A SEARCH BASED ANALYSIS OF DECISION MAKING IN SIMPLE ALLOCATION PROBLEMS

by

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ABSTRACT

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In this thesis, we focus on the analysis of rationality for simple allocation problems by interpreting solution rules as data on the choices of a policy maker. For an inventory of bankruptcy rules, we show that only constrained equal awards rule satisfies contraction independence. In addition, we weaken the rationality axiom and formulate the weak WARP (Weak Axiom of Revealed Preferences) property for simple allocation problems. We conclude that among a class of well-known solutions to simple allocation problems, the constrained equal awards rule uniquely satisfies contraction independence and weak WARP. In order to see the implications of existence of behavioral constraints in making choice, we next construct a search based model in which the decision maker has to engage in a dynamic search to adjudicate the conflicting claims and chose a division. Finally, we show that all allocation rules can be rationalized with this simple search model.

ÖZET

BASİT DAĞITIM PROBLEMLERİNDE KARAR VERME SÜRECİNİN ARAMA TEMELİNDE İNCELENMESİ

RABİA TELLİ

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Anahtar Kelimeler: Rasyonellik, Daralmadan Bağımsızlık, Açığa Çıkan Tercihlerin Zayıf Axiomu, Arama, Değerlendirme Seti

Bu tezde, basit dağıtım problemlerinin çözüm kurallarını, bir karar merciinin seçimleri ile ilgili veri olarak yorumlayıp bu basit dağıtım problemlerinin rasyonelliği üzerine odaklandık. İflas problemlerinin çözümü amacıyla kullanılan bir takım kurallar için, daralmadan bağımsızlık özelliğini, sadece sınırlandırılmış eşit ödüllendirme kuralının sağladığını gösterdik. Ek olarak, rasyonellik aksiyomunu zayıflattık ve açığa çıkan tercihlerin zayıf aksiyomu'nun daha zayıf bir versiyonunu, basit dağıtım problemleri için formüle ettik. Basit dağıtım problemlerinin çözümü için tanımlanmış, tanınmış çözümler arasında, sınırlandırılmış eşit ödüllendirme kuralının daralmadan bağımsızlık ve açığa çıkan tercihlerin zayıf aksiyomu'nun daha zayıf bir versiyonunu sağlayan tek kural olduğu sonucuna vardık. Seçim yapma sürecinde, davranışsal kısıtların varlığının olası sonuçlarını görmek amacıyla arama temelli bir model oluşturduk. Bu modelde, karar mercii, çakışan hak taleplerini karara bağlamak ve bir bölüşüm yapmak için dinamik bir arama sürecine girmek zorundadır. Son olarak, bu modelle açıklanan bütün dağıtım kurallarının rasyonel olduğunu gösterdik.

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TABLE OF CONTENTS

LIST OF FIGURES

1 Introduction

Revealed preference theory relies on the idea that the individual's choice behavior reveals underlying preferences that govern it. Most of the earlier work on revealed preference theory is related to the applications of these ideas to classical demand theory (e.g. see Samuelson, 1938, 1948). However, the concept of revealed preference is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash, 1950) characterize bargaining rules which can be "rationalized" as maximizing the underlying preferences of an impartial arbitrator (or, depending on the interpretation, a social welfare function of the bargainers) (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 1999; Sánchez, 2000).

Revealed preference theory assumes that the decision maker is maximizing her preferences and her choices are the result of this maximizing procedure. As a consequence, revealed preference theory represents the empirical content of rational decision making behavior. The theory offers a condition for the choices to be consistent with preference maximization. The Weak Axiom of Revealed Preferences (hereafter, WARP) is a necessary and sufficient condition for rationality. When the choice is single-valued, this relevant condition is called Property α in Sen (1971) in the context of consumer choice, and as independence of irrelevant alternatives in Nash (1950) in the context of bargaining. This condition requires that the chosen element from a set also be chosen from every subset that contains it. However, real life choice procedures often violate these conditions. Instead of classifying this kind of choice procedures as irrational, alternative explanations for rationality are proposed by recent research that incorporates the behavioral approach to the rational decision making theory. Rationality is redefined as optimal behavior within the additional constraints such as, loss aversion, endowment effect, limited attention, status quo bias and temptation. This idea is labeled as bounded rationality. As in the case of rationality, it can be applied to a wide range of decision problems. This observation will be the starting point of this thesis.

In this thesis, we analyze concepts of rationality on a class of simple allocation problems. The implications of full rationality on these problems has been previously analyzed by Kıbrıs (2012, 2013) who carries out a revealed preference analysis. A simple allocation problem for a society N is an $|N|+1$ dimensional nonnegative real vector $(c_1, ...c_{|N|}, E) \in \mathbb{R}^N_+$ satisfying $\sum_N c_i \geq E$ where E, the endowment has to be allocated among agents in N who are characterized by c, the characteristic vector. By interpreting an allocation rule on simple allocation problems as representing the choices of a decision maker (e.g.

a policy maker or a bankruptcy judge), Kıbrıs (2012, 2013) analyzes the conditions under which an allocation rule can be rationalized as maximizing a binary relation. It states that contraction independence property is equivalent to the rationality of a rule.

The main purpose of the allocation literature is to determine well-behaved rules for associating with each problem a division between the claimants of the amount available. Simple allocation problems have a wide range of applications. These are analyzed in detail in Kıbrıs (2012) and we will give them in the subsection 1.1. Even though relative importance of the rules depends on the application, there are several rules that are commonly used in practice or discussed in theoretical work. Throughout this paper, we analyze the rules used in the bankruptcy literature. In the section 2, we will present these rules that are compiled in Thomson (2003, 2012).

The thesis is organized as follows. In Section 3, we analyze rationality of the inventory of rules described in the section 2. As in Kıbrıs (2012, 2013) we assume that the allocation rule represents the choice of a decision maker. We make our analysis based on the contraction independence property. We identify the rules that violate this property by giving an example for two and three agent case. We show that the only rational rule is constrained equal awards rule (also known as the equal gains rule) for the reason that its operation principle is based on equal division and thus, is independent of the agents' characteristics values. It treats them as constraints in the application of this principle.

In Section 4, we weaken the rationality axiom and introduce a new property called weak WARP proposed by Manzini and Mariotti (2007). This property requires that if a decrease in the characteristics values does not change the initially chosen allocation, this allocation has to be chosen for the characteristic values between the initial and decreased characteristic values. This property captures the existence of menu dependence in a consistent manner. Change of choice set may lead to change of preferences. However, if the larger set does not contain any reason for the choice reversal, no smaller menu contains such a reason either. We check which one of the rules satisfies this weak rationality axiom.

