# EXTENSION OF THE LOGISTIC EQUATION WITH PIECEWISE CONSTANT ARGUMENTS AND POPULATION DYNAMICS

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# EXTENSION OF THE LOGISTIC EQUATION WITH PIECEWISE CONSTANT ARGUMENTS AND POPULATION DYNAMICS

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Approval of the Graduate School of Applied Mathematics

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## EXTENSION OF THE LOGISTIC EQUATION WITH PIECEWISE CONSTANT ARGUMENTS AND POPULATION DYNAMICS

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Population dynamics is the dominant branch of mathematical biology. The first model for population dynamics was developed by Thomas Malthus. A more complicated model was developed by Pierre François Verhulst and it is called the logistic equation. Our aim in this thesis is to extend the models using piecewise constant arguments and to find the conditions when the models have fixed points, periodic solutions and chaos with investigation of stability of periodic solutions.

Keywords: Differential Equations with Piecewise Constant Arguments, Population Dynamics, Periodicity, Stability, Chaos.

# ÖZ

## LOJİSTİK DENKLEMİN PARÇALI SABİT ARGÜMANLAR İLE GENİŞLETİLMESİ VE NÜFUS DİNAMİKLERİ

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Nüfus dinamikleri matematiksel biyolojinin baskın olan alanıdır. Nüfus dinamikleri için ilk model Thomas Malthus tarafından geliştirilmiştir. Daha karmaşık bir model ise Pierre François Verhulst tarafından geliştirilmiştir ve bu denklem lojistik denklem olarak adlandırılmıştır. Bu tezde amacımız modelleri parçalı sabit argümanlar kullanarak genişletmek ve periyodik çözümlerinin kararlılıklarını araştırarak modellerin sabit noktalara, periyodik çözümlere ve kaosa sahip olması için gerekli koşulları bulmaktır.

Anahtar Kelimeler: Parçalı Sabit Argümanlar ile Differensiyel Denklemler, Nüfus Dinamikleri, Periyodiklik, Kararlılık, Kaos.

To my family

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# CHAPTER 1

# INTRODUCTION

Mathematical modeling of population dynamics is one of the important parts of applied mathematics. Predictions about the outcome of the population model are made by solving different type of differential equations.

## 1.1 Biological Background

In this part of the study, we want to introduce simple biological models commonly used in modeling of population dynamics.

### 1.1.1 Single Species Models

This section contains two models for the growth of single species. First model was developed by ***Thomas Maltus*** (1766-1834), so it is called ***Malthusian Model***. Second model was suggested by ***Pierre François Verhulst*** (1804-1849) and it is called ***Verhulsts' Model***. For both models $N(t)$ indicates the size of population of species at time $t$. The derivative $\frac{dN}{dt}$ shows the rate of increase of the population at time $t$. The ratio $\frac{1}{N}\frac{dN}{dt}$, where $N > 0$, is the rate of increase of the population per individual. If deaths and births are considered, then the rate of increase of the population at time $t$ is [5]

$$\frac{dN}{dt} = births - deaths. \tag{1.1.1}$$

**A. Malthusian Model**

In ***Malthusian Model***, the births and deaths terms in (1.1.1) are proportional to $N$. The Malthus's equation is

$$\frac{dN}{dt} = bN - dN = kN, \tag{1.1.2}$$

where $b, d$ are positive constants and $k$ is a constant. If we take the initial population $N(0) = N_0$, one can easily solve (1.1.2) to obtain

$$N(t) = N_0 e^{(b-d)\,t}. \tag{1.1.3}$$

Thus if $b > d$ then the population grows exponentially. That is why this model can be called as an ***unlimited growth model***. In the real world, however, there are restrictions for population growth such as food, space, competition and other necessities of species. Therefore, ***limited growth models*** were suggested [1, 20, 22, 28].

**B.Verhulsts' Model**

The simplest model which takes these limitations into account was introduced by Verhulst in 1844. In Verhulsts' model, the population has a maximum size $K$. When the size of population approaches to $K$, then the ratio $\frac{1}{N}\frac{dN}{dt}$ approaches to 0 and when the size of population is close to 0 it approaches to a constant number $r > 0$. The equation suggested by Verhulst is

$$\frac{dN}{dt} = rN\,(1 - \frac{N}{K}), \tag{1.1.4}$$

where $r$ is a positive constant and $K$ is the ***carrying capacity*** of the enviroment. The equation (1.1.4) is called ***logistic equation***. If $N(0) = N_0$ is taken as an initial population, the solution of the logistic equation is

$$N(t) = \frac{N_0 K e^{r\,t}}{[K + N_0(e^{r\,t} - 1)]}. \tag{1.1.5}$$

Here, the size of population tends to $K$, when $t$ becomes too large. Therefore, Verhulsts' approach is more realistic than Malthuss' approach.

From the solution (1.1.5) it is not difficult to make predictions about the size of population at any time $t$ when the initial value $N_0$ is given [5].

## 1.2 Mathematical Background

In this part, we will review a few mathematical terms which are used in this study. Discrete-time equations, continuous-time equations, stability and fixed points of maps will be reviewed.

### 1.2.1 Discrete-Time Equations

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ be the sets of all integers, natural numbers without or with 0 and the set of real numbers, respectively. Denote by $\mathbb{R}^n$ the $n-$dimensional Euclidean space where $n \in \mathbb{N}$.

A discrete-time equation is expressed by

$$x_{k+1} = F(x_k) \tag{1.2.6}$$

for some map $F$, where $k \in \mathbb{N}_0$. Here, $x_k$ stands for the state of the system at a discrete time $t_k \in \mathbb{R}$.

The values of $x_k$ and $x_{k+1}$ are usually limited to be in the same set $S$. Therefore, the function $F$ has $S$ both domain and codomain. Then we can write $F : S \to S$.

$F$ is called the ***right-handside map*** for the discrete-time equation (1.2.6). Conversely, each map $F : S \to S$ is the right-handside map for the discrete-time equation (1.2.6). Hence, it is possible to say that there is a one-to-one correspondence between discrete-time equations and maps.

A solution (1.2.6) is a sequence

$$(x_0, x_1, x_2, ...), \tag{1.2.7}$$

of $S$, such that (1.2.6) is satisfied for $k = 0, 1, 2, ....$

If we write the equation (1.2.6) for these values of $k$, then we obtain the following equations:

$$\begin{aligned} x_1 &= F(x_0) \\ x_2 &= F(x_1) \\ x_3 &= F(x_2) \\ &\vdots \end{aligned} \tag{1.2.8}$$

Thus, (1.2.7) is the solution of (1.2.6) if and only if it satisfies all of these equations.

In order to solve (1.2.6), we choose $x_0 \in S$ and use (1.2.8) to obtain successively

$$x_1, x_2, x_3, \ldots. \tag{1.2.9}$$

The sequence (1.2.9) is obtained uniquely by $x_0$ and it is called the solution of (1.2.6) whose initial value is $x_0$.

The process of calculating the iterates using (1.2.8) is called ***iteration***. When expressed in terms of maps, we say that the sequence obtained by this process is called the sequence of ***iterates of*** $x_0$ ***under*** $F$.

The repeated application of the function $F$ can be expressed in terms of composition of functions. Since $F$ transforms $S$ into $S$, so does each of the maps

$F^2,\ F^3,\ F^4 ...$, where

$$\begin{aligned} F^2 &= F \circ F, \\ F^3 &= F \circ F \circ F, \\ F^4 &= F \circ F \circ F \circ F, \\ &\vdots \end{aligned}$$

By using this notation (1.2.8) can be written as

$$\begin{aligned} x_1 &= F(x_0), \\ x_2 &= F(x_1) = F(F(x_0)) = F^2(x_0), \\ x_3 &= F(x_2) = F(F(F(x_0))) = F^3(x_0), \\ &\vdots \end{aligned} \tag{1.2.10}$$

Generally $F^n : S \to S$ is defined by

$$F^n = \underbrace{F \circ F \circ F \circ F \circ ... \circ F}_{\text{n copies of F}}$$

where $n \in \mathbb{N}$. $F^n$ is called ***the nth iterate of*** $F$. It is also possible to call it the ***nth power of*** $F$ ***under composition***. It is easy to see that $F^{n+1} = F \circ F^n$ and $x_n = F^n(x_0)$, $n \in \mathbb{N}$ (see [5] for details).

The set $(x_0, x_1, x_2, \ldots)$ is called the (*forward*) ***orbit*** of $x_0$ [15].

An orbit is said to be periodic of period $p \geq 1$ if $x_p = x_0$ and the smallest $p$ is called the ***period of the orbit*** [21].

### 1.2.2 Continuous-Time Equations

A continuous-time equation is expressed by

$$\dot{x} = G(x, t, \mu) \tag{1.2.11}$$

with $x \in \mathbb{U} \subset \mathbb{R}^n$, $t \in \mathbb{R}$, $\mu \in \mathbb{R}^p$ and $p, n \in \mathbb{N}$. Here, $\mu$ is a parameter and $\dot{x}$ is the derivative of $x$ with respect to $t$.

The solution of (1.2.11) is a map, $x$, from some interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$, which can be represented as follows

$$\begin{aligned} x : I &\rightarrow \mathbb{R}^n \\ t &\mapsto x(t) \end{aligned}$$

such that $\dot{x}(t) = F(x(t), t, \mu)$ [29].

### 1.2.3 Stability

**Discrete-Time Equations**

***Lyapunov Stability***

A solution, $x_k$, of (1.2.6) is said to be Lyapunov stable (or stable) if, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, m) > 0$ such that, for any other solution, $y_k$, of (1.2.6) satisfying $||x_m - y_m|| < \delta$ then $||x_k - y_k|| < \varepsilon$ for $k > m$ (here, $k$, $m$ are positive integers and $||\cdot||$ is a norm in $S$).

A solution that is not stable is said to be ***unstable***.

***Asymptotic Stability***

A solution, $x_k$, of (1.2.6) is said to be asymptotically stable if it is Lyapunov stable and there exists a constant $\Delta > 0$ such that, for any other solution, $y_k$, of (1.2.6) if $||x_m - y_m|| < \Delta$, then $\lim_{k \longrightarrow \infty} ||x_k - y_k|| = 0$ (here, $k$, $m$ are positive integers and $||\cdot||$ is the norm in $S$).

**Continous-Time Equations**

***Lyapunov Stability***

A solution, $\bar{x}(t)$, of (1.2.11) is said to be Lyapunov stable (or stable) if, given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon,\, t_0) > 0$ such that, for any other solution, $y(t)$, of (1.2.11) satisfying $||\bar{x}(t_0) - y(t_0)|| < \delta$, then $||\bar{x}(t) - y(t)|| < \varepsilon$ for $t > t_0$, $t_0 \in \mathbb{R}$ (here $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$).

A solution that is not stable is said to be ***unstable***.

***Asymptotic Stability***

Let $\bar{x}(t)$ be any solution of (1.2.11). $\bar{x}(t)$ is said to be asymptotically stable if it is Lyapunov stable and there exists a constant $\Delta > 0$ such that, for any other solution, $y(t)$, of (1.2.11) if $||\bar{x}(t_0) - y(t_0)|| < \Delta$, then $\lim_{t \longrightarrow \infty} ||x - y|| = 0$, $t_0 \in \mathbb{R}$ (here $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$) [29].

### 1.2.4 Fixed Points of Maps

Let $S$ be a set and $F : S \to S$ be a map. An element $x \in S$ is a fixed point of $F$ if it satisfies the following condition

$$x = F(x). \tag{1.2.12}$$

In other words if $x$ is a fixed point of $F$ then successive applications of $F$ send $x$ to itself.

In terms of discrete-time equations: $x$ ***is a fixed point for*** $F$ ***if and only if*** $x$ ***is the initial value for a constant solution of the discrete-time equation*** $x_{k+1} = F(x_k)$ [5].

Fixed points of $F$ can be obtained by using elementary algebra. But if $F(x)$ is not simple enough, then one can use graphical methods to find the fixed points of $F$. Graphical methods are more generally applicable when $S \subseteq \mathbb{R}$. A function $F$ has a fixed point at $x$ if and only if the graphs of $F$ and the line $y = x$ intersect on the vertical line through the point $x$.

An element $x \in S$ is said to be ***periodic point*** of $F$ if $F^n(x) = x$, for a fixed $n \in \mathbb{N}$. Here, $n$ is called ***a period*** of the point $x$. A point of period $n$ is said to be a ***period-n point***. In other words, a period-$n$ point of $F$ is a fixed point of $F^n$. It is trivial that a period-1 point of $F$ is a fixed point of $F$.

**Definition 1.1.** Let $F : S \to S$ where $S \subseteq \mathbb{R}$, $x \in S$ be a fixed point of $F$ and $F$ be differentiable at $x$. Then, $x$ is called:

- an attractor (or an attracting fixed point) if $|F^{'}(x)| < 1$,

- a repellor (or a repelling fixed point) if $|F^{'}(x)| > 1$,
- an indifferent fixed point if $|F^{'}(x)| = 1$.

The number $F^{'}(x)$ is said to be the ***multiplier*** of $x$.

Indifferent fixed points can have attracting behavior, repelling behavior or both of them. An attractor fixed point is said to be asymptotically stable, and a repellor fixed point is said to be unstable [5].

**Example 1.2.1.** Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x\, e^{a-b\,x}$ where $a,\, b \in \mathbb{R}$. Find the conditions on the parameter $a$ under which the fixed point 0 is an attractor, a repellor and an indifferent point.

**Solution:** Since $f'(x) = (1 - b\,x)\, e^{a-b\,x}$, then $f'(0) = e^{a}$. If $a < 0$, then $f'(0) < 1$; so $x = 0$ is an attractor. If $a > 0$, then $f'(0) > 1$; so $x = 0$ is a repellor. Finally, when $a = 0$, then it is an indifferent fixed point.

The following theorem describes the behaviour of a map near the fixed point.

**Theorem 1.1.** Let $F : S \to S$ where $S$ is an interval. Assume that $x \in S$ is a fixed point of $F$ and $F$ is differentiable at $x$ with $F'(x) \neq 1$.

- If $F'(x) < 1$, then there exists an open interval $I \subset S$ containing $x$ such that for every $x_0 \in I$, $F^n(x_0)$ $(n \in \mathbb{N})$ converges to $x$,
- if $F'(x) > 1$, then there exists an open interval $I \subset S$ containing $x$ such that all points in the interval that are not equal to $x$ must leave the interval under $F$ [16].

### 1.2.5 Lambert W Function

The inverse function of $w \mapsto \ w\, e^{w} = z$ is called the ***Lambert W*** function. This function $W(z)$, which verifies $W(z)\, e^{W(z)} = z$, has the following properties [8]:

- if $z < \frac{-1}{e}$, then $W(z)$ is multivalued which are complex;
- if $\frac{-1}{e} \leq z < 0$, there are two roots of $W(z)$ one of them is smaller than $-1$ while the other is bigger than $-1$;
- if $z \geq 0$, then there is a single real value for $W(z)$.

### 1.2.6 Chaos

There are different definitions of chaos but in this study we want to give the definition which was suggested by Robert L. Devaney [11]

Let $V$ be a set, $f : V \to V$ is said to be ***chaotic*** on $V$ if

- $f$ has sensitive dependence on initial conditions;
- $f$ has topologically transitive;
- periodic points are dense in $V$.

What are the meaning of these properties?

A map $f : V \to V$ has ***sensitive dependence*** on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood $N$ of $x$, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| \geq \delta$.

A map $f : V \to V$ is called ***topologically transitive*** if for any pair of open sets $W \subset V, J \subset V$ there exists $n > 0$ such that $f^n(J) \cap W \neq \emptyset$.

