# COOPERATIVE BARGAINING AND COALITION FORMATION

by

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## Abstract

In this study, I am working on the relationship between coalition formation and bargaining. More specifically, I use a baseline cooperative bargaining model in which a group of agents with symmetric single peaked preferences form coalitions to bargain with a principle. I use this model to study the effects of the underlying bargaining process on the structure of the coalition formed by the agents, and to classify the properties that form a grand coalition. Later on, I also introduce an alternative cooperative bargaining model to understand the connection between the bargaining process and coalition formation.

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## Özet

Bu tezde koalisyon oluşumu ve birlikte pazarlık arasındaki ilişki araştırılıyor. Her bir oyuncunun tek tepeli simetrik tercihleri olan temel bir pazarlık modeli tanımlanıyor. Oyunculardan koalisyon oluşturması isteniyor. Daha sonra bu koalisyonu temsil eden simetrik tek tepeli tercih, dışsal bir sosyal seçim kuralı ile oluşturuluyor. Önceden belirlenmiş bir yönetici ile koalisyonu temsil eden simetrik tek tepeli tercihin pazarlık yapılacağı belirtiliyor ve bu pazarlık sonucu dışsal bir pazarlık kuralı ile belirleniyor. Ortaya çıkan pazarlık sonucu her bir ajanın pazarlık sonucu oluyor. Çalışmanın amacı, pazarlık sürecinin koalisyon oluşumuna etkilerini ve bütün toplumun üyesi olduğu bir koalisyonun nasıl oluşturulabileceğini araştırmak. Tezin ilerleyen kısımlarında, temel pazarlık modelinde bazı değişiklikler yapılıyor. Bu değişikliğin amacı; farklı modeller üzerinde pazarlık sürecinin koalisyon oluşumuna etkilerini araştırmak.

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# Chapter 1

# Introduction

Bargaining takes place between two or more parties over an object or monetary amount or a policy. The result of the bargaining process is the agreement of all interested parties or disagreement. We can observe many examples of bargaining; social interactions, such as in government policies and international organizations' decision processes.

For example, recently there has been a debate over the arms embargo over Syria. French and British governments are lobbying with EU members about lifting the arms embargo to help rebels in Syria. UK prime minister, David Cameron, and French president, François Hollande's insistence to fellow leaders about the embargo most likely will not work. Germany's stand point is opposite to French and British lobbyers about supporting the rebels in the civil war of Syria that has caused the death of approximately 70.000 people. Opponents argue supplying rebels with arms may encourage the Assad's supporters such as Russia and Iran to pursue more aggressive policies. EU foreign policy chief Catherina Ashton said the EU needed to think "very carefully" about French and British arguments that lifting the embargo would encourage Assad to negotiate.

We see that this is a policy bargaining for the actions of EU. EU members form opposing coalitions that negotiate with each other so that they can determine a single policy that binds all EU members. In this example, we observe that there exists an interplay between bargaining and coalition formation.

Another example is a new legislation “Employee Free Choice Act” that was introduced into both chambers of the U.S. Congress on March 10, 2009. This legislation brings the old arguments about the National Labor Relations Act (NLRA). The debate is about the membership of employees to a union as a part of the employment contract. We should focus on the United States Supreme Court decisions in National Labor Relations Board (NLRB) versus General Motors. The decision was: “It is permissible to condition employment upon membership, but membership, insofar as it has significance to employment rights, may in turn be conditioned only upon payment of fees and dues.”[1] Therefore employees do not have to pay full union dues. The payment will be the portion of dues that covers the costs of collective bargaining, contract administration, grievance adjustments, but not the costs of political, ideological, non-representational activities. The new act EFCA brings some changes. The certification of the union as official will depend only on majority vote of employees. There will be no other additional ballot as a demand of employer. The act also increases penalties to employers who discourage workers to union involvement. EFCA is a significant and controversial bills facing the Congress. Its opponents have attempted to portray the bill as a radical, undemocratic and dangerous piece of legislation that would disenfranchise millions of American workers and damage an already fragile economy. And the supporters claims that EFCA can restore the economic stability and division of labor, giving more workers a chance to form unions and get better health care, job security, and benefits.

As you can see there are controversial ideas over these types of legislations (NLRA, EFCA) because its effects over unions (coalitions) is unobservable for the

[1] LABOR BOARD v. GENERAL MOTORS, 373 U.S. 734 (1963)- 373 U.S. 734- NATIONAL LABOR RELATIONS BOARD v. GENERAL MOTORS CORP. CERTIORARI TO THE UNITED STATES COURT OF APPEALS FOR THE SIXTH CIRCUIT.- No. 404.- Argued April 18, 1963. Decided June 3, 1963.

time being. Conventional wisdom suggests that collectivism and centralization is advantageous for employees and that the individualism of employees is more advantageous for employers. Part of this thesis shows that conventional wisdom fails since it is not always beneficial to form a grand coalition. This example also a good indication of the interplay between the bargaining and coalition formation process. The new legislation defines new rules of coalition formation, which in turn cause different coalitions.

To model the above issues, we will use a simple model of bargaining and coalition formation. Suppose we locate policy alternatives along a one-dimensional political spectrum. On the left is the communist party and on the right is the liberal one. This left-right axis or Downsian axis was first introduced by Downs (1957, Chapter 8) [10]. This model indicates that voters with single peaked preferences choose alternatives closest to their most preferred outcome. Hence the peaks of the people who vote for the same party will be close to each other. Since political parties construct their policies in order to get the maximum amount of vote at the corresponding political spectrum, Downsian model suggests that policies will converge to the position of median voter. The idea of this thesis comes from the question: "What if we impose an exogenous bargaining process in a Downsian model?". This exogenous bargaining process consists of two stages. At the first stage, an exogenous social welfare function (check definition 1) will determine a representative preference of the agents inside the coalition. Agents outside the coalition is bound by the representative preference of the coalition. At the second stage, the representative agent (representative preference of the coalition) and principal will bargain. An exogenous bargaining rule (check definition 2) will determine the outcome of bargaining between the representative agent and the principal. It is crucial that the coalition outcome binds each agent inside the society and there may not be unique coalition. I will then alter the model by allowing individual bargaining along with coalitional bargaining with the principal.. With these models, I aim to characterize the conditions over bargaining and social welfare function to form a grand coalition.

We have two models in this thesis; representative coalition and non-representative coalition models. In representative coalition model we allow only one coalition. This single coalition has the power to dictate its agreement with the principal to all agents. In this model, every agent decides whether to be a member of the coalition or not.The formed coalition then bargains with the principal. In non-representative coalition model we allow individual bargainers along with a single coalition. This model is an alteration of the restriction on coalition formation at representative coalition model. For detailed explanations check the subsections (4.0.1) and (4.0.2).

As I mentioned before, we aim to characterize the assumptions over bargaining and social welfare functions to form a grand coalition. In representative coalition model, we have classified the assumptions over bargaining and social welfare function to form a grand coalition. Theorem 1 and Theorem 2 together show that grand coalition can be achieved, under certain assumptions over bargaining rule and social welfare function. As a by product of this classification process, we classify the assumptions over bargaining and social welfare functions to form an unconnected coalition by Theorem 3. Unconnected coalition refers to a coalition which is not connected (check definition 5). Connected coalition is a coalition with agents that have consecutively ordered peaks. If two agents $i_1$ and $i_2$ are inside a connected coalition S and there is another agent $i_3$ with $p_{i_1} \leq p_{i_2} \leq p_{i_3}$, then agent $i_3$ is a member of the coalition too. Note that Theorem 3 indicates that conventional wisdom fails even under strong assumptions such as Pareto efficiency and monotonicity. And finally in non-representative coalition model, Theorem 4 shows that we can not produce a grand coalition if we allow individual bargainers to the bargaining process.

# Chapter 2

# Literature Review

## Cooperative Bargaining

Before we mention the literature on coalition formation, we need to focus on the cooperative bargaining literature. Cooperative bargaining theory is originated on paper by Nash (1950) [21]. Nash modeled the negotiation processes and defined an axiomatic methodology to analyze that sort of models. The modeling of negotiation process consists of identifying the alternative agreements and their values for the negotiators that is, the implications of each agreement and disagreement. Cooperative bargaining theory focuses on producing methods to identify and determine desirable bargaining rules. In his paper, Nash proposed the Nash bargaining rule which maximizes the product of each negotiators' utility gain with respect to their disagreement payoffs.

Kıbrıs (2010) [16] provides an extensive review of cooperative bargaining theory. He summarizes and surveys the cooperative bargaining literature starting with Nash (1950) [21] to more recent studies. With the guidance of Kıbrıs (2010) [16] paper, we are going to focus on the most well-known bargaining rules which are Nash, Kalai-Smorodinsky, Egalitarian, and Utilitarian.

Let's start with different characterizations of Nash bargaining rule. Nash showed that his bargaining rule uniquely satisfies Pareto optimality[1], symmetry[2], scale invariance[3], and independence of irrelevant alternatives[4]. There are several studies on Nash bargaining rule. Some of them alter Nash's model such as changing the structure of the feasible set or disagreement point. Others search for new properties for the characterization of Nash bargaining rule without changing the model. Roth (1979) [24] works on both types. He studies n-person games in which agents try to reach a unanimous agreement. Each agent has a veto right. If there is no unanimous decision then the result of the game will be some ex-ante disagreement point. In this context, Roth works on Nash's model of bargaining (formal model, risk posture) and other models of bargaining. He introduces different properties over Nash bargaining rule at formal model chapter. Roth indicates that Pareto optimality is the strongest assumption among other assumptions of Nash's. Pareto optimality requires that the selection of the solution will be a "good" outcome in every bargaining game. Therefore, Pareto optimality eliminates most of the potential outcomes, including the occurrence of a disagreement. Instead of the collective choice assumption Pareto optimality, Roth imposes individual rationality[5]. He shows that it is essentially unnecessary to impose the requirement of Pareto optimality in order to derive Nash's solution. Then Roth imposes the property strong Individual rationality [6]. He shows that strong Individual rationality together with other properties except symmetry implies strong Pareto optimality [7]. In risk posture chapter, he proposes a risk component to utility function of bargainers. He shows that a utility function is risk averse if it is strictly concave. Then he compares the risk aversions of utility functions for

[1]Pareto optimality of an agreement means that not all bargainers benefit from altering to another agreement

[2]A bargaining rule F is symmetric if for each permutation $\pi$ of negotiators, $\pi(S) = S$ and $\phi(d) = d$ implies $F_1(S,d) = \ldots = F_n(S,d)$

[3]A bargaining rule F is scale invariant if for each (S,d) and for each positive affine function $\lambda$ $F(\lambda(S), \lambda(d)) = \lambda(F(S,d))$

[4]Let $S_1 \subset S_2$ and ($S_1$,d),($S_2$,d) $\in \mathcal{B}$ such that if $F(S_2,d) \in S_1$ then $F(S_1,d) = F(S_2,d)$. The agreement of a bargaining problem would not change if we decrease the size of the feasible set.

[5]For any bargaining game (S,d), $F(S,d) \geq d$

[6]For any bargaining game (S,d), $F(S,d) > d$

[7]For any bargaining game (S,d), if x and y are distinct elements of S such that $x \geq y$, then $F(S,d) \neq y$

money. Major point here is to see how risk aversion enters into Nash's model of the bargaining problem. He deduces that, "In a two player bargaining game, making player 2 more risk averse has the effect of making Pareto optimal set of utility payoffs more concave as a function of player 1's utility." Hence, as player 1 becomes more risk averse the utility of player 2 increases which is assigned by Nash solution. He also defines risk sensitivity [8]. He shows that: " The Nash solution is the unique solution for two players games which possesses the properties symmetry, independence of irrelevant alternatives, Pareto optimality and risk sensitivity.". Boldness [9] and fear of ruin[10] are other two definitions Roth proposed. He finds that: "The player who is bolder with respect to an equal division of the available money obtains the larger share according to Nash solution." Roth suggests that players which are completely informed of one another's preferences as captured by their utility function is not always the case. Suppose players know one another's preferences only over riskless events, but not over lotteries. Even in this case, the player's attitude towards risk would influence the bargaining process only indirectly or even not at all. Therefore, we need a wider class of transformations than these required by scale invariance property. In this context, Roth construct a theory of bargaining which depends only on the ordinal transformations [11] contained in the players' utility functions. Then he defines a new property Independence of ordinal transformations[12]. Independence of ordinal transformations is a stronger property than scale invariance. He shows that no solution which possesses Independence of ordinal transformation can also possess independence of irrelevant alternatives and strong individual rationality. Now in this chapter, he considers a model with less information than the previous models of this book. And it is possible in the class of monetary games to identify the outcome

[8] If a two person game (S,d) is transformed into a game (S',d') by replacing player i with a more risk averse player, then $F_j(S', d') \geq F_j(S, d)$

[9] Consider the game (S,d) with two players such that $w_1$ and $w_2$ are their initial wealths and players bargain how to split Q dollars. A feasible proposal is $(c_1, c_2)$ such that $c_1 + c_2 \leq Q$. A player's boldness with respect to $(c_1, c_2)$ is $b_i(w_i, c_i) := \frac{u_i'(w_i+c_i)}{u_i(w_i+c_i)-u_i(w_i)}$

[10] Inverse of boldness

[11] Each player's preference ordering over riskless alternatives.

[12] For any bargaining gam (S,d) in $B^*$ and any continuous, order preserving functions $m_i$, $i = 1, \ldots, n$ , let the bargaining game (S',d') be defined by $S' = m(S) \equiv \{y \in \mathcal{R}^n | y = m(x) \; for\ some\ x \in S\}$ and $d' = m(d)$. Then $f_i(S', d') = m_i(f_i(S, d))$ for any $i = 1, \ldots, n$

of equal division of available money, and this is the unique outcome selected by an ordinally independent solution which is symmetric and Pareto optimal on the class of monetary games. There are other papers who works on new properties. Peters (1986) [22] works on simultaneous bargaining situations on different issues by two bargainers. The axiomatic approach is the same as Nash indicated. He shows that (Partial) Superadditivity[13], homogeneity[14], weak Pareto optimality [15] characterize a family of proportional solutions [16]. He also shows that, in addition to individual rationality and Pareto continuity[17], the axioms of restricted additivity[18], scale transformation invariance[19], and Pareto optimality gives an alternative characterization of a family of solutions consisting of all non-symmetric extensions of Nash solution. Lensberg (1988) [19] shows that the Nash solution is the only one to satisfy Pareto optimality, anonymity, scale invariance, and stability. He also weakens the Pareto optimality by using stability axiom and still characterizes the Nash solution. Dagan, Volij, and Winter E. (2002) [8] provides an alternative characterization of Nash bargaining solution by replacing Independence of Irrelevant Alternatives with three axioms which are Independence of Non-Individually Rational Alternatives[20], Twisting[21], and Disagreement Point Convexity[22].

[13] $\sigma(S+T) \geq \sigma(S) + \sigma(T)\ for\ all\ S,T \in B$. Partial Super additivity: $\sigma(S+T) \geq \sigma(S)$ and $\sigma(S+T) \geq \sigma(T)\ for\ all\ S,T \in B$

[14] $\sigma(xS) = x\sigma(S)\ \forall S \in B,\ x \in \mathcal{R}^+$

[15] $\sigma(S) \in W(S)\ for\ all\ S \in B$

[16] For every $p \in \mathcal{R}^2$ with $p \geq 0$ and $p_1 + p_2 = 1$, the bargaining solution $E^p : B \to \mathcal{R}^2$ is defined by $\{E^p(S)\} = W(S) \cap \{xp | x \in \mathcal{R},\ x > 0\}$ for all $S \in B$. $E^p$ is called the egalitarian or proportional solution with weighted vector p

[17] $\sigma$ is continuous on $(B, \pi)$ where $\pi$ is the metric on B defined by $\pi(S,T) := d_H(P(S), P(T))$ and $d_H$ is the Hausdorff metric .Let X,Y be non-empty sets such that $d(x,Y) := inf\{d(x,y) | y \in Y\}$ and $d(X,Y) := sup\{d(x,Y) | x \in X\}$ and hausdorff metric $d_H(X,Y) := max\{d(X,Y), d(Y,X)\}$

[18] $S \in B$ is called smooth at $x \in S$ if there exists a unique line of support of S at x, and where $\sigma$ is a bargaining solution. Restricted additivity: For all S and T in B, if S and T are smooth at $\sigma(S)$ and $\sigma(T)$ respectively, and $\sigma(S) + \sigma(T) \in P(S+T)$, then $\sigma(S+T) = \sigma(S) + \sigma(T)$

[19] A scale transformation $a = (a_1, a_2)$ is a vector in $\mathcal{R}^2_{++} := \{x \in \mathcal{R}^2 | x > 0\}$. Scale transformation Invariance: $\sigma(xS) = x\sigma(S)$ for all $S \in B$, $x \in \mathcal{R}^2_{++}$

[20] A bargaining solution satisfies independence with respect to non-individually rational alternatives if for every two problems (S,d) and (S',d) such that $\mathcal{IR}(S,d) = \mathcal{IR}(S',d)$ we have $f(S,d) = f(S',d)$. ($\mathcal{IR}(S,d)$ is the set of individually rational points in (S,d))

[21] Let (S,d) be a bargaining problem and let $(\bar{s}_1, \bar{s}_2) \in f(S,d)$. Let (S',d) be another bargaining problem such that for some agent $i = 1,2$ $S \setminus S' \subseteq \{(s_1,s_2) | s_i > \bar{s}_i\}$ and $S' \setminus S \subseteq \{(s_1,s_2) | s_i < \bar{s}_i\}$. Then there is $(s'_1, s'_2) \in f(S',d)$ such that $s'_i \leq \bar{s}_i$

[22] For every bargaining problem B=(S,d), for all $s \in f(S,d)$ and for every $\lambda \in (0,1)$ we have $s \in f(S, (1-\lambda)d + \lambda s)$

Papers which alter the Nash's model such as changing the structure of the feasible set or disagreement point are also part of the cooperative bargaining literature. Chun (1988) [5] studies bargaining processes which are constructed by unknown feasible sets, and known disagreement points. The reason he works on this subject is to formulate axioms which specifies the effects of the characterization of feasible set and disagreement points over bargaining solution. Peters and Van Damme (1991) [31] provides a new characterization of n-person Nash bargaining solution without independence of irrelevant alternatives. They also characterize continuous Raiffa solution[23]. Different from Chun (1988) [5], they mainly focus on axioms which acts on the changes in the disagreement point and leave the feasible set fixed. Chun and Thomson (1991) [7] introduce a claims (expectations) point to disagreement point and feasible set. Agents may have these claims when they bargain. They assume that the claims point is not an element of feasible set. And they investigate the response of bargaining solution by changing the feasible set, the disagreement point and the claims point, the number of agents. Each change leads to the proportional solution which is the maximal point of the feasible set on the line segment connecting the disagreement point to the claims point.