In Section 5, we characterize a search model in which the choice process is generated by a time-continuous dynamic search. The decision maker looks through alternatives continuously and uses them to construct consideration sets. Then, she chooses the best alternative in order to maximize utility on the intersection of the choice set and the consideration set. Our main assumption is that the decision maker cannot consider all alternatives due to lack of information or unawareness. Therefore, she must actively search for alternatives. She starts the search with origin, at that point each claimant gets nothing and terminates search when the characteristic vector is considered. We identify preferences and search paths for most commonly used rules: Proportional rule, constrained equal awards rule, constrained equal losses rule and the Talmud rule. We then, show that every rule can be rationalizable by this search model.

1.1 Applications of Simple Allocation Problems

A simple allocation problem for a society N is an $|N| + 1$ dimensional nonnegative real vector $(c_1, ...c_{|N|}, E)$, which, with the exception of the last application below, is interpreted as follows. A social endowment E of a perfectly divisible commodity is to be allocated among members of N. Each agent $i \in N$ is characterized by an amount c_i of the commodity. Next, we present the alternative interpretations of c and E at various applications. These applications are discussed in detail in Kıbrıs (2012). Therefore, we present them as in that paper.

1. **Taxation:** A public authority is to collect an amount E of tax from a society N. Each agent i has income c_i. This is a central and very old problem in public finance. For example, see Edgeworth (1898) and the following literature. Young (1987) proposes a class of "parametric solutions" to this problem.

2. **Bankruptcy:** A bankruptcy judge is to allocate the remaining assets E of a bankrupt firm among its creditors, N. Each agent i has credited c_i to the bankrupt firm and now, claims this amount. For example, see O'Neill (1982) and the following literature. For a detailed review of the extensive literature on taxation and bankruptcy problems, see Thomson (2003 and 2007).

3. **Permit Allocation**: The Environmental Protection Agency is to allocate an amount E of pollution permits among firms in N (such as CO_2 emission permits allocated among energy producers). Each firm i, depending on its location, is imposed by the local authority an emission constraint c_i on its pollution level. For more on this application, see Kıbrıs (2003) and the literature cited therein.

4. **Single-peaked or Saturated Preferences**: A social planner is to allocate E units of a perfectly divisible commodity among members of N. Each agent i is known to have preferences with peak (saturation point) c_i. The rest of the preference information is disregarded as typical in several well-known solutions to

this problem, such as the Uniform rule or the Proportional rule. For example, see Sprumont (1991) and the following literature.

5. **Demand Rationing**: A supplier is to allocate its production E among demanders in N. Each demander i demands c_i units of the commodity. The supply-chain management literature contains detailed analysis of this problem. For example, see Cachon and Lariviere (1999) and the literature cited therein.

6. **Bargaining with Quasilinear Preferences and Claims**: An arbitrator is to allocate E units of a numeriare good among agents who have quasilinear preferences with respect to it. Each agent holds a claim c_i on what he should receive. For examples of bargaining problems with claims, see Chun and Thomson (1992) and the following literature. For bargaining problems with quasilinear preferences, see Moulin (1985) and the following literature.

7. **Surplus Sharing:** A social planner is to allocate the return E of a project among its investors in N. Each investor i has invested s_i. The project is profitable, that is, $\sum_N s_i \leq E$. Using the principal that no agent should receive less than his investment, define the maximal share of an agent i as $c_i = E - \sum_{N\setminus\{i\}} s_j$. Note that $\sum_N c_i \geq E$. The surplus sharing problem can now be analyzed as a simple allocation problem. For more on surplus-sharing problems, see Moulin (1985 and 1987) and the following literature.

8. **Consumer Choice under fixed prices and rationing**: A consumer has to allocate his income E among a set N of commodities. The prices of the commodities are fixed and thus, do not change from one problem to another. (With appropriate choice of consumption units, normalize the price vector so that all commodities have the same price.) As typical in the fixed-price literature, the consumer also faces "rationing constraints" on how much he can consume of each commodity. Let c_i be the agent's consumption constraint on commodity i. See Benassy, 1993 or Kıbrıs and Küçüksenel, 2008, for more on rationing rules.

2 Literature Review

Revealed preference theory is described firstly in Samuelson (1938). By using the "selected over" expression, he actually defined the well known Weak Axiom of Revealed

Preference. Samuelson's paper stimulated a significant amount of theoretical and empirical work and revealed preference literature has grown rapidly. Varian (2005) presents a detailed survey starting from Samuelson's seminal work. His paper offers an understanding about the development of the literature. However, in recent years, the standard theory of individual decision making evolved into an area which suggests that the study of choice procedures may yield better understanding of choice behavior. The experimental and theoretical work of Kahneman and Tversky (1979) led to take into consideration the behavioral analysis of economic decision making.

Experimental evidence discussed in Kahneman and Tversky (1979) shows that the individuals make their choices by taking into account the status quo option. By using the current option as a reference point, they determine their preferences. Thus, change of reference option leads to change of preferences. In light of this information, the paper tries to construct a reference dependent choice theory based on the reference dependence, loss aversion and diminishing sensitivity assumptions which are ignored by standard rational choice models.

Masatlıoğlu and Ok (2004) formulates a rational choice theory which allows for the presence of the status quo bias. Their axiomatic choice model incorporates the standard choice theory as a special case (the absence of a status quo). Their model allows for choice reversals conditional on default option in the sense that a status quo point may alter the individual's choices even if it is not chosen.

Dean, Kıbrıs and Masatlıoğlu (2015) develops a model that captures both status quo bias and limited attention phenomenon. They construct their model based on the following assumptions. First of all, decision maker has limited attention and status quo always receives attention. Moreover, status quo bias becomes more prevalent when the choice set expands. This pattern is called choice overload. Secondly, they assume that a status quo option may cause the decision maker to eliminate some alternatives by constructing some sort of consideration sets. They also provide experimental evidence to show that their assumptions are necessary to explain status quo biased choice behavior.

Manzini and Mariotti (2007) defines a procedure in which the decision maker uses sequentially two asymmetric binary relations (rationales) to account for cyclical choice patterns. The rationales are applied in a fixed order. While the first rational removes inferior alternatives, the second rational determines the chosen alternatives from the narrowed set. Their elimination approach can explain a limited form of menu dependence.

