

YIELD CURVE ESTIMATION BY SPLINE-BASED MODELS

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ABSTRACT

YIELD CURVE ESTIMATION BY SPLINE-BASED MODELS

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This thesis uses Spline-based model, which was developed by McCulloch, and parsimonious model, which was developed by Nelson-Siegel, to estimate the yield curves of zero-coupon bonds in Turkey. In this thesis, we construct the data by using Turkish secondary government zero-coupon bond data, which contain the data from January 2005 to June 2005. After that, relative performances of models are compared using in-sample goodness of fit. As a result, we see that performance of McCulloch model in fitting yield is better than that of Nelson-Siegel model.

Keywords: Yield curves, zero-coupon bonds, B-spline, The McCulloch and Nelson-Siegel Model

ÖZ

GETİRİ EĞRİLERİNİN SPLINE BAZLI MODELLERLE BELİRLENMESİ

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Bu çalışmada, Türkiye'deki kuponsuz bonoların getiri eğrilerini belirlemek için, McCulloch tarafından geliştirilen Spline bazlı modelini ve Nelson ve Siegel tarafından geliştirilen, parsimonious modelini kullandık. Bu çalışmadaki verimizi Ocak 2005'ten Haziran 2005'e kadar olan Türkiye'deki kuponsuz bonoları kullanarak oluşturduk. Daha sonra, kullandığımız modellerin performansları bu veriye göre karşılaştırıldı. Sonuç olarak, McCulloch modeli, getirilerin tahmin edilmesinde Nelson-Siegel modelinden daha iyi performans göstermektedir.

Anahtar Kelimeler: Getiri eğrileri, kuponsuz bonolar, B-spline, The McCulloch ve Nelson-Siegel modeli

To my family

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CHAPTER 1

INTRODUCTION

The term structure modelling play a major role in fixed income attribution analysis, since movements in yield curves is crucial for pricing and the return of fixed income securities. Therefore, the modelling the term structure is one of the major subject in economics and in finance. There are many models proposed to construct term structure. There are two class of models used in estimating term structure, these are the equilibrium models and the empirical models. The equilibrium models are used by Vasicek (1977), Dothan (1978) and Schwartz (1979). If multi-factor models are not used, the performance of the equilibrium models is not good in fitting observed data. However, the empirical models use observed government bond prices to construct the term structure and have a good performance in the estimation of term structure. There are many examples of the empirical models, such as, models developed by McCulloch (1971), Vasicek and Fong (1982) and Nelson and Siegel (1987). The main purpose of these empirical models is to fit data sufficiently well and sufficiently smooth. Moreover, the resulting function must be continuous.

In this thesis we use Spline-based model, which was developed by McCulloch, and parsimonious model, which was developed by Nelson-Siegel. The parsimonious model specify a functional form for implied forwards rate, in other words, we can define implied forward rate with a function which has some parameters. However, in McCulloch model we use B-splines for fitting discount

function. Since B-splines are defined piecewise in small subintervals, they give stable and reliable results in fitting discount function. B-splines was firstly used in this thesis for fitting Turkish government zero-coupon bond data and we got reliable results in the application. In this study, we construct our data by using Turkish secondary government zero-coupon bond data, which contain the data from January 2005 to June 2005, and relative performances of models are compared using in-sample goodness of fit.

We proceed as follows. Chapter 2 provides basic definitions of the term structure theory, Chapter 3 presents basics of interpolation and spline methods and their applications, Chapter 4 contain descriptions of Mc Culloch and Nelson-Siegel model respectively and also applications of these models to the Turkish government zero-coupon bond data. Finally, the conclusions of this study are presented in Chapter 5.

CHAPTER 2

THE GENERAL THEORY OF THE TERM STRUCTURE MODELS

In this chapter we present a general framework for the term structure models which describe the dynamics of bond prices. The main purpose of this chapter is to discuss the common features of the different models presented in the next sections [1].

2.1 AN INTRODUCTION TO PRICING OF BONDS

The term structure of interest rates can be characterized either in term of discount bond prices, interest rates or forward rates, so we begin this section by defining the relationship between these concepts. In the following, t is the present time, $T \geq t$ is the time of maturity of the bond, while $\tau = T - t$ is the time to maturity of the bond.

From the point of view of the holder a discount bond is a claim to one certain unit of account at some future point of time T . Since this bond only has one payment it is often called a zero coupon bond, but we will use the first term. We denote its price at time t by $P(t, T)$. If $P(t, T)$ is viewed as a function of T it is called the discount function which is denoted by $d(t, T)$. The instantaneously

compounded interest rate $R(t, T)$ is the yield to maturity at time t for investment in the bond maturing at time T . Since

$$P(t, T) = e^{-R(t, T)(T-t)} \quad (2.1.1)$$

Then we get the following equation for $R(t, T)$.