Now I will provide the literature on Kalai-Smorodinsky rule which focuses on different characterizations of it. In most cases, we can consider bargaining process as step-by-step interim settlements such that each settlement is a start point for new negotiations. We can thus construct interim settlement approach in Nash's bargaining framework. For two player bargaining problem, Raiffa (1953) [17] proposed two different solution methods that use this idea. The first one is considering the interim agreement discrete. The outcome that gives a player her maximal utility while keep-

[23]Let CR denotes the continuous raiffa solution. Let $(S, d) \in B$ and let h(S,d) denote the utopia point of (S,d), where $h_i(S, d) := \{x_i| \ x \geq d\}$for i=1,2. If $d < h(S, d)$, then let $R_S$ be the unique solution of the differential equation $(dx_1/dx_2) = r_S(x)$ (x in the interior of S) with $R_S(d_1) = d_2$, where $r_S(x)$ is the slope of the straight line through x and h(S,x). For this case $CR(S, d) \in P(S)$ is defined to be the limit point of the graph of $R_S$. Otherwise $CR(S, d)$ be equal to the unique Pareto optimal point weakly dominating d.

ing the other player at her disagreement is the most preferred outcome. The interim agreement is the average of most preferred outcomes of the two players. By using this outcome as a disagreement point in each step, the process converges to a Pareto optimal point of the bargaining set. In his second solution, Raiffa proposed that the process is continuous in the direction of the average of the two most preferred points. Kalai and Smorodinsky (1975) [15] focused on two person bargaining problems. They showed that taking monotonicity axiom [24](For every utility level player 1 will demand, the maximum feasible utility level player 2 can simultaneously reach is increased, then the utility level of player 2 is increased at the solution.) instead of IIA, there is a unique solution which is different from the Nash solution called the KS solution. We can observe that both Nash's solution and the KS solution are continuous functions of the pairs (S,d). Thomson (1980) [29] shows how to generalize two person bargaining solution of Raiffa to n-person bargaining solution. There are two characterizations under two new monotonicity definitions along with the usual axioms Pareto optimality, Symmetry and Invariance. Dubra (2001) [11] works on standard two person bargaining problems, and defines a restricted Independence of Irrelevant Alternatives[25](If the ratio of the utopia points is fixed as we passing to a smaller feasible set and original choice remains in the smaller feasible set, then they would choose again the same point.) along with other familiar axioms except symmetry and shows an asymmetric version of KS solution. He also observes that restricted version of IIA is compatible with Individual Monotonicity. [26]

We focused on the literature review of classifications of Nash and KS solution

[24]For a pair $(a,S) \in B$, let $b(S) = (b_1(S), b_2(S))$ such that $b_1(S) := sup\{x \in \mathcal{R} |\ for\ some\ y \in \mathcal{R},\ (x,y) \in S\}$ and $b_2(S) := sup\{y \in \mathcal{R} |\ for\ some\ x \in \mathcal{R},\ (x,y) \in S\}$. Let $g_S(x)$ be a function defined for $x \leq b_1(S)$ such that $g_S(x) := \begin{cases} y & if\ (x,y)\ is\ the\ Pareto\ (a,S) \\ b_2(S) & if\ there\ is\ no\ such\ y \end{cases}$.
Here $g_S(x)$ function indicates the maximum player 2 can get whenever player 1 gets at least x. Axiom of Monotonicity: If $(a, S_2)$ and $(a, S_1)$ are bargaining pair such that $b_1(S_1) = b_2(S_2)$ and $g_{S_1} \leq g_{S_2}$, then $f_2(a, S_1) \leq f_2(a, S_2)$ where $(f(a,S) = f(a,S_1) = f(a,S_2))$

[25]S is comprehensive iff $y \in S$ whenever $x \in S$ and $x \geq y \geq 0$. Let $\Sigma$ be the class of compact and comprehensive sets $S \subseteq \mathcal{R}^2_+$ for which there is an x such that $x \gg 0$. A utopia point $\alpha_i(S) \equiv max\{x_i |\ (x_1,x_2) \in S\}$, $i = 1,2$. Restricted Independence of Irrelevant Alternatives: For all $T, S \in \Sigma$, is $S \subseteq T$ $F(T) \in S$ and $\beta\alpha(S) = \alpha(T)$ for $\beta \in \mathcal{R}_{++}$ hold, then $F(T) = F(S)$

[26]If $S \subseteq T$, $\alpha_i(T) = \alpha_i(S)$ and $\alpha_j(T) \geq \alpha_j(S)$ then $F_j(T) \geq F_j(S)$ for $i,j \in \{1,2\}$ and $i \neq j$.

methods until now. Let us survey the literature of Egalitarian bargaining solution. Kalai (1977) [14] works on a n-person bargaining situations where bargainers may encounter. An encounter is a situation described by two components. The first component is the feasible outcome of cooperation and the second component is the outcome of disagreement. He uses the axiomatic method as in Nash. He shows that after the suitable normalization of the utilities, the players will maximize their utilities with the restriction of equality, in other words they all gain "equally" in the given situation. Myerson (1981) [20] investigates properties of social welfare functions which are related to utilitarianism(favors maximal total welfare) and egalitarianism(favors maximal welfare constrained by individual members of the society should enjoy equal benefits from the society). He proves two theorems. Theorem 1 shows that a linearity condition[27] and Pareto optimality implies such social choice funstions are utilitarian. For theorem 1 we suppose $CP = CP^0$. The second theorem indicates that concavity condition, regularity condition[28], Pareto optimality and Independence of irrelevant alternatives implies that a social choice function is either utilitarian or egalitarian. The main purpose is to explain the role of these two principals in the development of ethical theories and in practical social decision making. Chun and Thomson (1990a) [6] describe the bargaining problem as a pair of feasible set and disagreement point. Different from Nash, they assumed that only the feasible set is known. Their aim is to evaluate a new solution method to the bargaining problem of known feasible sets and uncertain disagreement points. They propose the concavity of disagreement points to guarantee compromise among agents before resolving the uncertainty regarding the disagreement point. They show that disagreement point concavity together with weak Pareto optimality, independence of non-individually rational points[29] and continuity [30] is enough to characterize the

[27]For any finite collection of vectors $\{x^1, x^2, \ldots, x^n\} \subseteq \mathcal{R}^n$, $H(x^1, x^2, \ldots, x^n)$ be the comprehensive convex hull of $(x^1, x^2, \ldots, x^n)$ which is the smallest convex and comprehensive set containing the set $(x^1, x^2, \ldots, x^n)$. Let CP be the set of choice problems to be studied, choice problem is nonempty, closed, convex, and comprehensive subset of $\mathcal{R}^n$ and $CP^0 := \{H(x^1, x^2, \ldots, x^n) | (x^1, x^2, \ldots, x^n) is finite\}$. Linearity Condition: A function $F : CP \rightarrow \mathcal{R}^n$ is linear iff $F(\lambda S + (1-\lambda)T) = \lambda F(S) + (1-\lambda)F(T)$

[28]$CP^0 \subseteq CP$

[29]If $S' := \{x' \in \mathcal{R}^n | \ \exists x \in S \ with \ d \leq x \ and \ x' \leq x\}$ then $F(S', d) = F(S, d)$

[30]Let $\Sigma$ be the class of all n-person problems. For all sequences $\{(S^v, d^v)\}$ in $\Sigma$, if $S^v \rightarrow S$ in the

one parameter family of weighted Egalitarian solution.

Now we are going to survey the literature on Utilitarian Bargaining solution. Thomson and Myerson (1980) [28] provides a strongly monotonic[31] bargaining solution. To achieve this characterization, first they provide intuitive axioms such as cutting[32]), adding[33], et cetera. Then they provide counterintuitive axioms which they call "preserve" (preserve adding[34], preserve cutting[35], et cetera). Finally they deduce that all these axioms are logical consequences of strong monotonicity. Hence they provide a characterization of choice functions satisfying it. Thomson(1981) [30] characterizes both the Nash solution and the utilitarian choice rules by replacing independence of irrelevant axiom with Independence of irrelevant expensions on $\Sigma'$ [36]. Blackorby, Bossert, and Donaldson (1994) [3] provide generalized Gini orderings [37] and on the agents' utility gains which are quasi-concave, non-decreasing functions, an linear in ranked subspaces of n dimensional Euclidean spaces. And They characterize the generalized Gini class of bargaining solutions.

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Hausdorff topology and $d^v = d$ for all v, then $F(S^v, d^v) = F(S, d)$

[31] $\forall\ S, T$, if $T \subseteq S$, then $f(S) = f(T)$ or $f(S) > f(T)$

[32] Given S,T and player i, we say that $P_i(S,T)$ iff $\{x|x_i \leq f_i(S)\} \cap S = \{x|x_i \leq f_i(S)\} \cap S$ Cutting: $\forall\ S, T$, if $P_i(S,T)$ and $T \subseteq S$ then either $f_j(T) > f_j(S)\ \forall\ j \neq i$ or $(f_j(T) = f_j(S)\ \forall\ j \neq i$ and $f_i(T) \leq f_i(S)$

[33] $\forall\ S, T$, if $P_i(S,T)$, $S \subseteq T$, and $f(S) \in \partial S$ $(Boundry\ set\ of\ S)$ then either $f_i(T) > f_i(S)$ or $(f_i(T) = f_i(S)\ and\ f_j(T) \leq f_j(S)\ \forall\ j \neq i\ )$

[34] $\forall\ S, T$, if $P_i(S,T)$, $S \subseteq T$, and $f(S) \in \partial S$ $(Boundry\ set\ of\ S)$ then either $f_j(T) > f_j(S)\ \forall\ j \neq i$ or $(f_j(T) = f_j(S),\ \forall\ j \neq i\ and\ f_i(T) \leq f_i(S)\ \forall\ j \neq i\ )$

[35] $\forall\ S, T$, if $P_i(S,T)$ and $T \subseteq S$ then either $f_i(T) > f_i(S)$ or $(f_i(T) = f_i(S)\ and\ f_i(T) \leq f_i(S)\ \forall\ j \neq i$

[36] $(\triangle := \{p \in \mathcal{R}^2 | \|p\| = 1\}$, $W(S,x) := \{p \in \triangle | \forall\ y \in S, py \leq px\}.)$ $\forall\ S' = (S,d) \in \Sigma'$ with $x = f(S')$, $\exists p^{S'} \in W(S,x)$ such that $\forall T' = (T,d) \in \Sigma'$ with (a) $S \subset T$ and (b)$p^{S'} \in W(S,x)$, then $f(S') = f(T')$

[37] Let $x^r$ denote a rank-ordered permutation of $x \in \mathcal{R}^n$ such that $x_1^r \geq x_2^r \geq \ldots \geq x_n^r$. A generalized Gini ordering is represented by a function $g_a^n : \mathcal{R}^n \rightarrow \mathcal{R}$ such that $g_a^n(x) = \sum_{i=1}^n a_i^n x_i^r$ $\forall x \in \mathcal{R}^n$, $a = (a_1^n, .., a_n^n)$ with $0 \leq a_1^n \leq a_2^n \leq .. \leq a_n^n$, $a_n^n > 0$

## Coalition Formation

We can decompose the literature of cooperative bargaining and coalition formation into two distinct strands; one group supports the conventional wisdom, and the other group claims that collective action does not have to be advantageous. Conventional wisdom supports an intuitive claim: "Collectivism and centralization is advantageous for employees and that the individualism of employees is more advantageous for employers." This controversial claim has been supported and opposed by several author with different models. Some authors construct models that are centered around substitute or complementary agents which will bargain with a firm and there is a production process. At some other papers, the model is constructed over bargaining between downstream and upstream firms.

The conventional wisdom states that size has a bargaining advantage. There are several studies that support this claim. Galbraith (1952) [12] states that economies give power to large corporations, and so they exploit this power. In this context, countervailing power arises in the form of trade unions or civil organizations to reduce the advantage of corporations. Scherer and Ross (1990, Chapter 14) [25] investigates the structures of industries of US and abroad to focus on the motives for mergers and their effects and supports the conventional wisdom.

There is a huge amount of work that opposes the conventional wisdom. Theoretical analysis starts with Auman (1973) [2] who finds examples in which a monopoly is not always at an advantage. He proves that in some cases the monopolist would do well if he splits himself to many competing small traders. He provides an abstract example such that the core is quite large and there is a unique competitive allocation and for the monopolist competitive allocation is the best in the core. Hence the monopolist would do better if he split himself into many competing small traders. Postlewaite and Rosenthal (1974) [23] investigate Auman's (1973) [2] paper, and ask the question "Can all the members of a coalition be in some sense worse off if they form a syndicate than if they don't in an economically motivated setting ?". They give some examples to show that syndicate may disadvantageous. They also construct an example to show that if agents are a set of individually small agents relative to the market then Aumann's phenomenon disappears. Legros (1987) [18] works on bilateral markets with two complementary commodities. He shows that if the two sides of the market are equal regarding the endowments then every syndicate is strongly stable. Davidson (1988) [9] works on a wage determination model. Consider a unionized oligopolistic industry with two different bargaining structures. One is where the workers of each firm is represented by different unions and the other is an industry wide union. In this context, Davidson investigates collective bargaining in two different union types. He uses a noncooperative bargaining structure for contracts in oligopolistic industries. The result is that the industry wide bargaining leads to higher wages. For multiple unions, if a firm offers a higher wage, then its' competitors will increase the employment of workers as a response. This externality is internalized when an industry wide union forms. Stole and Zwiebel (1996a) [27] works on within firm bargaining where employees and the firm faces a wage bargaining. They consider a wide range of economic applications regarding labor decisions, technological choice, and organizational design using a novel bargaining methodology. In this context they investigate preference for unionization, along with hiring and capital decisions, training and cross-training, the importance of labor and asset specificity, managerial hierarchies. The results they find is that desirability of a union for the employees' point of view depends on the underlying technology. If it is concave, then union is desirable for the employee. And the reverse holds for a convex technology. Horn and Wolinski (1998) [13] works on a bargaining process where there are two firms whose product is either substitutive or complementary. There is a unique input for the firms and its price is determined at the bargaining process with the supplier. There are two cases for upstream industry, a monopolistic supplier or separate suppliers for each firm. Main results of these two upstream industry definitions are significantly different from the related models where input prices are not determined in the bargaining process. For example, the profit of upstream firm is not necessarily maximized when the industry is monopolized. In their paper, Chipty and Snyder (1999) [4] examine and construct an abstract model of the cable television industry to explain why large buyers may receive lower transfer prices from bargaining with suppliers (Downstream firms bargaining with an upstream firm). They allow buyer merger and characterize all buyer-supplier transactions as bilateral bargaining process. The suppliers bargain simultaneously with each of the buyers separately, and the bargaining outcome is the quantity to be traded and the tariff for the bundle which is characterized by the Nash bargaining solution. They characterize the buyer merger effect over three categories: downstream efficiencies, upstream efficiencies, and bargaining effects. They do not investigate over all alternative mechanisms through which buyer size can affect market outcomes. Segal (2003) [26] examines the profitability of integrations in a cooperative game solved by a random-order value[38] and shows that if the complementarity of the colluding players is reduced by other players then collusion is profitable. The same logic yields for unprofitability whenever complementarity is increased. Segal also shows that different types of integration have different bargaining effects. Atakan(2008) [1] search for the conditions over economic agents that will cause to bargain collectively instead of individually with a principal. Previous work imposed exogenously determined bargaining sequences and the result is the common intuition (substitutability cause collectivism). Atakan imposes an endogenously determined bargaining sequence. The results show that the previous work is not robust for substitute agents. For example, sufficiently patient heterogeneous[39] substitute agents[40] prefer individual bargaining to collective bargaining.

[38] For each player $i \in N$, $[\Delta_i v](S) = v(S \cup i) - v(S \setminus i)$ for all $S \subset N$. Let $\Pi$ denote the set of orderings of N. Let $\pi(i)$ denote the rank of player $i \in N$ in ordering $\pi \in \Pi$, $\pi^i := \{j \in N | \pi(j) \leq \pi(i)\}$. Let $P(\Pi) := \{\alpha \in \mathcal{R}_+^{\Pi} | \sum_{\pi \in \Pi} \alpha_\pi = 1\}$ denote the set of probability distrubitions over $\Pi$. For each $\alpha \in P(\Pi)$, $\exists$ a random value order $f^\alpha(v)$ such that for each $i \in N$ $f_i^\alpha(v) := \sum_{\pi \in \Pi} \alpha_\pi \Delta_i v(\pi^i)$

[39] Consider a production process. There exists a principal and agents. Each agent is an input for the production, they bargain with the principal about the wage. If agent i is employed, then his contribution to the production is $v_i$. Hence after the employment of agent i, if principal hire agent j then his contribution to the production is $1 - v_i$. Let $v = v_1$ and $d = v - v_2$, here d denotes the degree of heterogeneity. If d=0 then then the agents are homogeneous.