A different approach to explain the choice procedures that are inconsistent with standard choice theory is pursued in Masatlıoğlu and Nakajima (2012). They eliminate one of its main assumptions. They assume that decision maker may not evaluate simultaneously all alternatives in the choice set. As well as, they formulate a behavioral search model by using consideration sets. These sets evolve during the course of search and provide a dynamic decision procedure. The major novelty of the paper is the explicit formalization of the evolution of the consideration set over time.

Thomson (2003, 2013) provides a detailed review of the literature on taxation and bankruptcy problems. He presents the rules and their properties when the number of agent is fixed or varying; compares the rules on the basis of these properties. His surveys cover an axiomatic and game theoretic modeling of allocation problems and discuss experimental testing of the theory devoted to adjudicate the conflicting claims.

Kıbrıs (2012) analyzes rationality and transitive rationality notions for simple allocation problems. Rationality of a rule is about whether its choices can be modeled as maximization of a binary relation. That is, a rule is said to be rational (transitive rational) if its choices coincide with maximization of a (transitive) binary relation on the allocation space. He shows that rationalizability is equivalent to WARP. Additionally, contraction independence and WARP imply each other. Kıbrıs also introduces a weak rationality property which allows a rule to maximize a different binary relation for each characteristic vector. However, he shows that every rule satisfies weak rationality. In the same spirit, Kıbrıs (2013) characterizes a family of rational rules named recursive rules by using other well known axioms in the literature.

Stovall (2014) characterized the family of asymmetric parametric rules on the basis of the family of symmetric parametric rules (Young, 1987). In that paper, Young's characterization becomes a special case of the family of asymmetric parametric rules. Moreover, Stovall (2014) characterizes a family of rules which can be described in three different ways by imposing Independence of Irrelevant Alternatives, Consistency and Resource Monotonicity on the rules. He states that these three axioms characterize each of following families of rules, and thus these families are in fact one and the same. He refers to the first solution concept as monotone path rules that are identified with the path of awards for a given claims vector. He gives the name of claims independent parametric rules to second family of rules which are identified with a set of parametric functions. Collectively rational additively separable (CRAS) rules are the third one that is identified with an additively separable, strictly concave social welfare function.

3 Simple Allocation Problems

Let $N = \{1, ..., n\}$ be the set of agents. For $i \in N$, let e_i be the i^{th} unit vector in $\mathbb{R}^N_+$. Let $e = \sum_N e_i$. We use the vector inequalities, $\leqq, \leq, <$.[1].

A simple allocation problem for N is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$ (please see Figure 1). We call E the **endowment** and c the **characteristic vector**. As discussed at the end of Section 1; depending on the application, E can be an asset or a liability and c can be a vector of incomes, claims, demands, preference peaks, or consumption constraints. Let C be the set of all simple allocation problems for N. Given a simple allocation problem $(c, E) \in C$, let $X(c, E) = \{x \in \mathbb{R}^N_+ \mid x \leqq c$ and $\sum_N x_i \leqq E\}$ be **the choice set of** (c, E).

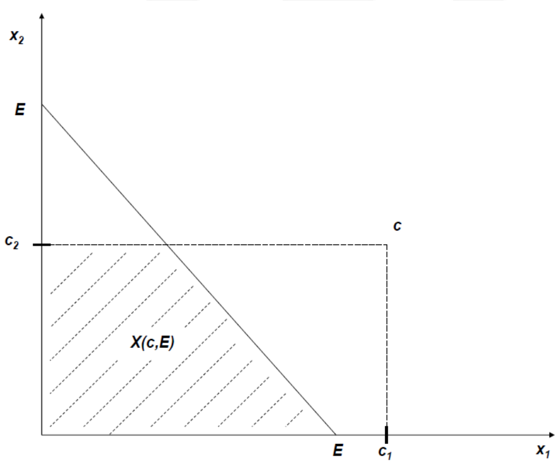

Figure 1: A two-agent simple allocation problem

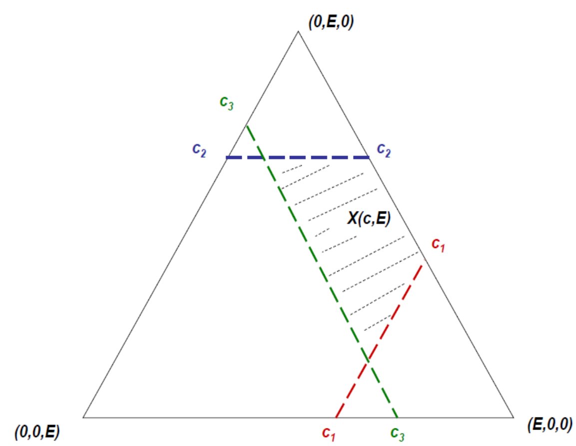

Figure 2: A three-agent simple allocation problem

An allocation rule $F : C \to \mathbb{R}^N_+$ assigns each simple allocation problem (c, E) to an allocation $F(c, E) \in X(c, E)$ such that $\sum_N F_i(c, E) = E$. Each rule F satisfies $F(c, E) \leqq c$ which, depending on the application, might be interpreted as a consumption constraint (as in permit allocation) or an efficiency requirement (as in single-peaked preferences). Also, $\sum_N F_i(c, E) = E$ can be interpreted as an efficiency property (as in permit allocation) or feasibility requirement (as in taxation). In consumer choice, this condition is equivalent to the Walras law.

$F(c, E)$ is called also an **awards vector for** (c, E). Given a claims vector, the graphical

[1] That is, $x \leqq y$ if and only if $x_i \leqq y_i$ for each $i \in N$; $x \leq y$ if and only if $x \leqq y$ and $x \neq y$; $x < y$ if and only if $x_i < y_i$ for each $i \in N$.

location of the awards vector chosen by a rule as the endowment varies from 0 to the sum of the claims $\sum c_i$, is the **path of awards of the rule** for the claims vector.

