{sim} > PRO > EA_{12} > EA_{21} > EA_{sim}$

(ii) According to our numerical analysis, PRO always induces a higher egalitarian social welfare than both EA and EL in an interior equilibrium, independent of both the setting used and identity of the first mover. Also, our numerical analysis shows that EA exceeds EL independent of setting used. According to the numerical analysis in terms of egalitarian social welfare induced, rules are ordered as;

Egalitarian social welfare: $PRO > EA_{21} > EA_{sim} > EA_{12} > EL_{21} > EL_{12} > EL_{sim}$.

(iii) According to our numerical analysis, PRO induces a higher utilitarian social welfare than both EA and EL in an interior equilibrium, independent of both the setting used and identity of the first mover. Also, our numerical analysis shows that EA exceeds EL independent of setting used. According to the numerical analysis in terms of utilitarian social welfare induced, rules are ordered as;

Utilitarian social welfare: $PRO > EA_{12} > EA_{21} > EA_{sim} > EL_{sim} > EL_{21} > EL_{12}$.

(iv) There always is a unique dominant strategy equilibrium under PRO

(where agents play strictly dominant strategies). No other rule induces dominant strategies. However, under both EA and EL, a unique subgame perfect Nash equilibrium always exists independent of the setting.

Overall, the almost universal principle of proportionality does not maximize total investment in the economy. By switching from proportionality to equal losses, it is possible to increase total investment. However, this switch may cause a welfare loss in the society, both according to the egalitarian and utilitarian social welfare functions. A switch from proportionality to the equal awards principle always decreases total investment. It is also interesting to note that it also may lower egalitarian social welfare.

Moreover, comparing equilibria at different risk-aversion profiles shows us that the three principles are very different in terms of the incentives that they provide for big versus small investors. The equal losses principle offers relatively better protection to the bigger (i.e. less risk averse) investors. The equal awards principle does the opposite. The proportionality principle strikes a compromise by offering the same proportion of their investment to every agent.

Finally, a transition from a simultaneous case to a sequential case always increases total investment independent of the order of moves. Also, a transition from a simultaneous case to a sequential case may increase egalitarian social welfare, however, due to the rule used it may decrease (under EL) or increase (under EA) social welfare. Similarly according to our numerical analysis, a transition from a simultaneous setting to a sequential setting will increase (decrease) utilitarian social welfare under EA (EL) independent of the order of moves.

Note that our social welfare measures only consider the investors in the game. They do not take into account the welfare implications of investment in the rest of the economy (such as welfare effects of investment on consumers or future generations). This is an interesting question which is, unfortunately, out of the scope of our current model.

For tractability of the model, we use CARA utility functions. While the CARA family is widely used in economic modeling as well as finance, it is an open question whether our findings are replicated with other families of utility functions.

In our model, the rate of return r and the probability of success p are independent of the agents' investment levels. It might be interesting to look at extensions of the model where these parameters, in some way, depend on the investment levels.

Finally an extension of these work can be done by allowing heterogeneity in investment type, such as, secured vs. nonsecured investment. It will be interesting to analyze such game and check whether if our findings are consistent with such case.

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7 Appendix: Proofs

Proof. **(Proposition 1)** PRO, as a function of the investment levels, is defined as $PRO_i(s) = \beta s_i$. Then, by Equation 1, agent i's utility under PRO can be written as

$$U_i^{PRO}(s) = -pe^{-a_i r s_i} - (1-p)e^{a_i s_i (1-\beta)}.$$

Note that the payoff function of agent i is independent of the other agents' investment levels. The unconstrained maximizer of this expression is

$$\sigma_i\left(s_{-i}\right) = \frac{1}{a_i\left(r+1-\beta\right)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right).$$

The best response function of agent i can then be written as $b_i(s_{-i}) = \max\left\{0, \sigma_i\left(s_{-i}\right)\right\}$. Note that this expression is independent of s_{-i}. So it in fact defines a *strictly dominant strategy* for each agent i. ■

Proof. (Proposition 2) By Equation 1, agent i's utility under EA_{12} becomes

$$\begin{aligned}
U_i^{EA_{12}}\left(s_i, s_j\right) &= -pe^{-a_i r s_i} - (1-p)e^{-a_i\left(\frac{\beta}{2}(s_i+s_j)\right)+a_i s_i} \\
U_1^{EA_{12}}\left(s_1, s_2\right) &= -pe^{-a_1 r s_1} - (1-p)e^{a_1 s_1\left(\frac{2-\beta}{2}\right)-a_1\frac{\beta}{2}s_2} \\
U_2^{EA_{12}}\left(s_1, s_2\right) &= -pe^{-a_2 r s_2} - (1-p)e^{a_2 s_2\left(\frac{2-\beta}{2}\right)-a_2\frac{\beta}{2}s_1}
\end{aligned}$$

The unconstrained maximizer of $U_2^{EA_{12}}\left(s_1, s_2\right)$ is

$$\sigma_2\left(s_1\right) = \frac{2\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{a_2\left(2\left(1+r\right)-\beta\right)} + \frac{\beta s_1}{\left(2\left(1+r\right)-\beta\right)}$$

Thus, agent 2's best response is $b_2(s_1) = \begin{cases} \sigma_2(s_1) \quad if & \begin{array}{l} \frac{2pr}{(1-p)(2-\beta)} \geq 1 \\ or \\ \frac{2pr}{(1-p)(2-\beta)} < 1 \text{ and } s_1^* > -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_2} \end{array} \\ 0 & otherwise \end{cases}$

Case 1: $\frac{2pr}{(1-p)(2-\beta)} \geq 1$ then given $b_2(s_1)$ the maximizer of $U_1^{EA_{12}}(s_1, \sigma_2(s_1))$ is

$$s_1^* = \frac{\left(a_2(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_2a_1(r+1)(r-\beta+1)}$$

Note that, $s_1^* > 0$ if the numerator is positive and by assumption we have following inequality

$$\frac{2pr}{(1-p)(2-\beta)} \geq 1$$

Moreover, it is easy to show that

$$\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)} > \frac{2pr}{(1-p)(2-\beta)}$$

So if $\frac{2pr}{(1-p)(2-\beta)} \geq 1$ then $\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)} > 1$. Therefore, $s_1^* > 0$
Given $s_1^*, \sigma_2(s_1)$ becomes

$$s_2^* = \frac{\left(a_2\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)}.$$

Hence under Case 1 agents' investment levels are according to the formulas (s_1^*, s_2^*)
Case 2: $\frac{2pr}{(1-p)(2-\beta)} < 1$ and $s_1^* > -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_2}$
Then, given $b_2(s_1)$ the maximizer of $U_1^{EA_{12}}(s_1, \sigma_2(s_1))$ is

$$s_1^* = \frac{\left(a_2\left(2+2r-\beta\right)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_2a_1\left(r+1\right)\left(r-\beta+1\right)}$$

Note that, $s_1^* > 0$, if the numerator is positive, that means $s_1^* > 0$, if the following inequality is true

$$a_2\left(2+2r-\beta\right)\ln\left(\frac{pr\left(2+2r-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}\right) > -a_1\beta\ln\left(\frac{2pr}{\left(1-p\right)\left(2-\beta\right)}\right)$$

Now, substituting s_1^* into the inequality $s_1^* > -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_2}$ and rearranging it gives us the following inequality

$$\beta a_2\ln\left(\frac{pr\left(2+2r-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}\right) > -a_1(2+2r-\beta)\ln\left(\frac{2pr}{\left(1-p\right)\left(2-\beta\right)}\right)$$

