BARGAINING WITH EXIT THREAT

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Abstract

We study the effects of exit threat in continuous two person bargaining games. Players try to establish reputation of being irrational type who never accepts an offer below his demand and exits the game at the time he announced in the beginning of the game. We show that a player becomes advantageous if he is able to threaten with exit time compared to the case where no one can choose exit time. However, this advantage becomes smaller if his opponent can also choose exit time to threaten. Moreover, we show that whether players can choose exit time or not, a player's payoff is decreasing with his discount rate and the initial probability of his opponent's irrationality and increasing with the discount rate of his opponent and the initial probability of his irrationality.

In this thesis we use Matlab program for computation. Detailed information and program codes can be found in the file named *codes*.

TERK ETME TEHDİDİ İLE PAZARLIK

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Özet

Bu tezde, terk etme tehdidinin iki kişilik sürekli zaman pazarlık oyununa etkileri analiz edilmektedir. Oyuncular kendilerininin irrasyonel tipler olarak tanınması için çaba sarfederler. İrrasyonal tipler kendi taleplerinin altında hiçbir öneriyi kabul etmeyenler ve oyunun başında açıkladıkları zaman geldiğinde oyunu terk eden tiplerdir. Bu tezde, sadece bir oyuncu terk etmekle tehdit edebildiği zaman, o oyuncu tehdit olmayan duruma göre daha avantajlı olduğunu gösteriyoruz. Fakat, diğer oyuncu da tehdit etme hakkına sahip olduğu zaman bu avantaj küçülmektedir. Ayrıca dengede bir oyuncunun alacağı beklenen getirisinin o oyuncunun geleceği iskonto oranı arttıkça veya diğer oyuncunun irrasyonel tip olma olasılığı arttıkça azalmakta olduğunu, fakat bu getirinin rakip oyuncunun geleceği iskonto oranı arttıkça veya kendisinin irrasyonel olma ihtimali arttıkça artmakta olduğunu gösteriyoruz.

Bu tezde hesaplamalar için Matlab programı kullanılmaktadır. Ayrıntılı bilgi ve program kodları *codes* isimli dosyada bulunmaktadır.

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1 Introduction

Division of surplus is one of the most important issues in economics. It is a central issue for human beings, firms, states, even for animals. While some bargaining problems may be resolved immediately, some may continue for a while, even forever. In the bargaining which is not resolved immediately, one of the agents who bargain may threaten the others by saying that they will stop bargaining at a certain time if a solution is not reached by that time. Such a threat is often encountered in bargaining problems. The one who threatens to exit may gain advantage of from this or he may lose advantage. A rational player, who wants to maximize his payoff never leaves the game. So, for such a threat to work a player must either be a type who certainly leaves the game at the specified time when he threatens or a rational player who mimic the former type. So, we use irrational types our model who leaves the bargaining game at the time he specified to leave the game and never accepts an offer which gives him a share less than his demand. The probability of one's being a irrational type is the main determinant of the outcome.

There is huge literature for bargaining. It generally asks the following question. One unit of surplus will be shared by two agents, how will they share it? Many factors may affect the bargaining outcomes, namely the way they share the surplus. We want to show what happens if players may use the threat of leaving the game by using their reputation. Rubinstein (1982), in his seminal paper, explains how players' impatience determines bargaining outcomes. Abreu&Gul (2000) and Kambe (1999) emphasize the role of reputation along with impatience. In their models each player is an irrational type with a small probability. They use simple irrational types; an irrational type never accepts an offer below his demand. Our model, in essence, is similar to the models in Abreu&Gul (2000) and Kambe (1999), but the main difference in our model is that an irrational player announces an exit time and he really exits at this time and rational players may mimic the irrational types by pretending they have a fixed acceptance rule like irrational types and by threatening their opponents with an exit time.

In Section 2 we explain the model. In Section 3, we investigate the equilibrium of the game where players do not choose exit time. In Section 4, we investigate the case where only player 1 can choose an exit time to threat. In Section 5, we show the complete model: both players can choose exit time. In, Section 6 we compare the results of the models. In Appendix, there are proofs that are not given in the main text.

2 The Model

We investigate a continuous time bargaining game with threat of exit. There are two players, $i = 1, 2$, who try to split one unit of surplus (From now on player i's opponent will be called as player j). In the beginning of the game each player i chooses a demand $\alpha_i \in (0, 1)$ for $i \in \{1, 2\}$. We call (α_1, α_2) as a demand profile. Then players' choices are publicly observed. If $\alpha_1 + \alpha_2 \leq 1$ each player gets his demand and then the remaining surplus is divided equally, i.e., each player i gets $(1 + \alpha_i - \alpha_j)/2$. If $\alpha_1 + \alpha_2 > 1$, bargaining will continue as a continuous time war of attrition game on $[0, \infty)$. Before the war of attrition starts, player i chooses a quitting time $K_i \in [0, \infty]$ for $i \in \{1, 2\}$. Normally players are rational, that is, each player maximize his discounted expected utility. But, with probability $z_i \in (0, 1)$, player i is replaced by an "irrational" type. An irrational type always offers his initial demand, accepts any offer weakly more than his initial demand and never accepts a lower offer. Moreover, he leaves the bargaining game at time K_i, he announced at time 0 (a rational player never leaves the game). Then the war off attrition starts. In the war of attrition game, if a player accepts and his opponent waits he gets what his opponent offers and his opponent gets what he had demanded. In case of simultaneous acceptance each player gets what his opponent offer and the remaining surplus is shared equally. If a player leaves the bargaining game without an agreement, both players get their outside option which is 0 for both player. The game ends when one player accepts his opponent's offer or when one of the players leaves the game. If player i choose to accept player j's offer and player j choose to leave the game at the same, game will be over by the acceptance of player i. Finally each player i discounts time by rate $r_i \in (0, \infty)$. So, if the game ends by acceptance of player i, he will get $(1 - \alpha_j)$ and his payoff will be $e^{-r_i t}(1 - \alpha_j)$ and player j will get α_j and his payoff will be $e^{-r_j t}\alpha_j$. Note that $r_i < r_j$ means player j discounts future more, that is he is more impatient than player i.

Since the game is continuous there are measure theoretic problems in defining the strategies. To overcome such technical issues we introduce 2 stages at time K_i. In stage 1 both players can accept their opponents' offer or wait. Moreover player i can leave at this time. (If player j chooses to accept and player i chooses to leave, game will be over by player j's acceptance). If player j chooses to wait and player i chooses to leave, then the game will be over by leaving of player i (both player get zero payoff). Irrational player i leaves the bargaining game at stage 1 of K_i . Therefore at the beginning of stage 2, rational player i's rationality will be common knowledge. (from now on, K_i^1 and K_i^2 will represent stage 1 of K_i and stage 2 of K_i, respectively). If player i is rational, he does not leave the game at his exit time K_i since his outside option is zero.

Strategy of player i is defined by a demand $\alpha_i \in (0, 1),$ exit time $K_i^{\alpha_1,\alpha_2}$ for each (α_1, α_2), cumulative distribution functions $F_i^{\alpha_1,\alpha_2,K_1,K_2}(t)$ on $[0, K^1]$ where $K = \min\{K_1, K_2\}$ for each $(\alpha_1, \alpha_2, K_1, K_2)$ and acceptance behavior after time K . The value $F_i(t)$ gives the probability of player i's acceptance by time t (inclusive). (From know on we will hide the dependence of the strategies, that is K_i and F_i will denote $K_i^{\alpha_1,\alpha_2}$ and $F_i^{\alpha_1,\alpha_2,K_1,K_2}$ respectively). F_i is a weighted avarage of rational player i's strategy and irrational l player i's strategy, it is actually the acceptance behavior of player i which player j believes. Rational player i's acceptance behavior is described by $F_i(t)/(1 - z_i)$ on $[0, K^1]$, he never accepts after time K^1 if $K < K_i$ and he fully accept at time K^2 if $K = K_i$. A player's acceptance behavior after time K is always the same, so from now we will not explicitly write it.

3 No Exit Time

For now suppose that no player chooses an exit time. This is the case in Abreu&Gul(2000) and Kambe(1999). Players choose their demands in the beginning of the game. If the demands compatible each player gets his demand and the remaining surplus is shared equally. If demands are not compatible, then war of attrition starts.

3.1 War Of Attrition Stage

Suppose demands $\alpha_1,\ \alpha_2$ given. If $\alpha_1 + \alpha_2 \leq 1,$ the games ends at time 0 and player i gets a share $(1 + \alpha_i - \alpha_j)/2$ for $i \in \{1, 2\}$. So, we assume $\alpha_1 + \alpha_2 > 1$. Define $\lambda_i = r_j(1 - \alpha_i)/(\alpha_1 + \alpha_2 - 1)$, $T_i = -\log(z_i)/\lambda_i$ and $T_0 = \min\{T_1, T_2\}$ for $i \in \{1, 2\}$ and $i \neq j$. Now let $\hat{F}_i = 1 - c_i e^{-\lambda_i t}$ where $c_i = z_i e^{\lambda_i T_0}$ for $t \leq T_0$ and $\hat{F}_i(t) = 1 - z_i$ for $t \geq T_0$ for $i \in \{1, 2\}$.

We use Abreu and Gul's result for this case. Proof can be found in Abreu and Gul, 2000.

Lemma 1 (Abreu and Gul, 2000) . *Given demand profile (α_1, α_2) such that $\alpha_1 + \alpha_2 > 1$, $(\hat{F}_1, \hat{F}_2)$ is the unique equilibrium in the war of attrition stage.*

The strategies $\hat{F}_1$ and $\hat{F}_2$ imply that *i)* at most one player accepts with positive probability at time zero, *ii)* each player i accepts at rate λ_i until time T_0 and *iii)* after time T_0 no player accepts. Player i accepts with positive probability at time zero if $T_i > T_0$ and accepts with zero probability at time zero if $T_i = T_0$.