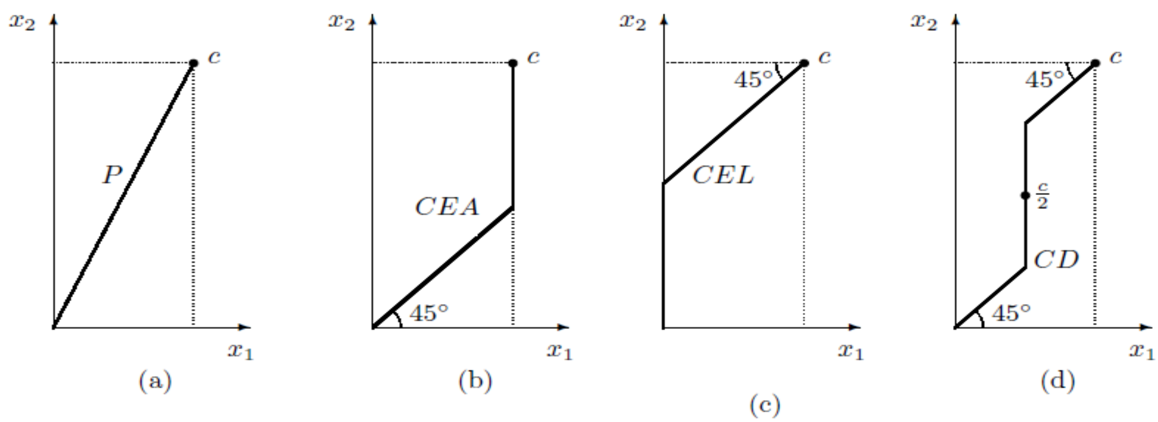

Figure 3: Paths of awards of four central rules for $N = \{1, 2\}$. (a) Proportional rule. (b) Constrained equal awards rule (c) Constrained equal losses rule. (d) Talmud Rule

3.1 An Inventory of Rules

We introduce an inventory of most commonly used rules in literature as defined in the surveys of Thomson (2003 & 2015). They are defined for a fixed N.

Proportional Rule: The proportional rule is one of the best known rules. It allocates the endowment proportional to the claims.

- For each $(c, E) \in \mathcal{C}$, $P(c, E) = \lambda c$ where λ is chosen so that $\sum \lambda c_i = E$.

Constrained Equal Awards Rule: This rule allocates equal amount to all claimants subject to no agent receiving more than his claim. It involves the idea of equality and favors the agents who have the smallest claims. Because of these properties, it has a central role in the literature.

- For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CEA_i = \min\{c_i, \lambda\}$, where λ is chosen so that $\sum \min\{c_i, \lambda\} = E$.

Piniles' Rule: It is based on a double application of the constrained equal awards rule. It uses the half-claims instead of the claims themselves.

- For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $Pin_i(c, E) = CEA_i(c/2, E)$ if $\sum(c_j/2) \geq E$, and $Pin_i(c, E) = c_i/2 + CEA_i(c/2, E - \sum(c_j/2))$ otherwise.

Constrained Egalitarian Rule: This rule recommends an egalitarian division concept and proposes the constrained equal awards rule for the half claims until the endowment reaches the sum of the half claims. When the endowment is more than the half sum of the claims, each agent receives the maximum of his half claim and $min\{c_i, \lambda\}$ where λ is set so that awards add up to E.

- For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CE_i(c, E) = min\{c_i/2, \lambda\}$ if $E \leq \sum(c_j/2)$ and $CE_i(c, E) = max\{c_i/2, min\{c_i, \lambda\}\}$ otherwise, where in each case, λ is chosen so that $\sum CE_i(c, E) = E$.

Constrained Equal Losses Rule: This rule proposes an awards vector which equalizes the losses imposed to the agents subject to no agent receiving a negative amount. In opposition to constrained equal awards rule, it favors the agents who have the largest claims.

- For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CEL_i(c, E) = max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum max\{0, c_i - \lambda\} = E$.

Concede-and-divide: This rule is defined only for the two-claimant case. It first assigns to each claimant the difference between the endowment and the other agent's claim (or 0 if this difference is negative), and divides the remainder equally.

- For $|N| = 2$. For each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$CD_i(c, E) = \max\{E - c_j,\, 0\} + \frac{E - \sum \max_N\{E - c_k,\, 0\}}{2}.$$

Talmud Rule: This rule is a mixture of constrained equal awards and constrained equal losses. For an endowment less than the half-sum of the claims, the constrained equal awards rule is applied; if there is more, the constrained equal losses rule is utilized.

- For each $(c, E) \in \mathcal{C}$ and each $i \in N$,

    1. If $\sum(c_i/2) \geq E$, then $T_i(c, E) = min\{c_i/2, \lambda\}$, where λ is chosen so that $\sum min\{c_i/2, \lambda\} = E$.

    2. If $\sum(c_i/2) \leq E$, then $T_i(c, E) = c_i - min\{c_i/2, \lambda\}$, where λ is chosen so that $\sum[c_i - min\{c_i/2, \lambda\}] = E$.

Random Arrival Rule: This rule uses the following pattern: It compensates fully each claimant with respect to the order of the claimant's arrival until the endowment runs out. All orders are given equal probabilities and the average of the awards vectors obtained by this pattern is taken to remove the unfairness associated with a particular order. In the following formal definition, Π^N refers to the class of bijections from N into itself.

- For each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$RA_i(c, E) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \min \left\{ c_i, \max \left\{ E - \sum_{j \in N,\ \pi(j) < \pi(i)} c_j,\ 0 \right\} \right\}.$$

ICI Family: This rule is described as in Thomson (2012).

- The ICI family (Thomson, 2000, 2008b) generalizes the Talmud rule. The pattern of distribution is the same but the definition allows the critical values of the endowment at which claimants come in and out of the distribution to differ from the half-claims, and moreover, to depend on the claims vector. To specify a rule in the family, we need lists $F \equiv (F_k)_{k=1}^{k=n-1}$and $G \equiv (G_k)_{k=1}^{k=n-1}$ (where $n \equiv |N|$) of functions from $\mathbb{R}_+^N$ to $\mathbb{R}_+$ such that for each pair k, $k' \in \{1, ..., n-1\}$ with $k < k'$, $F_{k\prime} \leq F_{k'}$ and $G_{k\prime} \leq G_k$. Let c $\in \mathbb{R}_+^N$ be given, and let E grow from 0 to $\sum c_i$. The distribution is as follows. The first units are divided equally until E reaches $F_1(c)$, at which point the smallest claimant drops out for a while. The next units are divided equally among the others until E reaches $F_2(c)$, at which point the second smallest claimant also drops out for a while. This goes on until E reaches $F_{n-1}(c)$, at which point only the largest claimant is left; he receives each additional unit until E reaches $G_{n-1}(c)$. The other claimants return for more, one at a time, in the reverse order of their departure. As E increases from $G_{n-1}(c)$ to $G_{n-2}(c)$, each increment is divided equally between the two largest claimants, and so on. The process continues until E reaches $G_1(c)$, at which point each increment is divided equally among all claimants, and until the end. To guarantee that then, each agent receives exactly his claim, the lists $F(c) \equiv (F_k(c))_{k=1}^{k=n-1}$ and $G(c) \equiv (G_k(c))_{k=1}^{k=n-1}$ have to satisfy certain linear relations, the ICI relations.