Lastly, let $V$ be a set and $J$ be a subset of $V$. Then $J$ is ***dense*** in $V$ if every points in $V$ is an accumulation point of $J$, a point of $J$ or both [16].

# CHAPTER 2

# DEVELOPMENT OF A SINGLE SPECIES POPULATION MODEL

It was in 1798 when T. Maltus introduced the first population model for single species. This model

$$\frac{dN}{dt} = mN \tag{2.0.1}$$

is called ***Malthusian Model***. The coefficient $m$ is a positive constant and $N(t)$ is the size of population at time $t$. The solutions of (2.0.1) increase exponentially. Therefore, this model is said to be ***unlimited growth model***. However, the restrictions in the real world do not allow such kind of an unlimited growth model, so ***limited growth models*** were suggested. To obtain limited growth models, the following modifications can be performed [1, 22]:

(A) : changing $m$ with a coefficient which is dependent to $x(t)$,

(B) : introducing a deviation on $t$.

We will develop both approaches in our study. In this study, we consider $x(t) = N(t) - k$, where $N(t)$ is the size of population at time $t$ and $k$ is a positive constant [1].

The following figure shows all extensions of Malthusian model which will be studied in this thesis.

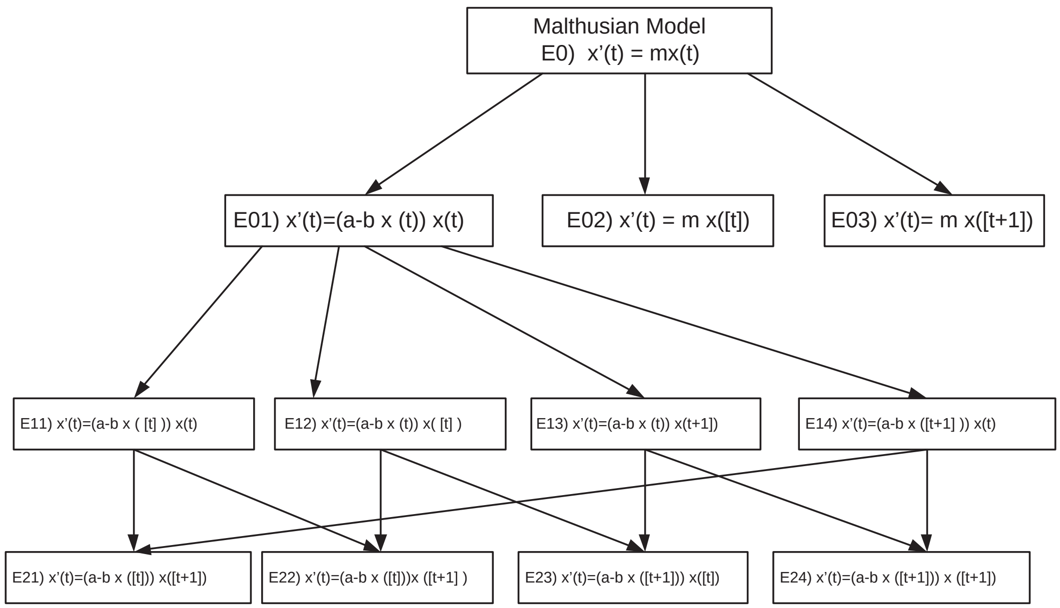


Figure 2.1: Extensions of the Malthusian model which will be studied this thesis.

## 2.1 1-Periodic Points of The Extensions of The Malthusian Model

### 2.1.1 Equation E01

The most important suggestion regarding the first approach, (A), was changing the coefficient as follows [28]:

$$\frac{dx}{dt} = (a - bx(t))\, x(t). \tag{2.1.2}$$

If $a = r$, $b = \frac{r}{K}$, where $r$ is a positive constant and $K$ is the carrying capacity of the environment, then one obtains Verhulst equation (1.1.4).

### 2.1.2 Equation E02

Regarding the second approach, (B), was introducing a deviation from $t$ in the following way

$$\frac{dx}{dt} = m\, x([t]), \tag{2.1.3}$$

where [.] denotes the greatest integer function and $m$ is a constant. This kind of equation has been considered in many papers [1, 9]. Let us use the notation $x_k = x(k)$ $(k \in N_0)$. For $t \in [0, 1)$, (2.1.3) has the form

$$\frac{dx}{dt} = m\, x_0;$$

then,

$$x(t) = x_0 + m\, x_0\, t,$$

so $x_1 = x_0\,(1+m)$. Consider $t \in [1,2)$, it is easily seen that (2.1.3) takes the following form

$$\frac{dx}{dt} = m\, x_1$$

and, hence,

$$x(t) = x_1 + m\, x_1\, (t-1)$$

for $t = 2$, $x_2 = x_1\,(1+m)$. If $t \in [2,3)$, then

$$\frac{dx}{dt} = m\, x_2$$

and

$$x(t) = x_2 + m x_2\, (t-2)$$

so $x_3 = x_2\,(1+m)$.

It is easy to find

$$x_{k+1} = F(x_k) = x_k\,(1+m). \tag{2.1.4}$$

Now it is simple to calculate that

$$\begin{aligned} x_1 &= (1+m)\, x_0, \\ x_2 &= (1+m)\, x_1 = (1+m)^2\, x_0, \\ x_3 &= (1+m)\, x_2 = (1+m)^3\, x_0, \\ &\vdots \end{aligned}$$

Thus, it can be seen $x_{k+1} = (1+m)^{k+1}x_0$, where $x_0$ is a constant. If $x_0 = 0$, then (2.1.4) has a constant solution, $x \equiv 0$. In other words, $x_* = 0$ is the fixed point of $F$. Then one can find that

- if $|1+m| < 1$, $x_* = 0$ is asymptotically stable;
- if $|1+m| > 1$, $x_* = 0$ is unstable;
- if $|1+m| = 1$, $x_* = 0$ is stable since $m = 0$.

The following three figures illustrate results of numerical analysis. In the first figure, we fix $m = -1/2$. It is not surprising to have negative $m$, because $x(t)$ is not the size of population at time $t$, but it is equal to $N(t) - k$. Since $|1+m| < 1$, the sequence of iterates of each given $x_0$ converges to 0. In the second one, we take $m = 2$. And since $|1+m| > 1$, the sequence of iterates of each given $x_0$ goes far away from the fixed point. In the third one, we consider $x_0 = 1$. One can see that when $|1+m| < 1$, the orbit of $x_0$ converges to 0, however, it goes far away from it when $|1+m| > 1$.

In order to make simplification in our computations we will talk not about populations but about the percentage of it in our figures.

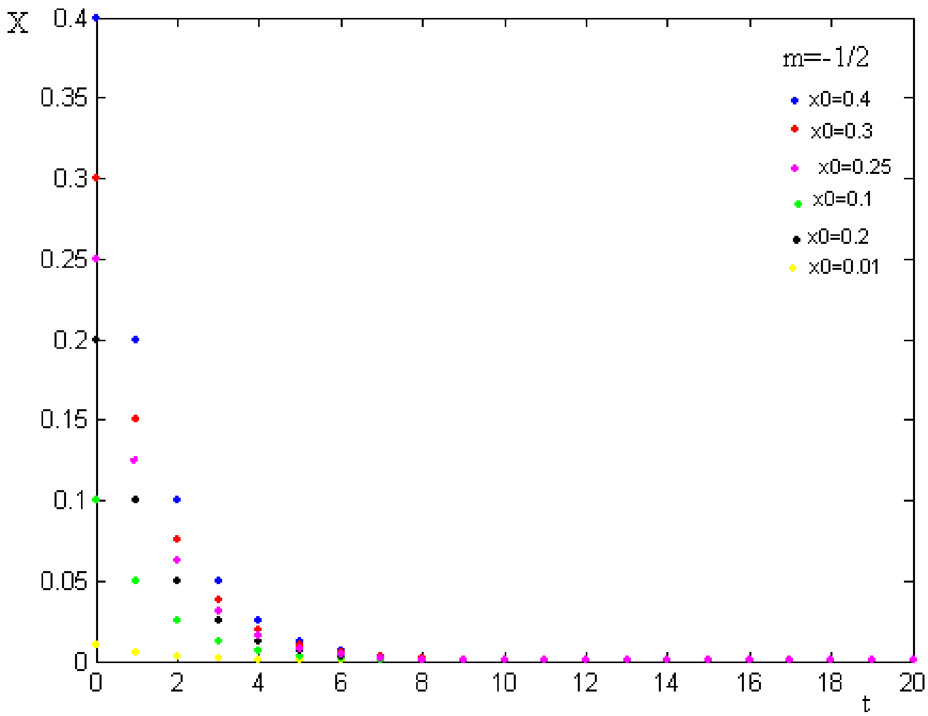


Figure 2.2: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1+m)$ with $m = -1/2$.

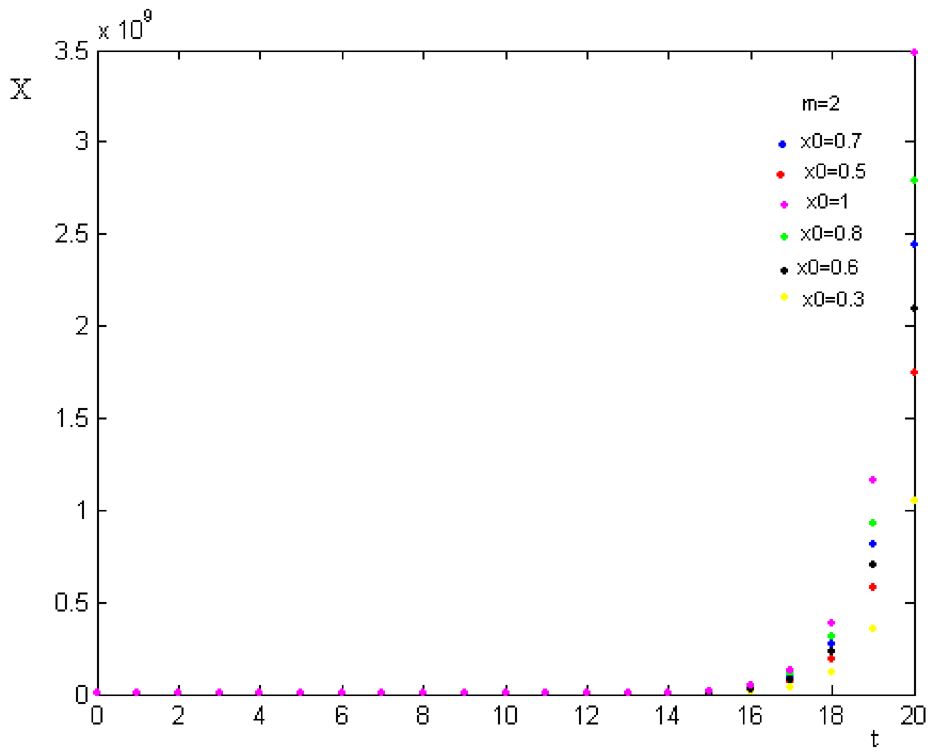


Figure 2.3: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1+m)$ with $m = 2$.

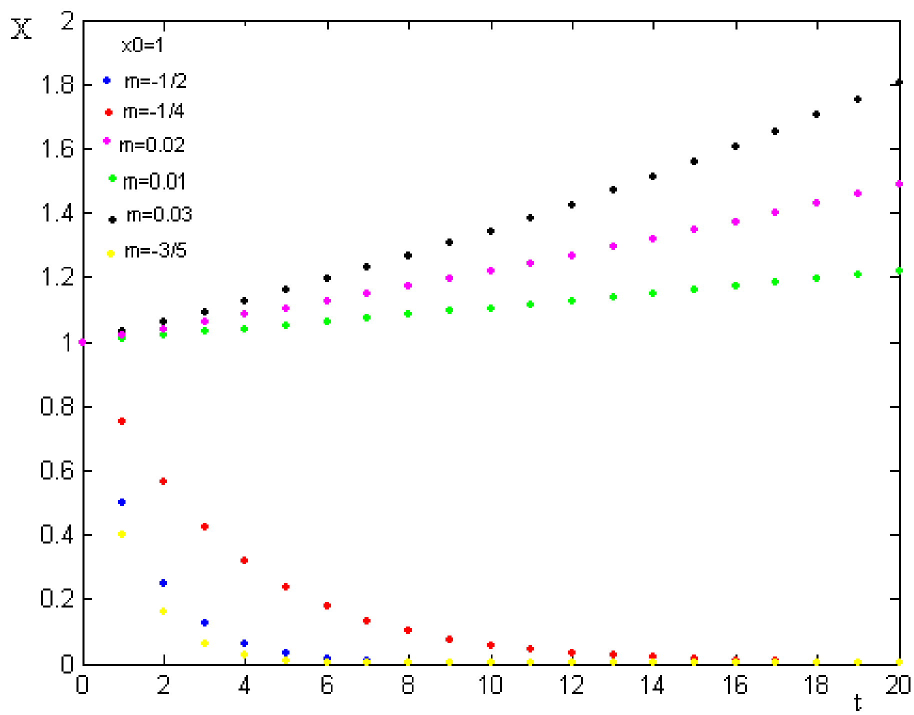


Figure 2.4: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1+m)$ with $x_0 = 1$.

### 2.1.3 Equation E03

Concerning the second approach, (B), we suggest to consider advances in our models. Thus, using anticipatory systems can be effective in population models [1, 6, 12, 26]. In this study, it is supposed that involving anticipation in a model means that in addition to objective external conditions ***a will, a wish, an anticipation*** is taken into account. Thus, we can assume that anticipation means a kind of prediction arrived with the decisions made at the present time. Although there are several ways to introduce anticipation in models, firstly we will consider the following equation:

$$\frac{dx}{dt} = m\, x([t+1]), \tag{2.1.5}$$

where $m$ is a constant different than 1 [1]. At every point $t$, $\frac{dx}{dt}$ depends on the value $x([t+1])$ of next period. Let $t \in [0,1)$, then (2.1.5) has the form

$$\frac{dx}{dt} = m\, x_1.$$

Thus,

$$x(t) = x_0 + m\, x_1 t$$

so it is obvious that $x_1 = x_0\,(1-m)^{-1}$. For $t \in [1,2)$, (2.1.5) takes the following form:

$$\frac{dx}{dt} = m\, x_2.$$

It follows immediately that

$$x(t) = x_1 + m\,x_2\,(t-1),$$

hence, $x_2 = x_1\,(1-m)^{-1}$. Similarly, when $t \in [2,3)$, then

$$\frac{dx}{dt} = m\,x_3,$$

and, hence,

$$x(t) = x_2 + m\,x_3\,(t-2);$$

of course clearly, $x_3 = x_2\,(1-m)^{-1}$. Repeated application of this process enables us to write ;

$$x_{k+1} = F(x_k) = x_k\ (1-m)^{-1}. \qquad (2.1.6)$$

Applying this result for $k \in \mathbb{N}_0$ gives

$$\begin{aligned}
x_1 &= (1-m)^{-1}\,x_0,\\
x_2 &= (1-m)^{-1}\,x_1 = (1-m)^{-2}\,x_0,\\
x_3 &= (1-m)^{-1}\,x_2 = (1-m)^{-3}\,x_0,\\
&\vdots
\end{aligned}$$

then one can see that $x_{k+1} = (1-m)^{-(k+1)}\,x_0$. The fixed point of $F$ is $x_* = 0$. Since $F'(0) = (1-m)^{-1}$, the following results can be written:

- if $|\ (1-m)^{-1}\ | < 1$, $x_* = 0$ is asymptotically stable;
- if $|\ (1-m)^{-1}\ | > 1$, $x_* = 0$ is unstable.