Claim 1 . *In this equilibrium player j's expected payoff is given by*

$$U_j = (1 - c_i)\alpha_j + c_i(1 - \alpha_i) \tag{1}$$

Proof: At time 0 player j gets his demand with probability $F_i(0)$ so he has an expected payoff equal to $F_i(0)\alpha_j$ for time 0. After time 0 player j expects a payoff is equal to $(1 - \alpha_i)$ since player j accepts his opponent offer continuously including time 0 and the game continues to after time 0 with probability $(1 - F_i(0))$. Thus, player j's expected payoff is $U_j = F_i(0)\alpha_j + (1 - F_i(0))(1 - \alpha_i) = (1 - c_i)\alpha_j + c_i(1 - \alpha_i$ Note that U_j is equal to $(1 - \alpha_i)$ if $T_j \geq T_i$ and more than $(1 - \alpha_i)$ if $T_j < T_i$.

3.2 Demand Stage

Now we investigate equilibrium demands α_1 and α_2. Players choose their demands simultaneously knowing what will be continuation strategies F_1, F_2 thereafter. Define

$$\bar{\alpha}_i = \frac{r_j \log(z_j)}{r_i \log(z_i) + r_j \log(z_j)} \tag{2}$$

for $i \in \{1, 2\}$.

Proposition 1 . *The demand profile* $(\bar{\alpha}_1,\bar{\alpha}_2)$ *forms an equilibrium.*

Proof: See Appendix.

Since $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$, each player gets his demand at the beginning of the game and war of attrition does not occur. So, the equilibrium is efficient. Player i's share, $\bar{\alpha}_i$ is increasing with z_i and r_j, and decreasing with r_i. That means player i becomes better off as his probability of irrationality increases. Also he becomes better off if he becomes more patient or his opponent becomes less patient.

3.2.1 Other equilibria

Although $(\bar{\alpha}_1, \bar{\alpha}_2)$, there are many other equilbria. To give the idea we give an example:

Example 3.1: Let $z_1 = 0.001, z_2 = 0.1,\ r_1 = 0.7$ and $r_2 = 0.7$. Then, in the equilibrium of Proposition 1, player 1 demands 0.25 and player 2 demands 0.75 and their payoffs are the same as their demands. However, there are other equilibria in which player $1'$s payoff varies between 0.25 and 0.259 and player $2'$s payoff varies between 0.741 and 0.75. The equilibrium demands are illustrated in figure 1.

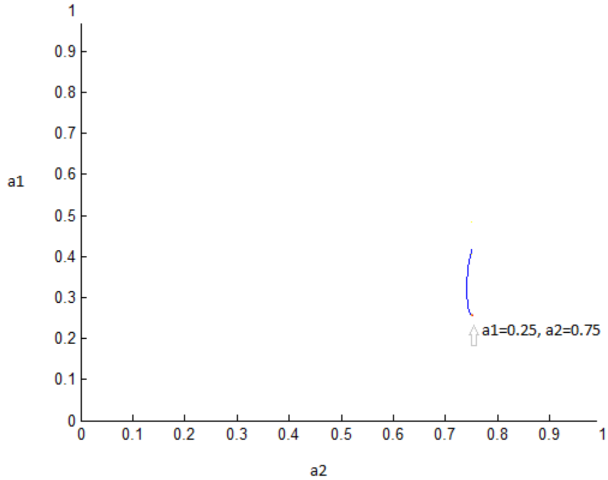

Figure 1: Equilibrium Demands for z1=0.001,z2=0.1,r1=0.7,r2=0.7

4 Only Player 1 Can Choose Exit Time

Suppose now that only player 1 can choose exit time to threaten his opponent. From now on we simply denote K as player $1'$s exit time. So, F_1 and F_2 are functions on $[0, K^1]$. If $\alpha_1 + \alpha_2 \leq 1$, there is no dispute so the game ends with the payoffs described above. Now we try to find equilibrium $\alpha_1, \alpha_2, K^{\alpha_1,\alpha_2}$ for each (α_1, α_2) and $F_1^{(\alpha_1,\alpha_2)}$, $F_2^{(\alpha_1,\alpha_2)}$ for each (α_1, α_2, K).

4.1 War of Attrition Stage

Here, we investigate the strategies in the war of attrition game. The demand profile $(\alpha_1,\ \alpha_2)$ and exit time K are given. Throughout this section we assume that $\alpha_1 + \alpha_2 > 1$.

We first look 3 main cases: $K = 0,\ 0 < K < T_0$ and $K \geq T_0$.

Case 1: $K = 0$

Suppose $(\alpha_1,\ \alpha_2)$ has been announced and then player 1 announces that he will exit at time 0. If no agreement is reached at time 0^1, player 1, if irrational, exits the game at this time and if rational he accepts his opponent's offer at time 0^2, since his rationality is revealed and a rational player does want to delay the acceptance. So, the game will be over at time 0^1 or 0^2. To simplify the strategies, define $\mu_i(t)$ be the acceptance rate of rational player i at time t. In this subsection will use the function μ to show the strategies. At time 0^1 and time 0^2 a rational player chooses to accept or wait.

Lemma 2 . *Suppose (α_1, α_2) given such that $\alpha_1 + \alpha_2 > 1$ and $K = 0$. Then, in any equilibrium $\mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$*

Proof: It is obvious by the above analysis.

Now the players' actions in the stage 2 is determined. So,players$'$ expected payoff by choosing "accept" or "wait" are given by the following equations.

$$U_1(accept) = (1-z_2)\mu_2(0^1)\frac{(1+\alpha_1-\alpha_2)}{2} + (1-z_2)(1-\mu_2 0^1))(1-\alpha_2) + z_2(1-\alpha_2) \quad (3)$$

$$U_1(wait) = (1-z_2)\mu_2(0^1)\alpha_1 + (1-z_2)(1-\mu_2 0^1))(1-\alpha_2) + z_2(1-\alpha_2) \quad (4)$$

$$U_2(Accept) = (1-z_1)\mu_1(0^1)\frac{(1+\alpha_2-\alpha 1)}{2} + (1-z_1)(1-\mu_1 0^1))(1-\alpha_1) \quad (5)$$

$$U_2(Wait) = (1-z_1)\alpha_2 \quad (6)$$

Define $z_1^* = (\alpha_1+\alpha_2-1)/\alpha_2$. This is the probability that is just enough to make player 2 to accept player 1 offer at the exit time. Suppose player 1 is irrational with probability z_1 in the beginning of the game and $K = 0$. So, if $z_1 > z_1^*$, player 2 accepts the offer, if $z_1 = z_1^*$, player 2 is indifferent between accepting and waiting. Finally if $z_1 < z_1^*$, player 2 does not accept the offer. Here are the details.

Case 1.1: $z_1 > z_1^*$

Lemma 3. *If $z_1 > z_1^*$, there is a unique equilibrium in which $\mu_1(0^1) = 0$, $\mu_1(0^2) = 1$, $\mu_2(0^1) = 1$ and $\mu_2(0^2) = 0$.*

Proof: See Appendix. In this equilibrium, expected payoffs:

$$U_1 = (1-z_2)\alpha_1 + z_2(1-\alpha_2) > (1-\alpha_2) \qquad \text{and} \qquad U_2 = (1-\alpha_1) \quad (7)$$

Case 1.2 $z_1 < z_1^*$

Lemma 4. *If $z_1 < z_1^*$, the set of equilibrium is $\{(\mu_1(0^1), \mu_2(0^1), \mu_1(0^2), \mu_2(0^2)) \in [0,1]^4 : \mu_1(0^1) \leq 2[1 - \frac{z_1(1-\alpha_1)}{(1-z_1)(\alpha_1+\alpha_2-1)}], \mu_2(0^1) = 0, \mu_1(0^2) = 1$ and $\mu_2(0^2) = 0\}$*

Proof: See Appendix.

In this equilibria, expected payoffs

$$U_1 = (1-\alpha_2) \qquad \text{and} \qquad U_2 = (1-z_1)\mu_1(0^1)\alpha_2 \quad (8)$$

Case 1.3 $z_1 = z_1^*$

Lemma 5. *If $z_1 = z_1^*$, the set of equilibrium is* $\{(\mu_1(0^1), \mu_2(0^1), \mu_1(0^2), \mu_2(0^2) \in [0,1]^4 : \mu_1(0^1) = 0, \mu_1(0^2) = 1$ *and* $\mu_2(0^2) = 0\}$

Proof: See Appendix.

In this equilibria, expected payoffs

$$U_1 = (1-z_2)\mu_2(0^1)\alpha_1 + [(1-z_2)(1-\mu_2(0^1)) + z_2](1-\alpha_2) \quad \text{and} \quad U_2 = (1-\alpha_1) = (1-z_1)\alpha_2 \tag{9}$$

Define $\hat{z}_i(t) = z_1/(1-F_i(t))$. This shows the way the reputations update. $\hat{z}_i(t)$ is probability of irrationality of player i by time t, that implies that at any time $t' > t$, player j believes that player i is irrational with probability $\hat{z}_i(t)$. At time 0 player j believes that player i is irrational with probability z_1. Now suppose that exit time is K. At time K^1, player 2 gets a share of $(1-\alpha_1)$ if the accepts, and gets an expected share of $(1-\hat{z}_i(K^1))\alpha_2$ if he waits. He weakly prefers accepting if and only if the former is not less the latter, i.e., $\hat{z}_i(K^1) \geq (\alpha_1+\alpha_2-1)/\alpha_2$. Recall that $z_1^* = (\alpha_1+\alpha_2-1)/\alpha_2$, so if player 1's reputation reach at least z_1^* by the exit time K^1, player 2 weakly prefers accepting to waiting at the exit time (if the inequality is strict, player 2 strictly prefers accepting). Let $T^* = -\log(z_1/z_1^*)/\lambda_1$.