[40] If $v_1 \geq 1 - v_2$ then the agents are substitute agents. If $v_1 \leq 1 - v_2$ then the agents are complements.

To sum up, we have discussed the literature review of cooperative bargaining and coalition formation. We observe that the studies focus on production processes or wage determinations. They all are private goods. In this thesis, we are going to focus on pure public good bargaining situations.

We are going to define two different model; representative coalition model and non-representative coalition model. In representative coalition model, there is a single coalition which has the power to dictate its agreement with the principal to all agents. Every agent decides whether to be a member of the coalition or not. Even if an agent is not a member of that coalition, the bargaining outcome of that coalition also binds the agent. Hence the bargaining outcome is non-excludable and non-rival. Therefore, the bargaining outcome is a pure public good. In non-representative coalition model, there is a single coalition and bargaining outcome binds only the members of the coalition. Agents who prefers not to join the coalition, individually bargains with the principal and receives the corresponding outcome. Again for agents who bargain as a coalition, the bargaining outcome is non-excludable and non-rival. Hence, the bargaining outcome is a pure public good. Therefore, we are going to focus on pure public good bargaining situations.

# Chapter 3

# Model

There exists a principal with single peaked preferences. Let $p_0 = 0$ be the peak of the principal. Let N:={1,2,...,n} be the set of agents. Each agent has symmetric single peaked preferences. For each $i \in N$ , $p_i \in [0,1]$ is the peak of agent i. Let $d$ be the disagreement point. Assume that for all $i \in N$ and for all $x \in \mathcal{R}$, $u_i(x) \geq u_i(d)$ . The Euclidean utility function of each agent is:

$$u_i : \mathcal{R} \cup \{d\} \rightarrow \mathcal{R} \text{ such that } u_i(x) = -|x - p_i| \qquad \forall x \in \mathcal{R} \cup \{d\} \text{ and } \forall i \in N.$$

The set of all utility functions is $\mathcal{U} := \{-|x - p_i| \mid p_i \in [0,1]\}$. Note that there exists a one-to-one correspondence between $u_i$ and $p_i$. Thus whenever there is no risk of confusion, we will use $u_i$ and $p_i$ interchangeably.

I have defined the preferences of the society. Now I will define a choice rule which will give us the answer to the question: "How would a society decide on a cooperative action?". Therefore we need a function that will show us how a coalition of agents aggregate their preferences. We are talking about a choice rule which will take the utilities of the agent as variables and produce a representative utility.

**Definition 1.** *A social welfare function is;*

$$\phi := \bigcup_{S \in P(N)} \mathcal{U}^{|S|} \rightarrow \mathcal{U}$$

We can provide some social welfare function examples;

$i)$ Mean of the coalitions is $\phi(u_S) = \frac{1}{|S|}\sum_{i\in S} p_i$

$ii)$ Median of the coalitions is $\phi(u_S) = \begin{cases} (\frac{|S|+1}{2})'th\ peak & if\ |S|\ is\ odd \\ (\frac{\frac{|S|}{2}'th\ peak + \frac{|S|+1}{2}'th\ peak}{2}) & if\ |S|\ is\ even \end{cases}$

Both rules are Pareto efficient (check definition 7) and population monotonic (check definition 9). We can check the properties of these social welfare functions from Table 5.1, presented in Chapter 5.

Now I will define a rule that will show us how a coalition bargains with the principal. This bargaining rule will take the principal's utility and representative utility as variables.

**Definition 2.** *A bargaining rule is a function of two variables,*

$$\mu := \mathcal{U}^2 \rightarrow \mathcal{R} \cup \{d\}$$

Let's give some bargaining rule examples;

$i)$ $\mu(p_0, \phi(p_S)) := \frac{(p_0+\phi(p_S))}{n}$, $(n \geq 0)$.

$$ii)\ \mu(p_0, \phi(p_S)) := \begin{cases} p_0 + \phi(p_S) & if\ \phi(p_S) < 0.4 \\ \phi(p_S)/2 & if\ 0.6 \neq \phi(p_S) \geq 0.4 \\ (\phi(p_S)+2)/2 & if\ \phi(p_S) = 0.6 \end{cases}$$

The bargaining rule $\mu(p_0, \phi(p_S)) := \frac{(p_0+\phi(p_S))}{n}$ produce bargaining outcomes between $p_0 = 0$ and $\phi(p_S)$. Hence it is Pareto efficient (check definition 8). This bargaining rule is also preference monotonic (check definition 11). The second bargaining rule particularly designed for not satisfying preference monotonicity. We can check the properties of these bargaining rules from Table 5.2, presented in Chapter 5.

We will analyze the implications of two alternative assumptions regarding the coalition formation process and representativeness of a coalition. They are detailed below:

### (i) Representative coalition

There is a single coalition which has the power to dictate its agreement with the principal to all agents. In this process every agent decides whether to be a member of the coalition or not. This is the only coalition that forms. The formed coalition then bargains with the principal. The bargaining outcome is binding for all agents, independent of whether they decided to join the coalition or not in the first place.

**Definition 3.** *A stable representative coalition $S$ is such that any member of the coalition will not be better off by leaving the coalition and any agent outside the coalition will not be better off by joining the coalition. $S \subseteq N$ is a stable coalition if and only if $u_i(\mu(u_0, \phi(u_S))) \geq u_i(\mu(u_0, \phi(u_{S\setminus\{i\}})))$ and $u_j(\mu(u_0, \phi(u_{S\cup\{j\}}))) \leq u_j(\mu(u_0, \phi(u_S)))$ for all $i \in S$ and for all $j \in N \setminus S$.*

### (ii) Non-Representative coalition

While as in the previous item, only a single coalition can form, this coalition is not a representation of the agents who prefers not to join it. Instead each such agent individually bargains with the principal and recieves the corresponding outcome. The coalition formation process is similar to the previous item; each agent declares whether she wants to be a member of the coalition or not. The important difference is that, now, an agent who chooses not to join the coalition bargains for himself (rather than being represented by the coalition as in the previous case).

**Definition 4.** *A stable non-representative coalition $S$ is such that any member of the coalition will not be better off by leaving the coalition and any agent outside the coalition will not be better off by joining the coalition. $S \subseteq N$ is a stable coalition if and only if $u_i(\mu(u_0, \phi(u_S))) \geq u_i(\mu(u_0, \phi(u_{\{i\}})))$ and $u_j(\mu(u_0, \phi(u_j))) \geq u_j(\mu(u_0, \phi(u_{S\cup\{j\}})))$ for all $i \in S$ and for all $j \in N \setminus S$.*

**Definition 5.** *A coalition $S \in \mathcal{P}(N)$ is connected; if there exists $i,\ j \in S$ such that $p_i \leq p_k \leq p_j$ then $k \in S$.*

Suppose we locate alternatives along a one-dimensional political spectrum. It is certain that voters with single peaked preferences choose alternatives closest to their most preferred outcome. Since political parties construct their policies in order to get the maximum amount of vote at the corresponding political spectrum, the Downsian model suggests that policies will converge to the position of the median voter, and agents who vote for the same party construct connected coalitions. In other words, party policies will be dependent to the distribution of voters, and coalitions will be connected at the Downsian axis. We can provide an example. Consider a normal distribution of voters with mean 1/2 on a [0,1] Downsian axis. Suppose there are only two political parties on 0 and 1, such as a communist party and a liberal party. In order to win the elections, one of them should get more votes then the other party. To achieve this goal, they will change their party policies. And trivially they will converge to the mean of the distribution which is 1/2. In other words, these two parties will look alike after a while. In this paper, we suggest a similar downsion axis. Agents with single peaked preferences have preferences on [0,1] interval. Different from the Downsian model we propose a second stage to the choice process which is the bargaining with a principal. Here an important question rises: "Will we still observe connected coalitions?".

We are going to impose two major definitions; Pareto efficiency and monotonicity with respect to our models. Both definitions are standard in bargaining literature.

**Definition 6.** *An agreement $x \in [0,1]$ is Pareto efficient with respect to $S$ and $u_S$ if for all $y \neq x$ there exists $i \in S$ such that $u_i(x) > u_i(y)$*

Pareto efficiency definition is applicable to both bargaining and social welfare functions.

**Definition 7.** *A social welfare function $\phi$ is Pareto efficient iff $\phi(u_S)$ is Pareto efficent with respect to for all $S \subseteq N$, for all $u_S$.*

**Definition 8.** *A bargaining rule $\mu$ is Pareto efficient iff $\mu(u_0, \phi(u_S))$ is Pareto efficient with respect to for all $S \subseteq N$, for all $\phi(u_S)$.*

Now, I am going to define two different types of monotonicity. Since social welfare function and bargaining rule possess different properties, we need different monotonicity definitions. Population monotonicity is defined for social welfare function. If an agent decides to join a coalition S whenever social welfare function is population monotonic, then agent is better of by joining the coalition S.

**Definition 9.** *A social welfare function $\phi$ is population monotonic, if for all $S \subseteq N$, for all $u_S$, for all $i \notin S$ and for all $u_i \in \mathcal{U}$ we have $u_i(\phi(u_S)) \leq u_i(\phi(u_{S\cup\{i\}}))$*

*A social welfare function $\phi$ is* ***strictly*** *population monotonic, if for all $S \subseteq N$, for all $u_S$, for all $i \notin S$ and for all $u_i \in \mathcal{U}$ we have $u_i(\phi(u_S)) < u_i(\phi(u_{S\cup\{i\}}))$ whenever $\phi(u_S) \neq p_i$ and $u_i(\phi(u_S)) = u_i(\phi(u_{S\cup\{i\}}))$ whenever $\phi(u_S) = p_i$.*

Population monotonicity is applicable to bargaining rule. But assuming this property on a bargaining rule give us triviality. If an agent will be better of by joining the coalition, then trivially she will join the coalition. Hence we need other means of definition for the montonicity of bargaining rule. Preference monotonicity is a standard monotonicity definition which aims to preserve order.

**Definition 10.** *A social welfare function $\phi$ is preference monotonic, if for all $S \subseteq N$, for all $u_S$, for all $i \in S$, for all $u_i^{'}$ such that $p_i < p_i^{'}$ we have $\phi(u_S) \leq \phi((u_{S-i}, u_i^{'}))$.*

**Definition 11.** *A bargaining rule $\mu$ is preference monotonic if $p_i \leq p_j$ implies $\mu(u_0, u_i) \leq \mu(u_0, u_j)$ for all $i, j \in N$, and for all $u_i,\ u_j$.*

*A bargaining rule $\mu$ is* ***strictly*** *preference monotonic if $p_i < p_j$ implies $\mu(u_0, u_i) < \mu(u_0, u_j)$ and $p_i = p_j$ implies $\mu(u_0, u_i) = \mu(u_0, u_j)$ for all $i, j \in N$.*

The next property, socially boundedness is defining a relation between social welfare function and bargaining rule. This property limits bargaining rule shifts with social welfare function shifts.

**Definition 12.** *A bargaining rule $\mu$ is called socially bounded by $\phi$ with respect to $S$ and $u_S$ if and only if there exists $i \in N \setminus S$ and $u_i \in \mathcal{U}$ such that $|\mu(p_0, \phi(u_S)) - \mu(p_0, \phi(u_{S\cup\{i\}}))| \leq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$.*

# Chapter 4

# Results

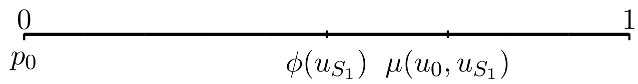
### 4.0.1 Representative Coalition

As we mentioned at the Chapter 3, representative coalition is the baseline model. There is a single coalition which has the power to dictate its' agreement with the principal to all agents. In this process every agent decides whether to be a member of the coalition or not. Stable coalition S is a coalition such that any member of the coalition will not be better off by leaving the coalition and any agent outside the coalition will not be better off by joining the coalition. I should point out that, each agent can observe only the results of the actions of one step forward. If an agent prefers to leave the coalition, then the agent will know the outcomes of the bargaining processes when she is inside and outside the coalition.

I will start with a *lemma* that shows us the Pareto efficient bargaining rules with respect to preferences of the agents'.

**Lemma 1.** *A bargaining rule $\mu$ is Pareto efficient if and only if* $\forall S \subseteq \mathcal{P}(N)$ *and* $\forall u_S$, $min\{p_0, \phi(u_S)\} \leq \mu((u_0, u_S)) \leq max\{p_0, \phi(u_S)\}$ *where* $p_0 = 0$

*Proof.* ($\Rightarrow$) Assume that a bargaining rule $\mu$ is Pareto efficient. I will show that $\forall S$ and $\forall u_S$ $min\{p_0, \phi(u_S)\} \leq \mu(u_0, u_S) \leq max\{p_0, \phi(u_S)\}$. Assume not, assume that $\exists\, S_1 \subset \mathcal{P}(N)$ and $\exists u_{S_1}$ such that $\mu((u_0, u_{S_1})) \notin [min\{p_0, \phi(u_{S_1})\}, max\{p_0, \phi(u_1)\}]$. Without loss of generality, suppose $\mu(u_0, u_{S_1}) > max\{p_0, \phi(u_{S_1})\}$.




It is certain that $\forall y \in [p_0, \mu(u_0, u_{S_1}))$ $u_0(y) > u_0(\mu(u_0, u_{S_1}))$ and $u_{S_1}(y) > u_{S_1}(\mu(u_0, u_{S_1}))$. But this contradicts with the fact that $\mu$ is Pareto efficient .

($\Leftarrow$) Consider a bargaining rule $\mu$ such that $\forall u_0$ and $\forall S$ , $min\{p_0, \phi(u_S)\} \leq \mu((u_0, u_S)) \leq max\{p_0, \phi(u_S)\}$. I will show that $\mu$ is Pareto efficient. Since $\forall x \neq \mu((u_0, u_S))$ implies $u_0(\mu(u_0, u_S)) > u_0(x)$ or $u_S(\mu(u_0, u_S)) > u_S(x)$ , $\mu$ is Pareto efficient with respect to $\forall S$ and $\forall u_S$. $\square$

We can observe by *Lemma* 1 that in one dimensional policy spectrum Pareto efficient points of two agents is the points between the peaks of the agents'.

*Lemma* 1 is related with bargaining rules, and we can impose the same logic to social welfare functions.

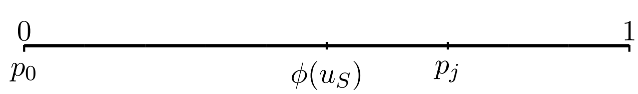
**Corollary 1.** *A social welfare function is Pareto efficient iff its range is a subset of* $[p_{min(S)}, p_{max(S)}]$ *where* $p_{min(S)}$ *and* $p_{max(S)}$ *stands for minimum and maximum peaks of the agents' which are members of the coalition S.*

*Proof.* By lemma 1

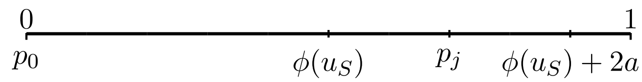
*Theorem* 1 will show us under specific circumstances there exist agents who will join the coalition.

**Theorem 1.** *Let* $\mu$ *be a preference monotonic, Pareto efficient bargaining rule and let* $\phi$ *be a population monotonic social welfare function. If* $\exists S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ *and* $\exists i \in N \setminus S$, $u_i \in \mathcal{U}$ *such that* $\phi(u_S) < p_i$ *then agent i will be better off or indifferent by joining the coalition.*

*Proof.* Suppose $\exists S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ and $\exists i \in N \setminus S$, $u_i \in \mathcal{U}$ such that $\phi(u_S) < p_j$. Without loss of generality, consider this is the case;



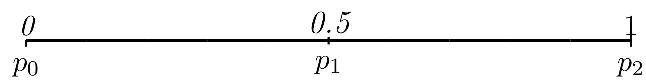
Since $\phi$ is population monotonic, $\phi(u_{S\cup\{j\}}) \in [\phi(u_S), \phi(u_S)+2a]$ where $a := |p_j - \phi(u_S)|$.




Since $\mu$ is Pareto efficient, by lemma 1 $min\{p_0, \phi(u_S)\} \leq \mu(u_0, \phi(u_S)) \leq max\{p_0, \phi(u_S)\}$ for all $S$ and for all $u_S$. Since $\mu$ is preference monotonic , $\mu(u_0, \phi(u_S)) \leq \mu(u_0, \phi(u_{S\cup\{j\}}))$. Since $0 = p_0 \leq \mu(u_0, \phi(u_S)) \leq \phi(u_S)$ and $\mu(u_0, \phi(u_S)) \leq \mu(u_0, \phi(u_{S\cup\{j\}}))$ for all $S$ and $\phi(u_{S\cup\{j\}}) \in [\phi(u_S), \phi(u_S)+2a]$, $u_j(\mu(u_0, \phi(u_S))) \leq u_j(\mu(u_0, \phi(u_{S\cup\{j\}})))$. Therefore agent j will be better off or indifferent by join the coalition. $\square$

In the following example, we will construct a social welfare function that satisfies all assumptions of theorem 1 except population monotonicity. We show that for the rule, the conclusion of Theorem 1 fail. Hence agent 2 does not join the coalition $\{1\}$.

**Example 1.** *Consider the case;*




*Let $p_1 = 0.5$, $p_2 = 1$. Suppose that the social welfare function is*

$$\phi(u_S) = \begin{cases} 0.4 & if|N| = 2, S = \{1\}, p_1 = 0.5,\ \ p_2 = 1 \\ 0.3 & if|N| = 2, S = \{1,2\}, p_1 = 0.5,\ \ p_2 = 1 \\ 0.2 & if|N| = 2, S = \{2\}, p_1 = 0.5,\ \ p_2 = 1 \\ \frac{1}{|S|} \cdot \sum_{i\in S} p_i & otherwise \end{cases}$$

*, and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S)+p_0}{2}$.