These results are exemplified in the following three figures. In the first figure, we consider $m = 4$. Then $x_* = 0$ is asymptotically stable which means that the sequence of iterates of each given $x_0$ approaches to 0. In the second figure, we take $m = 1/3$, so the sequence of iterates of each given $x_0$ goes far away from the fixed point. In the third figure, one can observe that when $\mid (1-m)^{-1} \mid < 1$, the orbit of 0.3 approaches to 0, and when $\mid (1-m)^{-1} \mid > 1$, it goes far away from 0.

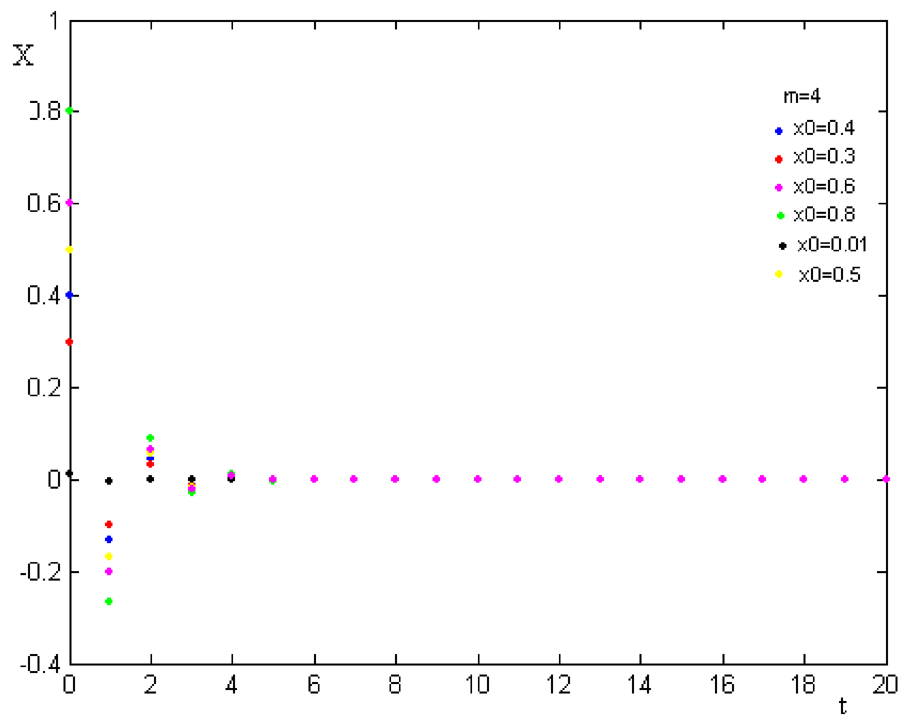


Figure 2.5: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = (1-m)^{-1}x_k$ with $m = 4$.

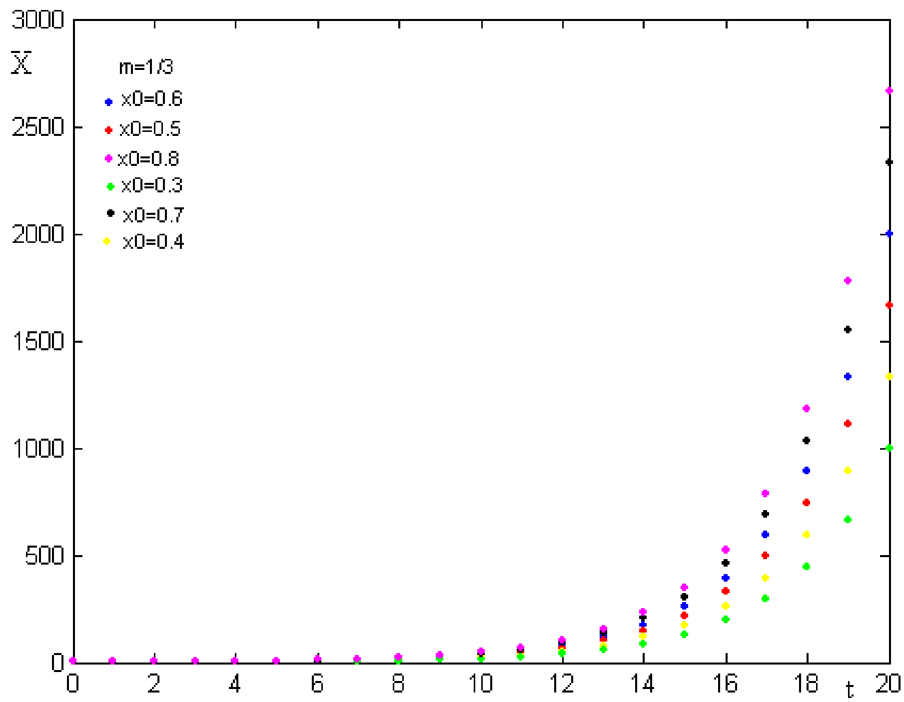


Figure 2.6: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = (1-m)^{-1}x_k$ with $m = 1/3$.

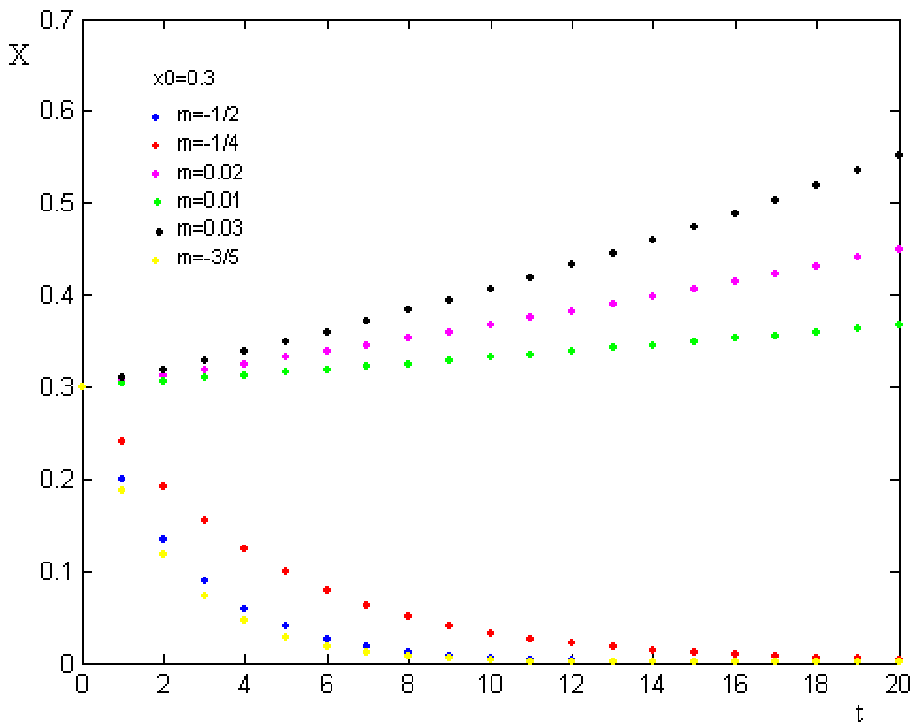


Figure 2.7: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = (1-m)^{-1}x_k$ with $x_0 = 0.3$.

### 2.1.4 Equation E11

Let us consider another equation:

$$\frac{dx}{dt} = (a - b\,x([t]))\,x(t). \tag{2.1.7}$$

For $t \in [0, 1)$, it is a simple matter to see that (2.1.7) has the form

$$\frac{dx}{dt} = (a - b\,x_0)\,x(t),$$

then one obtains

$$x(t) = x_0\,e^{(a-b\,x_0)(t-0)};$$

by continuity, it follows easily that $x_1 = x_0\,e^{a-b\,x_0}$. Consider $t \in [1, 2)$, then

$$\frac{dx}{dt} = (a - b\,x_1)\,x(t);$$

so,

$$x(t) = x_1\,e^{(a-b\,x_1)(t-1)},$$

and $x_2 = x_1\,e^{a-b\,x_1}$. Similarly, if $t \in [2, 3)$, then (2.1.7) takes the following form

$$\frac{dx}{dt} = (a - b\,x_2)\,x(t).$$

Hence,

$$x(t) = x_2\,e^{(a-b\,x_2)\,(t-2)},$$

and $x_3 = x_2\, e^{a-b\,x_2}$. It is not so difficult to find that

$$x_{k+1} = F(x_k) = x_k\, e^{a-b\,x_k}. \tag{2.1.8}$$

It is easy to check that $F$ has two fixed points which are $x_* = 0$ and $x_*^{'} = \frac{a}{b}$; in other words, these are the initial values for constant solutions of (2.1.8). Since $F'(\frac{a}{b}) = 1 - a$ and $F'(0) = e^a$, then the following results can be written

- if $\mid 1 - a \mid < 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $\mid 1 - a \mid > 1$, $x_*^{'} = \frac{a}{b}$ is unstable,

and

- if $e^a < 1$, $x_* = 0$ is asymptotically stable,
- if $e^a > 1$, $x_* = 0$ is unstable.

One can see these results in the following two figures. In the first figure, we consider $a = 1/2$ and $b = 2$. Since $\mid 1 - a \mid < 1$, the sequence of iterates of each given $x_0$ approaches to the fixed point $\frac{a}{b}$. In the second one, it is easily seen that if $a = -2$ and $b = -1$, then $e^a < 1$, so the sequence of iterates of each given $x_0$ converges to the fixed point 0.

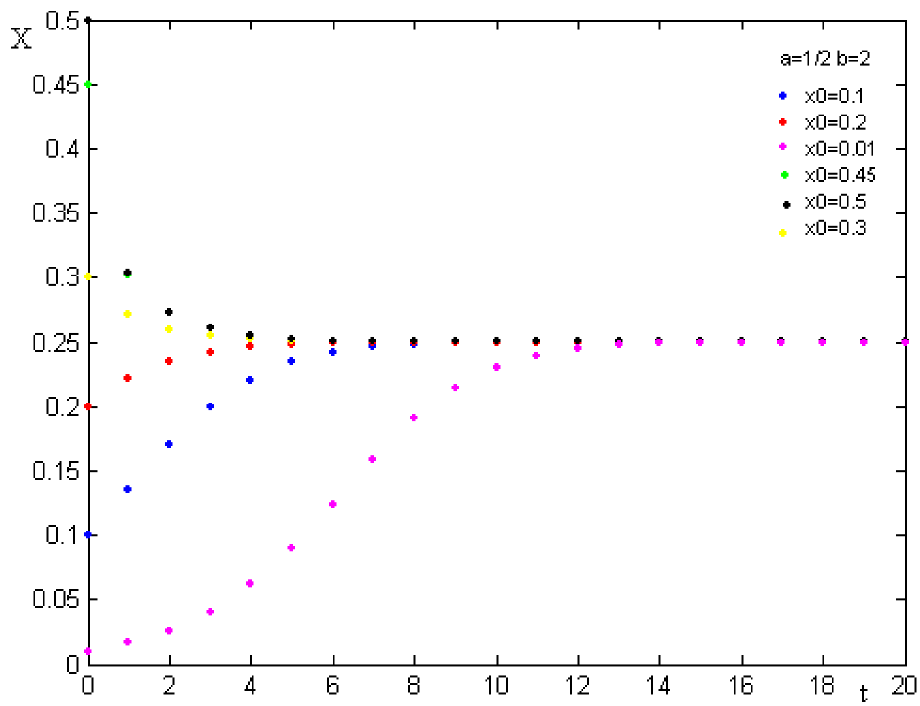


Figure 2.8: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\, e^{a-bx_k}$ with $a = 1/2,\ b = 2$.

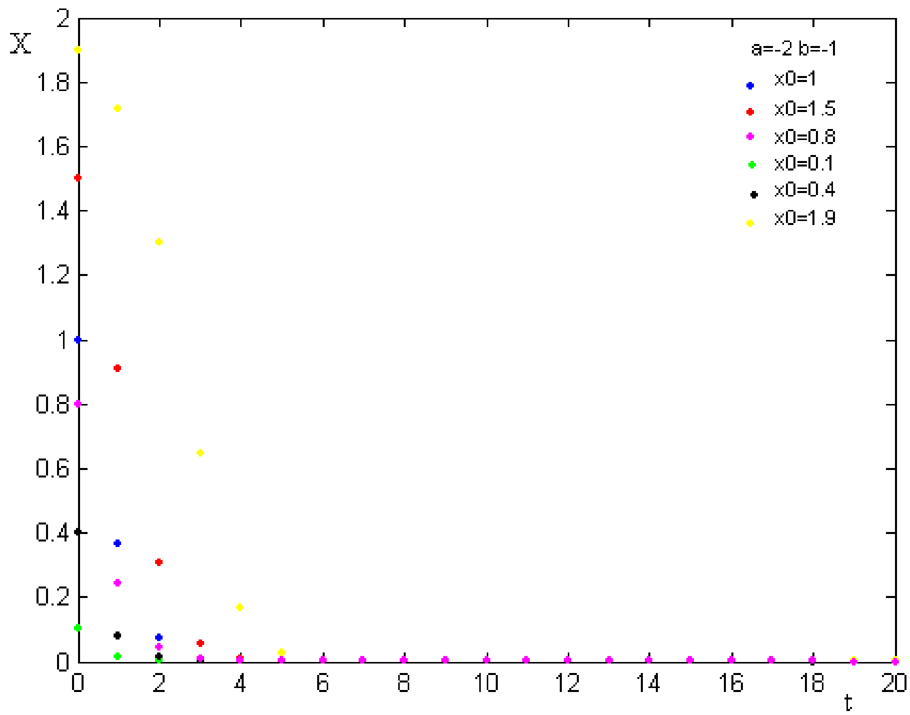


Figure 2.9: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\, e^{a-bx_k}$ with $a = -2,\ b = -1$.

### 2.1.5 Equation E12

Now, we will consider the following system:

$$\frac{dx}{dt} = (a - bx(t))\, x([t]). \tag{2.1.9}$$

If $t \in [0, 1)$, then (2.1.9) has the form

$$\frac{dx}{dt} = (a - b\, x(t))\, x_0;$$

and, hence,

$$x(t) = x_0\, e^{-b\, x_0\, t} + \frac{a}{b}(1 - e^{-b\, x_0\, t}).$$

For $t = 1$, one gets

$$x_1 = e^{-b\, x_0}(x_0 - \frac{a}{b} + e^{b\, x_0} \frac{a}{b}).$$

Let $t \in [1, 2)$, it is clear that (2.1.9) has the form

$$\frac{dx}{dt} = (a - b\, x(t))\, x_1,$$

so,

$$x(t) = x_1\, e^{-b\, x_1\, (t-1)} + \frac{a}{b}(1 - e^{-b\, x_1\, (t-1)}).$$

Then

$$x_2 = e^{-b\, x_1}(x_1 - \frac{a}{b} + e^{b\, x_1} \frac{a}{b}).$$

Consider $t \in [2, 3)$, then (2.1.9) takes the following form

$$\frac{dx}{dt} = (a - b\, x(t))\, x_2,$$

herewith,

$$x(t) = x_2\, e^{-b\, x_2\, (t-2)} + \frac{a}{b}(1 - e^{-b\, x_2\, (t-2)}),$$

for $t = 3$

$$x_3 = e^{-b\, x_2}(x_2 - \frac{a}{b} + e^{b\, x_2}\frac{a}{b}).$$

Then, by induction one obtains the following discrete-time equation

$$x_{k+1} = F(x_k) = e^{-b\, x_k}(x_k - \frac{a}{b} + e^{b\, x_k}\frac{a}{b}). \tag{2.1.10}$$

Fixed points of $F$ are $x_* = 0$ and $x_*^{'} = \frac{a}{b}$, so it can be said that these points are period-1 points of $F$ and they are also the initial values for constant solutions of (2.1.10). To talk about the behavior of $F$ near fixed points we must look at the derivative of $F$ at these points. Then, we obtain the following results:

- if $\mid a + 1 \mid < 1$, $x_* = 0$ is asymptotically stable,
- if $\mid a + 1 \mid > 1$, $x_* = 0$ is unstable,
- if $e^{-a} < 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $e^{-a} > 1$, $x_*^{'} = \frac{a}{b}$ is unstable.