Case 2: $0 < K < T_0$

Lemma 6 . *F_1, F_2 are equilibrium strategies in the war of attrition game only if $F_i(t) = 1 - c_i e^{-\lambda_i t}$ where $c_i \in [0,1]$ with $(1-c_1)(1-c_2) = 0$, for $t \in [0, K)$ and $i \in \{1, 2\}$.*

Proof: See Weiss at all (1988).

At time 0, F_i can jump for at most one player i by $(1-c_1)(1-c_2) = 0$ condition. So, F_1 and F_2 are continuous and strictly increasing on $[0, K)$. By claim 1, in any equilibirum expected payoff of player j is given by

$$U_j = (1-c_i)\alpha_j + c_i(1-\alpha_i) \tag{10}$$

Case 2.1 $K < T^*$

Lemma 7. *If $0 < K < T^*$ and $K < T_0$, then in equilibrium $F_1(t) = 1 - \frac{z_1}{z_1^*} e^{\lambda_1(K-t)}$ for $t \in [\in 0, K^1]$ and accepting with probability 1 at time K^2, and $F_2(t) = 1 - e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ and never accepting starting from time K^2.*

Proof: See Appendix.

Lemma 6 and equation 3 implies that expected payoffs are such that

$$U_1 = (1-\alpha_2) \tag{11}$$

$$U_2 = (1 - \frac{z_1}{z_1^*} e^{\lambda_1 K})\alpha_2 + \frac{z_1}{z_1^*} e^{\lambda_1 K}(1 - \alpha_1) \tag{12}$$

the latter is greater than $(1 - \alpha_1)$ since $\frac{z_1}{z_1^*} e^{\lambda_1 K} < 1$.

Case 2.2 $K > T^*$

Lemma 8. *If $K > T^*$ and $0 < K < T_0$, in equilibrium player 1's strategy is $F_1(t) = 1 - c_1 e^{-\lambda_1 t}$ for $t \in [\in 0, K^1]$ where $c_1 = 1$ and accepting with probability 1 at time K^2, and player 2's strategy is $F_2(t) = 1 - c_2 e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ where $c_2 = z_2 e^{\lambda_2 K}$ and never accepting starting from time K^2.*

Proof: See Appendix

Lemma 6 and equation 3 implies that expected payoffs are such that

$$U_1 = (1 - z_2 e^{\lambda_2 K})\alpha_1 + z_2 e^{\lambda_2 K}(1 - \alpha_2) \tag{13}$$

$$U_2 = (1 - \alpha_1). \tag{14}$$

the former is more than $(1 - \alpha_2)$ since $K < T_0$ and it less than $(1 - z_2)\alpha_1 + z_2(1 - \alpha_2)$ since $K > 0$.

Case 2.3 $K = T^*$

Lemma 9 . *If $K = T^*$ and $K > 0$, then in equilibrium player 1's strategy is $F_1(t) = 1 - c_1 e^{-\lambda_1 t}$ for $t \in [\in 0, K^1]$ where $c_1 = 1$ and accepting with probability 1 at time K^2, and player 2's strategy is $F_2(t) = 1 - c_2 e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ where $c_2 \in [z_z(\frac{z_1^*}{z_1})^{\frac{\lambda_2}{\lambda_1}}, 1]$ and never accepting starting from time K^2.*

Proof: See Appendix.

Lemma 6 and equation 3 implies that expected payoffs are such that

$$U_1 = (1 - c_2)\alpha_1 + c_2(1 - \alpha_2) \tag{15}$$

$$U_2 = (1 - \alpha_1) \tag{16}$$

the former is more than $(1 - \alpha_2)$ if $c_2 < 1$ and equal to $(1 - \alpha_2)$ if $c_2 = 1$.

Case 3: $K \geq T_0$

By lemma 6, in any equilibrium, $F_i(t) = 1 - c_i e^{-\lambda_i t}$ for $t \in [0, K)$ and $i \in \{1, 2\}$, where $c_i \in [0, 1]$ with $(1 - c_1)(1 - c_2) = 0$

Lemma 10 . *If $K \in [T_0, \infty)$, in equilibrium player i's strategy is $F_i(t) = 1 - c_i e^{-\lambda_i t}$ for $t \in [\in 0, K^1]$ where $c_i = z_1 e^{\lambda_i T_0}$ and never accepting starting from time T_0 for $i \in \{1, 2\}$.*

Proof: See Appendix.

To summarize the payoffs in this equilibria, consider two cases:

Case 3.1 $T_2 \leq T_1$

In this case $c_2 = 1$ since $T_2 = T_0$. Then, $c_1 = z_1 z_2^{\frac{-\lambda_1}{\lambda_2}}$ by lemma 10. So, by equation 3 expected payoffs are such that

$$U_1 = (1-\alpha_2) \qquad \text{and} \qquad U_2 = (1 - z_1 z_2^{\frac{-\lambda_1}{\lambda_2}})\alpha_2 + z_1 z_2^{\frac{-\lambda_1}{\lambda_2}}(1-\alpha_1) \tag{17}$$

Case 3.2 $T_2 > T_1$

In this case $c_2 = z_2 z_1^{\frac{-\lambda_2}{\lambda_1}}$, which is less than 1 since $T_2 > T_0$. So, by equation 3 expected payoffs are such that

$$U_1 = (1 - z_2 z_1^{\frac{-\lambda_2}{\lambda_1}})\alpha_1 + z_2 z_1^{\frac{-\lambda_2}{\lambda_1}}(1-\alpha_2) \qquad \text{and} \qquad U_2 = (1-\alpha_1) \tag{18}$$

The next table summarizes the expected payoff of player 1 in all the equilibria shown above.

Case	U_1
$1.1:\ K=0$ and $z_1 > z_1^*$	$(1-z_2)\alpha_1 + z_2(1-\alpha_2)$
$1.2:\ K=0$ and $z_1 < z_1^*$	$(1-\alpha_2)$
$1.3:\ K=0$ and $z_1 = z_1^*$	$(1-z_2)\mu_2(0^1)\alpha_1 + [(1-z_2)(1-\mu_2(0^1)) + z_2](1-\alpha_2)$
$2.1:\ 0 < K < T_0$ and $K < T^*$	$(1-\alpha_2)$
$2.2:\ 0 < K < T_0$ and $K > T^*$	$(1 - z_2 e^{\lambda_2 K})\alpha_1 + z_2 e^{\lambda_2 K}(1-\alpha_2)$:
$2.3:\ 0 < K < T_0$ and $K = T^*$	$(1-c_2)\alpha_1 + c_2(1-\alpha_2) : c_2 \in [z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}, 1]$
$3.1:\ K \geq T_0$ and $T_2 \leq T_1$	$(1-\alpha_2)$
$3.2:\ K \geq T_0$ and $T_2 > T_1$	$(1 - z_2 z_1^{\frac{-\lambda_2}{\lambda_1}})\alpha_1 + z_2 z_1^{\frac{-\lambda_2}{\lambda_1}}(1-\alpha_2)$

4.2 Exit Time Stage

Now, given $\alpha_1, \alpha_2 \in (0,1)$ such that $\alpha_1 + \alpha_2 > 1$, we investigate the equilibrium exit time K which is chosen by player 1. In section 4.1 we've determined the continuation strategies F_1, F_2 after α_1, α_2 and K are announced. So, when player 1 chooses K, the demands α_1, α_2 has already been announced and the continuation strategies of strategies F_1, F_2 which will be in the war of attrition game is known .

Exit time chosen by player 1 depends on the initial probability of player $1's$ irrationality, z_1. We have four different cases.

Case A.1: $z_1 > z_1^*$. In this case, initial probability player 1's irrationality is high enough to make player 2 accept at any exit time. He optimally chooses not to delay the exit time.

Proposition 2 . *If $z_1 > z_1^*$, in equilibrium player 1 chooses $K = 0$.*

Proof: See Appendix.

In this case, equilibrium expected payoffs:

$$U_1 = (1 - z_2)\alpha_1 + z_2(1 - \alpha_2) \qquad \text{and} \qquad U_2 = (1 - \alpha_1) \tag{19}$$

Case A.2: $z_1 = z_1^*$. In this case, initial probability player 1's irrationality is just enough to make player 2 indifferent between accepting or waiting at exit time 0 . So, he chooses exit time 0 only if player 2 accepts his offer at time 0.

Proposition 3 . *If $z_1 = z_1^*$, in equilibrium player 1 chooses $K = 0$ and in equilibrium only if player $2's$ strategy $\mu_2(0^1) = 1$.*

Proof: See Appendix. $\mu_1(0^1) = 0$, $\mu_2(0^1) = 0$, $\mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$.

In this case, equilibrium expected payoffs: $U_1 = (1-z_2)\alpha_1+z_2(1-\alpha_2)$ and $U_2 = (1-\alpha_1)$

Case A.3: $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} < z_1 < z_1^*$. In this case, in this case, initial probability player 1's irrationality is not enough to make player 2 to accept at exit time 0, so player 1 chooses time $K = T^*$ which is enough time for player 1 to build a reputation that makes player 2 not to wait at the exit time.

