ONE–MEMORY IN MULTIPERSON BARGAINING

by

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Abstract

In Rubinstein's (1982) 2–player discounted alternating offers bargaining game, the subgame perfect equilibrium outcome is unique and equivalent to the Nash bargaining solution. However, when there are more than 2 players, every feasible partition can be sustained in subgame perfect equilibrium with a sufficiently high discount factor (Shaked 1986). We prove that when the restriction to one–memory strategies is employed in the multiplayer version of the game, the subgame perfect equilibrium is unique and equivalent to the multiplayer generalization of Rubinstein's. This also implies that the unique subgame perfect equilibrium outcome corresponds to the Nash solution in the multiplayer cooperative game.

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Özet

Rubinstein, 1982 tarihli makalesinde, 2–oyunculu iskonto edilmiş sıralı–teklif pazarlık oyununda, sadece tek bir alt-oyun-yetkin denge olduğunu, ve bu dengede elde edilen dağılımın, iskonto değeri 1'e yaklaştığında, Nash pazarlık çözümüne denk olduğunu kanıtlamaktadır. Fakat, oyuncu sayısının 2'den fazla olduğu durumlarda, yeteri kadar yüksek bir iskonto faktörü ile mümkün olan her dağılım, bir alt-oyun-yetkin denge sonucu olarak elde edilebilmektedir (Shaked 1986). Biz, bu çalışmada, bahsi geçen oyunun çok-kişili versiyonunda izin verilen stratejilere bir–hafıza kısıtlaması getirmekte, ve bu kısıt altında alt-oyun-yetkin dengenin tek, ve Rubinstein'ın 2–kişili oyunda elde ettiği dengenin çok-kişi versiyonları için yapılan genellemesine eşit olduğunu ispatlamaktayız. Nitekim bu, aynı zamanda tek denge dağılımının ilgili çok-kişili işbirlikçi oyundaki Nash çözümüne de yüksek iskonto değerlerinde denk olduğu anlamına gelmektedir.

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1 Introduction

A bargaining situation is described as the interaction between two or more players trying to reach an agreement among multiple alternatives, in which all of them would be at least as well off as the case of no agreement. Many interactions that economists are interested in, like negotiations between labor unions and employers, or actually almost any kind of trade between parties, can be modeled in this framework. This is why bargaining has continued to draw the attention of researchers over the years.

It was John Nash, who introduced game theorists to the subject with his pinoeering paper "The Bargaining Problem" (Nash 1950), followed by "Two-player cooperative games" (Nash 1953). The significance of John Nash's contribution is that he laid the groundwork for two different approaches to analyzing bargaining situations, which complement each other.

In the article titled "The Bargaining Problem", Nash proposes an axiomatic approach to bargaining, which focuses on the "cooperative game". To be more precise, he lists some axioms that seem natural for a solution to have, without specifying any details about the bargaining procedure. Then, taking as given only the set of all possible payoff profiles that can be obtained as the outcome of a bargaining situation, he finds out that the payoff profile satisfying all of these axioms is unique, which came to be known as the Nash bargaining solution.

In his latter paper, "Two-player cooperative games", Nash describes a non-cooperative bargaining game (known as Nash's simultaneous-move demand game) specifying all the details of the bargaining process, and analyzes the equilibrium outcome, which turns out to be the same outcome as the cooperative solution. His intuition is that non–cooperative games are more basic and analyzing cooperative games by solving for the equilibria of a corresponding non–cooperative game might provide a worthy insight:

> We give two independent derivations of our solution of the two–person cooperative game. In the first, the cooperative game is reduced to a non-cooperative game. To do this, one makes the players' steps in negotiations in the cooperative game become moves in the non-cooperative model. Of course, one cannot represent all possible bargaining devices as moves in the non-cooperative game. The negotiation process must be formalized and restricted, but in such a way that each participant is still able to utilize all the essential strength of his position. The second approach is by the axiomatic method. One states as axioms several properties that would seem natural for the solution to have, and then one discovers that the axioms actually determine the solution uniquely. The two approaches to the problem, via the negotiation model or via the axioms, are complementary. Each helps to justify and clarify the other. (Nash (1953, p.128))

This attempt to bridge the gap between the two approaches was later called the Nash program. The unification of the two approaches provides justification for both, which they separately lack. As Sutton (1986) argues, the non–cooperative approach has the problem that the rules of a non–cooperative bargaining game might differ so extensively from case to case, that in order to obtain any useful insight into the theory, some principles that hold over a wide range of possible bargaining processes need to be prescribed. The axiomatic approach is helpful in this respect. However, "prescribing" such principles, or axioms, normatively is also problematic in turn. In order to establish the reasonableness of these axioms, one must examine whether or not they hold in some plausible bargaining processes, and this is done by formulating some non–cooperative bargaining games and examining their equilibria.

As Serrano (2005) suggests, the basic motivation behind the Nash program is that the relevance of a concept is enhanced if one arrives at it from different approaches. Thus, in the Nash program, the most significant research agenda is to find non–cooperative bargaining games that yield the same, unique equilibrium outcome as the cooperative Nash bargaining solution. In this regard, one of the most important contributions was made by Rubinstein (1982), which examines a 2–player discounted alternating offers bargaining game that uniquely sustains the Nash bargaining solution in subgame perfection. What makes Rubinstein's result significant is that unlike Nash's simultaneous-move demand game which relies on uncertainty in obtaining the uniqueness of Nash equilibrium, Rubinstein's uniqueness result comes from the use of credible threats, together with the assumption of impatience (Serrano 2005). However, later Shaked (1986) proves that these two key features, the use of subgame perfection and discounting, do not suffice to obtain a unique equilibrium in the 3–player version of the same game (restated in Osborne and Rubinstein (1990)). In fact, he shows that every feasible outcome can be obtained in subgame perfect equilibrium when there are more than 2 players.

This striking result of Shaked displays that the game theoretic prediction power of the $n \geq 3$ player generalization of Rubinstein's game, collapses. Moreover, there is no link between the non–cooperative and cooperative solutions in the $n \geq 3$ version of the game. The multiplicity of equilibria was tried to be eliminated using various extensions to the game. In Jun (1987), Chae and Yang (1988) and Fershtman and Seidmann (1993) players are restricted by the offers that they have accepted or rejected before. In Chae and Yang (1994), Krishna and Serrano (1996) and Suh and Wen (2006) the bargaining procedure encompasses bilateral negotiations. Yang (1992) modifies the game to make the offers include only the share of the last player in the responding order. In Merlo and Wilson (1995), the identity of the proposer is determined by a stochastic process. Chatterjee and Samuelson (1990) and Stahl (1990) consider simultaneous offers in the bargaining game, whereas Perry and Reny (1993) and Sakovics (1993) assume there might be lags before players recognize and respond to offers. In Baron and Ferejohn (1989), agreement does not require unanimous consent. Haller and Holden (1990) imposes costs on rejecting an offer. Binmore, Shaked, and Sutton (1989) and Huang (2002) allow for outside options. Asheim (1992) employs the notion of "acceptable paths", where a path is acceptable if and only if a player cannot profit by rejecting an offer an proposing another one herself. Baliga and Serrano (2001) and Vannetelbosch (1999) allow players to exit with partial agreements and the latter analyzes the equilibria using the refinement of trembling-hand rationalizability.

One method aiming to eliminate the multiplicity of equilibria in the multiplayer alternating offers bargaining game concerns the use of bounded rationality. This method is based on the idea that players prefer to use less "complex" strategies since learning and implementing them is easier. Different definitions of "complexity" have been used in this regard. The most basic, but just as debated, way of using "non–complex" strategies is imposing stationarity. When players' strategies are independent of the past, Herrero (1985) shows that the subgame perfect equilibrium is unique, and corresponds to the cooperative solution in the multiplayer cooperative game. However, using stationary strategies in order to formulate bounded rationality is disputed due to its extremity. We will elaborate more on this subject in Chapter 4.

Another, less extreme, formulation is based on the number of states of an automaton representing a strategy. We refer the reader to Aumann (1981) for a detailed discussion about the use of strategies described by finite–state automata in the analysis of bounded rationality. This particular "counting states" measure, on the other hand, does not provide a remedy. This is because, the strategy Shaked uses to prove that every feasible allocation can be obtained in subgame perfect equilibrium is a finite–state automaton (see Theorem 3 of the current thesis). It only uses $n + 1$ number of states, where n denotes the number of players. That is, this method does not solve the multiplicity of equilibria. Hence, Chatterjee and Sabourian (2000) uses a different formulation of complexity, which they define as: "...if two strategies are otherwise identical except that in some instances the first strategy uses more information than that available in the current "stage" of bargaining and the second uses only the information available in the current "stage", then the first strategy is more complex than the second."(Chatterjee and Sabourian (2000, p.1493)). However, although shrinking the set of Nash equilibria to some extent, they do not obtain a unique equilibrium either[1].

In this study, we use an alternative and widely accepted notion of bounded ra-

[1]Because that complexity is associated with some costs, they are obliged to employ modified notions of subgame perfection as in Rubinstein (1986) and Abreu and Rubinstein (1988).

tionality in the n–player discounted alternating offers bargaining game. Specifically, each player is restricted to use "one–memory" strategies. That is, players can base their actions only on yesterday's history, and cannot recall the history prior to that. Employing the restriction to one–memory strategies, we establish that the subgame perfect equilibrium outcome is unique and equivalent to the multiplayer generalization of Rubinstein's. This also implies that the unique subgame perfect equilibrium outcome corresponds to the Nash solution in the multiplayer cooperative game. Therefore, the main finding of this thesis constitutes a significant contribution to the Nash program.

It is important to note that, the notion of one–memory does not imply or is not implied by any of the other complexity formulations mentioned above. In fact, we show that the strategy used in Shaked's Theorem (presented as Theorem 3 in the current thesis which establishes the multiplicity of subgame perfect equilibria) is not one–memory, although being associated with a finite–state automaton. Moreover, Chatterjee and Sabourian (2000) is about complexity costs and not restricting history–dependence.

An interesting observation is that in the proof of our main result, we establish that one–memory subgame perfect equilibrium strategies have to be stationary. Indeed, the rest of our proof follows Herrero (1985, Proposition 4.2, p.90), which is restated (and proven for purposes of completeness) as Theorem 4 in the current thesis.