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{array}{lll} \textit{if } S = \{1,2\} & \textit{then} & \phi(u_S) = 0.4,\ \mu(u_0, \phi(u_S)) = 0.2 \\ \textit{if } S = \{1\} & \textit{then} & \phi(u_S) = 0.3,\ \mu(u_0, \phi(u_S)) = 0.15 \\ \textit{if } S = \{2\} & \textit{then} & \phi(u_S) = 0.2,\ \mu(u_0, \phi(u_S)) = 0.1 \end{array}$$

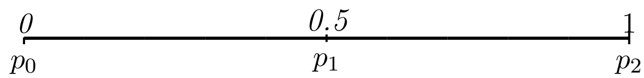
- *$S = \{1,2\}$ is not stable because agent 2 will leave the coalition to move the bargaining outcome from 0.15 to 0.2 which is closer to agent 2's peak.*

- *$S = \{2\}$ is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.1 to 0.15 which is closer to agent 1's peak.*

*Hence $\{1\}$ is the stable coalition.*

In the following example, we will construct a bargaining rule that satisfies all assumptions of theorem 1 except preference monotonicity. We show that for the rule, the conclusion of Theorem 1 fail. Hence agent 2 does not join the coalition $\{1\}$.

**Example 2.** *Consider the case;*




*Let $p_1 = 0.5$, $p_2 = 1$. Suppose that social welfare function is $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i \in S} p_i$, and the bargaining rule is*

$$\mu(p_0, \phi(p_S)) := \begin{cases} \phi(p_S) & if\ \phi(p_S) = 0.5 \\ \phi(p_S)/10 & otherwise \end{cases}$$

*Now let's check the possible coalitions and bargaining outcomes:*

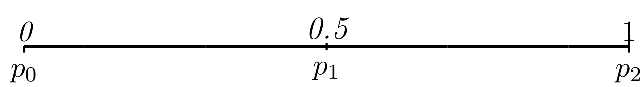
$$\text{if } S = \{1,2\} \quad \text{then} \quad \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 0.075$$
$$\text{if } S = \{1\} \quad \text{then} \quad \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 0.5$$
$$\text{if } S = \{2\} \quad \text{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 0.1$$

- *$S = \{1,2\}$ is not stable because agent 2 will leave the coalition to move the bargaining outcome from 0.075 to 0.5 which is closer to agent 2's peak.*

*$\{1\}$ and $\{2\}$ are stable coalitions.*

In the following example, we will construct a bargaining rule that satisfies all assumptions of theorem 1 except Pareto efficiency. We show that for the rule, the conclusion of Theorem 1 fail. Hence agent 2 does not join the coalition $\{1\}$.

**Example 3.** *Consider the case;*



*Let $p_1 = 0.5$, $p_2 = 1$. Suppose that social welfare function is $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i \in S} p_i$, and the bargaining rule is $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{2} + 1$.*

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{aligned} &\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 1.375 \\ &\textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 1.25 \\ &\textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 1.5 \end{aligned}$$

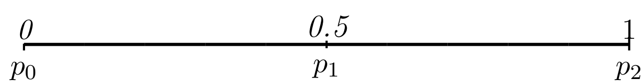
- $S = \{1,2\}$ *is not stable because agent 2 will leave the coalition to move the bargaining outcome from 1.375 to 1.25 which is closer to agent 2's peak.*

- $S = \{2\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 1.5 to 1.25 which is closer to agent 2's peak.*

*Hence* $\{1\}$ *is the stable coalitions.*

In the following example, we will construct a bargaining rule that satisfies all assumptions of theorem 1 except $\phi(u_S) \leq p_i$. We show that for the rule, the conclusion of Theorem 1 fail. We observe that $\phi(u_{\{2\}}) > p_1$, and agent 1 does not join the coalition whenever the social welfare function is population monotonic and the bargaining rule is preference monotonic and Pareto efficient.

**Example 4.** *Consider the case;*




*Let $p_1 = 0.5$, $p_2 = 1$. Suppose that social welfare function is $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i \in S} p_i$, and the bargaining rule is $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{2}$.*

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{aligned} &\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 0.375 \\ &\textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 0.25 \\ &\textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 0.5 \end{aligned}$$

- $S = \{1, 2\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0.375 to 0.5 which is closer to agent 1's peak.*
- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0.25 to 0.375 which is closer to agent 2's peak.*

*Hence* $\{2\}$ *is the stable coalitions.*

$\star$ Examples 1-2-3-4 together show the necessity of the assumptions of Theorem 1. These examples also show that the assumptions of theorem 1 does not imply each other.

Now I will provide some corollaries to Theorem 1. The first one provides a case that we always reach a connected coalition whenever $|N| = 3$. Since we are interested in connected coalitions especially grand coalition in this thesis, corollary 2 is an important case.

**Corollary 2.** *If the bargaining rule* $\mu$ *is strictly preference monotonic and Pareto efficient and the social welfare function* $\phi$ *is strictly population monotonic, not Pareto efficient and* $N = \{1, 2, 3\}$ *such that* $\phi(u_{\{1,3\}}) < p_2$ *then any stable coalition will be connected.*

*Proof.* By Theorem 1, agent 2 will be better of by joining the coalition $\{1, 3\}$. Since $\{1, 3\}$ is the only unconnected coalition type, any stable coalition will be connected. $\square$

The next corollary provides a case that we always reach a grand coalition. But the assumptions are strong.

**Corollary 3.** *If the bargaining rule* $\mu$ *is strictly preference monotonic, Pareto efficient, and the social welfare function* $\phi$ *is strictly population monotonic, not Pareto efficient and* $\phi(u_{N-\{i\}}) < p_i$ *for any* $i \in N$ *then* $N$ *will be the unique stable coalition.*

*Proof.* Assume that for any $i \in N$, $\phi(u_{N-\{i\}}) < p_i$. Since $\mu$ is strictly preference monotonic, Pareto efficient and $\phi$ is strictly population monotonic, Theorem 1 satisfies. And from Theorem 1 , any agent who satisfies $\phi(u_{N-\{i\}}) < p_i$ will join the coalition. Since each agent in N satisfies this property, it is clear that N will be the unique stable coalition. □

Theorem 1 provides an intuition to search for agents' peaks greater then social choice outcome of a coalition which they are not part of it. First of all, we need to impose the assumptions of theorem 1 on social welfare function and bargaining rule. But these assumptions will not be enough. We are going to impose also Pareto efficiency of social welfare function. Because without Pareto efficieny, social choice outcomes does not have to be inside the interval $[p_{min(S)}, p_{max(S)}]$, so we may not find an agent peak which is greater then social choice outcome.

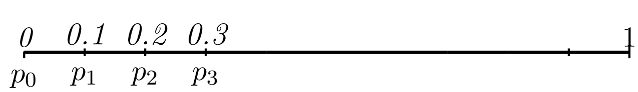
**Proposition 1.** *If the bargaining rule $\mu$ is preference monotonic, Pareto efficient and the social choice rule $\phi$ is population monotonic, Pareto efficient then the agent with the largest peak will be a member of the stable coalition.*

*Proof.* Let $S \subseteq N$ be any coalition and $p_n = max\{p_i | \ i \in N\}$ *and* $p_{min} = min\{p_i | \ i \in N\}$. Since $\phi$ is Pareto efficient and population monotonic, $p_{min} \leq \phi(u_S) \leq \phi(u_{S \cup \{n\}}) \leq p_{max}$. Since $\mu$ is Pareto efficient and preference monotonic, $0 = p_0 \leq \mu(u_0, \phi(u_S)) \leq \mu(u_0, u_{S \cup \{n\}}) \leq \phi(u_{S \cup \{n\}}) \leq p_{max}$. Hence agent n will be better off or indifferent by join the coalition. I am done. □

By imposing Pareto efficieny to social welfare function, we deduce a nice result. This also shows the importance of Pareto efficiency.

In the following example, we observe that agent 3 will join any coalition S whenever $\phi(u_S) < p_3$ such that $3 \notin S$. For this example, we observe an ambiguity. Consider the inequality: $\phi(u_{\{3\}}) = 0.3 > 0.1 = p_1$, despite this inequality agent 1 prefers to join the coalition. Since *Theorem* 1 is not an 'if and only if' statement, we can not suppose any action regarding agent 1. Now lets provide another example to reveal the ambiguity: "What will be the action of agent i if $\phi(u_{\{S\}}) > p_i$ for all $S$ such that i is not an element of $S$?".

**Example 5.** *Consider the case;*



*Let $p_1 = 0.1$, $p_2 = 0.2$ and $p_3 = 0.3$. Suppose that social welfare function is the mean of the peaks that is for all $S$, $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i \in S} p_i$, and the bargaining rule is*

$$\mu(u_0, \phi(u_S)) = \frac{\phi(S) + p_0}{2} \ .$$

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{aligned}
&\text{if } S = \{1,2,3\} \quad \text{then} \quad \phi(u_S) = 0.2,\ \mu(u_0, \phi(u_S)) = 0.1\\
&\text{if } S = \{1,3\} \quad \text{then} \quad \phi(u_S) = 0.2,\ \mu(u_0, \phi(u_S)) = 0.1\\
&\text{if } S = \{2,3\} \quad \text{then} \quad \phi(u_S) = 0.25,\ \mu(u_0, \phi(u_S)) = 0.125\\
&\text{if } S = \{1,2\} \quad \text{then} \quad \phi(u_S) = 0.15,\ \mu(u_0, \phi(u_S)) = 0.075\\
&\text{if } S = \{1\} \quad \text{then} \quad \phi(u_S) = 0.1,\ \mu(u_0, \phi(u_S)) = 0.05\\
&\text{if } S = \{2\} \quad \text{then} \quad \phi(u_S) = 0.2,\ \mu(u_0, \phi(u_S)) = 0.1\\
&\text{if } S = \{3\} \quad \text{then} \quad \phi(u_S) = 0.3,\ \mu(u_0, \phi(u_S)) = 0.15
\end{aligned}$$

- $S = \{1,2\}$ *is not stable because* $\phi(u_{\{1,2\}}) = 0.15 < 0.3 = p_3$ *which means agent 3 will join the coalition to move the bargaining outcome from 0.075 to 1 which is closer to agent 3's peak.*

- $S = \{1\}$ *is not stable because* $\phi(u_{\{1\}}) = 0.1 < 0.3 = p_3$ *which means agent 3 will join the coalition to move the bargaining outcome from 0.05 to 0.1 which is closer to agent 3's peaks.*

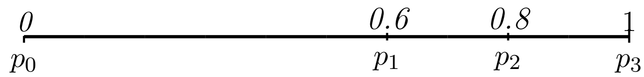
- $S = \{2\}$ *is not stable because* $\phi(u_{\{2\}}) = 0.2 < 0.3 = p_3$ *which means agent 3 will join the coalition to move the bargaining outcome from 0.1 to 0.125 which is closer to agent 3's peak.*

- $S = \{3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.15 to 0.1 which is closer to agent 1's peak.*

*Hence there are two stable coalitions which are* $\{1,2,3\}$ *and* $\{1,3\}$.

In the following example, we observe that $\phi(u_{\{3\}}) = 1 > 0.6 = p_1$ and agent 1 prefers not to join the coalition $\{3\}$.

**Example 6.** *Consider the case;*




*Let $p_1 = 0.6$, $p_2 = 0.8$ and $p_3 = 1$. Suppose that social welfare function is the mean of the peaks which means for all $S$, $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i \in S} p_i$, and the bargaining rule is*

$$\mu(u_0, \phi(u_S)) = \frac{\phi(S) + p_0}{2} \ .$$

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{gathered}
\textit{if } S = \{1,2,3\} \quad \textit{then} \quad \phi(u_S) = 0.8,\ \mu(u_0, \phi(u_S)) = 0.4 \\
\textit{if } S = \{1,3\} \quad \textit{then} \quad \phi(u_S) = 0.8,\ \mu(u_0, \phi(u_S)) = 0.4 \\
\textit{if } S = \{2,3\} \quad \textit{then} \quad \phi(u_S) = 0.9,\ \mu(u_0, \phi(u_S)) = 0.45 \\
\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 0.7,\ \mu(u_0, \phi(u_S)) = 0.35 \\
\textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.6,\ \mu(u_0, \phi(u_S)) = 0.3 \\
\textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 0.8,\ \mu(u_0, \phi((p_2))) = 0.4 \\
\textit{if } S = \{3\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 0.5
\end{gathered}$$

- $S = \{1,2,3\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0.4 to 0.45 which is closer to agent 1's peak.*

- $S = \{2,3\}$ *is not stable because agent 2 will leave the coalition to move the bargaining outcome from 0.45 to 0.5 which is closer to agent 2's peaks.*

- $S = \{1,3\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0,4 to 0,5 which is closer to agent 1's peak.*

- $S = \{2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0,4 to 0,45 which is closer to agent 3's peak.*

- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0,3 to 0,435 which is closer to agent 2's peak.*

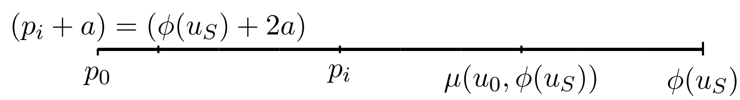
*Hence the only stable coalition is* $\{3\}$.

⋆ Examples 5-6 together show that theorem 1 assumptions are not enough to predict the action of agent i whenever $\phi(u_{\{S\}}) \geq p_i$ for any $S \subseteq N$ and for any $i \in N \setminus S$.

The next lemma will be necessary to prove Theorem 2. The lemma is about the movement of location of the social choice outcome whenever an agent enters a coalition.

**Lemma 2.** *If $p_i \leq \mu(u_0, \phi(u_S))$ and $|\mu(u_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ for some $S \subseteq N$ and $i \in N \setminus S$, and $\mu$ is Pareto efficient and $\phi$ is population monotonic then $p_i \leq \phi(u_{S\cup\{i\}}) \leq \phi(u_S)$*

*Proof.* Suppose there exists $S \subseteq N$ and $i \in N \setminus S$ such that $p_i \leq \mu(u_0, \phi(u_S))$ and $|\mu(u_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. And suppose also $\mu$ is Pareto efficient and $\phi$ is population monotonic. I will show that $p_i \leq \phi(u_{S\cup\{i\}}) \leq \phi(u_S)$. Since there exists $S \subseteq N$ and $i \in N \setminus S$ such that $p_i \leq \mu(u_0, \phi(u_S))$ and $\mu$ is Pareto efficient, $p_i \leq \mu(u_0, \phi(u_S)) \leq \phi(u_S)$. Since $\phi$ is population monotonic, $\phi(u_{S\cup\{i\}}) \in [\phi(u_S), \phi(u_S) + 2a]$ where $a = |p_i - \phi(u_S)|$.



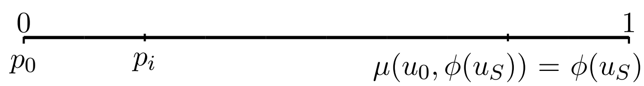

Since we have $S \subseteq N$ and $i \in N \setminus S$ such that $|\mu(u_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ and $\phi(u_{S\cup\{i\}}) \in [\phi(u_S), \phi(u_S)+2a]$ where $a = |p_i - \phi(u_S)|$, $p_i \leq \phi(u_{S\cup\{i\}}) \leq \phi(u_S)$. □

So far, we have discussed the action of agent i whenever $\phi(u_{\{S\}}) < p_i$ for any $S \subseteq N$ and for any $i \in N \setminus S$. We showed that if the bargaining rule is Pareto efficient, preference monotonic, and the social welfare function is population monotonic, and $\phi(u_{\{S\}}) < p_i$ for $S \subseteq N$, $i \in N \setminus S$ then agent i joins the coalition S. Now we are interested in the action agent i whenever $\phi(u_{\{S\}}) \geq p_i$ for any $S \subseteq N$ and for any $i \in N \setminus S$. By examples 5-6, we know that the assumptions of theorem 1 will not be enough to classify the action of agent i whenever $\phi(u_{\{S\}}) \geq p_i$ for any $S \subseteq N$ and for any $i \in N \setminus S$.

**Theorem 2.** *Let $\mu$ be preference monotonic, Pareto efficient, and let $\phi$ population monotonic. If $\exists S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ and $\exists i \in N \setminus S$, $u_i \in \mathcal{U}$ such that $\mu$ is socially bounded by $\phi$, $\mu(p_0, \phi(u_S)) \geq p_i$, and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ then $u_i(\mu(u_0, \phi(u_{S\cup\{i\}}))) \geq u_i(\mu(u_0, \phi(u_S)))$.*

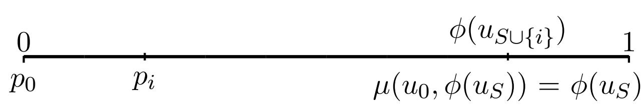
*Proof.* Suppose that there exist a coalition $S \subseteq N$, with $u_S \in \mathcal{U}^{|S|}$ and an agent $i \in N - S$ with $u_i \in \mathcal{U}$ such that $\mu$ is socially bounded by $\phi$, $\mu(p_0, \phi(u_S)) \geq p_i$, and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. Since $\mu$ is Pareto efficient, $\phi(u_S) \geq \mu(p_0, \phi(u_S))$.