Parametric Rule of Representation: This rule is described as in Thomson (2012).

- Let Φ be the family of functions $f : \mathbb{R}_+ \times [\underline{\lambda}, \overline{\lambda}] \rightarrow \mathbb{R}_+$, where $-\infty \leq \underline{\lambda} \leq \overline{\lambda} \leq \infty$ that are continuous, nowhere decreasing with respect to their second argument, and such that for each $c_0 \in \mathbb{R}_+$, we have $f(c_0, \underline{\lambda}) = 0$ and $f(c_0, \overline{\lambda}) = c_0$. The parametric rule of representation $f \in \Phi$, $\mathcal{S}^f$, is defined as follows: for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\mathcal{S}^f(c, E)$ is the awards vector x such that for some $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, and for each $i \in N$, $x_i = f(c_i, \lambda)$.

4 Analysis of Rationality For Division Rules

An allocation rule on simple allocation problems can be interpreted as data on the choices of a decision maker. In that context, a rule can be qualified as rational if there is a binary relation defined on $\mathbb{R}_+^N$ for a given rule such that for each problem, the awards vector chosen by the rule is the unique maximizer of the relation over the choice set of the problem. Rationalizability is equivalent to the **Weak Axiom of Revealed Preference (WARP)** which can be equivalently stated as follows: for each pair $(c, E), (c', E) \in \mathcal{C}$, $F(c, E) \in X(c', E)$ and $F(c, E) \neq F(c', E)$ implies $F(c', E) \notin X(c, E)$. WARP requires the binary relation to be antisymmetric[2] (Kıbrıs, 2012).

The counterpart of WARP for allocation rules can be defined as follows: A rule F satisfies **contraction independence** if a chosen alternative from a set is still chosen from subsets (contractions) that contain it: For each pair $(c, E), (c', E) \in C$, $F(c, E) \in X(c', E) \subseteq X(c, E)$ implies $F(c', E) = F(c, E)$. For an allocation rule, WARP and contraction independence imply each other. As a result, rationality is equivalent to contraction independence. The following lemma provides a simple way of controlling whether a rule satisfies the contraction independence (Kıbrıs, 2012).

Lemma 1 *A rule F satisfies contraction independence if and only if for each (c, E), $(c', E) \in \mathcal{C}$ it satisfies the following properties*

[2] A binary relation B on $\mathbb{R}_+^N$ is antisymmetric if for each x, $y \in \mathbb{R}_+^N$, xBy and yBx imply $x = y$.

Property (i). if for each $i \in N$, $min\{c_i, E\} = min\{c'_i, E\}$, then $F(c,E) = F(c',E)$,

Property (ii). if $F(c,E) \leq c' \leq c$, then $F(c',E) = F(c,E)$.

Constrained equal awards rule is the only rule that satisfies contraction independence among the rules presented in the Section 3. The following proposition provides a general proof.

Proposition 2 *Constrained Equal Awards Rule satisfies contraction independence.*

Proof. For each $(c,E) \in \mathcal{C}$, $F_i(c,E) = \min\{c_i, \lambda(c,E)\}$, where $\lambda(c,E)$ is chosen so that $\sum \min\{c_i, \lambda(c,E)\} = E$. Let $(c,E),(c',E) \in \mathcal{C}$ be such that $F(c,E) \in X(c',E) \subseteq X(c,E)$. We want to show that $F(c,E) = F(c',E)$. If $X(c',E) \subseteq X(c,E)$, then for each $i \in N$, either $c'_i \leqq c_i$ or $min\{c'_i, E\} = min\{c_i, E\}$. If $F(c,E) \in X(c',E)$, then $F_i(c,E) \leqq c'_i$. Assume initially that for each $i \in N$, $F_i(c,E) \leqq c'_i \leqq c_i$.

Let $F(c,E)$ be such that $F_i(c,E) = c_i$ for all $i \in \{1,...k\}$ and $F_j(c,E) < c_j$ for all $j \in \{k+1,...,n\}$. Suppose $F(c',E) \neq F(c,E)$. Then, there exists $\{i,j\} \in N$ such that $F_i(c',E) < F_i(c,E)$ and $F_j(c',E) > F_j(c,E)$. Now, $j \notin \{1,...,k\}$ since otherwise, $F_j(c,E) = c_j = c'_j$. So $j \in \{k+1,...,n\}$. That is $F_j(c,E) < c_j$. Then, there exists two cases.

Case 1: $i \in \{1,...,k\}$. Then, $F_i(c',E) < F_i(c,E) = c_i = c'_i$ and $F_i(c',E) = \lambda(c',E)$.We know that $F_j(c',E) = \min\{c'_j, \lambda(c',E)\} \leq \lambda(c',E)$. However, we claim that $F_j(c',E) > F_j(c,E) = \lambda(c,E) > F_i(c,E) > F_i(c',E) = \lambda(c',E)$. Then, we obtain $F_j(c',E) > \lambda(c',E)$, a contradiction.

Case 2: $i \in \{k+1,...,n\}$. Then, $F_i(c,E) = F_j(c,E) = \lambda(c,E)$. We have $F_i(c',E) < F_i(c,E) \leq c'_i$ and $F_j(c',E) > F_j(c,E)$, $F_j(c,E) < c'_j$. $F_i(c',E) < c'_i$, then $F_i(c',E) = \lambda(c',E)$. Altogether, these imply $F_j(c',E) > F_j(c,E) = \lambda(c,E) = F_i(c,E) > F_i(c',E) = \lambda(c',E)$. Then, we obtain $F_j(c',E) > \lambda(c',E)$, a contradiction.

Secondly, for the case that $min\{c'_i, E\} = min\{c_i, E\}$, the proof follows the same construction. ■

In the following subsections, we focus on the rules other than the contrained equal awards rule for two and three agent case.

4.1 Two-Agent Problems

In this section, we show that the rules other than the constrained equal awards, all violate contraction independence and thus, are not rational for two-agent problems.