In the next chapter, we introduce our model and make the necessary definitions. In Chapter 3, we provide an overview of the axiomatic approach to bargaining and elaborate on some important theorems in non–cooperative bargaining that are relevant to our main result, specifically those of Rubinstein's and Shaked's. In Chapter 4, we present our main result, in addition to some auxiliary results.

2 The Model

The set of *players* is given by $N = \{1, \ldots, n\}$ and they are bargaining on the partition of a pie of size one. Only if they unanimously agree upon a partition will they get their respective shares. Otherwise, each will get none. The *bargaining process* is as follows: The game starts in period 0 with an *offer*

$$x = \left(x^{(i)}\right)_{i \in N} \in X$$

by player 1, where

$$X \equiv \left\{y \in [0,1]^n : y^{(1)} + \ldots + y^{(n)} \leq 1\right\} \subset \mathbb{R}^n_+$$

is *the set of all offers*. Then all other players respond one by one in the order $2, 3, \ldots, n$, by either *accepting* (Y) or *rejecting* (N) the offer. If all of them choose Y, the game ends and everybody gets her agreed share. If any of the players choose N, the game goes on to the next period, where the same process starts again with an offer by player 2 and responds by other players in the order $3, 4, \ldots, n, 1$. This goes on until there is agreement on an offer.

The *set of all histories* is denoted by H. The *empty history* is included in the set of all histories, i.e. $\emptyset \in H$. A *one–period history* is defined as a history ending in the same period at the beginning of which it started. The set of all one-period histories is denoted by $H^1 \equiv \{\{(x), (x, \mathbf{R})\} : \mathbf{R} = (r_1, \ldots, r_m)$ where $r_k \in \{Y, N\}$ for $k = 1, \ldots, m$ and $m \leq n-1\}$. Similarly, we define $H^1_1 \equiv \{\{(x), (x, \mathbf{R})\} : \mathbf{R} = (r_1, \ldots, r_m)$ where $r_k \in \{Y, N\}$ for $k = 1, \ldots, m$ and $m \leq n-2\}$, and $H^1_2 \equiv$

$\{\{(x, \mathbf{R})\} : \mathbf{R} = (r_1, \ldots, r_{n-1})$ where $r_k \in \{Y, N\}$ for $k = 1, \ldots, n-1$ and $r_{k'} = N$ for at least one $k' = 1, \ldots, n-1\}$, and finally $H_3^1 \equiv \{\{(x, \mathbf{R})\} : \mathbf{R} = (r_1, \ldots, r_{n-1})$ where $r_k = Y$ for all $k = 1, \ldots, n-1\}$. In words, H_1^1 is the set of all one-period histories after which one or more players are yet to play for that period to end. H_2^1 is the set of all one-period histories after which the period ends, and the next one starts. H_3^1 is the set of all terminal 1–period histories. A *t–period history* is denoted by $h^t = (h_0, h_1, ..., h_{t-1})$, where h_s denotes the one-period history covering period s with $h_s \in H_2^1$ for all $s \in \{0, 1, \ldots, t-2\}$ and $h_{t-1} \in H^1$. H^t is the set of all t-period histories. Following the above, we let $H_1^t \equiv \{h^t \in H^t : h_{t-1} \in H_1^1\}$, and $H_2^t \equiv \{h^t \in H^t : h_{t-1} \in H_2^1\}$, and finally $H_3^t \equiv \{h^t \in H^t : h_{t-1} \in H_3^1\}$. The *set of all terminal histories* is denoted by $Z = \bigcup_{t=1}^{\infty} H_3^t$.

The *concatenation* of two histories $h^t = (h_0, \ldots, h_{t-1}) \in H_2^t$ and $h^s = (h'_0, \ldots, h'_{s-1}) \in H^s$ is the operation defined as $h^t \circ h^s = (h_0, \ldots, h_{t-1}, h'_0, \ldots, h'_{s-1}) \in H^{t+s}$.

The *1–tail of a history* h^t is the history starting from and including the last complete one–period history in h^t. Formally, the 1–tail of $h^t = (h_0, h_1, ..., h_{t-1})$ is defined as

$$T^1(h^t) = \begin{cases} h_{t-2} \circ h_{t-1} & \text{if } h^t \in H_1^t \\ h_{t-1} & \text{if } h^t \in H_2^t \cup H_3^t. \end{cases}$$

In this setting, a *strategy* for player i is a function f_i mapping any non–terminal history in which she decides (denoted by $H(i)$) into the actions allowed for her at that particular history. The set of all strategies of player i is denoted by F_i. As usual, $F = \times_{i \in N} F_i$ and $F_{-i} = \times_{j \neq i} F_j$. It should be emphasized that $f_j^{(i)}(h^t)$ denotes the share of player i in the partition offered by player j in history h^t, $i, j \in N$ and $h^t \in H_2^t$ and $t \in \mathbb{N}$ where $(t \bmod n) + 1 = i$. A strategy f_i is *one–memory* if for all (h^t, h^s) with $t, s \in \mathbb{N}$ where $(t \bmod n) = (s \bmod n)$ satisfying $T^1(h^t) = T^1(h^s)$ we have $f_i(h^t) = f_i(h^s)$. The set of one–memory strategy profiles is denoted by $F^1 = \times_{i \in N} F_i^1$. Hence, a one–memory strategy assigns the same actions to two histories that are different but have the same 1–tail. Notice that the two histories need not be of same length, i.e. the one–memory restriction does not allow time–dependence.

Given a strategy $f_i \in F_i$ and a history $h \in H$ we denote the *strategy induced by* f_i *at* h by $f_i|h$. Thus, $(f_i|h)(\bar{h}) = f_i(h \circ \bar{h})$ for every $(h \circ \bar{h}) \in H(i)$. Let $f|h = (f_i|h)_{i \in N}$ and $F|h$ be the set of all such strategy profiles. A strategy profile f induces two types of terminal histories, the first where players agree upon some partition in a finite time period, and the second where this does not hold. Players use a common discount factor $\delta \in (0,1)$. In the first type of terminal histories that involve an allocation $x = \left(x^{(i)}\right)_{i \in N} \in X$ agreed upon in period t, the *utility* of player i, $i \in N$, is a function $U_i : F \to \mathbb{R}$ where $U_i(f)$ is given by $u_i : X \times \mathbb{N} \to [0,1]$ which is defined by $u_i(x,t) = \delta^t x^{(i)}$. In case of no agreement –the second type of terminal histories– player i gets a utility $U_i(f)$ given by $\lim_{t \to \infty} u_i(x,t) = \lim_{t \to \infty} \delta^t x^{(i)} = 0$. A strategy vector $f \in F$ is a *Nash equilibrium* if $U_i(f) \geq U_i(\hat{f}_i, f_{-i})$ for all $i \in N$ and $\hat{f}_i \in F_i$. Also, $f \in F$ is a *subgame perfect equilibrium (SPE)* if $f|h \in F|h$ is a Nash equilibrium for all $h \in H$. The set of all subgame perfect equilibrium strategy profiles is denoted by E and those with one-memory by E^1, i.e. $E^1 = E \cap F^1$. It should be pointed out that, in the current study, the definition of one–memory subgame perfect equilibrium does not involve only deviations that are one–memory. In fact, there are no limitations on the magnitude of memory of a strategy that constitutes a deviation from a one–memory strategy.

3 The Bargaining Problem

In this chapter, first we will discuss Nash's cooperative game theoretic solution to convex bargaining problems employing an axiomatic approach. Then, the well–known result due to Rubinstein (1982), establishing the uniqueness of subgame perfect equilibrium in two player alternating offers bargaining games (with discounting) will be presented. Moreover, it should be emphasized that the subgame perfect equilibrium outcome of two player alternating offers bargaining games (with discounting) converges to the unique Nash solution of the underlying bargaining problem with sufficiently patient players (i.e. when the common discount factor tends to 1).

In general, following Serrano (2005), *the Nash program* "is an attempt to bridge the gap between the two counterparts of game theory (cooperative and non-cooperative)". In fact, obtaining the Nash bargaining solution in a plausible non-cooperative game theoretic formulation as the unique (non-cooperative game theoretic) equilibrium constitutes a noteworthy contribution to the Nash program. Therefore, Rubinstein's result is, indeed, a corner stone in the Nash program.

However, when Rubinstein's setting is generalized to more than two players, the above discussed and well celebrated result no longer holds. In fact, Shaked (1986) shows that in the multiplayer bargaining game, also known as the Shaked's game, every feasible allocation can be sustained in subgame perfect equilibrium. To make things worse, this striking result weakening game theoretic prediction power, is obtained using finite automata strategies, which are the standard modeling tool to capture bounded rationality. That is, the result in the Nash program with more than two players is not established even when attention is restricted to the use of non–complex (finite automata) strategies.

3.1 Nash's Barganing Solution

According to the framework established by Nash (1950), a bargaining situation consists of the set of players $N = \{1, 2\}$, the set of possible agreements A, the disagreement outcome D, and a preference relation $\succeq_i$ for each player $i \in N$ on lotteries over the set of possible agreements and the disagreement outcome, i.e. over $A \cup D$.

Let the set of lotteries over $A \cup D$ be denoted by $\mathcal{L} = \{\ell \in [0,1]^{|A \cup D|} : \sum_{i=1}^{|A \cup D|} \ell_i = 1\}$. Players' preference relations satisfy von Neumann and Morgenstern assumptions. Thus, the preference relation for each player i can be represented by a utility function $v_i : \mathcal{L} \to \mathbb{R}$, where a lottery is preferred by player i to another if and only if the expected utility of that lottery is greater than the expected utility of the other lottery according to v_i.

Let $S = \{s \in \mathbb{R}^2 : \exists \ell \in \mathcal{L} \text{ with } (v_1(\ell), v_2(\ell)) = (s_1, s_2)\}$ denote the set of all possible utility pairs and let $d = (v_1(D), v_2(D))$. Nash defines a *bargaining problem* as the pair $\langle S, d \rangle$, where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$, and there exists $s \in S$ with $s_i > d_i$ for $i = 1, 2$. Letting $\mathcal{B}$ denote the set of all bargaining problems, a *bargaining solution* is defined as a function $f : \mathcal{B} \to \mathbb{R}^2$, assigning a unique element of S to each bargaining problem $\langle S, d \rangle$. [1]

The bargaining problem $\langle S, d \rangle$ does not specify any details as to the nature of the bargaining process. Instead of dealing with the details of a given bargaining process, Nash employs an axiomatic approach. That is, he specifies four properties that he expects a natural solution to have, and defines the subset of S satisfying all four of these properties as the solution of the bargaining problem $\langle S, d \rangle$. We state these four axioms below (Osborne and Rubinstein 1990):

1. *Invariance to equivalent utility representations*: Suppose that $\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ using the transformation $s'_i \mapsto \alpha_i s_i + \beta_i$ for some $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$, for $i = 1, 2$, i.e. $S' = \{(\alpha_1 s_1 + \beta_1,\ \alpha_2 s_2 + \beta_2) : (s_1, s_2) \in S\}$, and $d' = (\alpha_1 d_1 + \beta_1,\ \alpha_2 d_2 + \beta_2)$. Then $f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$, for $i = 1, 2$.