(*Case* 1) $\quad p_i < \mu(u_0, \phi(u_S)) = \phi(u_S)$



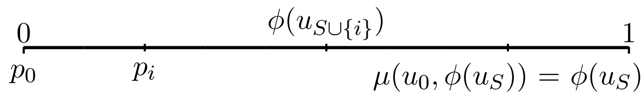

Since there exists $S \subseteq N$ and $i \in N \setminus S$ such that $p_i \leq \mu(u_0, \phi(u_S))$ and $|\mu(u_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ and $\mu$ is Pareto efficient and $\phi$ is population monotonic, $p_i \leq \phi(u_{S\cup\{i\}}) \leq \phi(u_S)$ by *Lemma* 2. Now we are going to check all possible locations of $\phi(u_{S\cup\{i\}})$ by cases;

(*Subcase* 1.1) $\quad \phi(u_{S\cup\{i\}}) = \phi(u_S)$



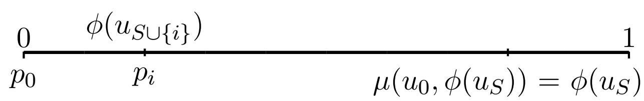

It is trivial that $|\phi(u_{S\cup\{i\}}) - \phi(u_S)| = 0$. Since $\mu$ is socially bounded by $\phi$, $0 = |\phi(u_{S\cup\{i\}}) - \phi(u_S)| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))|$. Hence $|\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))| = 0$, so $\mu(u_0, \phi(u_{S\cup\{i\}})) = \mu(u_0, \phi(u_S))$ which means $u_i(\mu(u_0, \phi(u_{S\cup\{i\}}))) = u_i(\mu(u_0, \phi(u_S)))$. Therefore agent i is indifferent of being inside or outside the coalition, so I am done.

$(Subcase\ 1.2)\quad p_i < \phi(u_{S\cup\{i\}}) < \phi(u_S)$



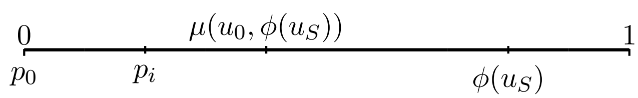

Now we have $p_i < \phi(u_{S\cup\{i\}}) < \phi(u_S) = \mu(u_0, \phi(u_S))$. Since $\mu$ is socially bounded by $\phi$ and Pareto efficient, $\phi(u_{S\cup\{i\}}) = \mu(u_0, \phi(u_{S\cup\{i\}}))$. Hence $p_i < \phi(u_{S\cup\{i\}}) = \mu(u_0, \phi(u_{S\cup\{i\}})) < \phi(u_S) = \mu(u_S)$. Therefore agent i will join the coalition S.

$(Subcase\ 1.3)\quad p_i = \phi(u_{S\cup\{i\}}) < \phi(u_S)$



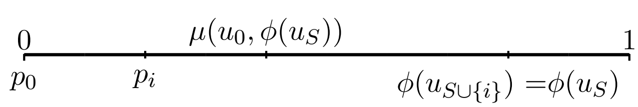

Since $\mu$ is socially bounded by $\phi$ and Pareto efficient, $p_i = \phi(u_{S\cup\{i\}}) = \mu(u_0, \phi(u_{S\cup\{i\}})) < \phi(u_S) = \mu(u_S)$. Hence agent i will join the coalition.

$(Case\ 2)\quad p_i < \mu(u_0, \phi(u_S)) < \phi(u_S)$



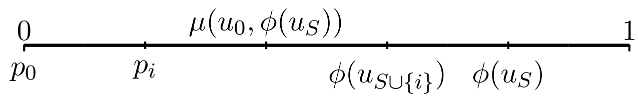

Since there exists $S \subseteq N$ and $i \in N \setminus S$ such that $p_i \leq \mu(u_0, \phi(u_S))$ and $|\mu(u_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ and $\mu$ is Pareto efficient and $\phi$ is population monotonic, $p_i \leq \phi(u_{S\cup\{i\}}) \leq \phi(u_S)$ by *Lemma* 2. Again we are going to check all possible locations of $\phi(u_{S\cup\{i\}})$ by cases;

$(Subcase\ 2.1)\quad \phi(u_{S\cup\{i\}}) = \phi(u_S)$



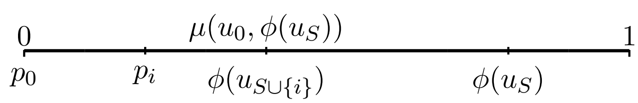
We have $p_i < \mu(u_0, \phi(u_S)) < \phi(u_S) = \phi(u_{S\cup\{i\}})$. Since $\mu$ is socially bounded by $\phi$ and $0 = |\phi(u_S) - \phi(u_{S\cup\{i\}})|$, $|\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))| = 0$. Hence $u_i(\mu(u_0, \phi(u_{S\cup\{i\}}))) = u_i(\mu(u_0, \phi(u_S)))$, so agent i is indifferent of being inside or outside the coalition. I am done.

(*Subcase* 2.2) $\quad p_i < \mu(u_0, \phi(u_S)) < \phi(u_{S\cup\{i\}}) < \phi(u_S)$



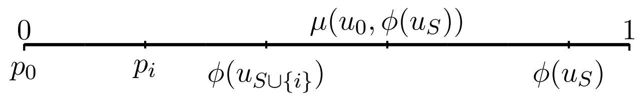

Recall the assumption $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. Since $\mu$ is socially bounded by $\phi$, $|\phi(u_S) - \phi(u_{S\cup\{i\}})| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))|$. Hence we have $|\mu(p_0, \phi(u_S)) - p_i| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))|$. Since $\mu$ is preference monotonic and $\phi(u_{S\cup\{i\}}) < \phi(u_S)$ and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))|$, $p_i \leq \mu(u_0, \phi(u_{S\cup\{i\}})) \leq \mu(u_0, \phi(u_S))$. Agent i will be better off or indifferent by joining the coalition.

(*Subcase* 2.3) $\quad p_i < \mu(u_0, \phi(u_S)) = \phi(u_{S\cup\{i\}}) < \phi(u_S)$



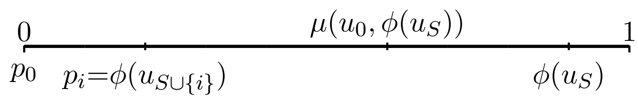

Recall the assumption $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. Since $\mu$ is socially bounded by $\phi$ and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$, $|\mu(p_0, \phi(u_S)) - p_i| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))|$. Since $\mu$ is preference monotonic and $\phi(u_{S\cup\{i\}}) < \phi(u_S)$ and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))|$, $p_i \leq \mu(u_0, \phi(u_{S\cup\{i\}})) \leq \mu(u_0, \phi(u_S))$. Agent i will be better off or indifferent by joining the coalition.

(*Subcase* 2.4) $\quad p_i < \phi(u_{S\cup\{i\}}) < \mu(u_0, \phi(u_S)) < \phi(u_S)$



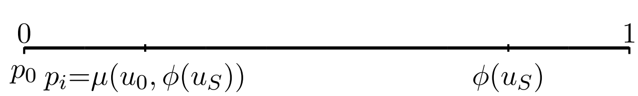

Lets call $a := |\phi(u_{S\cup\{i\}}) - p_i|$, $b := |\phi(u_{S\cup\{i\}}) - \mu(u_0, \phi(u_S))|$, and $c := |\mu(u_0, \phi(u_S)) - \phi(u_S)|$.
So $|\mu(u_0, \phi(u_S)) - p_i| = a + b$ and $|\phi(u_{S\cup\{i\}}) - \phi(u_S)| = b + c$. Since $|\mu(u_0, \phi(u_S)) - p_i| \geq |\phi(u_{S\cup\{i\}}) - \phi(u_S)|$, $a + b \geq b + c$ which implies $a \geq c$. Since $\mu$ is Pareto efficient, $\mu(u_0, \phi(u_{S\cup\{i\}})) \leq \phi(u_{S\cup\{i\}})$. So we reach $\mu(u_0, \phi(u_{S\cup\{i\}})) \leq \phi(u_{S\cup\{i\}}) < \mu(u_0, \phi(u_S)) < \phi(u_S)$, lets call $d := |\mu(u_0, \phi(u_{S\cup\{i\}})) - \phi(u_{S\cup\{i\}})|$. From social boundedness we know that:
$d + b = |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))| \leq |\phi(u_{S\cup\{i\}}) - \phi(u_S)| = b + c$, which gives us $c \geq d$ and we showed that $a \geq c$ so by transitivity $a \geq d$ which means $a := |\phi(u_{S\cup\{i\}}) - p_2| \geq d := |\mu(u_0, \phi(u_{S\cup\{i\}})) - \phi(u_{S\cup\{i\}})|$. Hence $p_i \leq \mu(u_0, \phi(u_{S\cup\{i\}})) \leq \phi(u_{S\cup\{i\}})$ and $u_i(\mu(u_0, \phi(u_{S\cup\{i\}}))) \geq u_i(\mu_0, \phi(u_S)))$. Hence agent i will be better off or indifferent by joining the coalition.

$(Subcase\ 2.5) \quad p_i = \phi(u_{S\cup\{i\}}) < \mu(u_0, \phi(u_S)) < \phi(u_S)$




Since $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ by assumption and $|\mu(u_0, u_S) - \phi(u_S)| > 0$, there will be no such a case.

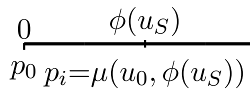
$(Case\ 3) \quad p_i = \mu(u_0, \phi(u_S)) < \phi(u_S)$




From the assumption we know that $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. Hence $0 = |\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. So we have $0 = |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. Since $\mu$ is socially bounded by $\phi$, $|\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))| \leq |\phi(u_{S\cup\{i\}}) - \phi(u_S)| = 0$.

So $|\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))| = 0$. Hence $p_i = \mu(u_0, \phi(u_{S\cup\{i\}})) = \mu(u_0, \phi(u_S))$. Agent i is indifferent of being inside or outside the coalition.

$(Case\ 4)\quad p_i = \mu(u_0, \phi(u_S)) = \phi(u_S)$

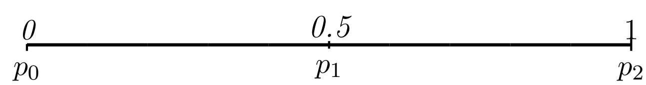



Since $\phi$ is population monotonic, $\phi(u_{S\cup\{i\}}) = \phi(u_S)$. Since $\mu$ is socially bounded by $\phi$, $0 = |\phi(u_{S\cup\{i\}}) - \phi(u_S)| \geq |\mu(u_0, \phi(u_{S\cup\{i\}})) - \mu(u_0, \phi(u_S))| = 0$. Agent i is indifferent of being inside or outside the coalition.

By cases 1-4, we have completed the proof. $\square$

In the following example, we will construct a bargaining rule and a social welfare function that satisfies all assumptions of theorem 2 except Pareto efficiency of bargaining rule We show that for the rules, the conclusion of Theorem 2 fail.

**Example 7.** *Consider the case;*




*Let* $p_1 = 0.5$*,* $p_2 = 1$*. Suppose that the social welfare function is* $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i\in S} p_i$*, and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{2} + 2$*.*

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{aligned} &\text{if } S = \{1,2\} \quad \text{then} \quad \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 2.375 \\ &\text{if } S = \{1\} \quad \text{then} \quad \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 2.25 \\ &\text{if } S = \{2\} \quad \text{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 2.5 \end{aligned}$$

- $S = \{1,2\}$ *is not stable because agent 2 will leave the coalition to move the bargaining outcome from 2.375 to 2.25 which is closer to agent 2's peak.*

- $S = \{2\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 2.5 to 2.375 which is closer to agent 1's peak.*

*Hence* $\{1\}$ *is the stable coalition.*

In the following example, we will construct a bargaining rule and a social welfare function that satisfies all assumptions of theorem 2 except social boundedness of bargaining rule by social welfare function for some S, $u_S$ and $i \in N \setminus S$, $u_i$. We show that for the rules, the conclusion of Theorem 2 fail.

**Example 8.** *Consider the case;*


*Let* $p_1 = 0.5$, $p_2 = 1$. *Suppose that the social welfare function is* $\phi(u_S) = \frac{1}{|S|} \cdot \sum_{i \in S} p_i$, *and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \begin{cases} \phi(p_S)/10 & if\ \phi(u_S) \leq 0.9 \\ \phi(p_S) & otherwise \end{cases}$

*Now let's check the possible coalitions and bargaining outcomes:*

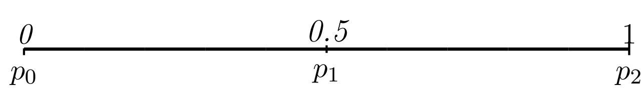
$$\begin{gathered} \textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 0.9,\ \mu(u_0, \phi(u_S)) = 0.09 \\ \textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.8,\ \mu(u_0, \phi(u_S)) = 0.08 \\ \textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 1 \end{gathered}$$

- $S = \{1,2\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0.09 to 1 which is closer to agent 1 's peak.*
- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0.08 to 0.09 which is closer to agent 2's peak.*

*Hence* $\{2\}$ *is the stable coalition.*

In the following example, we will construct a bargaining rule and a social welfare function that satisfies all assumptions of theorem 2 except preference monotonicity of bargaining rule. We show that for the rules, the conclusion of Theorem 2 fail.

**Example 9.** *Consider the case;*



*Let* $p_1 = 0.5$*,* $p_2 = 1$*. Suppose that the social welfare function is* $\phi(u_S) = \frac{1}{10|S|} \cdot \sum_{i \in S} p_i$*, and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \begin{cases} 0.05 & if\ \phi(u_S) = 0.05 \\ 0.074 & if\ \phi(u_S) = 0.075 \\ 0.073 & if\ \phi(u_S) = 0.1 \\ 0 & otherwise \end{cases}$

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{aligned} &\text{if } S = \{1,2\} \quad \text{then} \quad \phi(u_S) = 0.075,\ \mu(u_0, \phi(u_S)) = 0.074 \\ &\text{if } S = \{1\} \quad \text{then} \quad \phi(u_S) = 0.05,\ \mu(u_0, \phi(u_S)) = 0.05 \\ &\text{if } S = \{2\} \quad \text{then} \quad \phi(u_S) = 0.1,\ \mu(u_0, \phi(u_S)) = 0.073 \end{aligned}$$

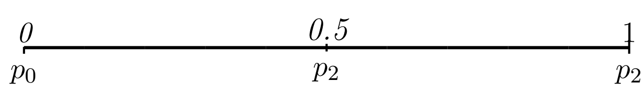
- $S = \{1,2\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0.074 to 0.073 which is closer to agent 1's peak.*

- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0.073 to 0.074 which is closer to agent 2's peak.*

*Hence* $\{2\}$ *is the stable coalition.*

In the following example, we will construct a bargaining rule and a social welfare function that satisfies all assumptions of theorem 2 except population monotonicity of social welfare function. We show that for the rules, the conclusion of Theorem 2 fail.

**Example 10.** *Consider the case;*




*Let* $p_1 = 0.5$*,* $p_2 = 1$*. Suppose that the social welfare function is*

$$\phi(u_S) := \begin{cases} \phi(u_{\{1,2\}}) = 0.6 & if\ |N| = 3,\ \ p_1 = 0.2,\ p_2 = 0.5,\ p_3 = 1 \\ \phi(u_{\{1\}}) = 0.75 & if\ |N| = 3,\ \ p_1 = 0.2,\ p_2 = 0.5,\ p_3 = 1 \\ \phi(u_{\{2\}}) = 0.56 & if\ |N| = 3,\ \ p_1 = 0.2,\ p_2 = 0.5,\ p_3 = 1 \\ \qquad \frac{1}{|S|} \sum_{i \in S \subseteq N} p_i & otherwise \end{cases},$$

*and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \phi(u_S)$

*Now let's check the possible coalitions and bargaining outcomes:*

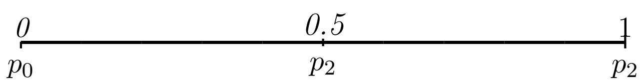
$$\begin{aligned}
&\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 0.6,\ \mu(u_0, \phi(u_S)) = 0.6 \\
&\textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 0.75 \\
&\textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 0.56,\ \mu(u_0, \phi((p_2))) = 0.56
\end{aligned}$$

- $S = \{1,2\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0,6 to 0,56 which is closer to agent 1's peak.*

*Hence* $\{2\}$*,*$\{1\}$ *are stable coalitions.*

In the following example, we will construct a bargaining rule and a social welfare function that satisfies all assumptions of theorem 2 except $\mu(u_0, \phi(u_{\{S\}})) \geq p_i$ for some S, $u_S$ and $i \in N \setminus S$, $u_i$. We show that for the rules, the conclusion of Theorem 2 fail.

**Example 11.** *Consider the case;*




*Let* $p_1 = 0.5$*,* $p_2 = 1$*. Suppose that the social welfare function is* $\phi(u_S) = \frac{1}{|S|} \sum_{i \in S \subseteq N} p_i$*, and the bargaining rule* $\mu$ *is* $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{5}$*.*

*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{aligned}
&\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 0.15 \\
&\textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 0.1 \\
&\textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi((p_2))) = 0.2
\end{aligned}$$

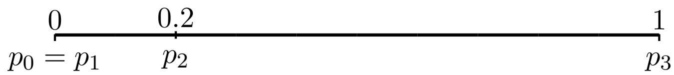
- $S = \{1,2\}$ *is not stable because agent 1 will leave the coalition to move the bargaining outcome from 0,15 to 0,2 which is closer to agent 1's peak.*

- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0,1 to 0,15 which is closer to agent 2's peak.*

*Hence* $\{2\}$ *is the stable coalition.*

In the following example, we will construct a bargaining rule and a social welfare function that satisfies all assumptions of theorem 2 except $|\mu(p_0, \phi(u_{\{S\}})) - p_i| < |\phi(u_{\{S\}}) - \phi(u_{S\cup\{i\}})|$ for some S, $u_S$ and $i \in N \setminus S$, $u_i$. We show that for the rules, the conclusion of Theorem 2 fail.