By assumption this inequality is true and it implies the condition for $s_1^* > 0$
Given $s_1^*, \sigma_2\left(s_1\right)$ becomes

$$s_2^* = \frac{\left(a_2\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1\left(2+2r-\beta\right)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2\left(r+1\right)\left(r-\beta+1\right)}.$$

Therefore under Case 2, agents' investment levels are according to the formulas $\left(s_1^*, s_2^*\right)$
Case 3: $\frac{2pr}{(1-p)(2-\beta)} < 1$ and $s_1^* < -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_2}$
then $b_2\left(s_1\right) = 0$ and the maximizer of $U_1^{EA_{12}}\left(s_1, 0\right)$ is

$$\frac{2\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{a_1\left(2\left(1+r\right)-\beta\right)}$$

Which is negative since $\frac{2pr}{(1-p)(2-\beta)} < 1$ therefore under we assume agent 1 makes zero investment

Note that our assumption is not violated under this case since

$$0 < -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_2}$$

Therefore, under Case 3 agent's investment levels are $(0,0)$

■

Proof. (Proposition 3) By Equation 1, agent i's utility under EL_{12} becomes

$$\begin{aligned}
U_i^{EL_{12}}(s_i, s_j) &= -pe^{-a_i r s_i} - (1-p)e^{\left(\frac{(1-\beta)}{2}\right)a_i s_i + \frac{(1-\beta)}{2}a_i \sum_{N\setminus i} s_j}. \\
U_1^{EL_{12}}(s_1, s_2) &= -pe^{-a_1 r s_1} - (1-p)e^{\left(\frac{(1-\beta)}{2}\right)a_1(s_1+s_2)} \\
U_2^{EL_{12}}(s_1, s_2) &= -pe^{-a_2 r s_2} - (1-p)e^{\left(\frac{(1-\beta)}{2}\right)a_2(s_1+s_2)}
\end{aligned}$$

The unconstrained maximizer of $U_2^{EL_{12}}(s_1, s_2)$ is

$$\sigma_2(s_1) = \frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_2(2r + (1-\beta))} - \frac{(1-\beta)s_1}{(2r + (1-\beta))}.$$

Thus, agent 2's best response is $b_2(s_1) = \begin{cases} \sigma_2(s_1) \quad if & \begin{array}{c} \frac{2pr}{(1-p)(1-\beta)} > 1 \\ and \\ s_1^* < \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_2(1-\beta)} \end{array} \\ 0 & otherwise \end{cases}.$

Case 1: $\frac{2pr}{(1-p)(1-\beta)} > 1$ and $s_1^* < \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_2(1-\beta)}$ then given $b_2(s_1) = \sigma_2(s_1)$ the maximizer of $U_1^{EL_{12}}(s_1, \sigma_2(s_1))$ is

$$s_1^* = \frac{\left(a_2(2r + (1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) - a_1(1-\beta)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 r a_1 (r + 1 - \beta) 2}$$

Note that, $s_1^* > 0$ since

$$a_2 (2r + (1-\beta)) \ln \left(\frac{p (2r+1-\beta)}{(1-p)(1-\beta)} \right) > a_1 (1-\beta) \ln \left(\frac{2pr}{(1-p)(1-\beta)} \right)$$

Then inserting s_1^* into $\sigma_2 (s_1)$ gives us second agent's equilibrium investment

$$s_2^* = \frac{\left(a_1 (2r + (1-\beta)) \ln \left(\frac{2pr}{(1-p)(1-\beta)} \right) - a_2 (1-\beta) \ln \left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)} \right) \right)}{a_2 r a_1 (r+1-\beta) 2}.$$

Hence under Case 1 agents' investment levels are according to the formulas (s_1^*, s_2^*)

Before moving to the second case, note that substituting s_1^* into the inequality $s_1^* < \frac{2 \ln \left(\frac{2pr}{(1-p)(1-\beta)} \right)}{a_2 (1-\beta)}$ and rearranging it gives us the following inequality;

$$\left(\frac{p (2r+1-\beta)}{(1-p)(1-\beta)} \right) < \left(\frac{2pr}{(1-p)(1-\beta)} \right)^{\frac{a_1 (1+2r-\beta)}{(1-\beta) a_2}}$$

Case 2: $\frac{2pr}{(1-p)(1-\beta)} > 1$ and $\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)} \right) < \left(\frac{2pr}{(1-p)(1-\beta)} \right)^{\frac{a_1 (1+2r-\beta)}{(1-\beta) a_2}}$ then given $b_2 (s_1) = 0$ the maximizer of $U_1^{EL_{12}} (s_1, 0)$ is

$$\frac{2 \ln \left(\frac{2pr}{(1-\beta)(1-p)} \right)}{a_1 (2r + (1-\beta))}$$

Which is positive by assumption $\frac{2pr}{(1-\beta)(1-p)} > 1$

Note that, $b_2 (s_1) = 0$ if the following holds

$$\frac{2 \ln \left(\frac{2pr}{(1-\beta)(1-p)} \right)}{a_1 (2r + (1-\beta))} > \frac{2 \ln \left(\frac{2pr}{(1-p)(1-\beta)} \right)}{a_2 (1-\beta)}$$

And this inequality holds if the following holds

$$a_1 (2r + (1-\beta)) < a_2 (1-\beta)$$

Which is true by assumption

Therefore under Case 2 agents make following investment levels: $\left(\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_1(2r+(1-\beta))}, 0\right)$

Case 3: $\frac{2pr}{(1-p)(1-\beta)} < 1$ then given $b_2(s_1) = 0$ the maximizer of $U_1^{EL_{12}}(s_1, 0)$ is

$$\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_1 (2r + (1-\beta))}$$

Which is negative by assumption $\frac{2pr}{(1-\beta)(1-p)} < 1$

Therefore under Case 3 agents make following investment levels.$(0, 0)$. ■

Proof. (Proposition 4) By Equation 1, agent i's utility under EA_{12} becomes

$$\begin{aligned} U_i^{EA_{21}}(s_i, s_j) &= -pe^{-a_i r s_i} - (1-p)e^{-a_i\left(\frac{\beta}{2}(s_i+s_j)\right)+a_i s_i} \\ U_1^{EA_{21}}(s_1, s_2) &= -pe^{-a_1 r s_1} - (1-p)e^{a_1 s_1\left(\frac{2-\beta}{2}\right)-a_1\frac{\beta}{2}s_2} \\ U_2^{EA_{21}}(s_1, s_2) &= -pe^{-a_2 r s_2} - (1-p)e^{a_2 s_2\left(\frac{2-\beta}{2}\right)-a_2\frac{\beta}{2}s_1} \end{aligned}$$

The unconstrained maximizer of $U_1^{EA_{21}}(s_1, s_2)$ is

$$\sigma_1(s_2) = \frac{2\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{a_1(2(1+r)-\beta)} + \frac{\beta s_2}{(2(1+r)-\beta)}$$

Thus, agent 1's best response is $b_1(s_2) = \begin{cases} \sigma_1(s_2) \quad if & \begin{array}{l} \frac{2pr}{(1-p)(2-\beta)} \geq 1 \\ or \\ \frac{2pr}{(1-p)(2-\beta)} < 1 \text{ and } s_2^* > -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_1} \end{array} \\ 0 & otherwise \end{cases}$