Proposition 4 . *If $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$, Then, in equilibrium $K = T^*$, $F_1(t) = 1 - e^{-\lambda_1 t}$ for $t \in [\in 0, K^1]$ and $F_2(t) = 1 - z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}} e^{-\lambda_2 t}$ for $t \in [0, K)$.*

Proof: See Appendix.

In this case, Case 2.3 happens since $T^* \in (0, T_0)$. So equilibrium expected payoffs:

$$U_1 = (1 - z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}})\alpha_1 + z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}(1 - \alpha_2) \qquad \text{and} \qquad U_2 = (1 - \alpha_1) \tag{20}$$

Case A.4 : $z_1 \leq z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$. In this case, in this case, initial probability player 1's irrationality is so low that he must choose $K \geq T_0$ to build a reputation that makes player 2 not to wait at the exit time. But then player 1 is no longer advantageous.

Proposition 5 . *If $z_1 \leq z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$, then in equilibrium we have $K \in [0, \infty]$ is an equilibrium.*

Proof: In this case $T^* \geq T_0$ and $T_1 > T_2$. If player 1 chooses $K = 0$, it will fall into *case 1.2* so payoff of player 1 will be $(1-\alpha_2)$. If he chooses $K < T_0$, it will fall into *case 2.1* so the payoff is $(1-\alpha_2)$. And finally if he chooses $K \geq T_0$, it will fall into *case 3.1* so again the payoff is $(1-\alpha_2)$. Thus, if $z_1 \leq z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$ any $K \in [0,\infty]$ is an equilibrium. In this case, equilibrium expected payoffs:

$U_1 = (1-\alpha_2)$ and $U_2 = (1 - z_1 z_2^{\frac{-\lambda_1}{\lambda_2}})\alpha_2 + z_1 z_2^{\frac{-\lambda_1}{\lambda_2}}(1-\alpha_1)$

4.3 Demand Stage

Now we investigate equilibrium demands α_1 and α_2. We've determined continuation strategies F_1, F_2 in section 4.1 and K in section 4.2. So, when players choose the demands, then they know the continuation strategies F_1, F_2 and K.

We have defined before $z_1^* = \frac{\alpha_1+\alpha_2-1}{\alpha_2} =$ and $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} = \frac{\alpha_1+\alpha_2-1}{\alpha_2} z_2^{\frac{r_2(1-\alpha_1)}{r_1(1-\alpha_2)}}$ both z_1^* and $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$ depend on α_1 and α_2, so in the equilibrium path exit time depends on these demands (see Section 4.2). Thus, players also consider this while they are choosing their demands.

Lemma 11 . *In any equilibrium player,* 1 *always demands* α_1 *such that* $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$.

Proof: See Appendix.

Proposition 6. *Suppose* (α_1^*, α_2^*) *is the equilibrium demand profile. Then, the following conditions are satisfied:*

i) $\alpha_1^* = \arg\max V_1^*(\alpha_1, \alpha_2^*) = (1 - z_z(\frac{\alpha_1+\alpha_2^*-1}{z_1\alpha_2^*})^{\frac{r_1(1-\alpha_2^*)}{r_2(1-\alpha_1)}})\alpha_1 + z_z(\frac{\alpha_1+\alpha_2^*-1}{z_1\alpha_2^*})^{\frac{r_1(1-\alpha_2^*)}{r_2(1-\alpha_1)}}(1-\alpha_2^*)$

ii) $\alpha_2^* = \arg\max V_2^*(\alpha_1^{*,\alpha_2}) = \arg\max\,(1-\alpha_1^*)$

iii) $z_1 \in (z_1^* z_2^{\frac{\lambda_1^*}{\lambda_2^*}}, z_1^*)$.

Proof: See Appendix.

Remark 1. *In any equilibrium* $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$*, then we have* $K = T^*$ *which is greater than zero. Thus, equilibrium exhibits delay*

Condition *ii) and iii)* of proposition 6 implies that in equilibrium player $1'$s demand must be such that player 2 cannot gain more than player 1 offers to him. This prevent player 1 to make excessive demands. The next remark gives the boundary.

Remark 2 . *In any equilibrium, there is a level* $\alpha_1^u \in (0,1)$ *such that* $\alpha_1^* < \alpha_1^u$.

Proof: As α_1 goes to 0, $\sup_{\alpha_2 \in (1-\alpha_1,1)} z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} < 0$ and as α_1 goes to 1,$\sup_{\alpha_2 \in (1-a_1,1)}$ $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} = 1$.Then, since $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$ continuously increasing in α_1, there exists $\alpha_1' \in (0,1)$ such that when $\alpha_1 < \alpha_1'$, $\sup_{\alpha_2 \in (1-\alpha_1,1)} z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} < z_1$. So, when $\alpha_1 < \alpha_1'$, for any $\alpha_2 \in (0,1)$, $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, 1)$. Define $\alpha_1^u = \sup \alpha_1'$. If $\alpha_1^* > \alpha_1^u$, then by definition of α_1^u, there exists some $\alpha_2 \in (0,1)$ such that $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} \geq z_1$. But, this contradicts with the proposition.

Remark 3. *When z_2 is constant, as z_1 goes to zero, α_1^u goes to 0, then expected payoff of player 1 goes to 0. And conversely when z_1 is constant, as z_2 goes to zero, α_1^u goes to 1, then expected payoff of player 1 goes to 1.*

Proof: α_1^u is defined such that when $\alpha_1 < \alpha_1^u$, $\sup_{\alpha_2 \in (1-\alpha_1,1)} z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} < z_1$. Take some $\alpha_1 \in (0,1)$, if z_1 becomes small enough, $\sup_{\alpha_2 \in (1-\alpha_1,1)} z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} \geq z_1$. But for some smaller α_1, $\sup_{\alpha_2 \in (0,1)} z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} < z_1$ since $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$ is increasing in α_1. This proves the first part. Second part is proven similarly.

4.4 Examples

Example 4.1: An Illustration of the Equilibria (Figure 2)

There are multiple equilibrium for demands. So we give an illustration of the equilibrium demands. I set parameters such that $z_1 = z_2 = 0.1$ and $r_1 = r_2 = 0.7$ .The demands were generated using Matlab program. In the figure $BR_i(\alpha_2)$ is best response function of player i which gives optimal α_i for each value of α_j for $i, j \in \{1,2\}$. For each $\alpha_2 \in (0,1)$, $BR_1(\alpha_2)$ is found by the program by choosing α_1 such that $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$ and it maximizes $V_1 = (1 - z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}})\alpha_1 + z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}(1-\alpha_2)$. V_1 is $1'$s equilibrium expected payoff for Case A.3 in which 1 get more payoff than any other case and $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$ implies Case A.3 will happen. Similarly for each $\alpha_1 \in (0,1)$, $BR_2(\alpha_1)$ is found by the program by choosing $\alpha_2 \in (0,1)$ that makes $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} \geq z_1$ and maximizes $U_2 = (1 - z_1 z_2^{\frac{-\lambda_1}{\lambda_2}})\alpha_2 + z_1 z_2^{\frac{-\lambda_1}{\lambda_2}}(1-\alpha_1)$, where V_2 is $2'$s payoff in Case A.4 in which 2 gets more payoff than any other case. If there is no $\alpha_2 \in (0,1)$ such that $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} \geq z_1$, then $BR_2(\alpha_1) = (0,1)$ since if $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} < z_1$ player 2 is indifferent between all of his demands. The red curve in the figure shows the points where the best response functions intersects, thus any point (α_2, α_1) in the red curve represents an equilibrium for demands. In this example, $\alpha_1^u = 0.7$, so by the remark 2, in any equilibrium $\alpha_1^* < 0.7$.

Example 4.2: Let $z_1 = z_2 = 0.1$ and $r_1 = r_2 = 0.7$. Here, one of the equilibrium demands are such that: $\alpha_1 = 0.6143$ and $\alpha_2 = 0.562$. Then $z_1^* z_2^{\frac{\lambda_1}{\lambda_2}} = 0.0412$ and $z_1^* = 0.3137$, so $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$ as Lemma suggests. Then we are in Case A.3, so in equilibrium $K = T^* = 0.7465$. F_1, F_2 are as described by Case 2.3. Until time 0.7465 at least one of

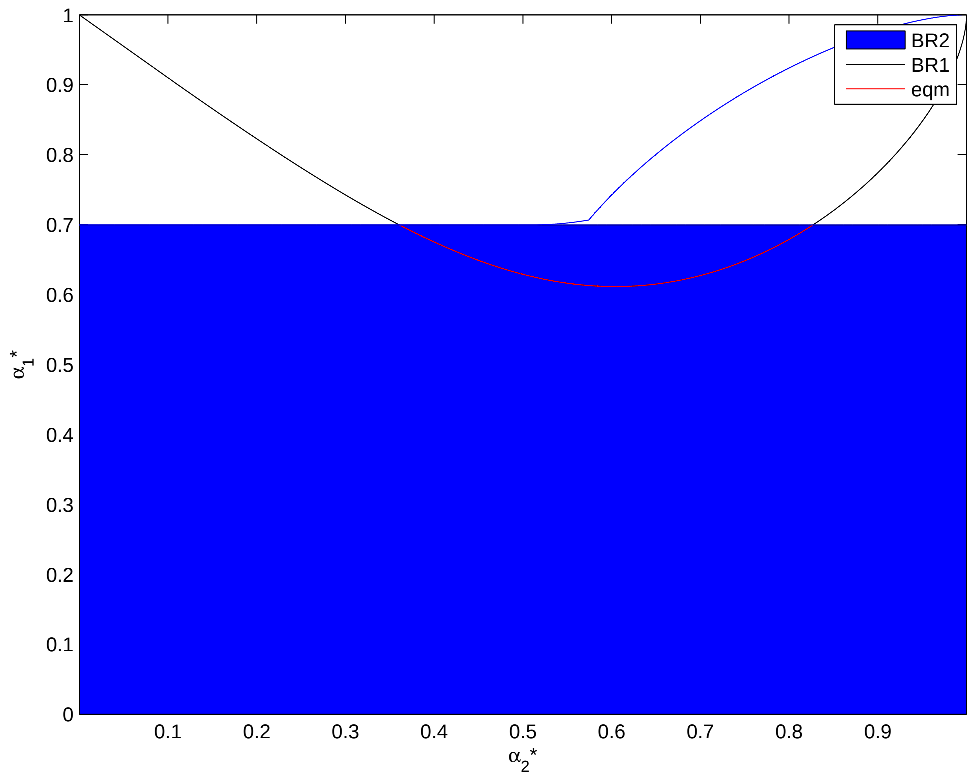

Figure 2: Equilibrium Demands for z1=0.1,z2=0.1,r1=0.7,r2=0.7

the players accepts his opponents offer, or player 1 leaves the game at time 0.7465. The game ends with player $1'$s left only if both players are irrational which happens with probability 0.01. Expected utilities are $U_1 = 0.5497$ and $U_2 = 0.3857$. Since $U_1 + U_2 = 0.9354 < 1$, there is an inefficiency of amount 0.0646.