[1] Note that, f here is not the same as the f we used to denote strategies in our model, in Chapter 2. Here, for the sake of convention, f denotes the bargaining solution. However, outside this section, f will continue to stand for strategies.

2. *Symmetry*: If the bargaining problem is symmetric, i.e. $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$, then $f_1(S, d) = f_2(S, d)$.

3. *Independence of irrelevant alternatives*: If $\langle S, d\rangle$ and $\langle T, d\rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$, then $f(S, d) = f(T, d)$.

4. *Pareto efficiency*: Suppose $\langle S, d\rangle$ is a bargaining problem and $s, t \in S$ are such that $t_i > s_i$ for $i = 1, 2$. Then $f(S, d) \neq s$.

In his seminal paper (Nash 1950), Nash establishes that the bargaining solution satisfying all four of these axioms is unique. We next present, without proof, this very well–known result:

Theorem 1 (Nash (1950)) *There is a unique bargaining solution* $f^N : \mathcal{B} \to \mathbb{R}^2$ *satisfying the axioms 1-4, which is given by*

$$f^N(S, d) = \underset{(d_1,d_2)\leq(s_1,s_2)\in S}{\arg\max} (s_1 - d_1)(s_2 - d_2). \tag{3.1}$$

It should be emphasized that when the Axioms 1 – 4 are generalized to the case of n–players, the bargaining solution satisfying all four axioms remains to be unique. In fact, it is the n–player generalization of 3.1:

$$\underset{d\leq s\in S}{\arg\max} \prod_{i=1}^{n} (s_i - d_i). \tag{3.2}$$

Moreover, it is important to note that, 3.2 gives the solution as equal division due to Axiom 2 when the bargaining problem is symmetric, which is the case with the underlying bargaining problem in our model.

3.2 Rubinstein's Result

We now turn our attention back to non-cooperative bargaining games, which are the focus of this thesis, as described in our model. In this section, we present the well-known result of Rubinstein (1982) and its proof, establishing the uniqueness of subgame perfect equilibrium in the 2–player discounted alternating offers bargaining game:

Theorem 2 (Rubinstein (1982)) *The bargaining game with two players has a unique subgame perfect equilibrium f^*, where (regardless of the history) player 1 proposes a sharing scheme given by $x^* = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ whenever it is her turn to propose, and player 2 proposes $y^* = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$. Each player accepts an offer if and only if it delivers a share not less than $\frac{\delta}{1+\delta}$.*

Proof. First, we need to show that the strategy profile denoted by f^* and specified in the statement of the Theorem, constitutes a subgame perfect equilibrium.

Consider a subgame starting with an offer by player 1. If player 1 proposes $x \neq x^*$, then either $x^{(2)} > \frac{\delta}{1+\delta}$, and x gets accepted, where player 1 will receive $x^{(1)} < \frac{1}{1+\delta}$, or $x^{(2)} < \frac{\delta}{1+\delta}$, which will result in the rejection of x and acceptance of y^* the next period, providing player 1 with $\frac{\delta}{1+\delta}$ with one period of delay. Thus, the highest utility player 1 can get, by deviating from x^* is less than $x^{*(1)}$. Now suppose player 2 deviates from choosing Y in the subgame starting after the offer x^* by player 1. After rejecting x^*, player 2 will propose y^*, and it will be accepted. Player 2 will get $\frac{1}{1+\delta}$ with one period of delay, which gives her a utility equal to getting $x^{*(2)}$ with no delay. Hence, no player has a profitable deviation. Clearly, the same arguments apply to the subgames starting with an offer by player 2, or a response by player 1, establishing that f^* is a subgame perfect equilibrium.

Now, we need to show that f^* is the unique subgame perfect equilibrium. Let G_i denote a subgame starting with an offer by player i. Since it is shown above that the set of subgame perfect equilibria is not empty, we can define:

$$M_i \equiv \sup\{\delta^t x^{(i)} : \text{ there is an SPE of } G_i \text{ with outcome } (x,t)\}$$
$$m_i \equiv \inf\{\delta^t x^{(i)} : \text{ there is an SPE of } G_i \text{ with outcome } (x,t)\}.$$

In the following steps, we will establish that the present utility that player 1 gets from every SPE outcome of G_1 is $x^{*(1)}$, and the present utility that player 2 gets from every SPE outcome of G_2 is $y^{*(2)}$, i.e.

$$M_1 = m_1 = x^{*(1)} \text{ and } M_2 = m_2 = y^{*(2)}. \tag{3.3}$$

Step 1 $m_2 \geq 1 - \delta M_1$.

Proof. Suppose player 2 proposes y with $y^{(1)} > \delta M_1$. Player 1 will get at most M_1 with one period of delay if she rejects, so she will accept any such offer y. Thus, the minimum utility that player 2 gets in a SPE of G_2 must be $1 - \delta M_1$. ■

Step 2 $M_1 \leq 1 - \delta m_2$.

Proof. By rejecting an offer by player 1 in G_1, Player 2 can guarantee getting at least m_2 the next period. Thus, in any subgame perfect equilibrium of G_1 with no delay, player 2 must get at least δm_2, which means player 1 can get at most $1-\delta m_2$. If player 2 rejects the offer by player 1 and agreement is delayed, player 1 gets at most $1-m_2$ with one period of delay. Since $\delta(1-m_2) \leq 1-\delta m_2$, the result follows. ■

Multiplying both sides of Step 1 by δ, and then subtracting both sides from 1 gives $1-\delta m_2 \leq 1-\delta+\delta^2 M_1$. This, together with Step 2 implies $M_1 \leq 1-\delta+\delta^2 M_1$, which is equivalent to $M_1 \leq \frac{1}{1+\delta}$. We know that $M_1 \geq x^{*(1)} = \frac{1}{1+\delta}$ since f^* is SPE. Hence, it must be $M_1 = x^{*(1)}$. Similarly, multiplying both sides of Step 2 by δ and subtracting from 1 gives $1-\delta M_1 \geq 1-\delta+\delta^2 m_2$, which, together with Step 1 implies $m_2 \geq \frac{1}{1+\delta}$. Since it must be that $m_2 \leq y^{*(2)} = \frac{1}{1+\delta}$, we get $m_2 = y^{*(2)}$.

Replacing player 1 with player 2 in the steps and the following argument above will give $m_1 = x^{*(1)}$ and $M_2 = y^{*(2)}$.

The remaining part is to show that in any subgame perfect equilibrium, the first offer is accepted. Suppose, to the contrary, player 2 rejects player 1's first offer. The next period, an SPE of G_2 will be followed. By 3.3, the present value (in G_2) of such an outcome to player 2 is $y^{*(2)}$. Thus, the present value (in G_2) of the same outcome to player 1 is no more than $y^{*(1)}$, which makes the present value to player 1 in G_1, $\delta y^{*(1)} < x^{*(1)}$ resulting in a contradiction with 3.3. Hence, in any SPE, the first offer must be accepted, finishing the proof. ■

Formally, the strategy profile f^* employed in the proof of Rubinstein's result is defined as the following:

Let z_i, $i \in N$ be such that $z_i^{(i)} = \frac{1}{1+\delta}$ and $z_i^{(-i)} = \frac{\delta}{1+\delta}$, i.e. $z_1 = x^*$ and $z_2 = y^*$; x^*, y^* as specified in the statement of Theorem 2. Then for $i \in N$ and for any given history $h^t = (h_0, \ldots, h_{t-1})$,

$$f_i^*(h^t) = \begin{cases} z_i & \text{if } i = t \bmod 2 + 1, \\ Y & \text{if } i = t \bmod 2, \\ & \text{and } h_{t-1} = z \text{ with } z^{(i)} \geq \frac{\delta}{1+\delta} \\ N & \text{otherwise.} \end{cases} \tag{3.4}$$

The remark below emphasizes an important observation about f^*.

Remark 1 *f^* defined by 3.4 is stationary.*

3.3 The Multiplayer Case

The result of Rubinstein (1982) (Theorem 2) does not hold when the number of players is more than 2. In fact, Shaked (1986) shows that in the 3–player bargaining game, also known as the Shaked's game, every feasible allocation can be sustained in subgame perfect equilibrium. This is presented in the following theorem by Shaked, restated as Theorem 3.4 in Osborne and Rubinstein (1990):

Theorem 3 (Shaked (1986)) *When $\delta \geq \frac{1}{2}$, for any partition $x^* \in X$, there exists a subgame perfect equilibrium of the 3–player bargaining game, where x is agreed upon in period 0.*

Proof. Following the same notation with Osborne and Rubinstein (1990), we consider the strategy profile σ^* represented by automata in Table 3.3. In words, the strategy profile σ^* prescribes each player to propose y at state y and accept an

		$\mathbf{x}^*$	$\mathbf{e^1}$	$\mathbf{e^2}$	$\mathbf{e^3}$
1	proposes	x^*	e^1	e^2	e^3
	accepts	$x_1 \geq \delta x_1^*$	$x_1 \geq \delta$	$x_1 \geq 0$	$x_1 \geq 0$
2	proposes	x^*	e^1	e^2	e^3
	accepts	$x_2 \geq \delta x_2^*$	$x_2 \geq 0$	$x_2 \geq \delta$	$x_2 \geq 0$
3	proposes	x^*	e^1	e^2	e^3
	accepts	$x_3 \geq \delta x_3^*$	$x_3 \geq 0$	$x_3 \geq 0$	$x_3 \geq \delta$
Transitions		If in any state y, any Player i proposes x with $x_i > y_i$, then go to state e^j, where $j \neq i$ is the player with the lowest index for whom $x_j < 1/2$.			