$$R(t, T) = -\frac{1}{T-t} \log P(t, T) \quad (2.1.2)$$

The general definition of the term structure at time t is $R(t, t+\tau)$ as a function of the time to maturity τ , and in this sense $R(t, t+\tau)$ is often called the yield curve. The spot interest $r(t)$ plays a crucial role in all term structure models, and it can be obtained by letting T approach t in (2.1.2)

$$r(t) = R(t, t) = \lim_{T \rightarrow t} R(t, T) \quad (2.1.3)$$

The spot rate is sometimes called the instantaneous interest rate or risk free interest rate since it is the guaranteed return for placements between time t and $t+dt$. The final way to characterize the term structure is in terms of forward rates. The continuously compounded forward rate $f(t, T)$ is defined as

$$f(t, T) = \frac{-\partial P(t, T)/\partial T}{P(t, T)} = \frac{-\partial \log P(t, T)}{\partial T} \quad (2.1.4)$$

The forward rate $f(t, T)$ is the guaranteed return for investments between time T and $T+dt$. It can be secured by selling one bond maturing at time T and using the proceeds to buy bonds maturing at time $T+dt$. Alternatively, $f(t, T)$ can be viewed as the marginal rate of return between T and $T+dt$. Since the forward rate is an interest rate it must be non-negative, and from (2.1.4) this is seen to be equivalent

to a monotonically declining discount function. By solving the differential equation (2.1.4) we can express bond prices in terms of forward rates

$$\log P(t, T) - \log P(t, t) = \int_t^T \frac{\partial \log P(t, s)}{\partial s} ds = \int_t^T -f(t, s) ds \quad (2.1.5)$$

since $P(t, t) = 1$, so we get

$$P(t, T) = \exp\left(-\int_t^T f(t, s) ds\right) \quad (2.1.6)$$

From the definition of the forward rate we must obtain the spot rate if we let T approach t , i.e.,

$$r(t) = f(t, t) = \lim_{T \rightarrow t} f(t, T) \quad (2.1.7)$$

The traditional arbitrage-free models of the term structure are all based on the spot interest rate, whereas the recent arbitrage-free models based on the martingale measure technique use the discount function or the forward rates.

Most bonds have standardized repayment schemes e.g. straight bonds (fixed loans), annuities, serial bonds or consols, but we shall not be concerned with the exact payment profile, and we will therefore use the term coupon bond for all bonds with more than one payment. For a coupon bond with M payments $c(T_j)$ denote the payment at time T_j with $j=1, 2, \dots, M$, and we define the time to maturity of the bond as T_M , the date of the last payment. Corresponding to (2.1.1) we can define the yield to maturity $Y_c(t, T)$ for a coupon bond with price $P_c(t, T)$ as the positive root of

$$P_c(t, T) = \sum_{j=1}^M c(T_j) e^{-(T-t)Y_c(T, t)} \quad (2.1.8)$$

Two discount bonds maturing at the same date must obviously sell for the same prices, but this relationship does not hold for coupon bonds. The reason for this is quite obvious when we consider that two coupon bonds maturing at the same date may have quite different payment profiles. Contrary to discount bonds they are not identical, so we cannot use the law of one price. In general, unless the term structure is flat two coupon bond with different payment profiles will not sell for the same yield to maturity.

The solution to the problem is quite simple. A coupon bond is nothing else than a bundle of discount bonds, which suggests that it should command the price

$$P_c(t, T) = \sum_{j=1}^M d(t, T) c(T_j) = \sum_{j=1}^M c(T_j) e^{-R(t, T_j) (T_j - t)} \quad (2.1.9)$$

Note that we are using different interest rates to discount each payment. It is easily shown, that if coupon bonds are not priced in this way, arbitrage opportunities exist. For the moment, we define an arbitrage opportunity as a portfolio which generates a positive cash flow at time t and a non-negative cash flow at any future payment date. Let there be n bonds and m different payment dates, and let $C = \{c_{ij}\}$ be $n \times m$ payoff matrix. An arbitrage opportunity exists if we can find a vector w that solves the system of linear inequalities

$$\begin{aligned} w^T P &< 0 \\ C^T w &\geq 0 \end{aligned} \quad (2.1.10)$$

There is no need to formulate this as an mathematical optimization problem since once we have found an arbitrage opportunity, it can be carried out to an

unlimited scale. Stated otherwise, if w is a solution to (2.1.10), so is λw for any positive scalar λ . From Farka's lemma we know that (2.1.10) does not possess a solution (i.e. no arbitrage opportunities exist) if and only if the following problem has a solution vector d

$$\begin{aligned} Cd &= p \\ d &\geq 0 \end{aligned} \tag{2.1.11}$$

We can take d as the vector of discount factors for the M payment dates. In general though, the mere knowledge of bond prices will not be sufficient to determine whether the bond market is in arbitrage equilibrium. In the paper of Guo Chen (1999), it was proved that arbitrage-free equilibrium is unique. The unique solution is a linear combination of some exponential functions, which is referred to as the Exponential Polynomial (EP) model, because it is equivalent to the component function of the Exponential Spline model of Vasicek and Fong (1982) defined on a subinterval of the maturity range. In other words, EP model is equivalent to the Vasicek-Fong model without spline fitting, i.e. with the subinterval stretched to the entire maturity range $[0, \infty)$. Intuitively, EP model represents a term structure space that is linearly spanned by its state factors on an exponential basis that consists of a number of distinct exponential functions.