Example 4.3: First part of Remark 3 says at when z_1 decreases relative to $z_2,$ player 1 becomes less advantageous. In Figure 3, this point is illustrated: when z_2 is constant, as z_1 decreases to zero, α_1^u decreases to zero and equilibrium demands becomes very to zero, hence any payoff of player 1 in the equilibria becomes very close to zero and so any payoff of player 2 in the equilibria becomes very close to 1. In the figure, the dashed lines shows α_1^u for each z_1 and the solid lines gives equilibria for demands for each z_1. We find the numerical results and create the figure by using Matlab program. In this example parameters are such that $z_2 = 0.1,\ r_1 = 0.7$ and $r_2 = 0.7$ while z_1 takes four different values; $0.1, 0.01, 0.000001$ and 0.0000000001. Note that even if player 1's payoff is very close to zero, it is still strong, i.e., player 2's payoff is equal to $(1 - \alpha_1)$.

Example 4.4: The second part of Remark 3 says at when z_2 decreases relative to $z_1,$ player 1 become more advantageous. In Figure 4, this point is illustrated: when z_1 is constant, as z_2 decreases to zero, α_1^u goes to 1 and equilibrium demands equilibrium demands becomes

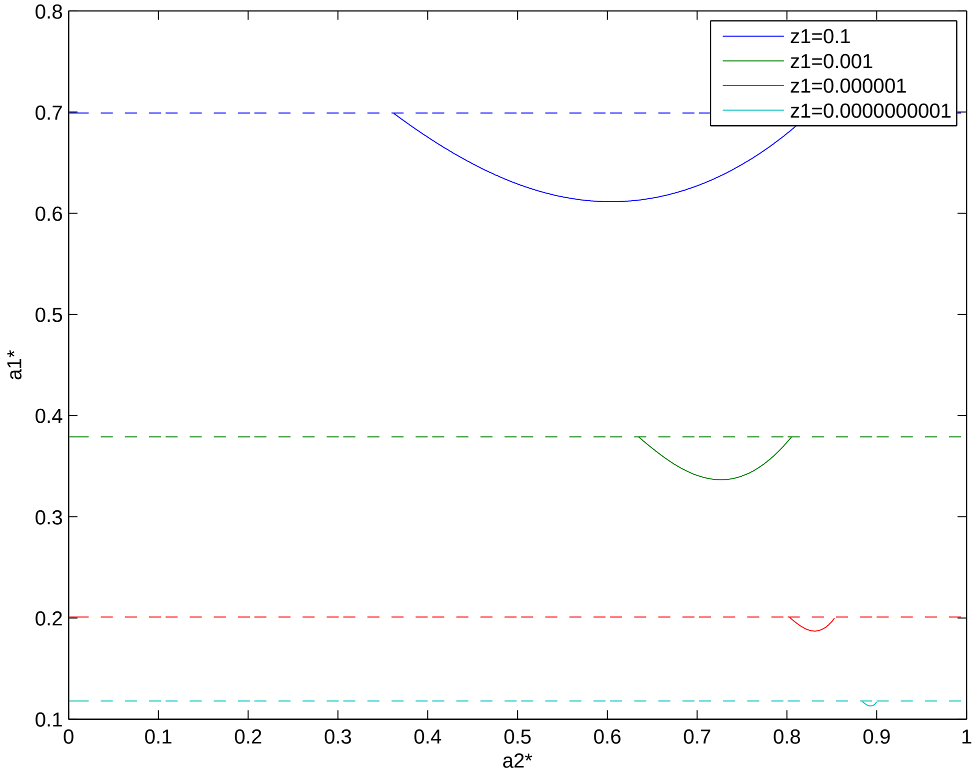

Figure 3: Eqm Demands and α_1^u for each z_1

very to 1, hence any payoff of player 1 in the equilibria becomes very close to 1 and so any payoff of player 2 in the equilibria becomes very close to 0. In the figure, the dashed lines shows α_1^u for each z_2 and the solid lines gives equilibria for demands for each z_2. We find the numerical results and create the figure by using Matlab program. In this example parameters are such that $z_1 = 0.1,\ r_1 = 0.7$ and $r_2 = 0.7$ while z_2 takes four different values; 0.1, 0.01, 0.000001 and 0.0000000001.

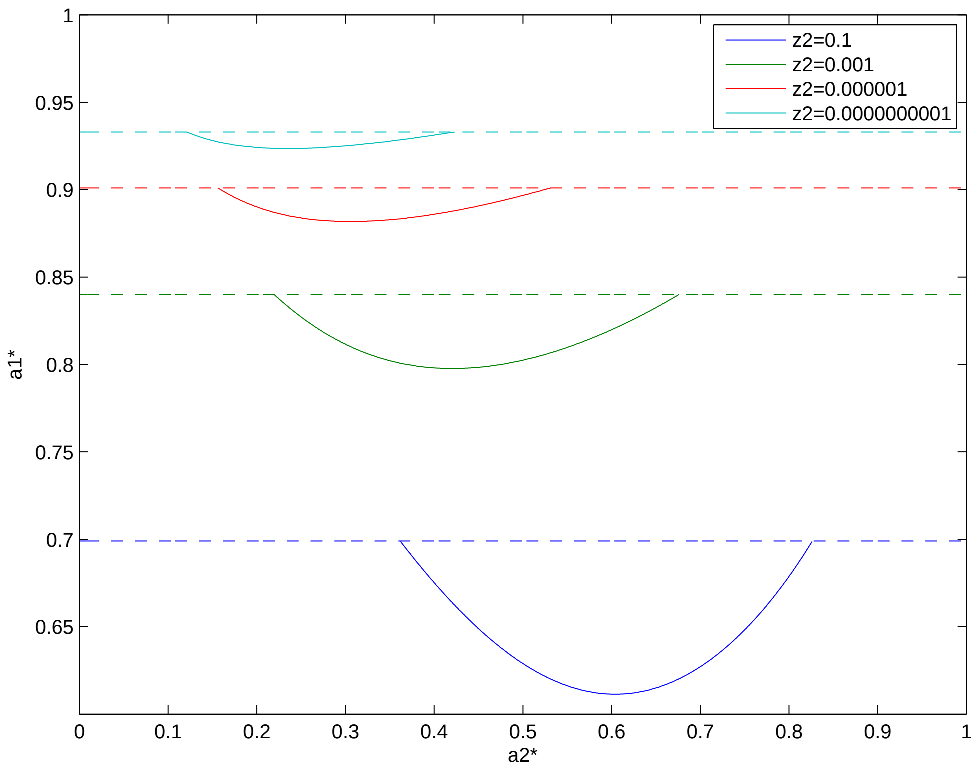

Figure 4: Eqm Demands and α_1^u for each z_2

5 Both Players Can Choose Exit Time

Now we allow both players to use the threat of exit. If $\alpha_1 + \alpha_2 \leq 1$, there is no dispute so the game ends immediately. Now we try to find equilibrium values of demands α_1, α_2, exit time $K_1^{\alpha_1,\alpha_2}$ for each $(\alpha_1\,, \alpha_2)$ and cumulative distribution functions $F_1^{(\alpha_1\,,\alpha_2,K_1,K_2)}$, $F_2^{(\alpha_1\,,\alpha_2,K_1,K_2)}$ for each $(\alpha_1\,, \alpha_2, K_1, K_2)$

If $K_i < K_j$, then player j's exit time threat is not effective since either player i is irrational and exits the game at time K_i or he is rational and fully accepts at time K_i. So, equilibrium strategies in the war of attrition game are the same as the strategies in the game with only player i can choose exit time (see Section 4.1).

When only player i can choose exit time, given the demands, his optimal exit time $K_i^* = \max\{0, T_i^*\}$ by section 4. The same is true for this case, that is if $T_i^* < T_j^*$, then $K_i^* = T_i^*$. So, if $(\alpha_1\,, \alpha_2)$ such that $T_i^* < T_j^*$, then player $i's$ exit threat is valid so he gets the payoff specified in section 4 and player j gets a payoff of $(1 - \alpha_i)$. In this case equilibrium demands are not unique so we give illustrations of the equilbria of demands.

Example 5.1: Let $z_1 = z_2 = 0.1,\ r_1 = 0.7$ and $r_2 = 0.7$. Then each player gets a payoff between 0.39 and 0.5. The equilibrium is illustrated at figure 5.