Table 3.1: Shaked's SPE strategy profile σ^*.

offer x if and only if $x_i \geq \delta y_i$. The game starts at (equilibrium) state x^*, and the state changes whenever (and immediately after) player i proposes z with $z_i > x_i^*$, i.e. whenever she proposes a share to herself exceeding the one she is supposed to obtain in that particular state. In that case, transition occurs to the new state e^j where j is the player with the smallest index for whom player i has proposed a share less than 1/2. Indeed, e^j constitutes the reward state of player j. Notice that, player j will reject z, since the transition to state e^j occurs immediately after z is offered, and the strategy of player j prescribes her to reject z with $z_j < \delta e_j^j = \delta$ at the new state e_j. The state does not change if a player deviates from accepting or rejecting an offer.

These strategies constitute a subgame perfect equilibrium. To see this, take any state y, $y \in \{x^*, e^1, e^2, e^3\}$ and player i, $i \in \{1, 2, 3\}$. At this state, if player i proposes x with $x_i > y_i$, the state changes to some e^j with $j \neq i$, $j \in \{1, 2, 3\}$, which means that x will be rejected and the next period e^j will be offered and accepted, giving player i a payoff of 0. If player i proposes x with $x_i < y_i$, either it will get accepted and she will get x_i, or it will get rejected and y will be accepted the next period, giving player i, y_i with one period of delay. In both cases, player i gets a present utility less than y_i, so it is not optimal for player i to deviate.

Now suppose at state y, it is player i's turn to respond to the offer. If she rejects y, the next period y will be offered again, and accepted. Player i will get y with one period of delay, making the deviation unprofitable. Hence, σ^* is a subgame perfect equilibrium. ■

3.4 Finite–State Automata and Bounded Rationality

It is important to note that the strategy profile σ^* described in Table 3.3 consists of a finite–state automaton for each player.

Finite–state automata have been used extensively in the literature as a means to model complexity of strategies and bounded rationality of players, an idea first proposed in the economics literature by Aumann (1981) following Simon's pioneering formulations. Using such a formulation, Neyman (1985) shows that cooperation can be sustained even in finitely repeated games when the strategies are restricted to finite–state automata. Rubinstein (1986) and Abreu and Rubinstein (1988) impose costs on maintaining an additional machine state, rather than taking finite–state automata as given. The reasoning behind using finite–state automata is that players face a tradeoff between maximizing their payoffs in the game and using as simple strategies as possible. The less number of states an automata has, the less likely it is to break down, the easier it is to learn, and it requires less time and effort to be implemented (Rubinstein 1998).

Hence, Shaked's Theorem (Theorem 3, employing a strategy profile described by the finite–state autamata given in Table 3.3) establishing that all feasible allocations can be sustained in subgame perfect equilibrium even when the strategies are restricted to finite–state automata bears a special significance. That is, the set of subgame perfect equilibria does not shrink even when players are assumed to be boundedly rational, bounded rationality being modeled as usual in the literature.

Counting the number of states of an automaton associated with a strategy is not the only method to measure bounded rationality. In fact, it should be emphasized that finite–state automata are less complex to implement compared to their infinite–state counterparts, because identifying what to do (i.e. the implied action) under such a strategy is simpler. However, in order to identify the state of the game, players may need to know the entire history. Therefore, finite–state automata do not restrict, in any way, the history–dependence of strategies, and may require players to "*recall*" the entire history of the game. Clearly, this results in an inconsistency with the essence of complexity and bounded rationality. Hence, another method of alleviating such complexity involves finite–memory strategies.

3.5 The Loss of Game Theoretic Prediction Power

Rubinstein's result (Theorem 2) that in 2–player alternating offers bargaining games with discounting, the subgame perfect equilibrium outcome is unique and converges to the equal division as the discount factor tends to 1 bears significance due to a number of facts. One of these is that the uniqueness of the subgame perfect equilibrium outcome raises the prediction power of the model. Another is that the unique outcome is plausible, and in fact, desirable. This is due to not only fairness concerns, but also the fact that, the game precisely implements the Nash bargaining solution in subgame perfection and hence provides it with a non–cooperative game theoretic foundation.

However, these desirable properties are not carried over to the game with $n \geq 3$ players. Shaked's result that every feasible allocation can be sustained in subgame perfect equilibrium means that we have no game theoretic prediction power as to what the equilibrium outcome of the game will be. Moreover, the Nash bargaining solution for $n \geq 3$ players, which is equal division of the pie, is not implemented in subgame perfection in an alternating offers bargaining game with $n \geq 3$ players. In turn, this means that the Nash program fails to deliver a desirable result with $n \geq 3$ players.

The emerging inconsistency between the conclusions of the 2–player and multiplayer versions of this game, has been well–known in the literature. For more on this subject, we refer the reader to Osborne and Rubinstein (1990). In fact, many studies consider modifications to the multiplayer version of the game at hand in order to obtain a unique equilibrium that can be associated with the Nash bargaining solution. In that regard, some of the related studies are (but not restricted to) Binmore, Rubinstein, and Wolinsky (1986), Jun (1987), Chae and Yang (1988), Baron and Ferejohn (1989), Binmore, Shaked, and Sutton (1989), Chae and Yang (1990), Stahl (1990), Chatterjee and Samuelson (1990), Asheim (1992), Yang (1992), Perry and Reny (1993), Chae and Yang (1994), , Krishna and Serrano (1996), Vannetelbosch (1999), Chatterjee and Sabourian (2000), Baliga and Serrano (2001), Huang (2002), Lee and Sabourian (2005), and Suh and Wen (2006). While some of these modifications feature plausible economic insight, only a few of them are concerned with the elimination of the inconsistency at hand by employing bounded rationality concerns.

When considering complexity formulations that are aimed to re-evaluate / alleviate these inconsistencies, the standard complexity formulations (based on counting the number of states of an automaton describing a player's strategy), do not have any bite, as the above discussion displays. Therefore, the search for the elimination of the inconsistencies between the 2–player and multiplayer versions based on bounded rationality concerns, necessitates considerations of nonstandard complexity formulations. Indeed, one such study is Chatterjee and Sabourian (2000).

It is useful to remind the reader that the current study considers a complexity formulation based on restricting players' *memory* (recall), the number of consecutive periods that a player can recall. While it is "a" particular measure of complexity, it needs to be emphasized that (to our knowledge) this is the pioneering work regarding the use of memory in alternating offers bargaining games.

4 One–memory

The sharp contrast between the results of the 2–player and multiplayer versions, i.e. the inconsistency presented in the previous Chapter, ruining game theoretic prediction power in the multiplayer version, is due to the fact that in the multiplayer case one of the players can be rewarded for rejecting a deviant offer. This is an observation that does not hold in the 2–player case. In fact, this interesting, yet problematic, contrast holds not only when players' strategies are unconstrained (i.e. players are fully rational) but also when players are boundedly rational (i.e. players' strategies are constrained to finite–state automata).

As mentioned in Section 3.4, the problem with using finite–state automata or complexity costs such as in Chatterjee and Sabourian (2000) to model bounded rationality is that they may require players to recall the entire history of the game: Even though each player's strategy (described by a finite–state automaton) is, by definition, a function from a finite set of states into that player's set of actions, the identification of the state that the game is in, may necessitate to employ the entire history of the game, requiring players to have unbounded memory (recall) in turn. This, clearly, is unappealing. For more on this subject, we refer the reader to Barlo, Carmona, and Sabourian (2012).

One remedy proposed to solve this issue consists of the use of stationary strategies. However, that particular method is arguably not that appealing. This is because, the difference between using strategies described by finite–state automata and stationary plans of actions would amount to going from the situation where a player is concerned about the number of states associated with the automaton describing her strategy, into one where she has to use a strategy that does not depend on any past information. Moreover, as Osborne and Rubinstein (1994) argue, it is difficult to justify the use of stationary strategies only based on their simplicity, considering that the equilibrium should also be an equilibrium in beliefs and there is no reason for a player to believe that others will choose the same action, even when many deviations have occurred before.

Thus, limited memory strategies present themselves as a natural way to model bounded rationality without being too restrictive; and, when extending the notion of stationarity to allow history dependence, the next immediate and evident formulation in the current setting involves the use of one–memory[1].

In this chapter, we consider a widely accepted method of restricting players' strategies on grounds of bounded rationality. Particularly, all players are assumed to use one–memory strategies, plans of actions that depend only on yesterday's profile of choices. It is appropriate to point out that in this setting, each player can condition her actions not only on the current offer, but also on the outcome of the last stage (period) of the game. This stage includes both the offer proposed in that particular period and associated responses made by others. It is also imperative to mention that deviations from one–memory strategy profiles are not restricted to be of one–memory. That is, (unlike Rubinstein (1986) and Abreu and Rubinstein (1988)) we do not use a modified notion of subgame perfection, but employ the standard one. For more about this distinction, we refer the reader to Kalai and Stanford (1988). Furthermore, when employing one–memory strategies, a player cannot condition her actions on calendar time. At this stage, it is appropriate to point out that the notion of memory employed in this study is the one–period version of those that are used in the following studies: Sabourian (1989), Sabourian (1998), Barlo and Carmona (2007), Barlo, Carmona, and Sabourian (2009), and Barlo, Carmona, and Sabourian (2012). And it differs from the time–dependent versions of Barlo (2003), Cole and

[1]Notice that the notion of Markov perfect equilibria, subgame perfect equilibria where players' strategies are restricted to be functions obeying the Markov property, are generally considered the next step in extending stationarity to allow for history dependence. However, a Markov strategy necessitates the use of a finite state space, hence, is nothing but a finite–state automaton. Therefore, as the Chapter3 has depicted, such an extension does not eliminate the inconsistency under consideration.

Kocherlakota (2005), Hörner and Olszewski (2006), Hörner and Olszewski (2009), Mailath and Olszewski (2011).

In what follows, first we show that the strategy profile σ^* (see Table 3.3) used in the proof of Theorem 3 is not one–memory. It is worthwhile to remind the reader that Theorem 3 establishes that for any feasible partition of the pie, there exists a subgame perfect equilibrium strategy profile where each component strategy is described by a finite–state automaton. That is, σ^* is described by finite–state automata, yet is not one–memory.

Following this, an auxiliary result due to Herrero (1985) is presented and proven to maintain completeness of this thesis: The stationary subgame perfect equilibrium is unique and in the multiplayer version, corresponds to the multiplayer generalization of Rubinstein's.