The results of numerical analysis are illustrated in the following two figures. In the first figure, we consider $a = 3$ and $b = 5$. Since $e^{-a} < 1$, the orbit of each given $x_0$ approaches to the fixed point $\frac{a}{b}$. In the second figure, we take $a = -1/2$ and $b = -1$. Since $\mid 1 + a \mid < 1$, the sequence of the iterates of each given $x_0$ converges to 0.

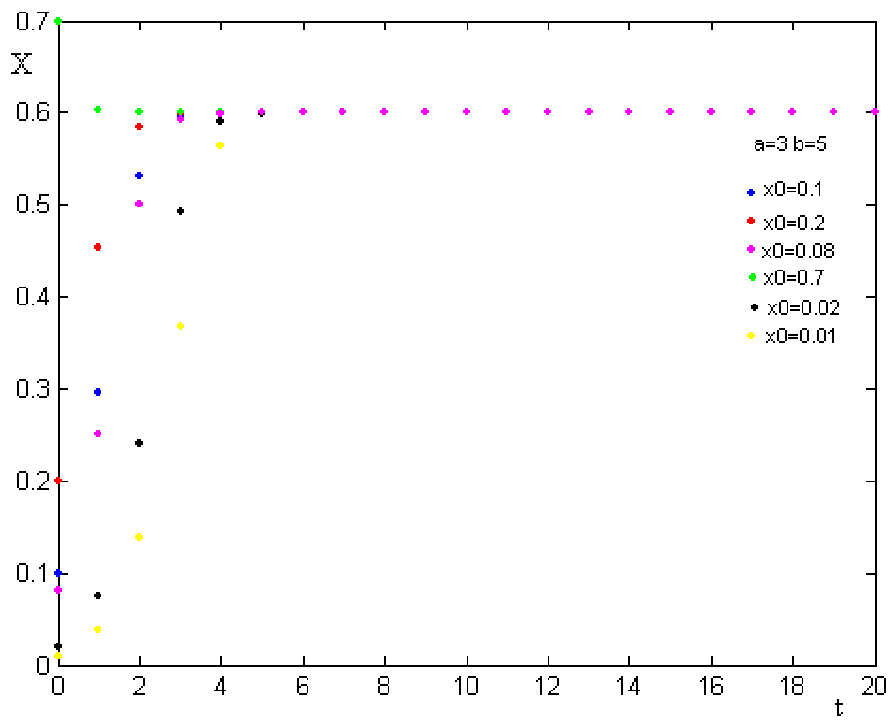


Figure 2.10: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = e^{-b\,x_k}(x_k - \frac{a}{b} + e^{b\,x_k}\frac{a}{b})$ with $a = 3,\ b = 5$.

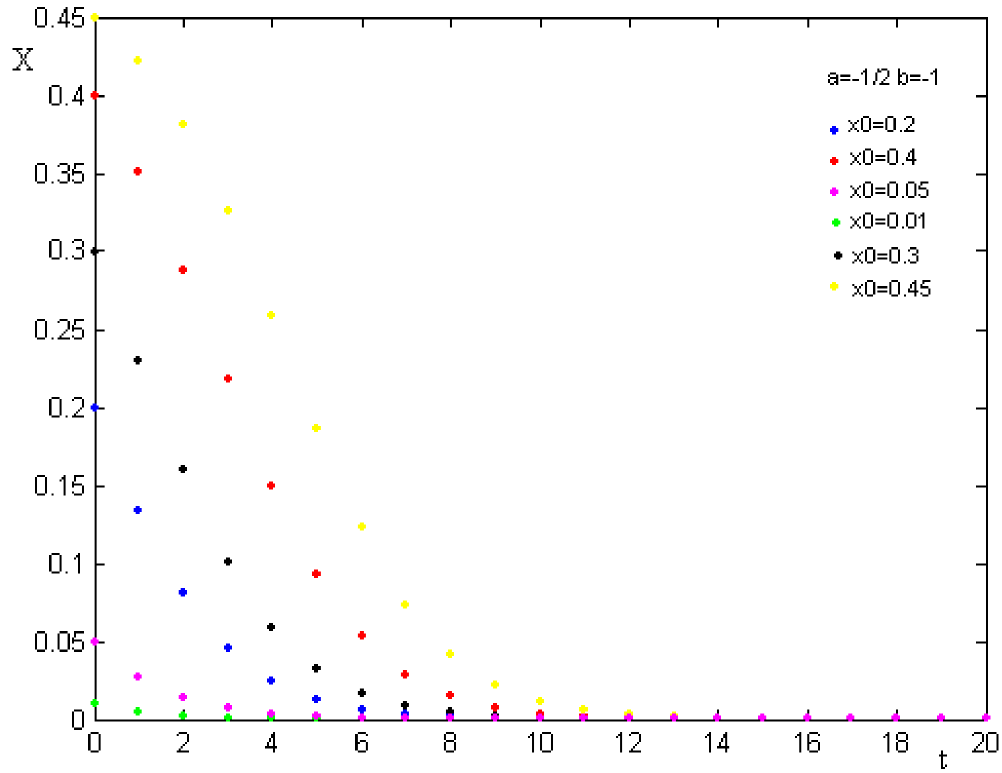


Figure 2.11: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = e^{-b\,x_k}(x_k - \frac{a}{b} + e^{b\,x_k}\frac{a}{b})$ with $a = -1/2,\ b = -1$.

### 2.1.6 Equation E13

Let us consider another equation:

$$\frac{dx}{dt} = (a - bx(t))\, x([t+1]). \tag{2.1.11}$$

If $t \in [0, 1)$, then the equation takes the following form:

$$\frac{dx}{dt} = (a - bx(t))\, x_1,$$

hence,

$$x(t) = x_0\, e^{-b\, x_1\, t} + \frac{a}{b}(1 - e^{-b\, x_1\, t}).$$

For $t = 1$, one obtains

$$x_1 = x_0\, e^{-b\, x_1} + \frac{a}{b}(1 - e^{-b\, x_1\, t}).$$

Similarly, when $t \in [1, 2)$, (2.1.11) has the form

$$\frac{dx}{dt} = (a - bx(t))\, x_2,$$

and, therefore,

$$x_2 = x_1\, e^{-b\, x_2} + \frac{a}{b}(1 - e^{-b\, x_2}).$$

Consider $t \in [2, 3)$, then

$$\frac{dx}{dt} = (a - b\, x(t))\, x_3;$$

so,

$$x_3 = x_2\, e^{-b\, x_3} + \frac{a}{b}(1 - e^{-b\, x_3}).$$

Repeated application of this process enables us to write the following equation

$$x_{k+1} = x_k\, e^{-b\, x_{k+1}} + \frac{a}{b}(1 - e^{-b\, x_{k+1}}). \tag{2.1.12}$$

Here, 0 and $\frac{a}{b}$ are the initial points for constant solutions of (2.1.12).

In equation (2.1.12) $x_{k+1}$ depends on both $x_{k+1}$, $x_k$. For simplicity of the notation, we write $x$ instead of $x_k$ and $y$ for $x_{k+1}$. Then (2.1.12) has the form

$$y = F(x, y) = x\, e^{-b\, y} + \frac{a}{b}(1 - e^{-b\, y}), \tag{2.1.13}$$

which can be written as $y - x\, e^{-b\, y} - \frac{a}{b}(1 - e^{-b\, y}) = 0$. It is a simple matter to find

$$\frac{dy}{dx} = \frac{-e^{-b\, y}}{(e^{-b\, y}(a - b\, x) - 1)}. \tag{2.1.14}$$

When $(x, y) = (\frac{a}{b}, \frac{a}{b})$, then one obtains $\frac{dy}{dx} = e^{-a}$, and when $(x, y) = (0, 0)$, then one gets $\frac{dy}{dx} = (1 - a)^{-1}$. Now, the following results can be written:

- if $e^{-a} < 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $e^{-a} > 1$, $x_*^{'} = \frac{a}{b}$ is unstable,
- if $\mid (1 - a)^{-1} \mid < 1$, $x_* = 0$ is asymptotically stable,
- if $\mid (1 - a)^{-1} \mid > 1$, $x_* = 0$ is unstable.

It is also possible to write the equation (2.1.12) in the following form

$$x_k = x_{k+1}\, e^{b\, x_{k+1}} - \frac{a}{b}\, e^{b\, x_{k+1}} + \frac{a}{b},$$

so,

$$x_k - \frac{a}{b} = x_{k+1}\, e^{b\, x_{k+1}} - \frac{a}{b}\, e^{b\, x_{k+1}}. \tag{2.1.15}$$

When we multiple both sides with $e^{-a}\, b$, then (2.1.15) changes to

$$e^{-a}\, b\, (x_k - \frac{a}{b}) = (b\, x_{k+1} - a)\, e^{b\, x_{k+1} - a}.$$

Set $u_{k+1} = bx_{k+1} - a$, then

$$e^{-a}\, b\, (x_k - \frac{a}{b}) = u_{k+1}\, e^{u_{k+1}};$$

and, hence,

$$u_{k+1} = W(e^{-a}\, b\ \ (x_k - \frac{a}{b})),$$

where $W$ shows the ***Lambert W function***. From the properties of Lambert W function it follows that

- if $e^{-a}\, b\, (x_k - \frac{a}{b}) < \frac{-1}{e}$, then $u_{k+1}$ is multivalued which are complex,
- if $\frac{-1}{e} \leq e^{-a}\, b\, (x_k - \frac{a}{b}) < 0$, $u_{k+1}$ has two values one of them is smaller than $-1$ while the other is bigger than it,
- if $e^{-a}\, b\, (x_k - \frac{a}{b}) \geq 0$, then there is a single real value for $u_{k+1}$.

Since $u_{k+1} = b\, x_{k+1} - a$, then $x_{k+1}$ has the form

$$x_{k+1} = \frac{W(e^{-a}\, b\,(x_k - \frac{a}{b})) + a}{b}.$$

The following finite sequences show $(x_1, x_2, x_3, \ldots, x_8)$ when $a = 1.5$, $b = 10$ and $x_0 = 0.2$, $x_0 = 0.4$, respectively. Since $e^{-a} < 1$, the corresponding sequences approach to $\frac{a}{b}$:

$$\begin{pmatrix} 0.1601 & 0.1522 & 0.1505 & 0.1501 & 0.1500 & 0.1500 & 0.1500 & 0.1500 \end{pmatrix},$$

$$\begin{pmatrix} 0.1881 & 0.1579 & 0.1517 & 0.1504 & 0.1501 & 0.1500 & 0.1500 & 0.1500 \end{pmatrix}.$$

Let $a = -1/2$ and $b = -1$. The following finite sequences show $(x_1, x_2, x_3, \ldots, x_8)$ for $x_0 = 0.4$ and $x_0 = 0.09$, respectively. Since $\mid (1-a)^{-1} \mid < 1$, the corresponding sequences approach to 0:

$$\begin{pmatrix} 0.3571 & 0.3059 & 0.2507 & 0.1965 & 0.1481 & 0.1080 & 0.0767 & 0.0535 \end{pmatrix},$$

$$\begin{pmatrix} 0.0632 & 0.0437 & 0.0299 & 0.0203 & 0.0137 & 0.0092 & 0.0061 & 0.0041 \end{pmatrix}.$$

### 2.1.7 Equation E14

Let us consider the following equation:

$$\frac{dx}{dt} = (a - b\,x([t+1]))\,x(t). \tag{2.1.16}$$

It is easy to check that when $t \in [0,1)$, then (2.1.16) has the form

$$\frac{dx}{dt} = (a - b\,x_1)\,x(t),$$

so,

$$x(t) = x_0\,e^{(a-b\,x_1)(t-0)},$$

and $x_1 = x_0\,e^{a-b\,x_1}$. If $t \in [1,2)$, then (2.1.16) has the form

$$\frac{dx}{dt} = (a - b\,x_2)\,x(t),$$

and

$$x(t) = x_1\,e^{(a-b\,x_2)\,(t-1)},$$

so $x_2 = x_1\,e^{a-b\,x_2}$. Similarly, when $t \in [2,3)$, one gets the following equation

$$\frac{dx}{dt} = (a - b\,x_3)\,x(t),$$

herewith,

$$x(t) = x_2\,e^{(a-b\,x_3)(t-2)},$$

for $t = 3$, $x_3 = x_2\, e^{a-b\,x_3}$. It is not so difficult to find that

$$x_{k+1} = x_k\, e^{a-b\,x_{k+1}}. \tag{2.1.17}$$

Again 0 and $\frac{a}{b}$ are the initial points for the constant solutions of (2.1.17). Since $x_{k+1}$ depends on both $x_k$, $x_{k+1}$, we apply same process with the previous equation. Then, we obtain the following results:

- if $e^{-a} > 1$, $x_* = 0$ is asymptotically stable,
- if $e^{-a} < 1$, $x_* = 0$ is unstable,
- if $\mid 1 + a \mid > 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $\mid 1 + a \mid < 1$, $x_*^{'} = \frac{a}{b}$ is unstable.

The recursive equation (2.1.17) can be written in the following form

$$x_k e^a = x_{k+1}\, e^{b\,x_{k+1}}.$$

If both sides are multiplied with $b$, one gets

$$b\,x_k\, e^a = b\,x_{k+1}\, e^{b\,x_{k+1}}.$$

Let $u_{k+1} = b\,x_{k+1}$, then

$$b\,x_k\, e^a = u_{k+1}\, e^{u_{k+1}};$$

hence,

$$u_{k+1} = W(b\,x_k\, e^a).$$

So,

$$x_{k+1} = \frac{W(b\,x_k\,e^a)}{b}.$$

Consider $a = -\frac{1}{2}$, $b = 1$. The following finite sequences show $(x_1, x_2, x_3, \ldots, x_8)$ for $x_0 = 0.1$ and $x_0 = 0.2$, respectively:

$$\begin{pmatrix} 0.0573 & 0.0336 & 0.0200 & 0.0120 & 0.0072 & 0.0044 & 0.0026 & 0.0016 \end{pmatrix},$$

$$\begin{pmatrix} 0.1088 & 0.0620 & 0.0363 & 0.0215 & 0.0129 & 0.0078 & 0.0047 & 0.0028 \end{pmatrix}.$$

Since $e^{-a} > 1$, each of the sequences approaches to 0. Now let us take $a = \frac{1}{5}$ and $b = 1$. The following finite sequences show $(x_1, x_2, x_3, \ldots, x_8)$ for $x_0 = 0.15$ and $x_0 = 0.01$, respectively:

$$\begin{pmatrix} 0.1566 & 0.1626 & 0.1679 & 0.1726 & 0.1767 & 0.1802 & 0.1832 & 0.1858 \end{pmatrix},$$

$$\begin{pmatrix} 0.0121 & 0.0145 & 0.0174 & 0.0209 & 0.0248 & 0.0295 & 0.0348 & 0.0408 \end{pmatrix}.$$

### 2.1.8 Equation E21

Now, let us consider another equation

$$\frac{dx}{dt} = (a - b\,x([t]))\,x([t+1]). \tag{2.1.18}$$

When $t \in [0,1)$, then (2.1.18) has the form

$$\frac{dx}{dt} = (a - b\,x_0)\,x_1$$

and, hence,

$$x(t) = (a - b\,x_0)\,x_1\,(t-0) + x_0;$$

thus, for $t = 1$ one obtains $x_1 = (1 - a + b\,x_0)^{-1}x_0$. Consider $t \in [1,2)$, then (2.1.18) takes the following form

$$\frac{dx}{dt} = (a - b\,x_1)\,x_2;$$

hence,

$$x(t) = (a - bx_1)\,x_2\,(t-1) + x_1$$

and $x_2 = (1 - a + b\,x_1)^{-1}\,x_1$. For $t \in [2,3)$, it is easily seen (2.1.18) has the following form

$$\frac{dx}{dt} = (a - b\,x_2)\,x_3.$$

So,

$$x(t) = (a - b\,x_2)\,x_3\,(t-2) + x_2;$$

then $x_3 = (1 - a + b\, x_2)^{-1}\, x_2$. By induction, it is easy to obtain the following discrete-time equation:

$$x_{k+1} = F(x_k) = (1 - a + b\, x_k)^{-1}\, x_k. \tag{2.1.19}$$

The fixed points of $F$ are $x_* = 0$, $x_*^{'} = \frac{a}{b}$. In other words, these points are the initial points for constant solutions of (2.1.19). The derivative of $F$ at $x_*^{'} = \frac{a}{b}$ is $(1 - a)$ and the derivative of $F$ at $x_* = 0$ is $(1 - a)^{-1}$. So the following results are obtained

- if $\mid 1 - a \mid < 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $\mid 1 - a \mid > 1$, $x_*^{'} = \frac{a}{b}$ is unstable,
- if $\mid (1 - a)^{-1} \mid < 1$, $x_* = 0$ is asymptotically stable,
- if $\mid (1 - a)^{-1} \mid > 1$, $x_* = 0$ is unstable at $x = 0$.