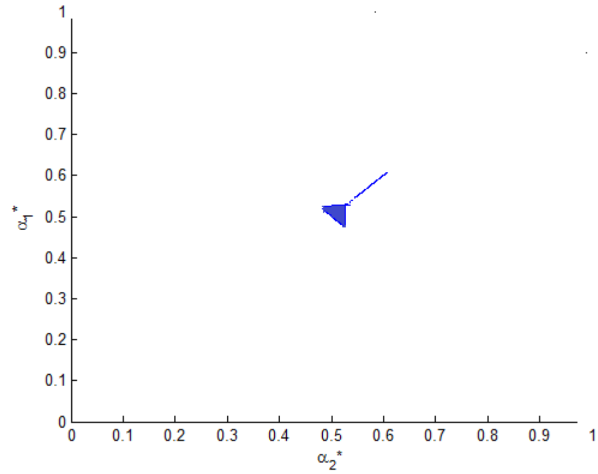

Figure 5: Equilibrium Demands for z1=0.1,z2=0.1,r1=0.7,r2=0.7

Example 5.2: Let $z_1 = z_2 = 0.1$, $r_1 = 5$ and $r_2 = 0.7$. Then player 2 has a payoff between 0.645 and 0.671 and player 1 has a payoff between 0.26 and 0.33. When we compare these results to example 5.1 we see that since r_1 has increased player 1's payoff decreases and player 2's payoff increases. The equilibrium is illustrated at figure 6.

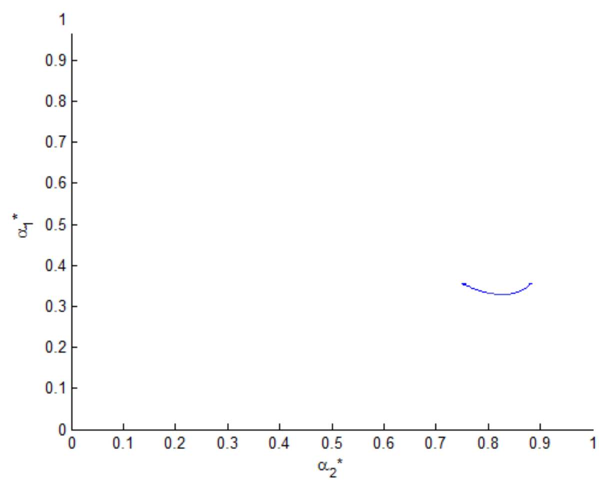

Figure 6: Equilibrium Demands for z1=0.1,z2=0.1,r1=5,r2=0.7

Example 5.3: Let $z_1 = 0.001, z_2 = 0.1$, $r_1 = 0.7$ and $r_2 = 0.7$. Then player 2 has a payoff between 0.772 and 0.796 and player 1 has a payoff between 0.188 and 0.202. When we compare these results to example 5.1 we see that since z_1 has decreased, player 1's payoff decreases and player 2's payoff increases.. The equilibrium is illustrated at figure 7.

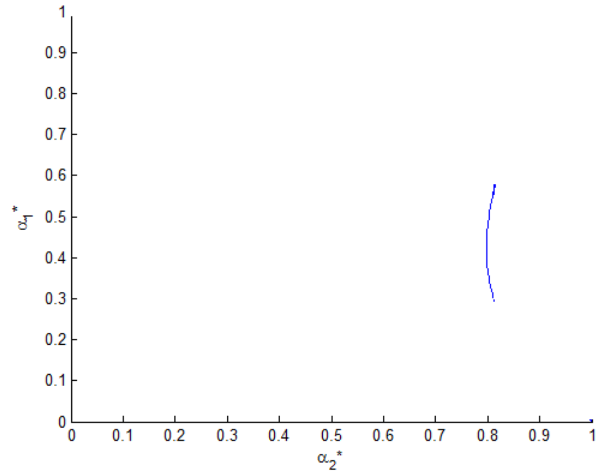

Figure 7: Equilibrium Demands for z1=0.001,z2=0.1,r1=0.7,r2=0.7

6 Comparison Between the Models

In this section we compare the models. We calculate expected payoff of players for each model and for different parameters. We compute the equilibriums by using the results of this paper and matlab code which helps to figure out multiple equilibria.

6.1 Symmetric Players

Example 6.1: Let $z_1 = z_2 = 0.1$ and $r_1 = r_2 = 0.7$. If no player can choose exit time as in Section 3, in equilibrium each player gets a payoff between 0.47 and 0.5 . When only player 1 can choose an exit time as in Section 4, his equilibrium payoff is at least 0.514 while payoff of player 2 is less than 0.5. Finally, when both players can choose exit time as in Section 5, each player's equilibrium payoff is between 0.39 and 0.5. These imply that when only player 1 choose exit time, he has a greater payoff compared to the case where no player chooses exit time. Thus, by being able to threaten his opponent to leave the game, player 1 gets an advantage. However, this advantage vanishes when both players can choose exit time to threat and both players become worse of compared to no exit time case.

6.2 Asymmetric Players

Example 6.2: Suppose $z_1 = z_2 = 0.1$ and $r_1 = 5$, $r_2 = 0.7$. Note that although players are symmetric in their initial reputation, but player1 is more impatient than player 2.In the first model, player $1's$ equilibrium payoff varies between 0.123 and 0.157 and player $2's$

equilibrium payoff varies between 0.76 and 0.88. Note that player 1 gets a very small share since he is more impatient. In the second model, where only player 1 can choose exit time, his payoff is at least 0.267. So, player 1 again takes the advantage of being able to choose exit time. In the third model, player 1 has a payoff between 0.188 and 0.202, so his advantage weakens because his opponent can also choose exit time, but again player 1 is better off compared to the first model.

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7 Appendix (Proofs)

Proof of Proposition 1: When the demand profile is $(\bar{\alpha}_1, \bar{\alpha}_2)$, $T_1 = T_2 = 0$. Suppose player 1 increases his demand, then T_1 will be more than T_2 causing player 1 to have an expected payoff of $(1 - \bar{\alpha}_2)$. However, this is equal to $\bar{\alpha}_1$. Thus, player 1 cannot increase his payoff by increasing his demand. On the other hand, if player 1 decreases his payoff, his new demand will be compatible with $\bar{\alpha}_2$, and the remaining surplus is divided equally. This causes player 1 to get an expected payoff less than $\bar{\alpha}_1$ since the initial demand profile is also compatible. Thus, player 1 cannot increase his payoff by changing his demand. Similarly player 2 cannot his payoff by changing his demand. So, the demand profile $(\bar{\alpha}_1, \bar{\alpha}_2)$ forms an equilibrium.

Lemma 3. *If $z_1 > z_1^*$, there is a unique equilibrium in which $\mu_1(0^1) = 0$, $\mu_1(0^2) = 1$, $\mu_2(0^1) = 1$ and $\mu_2(0^2) = 0$.*

Proof: In any equilibrium $\mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$ by lemma 2. Now suppose $\mu_2(0^1) > 0$, then $U_1(\text{Reject at time } 0^1) > U_1(\text{Accept at time } 0^1)$, so $\mu_1(0^1) = 0$. Then $U_2(\text{Accept at time } 0^1) = (1 - \alpha_1) > (1 - z_1)\, \alpha_2 = U_2(\text{Reject at time } 0^1)$ since $z_1 > z_1^*$. Then player 2 accepts his opponent's offer, i.e., $\mu_2(0^1) = 1$. Then since $\mu_2(0^1) = 1 > 0$, player 1 prefers to reject at time 0^1, i.e., $\mu_1(0^1) = 0$. Hence $\{\ \mu_1(0^1) = 0,\ \mu_2(0^1) = 1,\ \mu_1(0^2) = 1$ and $\mu_2(0^2) = 0\}$ is an equilibrium. To establish uniqueness, we need to show there is no equilibrium in which $\mu_2(0^1) = 0$. Suppose $\mu_2(0^1) = 0$,then it must be the case that $U_2(\text{Reject at time } 0^1) \geq U_2(\text{Accept at time } 0^1)$. But this is true only if $\mu_1(0^1) \leq 2[1 - \frac{z_1(1-\alpha_1)}{(1-z_1)(\alpha_1+\alpha_2-1)}]$ which is negative for $z_1 > z_1^*$, contradiction. So if $z_1 > z_1^*$, $\mu_2(0^1) = 0$ cannot be in equilibrium.

Lemma 4. *If $z_1 < z_1^*$, the set of equilibrium is $\{(\mu_1(0^1), \mu_2(0^1), \mu_1(0^2), \mu_2(0^2)) \in [0,1]^4 : \mu_1(0^1) \leq 2[1 - \frac{z_1(1-\alpha_1)}{(1-z_1)(\alpha_1+\alpha_2-1)}],\ \mu_2(0^1) = 0,\ \mu_1(0^2) = 1$ and $\mu_2(0^2) = 0\}$*

Proof: In any equilibrium $\mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$ by lemma 2. Now suppose $\mu_2(0^1) > 0$, then $U_1(\text{Reject at } t = 0^1) > U_1(\text{Accept at } t = 0^1)$, so $\mu_1(0^1) = 0$. But since $z_1 < z_1^*$, $U_2(\text{Reject at } t = 0^1) > U_2(\text{Accept at } t = 0^1)$ for any value of $\mu_1(0^1)$, which implies $\mu_2(0^1) = 0$. Thus, $\mu_2(0^1) > 0$ cannot be part of any equilibrium. So, in any equilibrium $\mu_2(0^1) = 0$. But then we must have $U_2(\text{Reject at } t = 0^1) \geq U_2(\text{Accept at } t = 0^1)$ which implies $\mu_1(0^1) \leq 2[1 - \frac{z_1(1-\alpha_1)}{(1-z_1)(\alpha_1+\alpha_2-1)}]$. Since $\mu_2(0^1) = 0$, $U_1(\text{Accept at } t = 0^1) = (1 - \alpha_2) = U_1(\text{reject at } t = 0^1)$, so any $\mu_1(0^1) \in [0,1]$ is a best response to $\mu_2(0^1) = 0$.Thus, any $(\mu_1(0^1), \mu_2(0^1), \mu_1(0^2), \mu_2(0^2))$ such that $\mu_1(0^1) \leq 2[1 - \frac{z_1(1-\alpha_1)}{(1-z_1)(\alpha_1+\alpha_2-1)}], \mu_2(0^1) = 0, \mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$ is an equilibrium and there is no other equilibrium.