McCulloch (1971) introduced the spline method because the observed cross-sectional samples of treasury securities usually contain far more short-term securities than the long-term ones. To accommodate the data structure, he suggested dividing the maturity range into subintervals, such that the number of securities in each subinterval is roughly the same. By dividing the maturity, he applied the same spline function, e.g., a cubic function, to all subintervals. Since each spline function is defined only on a subinterval, every pair of adjacent spline

functions have to carefully connected to ensure continuity and differentiability at the knot. Let us now look at the firstly interpolation methods and then spline methods, which will be used in Chapter 4 in McCulloch model.

CHAPTER 3

INTERPOLATION AND SPLINE METHODS

3.1 INTERPOLATION

Definition 3.1.1 For given data (k_i, y_i) $i=1, \dots, n$ with $k_1 < k_2 < \dots < k_n$, we determine function f such that $f(k_i) = y_i$ $i=1, \dots, n$ where f is called interpolating function, or interpolant for given data. Additional constraints might be imposed on interpolant, such as, smoothness, monotonicity or convexity. There are many purposes for interpolation [2].

- (i) Plotting smooth curve through discrete data points.
- (ii) Quick and easy evaluation of mathematical function.
- (iii) Replacing difficult function by easy one.
- (iv) Differentiating or integrating tabular data.

Interpolation is nonunique because there are arbitrarily many functions that interpolate given set of data points. Choice of interpolating function is based on:

- (a) How easy function is to work with,
 - (i) determining its parameters.
 - (ii) evaluating function.
 - (iii) differentiating or integrating function.
- (b) How well properties of function match properties of data to be fit (smoothness, monotonicity, convexity, etc.).

Some families of functions commonly used for interpolation include, polynomials, trigonometric functions and exponentials. Family of functions for interpolating given set of data points is spanned by set of basis functions $\phi_1(t), \dots, \phi_n(t)$. Interpolating function f is chosen as a linear combination of basis functions,

$$f(t) = \sum_{j=1}^n x_j \phi_j(t) \quad (3.1.1)$$

If we require that f interpolate data (k_i, y_i) , we mean that

$$f(k_i) = \sum_{j=1}^n x_j \phi_j(k_i) = y_i, \quad i = 1, \dots, n$$

which is system of linear equations $Ax=y$ for

parameters x_j , where entries of matrix A are given by $a_{ij} = \phi_j(k_i)$.

Simplest type of interpolation is polynomials. Unique polynomial of degree at most $n-1$ passes through n data points (k_i, y_i) , $i = 1, \dots, n$ where k_i are distinct. There are many ways to represent or compute polynomial, but in theory all must give same result. Let us look at types of basis are used in the interpolation.

3.1.1 Monomial Basis

We define monomial basis as follows,

$$\phi_j(t) = t^{j-1}, \quad j = 1, \dots, n$$

interpolating polynomial has form

$P_{n-1}(t) = x_1 + x_2 t + \dots + x_n t^{n-1}$ and its coefficients are determined by $n \times n$ linear system,

$$\begin{pmatrix} 1 & k_1 & \dots & k_1^{n-1} \\ 1 & k_2 & \dots & k_2^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & k_n & \dots & k_n^{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Matrix of this form is called Vandermonde matrix. For monomial basis, resulting matrix A is often ill-conditioned, especially for high-degree polynomials. Therefore, we should seek better alternatives.

3.1.2 Lagrange Interpolation

For given set of data points (k_i, y_i) , $i = 1, \dots, n$, Lagrange basis functions are given by

$$\ell_j(t) = \frac{\prod_{i=1, i \neq j}^n (t - k_i)}{\prod_{i=1, i \neq j}^n (k_j - k_i)}$$

For Lagrange basis,

$$\ell_j(k_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad j = 1, \dots, n$$

which means that matrix of linear system $Ax=y$ is identity. Thus, Lagrange polynomial interpolating data points (k_i, y_i) is given by,

$$P_{n-1}(t) = x_1 \ell_1(t) + x_2 \ell_2(t) + \dots + x_n \ell_n(t)$$

Lagrange interpolant is easy to determine but more expensive to evaluate for given argument, compared with monomial basis representation. Lagrangian form is also more difficult to differentiate and integrate.

3.1.3 Newton Interpolation

For a given set of data points (k_i, y_i) , $i = 1, \dots, n$ Newton interpolating polynomial has the form,

$$P_{n-1}(t) = x_1 + x_2(t-k_1) + x_3(t-k_1)(t-k_2) + \dots + x_n(t-k_1)(t-k_2)\dots(t-k_{n-1})$$

Basis functions for Newton interpolation are given by

$$\phi_j(t) = \prod_{i=1}^{j-1} (t-k_i), \quad j = 1, \dots, n$$

For $i < j$, we have $\phi_j(k_i) = 0$ so basis matrix A , with $a_{ij} = \phi_j(k_i)$ is lower triangular. Hence, solution x to system $Ax=y$ can be computed by forward-substitution. Newton interpolation has better balance between cost of computing interpolant and cost of evaluating it.

3.1.4 Disadvantages of Interpolation

Interpolating polynomials of high degree are expensive to determine and evaluate. Moreover, in some basis coefficients of polynomial may be poorly determined due to ill-conditioning of linear system to be solved. High-degree polynomial necessarily has lots of “wiggles” which may have not relation with the data to be fit. Polynomial goes through required data points, but it may oscillate wildly between data points.