Case 1: $\frac{2pr}{(1-p)(2-\beta)} \geq 1$ then given $b_1(s_2) = \sigma_1(s_2)$ the maximizer of $U_2^{EA_{21}}(\sigma_1(s_2), s_2)$ is

$$s_2^* = \frac{\left(a_1(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)}$$

Note that, $s_2^* > 0$, if the numerator is positive. That is,

$$\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)a_1(2+2r-\beta) > -\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)a_2\beta$$

By assumption we have following inequality

$$\frac{2pr}{(1-p)(2-\beta)} \geq 1$$

Moreover, it is easy to show that

$$\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)} > \left(\frac{2pr}{(1-p)(2-\beta)}\right)$$

So if $\frac{2pr}{(1-p)(2-\beta)} \geq 1$ then $\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}$. Hence, $s_2^* > 0$
Given s_2^*; $\sigma_1(s_2)$ becomes

$$s_1^* = \frac{\left(a_1\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)}.$$

Therefore, under Case 1 agents' investment levels are according to the formulas (s_1^*, s_2^*)
Case 2: $\frac{2pr}{(1-p)(2-\beta)} < 1$ and $s_2^* > -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_1}$ then given $b_1(s_2) = \sigma_1(s_2)$

the maximizer of $U_2^{EA_{12}}\left(\sigma_1\left(s_2\right), s_2\right)$ is

$$s_2^* = \frac{\left(a_1\left(2+2r-\beta\right)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2\left(r+1\right)\left(r-\beta+1\right)}$$

Note that $s_2^* > 0$ if the numerator is positive. That is,

$$\ln\left(\frac{pr\left(2+2r-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}\right)a_1\left(2+2r-\beta\right) > -\ln\left(\frac{2pr}{\left(1-p\right)\left(2-\beta\right)}\right)a_2\beta$$

Now, substituting s_2^* into the inequality $s_2^* > -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_1}$ and rearranging it gives us the following inequality;

$$\ln\left(\frac{pr\left(2+2r-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}\right)a_1\beta > -\ln\left(\frac{2pr}{\left(1-p\right)\left(2-\beta\right)}\right)a_2(2+2r-\beta)$$

By assumption this inequality is true and it implies the condition for $s_2^* > 0$
Hence, under Case 1 agents' investment levels are according to the formulas (s_1^*, s_2^*)
Case 3: $\frac{2pr}{(1-p)(2-\beta)} < 1$ and $s_2^* < -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_1}$ then given $b_1\left(s_2\right) = 0$ the maximizer of $U_2^{EA_{21}}\left(0, s_2\right)$ is

$$\frac{2\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{a_2\left(2\left(1+r\right)-\beta\right)}$$

Which is negative by assumption $\frac{2pr}{(1-p)(2-\beta)} < 1$, but we assume agent 1's investment is 0
Note that, our assumption is not violated under this case since

$$0 < -2\frac{\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)}{\beta a_1}$$

Therefore under Case 3 agent's agents' investment levels are $(0,0)$ ■

Proof. (Proposition 5) By Equation 1, agent i's utility under EL_{21} becomes

$$\begin{aligned}
U_i^{EL_{21}}(s_i, s_j) &= -pe^{-a_i r s_i} - (1-p)e^{\left(\frac{(1-\beta)}{2}\right)a_i s_i + \frac{(1-\beta)}{2}a_i \sum_{N\setminus i} s_j}. \\
U_1^{EL_{21}}(s_1, s_2) &= -pe^{-a_1 r s_1} - (1-p)e^{\left(\frac{(1-\beta)}{2}\right)a_1(s_1+s_2)} \\
U_2^{EL_{21}}(s_1, s_2) &= -pe^{-a_2 r s_2} - (1-p)e^{\left(\frac{(1-\beta)}{2}\right)a_2(s_1+s_2)}
\end{aligned}$$

The unconstrained maximizer of $U_1^{EL_{21}}(s_1, s_2)$ is

$$\sigma_1(s_2) = \frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_1(2r+(1-\beta))} - \frac{(1-\beta)s_2}{(2r+(1-\beta))}.$$

Thus, agent 1's best response is $b_1(s_2) = \begin{cases} \sigma_1(s_2) \quad if & \begin{array}{l} \frac{2pr}{(1-p)(2-\beta)} > 1 \\ and \\ s_2^* < \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_1(1-\beta)} \end{array} \\ 0 & otherwise \end{cases}$

Case 1: $\frac{2pr}{(1-p)(1-\beta)} > 1$ and $s_2^* < \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_1(1-\beta)}$ and $\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(2r+(1-\beta))} > \left(\frac{2pr}{(1-p)(1-\beta)}\right)^{a_2(1-\beta)}$ then given $b_1(s_2) = \sigma_1(s_2)$ the maximizer of $U_2^{EL_{21}}(\sigma_1(s_2), s_2)$ is

$$s_2^* = \frac{\left(a_1(2r+(1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) - a_2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_1 r a_2 (r+1-\beta)\,2}$$

Note that $s_2^* > 0$ if the numerator is positive, which is true by assumption.

Then, inserting s_2^* into $\sigma_1(s_2)$ gives us the agent 1's equilibrium investment

$$s_1^* = \frac{\left(a_2(2r+(1-\beta))\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) - a_1\ln\left(\frac{p(2r+(1-\beta))}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_2 r a_1 (r+1-\beta)\, 2}.$$

Hence under Case 1 agents' investment levels are according to the formulas (s_1^*, s_2^*)

Case 2: $\frac{2pr}{(1-p)(1-\beta)} > 1$ and $s_2^* < \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_1(1-\beta)}$ and $\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(2r+(1-\beta))} \leq \left(\frac{2pr}{(1-p)(1-\beta)}\right)^{a_2(1-\beta)}$ then given $b_1(s_2) = \sigma_1(s_2)$ the maximizer of $U_2^{EL_{21}}(\sigma_1(s_2), s_2)$ is

$$s_2^* = \frac{\left(a_1(2r+(1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) - a_2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_1 r a_2 (r+1-\beta)\, 2}$$

Which is negative, by assumption;

$$\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)^{a_1(2r+(1-\beta))} \leq \left(\frac{2pr}{(1-p)(1-\beta)}\right)^{a_2(1-\beta)}$$

Therefore, we will assume agent 2 is making 0 investment, given that, $\sigma_1(0)$ is equal to the following expression;

$$\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_1(2r+(1-\beta))}$$

Which is positive by assumption $\frac{2pr}{(1-p)(1-\beta)} > 1$.