These results are illustrated in the following three figures. In the first figure, we take $a = 9$ and $b = 4$. Because of $\mid (1 - a)^{-1} \mid < 1$, the sequence of iterates of each given $x_0$ converges to the fixed point 0. In the second one, we consider $a = 3/5$ and $b = 2$. Then the orbit of each given $x_0$ approaches to the fixed point $\frac{a}{b}$.

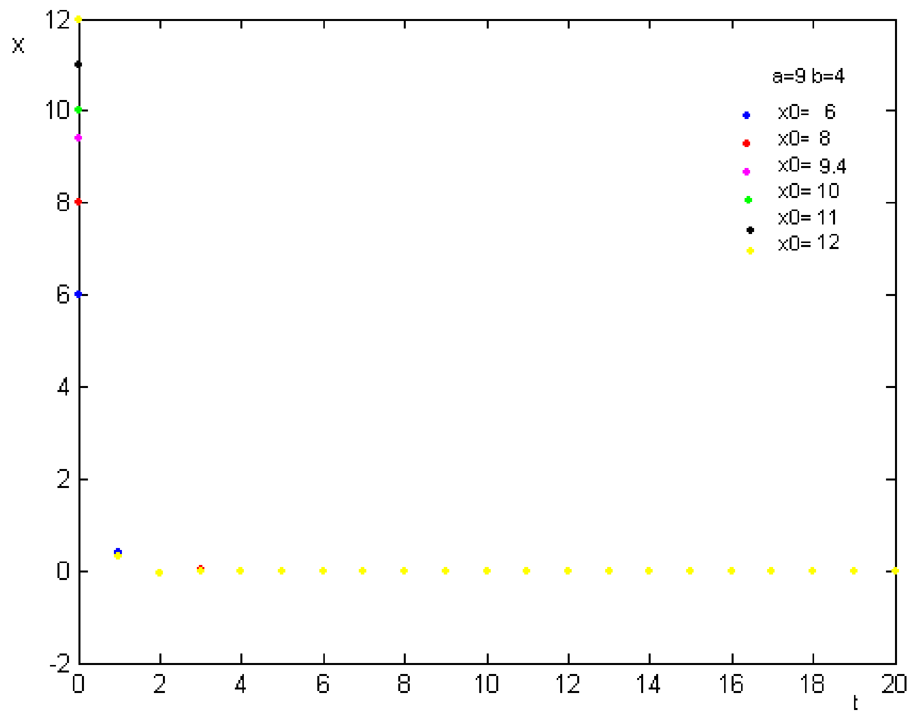


Figure 2.12: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = (1 - a + b\,x_k)^{-1}\,x_k$ with $a = 9,\ b = 4$.

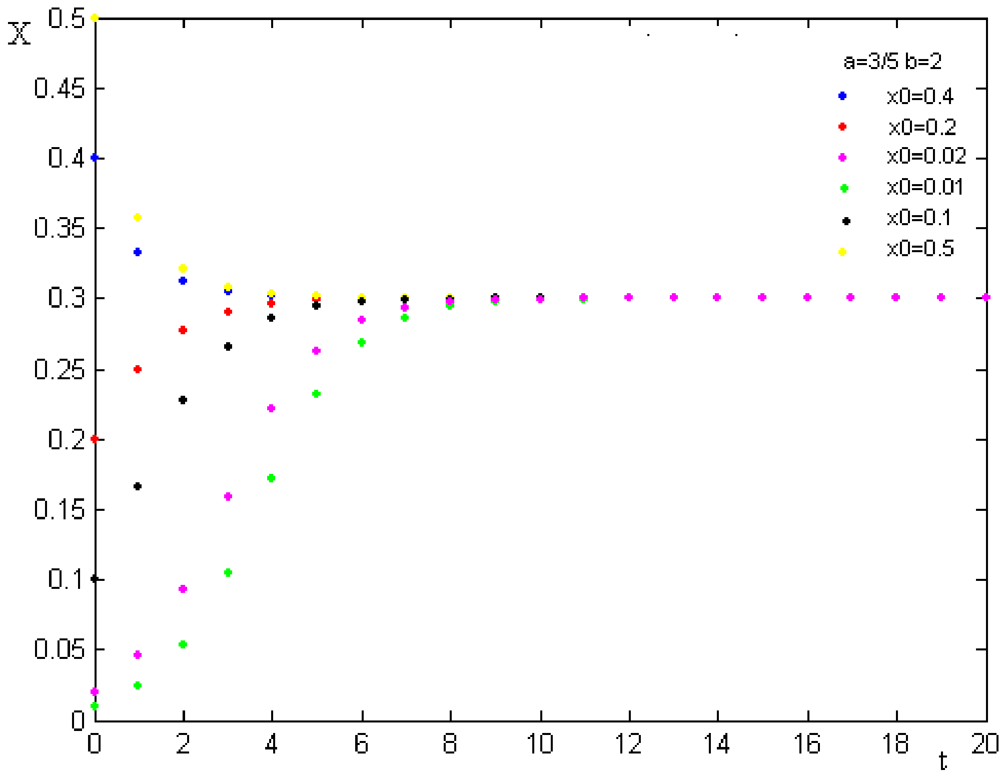


Figure 2.13: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = (1 - a + b\,x_k)^{-1}\,x_k$ with $a = 3/5,\ b = 2$.

### 2.1.9 Equation E22

In this part, we will consider the equation

$$\frac{dx}{dt} = (a - b\,x([t]))\,x([t]). \tag{2.1.20}$$

Let $t \in [0,1)$, then (2.1.20) has the form

$$\frac{dx}{dt} = (a - b\,x_0)\,x_0,$$

so,

$$x(t) = x_0 + x_0\ (a - b\,x_0)\,(t - 0).$$

Thus $x_1 = x_0\ (1 + a - b\,x_0)$. When $t \in [1,2)$, (2.1.20) takes the following form:

$$\frac{dx}{dt} = (a - b\,x_1)\,x_1,$$

and, hence,

$$x(t) = x_1 + x_1\,(a - bx_1)\,(t - 1).$$

It is a simple matter to obtain $x_2 = x_1\,(1 + a - b\,x_1)$. If $t \in [2,3)$, then (2.1.20) becomes

$$\frac{dx}{dt} = (a - b\,x_2)\,x_2.$$

Then,

$$x(t) = x_2 + x_2\,(a - b\,x_2)\,(t - 2)$$

and $x_3 = x_2\ (1 + a - b\, x_2)$. Finally,

$$x_{k+1} = F(x_k) = x_k\ (1 + a - b\, x_k). \tag{2.1.21}$$

The period-1 points of $F$ are 0 and $\frac{a}{b}$. Now, $F^{'}(x) = 1 + a - 2\, b\, x$, then we have $F^{'}(0) = 1 + a$ and $F^{'}(\frac{a}{b}) = 1 - a$. Thus,

- if $\mid 1 + a \mid < 1$, $x_* = 0$ is asymptotically stable,
- if $\mid 1 + a \mid > 1$, $x_* = 0$ is unstable,
- if $\mid 1 - a \mid < 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $\mid 1 - a \mid > 1$, $x_*^{'} = \frac{a}{b}$ is unstable.

These results can be seen in the following two figures. In the first figure, we take $a = -1/4$, $b = -1$. Since $\mid 1 + a \mid < 1$, the sequence of iterates of each given $x_0$ approaches to 0. In the second one, we consider $a = 1/3$, $b = 2/3$. Since $\mid 1 - a \mid < 1$, the sequence of iterates of each given $x_0$ converges to $\frac{a}{b}$.

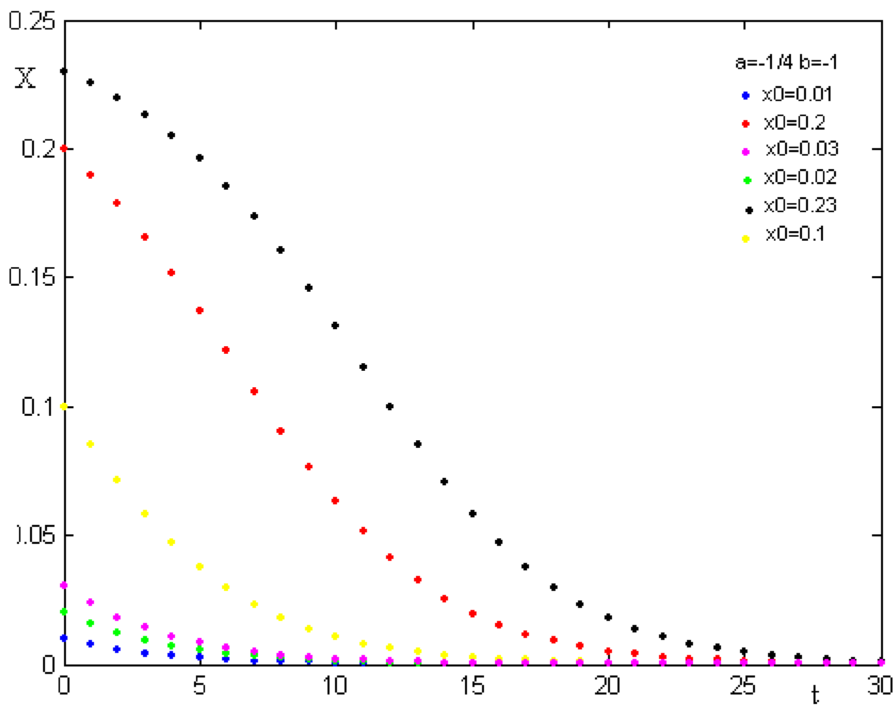


Figure 2.14: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1 + a - b\,x_k)$ with $a = -1/4$, $b = -1$.

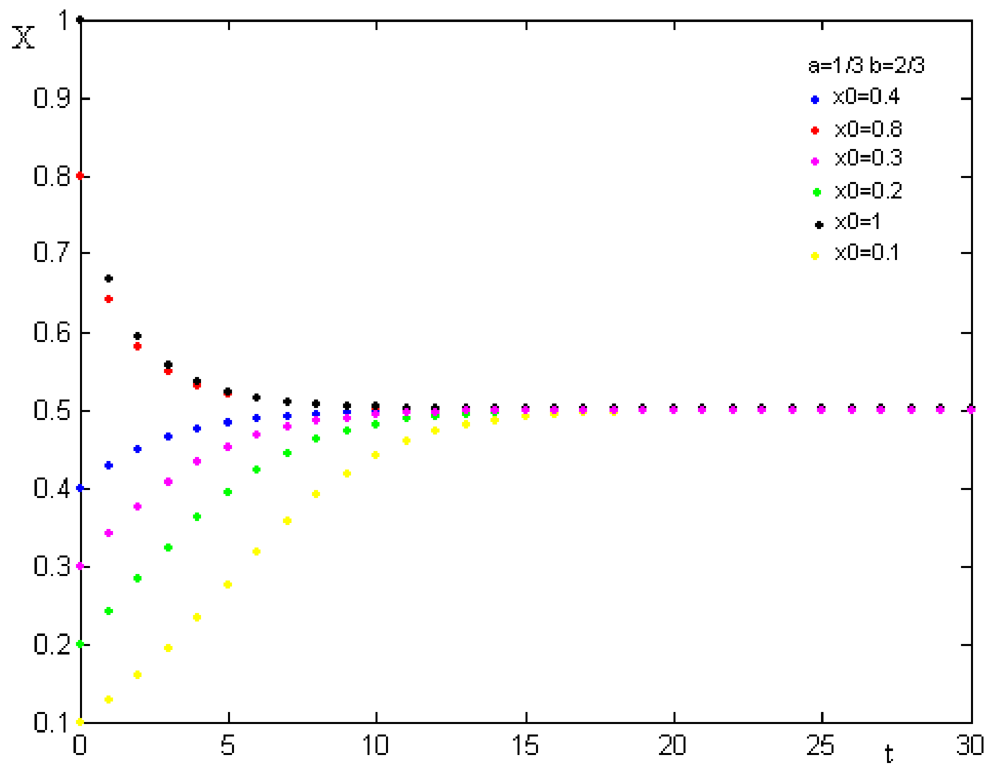


Figure 2.15: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1 + a - b\,x_k)$ with $a = 1/3,\ b = 2/3$.

### 2.1.10 Equation E23

In this part we will consider the following system

$$\frac{dx}{dt} = (a - b\,x([t+1]))\,x([t]). \tag{2.1.22}$$

Let $t \in [0, 1)$, then (2.1.22) takes the form

$$\frac{dx}{dt} = (a - b\,x_1)\,x_0,$$

then

$$x(t) = x_0 + (a - b\,x_1)\,x_0\,(t - 0),$$

for $t = 1$, $x_1 = x_0\,(1 + a)\,(1 + b\,x_0)^{-1}$. Consider $t \in [1, 2)$, it is not so difficult to see that (2.1.22) has the form

$$\frac{dx}{dt} = (a - b\,x_2)\,x_1,$$

it follows that

$$x(t) = x_1 + (a - b\,x_2)\,x_1\,(t - 1),$$

so $x_2 = x_1\,(1 + a)\,(1 + b\,x_1)^{-1}$. For $t \in [2, 3)$, (2.1.22) has the form

$$\frac{dx}{dt} = (a - b\,x_3)\ \ x_2,$$

so

$$x(t) = x_2 + (a - b\,x_3)\ \ x_2\,(t - 2),$$

then $x_3 = x_2\,(1+a)\,(1+b\,x_2)^{-1}$. When we iterate this process, we get

$$x_{k+1} = F(x_k) = x_k\,(1+a)\,(1+b\,x_k)^{-1}. \tag{2.1.23}$$

Again $x_* = \frac{a}{b}$ and $x_*^{'} = 0$ are the fixed points of $F$. And

- if $\mid 1 - \frac{a}{1+a} \mid < 1$, $x_*^{'} = \frac{a}{b}$ is asymptotically stable,
- if $\mid 1 - \frac{a}{1+a} \mid > 1$, $x_*^{'} = \frac{a}{b}$ is unstable,
- if $\mid 1 + a \mid < 1$, $x_* = 0$ is asymptotically stable,
- if $\mid 1 + a \mid > 1$, $x_* = 0$ is unstable.