Fitting single polynomial to large number of data points yield unsatisfactory oscillating behaviour in interpolant. However, fitting data piecewise has an advantage that large number of data points can be fit with low-degree polynomials and we do not have oscillation. Thus, let us look at B-splines which are used for fitting data piecewise.

3.2 B-SPLINES

In general a spline is a piecewise polynomial of degree k that is continuously differentiable $k-1$ times. A cubic B-spline is a piecewise cubic polynomial that is twice continuously differentiable. In this section we will collect the basic properties of splines and B-splines together for further reference. We will intuitively explain why B-splines are important. We will also give an impression of the computational complexity of the B-spline algorithms. In this section derivation of the cubic B-spline basis is involved, since it provides a useful tool for the general construction of cubic splines. A cubic B-spline is a piecewise cubic polynomial that takes positive values over only four adjacent subintervals in the overall partition. On all other subintervals, cubic B-spline vanishes. Moreover, any cubic spline on $[a,b]$ can be constructed as a linear combination of this sequence of cubic B-splines. Finally, because these cubic B-splines are defined piecewise, this linear combination is easy to compute and numerically stable [3].

Let us begin with general demonstration of cubic B-spline. In this thesis we denote the cubic B-spline as B_i . This B-spline defined on an arbitrary interval in the following manner.

$$B_i(x) = \begin{cases} 0 & \text{if } x \in (-\infty, k_i) \\ B_i & \text{if } x \in [k_i, k_{i+4}] \\ 0 & \text{if } x \in (k_{i+4}, \infty) \end{cases} \quad (3.2.1)$$

This is the important property of the B-spline but the question remains as to how we employ these mathematical objects in the construction of cubic splines. We must first discuss how we might construct a basis for the cubic splines on a given interval. We do not generally talk about a single B-spline, but rather consider a sequence of B-splines. For example, to create a basis for the knot sequence $\{k_0, k_1, \dots, k_N\}$, we would require the collection,

$$\{B_{-3}, B_{-2}, \dots, B_{N-1}\} \quad (3.2.2)$$

comprising $N+3$ B-splines defined on $\{k_{-3}, k_{-2}, \dots, k_{N+3}\}$.

By using this sequence of $N+3$ splines for a given knot sequence $\{k_0, k_1, \dots, k_N\}$ we can write,

$$S(x) = \sum_{i=-3}^{N-1} a_i B_i(x) \quad (3.2.3)$$

In other words, a cubic spline can be written as a linear combination of the B-spline basis. To be able to evaluate B-splines for an arbitrary point $x \in (k_i, k_{i+1})$ we need recursion formula for B-splines. To write our recursion formula, we need to introduce the idea of the degree of a B-spline basis. Technically, an n order B-spline with knot sequence $\{k_0, k_1, \dots, k_N\}$ is a $(n-1)$ th degree polynomial that is $(n-2)$ times continuously differentiable on the knot sequence. Thus, a cubic B-spline has order equal to four; moreover, we denote the i th B-spline of order n as, $B_{i,n}$. The order of the B-spline is important because the B-spline recursion formula is written in terms of B-splines of lesser order. It has the following form,

$$B_{i,n}(x) = \frac{x - k_i}{k_{i+n-1} - k_i} B_{i,n-1}(x) + \frac{k_{i+n} - x}{k_{i+n} - k_{i+1}} B_{i+1,n-1}(x) \quad (3.2.4)$$

for $i=-3, \dots, N-1$. To be able to use this formula, we need to know how to define $B_{i,1}$ because it is the final point in the recursion. If we know $B_{i,1}$, it is sufficient to determine any value of our cubic B-spline of interest, $B_{i,4}$. The first-order B-spline, defined as in the following form,

$$B_{i,1}(x) = \begin{cases} 0 & \text{if } x \in (-\infty, k_i) \\ 1 & \text{if } x \in [k_i, k_{i+1}] \\ 0 & \text{if } x \in (k_{i+1}, \infty) \end{cases} \quad (3.2.5)$$

Using equations (3.2.4) and (3.2.5), it is easy to evaluate a given cubic B-spline at any point $x \in (k_i, k_{i+1})$, for example, if we want to evaluate $B_{i,2}$, B-spline of order-2 (degree-1), we use the recursion formula and we get,

$$B_{i,2}(x) = \frac{x - k_i}{k_{i+1} - k_i} B_{i,1}(x) + \frac{k_{i+2} - x}{k_{i+2} - k_{i+1}} B_{i+1,1}(x) \quad (3.2.6)$$

To construct $B_{i,2}(x)$, we need to know $B_{i,1}(x)$ and $B_{i+1,1}(x)$. $B_{i,1}(x)$ is defined in equation (3.2.5), and using this equation we can determine $B_{i+1,1}(x)$, where

$$B_{i+1,1}(x) = \begin{cases} 0 & \text{if } x \in (-\infty, k_{i+1}) \\ 1 & \text{if } x \in [k_{i+1}, k_{i+2}] \\ 0 & \text{if } x \in (k_{i+2}, \infty) \end{cases} \quad (3.2.7)$$

Using equations (3.2.5), (3.2.6), and (3.2.7) we evaluate $B_{i,2}(x)$, which is a piecewise function defined on $[k_i, k_{i+2}]$.