Moreover, note that our assumption is not violated since

$$0 < \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_1\left(1-\beta\right)}$$

Therefore under Case 2 agents make following investment levels; $\left(\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_1(2r+(1-\beta))}, 0\right)$

Case 3: $\frac{2pr}{(1-\beta)(1-p)} > 1$ and $s_2^* > \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_1(1-\beta)}$ then given $b_1\left(s_2\right) = 0$ the maximizer of $U_2^{EL_{21}}\left(0, s_2\right)$ is

$$\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_2\left(2r+\left(1-\beta\right)\right)}$$

This expression is positive, since by assumption. $\left(\frac{2pr}{(1-\beta)(1-p)}\right) > 1$

Note that, again our assumption is not violated since

$$\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_2\left(2r+\left(1-\beta\right)\right)} > \frac{2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)}{a_1\left(1-\beta\right)}$$

Hence under Case 3 agents have the following investment levels, $\left(0, \frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_2(2r+(1-\beta))}\right)$

Case 4: $\frac{2pr}{(1-\beta)(1-p)} \leq 1$ then, given $b_1\left(s_2\right) = 0$, the maximizer of $U_2^{EL_{21}}\left(0, s_2\right)$ is

$$\frac{2\ln\left(\frac{2pr}{(1-\beta)(1-p)}\right)}{a_2\left(2r+\left(1-\beta\right)\right)}$$

This expression is negative, since by assumption. $\frac{2pr}{(1-\beta)(1-p)} \leq 1$. Therefore, we will assume agent 2's investment level is 0

Then, under Case 4, both agents make 0 investment. ■

Proof. (Proposition 6) First lets prove $s_1^{EA,12} > s_1^{EA,21}$

$$s_1^{EA,12} = \frac{\left(a_2\left(2+2r-\beta\right)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_1\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_2a_1\left(r+1\right)\left(r-\beta+1\right)}$$

$$s_1^{EA,21} = \frac{\left(a_1\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2\left(2+2r-\beta\right)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2\left(r+1\right)\left(r-\beta+1\right)}$$

We want to show that
$\frac{\left(a_2(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)+a_1\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_2a_1(r+1)(r-\beta+1)} > \frac{\left(a_1\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)+a_2(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)}$
By simplifying rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to
$a_1\beta\ln\underbrace{\left(\frac{2pr}{\left(1-p\right)\left(2-\beta\right)}\frac{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}{pr\left(2+2r-\beta\right)}\right)}_{\frac{1}{x}}+$
$a_2\left(2+2r-\beta\right)\ln(\underbrace{\frac{pr\left(2+2r-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}\frac{(1-p)(2-\beta)}{2pr}}_{x}) > 0$
Define
$$x = \frac{pr\left(2+2r-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)\left(1-p\right)}\frac{(1-p)(2-\beta)}{2pr}$$

Then by following property of ln: $\ln x = -\ln\frac{1}{x}$ the inequality is again reduced to

$$\left(a_2\left(2+2r-\beta\right)-a_1\beta\right)\ln\left(\frac{\left(2-\beta\right)\left(2r-\beta+2\right)}{\left(\left(2-\beta\right)r-2\beta+2\right)2}\right) > 0$$

which is true since $a_2\left(2+2r-\beta\right)-a_1\beta > 0$ and $\frac{(2-\beta)(2r-\beta+2)}{((2-\beta)r-2\beta+2)2} > 1$
Now lets prove that $s_1^{EA,21} > s_1^{EA,sim}$

$$s_1^{EA,21} = \frac{\left(a_1\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2\left(2+2r-\beta\right)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2\left(r+1\right)\left(r-\beta+1\right)}$$

$$s_1^{EA,sim} = \frac{2(1+r-\beta)+\beta+\beta a_1\frac{1}{a_2}}{a_1 2(1+r-\beta)(1+r)}\ln(\frac{2pr}{(1-p)\,(2-\beta)})$$

We want to show that
$\frac{\left(a_1\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)+a_2(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)} > \frac{2(1+r-\beta)+\beta+\beta a_1\frac{1}{a_2}}{a_1 2(1+r-\beta)(1+r)}\ln(\frac{2pr}{(1-p)(2-\beta)})$
By simplifying rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1\beta\ln\left(\frac{(2+2r-\beta)\,(2-\beta)}{(2+2r-(2+r)\,\beta)\,2}\right)+\underbrace{a_2\,(2+2r-\beta)\ln(\underbrace{\frac{2pr}{(1-p)\,(2-\beta)}\frac{(1-p)(2-\beta)}{2pr}}_{=1})}_{=0} > 0$$

Which is true since $a_1\beta > 0$ and $\ln\left(\frac{(2+2r-\beta)(2-\beta)}{(2+2r-(2+r)\beta)2}\right) > 0$ ■

Proof. (**Proposition 7)** First lets show $s_1^{EL,12} > s_1^{EL,sim}$

$$s_1^{EL,12} = \frac{\left(2a_2\left(r+\frac{(1-\beta)}{2}\right)\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)-a_1\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_2ra_1\,(r+1-\beta)\,2}$$

$$s_1^{EL,sim} = \frac{(2r-\beta+1)}{2r(r-\beta+1)}(\frac{1}{a_1}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_2})\ln(\frac{2pr}{(1-p)(1-\beta)})$$

Want to show
$\frac{\left(2a_2\left(r+\frac{(1-\beta)}{2}\right)\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)-a_1\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_2ra_1(r+1-\beta)2} > \frac{(2r-\beta+1)}{2r(r-\beta+1)}(\frac{1}{a_1}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_2})\ln(\frac{2pr}{(1-p)(1-\beta)})$
by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_2(2r+1-\beta)\ln\left(\frac{(2r+1-\beta)}{2r}\right)\underbrace{-a_1\,(1-\beta)\ln(\underbrace{\frac{2pr}{(1-p)\,(1-\beta)}\frac{(1-\beta)(1-p)}{2pr}}_{=1})}_{=0}>0$$

which is true since $a_2(2r+1-\beta)>0$ and $\ln\left(\frac{(2r+1-\beta)}{2r}\right)>0$

Secondly lets check $s_1^{EL,sim}>s_1^{EL,21}$

$$s_1^{EL,21}=\frac{\left(a_2(2r+(1-\beta))\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)-a_1\ln\left(\frac{p(2r+(1-\beta))}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_2ra_1\,(r+1-\beta)\,2}$$

$$s_1^{EL,sim}=\frac{(2r+(1-\beta))}{2r(r-\beta+1)}(\frac{1}{a_1}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_2})\ln(\frac{2pr}{(1-p)(1-\beta)})$$

Want to show

$\frac{(2r+(1-\beta))}{2r(r-\beta+1)}(\frac{1}{a_1}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_2})\ln(\frac{2pr}{(1-p)(1-\beta)})>\frac{\left(a_2(2r+(1-\beta))\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)-a_1\ln\left(\frac{p(2r+(1-\beta))}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_2ra_1(r+1-\beta)2}$

by simplifying, rearranging and using the following property of ln; $\ln x-\ln y=\ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1(1-\beta)\ln\left(\frac{(2r+1-\beta)}{2r}\right)\underbrace{-a_2(2r+(1-\beta))\ln(\underbrace{\frac{2pr}{(1-p)\,(1-\beta)}\frac{(1-\beta)(1-p)}{2pr}}_{=1})}_{=0}>0$$

Which is true since $a_1(1-\beta)>0$ and $\ln\left(\frac{(2r+1-\beta)}{2r}\right)>0$ ■

Proof. (Proposition 8) We check if the following inequality holds; $s_2^{EA,12}>s_2^{EA,sim}$

$$s_2^{EA,12}=\frac{\left(a_2\beta\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)+a_1\,(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2\,(r+1)\,(r-\beta+1)}$$

$$s_2^{EA,sim} = \frac{2(1+r-\beta)+\beta+\beta a_2 \frac{1}{a_1}}{a_2 2(1+r-\beta)(1+r)} \ln(\frac{2pr}{(1-p)\,(2-\beta)})$$