We prove that the one–memory subgame perfect equilibrium outcome displays the very same properties as the stationary one: It is unique, and in the multiplayer version, it corresponds to the associated generalization of Rubinstein's result.

Our main result constitutes a noteworthy contribution to the Nash program because it provides a remedy based on a widely accepted notion of bounded rationality, one–memory, eliminating the inconsistency between the conclusions of the 2–player and multiplayer versions.

In the following section, we elaborate on the main ingredients of our result, including a particular notion referred to as confusion by Barlo, Carmona, and Sabourian (2009). Moreover, Sections 4.2 and 4.3 provide some needed auxiliary results, while Section 4.4 presents our main contribution.

4.1 Confusion

When attention is restricted to the standard discounted repeated games, one–memory, the initial step in extending the notion of stationarity to allow history dependence, delivers striking results. In fact, an immediate observation in this setting is that the set of stationary subgame perfect equilibria consists of strategy profiles that involve the repetition of a given Nash equilibrium of the stage game. That is why stationary subgame perfect equilibria are not interesting in this setting. On the other hand, Barlo, Carmona, and Sabourian (2009) establishes that the subgame perfect Folk Theorem holds with one–memory strategies, when the set of actions in the stage game of the repeated game is sufficiently "large" for each player, so that each payoff profile is not isolated. Hence, in such games the set of subgame perfect equilibria gets considerably richer when players are allowed to use even one–memory. The next observations about the one–memory Folk Theorem of Barlo, Carmona, and Sabourian (2009) should be pointed out:

> The large action space assumption is critical in establishing these results because it allows players to encode the entire history of the past into the previous period's actions. More formally, with rich action sets any equilibrium strategy vector in which each player strictly prefers not to deviate at every history, can be perturbed so that each player chooses different actions at different histories. With such distinct plays of the game, at each date the players can use the outcome of the previous period to coordinate their actions appropriately. Thus, the original equilibrium can be approximated by another that has one period recall. (Barlo, Carmona, and Sabourian (2012, p.1))

While one–memory strategies are a considerably rich set of plans of actions, they also bring about an interesting problem identified in Barlo and Carmona (2007) and elaborated on Barlo, Carmona, and Sabourian (2009) and Barlo, Carmona, and Sabourian (2012): *Confusion.* With one–memory, problems of detecting the latest deviation and the identity of the deviator arise.

Indeed, the notion of subgame perfection is equivalent to *the principle of one–deviation*[2] (see Rubinstein (1982), Abreu (1988), and Osborne and Rubinstein (1994)) under certain assumptions which are satisfied in the setting considered in this thesis. In fact, the use of discounting in the current analysis provides the critical argument to that regard. It requires the following two critical properties: First, a single player deviation can be detected and second, the identity of the deviator can be revealed. If either of the above two properties were not to hold, there may be incentives for

[2]It is worthwhile to point out that the principle of one–deviation is also referred to as the principle of optimality or the single deviation principle.

some player to deviate and manipulate the path of future play. In Barlo, Carmona, and Sabourian (2009), this is referred to as "confusion-proofness".

These issues, do not arise with unbounded memory because one can use induction starting from beginning of the game to identify the latest deviation. With bounded memory, this inductive reasoning is not feasible. This, indeed, constitutes the main reason why a strategy described by a finite–state automaton needs not be finite–memory.

In order to see the issue of confusion in the scope of standard repeated games, the following example from Barlo and Carmona (2007) is appropriate:

> Consider a discounted repeated Prisoners' Dilemma in which at every date each player can either cooperate C or defect D, and suppose that we want to implement a cycle consisting of
>
> $$((C, D), (D, C))$$
>
> yielding an average payoff strictly higher than the repetition of (D, D) forever. The strategy inducing this cycle, denoted by $\pi = \{\pi_t\}_{t=1}^{\infty}$, and involving the play of (D, D) forever for any history inconsistent with the equilibrium path, is subgame perfect with unbounded memory and with sufficiently high discount factors. However, the limited–memory versions of this strategy is not subgame perfect regardless of the magnitude of memory. This is because, if players can remember at most M periods, one player prefers to deviate at a history with its last M entries equal to $(a_1, \pi_2, \ldots, \pi_M)$ with $a_1 \neq \pi_1$ instead of playing the punishment: If $\pi_M = (D, C)$, then player 1 can play C instead of D, which will make the play return to the equilibrium outcome in the next period. Notice that if $\pi_M = (C, D)$ by a similar argument, player 2 would deviate. Hence, player 1's continuation payoff in that history strictly exceeds the payoff he would receive by not deviating. (Barlo and Carmona (2007, p.3))

It is useful to point out that the issue of confusion, while creating cumbersome obstacles, is not sufficiently strong in the analysis of standard discounted repeated games to make the subgame perfect Folk Theorem fail: Barlo, Carmona, and Sabourian (2009) prove the subgame perfect Folk Theorem with one–memory strategies and with rich action spaces in the stage game.

An interesting observation about alternating offers bargaining games is that while not being a discounted repeated game in the standard sense, the stage game still involves a rich action space. In fact, in the alternating offers bargaining game (regardless of the number of players), the proposer has the option to encode the entire history of the game into his offer at a negligible cost. In turn, this also establishes the observation that the set of one–memory strategies is considerably richer than the set of stationary strategies[3].

In the analysis of the alternating offers bargaining game with discounted utilities, confusion resulting from the one–memory requirement eliminates the multiplicity of equilibria featured in Shaked's Theorem, Theorem 3. It is imperative to indicate that this observation holds even though the entire history can be recorded into players' offers.

In what follows, we elaborate on how confusion, brought about by the one–memory requirement, eliminates all but the stationary subgame perfect equilibrium. Indeed, what we prove is that one–memory subgame perfect strategies have to be stationary, which together with Herrero (1985) implies our main finding[4].

The first step which is formally presented in Lemma 3, shows that with subgame perfection and one–memory, each player always proposes the same share to herself regardless of the past. In order to see this in a 3–player version (for the sake of simplicity), consider the beginning of the game (the unique history with time period 0) and any history in time period 3. In both of them player 1 proposes. If player 1 were to obtain a strictly higher payoff in the 3–period history, she could deviate and offer the very same allocation (prescribed by her strategy for that particular 3–period history) in the beginning of the game. This deviant act can be identified by the other players as a single player deviation. Yet, if player 2 were to obtain a higher payoff by rejecting that offer at the beginning of the game, she would have rejected that very same offer in that 3–period history as well, which follows from

[3]To see this, consider a finite time period and a given nonterminal history in the alternating offers bargaining game for which all the past offers were given by rational numbers that end in finitely many digits. Then, the proposer at this stage has the option to come up with an offer into which the entire history of the game can be recorded after a sufficiently high digit, hence, at a negligible cost. Therefore, having a large action space may provide players with the capacity to record the whole history of the game by only using one–memory.

[4]Recall that Herrero (1985) establishes that the stationary subgame perfect equilibrium is unique and in the multiplayer version corresponds to the multiplayer generalization of Rubinstein's.

the one–memory requirement: The last stage of the history after rejecting player 1's deviant offer in the beginning of the game, is the same as the last stage of the history after rejecting player 1's equilibrium offer in that 3–period history. In other words, seeing player 1's deviant offer in the beginning of the game, player 2 does not find it profitable to punish her because she knows that tomorrow she will be *confused* as to which history they have been in, and hence, cannot be rewarded for punishing player 1. The same reasoning presented for player 2 holds for player 3 as well. Therefore, player 1 has to offer the same share to herself in both of these histories in any subgame perfect equilibrium. Clearly, following a similar reasoning for other histories and players establishes the observation.

The second step, presented in Lemma 4, displays that each player must propose the same offer in every history when employing the notion of subgame perfection and the requirement of one–memory. In other words, the part of the strategy involving offers is stationary, which easily implies that the part of the strategy involving responses is also stationary. Hence, establishing the second step finishes the proof with the help of Herrero (1985).

To see the intuition behind the second step in a 3–player version of our game, consider two histories both in which player 1 proposes, and in one of which player 3 gets a strictly higher payoff than the other one. Since both of these offers must provide player 3 with payoffs that exceed the level that she can obtain by waiting for 2 periods and proposing herself (which is fixed, as established by the first step), the offer giving player 3 a strictly higher payoff than the other, also must provide her a strictly higher payoff than the present value of the fixed amount that player 3 would get by waiting 2 periods and proposing herself. Now, consider the history where player 3 is supposed to get a strictly higher payoff than the other history and that player 1 cut player 3's slack off her share (in a sufficiently small manner) and distributed it between herself and player 2. At this history, observe that player 2 gets a strictly higher share than what she would get by rejecting. This is because, by the first step, the next period she will propose herself the same share no matter what the history has been. Since her share in the original offer (before player 1 cut the slack off player 3's share) has to be higher than the present value of what she would propose to herself in the next period, her share in this new offer must be strictly higher than what she would obtain by waiting one period and proposing herself. Thus, player 2 will be better off in this history if the offer gets accepted. For this to happen, player 3 must also accept, which will happen whenever she gets a lower payoff in the history after she rejects. However, since player 2 will be proposing in the next period, the best response property of her strategy (implied by subgame perfection) requires that she offers player 3 a lower payoff whenever player 3 rejects player 1's offer in which player 2 is strictly better off. This general observation is established in a separate lemma, Lemma 5, for $n \geq 3$ players.

It is also useful to notice that, the role that one–memory plays here, is that player 2 does not find it profitable to punish player 1 for deviating, because she knows that the next period she will be *confused* as to the identity of the deviator. In fact, as established in the first step, she will propose the same share for herself next period whatever the history of this period turns out to be. Player 3 also does not punish player 1, since the next period, player 2 will propose only based on the last period's history, which means that player 3 must make her decision considering how her actions will change that one–period history upon which player 2 will base her proposal the next period.

Therefore, by Lemma 5 the best response property of player 2's strategy requires the payoff that player 3 gets by rejecting player 1's offer to be less than player 3's share in player 1's offer, delivering the conclusion that in the history where player 1 cut the slack off player 3's share and distributed among herself and player 2, the offer must be accepted. Then, however, at the history where player 3 gets a strictly higher share, distributing player 3's slack between herself and player 2 is a profitable deviation for player 1, establishing that there cannot be a subgame perfect equilibrium where player 3 gets two different shares at two different histories both in which player 1 proposes. The same argument applies to player 2's shares, and to histories where player 2 or 3 propose. Hence, each player always proposes the same offer regardless of the history.