Lemma 5. *If $z_1 = z_1^*$, the set of equilibrium is $\{(\mu_1(0^1), \mu_2(0^1), \mu_1(0^2), \mu_2(0^2) \in [0,1]^4 : \mu_1(0^1) = 0,\ \mu_1(0^2) = 1$ and $\mu_2(0^2) = 0\}$*

proof: In any equilibrium $\mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$ by the lemma. Now suppose $\mu_2(0^1) > 0$, then $U_1(\text{reject at } t = 0^1) > U_1(\text{Accept at } t = 0^1)$, so $\mu_1(0^1) = 0$. Then $U_2(\text{Accept at } t = 0^1) = U_2(\text{Reject at } t = 0^1)$ since $z_1 = z_1^*$. Thus any $\mu_2(0^1) \in [0,1]$ is a best response. Thus, any $\mu_2(0^1) > 0$, $\mu_1(0^1) = 0$, $\mu_1(0^2) = 1$ and $\mu_2(0^2) = 0$ forms an equilibrium. Moreover, if $\mu_2(0^1) = 0$, then any $\mu_1(0^1) \in [0,1]$ is a best response since

$U_1(\text{Accept at } t=0^1) = U_1(\text{reject at } t=0^1)$. $\mu_2(0^1)=0$ is optimal for player 2 if $\mu_1(0^1) \leq 2[1-\frac{z_1(1-\alpha_1)}{(1-z_1)(\alpha_1+\alpha_2-1)}] = 0$, i.e., $\mu_2(0^1)=0$ if and only if $\mu_1(0^1)=0$. Thus, $\mu_1(0^1)=0$, $\mu_2(0^1)=0$, $\mu_1(0^2)=1$ and $\mu_2(0^2)=0$ also forms an equilibrium and there is no other equilibrium.

Facts 1: In any in equilibrium in which $0<K<T_0$, the followings must be satisfied

$a)$ From Weiss at all., we know that equilibrium strategy of player i, $F_i(t) = 1-c_ie^{-\lambda_i t}$ where $c_i \in [0,1]$ with $(1-c_1)(1-c_2)=0$ for $i \in \{1,2\}$ and $t \in [0,K)$ (i.e., player i concedes with probability $(1-c_i)$ at time 0 and at constant hazard rate $\lambda_i = \frac{r_j(1-\alpha_i)}{(\alpha_1+\alpha_2-1)}$ starting from time 0). So, F_1 and F_2 are continuous and strictly increasing on $[0,K)$. Note that $(1-c_1)(1-c_2)=0$ condition means that at time 0, F_i can jump for at most one player i.

$b)$ $F_2(t)$ does not jump at $t=K^1$, since if it jumps at $t=K^1$, F_1 is constant on $(K-\epsilon,K)$ for some $\epsilon>0$, this contradicts with $a)$.

$c)$ F_1 will not jump at K^1 because player 1 is always prefers to wait and make concession at time K^2.

$d)$ By $a),b)$ and $c)$, F_1 and F_2 are continuous on $[0,K^1]$. Thus, $F_i(t)=1-c_ie^{-\lambda_i t}$ for $i,j \in \{1,2\}$ and $t \in [0,K^1]$.

$e)$Player 1 concedes fully at time K^2 since player 1's rationality is revealed at time K^2 and a rational player does not delay conceding.

$f)$ Player 2 never concedes starting from time K^2, since at time K^2 player 1's rationality is revealed.

$g)$. In any equilibrium $\hat{z}_1(K^1) \geq z_1^*$. To see this, suppose $\hat{z}_1(K^1) < z_1^*$. Then for some $\epsilon>0$, player 2 prefers to wait for all $t \in (K-\epsilon,K)$, then F_2 is constant on $(K-\epsilon,K)$, which contradicts with $a)$.

Let $T^* = -\frac{\ln(\frac{z_1}{z_1^*})}{\lambda_1}$. Note that T^* is the time required for player 1 to build this reputation to z_1^* by conceding at constant hazard rate λ_1 starting from time 0 and without making a concession at time 0.

Lemma 7. *If* $0<K<T^*$ *and* $K<T_0$, *then in equilibrium* $F_1(t) = 1-\frac{z_1}{z_1^*}e^{\lambda_1(K-t)}$ *for* $t \in [\in 0,K^1]$ *and accepting with probability 1 at time* K^2, *and* $F_2(t) = 1-e^{-\lambda_2 t}$ *for* $t \in [\in 0,K^1]$ *and never accepting starting from time* K^2.

Proof: By $g)$, reputation of player 1 at time K, $\hat{z}_1(K^1) \geq z_1^*$. Since T^* is time required for player 1 to build this reputation to z_1^*, without making a concession at time 0. K is less than T^*, so player 1 has to make jump at time 0 to build reputation z_1^* at time K^1. Here $\hat{z}_1(K^1) > z_1^*$ is not possible in equilibrium, since if $\hat{z}_1(K^1) > z_1^*$, player 2 strictly prefers conceding before time K, to conceding at time K^2. Then player 2 has to concede fully before time K^2, which implies $F_2(K^1)=1-z_2$, but then F_2 must have a jump at time 0. Both F_1 and F_2 having jumps at time 0 contradicts $a)$. Thus, $\hat{z}_1(K^1)=z_1^*$. This condition determines player 1's strategy uniquely: $F_1(K^1) = 1-c_1e^{-\lambda_1 K^1}$ where c_1 is determined by the condition $\hat{z}_1(K^1)=z_1^*$. This condition implies $c_1 = \frac{z_1}{z_1^*}e^{\lambda_1 K}$ (which is clearly is less than 1). F_2 is also determined: since F_1 jumps at time 0, F_2 does not jump, i.e., $c_2=1$ in the formula of F_2.Thus in equilibrium, $F_i(t)=1-c_ie^{-\lambda_i t}$ for $t \in [\in 0,K^1]$ and $i \in \{1,2\}$ where $c_1 = \frac{z_1}{z_1^*}e^{\lambda_1 K}$and

$c_2 = 1$, player 1 accepts fully at time K^2 by e) and player 2 never accepts starting from time K^2 by f).

Lemma 8. *If $K > T^*$ and $0 < K < T_0$, in equilibrium player 1's strategy is $F_1(t) = 1 - c_1 e^{-\lambda_1 t}$ for $t \in [\in 0, K^1]$ where $c_1 = 1$ and accepting with probability 1 at time K^2, and player 2's strategy is $F_2(t) = 1 - c_2 e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ where $c_2 = z_2 e^{\lambda_2 K}$ and never accepting starting from time K^2.*

Proof: Since T^* is enough for player 1 to build reputation to z_1^*, reputation of player 1 at time K^1, , will be higher than z_1^* (i.e., $\hat{z}_1(K^1) > z_1^*$), since F_1 is strictly increasing on $[0, K^1]$. Then, player 2 prefers fully conceding before time K^2, so $F_2(K^1) = 1 - z_2$. Then, since $F_2(K) = 1 - c_2 e^{-\lambda_2 K}$, $c_2 = z_2 e^{\lambda_2 K}$ which is less than 1 since $K < T_0$. By a), $(1 - c_1)(1 - c_2) = 0$, so $c_2 < 1$ implies that $c_1 = 1$. Thus, we have unique equilibrium: $F_i(t) = 1 - c_i e^{-\lambda_i t}$ for $t \in [\in 0, K^1]$ and $i \in \{1, 2\}$ where $c_1 = 1$ and $c_2 = z_2 e^{\lambda_2 K}$, and player 1 accepts fully at time K^2 (by e)) and player 2 never accepts starting from time K^2 (by f))

Lemma 9 . *If $K = T^*$ and $K > 0$, then in equilibrium player 1's strategy is $F_1(t) = 1 - c_1 e^{-\lambda_1 t}$ for $t \in [\in 0, K^1]$ where $c_1 = 1$ and accepting with probability 1 at time K^2, and player 2's strategy is $F_2(t) = 1 - c_2 e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ where $c_2 \in [z_z(\frac{z_1^*}{z_1})^{\frac{\lambda_2}{\lambda_1}}, 1]$ and never accepting starting from time K^2.*

Proof: Here if $F_1(0) > 0$, then $\hat{z}_1(K^1) > z_1^*$. But then since conceding before K^2 is optimal for player 2, $F_2(K^1) = 1 - z_2$ which requires $F_2(0) > 0$. Both F_1 and F_2 cannot jump at time 0, contradiction implies that $F_1(0) = 0$, i.e., $c_1 = 1$. Thus, $\hat{z}_1(K^1) = z_1^*$, so player 2 is indifferent between conceding and waiting at time K^1. Thus it is possible that $F_2(K^1) < 1 - z_2$ but $F_2(K^1)$ cannot be more than $(1 - z_2)$ since an irrational player never concedes. Hence for any $c_2 \in [z_z(\frac{z_1^*}{z_1})^{\frac{\lambda_2}{\lambda_1}}, 1]$, $F_2(t) = 1 - c_2 e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ is part of the equilibrium. As a result, in equilibrium $F_1(t) = 1 - e^{-\lambda_1 t}$ for $t \in [\in 0, K^1]$, $F_2(t) = 1 - c_2 e^{-\lambda_2 t}$ for $t \in [\in 0, K^1]$ where $c_2 \in [z_z(\frac{z_1^*}{z_1})^{\frac{\lambda_2}{\lambda_1}}, 1]$. Moreover by e) and f), player 1 concedes fully at time K^2 and player 2 never accepts starting from time K^2.