Want to show
$\frac{\left(a_2\beta \ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)+a_1(2+2r-\beta)\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)} > \frac{2(1+r-\beta)+\beta+\beta a_2\frac{1}{a_1}}{a_2 2(1+r-\beta)(1+r)}\ln(\frac{2pr}{(1-p)(2-\beta)})$
by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1\,(2+2r-\beta)\underbrace{\ln\left(\underbrace{\frac{2pr}{(1-p)\,(2-\beta)}\frac{(1-p)(2-\beta)}{2pr}}_{=1}\right)}_{=0} + a_2\beta\ln(\frac{(2+2r-\beta)\,(2-\beta)}{(2+2r-(2+r)\,\beta)\,2} > 0$$

Which is true since $a_2\beta > 0$ and $\frac{(2+2r-\beta)(2-\beta)}{(2+2r-(2+r)\beta)2} > 1$
Now we check if the following is true $s_2^{EA,21} > s_2^{EA,sim}$

$$s_2^{EA,21} = \frac{\left(a_1\,(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right) + a_2\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2\,(r+1)\,(r-\beta+1)}$$

$$s_2^{EA,sim} = \frac{2(1+r-\beta)+\beta+\beta a_2 \frac{1}{a_1}}{a_2 2(1+r-\beta)(1+r)} \ln(\frac{2pr}{(1-p)\,(2-\beta)})$$

Want to show
$\frac{\left(a_1(2+2r-\beta)\ln\left(\frac{pr(2+2r-\beta)}{(2+2r-(2+r)\beta)(1-p)}\right)+a_2\beta\ln\left(\frac{2pr}{(1-p)(2-\beta)}\right)\right)}{2a_1a_2(r+1)(r-\beta+1)} > \frac{2(1+r-\beta)+\beta+\beta a_2\frac{1}{a_1}}{a_2 2(1+r-\beta)(1+r)}\ln(\frac{2pr}{(1-p)(2-\beta)})$
by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1\left(2+2r-\beta\right)\ln\left(\frac{\left(2+2r-\beta\right)\left(2-\beta\right)}{\left(2+2r-\left(2+r\right)\beta\right)2}\right)+\underbrace{a_2\beta\ln(\underbrace{\frac{2pr}{\left(1-p\right)\left(2-\beta\right)}\frac{(1-p)(2-\beta)}{2pr}}_{=1})}_{=0}>0$$

Which is true since $a_1\left(2+2r-\beta\right)>0$ and $\frac{(2+2r-\beta)(2-\beta)}{(2+2r-(2+r)\beta)2}>1$ ■

Proof. (**Proposition 9)**First we show that $s_2^{EL,21}>s_2^{EL,sim}$

$$s_2^{EL,21}=\frac{\left(a_1(2r+(1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)-a_2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_1ra_2\left(r+1-\beta\right)2}$$

$$s_2^{EL,sim}=\frac{(2r-\beta+1)}{2r(r-\beta+1)}\left(\frac{1}{a_2}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_1}\right)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)$$

Want to show
$\frac{\left(a_1(2r+(1-\beta))\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)-a_2\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(1-\beta)\right)}{a_1ra_2(r+1-\beta)2}>\frac{(2r-\beta+1)}{2r(r-\beta+1)}\left(\frac{1}{a_2}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_1}\right)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)$
by simplifying, rearranging and using the following property of ln; $\ln x-\ln y=\ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1(2r+1-\beta)\ln\left(\frac{(2r+1-\beta)}{2r}\right)\underbrace{-a_2\left(1-\beta\right)\ln(\underbrace{\frac{2pr}{\left(1-p\right)\left(1-\beta\right)}\frac{(1-\beta)(1-p)}{2pr}}_{=1})}_{=0}>0$$

Which is true since $a_1(2r+1-\beta)>0$ and $\ln\left(\frac{(2r+1-\beta)}{2r}\right)>0$

Now lets check if the following holds $s_2^{EL,sim}>s_2^{EL,12}$

$$s_2^{EL,12}=\frac{(2r+(1-\beta))\left(-a_2\left(1-\beta\right)\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)+\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)(2r+(1-\beta))\,a_1\right)}{2\left(2r+1-\beta\right)a_2ra_1\left(r+1-\beta\right)}$$

$$s_2^{EL,sim} = \frac{(2r-\beta+1)}{2r(r-\beta+1)}\left(\frac{1}{a_2} - \frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_1}\right)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)$$

Want to show

$\frac{(2r-\beta+1)}{2r(r-\beta+1)}\left(\frac{1}{a_2}-\frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_1}\right)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > \frac{2\left(r+\frac{(1-\beta)}{2}\right)\left(-\frac{a_2(1-\beta)}{2}\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)+\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\left(r+\frac{(1-\beta)}{2}\right)a_1\right)}{(2r+1-\beta)a_2 r a_1(r+1-\beta)}$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_2(1-\beta)\ln\left(\frac{(2r+1-\beta)}{2r}\right)\underbrace{-a_1\left(2r+(1-\beta)\right)\ln\Bigl(\underbrace{\frac{2pr}{(1-p)\left(1-\beta\right)}\frac{(1-\beta)(1-p)}{2pr}}_{=1}\Bigr)}_{=0} > 0$$

Which is true since $a_2(1-\beta) > 0$ and $\dfrac{(2r+1-\beta)}{2r} > 1$ ■

Proof. (Theorem 1) The theorem is basically saying that in terms of total investment $EL_{12} > PRO > EA_{12}$

Lets firstly show that in terms of total investment $EL_{12} > PRO$

$$PRO = \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)(a_2+a_1)}{a_2 a_1(1-\beta+r)}$$

$$EL_{12} = \frac{\left(a_2\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)+a_1\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1\left(r+1-\beta\right)}$$

Want to show

$$\frac{\left(a_2\ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right)+a_1\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1\left(r+1-\beta\right)} > \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)(a_2+a_1)}{a_2 a_1(1-\beta+r)}$$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_2 \ln(\frac{(2r+1-\beta)}{r}) + a_1 \ln(2) > 0$$

Which is true since all of the terms are greater than 0
Now lets show that in terms of total investment $PRO > EA_{12}$

$$EA_{12} = \frac{\left(a_2 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_1\right)}{a_2 a_1 \left(r-\beta+1\right)}$$

$$PRO = \frac{\ln(\frac{pr}{(1-p)(1-\beta)})(a_2+a_1)}{a_2 a_1 (1-\beta+r)}$$

Want to show

$$\frac{\ln(\frac{pr}{(1-p)(1-\beta)})(a_2+a_1)}{a_2 a_1 (r-\beta+1)} > \frac{\left(a_2 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_1\right)}{a_2 a_1 \left(r-\beta+1\right)}$$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_2 \ln(\frac{(2-\beta)r+2-2\beta}{(1-\beta)(2+2r-\beta)}) + a_1 \ln(\frac{2-\beta}{2-2\beta}) > 0$$

Which is true since all of the terms are greater than 0 ■
Proof. (Theorem 2) We want to prove that in terms of total investment $EL_{21} > PRO > EA_{21}$
Firstly lets show that in terms of total investment $EL_{21} > PRO$

$$PRO = \frac{\ln(\frac{pr}{(1-p)(1-\beta)})(a_2 + a_1)}{a_2 a_1 (1-\beta+r)}$$

$$EL_{21} = \frac{\left(a_1 \ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + a_2 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1 \left(r+1-\beta\right)}$$

Want to show

$$\frac{\left(a_1 \ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + a_2 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1 \left(r+1-\beta\right)} > \frac{\ln(\frac{pr}{(1-p)(1-\beta)})(a_2 + a_1)}{a_2 a_1 (1-\beta+r)}$$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1 \ln(\frac{(2r+1-\beta)}{r}) + a_2 \ln(2) > 0$$