In the rest of this chapter, we present this and some other results in a formal manner.

4.2 Shaked's Strategy is not One–Memory

We show that the subgame perfect equilibrium strategies in the multiplayer bargaining game, described in Theorem 3 are not one-memory. Again, as was the case in the statement of Theorem 3, the result and the proof are presented for the case of three players, in order to avoid nonfruitful technicalities. Moreover, it should be noted that generalizing these results to $n \geq 3$ players is trivial.

Proposition 1 *The subgame perfect equilibrium strategy profile σ^* given in Table 3.3 is not one-memory.*

Proof. Let $x^* \neq e^1$ and $x^* \neq e^2$. Take two histories

$$h = ((x^*, N, Y), (e^2, N, N)) \text{ and } h' = ((e^1, N, N), (e^2, N, N)).$$

After history h, the game is at state e^1, so $\sigma_3^*(h) = e^1$, whereas after h', the state is e^2 and $\sigma_3^*(h) = e^2$. Since $T^1(h) = T^1(h')$ but $\sigma_3^*(h) \neq \sigma_3^*(h')$, σ^* is not a one–memory strategy profile when $x^* \neq e^1$ and $x^* \neq e^2$.

We need to check for $x^* = e^1$ and $x^* = e^2$ separately. First, we will check for $x^* = e_1$. Let

$$h_1 = ((x^*, Y, N), (e^2, N, N), (e^3, N, N)),$$

and

$$h_2 = ((x^*, Y, N), ((1/2, 1/2, 0), N, N), (e^3, N, N)).$$

Although $T^1(h_1) = T^1(h_2)$, we have $\sigma_1^*(h_1) = e^1 \neq e^3 = \sigma_1^*(h_2)$, which means σ^* is not one–memory when $x^* = e^1$.

The last case to check is $x^* = e^2$. Take the histories

$$h_3 = ((x^*, Y, N), (x^*, N, N)) \text{ and } h_4 = (((1/2, 1/2, 0), N, Y), (x^*, N, N)).$$

Again, we have $T^1(h_3) = T^1(h_4)$, but $\sigma_3^*(h_3) = x^* \neq e^1 = \sigma_3^*(h_4)$.

Hence, σ^* is not a one-memory strategy profile for any $x^* \in X$. ■

4.3 Stationary Subgame Perfect Equilibrium

Next, we present a result by Herrero (1985), to be used in the proof of our main theorem, which establishes that there is a unique stationary subgame perfect equilibrium in the n–player bargaining game.

Theorem 4 (Proposition 4.2 of Herrero (1985).) *The multiplayer alternating offers bargaining game described in Chapter 2 has a unique stationary subgame perfect equilibrium $f^\star$. In this equilibrium, each player i always proposes $x_i^\star$ and accepts an offer y at period t if and only if $y^{(i)} \geq \delta x^{\star(i)}_{(t+1 \bmod n)+1}$, where*

$$x_i^\star = \left(x_i^{\star(j)}\right)_{j=1}^{n} = \left(\frac{\delta^{(j-i) \bmod n}}{1+\delta+...+\delta^{n-1}}\right)_{j=1}^{n}$$

and $x_i^{\star(j)}$ denotes player j's share in $x_i^\star$.

Proof. The proof follows from the next results.

Lemma 1 *$f^\star$ constitutes a subgame perfect equilibrium.*

Proof. Consider a subgame starting with an offer by player i. Suppose player i offers y with $y \neq x_i^\star$. If y gets rejected, player $(i+1) \bmod n$ will offer $x^\star_{(i+1) \bmod n}$ the next period, which will be accepted and player i will get the present value

$$\frac{\delta^n}{1+\delta+...+\delta^{n-1}} < x_i^{\star(i)}.$$

If y gets accepted, it means that $y^{(j)} \geq \delta x^{\star(j)}_{(i+1 \bmod n)}$ for all $j \in N$, $j \neq i$. Summing up $y^{(j)}$'s gives

$$\sum_{j\neq i, j\in N} y^{(j)} \geq \sum_{j\neq i, j\in N} \delta x^{\star(j)}_{(i+1) \bmod n} = \sum_{j\neq i, j\in N} x_i^{\star(j)}$$

and therefore $y^{(i)} \leq x_i^{\star(i)}$. Hence, player i cannot strictly benefit as a result of a deviation from $x_i^\star$ to y.

Now, suppose player $j \in N$, $j \neq i$ rejects $x_i^\star$. The next period, player $(i+1) \bmod n$ will propose $x^\star_{(i+1) \bmod n}$, and this will get player j the present value of $\delta x^{\star(j)}_{(i+1) \bmod n}$, which is equal to $x_i^{\star(j)}$. Thus, she does not profit from deviating.

Hence, $f^\star$ is subgame perfect. ■

The rest of the proof shows that the stationary SPE is unique. This is established in the following lemma:

Lemma 2 *The multiplayer alternating offers bargaining game described in Chapter 2 has a unique stationary subgame perfect equilibrium.*

Proof. Let $f = (f_i)_{i=1}^n$ be a stationary subgame perfect equilibrium and let $(x_t)_{t=0}^\infty$, where $x_t = f_{(t \bmod n)+1}(h^t)$, $h^t \in H^t$, denote the sequence of subgame perfect offers induced by f. It is worthwhile to mention that $(t \bmod n) + 1$ is the proposer at period t. Notice that if player i, $i \in N$, will accept an offer x_{t+1} at period $t+1$, she cannot refuse the present value of the same offer at period t. Then, for $i \in N$ with $i \neq (t \bmod n) + 1$, it must be:

$$x_t^{(i)} = \delta x_{t+1}^{(i)}. \tag{4.1}$$

Since

$$x_t^{(t \bmod n)+1} = 1 - \sum_{i \neq (t \bmod n)+1} x_t^{(i)}$$

it follows from equation 4.1 that:

$$x_t^{(t \bmod n)+1} = 1 - \sum_{i \neq (t \bmod n)+1} \delta x_{t+1}^{(i)}. \tag{4.2}$$

Moreover, observe that due to the stationarity of f, a player always proposes the same partition when it is her turn. That is, a proposing player always proposes the same share to herself in every history that she proposes. For player 1, this means:

$$x_n^{(1)} = x_0^{(1)}. \tag{4.3}$$

Equations 4.1–4.3 imply:

$$\begin{aligned} x_0^{(1)} &= 1 - \sum_{i=2}^{n} \delta x_1^{(i)} = 1 - \delta(1 - x_1^{(1)}) \\ &= 1 - \delta + \delta^2 x_2^{(1)} = \ldots = 1 - \delta + \delta^n x_n^{(1)} \\ &= 1 - \delta + \delta^n x_0^{(1)}. \end{aligned}$$

Hence,

$$x_0^{(1)} = \frac{1-\delta}{1-\delta^n} = \frac{1}{1+\delta+\ldots+\delta^{n-1}} = x_1^{\star(1)}. \tag{4.4}$$

This, together with 4.1–4.3 establishes that for all $i \in N$,

$$x_t^{(i)} = x_{(t \bmod n)+1}^{\star(i)}.$$

Hence, $f = f^\star$, completing the proof of the Lemma and the Theorem. ■ ■

4.4 One–Memory Subgame Perfect Equilibrium

The main result of this thesis is:

Theorem 5 *The multiplayer discounted alternating offers bargaining game has a unique subgame perfect equilibrium with one–memory, which approaches the equal split as δ tends to 1. In this equilibrium, each player i proposes $x_i^\star$ irrespective of the history when it is her turn to propose and accepts an offer y at period t if and only if $y^{(i)} \geq \delta x_{(t+1 \bmod n)+1}^{\star(i)}$, where*

$$x_i^\star = \left(x_i^{\star(j)}\right)_{j=1}^{n} = \left(\frac{\delta^{(j-i) \bmod n}}{1+\delta+\ldots+\delta^{n-1}}\right)_{j=1}^{n}$$

and $x_i^{\star(j)}$ denotes player j's share in $x_i^\star$.

Proof. The following Lemma states that every player proposes the same share for herself.

Lemma 3 *For all $f \in E^1$, and for all $h^t, h^s \in H_2$, such that $(t \bmod n) + 1 = (s \bmod n) + 1 = i$, $i \in N$, it must be that $f_i^{(i)}(h^t) = f_i^{(i)}(h^s)$.*

Proof. Let $f = (f_i)_{i \in N} \in E^1$ and let $x = \left(x^{(i)}\right)_{i \in N}$ be the partition offered by player 1 in period 0 according to f, i.e. $x = f_1(\emptyset)$. Since f is subgame perfect, x should be accepted in period 0. Thus, given that all the players acting before j have chosen Y, it must be more profitable for player j to choose Y compared to choosing N. Therefore, for all players to go for Y, the following needs to be satisfied for all $j \in N$, $j \geq 2$:

$$x^{(j)} \geq \delta f_2^{(j)} \left(x, \{R_i\}_{i=2}^n\right) \tag{4.5}$$

where for $i \geq 2$

$$R_i = \begin{cases} Y & \text{if } i < j, \\ N & \text{if } i = j, \\ f_i\left(x, \{R_k\}_{k<i}\right) & \text{if } i > j. \end{cases}$$

Now, suppose there is $h^t \in H_2^t$ in which player 1 proposes, i.e. $(t \bmod n) = 0$, and $f_1(h^t) = y \neq x$. Since f is subgame perfect, y has to be accepted when offered by player 1 after h^t. Again, for all players to choose Y, we have the following condition for all $j \in N$, $j \geq 2$:

$$y^{(j)} \geq \delta f_2^{(j)} \left(h^t \circ \left(y, \{R_i\}_{i=2}^n\right)\right) \tag{4.6}$$

where for $i \geq 2$

$$R_i = \begin{cases} Y & \text{if } i < j, \\ N & \text{if } i = j, \\ f_i\left(h^t \circ \left(y, \{R_k\}_{k<i}\right)\right) & \text{if } i > j. \end{cases}$$

The requirement of one-memory, i.e. $f \in F^1$, implies that for all $j \in N$

$$f_2^{(j)} \left(h^t \circ \left(y, \{R_i\}_{i=2}^n\right)\right) = f_2^{(j)} \left(y, \{R_i\}_{i=2}^n\right) \text{ for any } R_i \in \{Y, N\} \text{ and } i \geq 2. \tag{4.7}$$