Lemma 10 . *If $K \in [T_0, \infty)$, in equilibrium player i's strategy is $F_i(t) = 1 - c_i e^{-\lambda_i t}$ for $t \in [\in 0, K^1]$ where $c_i = z_1 e^{\lambda_i T_0}$ and never accepting starting from time T_0 for $i \in \{1, 2\}$.*

Proof: From Weiss at all. and Abreu&Gul, we know that equilibrium strategy of player i, $F_i(t) = 1 - c_i e^{-\lambda_i t}$ where $c_i = z_i e^{\lambda_i T_0}$ for $i \in \{1, 2\}$ and $t \in [0, T_0]$. So F_1 and F_2 are continuous and strictly increasing on $[0, T_0]$. Then, both players reputation becomes 1 at the time T_0 and rational players concede fully until T_0. Hence, if $K \geq T_0$, leaving threat has no effect on equilibrium since rational players already concedes fully until T_0 (See lemma 1). If $T_i > T_0$, then $c_i = z_i z_j^{\frac{-\lambda_i}{\lambda_j}} < 1$ which implies player i has to make a concession at time 0 and if $T_i = T_0$, then $c_i = 1$ which implies player i does not make a concession at time 0.(note that $\min\{T_1, T_2\} = T_0$ where $T_i = -\log(z_i)/\lambda_i$).

Facts 2:

i) all payoffs are weighted averages of α_1 and $(1 - \alpha_2)$. So, since $\alpha_1 > (1 - \alpha_2)$, a payoff increases as the coefficient of α_1 increases.

$ii)$ For $K > 0$, $z_2 e^{\lambda_2 K}$, $z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}$ and $z_2 z_1^{\frac{-\lambda_2}{\lambda_1}}$ are less than z_2, so the greatest payoff that player 1 can achieve is $(1-z_2)\alpha_1 + z_2(1-\alpha_2)$. It is the payoff of equilibrium in *case 1.1* and it is the payoff of an equilibrium in *case 1.3* in which $\mu_2(0^1) = 1$, and the payoffs in all other cases are less than $(1-z_2)\alpha_1 + z_2(1-\alpha_2)$.

$iii)$ $z_2 e^{\lambda_2 K} > z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}$ for $K > T^*$ and $z_2 e^{\lambda_2 K} = z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}$ for $K = T^*$.

$iv)$ $z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}} < z_2 z_1^{\frac{-\lambda_2}{\lambda_1}}$, so the payoff in $case$ 2.3 when $c_2 = z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}$ is greater than the payoff in $case$ 3.2

$v)$ If $K = T^* + \varepsilon$ for small enough $\varepsilon > 0$ and $K < T_0$, the equilibrium payoff of player 1 by choosing $K = T^* + \varepsilon$ is more than the equilibrium payoff by choosing $K \geq T_0$ by iv. Then if there exits such K, player 1 never chooses $K \geq T_0$. That is if $z_1 > z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$, then $K \geq T_0$ cannot be in equilibrium.

$vi)$ If $K < T_0$ and $K > T^*$, player 1 can increase his payoff by decreasing K a bit, so no K such that $K < T_0$ and $K > T^*$ be in equilibrium.

$vi)$ If $T^* < T_0$, $K > T^*$ is not optimal for player 1, since he increase his payoff by decreasing K to $T^* + \varepsilon$, for some small $\varepsilon > 0$. Thus, if $T^* < T_0$, $K > T^*$ cannot be in the equilibrium.

$vii)$ $0 < K < T_0$ and $K < T^*$ is also cannot be in equilibrium since player 1 increase his payoff by choosing $K > T^*$.

$vii)$ If $T^* < T_0$, $K < T^*$ also cannot be in equilibrium since player 1 increase his payoff by choosing $K = T^* + \varepsilon$ for some small $\varepsilon > 0$.

Proposition 2 . *If $z_1 > z_1^*$, in equilibrium player 1 chooses $K = 0$.*

Proof: If $z_1 > z_1^*$, The payoff player 1 achieve by choosing $K = 0$ is greater than any other payoff he can achieve by $ii)$. So, in equilibrium, player 1 chooses $K = 0$.

Proposition 3 . *If $z_1 = z_1^*$, in equilibrium player 1 chooses $K = 0$ and player $2'$strategy such that $\mu_2(0^1) = 1$.*

Proof: When $z_1 = z_1^*$, $T^* = 0$ and $T^* < T_0$. So by $vi)$, $K > 0$ is not possible in equilibrium. Then the only candidate for an equilibrium is $K = 0$. For $K = 0$, the payoff is equal to $(1-z_2)\mu_2(0^1)\alpha_1 + [(1-z_2)(1-\mu_2(0^1)) + z_2](1-\alpha_2)$ where $\mu_2(0^1) \in [0,1]$. $K = 0$ is possible in equilibrium only if it gives a payoff which is not less than the payoff with any $K > 0$. That is, $K = 0$ is possible in equilibrium only if $\mu_2(0^1) = 1$.

Proposition 4 . *If $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$, in equilibrium player 1 chooses $K = T^*$ and player $2'$strategy such that $c_2 = z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}$.*

Proof: In this case $T^* > 0$ and $T^* < T_0$. So $K > T^*$ or $K < T^*$ cannot be in equilibrium, by $vi)$ and $vii)$. Then it remains to choose $K = T^*$ but it is optimal only if player 1 must have a payoff greater than or equal to what he can get by any $K > T^*$. So $K = T^*$ is in the equilibrium only if $c_2 = z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}$.

In this case, Case 2.3 happens since $T^* \in (0, T_0)$.

Proposition 5 . *If* $z_1 \le z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$, *any* $K \in [0, \infty]$ *is an equilibrium.*

Proof: In this case $T^* \ge T_0$ and $T_1 > T_2$. If player 1 chooses $K = 0$, it will fall into *case 1.2* so payoff of player 1 will be $(1 - \alpha_2)$. If he chooses $K < T_0$, it will fall into *case 2.1* so the payoff is $(1 - \alpha_2)$. And finally if he chooses $K \ge T_0$, it will fall into *case 3.1* so again the payoff is $(1 - \alpha_2)$. Thus, if $z_1 \le z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$ any $K \in [0, \infty]$ is an equilibrium.

Lemma 11 . *In any equilibrium player,* 1 *always demands* α_1 *such that* $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$.

Proof: Suppose player 1 demands α_1 such that $z_1 > z_1^*$. Then player 1 chooses $K = 0$ which in turn gives him an expected payoff of $(1 - z_2)\alpha_1 + z_2(1 - \alpha_2)$ (see Case A). But if player 1 demanded $\alpha_1 + \varepsilon$, for some small enough $\varepsilon > 0$, still $z_1^* < z_1$, so his resulting expected payoff would be $(1 - z_2)(\alpha_1 + \varepsilon) + z_2(1 - \alpha_2)$ which is higher than the initial payoff. So, α_1 such that $z_1 > z_1^*$ is never demanded by player 1. Now, suppose that player 1 demands α_1 such that $z_1 \le z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$. Then it would fall into Case A.4 in which player 1's payoff is $(1-\alpha_2)$. But, if player 1 demands $(1-\alpha_2+\varepsilon)$ for some enough small $\varepsilon > 0$, $z_1^* < z_1$, then Case A.1 will happen, so player 1's expected payoff will be $(1 - z_2)(1 - \alpha_2 + \varepsilon) + z_2(1 - \alpha_2)$ which is greater than $(1 - \alpha_2)$. So, player 1 never demands α_1 such that $z_1 \le z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$. Now, so far we have that in any equilibrium α_1 such that $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*]$, so player 1 expected utility, $U_1 = (1 - z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}})\alpha_1 + z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}(1 - \alpha_2)$. This expected utitliy is always more than the utilities which player can get with any α_1 such that $z_1 > z_1^*$ or $z_1 \le z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}$. In any equilibrium, player 1 chooses α_1 which maximizes this utility for given α_2. We show computationally that $\alpha_1^*(\alpha_2)$ that maximizes this utility satisfy $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$. This proves the first part.

Proof of Proposition 6:

Suppose (α_1^*, α_2^*) is the equilibrium demand profile. Then condition *iii)* must be satisifed by lemma 11, that is we have $z_1 \in (z_1^* z_2^{\frac{\lambda_1}{\lambda_2}}, z_1^*)$. Then, expected utility of player 1 , $U_1 = V_1(\alpha_1, \alpha_2^*) := (1 - z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}})\alpha_1 + z_z(\frac{z_1}{z_1^*})^{\frac{-\lambda_2}{\lambda_1}}(1 - \alpha_2)$ by Proposition 4. Thus, in equilibrium, α_1^* must maximize $V_1(\alpha_1, \alpha_2^*)$ by optimality, so condition *i)* must be satisfied..Also by *iii)* player 2's payoff is always $(1 - \alpha_1^*)$, so condition *ii)* must be satisfied.