**Example 12.** *Consider the case;*




*Let $p_1 = 0$, $p_2 = 0.2$, $p_3 = 1$. Suppose that the social welfare function is $\frac{1}{|S|}\sum_{i\in S} p_i$, and $\mu(p_0, \phi(p_S)) := (p_0 + \phi(p_S))/0.4$, and $p_1 = 0$, $p_2 = 0.2$, $p_3 = 1$.*

*Lets check the possible coalitions and outcomes of bargaining:*

$$\begin{aligned}
&\textit{if } S = \{1,2,3\} \quad \textit{then} \quad \phi((p_1,p_2,p_3)) = 0.4 \textit{ and } \mu(p_0,\phi((p_1,p_2,p_3))) = 0.16\\
&\textit{if } S = \{2,3\} \quad \textit{then} \quad \phi((p_2,p_3)) = 0.6 \textit{ and } \mu(p_0,\phi((p_2,p_3))) = 0.24\\
&\textit{if } S = \{1,3\} \quad \textit{then} \quad \phi((p_1,p_3)) = 0.5 \textit{ and } \mu(p_0,\phi((p_1,p_3))) = 0.2\\
&\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi((p_1,p_2)) = 0.1 \textit{ and } \mu(p_0,\phi((p_1,p_2))) = 0.04\\
&\textit{if } S = \{1\} \quad \textit{then} \quad \phi((p_1)) = 0 \textit{ and } \mu(p_0,\phi((p_1))) = 0\\
&\textit{if } S = \{2\} \quad \textit{then} \quad \phi((p_2)) = 0.2 \textit{ and } \mu(p_0,\phi((p_2))) = 0.08\\
&\textit{if } S = \{3\} \quad \textit{then} \quad \phi((p_3)) = 1 \textit{ and } \mu(p_0,\phi((p_3))) = 0.4
\end{aligned}$$

- $S = \{1,2,3\}$ *is not stable because agent 2 will leave the coalition to move the bargaining outcome from 0.16 to 0.2 which is closer to agent 2's peaks.*

- $S = \{1,2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0.04 to 0.16 which is closer to agent 3's peak.*

- $S = \{2,3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.24 to 0.16 which is closer to agent 1's peak.*

- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0 to 0.04 which is closer to agent 2's peaks.*

- $S = \{2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0.08 to 0.24 which is closer to agent 3's peak.*

- $S = \{3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.4 to 0.2 which is closer to agent 1's peak.*

*There will be a deviation from each coalition except* $S = \{1, 3\}$.

⋆ Examples 7-8-9-10-11-12 together show the necessity of the assumptions of Theorem 2. These examples also show that the assumptions of theorem 2 does not imply each other.

Note that we do not reach unique stable coalitions in both theorems. We need stronger assumptions for unique stable coalitions. In other words, we need to impose stronger assumptions such as strict inequalities and strict properties of the rules to cancel out the indifference cases. **Indifferent cases** are the cases where there exists a coalition S and an agent outside of the coalition such that agent's utility is indifferent to being a member of a coalition or not. Once we achieve the uniqueness of the stable coalition, we will have the desired result; unique grand coalition. Because Theorem 1&2 together show that grand coalition can be achieved under certain assumptions. We have discussed the issues in Chapter 6, subcapter 6.0.3.

The next lemma will eliminate the dictatorship of the principal that is $\mu(u_0, \phi(u_S)) \neq 0$. And it will be necessary to prove *Theorem* 3.

**Lemma 3.** *If* $\mu$ *is strictly preference monotonic and Pareto efficient then*
$\mu(u_0, u_1) \neq 0 \quad \forall p_1 > 0$ *and for* $p_0 = 0$

*Proof.* Lets take any $u_1 \in \mathcal{U}$ such that $p_1 > 0$ and take $p_0 = 0$, so we have $p_0 < p_1$. Since $\mu$ is Pareto efficient, $\mu(u_0, u_0) = 0$. Since $\mu$ is strictly preference monotonic and $p_0 < p_1$ , $0 = \mu(u_0, u_0) < \mu(u_0, u_1)$. Hence $\mu(u_0, u_1) \neq 0 \quad \forall p_1 > 0$ and for $p_0 = 0$. □

Now I will provide some graphs about the numeric analysis of our representative coalition model. First I take bargaining rule $\mu(p_0, \phi(p_S)) := \frac{(p_0+\phi(p_S))}{2}$, and social welfare function $\frac{1}{|S|}\sum_{i\in S} p_i$ where $S \subseteq N := \{1,2,3\}$. The x-axis represents the peaks of agent 1 and the y-axis represents the peaks of agent 2, and the peak of agent 3 is equal to 1. Then I generate a MatLab code. To generate the code, first I defined the trivial condition which is $p_1 < p_2 < p_3 = 1$, then I defined the conditions of stability for each coalition. For example; if agent 1 prefers $\{1,3\}$ to $\{3\}$ and agent 2 prefers $\{1,3\}$ to $\{1,2,3\}$ and agent 3 prefers $\{1,3\}$ to $\{1\}$ then $\{1,3\}$ is the stable coalition. To draw the figures, I add satisfying points for each coalition on the same figure. The red area between the blue areas at the figure 4.1 represents the cases of unconnected coalitions.

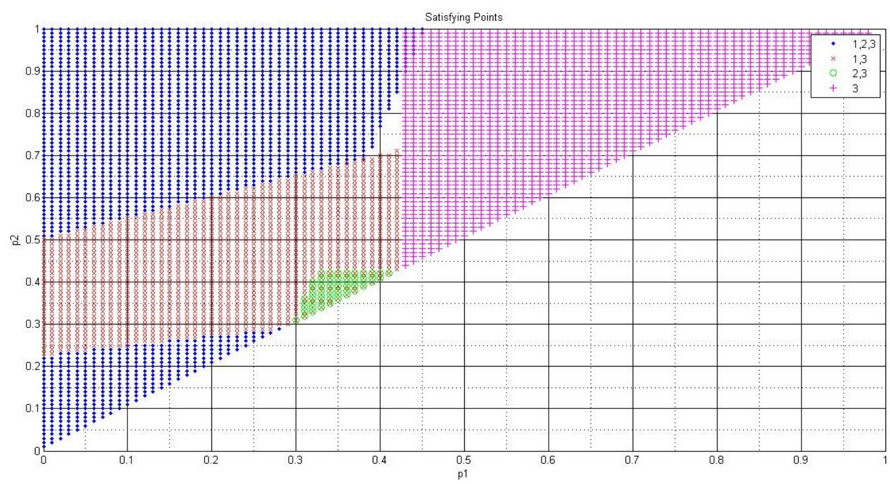



Figure 4.1: Two dimensional

The figure 4.2 represents a three dimensional example. The bargaining rule is $\mu(p_0, \phi(p_S)) := \frac{(p_0+\phi(p_S))}{2}$, and the social welfare function is $\frac{1}{|S|}\sum\limits_{i\in S} p_i$ where $S \subseteq N := \{1,2,3\}$. The coding is the same as in the previous figure except I do not take $p_3 = 1$. Here I change the trivial condition with $p_1 < p_2 < p_3$. And I add a z-axis which represents the peaks of agent 3. As in the previous figure, we again observe unconnected coalitions.

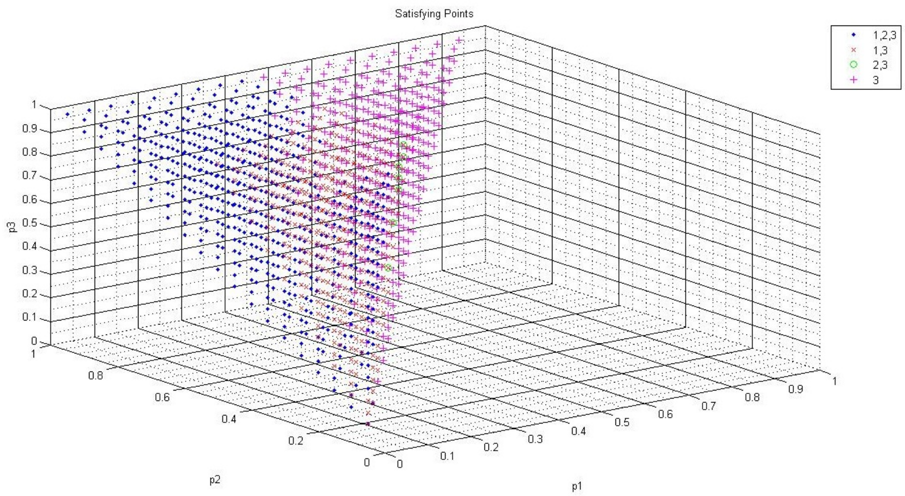



Figure 4.2: Three dimensional

Both figures provide unconnected coalitions, and give the indication of impossibility of connected coalition under certain assumptions. Because in both figures, there are preference profiles which only construct unconnected $\{1, 3\}$ coalition. Therefore if we want to reach an unconnected coalition, then we need to construct a preference profile that will give us unique unconnected stable coalition. We also need some restrictions over bargaining and social welfare functions.

**Theorem 3.** *Suppose that $N := \{1, 2, 3\}$. Let $\phi$ be a strictly population monotonic, Pareto efficient social welfare function, let $\mu$ be a Pareto efficient and strictly preference monotonic bargaining rule. If $\phi(u_{\{1,3\}}) \neq \mu(u_0, \phi(u_{\{1,3\}}))$, then there exists preference profile under which the unique stable coalition is unconnected.*

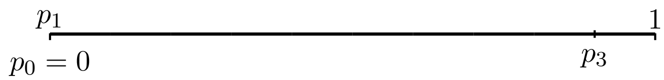
*Proof.* We are going to construct a $u \in \mathcal{U}^3$ that satisfies our claim. To construct a specific u, we need to define peaks for each agent that satisfies the properties;

$(i)$ Agent 2 prefers $\{1, 3\}$ to $\{1, 2, 3\}$

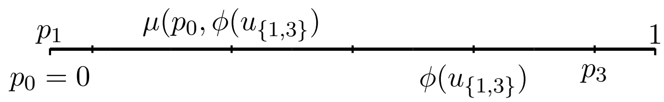
$(ii)$ Agent 1 prefers $\{1, 3\}$ to $\{3\}$

$(iii)$ Agent 3 prefers $\{1, 3\}$ to $\{1\}$

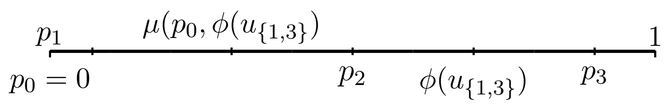
Lets start with agent 1 , and take $p_1 = 0$. We have located $p_1$, now we are going to locate $p_3$. Take any $p_3 \in [0, 1]$ such that $p_1 < p_3$. Now we have the case;



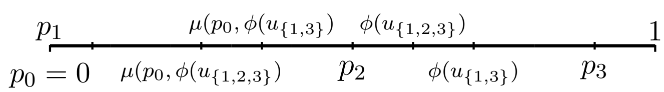

Since $\phi$ is strictly population monotonic, $\phi(u_{\{1,3\}}) > 0$. Since $\mu$ is strictly preference monotonic, Pareto efficient and $\phi(u_{\{1,3\}}) > 0$, by lemma 3
$\mu(p_0, \phi(u_{\{1,3\}}) > \mu(p_0, \phi(u_{\{1\}}) = 0$. Now we have $\phi(u_{\{1,3\}}) > 0$ and $\mu(p_0, \phi(u_{\{1,3\}}) > 0$. Since $\mu$ is Pareto optimal and $\phi(u_{\{1,3\}}) \neq \mu(u_0, \phi(u_{\{1,3\}}))$, $\mu(p_0, \phi(u_{\{1,3\}}) < \phi(u_{\{1,3\}})$.
So we have the case;




Now we need to locate the peak of agent 2. Take any $p_2 \in [\mu(p_0, \phi(u_{\{1,3\}}), \phi(u_{\{1,3\}}))$.




Since $\phi$ is strictly population monotonic, $\phi(u_{\{1,3\}}) > \phi(u_{\{1,2,3\}}) \geq p_2$. So from strict preference monotonicity of bargaining rule and $\phi(u_{\{1,3\}}) > \phi(u_{\{1,2,3\}})$ , it is clear that $\mu(p_0, \phi(u_{\{1,3\}}) > \mu(p_0, \phi(u_{\{1,2,3\}})$.




Hence $|\mu(p_0, \phi(u_{\{1,2,3\}}) - p_2| > |\mu(p_0, \phi(u_{\{1,3\}}) - p_2|$ which means agent 2 prefers $\{1,3\}$ to $\{1,2,3\}$. From Proposition 1, we know that agent 3 will join the coalition $\{1\}$ that is agent 3 prefers $\{1,3\}$ to $\{1\}$. Since $\phi$ is strictly population monotonic, $\phi(u_{\{1,3\}}) < \phi(u_{\{3\}})$. From strict preference monotonicity of bargaining rule and $\phi(u_{\{1,3\}}) < \phi(u_{\{3\}})$, $\mu(p_0, \phi(u_{\{1,3\}}) < \mu(p_0, \phi(u_{\{3\}})$. So agent 1 prefers $\{1,3\}$ to $\{3\}$. $\square$

### 4.0.2 Non-Representative Coalition

While as in the baseline model, only a single coalition can form, this coalition is not a representation of the agents who prefers not to join it. Instead each such agent individually bargain with the principal and receives the corresponding outcome. The coalition formation process similar to previous item; each agent declares whether she wants to be a member of the coalition or not.

**Theorem 4.** *If the bargaining rule $\mu$ is strict preference monotonic, Pareto efficient and the social welfare function $\phi$ is strict population monotonic, Pareto efficient and $\exists\ i \neq j \in N$ such that $p_i \neq p_j$ then each agent will bargain individually with the principal.*

*Proof.* Suppose there are $|N| = n$ agents and a principal, and suppose $\exists\ i \neq j \in N$ such that $p_i \neq p_j$. Since the social welfare function is strictly population monotonic, Pareto efficient and $\exists\ i \neq j \in N$ such that $p_i \neq p_j$, $\phi(u_{S\cup\{n\}}) < \phi(u_n), \quad \forall S \subseteq N$ with $n \notin S$ and $p_n$ is the largest peak. Since the bargaining rule is strictly preference monotonic, Pareto efficient and $\phi(u_{S\cup\{n\}}) < \phi(u_n)$, $\mu(p_0, \phi(u_{S\cup\{n\}})) < \mu(p_0, \phi(u_n)) < p_n, \quad \forall S \subseteq N$ with $n \notin S$ and $p_n$ is the largest peak. Hence $u_n(\mu(p_0, \phi(u_n))) > u_n(\mu(p_0, \phi(u_{S\cup\{n\}})))$ $\forall S \subseteq N$ with $n \notin S$ and $p_n$ is the largest peak. Therefore agent n will bargain individually. Since agent n will bargain individually, agent n-1 (agent with the second largest peak) will bargain individually too because of the same logic as agent n. By repeating this sequence n times, we can see that each agent will bargain individually. □

# Chapter 5

# Conclusions

The literature over cooperative bargaining and coalition formation consist of models designed on private goods. In this thesis, we focus on public good bargaining situations. And we ask the following questions :"What would be the interplay between bargaining and coalition formation? Under what conditions is the coalition of all agents, the grand coalition, stable?" Therefore the objectives of this study is mainly studying the interplay between the bargaining and coalition formation processes, and investigating the incentives that will lead to the grand coalition. Theorem 1 and Theorem 2 together shows that the grand coalition can be achieved, under certain assumptions over the bargaining rule and the social welfare function. In representative coalition model, Theorem 1 and Theorem 2 together shows that grand coalition can be constructed.

The third theorem is an impossibility theorem for connected coalitions. Theorem 3 shows the impossibility of grand coalition for $|N| = 3$. It's assumptions gives us the conditions for unconnected coalitions: Strict population monotonicity of the social welfare function $\phi$, and Pareto efficiency and strict preference monotonicity of the bargaining rule $\mu$, and $\phi(u_{\{1,3\}}) \neq \mu(u_0, \phi(u_{\{1,3\}}))$. Moreover this theorem indicates that conventional wisdom fails even under strong assumptions such as Pareto efficiency and monotonicity.

And Finally Theorem 4 shows that we can not produce a grand coalition if we allow agents a choice between individual and collective bargaining. whenever the bargaining rule $\mu$ is preference monotonic, Pareto efficient and the social welfare function $\phi$ is population monotonic, Pareto efficient.

Throughout the thesis, we see different types of social welfare functions and bargaining rules in the examples of Results Chapter and Appendix. Now we are going to analyze the properties of these rules. Population monotonicity was specifically defined for social welfare functions. If an agent decides to join a coalition S whenever the social welfare function is population monotonic, then the agent is better of by joining the coalition S. Hence population monotonicity categorizes social welfare functions with respect to agents preferences. Strictly population monotonicity definition eliminates some indifference cases which means an agent can be indifferent between being a coalition member or not. Since indifference eliminates uniqueness, we have defined the strict version.

Population monotonicity is applicable to bargaining rule, but if we suppose such a property then our model becomes trivial. If an agent will be better off by joining the coalition, then trivially she will join the coalition. Hence we need a different type of property. Preference monotonicity is a monotonicity definition which aims to preserve order. And the strict version is a standard strict monotonicity definition. Pareto efficiency is a standard property in bargaining literature. And it is applicable to both rules.

Now I am going to analyze each social welfare function and bargaining rule that has been used in this thesis. There are two tables in the next pages. Table 5.1 is designed to analyze the social welfare functions. And table 5.2 is designed to analyze the bargaining rules. In the first table, the first column is the column of social choice and in the second table, the first column is the column of bargaining rules. And rest of the columns are properties of these rules. The tables are rotated to fit the pages.