Which is true since all of the terms are greater than 0.
Now lets check if in terms of total investment the following holds; $PRO > EA_{21}$

$$EA_{21} = \frac{\left(a_1 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_2\right)}{a_2 a_1 \left(r-\beta+1\right)}$$

$$PRO = \frac{\ln(\frac{pr}{(1-p)(1-\beta)})(a_2 + a_1)}{a_2 a_1 (1-\beta+r)}$$

Want to show

$$\frac{\ln(\frac{pr}{(1-p)(1-\beta)})(a_2+a_1)}{a_2 a_1(r-\beta+1)} > \frac{\left(a_1 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_2\right)}{a_2 a_1\left(r-\beta+1\right)}$$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1 \ln(\frac{(2-\beta)r+2-2\beta}{(1-\beta)(2+2r-\beta)}) + a_2 \ln(\frac{2-\beta}{2-2\beta}) > 0$$

Which is true since all of the terms are greater than 0. ■

Proof. (Theorem 3) At first lets show that in terms of total investment $EL_{12} > EL_{21}$

$$EL_{12} = \frac{\left(a_2 \ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + a_1 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1\left(r+1-\beta\right)}$$

$$EL_{21} = \frac{\left(a_1 \ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + a_2 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1\left(r+1-\beta\right)}$$

Want to show

$$\frac{\left(a_2 \ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + a_1 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1\left(r+1-\beta\right)} > \frac{\left(a_1 \ln\left(\frac{p(2r+1-\beta)}{(1-p)(1-\beta)}\right) + a_2 \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)\right)}{a_2 a_1\left(r+1-\beta\right)}$$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y = \ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_2 \ln(\underbrace{\frac{(2r+1-\beta)}{2r}}_{x}) + a_1 \ln(\underbrace{\frac{2r}{(2r+1-\beta)}}_{\frac{1}{x}}) > 0$$

Define

$$x = \frac{(2r+1-\beta)}{2r}$$

then by following property of ln: $\ln x = -\ln \frac{1}{x}$ the inequality is again reduced to

$$(a_2 - a_1)\ln(\frac{(2r+1-\beta)}{2r}) > 0$$

which is true since by assumption $a_2 > a_1$

Secondly lets show that in terms of total investment $EA_{12} > EA_{21}$

$$EA_{12} = \frac{\left(a_2 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_1\right)}{a_2 a_1 (r-\beta+1)}$$

$$EA_{21} = \frac{\left(a_1 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_2\right)}{a_2 a_1 (r-\beta+1)}$$

Want to show

$$\frac{\left(a_2 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_1\right)}{a_2 a_1 (r-\beta+1)} > \frac{\left(a_1 \ln\left(\frac{pr(2+2r-\beta)}{(1-p)((2-\beta)r-2\beta+2)}\right) + \ln\left(\frac{2pr}{(1-p)(2-\beta)}\right) a_2\right)}{a_2 a_1 (r-\beta+1)}$$

by simplifying, rearranging and using the following property of ln; $\ln x - \ln y =$

$\ln(\frac{x}{y})$ the inequality we want to show is reduced to

$$a_1 \ln(\underbrace{\frac{2\left(\left(2-\beta\right)r-2\beta+2\right)}{\left(2-\beta\right)\left(2r-\beta+2\right)}}_{\frac{1}{x}}) + a_2 \ln(\underbrace{\frac{\left(2-\beta\right)\left(2r-\beta+2\right)}{\left(\left(2-\beta\right)r-2\beta+2\right)2}}_{x}) > 0$$

Define $x = \frac{(2-\beta)(2r-\beta+2)}{((2-\beta)r-2\beta+2)2}$ then by following property of ln: $\ln x = -\ln\frac{1}{x}$ the inequality is again reduced to

$$\left(a_2 - a_1\right)\ln(\frac{\left(2-\beta\right)\left(2r-\beta+2\right)}{\left(\left(2-\beta\right)r-2\beta+2\right)2}) > 0$$

which is true since by assumption $a_2 > a_1$. ■

8 Figures

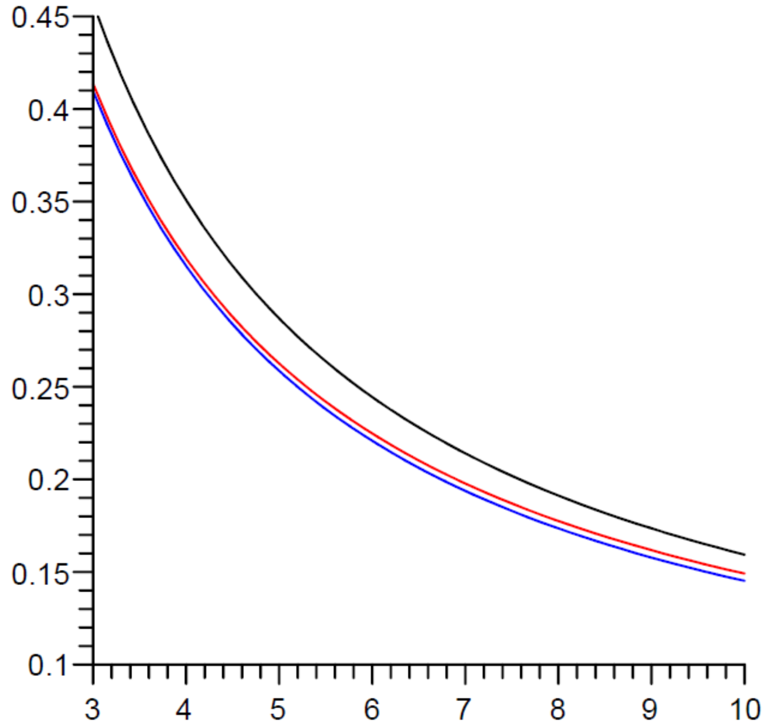

Figure 1: Agent 1's equilibrium investment s_1(vertical axis) as a function of his risk aversion a_1 (horizontal axis) under EA_{12} (black), EA_{21} (red), and EA_{sim} (blue) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_2 = 10$.

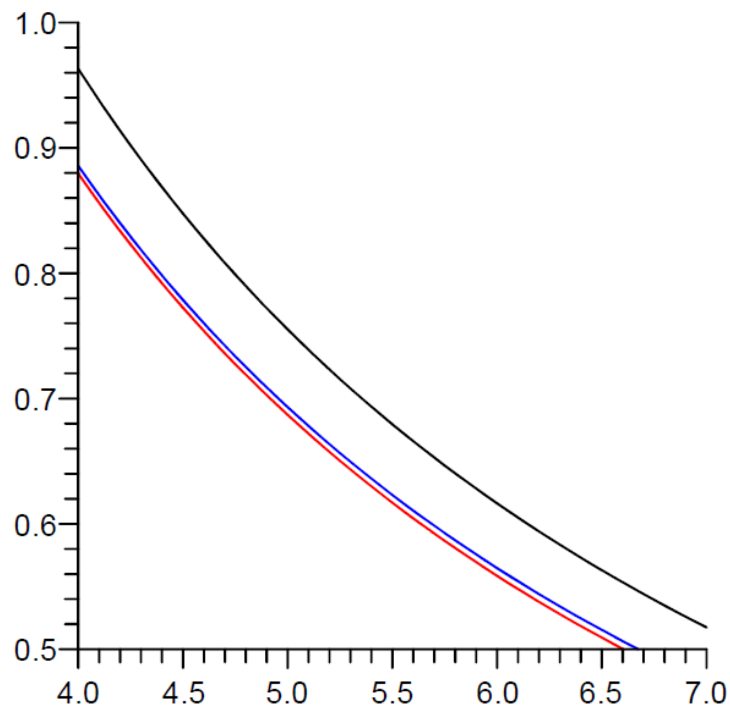

Figure 2: Agent 1's equilibrium investment s_1(vertical axis) as a function of his risk aversion a_1 (horizontal axis) under EL_{12} (black), EL_{21} (red), and EL_{sim} (blue) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_2 = 10$.