One can see these results in the following two figures. In the first figure, we take $a = 3, b = 5$. It can be seen that the sequence of iterates of each given $x_0$ converges to $\frac{a}{b}$. And in the second figure, one can see that when $a = -1/2$, $b = -1/3$, the orbit of each given $x_0$ approaches to 0.

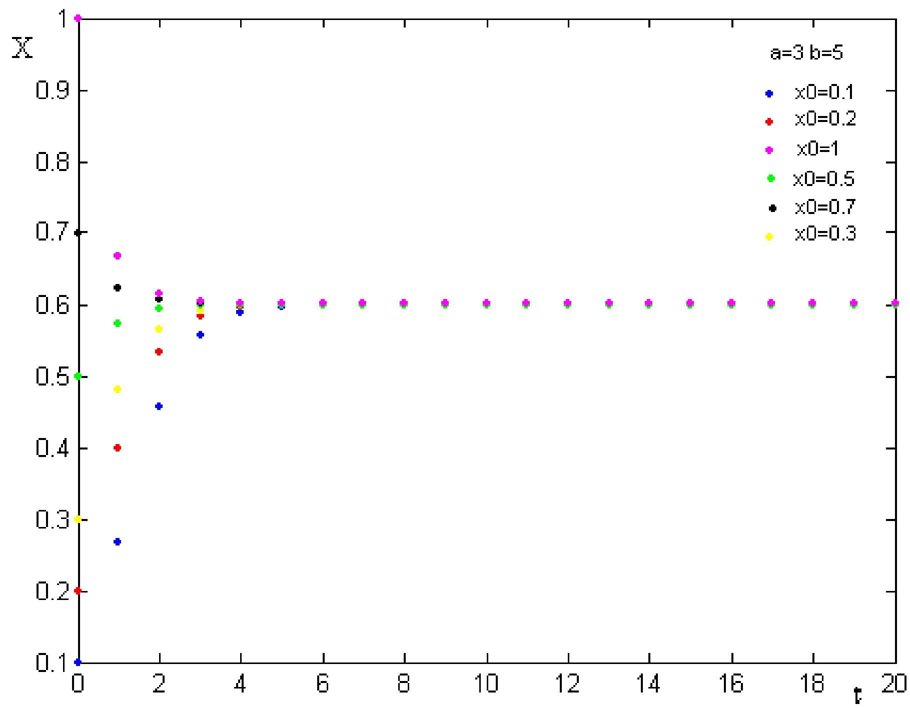


Figure 2.16: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1 + a)\,(1 + bx_k)^{-1}$ with $a = 3,\ b = 5$.

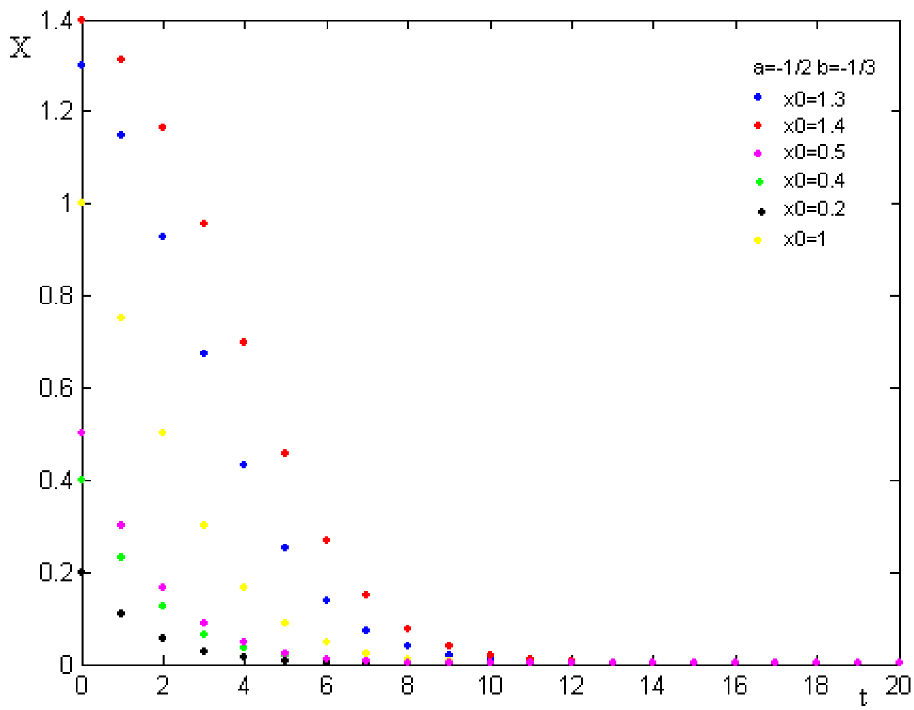


Figure 2.17: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1 + a)\,(1 + bx_k)^{-1}$ with $a = -1/2,\ b = -1/3$.

### 2.1.11 Equation E24

Let us consider the following system:

$$\frac{dx}{dt} = (a - b\,x([t+1]))\,x([t+1]). \qquad (2.1.24)$$

Let $t \in [0, 1)$, then

$$\frac{dx}{dt} = (a - b\,x_1)\,x_1,$$

it is easy to check that the solution of this equation is

$$x(t) = x_0 + (a - b\,x_1)\,x_1\,(t - 0)$$

when $t = 1$, $x_1 = x_0\ (1 - a + b\,x_1)^{-1}$. Consider $t \in [1, 2)$, then (2.1.24) has the form

$$\frac{dx}{dt} = (a - b\,x_2)\,x_2.$$

Hence

$$x(t) = x_1 + (a - b\,x_2)\,x_2\,(t - 1);$$

it is trivial that $x_2 = x_1\,(1 - a + b x_2)^{-1}$. For $t \in [2, 3)$, (2.1.24) takes the following form

$$\frac{dx}{dt} = (a - b\,x_3)\,x_3;$$

so,

$$x(t) = x_2 + (a - b\,x_3)\,x_3\,(t - 2)$$

and $x_3 = x_2\,(1 - a + b\,x_3)^{-1}$. By induction, one obtains that

$$x_{k+1} = F(x_k) = x_k\;\,(1 - a + b\,x_{k+1})^{-1}. \tag{2.1.25}$$

Now we can write the following results:

- if $|\,(1-a)^{-1}\,| < 1$, $x_* = 0$ is asymptotically stable,
- if $|\,(1-a)^{-1}\,| > 1$, $x_* = 0$ is unstable at $x_* = 0$,
- if $|\,(1+a)^{-1}\,| < 1$, $x_* = \frac{a}{b}$ is asymptotically stable,
- if $|\,(1+a)^{-1}\,| > 1$, $x_* = \frac{a}{b}$ is unstable.

Equation (2.1.25) can be written as follows:

$$x_k = x_{k+1}(1 - a + b\,x_{k+1}),$$

which rearranges to

$$b\,x_{k+1}^2 + (1-a)\,x_{k+1} - x_k = 0.$$

Then,

$$x_{k+1} = \frac{-(1-a) \mp \sqrt{(1-a)^2 + 4\,b\,x_k}}{2b}.$$

If $(1-a)^2 + 4bx_k > 0$, then $x_{k+1}$ has two different real values. This result can be explained with anticipation. As we have said before, anticipation is a *wish* depending on the current state so it is possible to have different wishes from the current state. Let us try to explain this result with an example from our daily life, a mathematics student can be an economist or a mathematics teacher in the future, these are our $x_1$ values, but to have these jobs one must be graduated from mathematics department, and this is our $x_0$ value.

## 2.2 2-Periodic Points of The Extensions of The Malthusian Model

In this part of our study, we mainly deal with period-2 points of the extensions of Malthusian Model.

We know that, a point $x \in S$ is a period-2 point of $F : S \longrightarrow S$ if $F^2(x) = x$. In other words, a period-2 point of $F$ is a fixed point of $F^2$.

The first equation is

$$\frac{dx}{dt} = m\, x([t]),$$

and the reduced discrete equation is

$$x_{k+1} = F(x_k) = x_0 \ (1+m)^{k+1} = x_k \ (1+m).$$

Investigating this equation, one can easily see that if $m = -2$ and $m = 0$, then all points are period-2 points of $F$. It is possible to have negative $m$ because $x$ is not the number of population, but it is equal to $N - k$. If $m \neq -2$ and $m \neq 0$ , then $x_* = 0$ is the only fixed point of $F^2$ (see [1] for details).

In the following figure one can see that when $m = -2$, then $x_0 = 6,\ x_0 = 5,\ x_0 = 3.7$ are period-2 points of $F$.

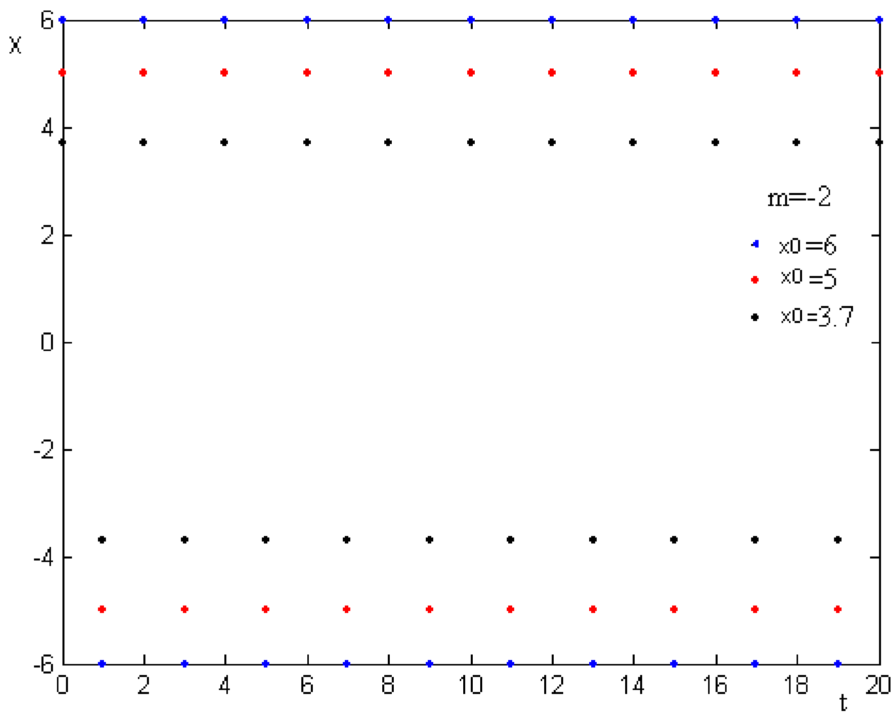


Figure 2.18: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k\,(1+m)$ with $m = -2$.

The second equation is

$$\frac{dx}{dt} = m\,x([t+1])$$

and the reduced discrete equation has the following form

$$x_{k+1} = F(x_k) = x_0(1-m)^{-(k+1)} = x_k(1-m)^{-1}.$$

When $m = 2$ and $m = 0$, then all points are period-2 points of $F$. However, when $m \neq 2$ and $m \neq 0$, $x_* = 0$ is the only fixed point of $F^2$.

In the following figure, it can be seen when $m = 2$, then $x_0 = 8$, $x_0 = 0.5$, $x_0 = 2.8$ are period-2 points of $F$.

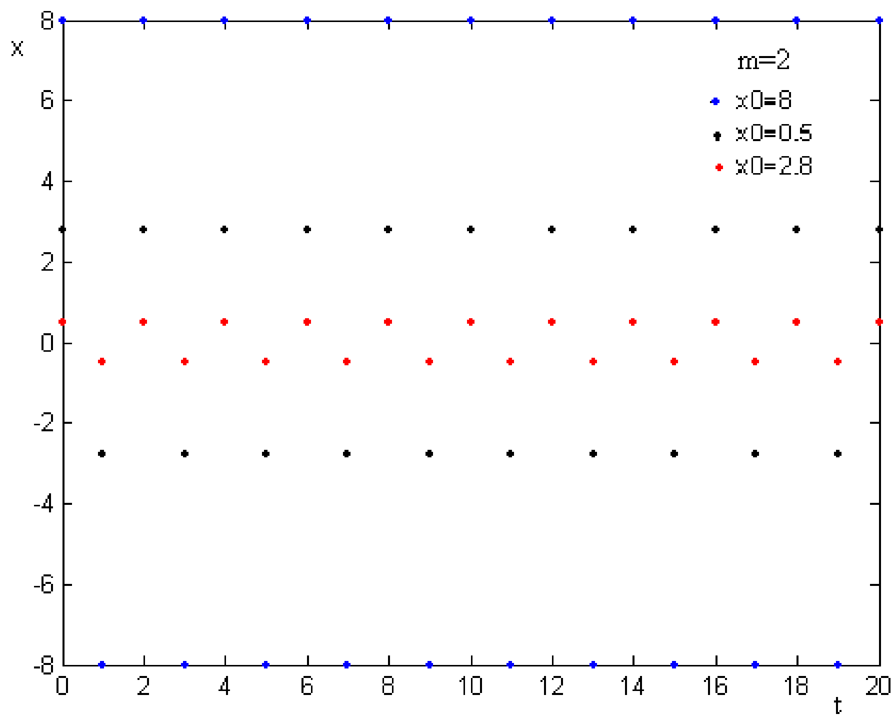


Figure 2.19: Graphs of orbits of the discrete equation $x_{k+1} = F(x_k) = x_k(1-m)^{-1}$ with $m = 2$.

Now, let us consider the following equations:

$$\begin{aligned}
\frac{dx}{dt} &= (a - bx(t))\, x([t]), \\
\frac{dx}{dt} &= (a - bx([t]))\, x([t+1]), \\
\frac{dx}{dt} &= (a - bx(t))\, x([t+1]), \\
\frac{dx}{dt} &= (a - bx([t+1]))\, x([t]).
\end{aligned}$$

The corresponding reduced discrete equations are

$$x_{k+1} = F_1(x_k) = e^{-b\,x_k}\,(x_k - \frac{a}{b} + e^{b\,x_k}\frac{a}{b}), \tag{2.2.26}$$

$$x_{k+1} = F_2(x_k) = x_k\,(1 - a + b\,x_k)^{-1}, \tag{2.2.27}$$

$$x_{k+1} = F_3(x_k) = x_k e^{-b\,x_{k+1}} + \frac{a}{b}(1 - e^{-b\,x_{k+1}}), \tag{2.2.28}$$

$$x_{k+1} = F_4(x_k) = x_k(1+a)(1+bx_k)^{-1}, \tag{2.2.29}$$

$$\tag{2.2.30}$$

respectively.

One can find that

$$
\begin{aligned}
F_1^2(x_k) &= e^{-b(e^{-bx_k}(x_k-\frac{a}{b})+\frac{a}{b}+x_k)}(x_k-\frac{a}{b})+\frac{a}{b} \\
F_2^2(x_k) &= \frac{x_k}{2\,b\,x_k - a\,b\,x_k + a^2 - 2a + 1} \\
F_3^{-2}(x_k) &= e^{b(e^{bx_k}(x_k-\frac{a}{b})+\frac{a}{b}+x_k)}(x_k-\frac{a}{b})+\frac{a}{b} \\
F_4^2(x_k) &= \frac{(1+a)^2\,x_k}{1+2\,b\,x_k+a\,b\,x_k}
\end{aligned}
$$

We evaluated that only fixed points of $F_j^2$ for $j = 1, 2, 3, 4$ are $x_* = 0$ and $x_*^{'} = \frac{a}{b}$ but these points are also period-1 points of $F_j$ for $j = 1, 2, 3, 4$, hence there is not period-2 point of the maps except the fixed points.