$$B_{i,2}(x) = \begin{cases} \frac{x - k_i}{k_{i+1} - k_i} & \text{for } k_i \leq x < k_{i+1} \\ \frac{k_{i+2} - x}{k_{i+2} - k_{i+1}} & \text{for } k_{i+1} \leq x < k_{i+2} \\ 0 & \text{otherwise} \end{cases} \quad (3.2.8)$$

Continuing in this way, we determine our cubic B-spline, $B_{i,4}(x)$, takes the following form,

$$B_{i,4}(x) = \begin{cases} \frac{(x-k_i)^3}{(k_{i+3}-k_i)(k_{i+1}-k_i)(k_{i+2}-k_i)} & \text{for } k_i \leq x < k_{i+1} \\ \frac{(x-k_i)^2(k_{i+2}-x)}{(k_{i+3}-k_i)(k_{i+2}-k_i)(k_{i+2}-k_{i+1})} + \frac{(x-k_i)(k_{i+3}-x)(x-k_{i+1})}{(k_{i+3}-k_i)(k_{i+3}-k_{i+1})(k_{i+2}-k_{i+1})} + \dots \\ \frac{(k_{i+4}-x)(x-k_{i+1})^2}{(k_{i+4}-k_{i+1})(k_{i+2}-k_{i+1})(k_{i+3}-k_{i+1})} & \text{for } k_{i+1} \leq x < k_{i+2} \\ \frac{(x-k_i)(k_{i+3}-x)^2}{(k_{i+3}-k_i)(k_{i+3}-k_{i+1})(k_{i+3}-k_{i+2})} + \frac{(k_{i+4}-x)}{(k_{i+4}-k_{i+1})} \dots & (3.2.9) \\ \left\{ \frac{(x-k_{i+1})(k_{i+3}-x)}{(k_{i+3}-k_{i+1})(k_{i+3}-k_{i+2})} + \frac{(k_{i+4}-x)(x-k_{i+2})}{(k_{i+4}-k_{i+2})(k_{i+3}-k_{i+2})} \right\} & \text{for } k_{i+2} \leq x < k_{i+3} \\ \frac{(k_{i+4}-x)^3}{(k_{i+4}-k_{i+1})(k_{i+4}-k_{i+2})(k_{i+4}-k_{i+3})} & \text{for } k_{i+3} \leq x < k_{i+4} \\ 0 & \text{otherwise} \end{cases}$$

Moreover, B-splines also have the interesting property that for any arbitrarily selected knot k_i ,

$$\sum_{j=-3}^0 B_{i+j}(x) = 1 \quad (3.2.10)$$

for all $x \in \{k_0, k_1, \dots, k_N\}$.

For example, if we apply this property in the equation (3.2.9), we get,

$$\begin{aligned}
B_{i-3}(k_i) &= \frac{1}{6} \\
B_{i-2}(k_i) &= \frac{2}{3} \\
B_{i-1}(k_i) &= \frac{1}{6} \\
B_i(k_i) &= 0
\end{aligned} \tag{3.2.11}$$

which confirms the our last property.

To set up the system of equations in the cubic case, we need the values of the B-splines B_i $i=-3, \dots, N-1$, at the knots k_0, \dots, k_n as well as the values of the derivatives B_i'' at k_0 for $i=-3, -2, -1$ and at k_n for $i=N-3, N-2, N-1$. Using equation (3.2.9) we have,

$$\begin{aligned}
B_{i-3}(x) &= \frac{(k_{i+1}-x)^3}{(k_{i+1}-k_{i-2})(k_{i+1}-k_{i-1})(k_{i+1}-k_i)} \\
B_{i-2}(x) &= \frac{(x-k_{i-2})(k_{i+1}-x)^2}{(k_{i+1}-k_{i-2})(k_{i+1}-k_{i-1})(k_{i+1}-k_i)} + \frac{(k_{i+2}-x)}{(k_{i+2}-k_{i-1})} \dots \\
&\quad \left\{ \frac{(x-k_{i-1})(k_{i+1}-x)}{(k_{i+1}-k_{i-1})(k_{i+1}-k_i)} + \frac{(k_{i+2}-x)(x-k_i)}{(k_{i+2}-k_i)(k_{i+1}-k_i)} \right\} \\
B_{i-1}(x) &= \frac{(x-k_{i-1})^2(k_{i+1}-x)}{(k_{i+2}-k_{i-1})(k_{i+1}-k_{i-1})(k_{i+1}-k_i)} + \frac{(x-k_{i-1})(k_{i+2}-x)(x-k_i)}{(k_{i+2}-k_{i-1})(k_{i+2}-k_i)(k_{i+1}-k_i)} + \dots \\
&\quad \left\{ \frac{(k_{i+3}-x)(x-k_i)^2}{(k_{i+3}-k_i)(k_{i+1}-k_i)(k_{i+2}-k_i)} \right\} \\
B_i(x) &= \frac{(x-k_i)^3}{(k_{i+3}-k_i)(k_{i+1}-k_i)(k_{i+2}-k_i)}
\end{aligned}$$