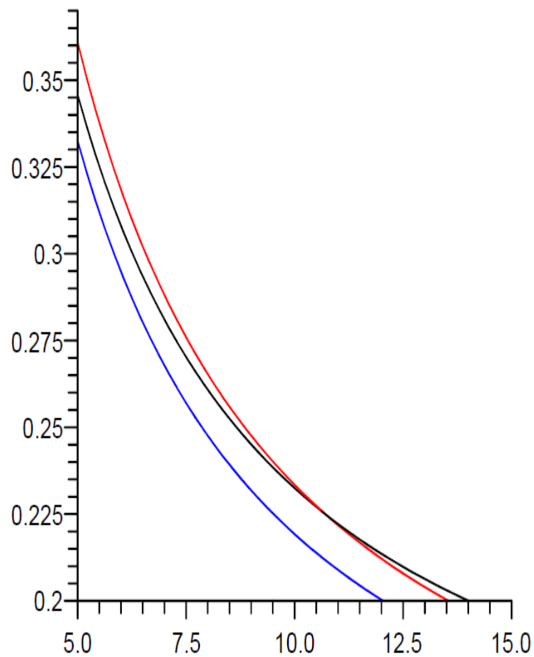

Figure 3: Agent 2's equilibrium investment s_2(vertical axis) as a function of his risk aversion a_2 (horizontal axis) under EA_{12} (black), EA_{21} (red), and EA_{sim} (blue) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 5$.

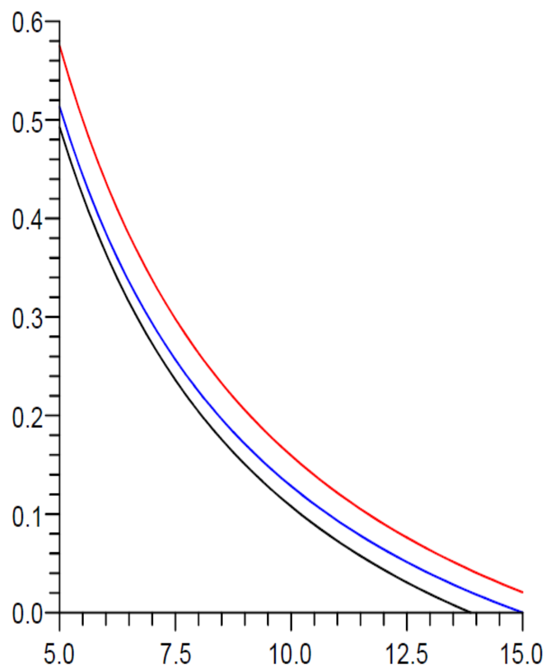

Figure 4: Agent 2's equilibrium investment s_2(vertical axis) as a function of his risk aversion a_2 (horizontal axis) under EL_{12} (black), EL_{21} (red), and EL_{sim} (blue) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 5$.

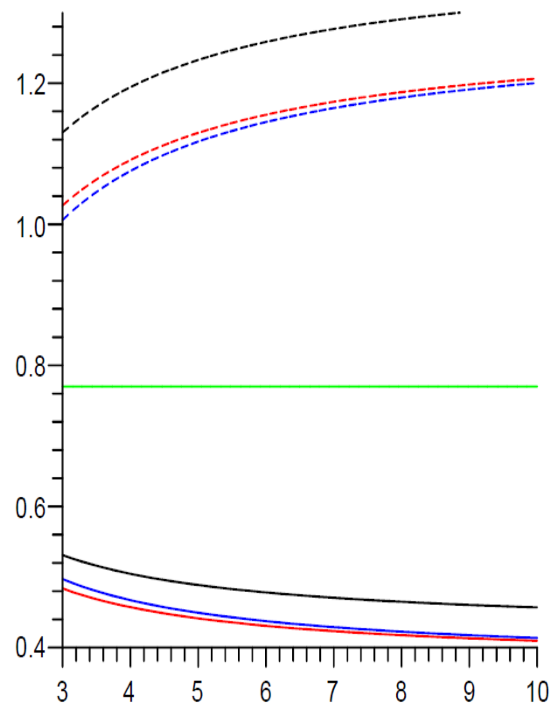

Figure 5: Agent 1's equilibrium investment s_1(vertical axis) as a function of agent 2's risk aversion a_2 (horizontal axis) under PRO(green solid line) EA_{12} (black solid line), EA_{21} (blue solid line), and EA_{sim} (red solid line),EL_{12} (black dashed line), EA_{21} (blue dashed line), and EA_{sim} (red dashed line) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$.

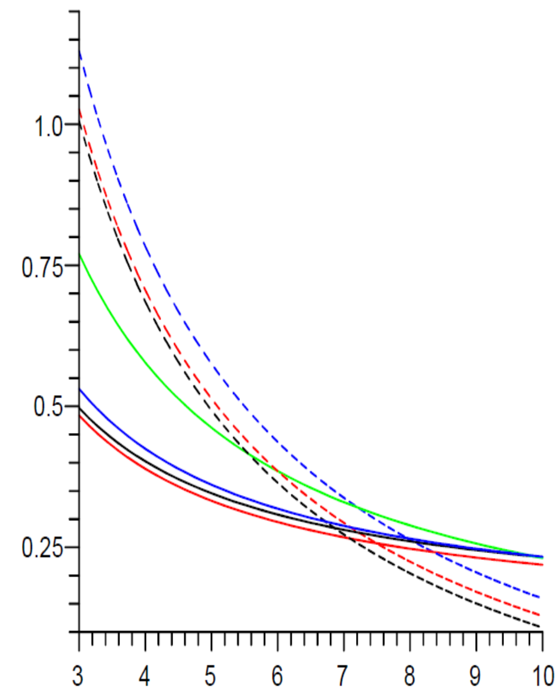

Figure 6: Agent 2's equilibrium investment s_2(vertical axis) as a function of his risk aversion a_2 (horizontal axis) under PRO(green solid line) EA_{12} (black solid line), EA_{21} (blue solid line), and EA_{sim} (red solid line),EL_{12} (black dashed line), EA_{21} (blue dashed line), and EA_{sim} (red dashed line) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$.