## 2.3 Chaos

In page 27, we reduced the differential equation

$$\frac{dx}{dt} = (a - b\,x([t]))\,x(t),$$

to the discrete equation

$$x_{k+1} = x_k\,e^{a-b\,x_k}. \tag{2.3.31}$$

It is easy to see that the right-handside of (2.3.31) is presented by the function

$$F(x) = x\,e^{a-b\,x}. \tag{2.3.32}$$

Also for the equation

$$\frac{dx}{dt} = (a - b\,x([t]))\,x([t]),$$

the reduced discrete equation is

$$x_{k+1} = x_k\,(1 + a - b\,x_k). \tag{2.3.33}$$

It is a simple matter to observe that the right-handside of (2.3.33) is a function of the form

$$K(x) = x\,(1 + a - b\,x). \tag{2.3.34}$$

If $q_k = \frac{b}{1+a}\,x_k$, then (2.3.33) becomes

$$q_{k+1} = (1 + a)\,q_k\,(1 - q_k). \tag{2.3.35}$$

The right-handside of (2.3.35) is a function of the form

$$G(q) = (1+a)\, q\, (1-q). \tag{2.3.36}$$

So, if $q = \frac{b}{1+a}\, x$, then $K(x) = G(q)$, where

$$G(q) = \mu\, q\, (1-q), \tag{2.3.37}$$

and $\mu = 1 + a$ [5].

It is well known that despite their simplicity the equations (2.3.32), (2.3.37) could generate very complex dynamics which is called ***chaos*** [5, 17]. We shall show it for our cases below.

Now, we study the changes occurring in the period-2 points of $F$ as $a$ varies. We know that the fixed points of $F$ are $\frac{a}{b}$ and 0. And it is also clear that when $0 < a < 2$, $x_*^{'} = \frac{a}{b}$ is a stable fixed point and $x_* = 0$ is an unstable fixed point. The fixed points of $F$ are also the fixed points of $F^n(x)$, where $n \in \mathbb{N}$. In this part of our study, we are interested in the fixed points of $F^2$ that are not the fixed points of $F$.

In the following figure, the graphs of $F$ and $F^2$ are plotted for some values of the parameter $a$.

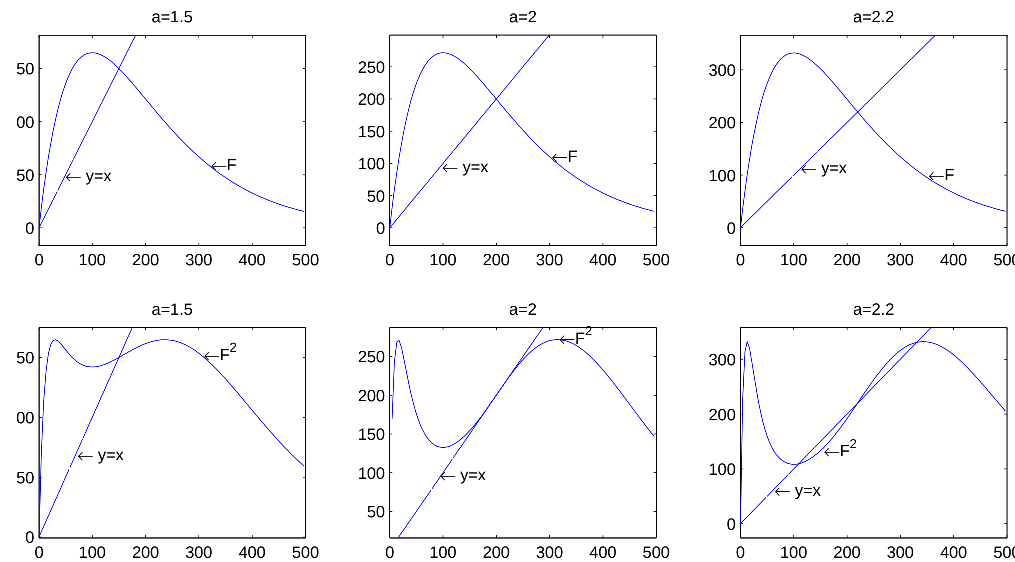


Figure 2.20: For $a > 2$, $F^2(x)$ has two extra fixed points and they are not the fixed points of $F(x)$.

It can be easily seen from the figure that when $a = 1.5$, $F$ and $F^2$ have the same fixed points which are 0 and $\frac{a}{b}$. When $a = 2$, then the straight line $y = x$ is tangent to $F^2$ at a fixed point. When $a = 2.2$, then $F^2$ has two extra fixed points. This means we have a ***period doubling bifurcation*** at $a = 2$ (see [5] for more details).

The following theorem gives the conditions for a one parameter family of one dimensional maps to undergo a period doubling bifurcation (see [29] for more details).

**Theorem 2.1.** Consider a map

$$z \mapsto f(z,\gamma) \quad z \in \mathbb{R}, \quad \gamma \in \mathbb{R}$$

having a fixed point, i.e.,

$$f(z_s,\gamma_s) = z_s, \tag{2.3.38}$$

$$\tfrac{\partial f}{\partial z}(z_s,\gamma_s) = -1, \tag{2.3.39}$$

$$\tag{2.3.40}$$

undergoes a period doubling bifurcation at $(z_s,\gamma_s)$ if

$$\tfrac{\partial f^2}{\partial \gamma}(z_s,\gamma_s) = 0, \tag{2.3.41}$$

$$\tfrac{\partial^2 f^2}{\partial z^2}(z_s,\gamma_s) = 0, \tag{2.3.42}$$

$$\tfrac{\partial^2 f^2}{\partial z \partial \gamma}(z_s,\gamma_s) \neq 0, \tag{2.3.43}$$

$$\tfrac{\partial^3 f^2}{\partial z^3}(z_s,\gamma_s) \neq 0. \tag{2.3.44}$$

Let us consider

$$F(x) = x\, e^{a-b\,x}, \tag{2.3.45}$$

where $b$ is a nonvanishing constant.

This function, $F$, has a fixed point at $(x_s, a_s) = (\frac{2}{b}, 2)$ with

$$F(x_s, a_s) = x_s,$$

$$\frac{\partial F}{\partial x}(x_s, a_s) = -1.$$

Further we note,

$$\frac{\partial F^2}{\partial a}(x_s, a_s) = 0,$$

$$\frac{\partial^2 F^2}{\partial x^2}(x_s, a_s) = 0,$$

$$\frac{\partial^2 F^2}{\partial x \partial a}(x_s, a_s) = 2,$$

$$\frac{\partial^3 F^2}{\partial x^3}(x_s, a_s) = -2\, b^2.$$

Hence, $F$ undergoes a period doubling bifurcation at $(x_s, a_s) = (\frac{2}{b}, 2)$.

Below we will follow F. C. Hoppensteadt and C. S. Peskin [17]. Let us describe the behavior of $F^k$ around the fixed points, if the orbit starts with and the same point $x_0 = 70$. Let us take three different values of $a$ which are $1.5, 2.2$ and 3.

| $k$ | $x\ (a=1.5)$ | $x\ (a=2.2)$ | $x(a=3)$ |
|---|---|---|---|
| 0 | 70.0000 | 70.0000 | 70.0000 |
| 1 | 155.7879 | 313.7182 | 698.1928 |
| 2 | 147.0271 | 122.8929 | 13.0211 |
| 3 | 151.4637 | 324.5321 | 229.6042 |
| 4 | 149.2629 | 114.0987 | 464.1997 |
| 5 | 150.3672 | 329.0063 | 89.8658 |
| 6 | 149.8161 | 110.6104 | 734.8449 |
| 7 | 150.0919 | 330.2699 | 9.4992 |
| 8 | 149.9540 | 109.6410 | 173.5070 |
| 9 | 150.0230 | 330.5644 | 614.7083 |
| 10 | 149.9885 | 109.4161 | 26.4185 |
| 11 | 150.0057 | 330.6291 | 407.4344 |
| 12 | 149.9971 | 109.3667 | 139.1477 |
| 13 | 150.0014 | 330.6431 | 695.1023 |
| 14 | 149.9993 | 109.3560 | 13.3703 |
| 15 | 150.0004 | 330.6461 | 234.9407 |
| 16 | 149.9998 | 109.3537 | 450.3054 |
| 17 | 150.0001 | 330.6467 | 100.1703 |
| 18 | 150.0000 | 109.3532 | 738.9045 |
| 19 | 150.0000 | 330.6469 | 9.1717 |
| 20 | 150.0000 | 109.3531 | 168.0743 |

Table 2.1: Data Generated by $F^k$ where $F(x) = x\, e^{a-bx}$, $x_0 = 70$, $b = 1/100$ and for $a = 1.5$, $a = 2.2$, $a = 3$.

One can see that when $a = 1.5$, the sequence of iterates of $x_0$ under $F$ approaches to the fixed point $x_*^{'} = \frac{a}{b}$. When $a = 2.2$, the iterates of $x_0$ oscillate between two numbers. However, if $a = 3$, the iterates of $x_0$ seem to be random [17].

The long term behavior of $F$ for some values of $a$ can be seen in the ***bifurcation diagram***. The bifurcation diagram (Figure 2.21) plots the $F$'s attractors as a function of $a$ [27].

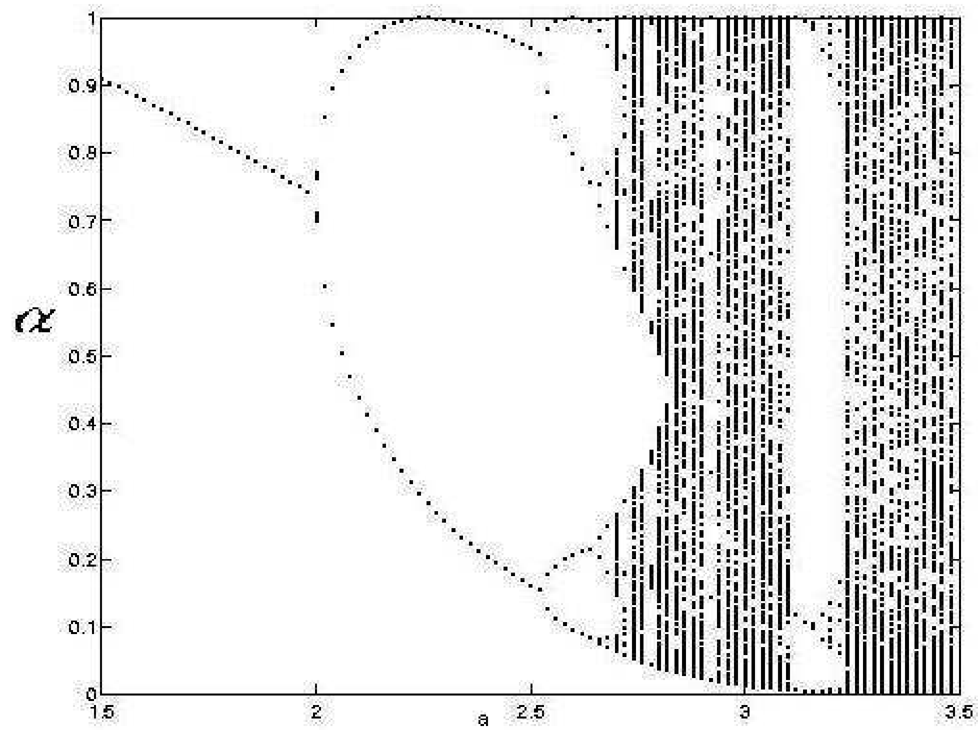


Figure 2.21: Bifurcation Diagram of $F(x) = x\, e^{a-b\,x}$

In order to obtain this diagram, we must use the following program [17]:

Since $F'(x)$ equals to zero when $x = 1/b$, so the range of this function is $0 \leq x \leq e^{a-1}/b$. Next we set $\alpha = b\,x/e^{a-1}$; then the range of $0 \leq \alpha \leq 1$. Substituting $x$ by $\alpha$ in (2.3.33), we get

$$\alpha_{k+1} = \alpha_k\, e^a\, e^{(-e^{a-1})\,\alpha_k}. \tag{2.3.46}$$

Then, we select an initial value from the interval $[0, 1]$ and we choose an $a$. Then, we iterate this initial value for $k = 0, 1, 2, \ldots, 299$, then plot many points say $x_{301}, x_{302}, x_{303}, \ldots, x_{600}$, move another $a$ and repeat same process again [27]. To obtain this bifurcation diagram, we take $a = 1.5$ and $x_0 = 0.4246$ as input values in chaos2.m whose Matlab codes can be seen in chapter 4.

At the Table 2.2, one can see $\alpha$ values corresponding to $x$ values at Table 2.1.

| $k$ | $\alpha$ $(a = 1.5)$ | $\alpha$ $(a = 2.2)$ | $\alpha$ $(a = 3)$ |
|---|---|---|---|
| 0 | 0.4246 | 0.2108 | 0.0947 |
| 1 | 0.9449 | 0.9449 | 0.9449 |
| 2 | 0.8918 | 0.3701 | 0.0176 |
| 3 | 0.9187 | 0.9775 | 0.3107 |
| 4 | 0.9053 | 0.3437 | 0.6282 |
| 5 | 0.9120 | 0.9909 | 0.1216 |
| 6 | 0.9087 | 0.3332 | 0.9945 |
| 7 | 0.9104 | 0.9948 | 0.0129 |
| 8 | 0.9095 | 0.3302 | 0.2348 |
| 9 | 0.9099 | 0.9956 | 0.8319 |
| 10 | 0.9097 | 0.3296 | 0.0358 |
| 11 | 0.9098 | 0.9958 | 0.5514 |
| 12 | 0.9098 | 0.3294 | 0.1883 |
| 13 | 0.9098 | 0.9959 | 0.9407 |
| 14 | 0.9098 | 0.3294 | 0.0181 |
| 15 | 0.9098 | 0.9959 | 0.3180 |
| 16 | 0.9098 | 0.3294 | 0.6094 |
| 17 | 0.9098 | 0.9959 | 0.1356 |
| 18 | 0.9098 | 0.3294 | 1.0000 |
| 19 | 0.9098 | 0.9959 | 0.0124 |
| 20 | 0.9098 | 0.3294 | 0.2275 |

Table 2.2: The Data in Table 2.1 after changing variable $x$ into $\alpha$.

We see in Figure 2.21 when $a < 2$, then all iterates go to a constant number. For $2 < a < 2.5$ all iterates oscillate between two numbers. This was illustrated in Table 2.2. Things get more complicated beyond $a = 2.5$.

Similarly, the equation

$$\frac{dx}{dt} = (a - b\,x([t+1]))\,x(t),$$

becomes reduced to the equation

$$x_{k+1} = x_k\,e^{a - b\,x_{k+1}},$$

which gives us

$$x_k = x_{k+1}\,e^{b\,x_{k+1} - a}. \tag{2.3.47}$$

If $x_k = -q_k$ then (2.3.47) becomes

$$q_k = q_{k+1}\,e^{-b\,q_{k+1} - a}.$$

For, $k = -n, n \in \mathbb{Z}$, the last equation is written as $q_{-n} = q_{1-n}e^{-b\,q_{1-n} - a}$. Finally, by means of the change of variable, $r_n = q_{1-n}$ for $n \in \mathbb{Z}$ we obtain

$$r_{n+1} = r_n\,e^{-b\,r_n - a}. \tag{2.3.48}$$

The right-handside of the equation (2.3.48) is a function of the form [17]

$$T(r) = r\,e^{-a - b\,r}. \tag{2.3.49}$$

Similar with the previous equation (2.3.32), the equation (2.3.49) could generate very complex dynamics, too.