for $k_i \leq x < k_{i+1}$. Let $k_{i+1}-k_i = h$, then we have,

$$B_{i-3}(k_i) = \frac{(k_{i+1} - k_i)^3}{(k_{i+1} - k_{i-2})(k_{i+1} - k_{i-1})(k_{i+1} - k_i)} = \frac{h^3}{(3h)(2h)(h)} = \frac{1}{6}$$

$$B_{i-2}(k_i) = \frac{(k_i - k_{i-2})(k_{i+1} - k_i)^2}{(k_{i+1} - k_{i-2})(k_{i+1} - k_{i-1})(k_{i+1} - k_i)} + \frac{(k_{i+2} - k_i)}{(k_{i+2} - k_{i-1})} \dots$$

$$\left\{ \frac{(k_i - k_{i-1})(k_{i+1} - k_i)}{(k_{i+1} - k_{i-1})(k_{i+1} - k_i)} + \frac{(k_{i+2} - k_i)(k_i - k_i)}{(k_{i+2} - k_i)(k_{i+1} - k_i)} \right\} = \frac{2}{3}$$

$$B_{i-1}(k_i) = \frac{(k_i - k_{i-1})^2(k_{i+1} - k_i)}{(k_{i+2} - k_{i-1})(k_{i+1} - k_{i-1})(k_{i+1} - k_i)} + \frac{(k_i - k_{i-1})(k_{i+2} - k_i)(k_i - k_i)}{(k_{i+2} - k_{i-1})(k_{i+2} - k_i)(k_{i+1} - k_i)} + \dots$$

$$\frac{(k_{i+3} - k_i)(k_i - k_i)^2}{(k_{i+3} - k_i)(k_{i+1} - k_i)(k_{i+2} - k_i)} = \frac{1}{6}$$

$$B_i(k_i) = \frac{(k_i - k_i)^3}{(k_{i+3} - k_i)(k_{i+1} - k_i)(k_{i+2} - k_i)} = 0$$

Similarly,

$$B_{-3}''(k_0) = 1 \quad B_{-2}''(k_0) = -2 \quad B_{-1}''(k_0) = 1$$

$$B_{N-3}''(k_N) = 1 \quad B_{N-2}''(k_N) = -2 \quad B_{N-1}''(k_N) = 1$$

Then, if we use natural end conditions for cubic splines, i.e.

$$S(k_i) = f(k_i) \quad i = 0, \dots, N$$

$$S''(k_0) = S''(k_N) = 0$$

Our linear system has the matrix form,

$$\begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & & 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_{-3} \\ a_{-2} \\ a_{-1} \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \\ f_1 \\ \vdots \\ f_{N-1} \\ f_N \\ 0 \end{pmatrix} \quad (3.2.12)$$

Equation (3.2.12) is also can be written as,

$$Va = f \quad (3.2.13)$$

Where $V \in \mathbb{R}^{(N+4) \times (N+3)}$, $a \in \mathbb{R}^{(N+4) \times 1}$ and $f \in \mathbb{R}^{(N+3) \times 1}$

As a result, solution of equation (3.2.12) is,

$$a = V^{-1}f \quad (3.2.14)$$

3.3 LEAST-SQUARES ESTIMATION

There is an important difference between interpolation and least-squares method. In the interpolation when we fit the data, resulting interpolation function passes through all data we are given, and thus we have oscillations if we have large data. In the interpolation method since the curve has oscillations, we can not understand the feature of our data. Therefore, we need to use least-squares method in which the curve do not need to passes through all given data, thus we may release our curve from oscillations. In the least-squares method, our goal is to find parameters of our cubic spline that provides the best fit to the observed data. To accomplish this task we will minimize sum of squared error, or l_2 -norm by using least-squares method, which is in the following form [3],

$$l^2(S) = \sum_{i=0}^N (S(k_i) - f_i)^2 \quad (3.3.1)$$

Here, we are trying to find the cubic B-spline, S , of the form,

$$S(x) = \sum_{j=-3}^m a_j B_j(x) \quad (3.3.2)$$

That is, we are trying to find the set of coefficients, a_j , $j=-3,-2,\dots, m$, that minimizes equation (3.3.1). The set of first-order conditions of the optimization problem requires that the partial derivatives of $l^2(S)$ with respect to the coefficients a_j $j=-3,-2,\dots, m$ must vanish. In other words,

$$\frac{\partial l^2(S)}{\partial a_j} = 0 \quad (3.3.3)$$

for $j=-3,-2,\dots,m$. We observe from equation (3.3.2), that each of these partial derivatives has the following form,

$$\frac{\partial S(x)}{\partial a_j} = B_j(x) \quad (3.3.4)$$

and using this to evaluate our set of first-order conditions,

$$\begin{aligned}
\frac{\partial l^2(S)}{\partial a_j} &= 0 \\
\frac{\partial}{\partial a_j} \left(\sum_{i=0}^N (S(k_i) - f_i)^2 \right) &= 0 \\
\sum_{i=0}^N 2 \left(\sum_{j=-3}^m a_j B_j(k_i) - f_i \right) \frac{\partial S(x)}{\partial a_j} &= 0 \\
\sum_{i=0}^N 2 \left(\sum_{j=-3}^m a_j B_j(k_i) - f_i \right) B_j(k_i) &= 0 \\
\sum_{i=0}^N \left[B_j(k_i) \left(\sum_{j=-3}^m a_j B_j(k_i) \right) - B_j(k_i) f_i \right] &= 0
\end{aligned} \tag{3.3.5}$$