Therefore, condition 4.6 becomes, for all $j \in N$, $j \geq 2$

$$y^{(j)} \geq \delta f_2^{(j)} \left(y, \{R_i\}_{i=2}^n\right) \tag{4.8}$$

where for $i \geq 2$

$$R_i = \begin{cases} Y & \text{if } i < j, \\ N & \text{if } i = j, \\ f_i \left(y, \{R_k\}_{k<i}\right) & \text{if } i > j. \end{cases}$$

This means, however, that if player 1 deviates and offers y instead of x in period 0, y will be accepted. This is because of the following: Other players know that if any one of them chooses N, it will be no different than choosing N to y after the history h^t. In either scenario, all they remember the next day will be the history of yesterday, which is the same whether it is offered and rejected in period 0 or after history h^t. Then if y gets accepted after history h^t according to f, it must be accepted when player 1 offers it after any history. Thus, for player 1 not to deviate to proposing y instead of x in period 0, y must give player 1 less than or equal to x, delivering

$$x^{(1)} \geq y^{(1)}. \tag{4.9}$$

Hence, any partition different than x that player 1 offers after some history must give player 1 less than or equal to $x^{(1)}$. The same argument also works in the other direction. Suppose player 1 deviates to offering x instead of y after the history h^t. So, since $f \in F^1$, i.e. one-memory, it must satisfy

$$f_2^{(j)} \left(x, \{R_i\}_{i=2}^n\right) = f_2^{(j)} \left(h^t \circ \left(x, \{R_i\}_{i=2}^n\right)\right) \text{ for any } R_i \in \{Y, N\} \text{ and } i \geq 2. \tag{4.10}$$

This, together with 4.5 implies

$$x^{(j)} \geq \delta f_2^{(j)} \left(h^t \circ \left(x, \{R_i\}_{i=2}^n\right)\right) \tag{4.11}$$

where for $i \geq 2$

$$R_i = \begin{cases} Y & \text{if } i < j, \\ N & \text{if } i = j, \\ f_i\left(h^t \circ (x, \{R_k\}_{k<i})\right) & \text{if } i > j. \end{cases}$$

Therefore, if player 1 deviates and offers x instead of y after history h^t, it will get accepted. But for f to be SPE, this must not be a profitable deviation for player 1, which implies

$$y^{(1)} \geq x^{(1)}. \tag{4.12}$$

Therefore $x^{(1)} = y^{(1)}$, meaning that player 1 always offers the same share for herself (in any history that she proposes). Clearly, the same arguments also apply to the other players, delivering the result that each player always offers the same share for herself irrespective of the history. ■

The next lemma shows that in any subgame perfect equilibrium, a player's offer vector does not depend on the history.

Lemma 4 *For all $f \in E^1$, and for all $h^t, h^s \in H_2$, such that $(t \bmod n) + 1 = (s \bmod n) + 1 = i$ for some $i \in N$, it must be that $f_i^{(j)}(h^t) = f_i^{(j)}(h^s)$, $j \in N$.*

Proof. By Lemma 3, we know players always propose the same share for themselves. Let these fixed shares be denoted by $\alpha^{(i)} = f_i^{(i)}(h^s)$, where $h^s \in H_2^s$ with $(s \bmod n) + 1 = i$, $i \in N$.

Now, take an arbitrary $f \in E^1$ and let $f_1(\emptyset) = \left(x^{(i)}\right)_{i \in N}$, and suppose there exists $h^t \in H_2^t$ such that $(t \bmod n) = 0$ and $f_1(h^t) = y \neq x$. Due to Lemma 3 and because that offers have to add up to 1, it must be that $x^{(1)} = y^{(1)} = \alpha^{(1)}$, $x^{(j)} < y^{(j)}$ and $x^{(i)} > y^{(i)}$ for some $i, j \in N$, $i \neq j \neq 1$. We will examine this in two cases, one with $i \neq 2$ and the other where $i = 2$, but first we wish to present the following result which will be used when handling these cases:

Lemma 5 *Let $x, z \in X$ be such that $f_1(\emptyset) = x$, where $f \in E^1$, and $z^{(2)} > x^{(2)}$ and $z^{(j)} \geq \delta^{j-1}\alpha^{(j)}$ for all $j \in N$. Then, for all $j = 3, \ldots, n$, it must be $\delta f_2^{(j)}\left(z, \{R_i\}_{i=2}^n\right) \leq z^{(j)}$, where R_i equals the following: Y if i is such that $2 \leq i < j$; N if $i = j$; and $f_i(z, \{R_k\}_{k<i})$ if $i > j$.*

Proof. Suppose not, i.e. there exists $j \in N$, $j > 2$ with $\delta f_2^{(j)}(z, \{R_i\}_{i=2}^n) > z^{(j)}$, where for $i \geq 2$

$$R_i = \begin{cases} Y & \text{if } i < j, \\ N & \text{if } i = j, \\ f_i(z, \{R_k\}_{k<i}) & \text{if } i > j. \end{cases} \tag{4.13}$$

Consider history $h^1 = z$, i.e. the history in which player 1 offers $z \neq x$; x, z as specified in the hypothesis. In what follows we will show that in this particular subgame there is a player with a profitable deviation, and so f is not subgame perfect, delivering the conclusion via counter-positive.

Let J be the set of players for whom $\delta f_2^{(j)}(z, \{R_i\}_{i=2}^n) > z^{(j)}$, where for $i \geq 2$ the responses R_i are as specified in condition 4.13. Notice that player 2 is not in J. Then, after history $h^1 = z$, at least one player in J will choose N. In the next period, by construction, player 2 will propose $f_2(h)$ where h is the last period's history. Because that $f \in E^1$, $f_2(h)$ must be accepted and player 2 will get a present utility of $\delta\alpha^{(2)}$.

We will show that in this subgame player 2 has a profitable deviation. Consider a deviation by player 2 to a one-memory strategy, g_2, which coincides with f_2 with the exception of the following cases:

$$\delta g_2^{(j)}(z, \{R_i\}_{i=2}^n) \leq z^{(j)}, \text{ for } j \in J, \tag{4.14}$$

$$g_2^{(1)} = f_2^{(1)} + \sum_{j \in J} f_2^{(j)}(z, \{R_i\}_{i=2}^n) - g_2^{(j)}(z, \{R_i\}_{i=2}^n), \tag{4.15}$$

where for $i \geq 2$ the responses R_i are as specified in condition 4.13. [5] In the subgame given by $h^1 = z$, player 2 will accept z since (by Lemma 3) $g_2^{(2)}(z, N, \{R_i\}_{i=3}^n) = f_2^{(2)}(z, N, \{R_i\}_{i=3}^n) = \alpha^{(2)}$ for all $R_i \in \{Y, N\}$ and $z^{(2)} > x^{(2)} \geq \delta\alpha^{(2)}$. For all players $k \neq 1, 2$, Y appears in their best responses. This is due to the following: (1) By construction of g_2 (in particular, condition 4.14) covering cases for all players in J; and, (2) the observation that for every player $k \notin J$ and $k \neq 1, 2$, we have

[5] It should be pointed out that, in the current study, deviations are not necessarily required to be one-memory, because our notion of one-memory does not restrict deviations considered to have the same magnitude of memory. On the other hand, identifying a one-memory deviation to show that a particular one-memory strategy is not subgame perfect, strengthens the execution to situations in which limited-memory equilibria were to be defined by considering only limited-memory deviations.

that $\delta f_2^{(k)}(z, \{R_i\}_{i=2}^n) \leq z^{(k)}$. Thus, Y resides in player k's best response, $k \neq 1, 2$ (regardless of whether or not k is in J) because player 2's deviation makes sure that every player (but player 1 who is making the offer) chooses Y given that the others choosing before do the same. Therefore, player 2 will get the utility of $z^{(2)} > \delta\alpha^{(2)}$, and hence, g_2 is a profitable deviation for player 2 in history $h^1 = z$. So, f is not subgame perfect. ■

Now, we are ready to handle the two cases discussed above.

Case 1 *$x^{(j)} < y^{(j)}$ and $x^{(k)} > y^{(k)}$ for some $j, k \in N$, $j \neq k \neq 1$, $k \neq 2$.*

Since any player can guarantee getting $\delta^{i-1}\alpha^{(i)}$ for herself by rejecting player 1's offer and all other offers until period $i - 1$, any subgame perfect offer by player 1 must give each player at least an amount of $\delta^{i-1}\alpha^{(i)}$. Thus,

$$x^{(i)}, y^{(i)} \geq \delta^{i-1}\alpha^{(i)}. \tag{4.16}$$

Since $x^{(k)} > y^{(k)}$, condition 4.16 implies $x^{(k)} > \delta^{k-1}\alpha^{(k)}$.

Define $\tilde{x} = \left(\tilde{x}^{(i)}\right)_{i \in N}$ where $\tilde{x}^{(i)} = x^{(i)} + \frac{x^{(k)} - \delta^{k-1}\alpha^{(k)}}{n-1}$ for all $i \in N$, $i \neq k$ and $\tilde{x}^{(k)} = \delta^{k-1}\alpha^{(k)}$. Suppose player 1 deviates and offers $\tilde{x}$ instead of x in period 0. Player 2 will accept since for all $R_i \in \{Y, N\}$, $x^{(2)} + \frac{x^{(k)} - \delta^{k-1}\alpha^{(k)}}{n-1} > \delta f_2^{(2)}\left(\tilde{x}, N, \{R_i\}_{i=3}^n\right) = \delta\alpha^{(2)}$ due to condition 4.16 and Lemma 3.

Here we use our earlier lemma: Lemma 5 implies that for all $j \in N$, $j > 2$, $\delta f_2^{(j)}\left(\tilde{x}, \{R_i\}_{i=2}^n\right) \leq \tilde{x}^{(j)}$, where for $i \geq 2$

$$R_i = \begin{cases} Y & \text{if } i < j, \\ N & \text{if } i = j, \\ f_i(\tilde{x}, \{R_\ell\}_{\ell < i}) & \text{if } i > j. \end{cases}$$

So, all other players will also accept $\tilde{x}$. Thus, proposing $\tilde{x}$ instead of x in period 0 is a profitable deviation for player 1, which contradicts the assumption that f is subgame perfect. □

Case 2 *$x^{(2)} > y^{(2)}$ and $x^{(j)} < y^{(j)}$ for some $j \in N$, $j \notin \{1, 2\}$.*

Because that $y^{(j)} > x^{(j)}$, condition 4.16 implies $y^{(j)} > \delta^{j-1}\alpha^{(j)}$.