Now, let us study the period-2 points of $G$. Recall that $G(q) = \mu\, q\,(1-q)$ where $q = \frac{b}{1+a}x$ and $\mu = 1+a$. Below we follow John Banks, Valentina Dragan, Arthur Jones [5].

The following figure shows the graphs of $G$ and $G^2$ for some typical values of the parameter $\mu$.

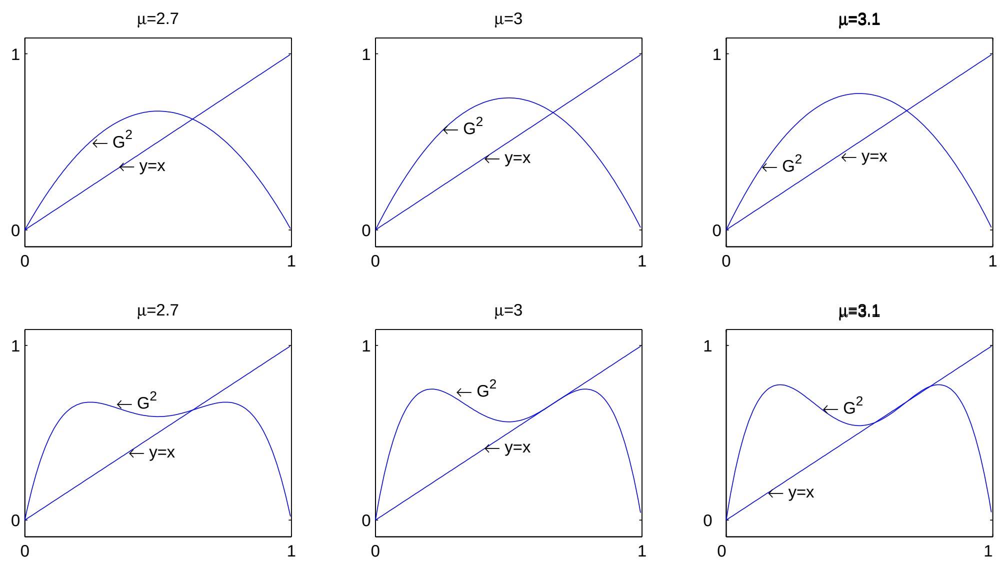


Figure 2.22: For $\mu > 3$, $G^2(q)$ has two extra fixed points and they are not the fixed points of $G(q)$.

When $\mu = 2.7$, both $G$ and $G^2$ have the same fixed points. When $\mu = 3$, the line $y = x$ is tangent to $G^2$ at a fixed point. And when $\mu = 3.1$, $G^2$ has two extra fixed points. Therefore, we can say that period doubling appears at $\mu = 3$.

Thus, $G$ has a fixed point at $(q_s, \mu_s) = (2/3, 3)$ with an eigenvalue $-1$,i.e.,

$$G(q_s, \mu_s) = q_s,$$
$$\frac{\partial G}{\partial q}(q_s, \mu_s) = -1.$$

Moreover,

$$\frac{\partial G^2}{\partial \mu}(q_s, \mu_s) = 0,$$
$$\frac{\partial^2 G^2}{\partial q^2}(q_s, \mu_s) = 0,$$
$$\frac{\partial^2 G^2}{\partial q \partial \mu}(x_s, a_s) = 2,$$
$$\frac{\partial^3 G^2}{\partial q^3}(q_s, \mu_s) = -108.$$

Thus, $G$ satisfies the properties of Theorem 2.1., so $G$ has a period doubling bifurcation at $(q_s, \mu_s) = (2/3, 3)$.

Figure 2.22 shows that if $\mu > 3$, then $G^2$ has two extra fixed points. Since the map, $G$, is a quadratic polynomial, this fact can be proved algebraically.

The fixed points of $G^2$ are the solutions of the following equation

$$G^2(q) = q \qquad (2.3.50)$$

Here $G(q) = \mu\, q\, (1 - q)$, so the equation (2.3.50) has the following form

$$\mu^2 q(1-q) - \mu^3 q^2 (1-q)^2 = q \qquad (2.3.51)$$

But $G$, and hence $G^2$, has two fixed points 0 and $1-\frac{1}{\mu}$. After factoring out $q$ and $q-(1-\frac{1}{\mu})$, one can obtain a pair of roots

$$q_1, q_2 = \frac{(\mu+1) \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu}, \tag{2.3.52}$$

when $\mu = \mu_s$, $q_1$, $q_2$ and $1-\frac{1}{\mu} = \frac{2}{3}$ coincide. For $\mu > \mu_s$, $q_1$ and $q_2$ are real and $1-\frac{1}{\mu}$ loses its stability. If $3 < \mu < 1+\sqrt{6}$, $q_1$ and $q_2$ are stable and if $1+\sqrt{6} < \mu \leq 4$ they are unstable (see [5] for details). This means that when $\mu > 1+\sqrt{6}$ the map, $G$, has extra period-4 points [24].

Recall that $K(x) = G(q)$, $q = \frac{b}{1+a}x$ and $\mu = 1+a$. The second iterate of $K$ has four fixed points 0, $\frac{a}{b}$, $x_1 = \frac{(a+2)+\sqrt{(a+2)(a-2)}}{2b}$ and $x_2 = \frac{(a+2)-\sqrt{(a+2)(a-2)}}{2b}$.

Notice that if $a = a_s = 2$, we have $x_1 = x_2 = \frac{a}{b} = \frac{2}{b}$. For $a > a_s$, $x_1$ and $x_2$ are real and $\frac{a}{b}$ is unstable. So, the map, $K$, has ***a period doubling bifurcation*** at $a_s = 2$.

If $2 < a < \sqrt{6}$, $x_1$ and $x_2$ are stable and if $\sqrt{6} < a \leq 3$ they are unstable. Hence, when $a > \sqrt{6}$, new fixed points of $K^4$ appears.

In the following table, one can see the behavior of $K^k$ around the fixed points when $x_0 = 40$.

| $k$ | $x\,(a = 1.7)$ | $x\,(a = 2.1)$ | $x\,(a = 2.9)$ |
|---|---|---|---|
| 0 | 40.0000 | 40.0000 | 40.0000 |
| 1 | 76.0000 | 92.0000 | 124.0000 |
| 2 | 89.6800 | 115.9200 | 176.0800 |
| 3 | 81.2860 | 90.6031 | 66.6287 |
| 4 | 87.3240 | 116.6912 | 171.0642 |
| 5 | 83.2652 | 89.4060 | 81.8911 |
| 6 | 86.1542 | 117.2899 | 185.2522 |
| 7 | 84.1654 | 88.4602 | 36.1159 |
| 8 | 85.5703 | 117.7225 | 114.7647 |
| 9 | 84.5943 | 87.7681 | 184.1636 |
| 10 | 85.2807 | 118.0163 | 39.9135 |
| 11 | 84.8019 | 87.2935 | 123.8009 |
| 12 | 85.1379 | 118.2067 | 176.2902 |
| 13 | 84.9031 | 86.9842 | 65.9671 |
| 14 | 85.0676 | 118.3260 | 170.2385 |
| 15 | 84.9526 | 86.7898 | 84.3073 |
| 16 | 85.0332 | 118.3990 | 186.6440 |
| 17 | 84.9768 | 86.6704 | 31.1918 |
| 18 | 85.0163 | 118.4431 | 102.1894 |
| 19 | 84.9886 | 86.5983 | 189.6852 |
| 20 | 85.0080 | 118.4694 | 20.1628 |

Table 2.3: Data Generated by $K^k$ where $K(x) = x\,(1 + a - b\,x)$ $x_0 = 40, b = 1/50$ and for $a = 1.7, a = 2.1, a = 2.9$.

It is easy to observe that when $a = 1.7$, the sequence of the iterates of $x_0$ approaches to the fixed point $\frac{a}{b}$. When $a = 2.1$, the iterates settle to simple oscillation between two numbers. However, when $a = 2.9$, the iterates seem to be random.

The following table shows the $q$ values corresponding to $x$ values in Table 2.3.

| $k$ | $q(a = 1.7)$ | $q(a = 2.1)$ | $q(a = 2.9)$ |
|---|---|---|---|
| 0 | 0.2963 | 0.2581 | 0.2051 |
| 1 | 0.5630 | 0.5935 | 0.6359 |
| 2 | 0.6643 | 0.7479 | 0.9030 |
| 3 | 0.6021 | 0.5845 | 0.3417 |
| 4 | 0.6468 | 0.7528 | 0.8773 |
| 5 | 0.6168 | 0.5768 | 0.4200 |
| 6 | 0.6382 | 0.7567 | 0.9500 |
| 7 | 0.6234 | 0.5707 | 0.1852 |
| 8 | 0.6339 | 0.7595 | 0.5885 |
| 9 | 0.6266 | 0.5662 | 0.9444 |
| 10 | 0.6317 | 0.7614 | 0.2047 |
| 11 | 0.6282 | 0.5632 | 0.6349 |
| 12 | 0.6307 | 0.7626 | 0.9041 |
| 13 | 0.6289 | 0.5612 | 0.3383 |
| 14 | 0.6301 | 0.7634 | 0.8730 |
| 15 | 0.6293 | 0.5599 | 0.4323 |
| 16 | 0.6299 | 0.7639 | 0.9571 |
| 17 | 0.6295 | 0.5592 | 0.1600 |
| 18 | 0.6298 | 0.7641 | 0.5240 |
| 19 | 0.6295 | 0.5587 | 0.9727 |
| 20 | 0.6297 | 0.7643 | 0.1034 |

Table 2.4: $q$ values corresponding $x$ values at Table 2.3.

The bifurcation diagram of $G(q) = \mu\, q(1-q)$ can be seen in the following figure. The Matlab codes of this diagram can be seen in chaos5.m in chapter 4. To obtain this diagram we take $\mu = 2$, $x_0 = 0.2963$ as input values in chaos5.m.

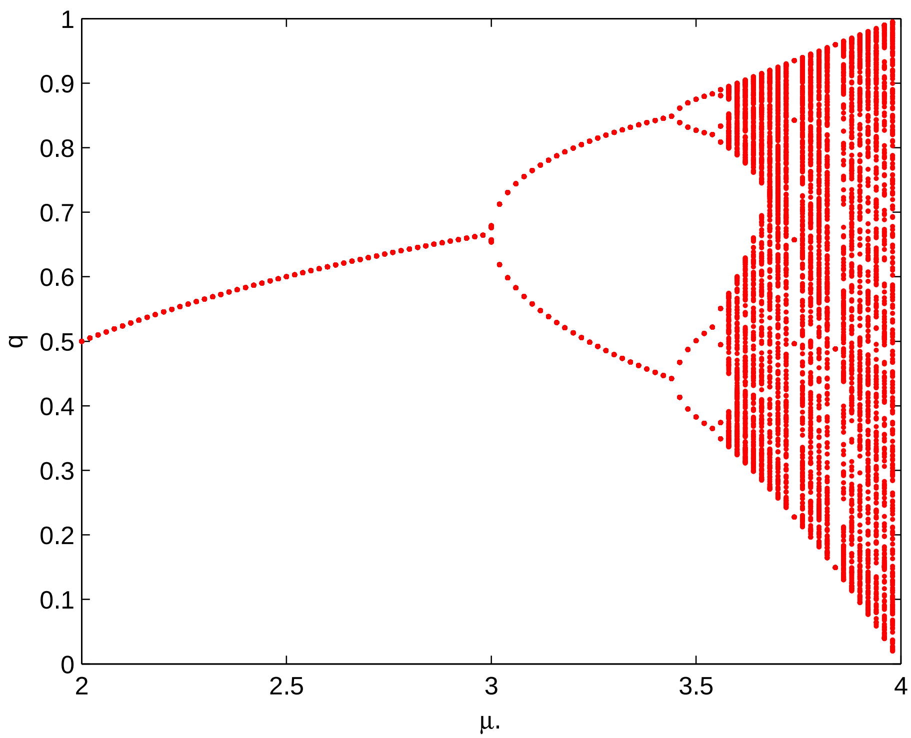


Figure 2.23: Bifurcation Diagram of $G(q) = \mu\, q\,(1-q)$.

We observed that at $\mu = 3$, period doubling appears, indicated by the new branches in Figure 2.23. And for certain $\mu$ values all branches in the diagram split into two new branches. And at $\mu = \mu_\infty \cong 3.57$, $G$ becomes chaotic and the attractor changes from a finite to an infinite set of points. One can see for $\mu > \mu_\infty$ the bifurcation diagram alternates between chaos and periodic orbits of other periods (see [5, 27] for more details). Since $K(x) = G(q)$, $q = \frac{b}{1+a}x$ and $\mu = 1 + a$, then at $a = a_\infty \cong 2.57$ the map, $K$, has a chaotic behavior, too.

Now, let us consider the following equation

$$\frac{dx}{dt} = (a - bx([t+1]))\, x([t+1]),$$

the corresponding discrete equation is

$$x_{k+1} = x_k \ (1 - a + b\, x_{k+1})^{-1}, \tag{2.3.53}$$

which gives us

$$x_k = x_{k+1} \ (1 - a + b\, x_{k+1}). \tag{2.3.54}$$

Now, the transformation $q_k = \frac{b}{a-1}x_k$ in (2.3.54) yields

$$q_k = (1 - a)\, q_{k+1}\, (1 - q_{k+1}).$$

For $k = -n$, $n \in \mathbb{Z}$, the last discrete-time equation takes the following form

$$q_{-n} = (1 - a)\, q_{1-n}\, (1 - q_{1-n}).$$

Set $r_n = q_{1-n}$ where $n \in \mathbb{Z}$, then

$$r_{n+1} = (1 - a)\, r_n\, (1 - r_n). \tag{2.3.55}$$

The right-handside of the equation is a function of the form

$$P(r) = \mu\, r\,(1 - r) \tag{2.3.56}$$

where $\mu = 1 - a$.

It is known that the relation determines chaos through period doubling [5].

# CHAPTER 3

# CONCLUSION

In this study we investigate the extensions of logistic equation with piecewise constant arguments which can be retarded as well as advanced.

Existence of fixed points; periodic solutions, chaos is proved for the different type of extensions.

Computer simulations for the solutions and bifurcation diagrams are given.

All results have a biological sense.

# CHAPTER 4

# APPENDIX

Below we present the Matlab m-files which are used to obtain bifurcation diagrams.

*chaos2.m*

*Bifurcation diagram of* $x_{k+1} = x_k e^{(a-bx_k)}$ *whose reduced form is* $\alpha_{k+1} = \alpha_k e^{(a)} e^{(-e^{a-1})\alpha_k}$ *where* $\alpha_{k+1} = bx_{k+1}/e^{a-1}$.

```
function t=chaos2(a,x0)
%a, x0  inputs
% t is the output of the function
m=x0;
for j=1:100
    x0=m;
  for i=1:300
    k=x0*exp(a)*exp(-exp(a-1)*x0);
    x0=k;
```

```
  end
 plot(a,x0,'r.')
 hold on
   for k=1:300
     t=x0*exp(a)*exp(-exp(a-1)*x0);
     x0=t;
     plot(a,t,'r.')
     hold on
   end
 a=a+0.02;
end
```

*chaos5.m*

*Bifurcation diagram of the logistic equation* $G(q) = \mu q(1-q)$.

```
function t=chaos5(n,x0)
%n, x0  inputs
%t is the output
m=x0;
for i=1:300
    x0=m;
    k=n*x0*(1-x0);
    x0=k;
end
 plot(n,x0,'r.')
 hold on
for k=1:300
    t=n*x0*(1-x0);
    x0=t;
    plot(n,t,'r.')
    hold on
  end
 n=n+0.02;
 end
```

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