Then, we get the normal equations in the matrix form,

$$\begin{pmatrix}
\sum_{i=0}^N B_{-3}^2(k_i) & \sum_{i=0}^N B_{-3}(k_i)B_{-2}(k_i) & \sum_{i=0}^N B_{-3}(k_i)B_m(k_i) \\
\sum_{i=0}^N B_{-2}(k_i)B_{-3}(k_i) & \sum_{i=0}^N B_{-2}^2(k_i) & \sum_{i=0}^N B_{-2}(k_i)B_m(k_i) \\
\vdots & \vdots & \vdots \\
\sum_{i=0}^N B_m(k_i)B_{-3}(k_i) & \sum_{i=0}^N B_m(k_i)B_{-2}(k_i) & \sum_{i=0}^N B_m^2(k_i)
\end{pmatrix}
\begin{pmatrix}
a_{-3} \\
a_{-2} \\
\vdots \\
a_{m-2} \\
a_{m-1} \\
a_m
\end{pmatrix}
=
\begin{pmatrix}
\sum_{i=0}^N B_{-3}(k_i)f_i \\
\vdots \\
\sum_{i=0}^N B_m(k_i)f_i
\end{pmatrix} \tag{3.3.6}$$

If we define,

$$V = \begin{pmatrix}
B_{-3}(k_0) & B_{-2}(k_0) & \dots & B_m(k_0) \\
B_{-3}(k_1) & B_{-2}(k_1) & \dots & B_m(k_1) \\
\vdots & \vdots & & \vdots \\
B_{-3}(k_N) & B_{-2}(k_N) & \dots & B_m(k_N)
\end{pmatrix} \tag{3.3.7}$$

where $V \in \mathbb{R}^{(N+1) \times (m+4)}$, and

$$a = (a_{-3} \quad a_{-2} \quad \dots \quad a_m)^T \tag{3.3.8}$$

$$f = (f_0 \quad f_1 \quad \dots \quad f_N)^T \tag{3.3.9}$$

where $a \in \mathbb{R}^{(m+4) \times 1}$ and $f \in \mathbb{R}^{(N+1) \times 1}$, respectively.

The definitions in equations (3.3.7) to (3.3.9) allow us to write equation (3.3.6) in the following form,

$$V^T V a = V^T f \quad (3.3.10)$$

which has the well known least-squares solution,

$$a = (V^T V)^{-1} V^T f \quad (3.3.11)$$

Here, $V^T V$ is symmetric and positive definite. Moreover, because of the nature of the B-splines, it has a large number of zero entries (i.e, it is a sparse matrix). Thus, solving this system is computationally straightforward and fast.

In the McCulloch model when we are modelling the discount function we use the least squares method. In the model, discount function is linear combination of B-splines, in other words, discount function vector is equal to multiplication of B-spline matrix with some parameter vector. To find the parameter vector, we use least squares method. In the McCulloch model, we solve the optimization problem in which, we find the parameters which minimizes the deviation of weighted theoretical prices from the weighted observed prices. Thus, in the following chapter we will go over the applications of least-squares method and Term structure models.

CHAPTER 4

TERM STRUCTURE MODELS

One of the study in Turkey about term structure model was done by C. Emre Alper, Aras Akdemir and Kazim Kazimov (2004). They used both spline based method of McCulloch and parsimonious model of Nelson-Siegel to estimate monthly yield curves. In McCulloch model, they used cubic splines (piecewise cubic polynomial) to model discount function. In their study secondary government securities data from 1992 to 2004 of Turkey was used to estimate monthly yield curves in Turkey. They constructed their data set by calculating monthly volume weighted average of price and maturity. They used both in-sample and out-of-sample analysis to compare McCulloch and Nelson-Siegel methods. From the results of their application, we understand that McCulloch method has superior in-sample properties, whereas Nelson-Siegel method has superior out-of-sample properties.

The other study was done by Derviş Bayazıt (2004). In his study, he estimated the zero-coupon bond yield curve of tomorrow by using Vasicek yield curve model with the zero-coupon bond yield data of today. The raw data in his study is the yearly simple spot rates of the Turkish zero-coupon bonds with different maturities of each day from July 1, 1999 to March 17, 2004. He completed the missing data by using Nelson-Siegel yield curve model and he estimated tomorrow yield curve with the discretized Vasicek yield curve model. He

concluded that by taking the Nelson-Siegel model as a benchmark he measured the performance of Vasicek model as a predictor and he found a considerable difference. Moreover, he concluded that all methods used in his study are applicable to affine models in which solution of bond price is explicitly obtained.

4.1 MC CULLOCH MODEL

In the Mc Culloch model we observe the set of coupon bond prices that are traded in the bond market at a given time, and in this model we try to get a cubic polynomial such that the resulting discount function fits these observed prices with the minimal error. Let's look at the necessary notations and formulas used in this model [3].