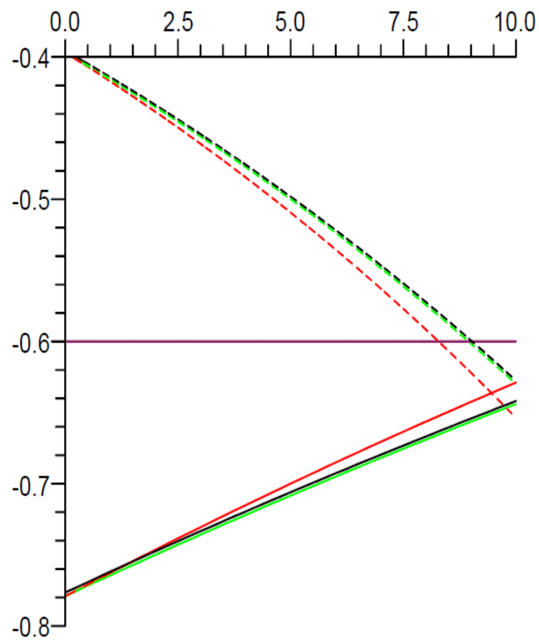

Figure 7: Agent 1's equilibrium utility (vertical axis) as a function of his risk aversion a_1(horizontal axis) underEA_{12}(black solid line) EA_{21}(red solid line) EA_{sim} (green solid line) $EL\ _{12}$(black dashed line), EL_{21} (red dashed line), and EL_{sim} (green dashed line) PRO(purple solid line) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_2 = 10$.

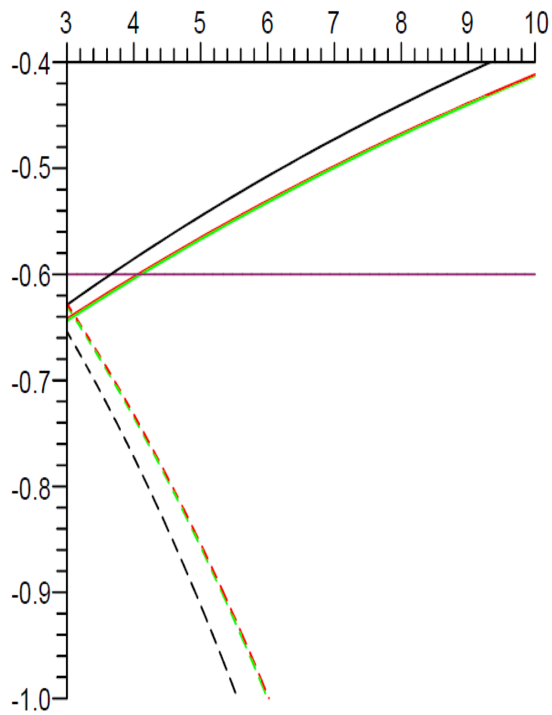

Figure 8: Agent 2's equilibrium utility (vertical axis) as a function of his risk aversion a_2(horizontal axis) underEA_{12}(black solid line) EA_{21}(red solid line) EA_{sim} (green solid line) $EL\ _{12}$(black dashed line), EL_{21} (red dashed line), and EL_{sim} (green dashed line) PRO(purple solid line) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$.

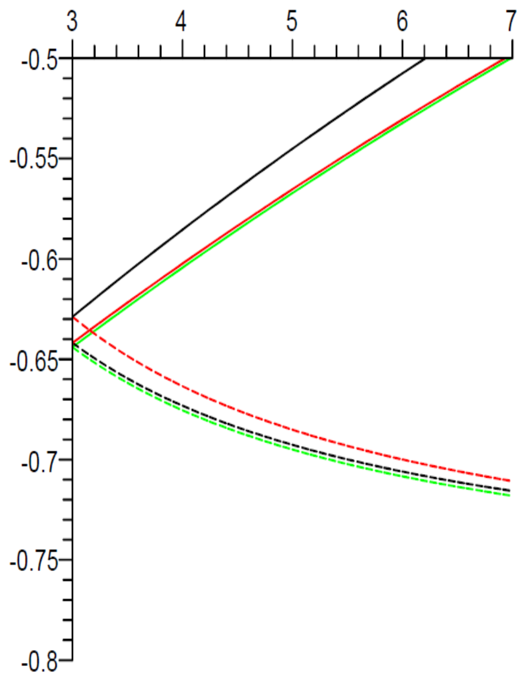

Figure 9: Both agent's equilibrium utility (vertical axis) as a function of agent 1's risk aversion a_1(horizontal axis) underEA_{12}(black) EA_{21}(red) EA_{sim} (green). Where dashed lines represent agent 1 and solid lines represent agent 2 and where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$.

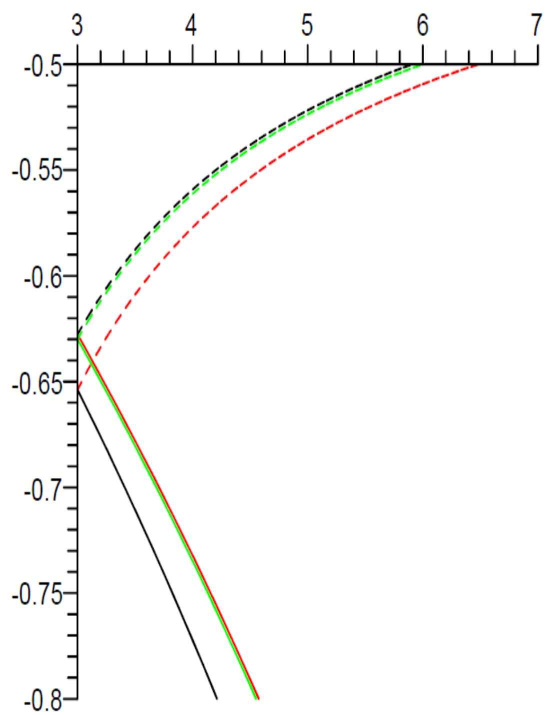

Figure 10: Both agent's equilibrium utility (vertical axis) as a function of agent 1's risk aversion a_1(horizontal axis) underEL_{12}(black) EL_{21}(red) EL_{sim} (green). Where dashed lines represent agen 1 and solid lines represent agent 2 and where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$.

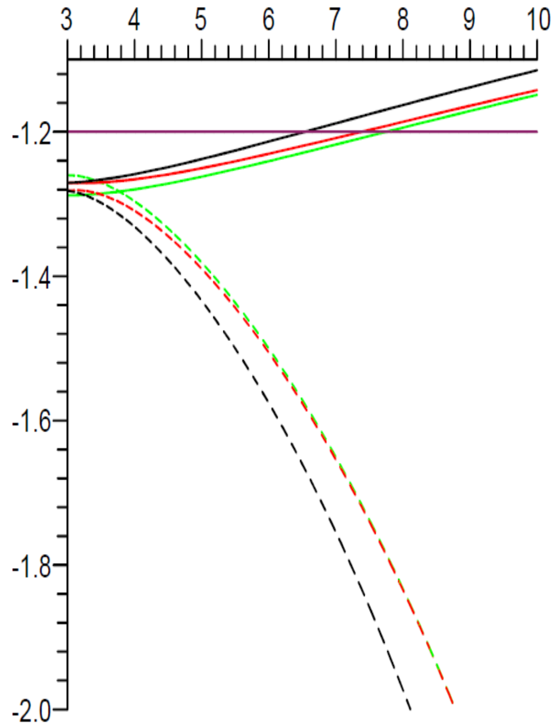

Figure 11: Total equilibrium utility (vertical axis) as a function of agent 2's risk aversion a_2(horizontal axis) underEA_{12}(black solid line) EA_{21}(red solid line) EA_{sim} (green solid line) $EL\ _{12}$(black dashed line), EL_{21} (red dashed line), and EL_{sim} (green dashed line) PRO(purple solid line) where $r = 0.6$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 3$.