Proposition 3 *Proportional Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c = (10, 2)$, $c' = (9, 2)$ and $E = 10$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (\frac{25}{3}, \frac{5}{3}) \neq F(c', E) = (\frac{90}{11}, \frac{20}{11})$, violating contraction independence. ■

Proposition 4 *Piniles' Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c = (14, 6)$, $c' = (12, 6)$ and $E = 10$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (7, 3) \neq F(c', E) = (6.5, 3.5)$, violating contraction independence. ■

Proposition 5 *Constrained Egalitarian Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c = (14, 6)$, $c' = (12, 6)$ and $E = 10$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (7, 3) \neq F(c', E) = (6, 4)$, violating contraction independence. ■

Proposition 6 *Constrained Equal Losses Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c = (10, 2)$, $c' = (10, 1)$ and $E = 10$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (9, 1) \neq F(c', E) = (8.5, 1.5)$, violating contraction independence. ■

Proposition 7 *Concede and Divide violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c = (10, 2)$, $c' = (10, 1)$ and $E = 10$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (9, 1) \neq F(c', E) = (9.5, 0.5)$, violating contraction independence. ■

Proposition 8 *Talmud Rule violates contraction independence.*

Proof. For the two-claimant case, concede and divide rule delivers the numbers proposed by the Talmud. ■

Proposition 9 *Random Arrival Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c = (10, 2)$, $c' = (10, 1)$ and $E = 10$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (9, 1) \neq F(c', E) = (9.5, 0.5)$ violating contraction independence. ■

4.2 Three-Agent Problems

In this section, we show that the rules other than the constrained equal awards, all violate contraction independence and thus, are not rational for three-agent problems.

Proposition 10 *Proportional Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c = (10, 8, 6)$, $c' = (10, 7, 5)$ and $E = 20$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (\frac{25}{3}, \frac{20}{3}, 5) \neq F(c', E) = (\frac{100}{11}, \frac{70}{11}, \frac{50}{11})$ violating contraction independence. ■

Proposition 11 *Piniles' Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c = (10, 8, 6)$, $c' = (8, 8, 6)$ and $E = 20$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (\frac{23}{3}, \frac{20}{3}, \frac{17}{3}) \neq F(c', E) = (7, 7, 6)$ violating contraction independence. ■

Proposition 12 *Constrained Egalitarian Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c = (10, 8, 6)$, $c' = (6, 4, 4)$, and $E = 12$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (5, 4, 3) \neq F(c', E) = (4, 4, 4)$ violating contraction independence. ■

Proposition 13 *Constrained Equal Losses Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c = (10, 8, 8)$, $c' = (10, 7, 6)$ and $E = 20$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (8, 6, 6) \neq F(c', E) = (9, 6, 5)$ violating contraction independence. ■

Proposition 14 *Talmud Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c = (10, 8, 8)$, $c' = (10, 7, 6)$ and $E = 20$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (8, 6, 6) \neq F(c', E) = (9, 6, 5)$ violating contraction independence. ■

Proposition 15 *Random Arrival Rule violates contraction independence.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c = (10, 8, 8)$, $c' = (10, 7, 6)$ and $E = 20$. Since $c' \leq c$, we have $X(c', E) \subseteq X(c, E)$. Also $F(c, E) \in X(c', E)$. However, $F(c, E) = (8, 6, 6) \neq F(c', E) = (9, 6, 5)$ violating contraction independence. ■

5 Weak WARP

From the previous section, we conclude that majority of the well known division rules violate rationality axioms. To explain choice patterns that are inconsistent with full rationality, alternative motivation and procedures of choice are proposed. One of these alternative explanations is related to menu dependence. Experimental evidence shows that a decision maker's preferences may depend on the set she confronts. Therefore, cyclical patterns of choice can be described by this choice procedure. In this sense, Manzini and Mariotti (2007) proposed a property called weak WARP which allows menu dependence but requires some consistency between choices. Suppose that alternative x is chosen over y in a small set and in a larger set including this small set. This condition reveals that there is no reason for choice reversal between x and y. As a result, in a subset of the larger set which includes the small set, x has to be chosen over y.

In the same spirit, we formulate a similar property with Manzini and Mariotti (2007) and we call it again weak WARP to capture bounded rationality and examine whether the rules satisfy it. Accordingly, weak WARP can be stated as follows: for the pairs $(c^1, E), (c^2, E), (c^3, E) \in \mathcal{C}$, such that $c^1 \leq c^2 \leq c^3$ and $x \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$, if $x = F(c^1, E) = F(c^3, E)$ then $x = F(c^2, E)$.

In what follows we will show that constrained equal awards is the only rule that satisfies weak WARP.

Proposition 16 *Constrained Equal Awards Rule satisfies weak WARP.*

Proof. Since the Constrained Equal Awards rule satisfies the stronger Contraction Independence axiom, it also satisfies weak WARP. ■

5.1 Two-Agent Problems

In this section, we show that the rules other than the constrained equal awards, all violate weak WARP for two-agent problems.

Proposition 17 *Proportional Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c^3 = (10, 4)$, $c^2 = (9, 3.5)$, $c^1 = (8, 3.2)$ and $E = 10$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (7.14, 2.86) \neq F(c^2, E) = (7.2, 2.8)$ violating weak WARP. ■

Proposition 18 *Piniles' Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c^3 = (10, 4)$, $c^2 = (8, 4)$, $c^1 = (6.5, 3.5)$ and $E = 10$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (6.5, 3.5) \neq F(c^2, E) = (6, 4)$ violating weak WARP. ■

Proposition 19 *Constrained Egalitarian Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c^3 = (10, 4)$, $c^2 = (8, 3)$, $c^1 = (5, 2)$ and $E = 7$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (5, 2) \neq F(c^2, E) = (4, 3)$ violating weak WARP. ■

Proposition 20 *Constrained Equal Losses Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c^3 = (10, 4)$, $c^2 = (9, 4)$, $c^1 = (9, 3)$ and $E = 10$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (8, 2) \neq F(c^2, E) = (7.5, 2.5)$ violating weak WARP. ■

Proposition 21 *Concede and Divide violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c^3 = (10, 4)$, $c^2 = (10, 3)$, $c^1 = (9, 3)$ and $E = 10$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (8, 2) \neq F(c^2, E) = (8.5, 1.5)$ violating weak WARP. ■

Proposition 22 *Talmud Rule violates weak WARP.*

Proof. Example: for the two-claimant case, concede and divide rule delivers the numbers proposed by the Talmud. ■

Proposition 23 *Random Arrival Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2\}$, $c^3 = (10, 4)$, $c^2 = (10, 2)$, $c^1 = (8, 2)$ and $E = 10$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (8, 2) \neq F(c^2, E) = (9, 1)$ violating weak WARP. ■

5.2 Three-Agent Problems

In this section, we show that the rules other than the constrained equal awards, all violate weak WARP for three-agent problems.