Table 5.1: Properties of Social Welfare Functions

| social welfare function ($\phi$) | Population Mon. | Strictly Population Mon. | Pareto E. |
|---|---|---|---|
| $\phi(u_S) = \frac{1}{\lvert S\rvert}\sum_{i\in S} p_i$ | Yes | Yes | Yes |
| $\phi(u_S) = \begin{cases} (\frac{\lvert S\rvert+1}{2})'th\ peak & if\ \lvert S\rvert\ is\ odd \\ (\frac{\frac{\lvert S\rvert}{2}'th\ peak + \frac{\lvert S\rvert+1}{2}'th\ peak}{2}) & if\ \lvert S\rvert\ is\ even \end{cases}$ | Yes | Yes | Yes |
| $\phi(u_S) = p_i\ where\ p_i \leq p_j\ for\ all\ j \in S$ | Yes | No | Yes |
| $\phi(u_S) = \frac{1}{\lvert S\rvert^2}\sum_{i\in S} p_i$ | No | No | No |
| $\phi(u_S) = \frac{1}{10\lvert S\rvert}\cdot\sum_{i\in S} p_i$ | Yes | Yes | Yes |
| $\phi(u_S) := \begin{cases} \phi(u_{\{1,2\}}) = 0.6 & if\ \lvert N\rvert = 3,\ \ p_1 = 0.2,\ p_2 = 0.5,\ p_3 = 1 \\ \phi(u_{\{1\}}) = 0.75 & if\ \lvert N\rvert = 3,\ \ p_1 = 0.2,\ p_2 = 0.5,\ p_3 = 1 \\ \phi(u_{\{2\}}) = 0.56 & if\ \lvert N\rvert = 3,\ \ p_1 = 0.2,\ p_2 = 0.5,\ p_3 = 1 \\ \frac{1}{\lvert S\rvert}\sum_{i\in S\subseteq N} p_i & otherwise \end{cases}$ | No | No | Yes |

Table 5.2: Properties of Bargaining Rules

| Bargaining Rule ($\mu$) | Preference Mon. | Strictly Preference Mon. | Pareto E. |
|---|---|---|---|
| $\mu(p_0, \phi(p_S)) := p_0$ | Yes | No | Yes |
| $\mu(p_0, \phi(p_S)) := (p_0 + \phi(p_S))$ | Yes | Yes | Yes |
| $\mu(u_0, \phi(u_S)) = \frac{\phi(S) + p_0}{2}$ | Yes | Yes | Yes |
| $\mu(p_0, \phi(p_S)) := \frac{(p_0+\phi(p_S))}{n}, (n \geq 1)$ | Yes | Yes | Yes |
| $\mu(p_0, \phi(p_S)) := (p_0 + \phi(p_S))/0.4$ | Yes | Yes | No |
| $\mu(p_0, \phi(p_S)) := \frac{(p_0+\phi(p_S))}{n}, (0 < n < 1)$ | Yes | Yes | No |
| $\mu(u_0, \phi(u_S)) = \begin{cases} 0.05 & if\ \phi(u_S) = 0.05 \\ 0.074 & if\ \phi(u_S) = 0.075 \\ 0.073 & if\ \phi(u_S) = 0.1 \\ 0 & otherwise \end{cases}$ | No | No | Yes |
| $\mu(p_0, \phi(p_S)) := \begin{cases} \phi(p_S) & if\ \phi(p_S) = 0.5 \\ \phi(p_S)/10 & otherwise \end{cases}$ | No | No | Yes |
| $\mu(p_0, \phi(p_S)) := \begin{cases} p_0 + \phi(p_S) & if\ \phi(p_S) < 0.4 \\ \phi(p_S)/2 & if\ 0.6 \neq \phi(p_S) \geq 0.4 \\ (\phi(p_S) + 2)/2 & if\ \phi(p_S) = 0.6 \end{cases}$ | Yes | Yes | No |
| $\mu(u_0, \phi(u_S)) = \frac{\phi(S) + p_0}{2} + 1$ | Yes | Yes | No |
| $\mu(u_0, \phi(u_S)) = \begin{cases} \phi(p_S)/10 & if\ \phi(u_S) \leq 0.9 \\ \phi(p_S) & otherwise \end{cases}$ | Yes | No | Yes |

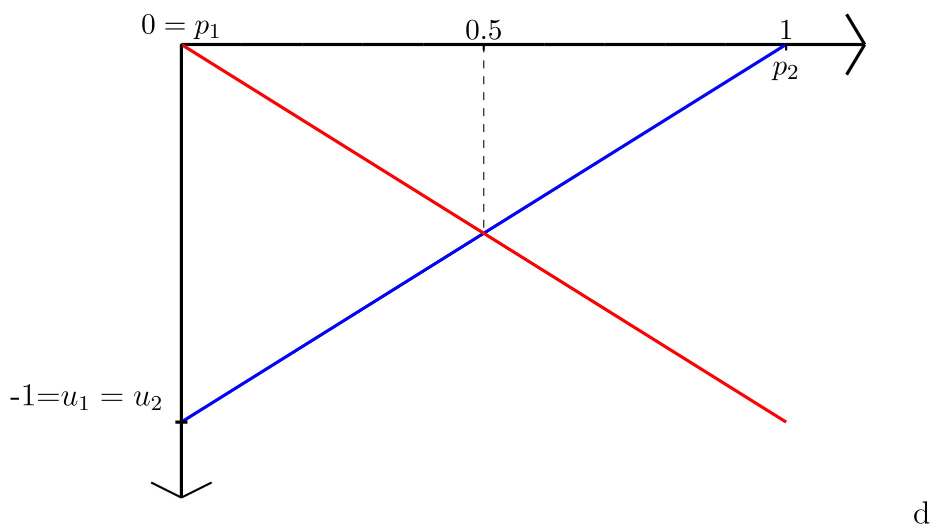
The most well known bargaining rule is **the Nash bargaining rule** which was proposed by Nash(1950) [21]. The Nash bargaining rule maximizes the product of each negotiators' utility gain with respect to their disagreement payoffs. Formally the Nash bargaining rule is $N : \mathcal{B} \to \mathcal{R}^N$ such that

$$N(S,d) := \underset{x \in I(S,d)}{\operatorname{argmax}} \prod_{i=1}^{n} (u_i(x) - u_i(d))$$

where $S \subset \mathcal{R}^N$ is the feasible payoff set, $d \in \mathcal{R}^N$ is the disagreement vector, $I(S,d) := \{x \in S \mid x \geq d\}$ (the individually rational set), and $\mathcal{B}$ is the set of $(S,d)$s or the set of bargaining problems $(S,d)$.

Consider a bargaining process of two agents with symmetric single peaked preferences as in the representative model in section 4.0.1 such that $p_1 = 0$ and $p_2 = 1$.


The utilities of the agents are $u_0(x) : [0,1] \to [0,1]$ such that $u_0(x) = -|x|$ and $u_1(x) : [0,1] \to [0,1]$ such that $u_1(x) = -|1-x|$ and $u_0(d) = u_1(d) = -1$. If agents face a disagreement then d is the bargaining outcome. So the utilities of the agents are;




The x-axis indicates the points agent 1 and 2 bargain about. The y-axis indicates the utility levels of bargainers for each point in [0,1]. Now we are going to determine the Nash solution with these assumptions.

$$\operatorname*{argmax}_{x\in I(S,d)} \prod_{i=0}^{1}(u_i(x)-u_i(d)) = \operatorname*{argmax}_{x\in I(S,d)}(u_0(x)-u_0(d))(u_1(x)-u_1(d))$$

By Lagrangian maximization we deduce that, $x = 0.5$. If we change $p_0$ and $p_1$ and do the same process, then we will observe that $x = \frac{p_0+p_1}{2}$. Hence in this thesis, the Nash bargaining rule is the bargaining rule $\mu$ such that $\mu(u_0, \phi(u_S)) = \frac{\phi(u_S)+u_0}{2}$.

One of the future work of this thesis can be changing the structure of the utilities or the preferences of the agents. Altering the agents' preferences to non-symmetric preferences may cause different formations of coalitions. What would be the properties that will form us the grand coalition in that environment? What would be the properties that will form us the unique unconnected coalition in that environment? The altering of the utility has different aspects. We can impose utility function with higher orders. Or we can impose a second dimension to our policy spectrum. Consider introducing money dimension to our policy spectrum. Now we have two dimensional spectrum. Each agent is endowed with some amount of money and they bargain over a tariff rate. The effect of money and tariff rate can be equal or different over agents' utilities.

Throughout the paper we assume that $p_0 = 0$ and the peaks of the agents' are greater then 0. We can alter this property by locating principal's peak ($p_0$) between agents' peak. For example, suppose that we have a society with three agents and a principal such as $p_1 \leq p_2 \leq p_0 \leq p_3$. Can we still achieve a grand coalition or an unconnected coalition?

We did not assume continuum of agents. We can observe that when we assume such a property, a single agent does not have an impact over bargaining outcome. And this assumption may change the results of the thesis.

# Chapter 6

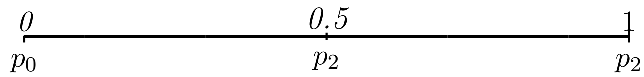
# Appendix

### 6.0.3 Discussion over Uniqueness and Grand Coalition

We did not impose strict property to preference monotonicity of bargaining rule and population monotonicity of social welfare function at Theorem 1 and Theorem 2. We proved more general cases of these theorems. But this generality also gives us indifferent cases. **Indifferent cases** are the cases where there exists a coalition S and an agent outside of the coalition such that agent's utility is indifferent to being a member of a coalition or not. Moreover, indifference cases gives us multiple stable coalitions. Imposing strict property to preference monotonicity of bargaining rule and population monotonicity of social welfare function eliminates some indifferent cases. We can still observe multiple stable coalitions even with strict preference monotonicity of the bargaining rule and strict population monotonicity of social welfare function. To examine the effects of strict properties, I will provide three examples. In the next three examples, for all $i \in N$, $u_i \in \mathcal{U}$ will be fixed that is peaks of the agents' will be fixed in all three examples. I will just change the bargaining and social welfare functions.

In the following example, I will provide an example such that the bargaining rule $\mu$ is strictly preference monotonic and the social welfare function is population monotonic but not strictly population monotonic.

**Example 13.** *Consider the case;*




*Let* $p_1 = 0.5,\ p_2 = 1.$ *Suppose that the social welfare function is* $\phi(u_S) = max\{p_i|\ i \in S\}$*, and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{2}$.

*Now let's check the possible coalitions and bargaining outcomes:*

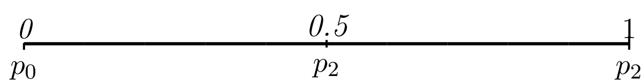
$$\begin{array}{lll} \textit{if } S = \{1,2\} & \textit{then} & \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 0.25 \\ \textit{if } S = \{1\} & \textit{then} & \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 0.5 \\ \textit{if } S = \{2\} & \textit{then} & \phi(u_S) = 1,\ \mu(u_0, \phi((p_2))) = 0.5 \end{array}$$

- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0,25 to 0,5 which is closer to agent 2's peak.*

*Hence* $\{1,2\}$*,* $\{2\}$ *are stable coalitions.*

In the following example, I will provide an example such that the bargaining rule $\mu$ is preference monotonic but not strictly preference monotonic and the social welfare function is strictly population monotonic.

**Example 14.** *Consider the case;*




*Let* $p_1 = 0.5,\ p_2 = 1.$ *Suppose that the social welfare function is* $\phi(u_S) = \frac{1}{|S|} \sum\limits_{i \in S \subseteq N} p_i$*, and the bargaining rule is* $\mu(u_0, \phi(u_S)) = p_0 = 0.$

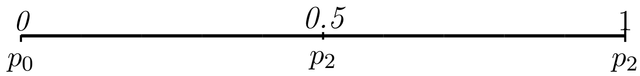
*Now let's check the possible coalitions and bargaining outcomes:*

$$\begin{array}{lll} \textit{if } S = \{1,2\} & \textit{then} & \phi(u_S) = 0.75,\ \mu(u_0, \phi(u_S)) = 0 \\ \textit{if } S = \{1\} & \textit{then} & \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 0 \\ \textit{if } S = \{2\} & \textit{then} & \phi(u_S) = 1,\ \mu(u_0, \phi((p_2))) = 0 \end{array}$$

*Hence all coalitions are stable coalitions.*

In the following example, I will provide an example such that the bargaining rule $\mu$ is preference monotonic and the social welfare function is population monotonic. I will drop the strict properties.

**Example 15.** *Consider the case;*




*Let* $p_1 = 0.5,\ p_2 = 1$. *Suppose that the social welfare function is* $\phi(u_S) = max\{p_i|\ i \in S\}$, *and the bargaining rule is* $\mu(u_0, \phi(u_S)) = p_0 = 0$.

*Now let's check the possible coalitions and bargaining outcomes:*

$$
\begin{aligned}
&\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi(u_S)) = 0 \\
&\textit{if } S = \{1\} \quad \textit{then} \quad \phi(u_S) = 0.5,\ \mu(u_0, \phi(u_S)) = 0 \\
&\textit{if } S = \{2\} \quad \textit{then} \quad \phi(u_S) = 1,\ \mu(u_0, \phi((p_2))) = 0
\end{aligned}
$$

*Hence all coalitions are stable coalitions.*

We see that there are indifference of utilities of the agents' in all the examples 13-14-15. As I mentioned before, we can still observe multiple stable coalitions even with strict preference monotonicity of the bargaining rule and strict population monotonicity of social welfare function. In the folloing example, I will provide a case shows that even strict preference monotonicity of the bargaining rule and strict population monotonicity of social welfare function is not enough for unique stable coalition happens to be grand.

**Example 16.** *Consider the case;*


*Let* $p_1 = 0.2,\ p_2 = 0.6,\ p_3 = 1$. *Suppose that the social welfare function is* $\frac{1}{|S|}\sum_{i \in S} p_i$, *and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{2}$.

*Now let's check the possible coalitions and outcomes of bargaining:*

$$
\begin{aligned}
&\textit{if } S = \{1,2,3\} \quad \textit{then} \quad \phi((p_1, p_2, p_3)) = 0.6 \textit{ and } \mu(p_0, \phi((p_1, p_2, p_3))) = 0.3 \\
&\textit{if } S = \{2,3\} \quad \textit{then} \quad \phi((p_2, p_3)) = 0.8 \textit{ and } \mu(p_0, \phi((p_2, p_3))) = 0.4 \\
&\textit{if } S = \{1,3\} \quad \textit{then} \quad \phi((p_1, p_3)) = 0.6 \textit{ and } \mu(p_0, \phi((p_1, p_3))) = 0.3 \\
&\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi((p_1, p_2)) = 0.4 \textit{ and } \mu(p_0, \phi((p_1, p_2))) = 0.2
\end{aligned}
$$

$$\begin{aligned} &\textit{if } S = \{1\} \quad \textit{then} \qquad \phi((p_1)) = 0.2 \textit{ and } \mu(p_0, \phi((p_1))) = 0.1 \\ &\textit{if } S = \{2\} \quad \textit{then} \qquad \phi((p_2)) = 0.6 \textit{ and } \mu(p_0, \phi((p_2))) = 0.3 \\ &\textit{if } S = \{3\} \quad \textit{then} \qquad \phi((p_3)) = 1 \textit{ and } \mu(p_0, \phi((p_3))) = 0.5 \end{aligned}$$

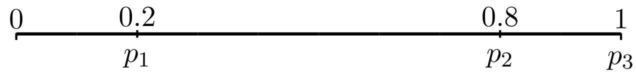
- $S = \{1, 2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0.2 to 0.3 which is closer to agent 3's peak.*
- $S = \{2, 3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.4 to 0.3 which is closer to agent 1's peak.*
- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0.1 to 0.2 which is closer to agent 2's peaks.*
- $S = \{2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0.3 to 0.4 which is closer to agent 3's peak.*
- $S = \{3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.5 to 0.3 which is closer to agent 1's peak.*

*There will be a deviation from each coalition except* $S = \{1, 3\}$, $S = \{1, 2, 3\}$.

$\star$ Examples 13-14-15-16 together show that strict population monotonicity of social welfare function and strict preference monotonicity of bargaining rule eliminates some indifference cases but not all of them. As I mentioned before, we need to impose some other properties. To discuss this issue, I will focus Theorem 1 and Theorem 2 properties seperately.

Theorem 1 has three assumptions; preference monotonicity and Pareto efficiency of bargaining rule, population monotonicity of social welfare function, and $p_i \geq \phi(u_S)$ where $i \notin S$. In Example 16, we observe that imposing strict property to preference monotonicity and population monotonicity is not enough to eliminate indifference cases. And we also observe that $0.6 = p_i \geq \phi(u_S) = 0.6$. Now I will give an example with $p_i > \phi(u_S)$ where $i \notin S$.

**Example 17.** *Consider the case;*




*Let* $p_1 = 0.2$, $p_2 = 0.8$, $p_3 = 1$. *Suppose that the social welfare function is* $\frac{1}{|S|}\sum_{i \in S} p_i$, *and the bargaining rule is* $\mu(u_0, \phi(u_S)) = \dfrac{\phi(S) + p_0}{2}$.