Let $\tilde{y} = \left(\tilde{y}^{(i)}\right)_{i \in N}$ where $\tilde{y}^{(i)} = y^{(i)} + \frac{y^{(j)} - \delta^{j-1}\alpha^{(j)}}{n-1}$ for all $i \in N$, $i \neq j$ and $\tilde{y}^{(j)} = \delta^{j-1}\alpha^{(j)}$. Suppose player 1 deviates and offers $\tilde{y}$ instead of y after history h^t. Player 2 will accept since for all $R_i \in \{Y, N\}$, $y^{(2)} + \frac{y^{(j)} - \delta^{j-1}\alpha^{(j)}}{n-1} > \delta f_2^{(2)}\left(\tilde{y}, N, \{R_i\}_{i=3}^n\right) = \delta\alpha^{(2)}$ due to condition 4.16 and Lemma 3.

By Lemma 5, we know that for all $j \in N$, $j > 2$, $\delta f_2^{(j)}\left(\tilde{y}, \{R_i\}_{i=2}^n\right) \leq \tilde{y}^{(j)}$, where $R_i = Y$ for all $2 \leq i < j$, $R_j = N$, and $R_i = f_i(\tilde{y}, \{R_\ell\}_{\ell < i})$ for all $i > j$, so all the other players will also accept $\tilde{y}$. Hence, proposing $\tilde{y}$ instead of y is a profitable deviation for player 1 in the subgame starting at h^t. □

Thus, it must be $x^{(j)} = y^{(j)}$ for all $j \in N$, contradicting our initial assumption. Moreover, clearly the same argument applies to the offers by other players. This finishes the proof of Lemma 4. ■

It is appropriate to point out that Lemmas 3 and 4 imply that the strategies of the proposing players have to be stationary. Then, this, clearly, implies that the strategies of the responding players must also be stationary. This finding is presented in the following Proposition, presented without proof:

Proposition 2 *One-memory subgame perfect equilibrium strategies are stationary.*

It is appropriate to remind the reader that Proposition 4.2 of Herrero (1985) (presented and proven in Theorem 4 in the current thesis) establishes that stationary subgame perfect equilibrium is unique and as given in the statement of our Theorem. Thus, Proposition 2 and Theorem 4 establish our result, finishing the proof of Theorem 5. ■

5 Concluding Remarks

To summarize, our result establishes that when we assume players can recall only yesterday and not the history prior to that, i.e. when we employ the restriction to one–memory strategies, the subgame perfect equilibrium of the multiplayer discounted alternating offers bargaining game is unique and equivalent to the multiplayer generalization of Rubinstein's. This also implies that the unique subgame perfect equilibrium outcome approaches the multiplayer Nash bargaining solution when the discount factor tends to 1. Hence, we provide a bridge between the multiplayer cooperative and non–cooperative bargaining games, contributing to the Nash program. Moreover, we show that the unique one–memory subgame perfect equilibrium is actually stationary.

Our use of one–memory strategies is an attempt to utilize bounded rationality to increase the prediction power of the multiplayer version of the game, while avoiding the "over-simplicity" of stationarity. We consider one–memory as the first step in allowing some history–dependence to strategies. The next step would be analyzing the restriction to M-memory strategies, which is our future avenue of research.

Bibliography

ABREU, D. (1988): "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica*, 56, 383–396.

ABREU, D., AND A. RUBINSTEIN (1988): "The Structure of Nash Equilibrium in Repeated Games with Finite Automata," *Econometrica*, 56, 1259–1281.

ASHEIM, G. B. (1992): "A Unique Solution to n–person Sequential Bargaining," *Games and Economic Behavior*, 4, 169–181.

AUMANN, R. (1981): "Survey of Repeated Games," in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*. Bibliographisches Institut, Mannheim.

BALIGA, S., AND R. SERRANO (2001): "Multilateral Negotiations with Private Side–Deals: A Multiplicity Example," *Economics Bulletin*, 3, 1–7.

BARLO, M. (2003): "Essays in Game Theory," Ph.D. thesis, University of Minnesota.

BARLO, M., AND G. CARMONA (2007): "Folk Theorems for the Repeated Prisoners' Dilemma with Limited Memory and Pure Strategies," Sabancı University, Universidade Nova de Lisboa and University of Cambridge.

BARLO, M., G. CARMONA, AND H. SABOURIAN (2009): "Repeated Games with One Memory," *Journal of Economic Theory*, 144(1), 312–336.

——— (2012): "The Bounded Memory Folk Theorem," Sabancı University, Universidade Nova de Lisboa and University of Cambridge.

Baron, D. P., and J. A. Ferejohn (1989): "Bargaining in Legislatures," *American Political Science Review*, 83, 1181–1206.

Binmore, K., A. Rubinstein, and A. Wolinsky (1986): "The Nash Bargaining Solution in Economic Modelling," *Rand Journal of Economics*, 17, 176–188.

Binmore, K., A. Shaked, and J. Sutton (1989): "An Outside Option Experiment," *Quarterly Journal of Economics*, 104, 753–770.

Chae, S., and J. Yang (1988): "The Unique Perfect Equilibrium of an n–person Bargaining Game," *Economics Letters*, 28(3), 221–223.

——— (1990): "An n–person Bargaining Process with Alternating Demand," *Seoul Journal of Economics*, 3, 255–261.

——— (1994): "An n–person Pure Bargaining Game," *Journal of Economic Theory*, 62(1), 86–102.

Chatterjee, K., and H. Sabourian (2000): "Multilateral Bargaining and Strategic Complexity," *Econometrica*, 68(6), 1491–1509.

Chatterjee, K., and L. Samuelson (1990): "Perfect Equilibria in Simultaneous Offer Bargaining," *International Journal of Game Theory*, 19, 237–267.

Cole, H., and N. Kocherlakota (2005): "Finite Memory and Imperfect Monitoring," *Games and Economic Behavior*, 53, 59–72.

Fershtman, C., and D. J. Seidmann (1993): "Deadline Effects and Inefficient Delay in Bargaining with Endogenous Commitment," *Journal of Economic Theory*, 60, 306–321.

Haller, H., and S. Holden (1990): "A Letter to the Editor on Wage Bargaining," *Journal of Economic Theory*, 52, 232–236.

Herrero, M. (1985): "A Strategic Bargaining Approach to Market Institutions," Ph.D. thesis, University of London.

Hörner, J., and W. Olszewski (2006): "The Folk Theorem for Games with Private Almost-Perfect Monitoring," *Econometrica*, 74, 1499–1544.

——— (2009): "How Robust is the Folk Theorem with Imperfect Public Monitoring?," *Quarterly Journal of Economics*, 124, 1773–1814.

Huang, C. Y. (2002): "Multilateral Bargaining: Conditional and Unconditional Offers," *Economic Theory*, 20, 401–412.

Jun, B. H. (1987): "A Structural Consideration on 3–person Bargaining," Ph.D. thesis, University of Pennsylvania.

Kalai, E., and W. Stanford (1988): "Finite Rationality and Interpersonal Complexity in Repeated Games," *Econometrica*, 56, 397–410.

Krishna, V., and R. Serrano (1996): "Multilateral Bargaining," *Review of Economic Studies*, 63, 61–80.

Lee, J., and H. Sabourian (2005): "Efficiency in Negotiation: Complexity and Costly Bargaining," Birkbeck Working Papers in Economics and Finance No.0505.

Mailath, G. J., and W. Olszewski (2011): "Folk Theorems with Bounded Recall under (Almost) Perfect Monitoring," *Games and Economic Behavior*, 71(1), 174–192.

Merlo, A., and J. Wilson (1995): "A Stochastic Model of Sequential Bargaining with Complete Information," *Econometrica*, 63(2), 371–399.

Nash, J. F. (1950): "The Bargaining Problem," *Econometrica*, 18, 155–162.

——— (1953): "Two-person Cooperative Games," *Econometrica*, 21, 128–140.

Neyman, A. (1985): "Bounded Complexity Justifies Cooperation in the Finitely Repeated Prisoner's Dilemma," *Economic Letters*, 9, 227–229.

Osborne, M., and A. Rubinstein (1994): *A Course in Game Theory.* MIT Press, Cambridge.

Osborne, M. J., and A. Rubinstein (1990): *Bargaining and Markets.* Academic Press, San Diego, CA.

Perry, M., and P. J. Reny (1993): "A Non-Cooperative Bargaining Model with Strategically Timed Offers," *Joumal of Economic Theory*, 59, 50–77.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50, 97–110.

Rubinstein, A. (1986): "Finite Automata Play the Repeated Prisoner's Dilemma," *Journal of Economic Theory*, 39, 83–96.

——— (1998): *Modeling Bounded Rationality.* The MIT Press, Cambridge, MA.

Sabourian, H. (1989): "The Folk Theorem of Repeated Games with Bounded Recall (One-Period) Memory," *Economic Theory Discussion Paper 143, University of Cambridge.*

——— (1998): "Repeated Games with M-period Bounded Memory (Pure Strategies)," *Journal of Mathematical Economics*, 30, 1–35.

Sakovics, J. (1993): "Delay in Bargaining Games with Complete Information," *Journal of Economic Theory*, 59, 78–95.

Serrano, R. (2005): "Fifty Years of the Nash Program, 1953-2003," *Investigaciones Economicas*, 29, 219–258.

Shaked, A. (1986): "A Three-Person Unanimity Game," talk given at the Los Angeles National Meetings of the Institute of Management Sciences and the Operations Research Society of America.

Stahl, D. O. I. (1990): "Bargaining with Durable Offers and Endogenous Timing," *Games and Economic Behavior*, 2, 173–187.

Suh, S. C., and Q. Wen (2006): "Multi-agent Bilateral Bargaining and Nash Bargaining Solution," *Journal of Mathematical Economics*, 41, 61–73.

SUTTON, J. (1986): "Non-Cooperative Bargaining Theory: An Introduction," *The Review of Economic Studies*, 53(5), 709–724.

VANNETELBOSCH, V. J. (1999): "Rationalizability and Equilibrium in n–person Sequential Bargaining," *Economic Theory*, 14, 353–371.

YANG, J. (1992): "Another n–person Bargaining Game with a Unique Perfect Equilibrium," *Economics Letters*, 38(3), 275–277.