Proposition 24 *Proportional Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1,2,3\}$, $c^3 = (10,8,6)$, $c^2 = (10,7,5)$, $c^1 = (\frac{25}{3}, \frac{20}{3}, 5)$ and $E = 20$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. Also $F(c^1,E) = F(c^3,E) \in X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. However, $F(c^1,E) = F(c^3,E) = (\frac{25}{3}, \frac{20}{3}, 5)) \neq F(c^2,E) = (\frac{100}{11}, \frac{70}{11}, \frac{50}{11})$ violating weak WARP. ■

Proposition 25 *Piniles' Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1,2,3\}$, $c^3 = (10,8,6)$, $c^2 = (8,8,6)$, $c^1 = (\frac{23}{3}, \frac{20}{3}, \frac{17}{3})$ and $E = 20$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. Also $F(c^1,E) = F(c^3,E) \in X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. However, $F(c^1,E) = F(c^3,E) = (\frac{23}{3}, \frac{20}{3}, \frac{17}{3}) \neq F(c^2,E) = (7,7,6)$ violating weak WARP. ■

Proposition 26 *Constrained Egalitarian Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1,2,3\}$, $c^3 = (10,8,6)$, $c^2 = (6,4,4)$, $c^1 = (5,4,3)$ and $E = 12$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. Also $F(c^1,E) = F(c^3,E) \in X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. However, $F(c^1,E) = F(c^3,E) = (5,4,3) \neq F(c^2,E) = (4,4,4)$ violating weak WARP. ■

Proposition 27 *Constrained Equal Losses Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1,2,3\}$, $c^3 = (10,8,8)$, $c^2 = (10,7,6)$, $c^1 = (8,6,6)$ and $E = 20$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. Also $F(c^1,E) = F(c^3,E) \in X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. However, $F(c^1,E) = F(c^3,E) = (8,6,6) \neq F(c^2,E) = (9,6,5)$ violating weak WARP. ■

Proposition 28 *Talmud Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1,2,3\}$, $c^3 = (10,8,8)$, $c^2 = (10,7,6)$, $c^1 = (8,6,6)$ and $E = 20$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. Also $F(c^1,E) = F(c^3,E) \in X(c^1,E) \subset X(c^2,E) \subset X(c^3,E)$. However, $F(c^1,E) = F(c^3,E) = (8,6,6) \neq F(c^2,E) = (9,6,5)$ violating weak WARP. ■

Proposition 29 *Random Arrival Rule violates weak WARP.*

Proof. Consider the following economy. Let $N = \{1, 2, 3\}$, $c^3 = (10, 8, 8)$, $c^2 = (10, 7, 6)$, $c^1 = (8, 6, 6)$ and $E = 20$. Since $c^1 \leq c^2 \leq c^3$, we have $X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. Also $F(c^1, E) = F(c^3, E) \in X(c^1, E) \subset X(c^2, E) \subset X(c^3, E)$. However, $F(c^1, E) = F(c^3, E) = (8, 6, 6) \neq F(c^2, E) = (9, 6, 5)$ violating weak WARP. ■

Before the presentation of our simple search model, we should mention that every distinct member of a family of rules may have different properties. In the same vein, ICI family (abbreviation of Increasing-Constant-Increasing expression) is a generalization of the Talmud rule and has infinitely many members. For this family, instead of the half-sum of the claims as in the Talmud rule, the points at which agents temporarily stop receiving additional units and the points at which they come back can be determined by the claims vector. The family contains constrained equal awards rule as well as the rules that violate contraction independence and weak WARP such as the constrained equal losses, and Talmud rules.

Parametric Rules also have infinitely many members and contains the rules that violate contraction independence and weak WARP such as the proportional, constrained equal losses, Talmud, and Piniles' rules. However, Stovall (2014) characterizes a sub-family of the asymmetric parametric rules. In that paper, this sub-family is formulated as follows: A parametric function is defined for each claimant and each parametric function depends only on a single parameter, in which it is weakly increasing. For any problem, each parametric function is truncated by the individual's claim, and a common parameter is found so that the sum of the truncated parametric functions evaluated at that parameter equals the endowment. These rules are called as claims independent parametric rules and satisfy the Independence of Irrelevant Alternatives (IIA) axiom.

6 Search Model for Simple Allocation Problems

Incomplete information about alternatives or lack of cognitive capacity may yield to failure in choosing the best available option. Hence, existence of such constraints is in contrast with the full rationality. In order to summarize choice behavior under these constraints, search based models are considered as an alternative theoretical framework. Since we think of a simple allocation problem as a choice problem, we design a simple search model in which the decision maker (bankruptcy judge or policy maker) makes a search to explore the choice set and to select a division.

Our choice process is generated by a time-continuous dynamic search. At time $t = 0$, no choice is to be made and each claimant gets nothing. For time $t > 0$, the decision maker searches continuously and constructs the consideration set with the options that are paid attention until time t. She stops searching when the characteristic vector is considered. At the end of the search, some considered options may not be available. Thus, the decision maker compares the alternatives in the intersection of the consideration set and the choice set. She reveals her final choice by maximizing a preference relation.

In our model, the search path reflects the consideration set formation process. The consideration set can be defined as the search history at time t. Because of the generality of our model, each rule performs adequately across the search model.

6.1 Model

Our search model consists of two components: a preference relation and the search path.

A preference relation denoted by $\succ$ is a strict order over the alternative space, $\mathbb{R}^N_+$ [3]. For a given (c, E), an alternative $x \in \mathbb{R}^N_+$ is $\succ$-best in $X(c, E)$, denoted $x = \arg\max_{\succ} X(c, E)$ if $x \succ y$ for each $y \in X(c, E)$.

The search path defines for a given $c \in \mathbb{R}^N_+$ the alternative that is considered at a particular time $t \in [0, 1]$ in the search process. The search starts with no division. Once the characteristic vector is considered, the decision maker finalizes the search process.

Definition 30 *A search path is a mapping $f : [0, 1] \times \mathbb{R}^N_+ \to \mathbb{R}^N_+$ such that for each $i \in N$ and $c \in \mathbb{R}^N_+$, $f_i(., c)$ is nondecreasing, $f(0, c) = 0$ and $f(1, c) = c$. At every time t, $f(t, c)$ represents the alternative considered at that time during the search process.*

The search history is the set of all alternatives that the decision maker have considered by the end of $t \in [0, 1]$. While search continues, the search history expands.

[3] A binary relation $\succ$ on $\mathbb{R}^N_+$ is a strict order over $\mathbb{R}^N_+$ if it is asymmetric ($x \succ y$ implies not $y \succ x$) and negatively transitive (not $x \succ y$ and not $y \succ z$ imply not $x \succ z$).

Definition 31 *The search history at time $t \in [0,1]$ and claims vector $c \in \mathbb{R}^N_+$ is* $A_{t(c)} = \bigcup_{s=0}^{t} f(s,c)$.