*Let's check the possible coalitions and outcomes of bargaining:*

$$\begin{aligned}
&\textit{if } S = \{1,2,3\} \quad \textit{then} \quad \phi((p_1,p_2,p_3)) = 0.\bar{6} \textit{ and } \mu(p_0, \phi((p_1,p_2,p_3))) = 0.\bar{3} \\
&\textit{if } S = \{2,3\} \quad \textit{then} \quad \phi((p_2,p_3)) = 0.9 \textit{ and } \mu(p_0, \phi((p_2,p_3))) = 0.45 \\
&\textit{if } S = \{1,3\} \quad \textit{then} \quad \phi((p_1,p_3)) = 0.6 \textit{ and } \mu(p_0, \phi((p_1,p_3))) = 0.3 \\
&\textit{if } S = \{1,2\} \quad \textit{then} \quad \phi((p_1,p_2)) = 0.5 \textit{ and } \mu(p_0, \phi((p_1,p_2))) = 0.25 \\
&\textit{if } S = \{1\} \quad \textit{then} \quad \phi((p_1)) = 0.2 \textit{ and } \mu(p_0, \phi((p_1))) = 0.1 \\
&\textit{if } S = \{2\} \quad \textit{then} \quad \phi((p_2)) = 0.8 \textit{ and } \mu(p_0, \phi((p_2))) = 0.4 \\
&\textit{if } S = \{3\} \quad \textit{then} \quad \phi((p_3)) = 1 \textit{ and } \mu(p_0, \phi((p_3))) = 0.5
\end{aligned}$$

- $S = \{1,3\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0.3 to* $0.\bar{3}$ *which is closer to agent 2's peak.*

- $S = \{1,2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0.25 to* $0.\bar{3}$ *which is closer to agent 3's peak.*

- $S = \{2,3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.45 to* $0.\bar{3}$ *which is closer to agent 1's peak.*

- $S = \{1\}$ *is not stable because agent 2 will join the coalition to move the bargaining outcome from 0.1 to 0.25 which is closer to agent 2's peaks.*

- $S = \{2\}$ *is not stable because agent 3 will join the coalition to move the bargaining outcome from 0.4 to 0.45 which is closer to agent 3's peak.*

- $S = \{3\}$ *is not stable because agent 1 will join the coalition to move the bargaining outcome from 0.5 to 0.3 which is closer to agent 1's peak.*

*There will be a deviation from each coalition except $S = \{1, 2, 3\}$.*

By imposing strict property, we only get rid off some cases that gives us multiple stable coalitions. To eliminate all the indifference cases, we need to impose more properties along with strict properties of population and preference monotonicity. In example 17, we observe no indifference cases when we impose $p_i > \phi(u_S)$ where $i \notin S$. Hence we reach a unique coalition. Therefore for the cases $p_i > \phi(u_S)$ where $i \notin S$ if bargaining rule is strictly preference monotonic, Pareto efficient, and social welfare function is strictly population monotonic, we reach a unique stable coalition. We also observe that the unique stable coalition is the grand coalition for example 17. To achieve unique stable coalition which is grand, we also need the assumptions of Theorem 2 because we can not know what would happen if $p_i < \phi(u_S)$ where $i \notin S$. Therefore, I will discuss the assumptions of Theorem 1 which covers the cases $p_i < \phi(u_S)$ where $i \notin S$. My aim is to understand the assumptions that lead us to the unique coalition happens to be grand.

Along with Theorem 1 assumptions, Theorem 2 has some other assumptions too: For $S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ and for $i \in N \setminus S$, $u_i \in \mathcal{U}$ that satisfies the sociall boundedness of $\mu$ by $\phi$, $\mu(p_0, \phi(u_S)) \geq p_i$, and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. And also remember, we should assume that bargaining rule is strictly preference monotonic, Pareto efficient, and social welfare function is strictly population monotonic to avoid some indifference cases (Check Examples 13-14-15-16). Now we have discussed the cases whenever $p_i \geq \phi(u_S)$ where $i \notin S$. So we are going to check the cases for $p_i < \phi(u_S)$ where $i \notin S$. Since $\phi$ is strictly population monotonic and $p_i < \phi(u_S)$ where $i \notin S$, $p_i < \phi(u_{S\cup\{i\}}) < \phi(u_S)$. Since $\mu$ is strictly preference monotonic and $p_i < \phi(u_{S\cup\{i\}}) < \phi(u_S)$, $\mu(p_0, \phi(u_{S\cup\{i\}})) < p_i$ whenever $\mu(p_0, \phi(u_S)) = p_i$. Hence agent i does not join the coalition S, so we have to assume $\mu(p_0, \phi(u_S)) > p_i$ where $i \notin S$. Now two property left; socially boundedness and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. We can not impose a strictness to social boundedness. So we need to investigate on $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. From social boundedness we know that there exists $S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ and there exists $i \in N \setminus S$, $u_i \in \mathcal{U}$ such that $|\phi(u_S) - \phi(u_{S\cup\{i\}})| \geq |\mu(u_0, \phi(u_S)) - \mu(u_0, \phi(u_{S\cup\{i\}}))|$, and for the same $S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ and $i \in N \setminus S$, $u_i \in \mathcal{U}$ we have $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$. Hence we have $|\mu(p_0, \phi(u_S)) - p_i| \geq |\mu(u_0, \phi(u_S)) - \mu(u_0, \phi(u_{S\cup\{i\}}))|$. This inequality can lead us indifference of agent i to be inside or outside of S. Since we impose $p_i < \mu(u_0, \phi(u_S))$, the indifference of agent i happens only whenever $|\mu(u_0, \phi(u_S)) - \mu(u_0, \phi(u_{S\cup\{i\}}))| = 0$. Since bargaining rule is strictly preference monotonic, Pareto efficient, and $\phi$ is strictly population monotonic and $p_i < \mu(u_0, \phi(u_S)) \leq \phi(u_S)$, $|\mu(u_0, \phi(u_S)) - \mu(u_0, \phi(u_{S\cup\{i\}}))| \neq 0$. Hence we eliminate all indifference cases. And if the bargaining rule is strictly preference monotonic, Pareto efficient, and the social welfare function is strictly population monotonic, and for $S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ and for $i \in N \setminus S$, $u_i \in \mathcal{U}$ such that $\mu$ is socially bounded by $\phi$, $\mu(p_0, \phi(u_S)) > p_i$, and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$ then agent i will better off by joining the coalition S.

To sum up, we have classified the assumptions to achieve a unique grand coalition:

- If $p_i > \phi(u_S)$ where $i \notin S$, then we need the assumptions strict preference monotonicity and Pareto efficiency of bargaining rule and strict population monotonicity of social welfare function.

- If $p_i < \phi(u_S)$ where $i \notin S$, then we need the assumptions strict preference monotonicity and Pareto efficiency of bargaining rule, and strict population monotonicity of social welfare function, and for $S \subseteq N$, $u_S \in \mathcal{U}^{|S|}$ with $i \in N \setminus S$ such that $\mu$ is socially bounded by $\phi$, $\mu(p_0, \phi(u_S)) > p_i$, and $|\mu(p_0, \phi(u_S)) - p_i| \geq |\phi(u_S) - \phi(u_{S\cup\{i\}})|$.

### 6.0.4 Matlab Code

I construct a numeric analysis for our representative model. As I mentioned before we can observe unconnected coalitions at the graphs. I both construct a 2 dimensional and 3 dimensional graphs in MatLab format. Here I will provide the codes;

**The code for 2-d**

```
pL1X = [];
pL1Y = [];
pL2X = [];
pL2Y = [];
pL3X = [];
pL3Y = [];
pL4X = [];
pL4Y = [];
pL5X = [];
pL5Y = [];
pL6X = [];
pL6Y = [];
pL7X = [];
pL7Y = [];
curInd1 = 1;
curInd2 = 1;
curInd3 = 1;
curInd4 = 1;
curInd5 = 1;
curInd6 = 1;
curInd7 = 1;

% Search the grid
for p1 = 0:0.01:1
   for p2 = 0:0.01:1

      % Other variables
      x1 = p1/2;
```

```
x2 = p2/2;
x3 = 0.5;
x4 = (p1+p2)/4;
x5 = (p1+1)/4;
x6 = (p2+1)/4;
x7 = (p1+p2+1)/6;

% Conditions for {1,2,3}

cond1 = p1 < p2;

% Agent 1 prefers {1,2,3} to {2,3}
cond2 = (abs(x7-p1) < abs(x6-p1));

% Agent 2 prefers {1,2,3} to {1,3}
cond3 = (abs(x7-p2) < abs(x5-p2));

% Agent 3 prefers {1,2,3} to {1,2}
cond4 = (abs(x7-1)  < abs(x4-1)) ;


if cond1 &&  cond2 && cond3 && cond4
        pL1X(curInd1) = p1;
        pL1Y(curInd1) = p2;
        curInd1 = curInd1+1;
end

% Conditions for {1,3}

cond1 = p1 < p2 && p2 < p3;
```

```
% Agent 1 prefers {1,3} to {3}
cond2 = (abs(x5-p1) < abs(x3-p1));

% Agent 2 prefers {1,3} to {1,2,3}
cond3 = (abs(x5-p2)-10^-15 <= abs(x7-p2));

% Agent 3 prefers {1,3} to {1}
cond4 = (abs(x5-1) < abs(x1-1));


    if cond1 &&  cond2 &&  cond3 &&  cond4
        pL2X(curInd2) = p1;
        pL2Y(curInd2) = p2;
        curInd2 = curInd2+1;
end

% Conditions for {2,3}

cond1 = p1 < p2;

% Agent 2 prefers {2,3} to {3}
cond2 = (abs(x6-p2) < abs(x3-p2));

% Agent 3 prefers {2,3} to {2}
cond3 = (abs(x6-1) < abs(x2-1));

% Agent 1 prefers {2,3} to {1,2,3}
cond4 = (abs(x6-p1) < abs(x7-p1));
```

```
if cond1 &&  cond2 && cond3 && cond4
        pL3X(curInd3) = p1;
        pL3Y(curInd3) = p2;
        curInd3 = curInd3+1;
end

% Conditions for {1,2}
cond1 = p1 < p2;

% Agent 1 prefers {1,2} to {2}
cond2 = (abs(x4-p1) < abs(x2-p1));

% Agent 2 prefers {1,2} to {1}
cond3 = (abs(x4-p2) < abs(x1-p2));

 % Agent 3 prefers {1,2} to {1,2,3}
cond4 = (abs(x4-1) < abs(x7-1));



if cond1 &&  cond2 && cond3 && cond4
        pL5X(curInd5) = p1;
        pL5Y(curInd5) = p2;
        curInd5 = curInd5+1;
end

% Conditions for {1}
cond1 = p1 < p2;

% Agent 2 prefers {1} to {1,2}
```

```
cond2 = (abs(x1-p2) < abs(x4-p2));

% Agent 3 prefers {1} to {1,3}
cond3 = (abs(x1-1) < abs(x5-1));



if cond1 &&  cond2 && cond3
        pL6X(curInd6) = p1;
        pL6Y(curInd6) = p2;
        curInd6 = curInd6+1;
end

% Conditions for {2}
cond1 = p1 < p2;

% Agent 1 prefers {2} to {1,2}
cond2 = (abs(x2-p1) < abs(x4-p1));

% Agent 3 prefers {2} to {2,3}
cond3 = (abs(x2-1) < abs(x6-1));

if cond1 &&  cond2 && cond3
        pL7X(curInd7) = p1;
        pL7Y(curInd7) = p2;
        curInd7 = curInd7+1;
end

% Conditions for {3}
cond1 = p1 < p2;
```

```
      % Agent 1 prefers {3} to {1,3}
      cond2 = (abs(x3-p1) < abs(x5-p1));

      % Agent 2 prefers {3} to {2,3}
      cond3 = (abs(x3-p2) < abs(x6-p2));

      if cond1 &&  cond2 && cond3
              pL4X(curInd4) = p1;
              pL4Y(curInd4) = p2;
              curInd4 = curInd4+1;
      end
    end
  end

% Plot the results
scatter(pL1X,pL1Y,'b.');
hold on;
scatter(pL2X,pL2Y,'rx');
scatter(pL3X,pL3Y,'go');
scatter(pL4X,pL4Y,'m+');
scatter(pL5X,pL5Y,'k*');
scatter(pL6X,pL6Y,'ys');
scatter(pL7X,pL7Y,'cd');

% Legends
legend('{1,2,3}','{1,3}','{2,3}','{3}','{1,2}','{1}','{2}');
xlabel('p1');
ylabel('p2');
title('Satisfying Points');
```

```
% Graph settings
xlim([0 1]);
ylim([0 1]);
grid on;
grid minor;
```

**The code for 3-d**

```
pL1X = [];
pL1Y = [];
pL1Z = [];
pL2X = [];
pL2Y = [];
pL2Z = [];
pL3X = [];
pL3Y = [];
pL3Z = [];
pL4X = [];
pL4Y = [];
pL4Z = [];
pL5X = [];
pL5Y = [];
pL5Z = [];
pL6X = [];
pL6Y = [];
pL6Z = [];
pL7X = [];
pL7Y = [];
pL7Z = [];
```

```
curInd1 = 1;
curInd2 = 1;
curInd3 = 1;
curInd4 = 1;
curInd5 = 1;
curInd6 = 1;
curInd7 = 1;
Search the grid
for p1 = 0:0.05:1
   for p2 = 0:0.05:1
      for p3 = 0:0.05:1

          % Other variables
          x1 = p1/2;
          x2 = p2/2;
          x3 = p3/2;
          x4 = (p1+p2)/4;
          x5 = (p1+p3)/4;
          x6 = (p2+p3)/4;
          x7 = (p1+p2+p3)/6;

          % Conditions for {1,2,3}

          cond1 = p1 < p2 && p2 < p3;

          % Agent 1 prefers {1,2,3} to {2,3}
          cond2 = (abs(x7-p1) < abs(x6-p1));

          % Agent 2 prefers {1,2,3} to {1,3}
          cond3 = (abs(x7-p2) < abs(x5-p2));
```

```
    % Agent 3 prefers {1,2,3} to {1,2}
    cond4 = (abs(x7-p3)  < abs(x4-p3)) ;


    if cond1 &&  cond2 && cond3 && cond4
            pL1X(curInd1) = p1;
            pL1Y(curInd1) = p2;
            pL1Z(curInd1) = p3;
            curInd1 = curInd1+1;
    end

% Conditions for {1,3}

cond1 = p1 < p2 && p2 < p3;

% Agent 1 prefers {1,3} to {3}
cond2 = (abs(x5-p1) < abs(x3-p1));

% Agent 2 prefers {1,3} to {1,2,3}
cond3 = (abs(x5-p2)-10^-15 <= abs(x7-p2));

% Agent 3 prefers {1,3} to {1}
cond4 = (abs(x5-p3) < abs(x1-p3));


    if cond1 &&  cond2 &&  cond3 &&  cond4
            pL2X(curInd2) = p1;
            pL2Y(curInd2) = p2;
            pL2Z(curInd2) = p3;
```

```
        curInd2 = curInd2+1;
end

% Conditions for {2,3}

cond1 = p1 < p2 && p2 < p3;

% Agent 2 prefers {2,3} to {3}
cond2 = (abs(x6-p2) < abs(x3-p2));

% Agent 3 prefers {2,3} to {2}
cond3 = (abs(x6-p3) < abs(x2-p3));

% Agent 1 prefers {2,3} to {1,2,3}
cond4 = (abs(x6-p1) < abs(x7-p1));


if cond1 &&  cond2 && cond3 && cond4
        pL3X(curInd3) = p1;
        pL3Y(curInd3) = p2;
        pL3Z(curInd3) = p3;
        curInd3 = curInd3+1;
end

% Conditions for {1,2}

cond1 = p1 < p2 && p2 < p3;

% Agent 1 prefers {1,2} to {2}
cond2 = (abs(x4-p1) < abs(x2-p1));
```

```
% Agent 2 prefers {1,2} to {1}
cond3 = (abs(x4-p2) < abs(x1-p2));

% Agent 3 prefers {1,2} to {1,2,3}
cond4 = (abs(x4-p3) < abs(x7-p3));


if cond1 &&  cond2 && cond3 && cond4
        pL5X(curInd5) = p1;
        pL5Y(curInd5) = p2;
        pL5Z(curInd5) = p3;
        curInd5 = curInd5+1;
end

% Conditions for {1}

cond1 = p1 < p2 && p2 < p3;

% Agent 2 prefers {1} to {1,2}
cond2 = (abs(x1-p2) < abs(x4-p2));

% Agent 3 prefers {1} to {1,3}
cond3 = (abs(x1-p3) < abs(x5-p3));


if cond1 &&  cond2 && cond3
        pL6X(curInd6) = p1;
        pL6Y(curInd6) = p2;
        pL6Z(curInd6) = p3;
```

```
            curInd6 = curInd6+1;
end

% Conditions for {2}

cond1 = p1 < p2 && p2 < p3;

% Agent 1 prefers {2} to {1,2}
cond2 = (abs(x2-p1) < abs(x4-p1));

% Agent 3 prefers {2} to {2,3}
cond3 = (abs(x2-p3) < abs(x6-p3));


if cond1 &&  cond2 && cond3
        pL7X(curInd7) = p1;
        pL7Y(curInd7) = p2;
        pL7Z(curInd7) = p3;
        curInd7 = curInd7+1;
end

% Conditions for {3}

cond1 = p1 < p2 && p2 < p3;

% Agent 1 prefers {3} to {1,3}
cond2 = (abs(x3-p1) < abs(x5-p1));

% Agent 2 prefers {3} to {2,3}
cond3 = (abs(x3-p2) < abs(x6-p2));
```

```
        if cond1 &&  cond2 && cond3
                pL4X(curInd4) = p1;
                pL4Y(curInd4) = p2;
                pL4Z(curInd4) = p3;
                curInd4 = curInd4+1;
        end
      end
  end
end

% Plot the results
scatter3(pL1X,pL1Y,pL1Z,'b.');
hold on;
scatter3(pL2X,pL2Y,pL2Z,'rx');
scatter3(pL3X,pL3Y,pL3Z,'go');
scatter3(pL4X,pL4Y,pL4Z,'m+');
scatter3(pL5X,pL5Y,pL5Z,'k*');
scatter3(pL6X,pL6Y,pL6Z,'ys');
scatter3(pL7X,pL7Y,pL7Z,'cd');

% Legends
legend('{1,2,3}','{1,3}','{2,3}','{3}','{1,2}','{1}','{2}');
xlabel('p1');
ylabel('p2');
zlabel('p3');
title('Satisfying Points');

% Graph settings
xlim([0 1]);
```

```
ylim([0 1]);
zlim([0 1]);
grid on;
grid minor;
```

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