We can now state that an allocation rule F is consistent with the search model if the following definition is satisfied:

Definition 32 *F is rationalized by a search model if there is a search path f and a preference relation $\succ$ such that for each c, E, and $t^* = \frac{E}{\sum c_i}$, $F(c,E) = \arg\max_{\succ} A_{t(c)^*}$.*

This simple search model rationalizes all bankruptcy rules, therefore it provides a general framework about the search behavior of the decision maker for simple allocation problems. Because of its generality, it can be taken as a base model. For further studies, additional structure and restriction can be imposed in order to construct a more informative model. We now state our main result.

Proposition 33 *Any allocation rule can be rationalized with a $(\succ, f(t,c))$ pair.*

Proof. *Define $\succ$ as represented by the function $U(x) = \sum x_i$. Define f for all c and for all s as $f(s,c) = F(c, s\sum c_i)$. Pick any c, E. Let $t^* = \frac{E}{\sum c_i}$. We now want to show that $F(c,E) = \arg\max_{\succ} A_{t(c)^*}$. Let $F(c,E) = x^*$. By construction, $A_{t(c)^*} = \{x \in \mathbb{R}^N_+ \mid x = F(c,E') \text{ for all } 0 \leq E' \leq t^* \sum c_i\}$. This implies $x^* \in A_{t(c)^*}$. Suppose there is $y \in A_{t(c)^*}$ such that $y \succ x^*$. By construction of $\succ$, $\sum y_i > \sum x_i^* = E$. In that case $y \notin A_{t(c)^*}$, a contradiction.* ■

By using the search path construction used in the proof, we can explicitly define search paths for the commonly used rules.

- A Search Path for Proportional Rule: The most-known rule is the proportional rule. One of the search path of this rule can be defined as follows:

$$f^P(s,c) = s \cdot c \text{ where } s \in [0,1] \tag{1}$$

- A Search Path for Constrained Equal Awards Rule (f^{CEA}): An important way of selecting a division between the claimants is constrained equal awards rule. By defining its explicit search path, the alternative considered at a particular time in the search process will be apparent.

$$
\text{f(s,c)} = \begin{cases} (\frac{\lambda s}{n}, \frac{\lambda s}{n}, ..., \frac{\lambda s}{n}) & if \quad s \in [0, \frac{nc_1}{\lambda}] \\ (c_1, \frac{\lambda s - c_1}{n-1}, ..., \frac{\lambda s - c_1}{n-1}) & if \quad s \in [\frac{nc_1}{\lambda}, \frac{(n-1)c_2 + c_1}{\lambda}] \\ (c_1, c_2, \frac{\lambda s - (c_1+c_2)}{n-2}, ..., \frac{\lambda s - (c_1+c_2)}{n-2}) & if \quad s \in [\frac{(n-1)c_2 + c_1}{\lambda}, \frac{(n-2)c_3 + c_1 + c_2}{\lambda}] \\ \vdots & \vdots \quad \vdots \\ \begin{array}{l} (c_1, c_2, ..., c_{n-1}, \\ \lambda s - (c_1 + c_2 + ... + c_{n-1}) \end{array} & if \quad s \in [\frac{2c_{n-1} + c_1 + ... + c_{n-2}}{\lambda}, \frac{c_1 + c_2 + ... + c_{n-1} + c_n}{\lambda}] \end{cases} \tag{2}
$$

where $\lambda = \sum c_i$

- Search Path for Constrained Equal Losses Rule (f^{CEL}): This rule can be considered as a counter part of the constrained equals awards rule in terms of losses. The following formula gives us its explicit search path.

$$
\text{f(s,c)} = \begin{cases} (0, 0, ..., 0, \lambda s) & if \quad s \in [0, \frac{c_n - c_{n-1}}{\lambda}] \\ (0, ...0, \frac{\lambda s - (c_n - c_{n-1})}{2}, \frac{\lambda s + (c_n - c_{n-1})}{2}) & if \quad s \in [\frac{c_n - c_{n-1}}{\lambda}, \frac{c_n + c_{n-1} - 2c_{n-2}}{\lambda}] \\ \begin{array}{l} (0, ..., \frac{\lambda s - (2c_{n-1} - c_n - c_{n-2})}{3}, \\ \frac{\lambda s - (2c_n - c_{n-1} - c_{n-2})}{3}) \end{array} & if \quad s \in [\frac{c_n + c_{n-1} - 2c_{n-2}}{\lambda}, \frac{c_n + c_{n-1} + c_{n-2} - 3c_{n-3}}{\lambda}] \\ \vdots & \vdots \quad \vdots \\ \begin{array}{l} (\frac{\lambda s + ((n-1)c_1 - c_2 - ... - c_n)}{n}, ..., \\ \frac{\lambda s + ((n-1)c_n - c_1 - ... - c_{n-1})}{n}) \end{array} & if \quad s \in [\frac{c_n + c_{n-1} ... - (n-1)c_1}{\lambda}, \frac{c_1 + c_2 + ... + c_{n-1} + c_n}{\lambda}] \end{cases} \tag{3}
$$

where $\lambda = \sum c_i$

- A Search Path for Talmud Rule: The following function is a compact formulation of the search path of the Talmud Rule.

$$
f^T = \begin{cases} f^{CEA}(s, \frac{c}{2}) & if \quad 0 \leq s \leq 0.5 \\ f^{CEL}(s, \frac{c}{2}) & if \quad 0.5 \leq s \leq 1 \end{cases} \tag{4}
$$

7 Conclusion

In this study, we first focus on the analysis of rationality for simple allocation problems. We take the contraction independence property as equivalent to rationality. For an inventory of bankruptcy rules, we show that only constrained equal awards rule satisfies the contraction independence. We then weaken the rationality axiom and formulate the weak WARP property for simple allocation rules. We see that constrained equal awards rule uniquely satisfies the contraction independence and weak WARP.

In real life choice problems, all available alternatives may not be observed and evaluated fairly by the decision maker. As a result, people in general follow a search process in order to figure out complicated decision problems. Since we treat simple allocation problems as choice problems, we develop a simple search model in which the decision maker (or policy maker) has to engage in a dynamic search to adjudicate the conflicting claims. We show that all simple allocation rules can be rationalized with this simple search model. Therefore, our model is not falsifiable. On the other hand, even if being not falsifiable is a major limitation, it provides a general framework to summarize the behavior of the decision maker in simple allocation problems. Hence, our work can be used as a starting point. For further research, alternative special models can be formulated to analyze choices under different restrictions on consideration sets.

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