The price of a zero-coupon bond is the discounted cash flow or,

$$P(\tau) = c(\tau) d(\tau) \quad (4.1.1)$$

$$\text{where } c(\tau) = \begin{pmatrix} c_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & c_N \end{pmatrix}$$

$$d(\tau) = [d_1(\tau), \dots, d_N(\tau)]^T \quad (4.1.2)$$

N = number of bonds

In this thesis, we use zero-coupon bond data, so we have only one coupon payment which is at the maturity. To fit our cubic splines to the observed bond price data we use B-spline basis. We define our knot sequence as,

$$\{k_m, m=1, \dots, \kappa: 0=k_1 < k_2 < \dots < k_{\kappa-1} < k_\kappa = M\}$$

and the augmented knot sequence required for our B-spline basis as,

$$\{0=k_{-2}=k_{-1}=k_0=k_1 < k_2 < \dots < k_{\kappa-1} < k_{\kappa}=k_{\kappa+1}=k_{\kappa+2}=k_{\kappa+3}=M\}$$

In total, we have $\kappa=K+2$ B-splines defined over the interval $[0, M]$. We can write any cubic spline as, $B(t) \theta$ for $t \in [0, M]$ where

$$\theta = [\theta_1, \dots, \theta_{\kappa}]^T \quad (4.1.3)$$

and

$$B(\tau) = \begin{pmatrix} B_1(\tau_{11}) & \dots & B_{\kappa}(\tau_{11}) \\ \vdots & \vdots & \vdots \\ B_1(\tau_{N1}) & \dots & B_{\kappa}(\tau_{N1}) \end{pmatrix} \quad (4.1.4)$$

After the notations, let us start to construct our model. We begin with an arbitrary function $g(t)$, which is identity function in the Mc Culloch model,

$$g(h(\cdot), t) = g(d(\cdot), t) = d(t) \text{ for } t \in [0, M] \quad (4.1.5)$$

The next step is to determine the form of the bond price function, $P(\theta)$ by using equation (4.1.5).

$$\begin{aligned} \hat{P}(\theta) &= c(\tau) g(h(\cdot), \tau) \\ &= c(\tau) d(\tau) = c(\tau) B(\tau) \theta \end{aligned} \quad (4.1.6)$$

This expression aids in the calculation of $X(\theta)$, where $X(\theta)$, is of the form

$$\begin{aligned} X(\theta) &= \frac{\partial P(\theta)}{\partial \theta^T} = \frac{\partial}{\partial \theta^T} (c(\tau)B(\tau)\theta) \\ &= c(\tau)B(\tau) \end{aligned} \quad (4.1.7)$$

As a result, our optimization problem becomes,

$$\min_{\theta} (P - X\theta)^T W (P - X\theta)$$

with the usual solution

$$\theta^* = (X^T W X)^{-1} X^T W P \quad (4.1.8)$$

where W is the weight matrix. Our choice for the weights is the reciprocal of the modified duration of the bond. Notice that this places less weight on longer-term, or equivalently higher-duration bonds. This is because we expect the observed prices for these bonds to exhibit greater variability. In other words, the longer duration, the less weight a bond has and the shorter duration, the more weight a bond has.

After the calculation of theta, we can evaluate theoretical prices for our data and deviation of these theoretical prices from observed prices. Moreover, since we know B-spline matrix and theta, we can calculate discount function for each bond by the formula,

$$d(\tau, \theta) = B(\tau)\theta \quad (4.1.9)$$

And also, we can write our discount function is of the form,

$$d(\tau_i) = e^{-\tau_i R_i} \quad (4.1.10)$$

where τ_i = time to maturity of i th bond

R_i = yield of i th bond

Thus,

$$R_i = \frac{-\log(d(\tau_i))}{\tau_i} \quad (4.1.11)$$

From equation (4.1.11), we can calculate theoretical yield for each bond and so deviation of the theoretical yields from the observed yields. Algorithm of McCulloch model is as follows,

Step 1: Construct B-spline matrix

$$B(\tau) = \begin{pmatrix} B_1(\tau_{11}) & \dots & B_\kappa(\tau_{11}) \\ \vdots & \vdots & \vdots \\ B_1(\tau_{N1}) & \dots & B_\kappa(\tau_{N1}) \end{pmatrix}$$

Step 2: Find θ from,

$$\min_{\theta} (P - X\theta)^T W (P - X\theta)$$

with the usual solution

$$\theta^* = (X^T W X)^{-1} X^T W P$$

Step 3: Evaluate the discount function,

$$d(\tau, \theta) = B(\tau)\theta$$

Step 4: Calculate the theoretical yields,

$$R_i = \frac{-\log(d(\tau_i))}{\tau_i}$$

Step 5: Find the yield curve by interpolating the theoretical yields by using cubic spline interpolation.

4.2 NELSON-SIEGEL MODEL

Nelson and Siegel Model (1987) is a model to obtain yield curves across maturities using zero-coupon government securities market data. The model has ability to describe variation in yield curves using only few parameters [4].

A class of functions that generates the yield curve shapes is that associated with solutions to differential or difference equations. The expectations theory of the term structure of interest rates provides motivation for that class of functions since, if spot rates are generated by a differential equation, we can get the forward rate function which is a solution of the differential equation. If instantaneous forward rate at maturity τ , denoted $f(\tau)$ is given by the solution to a second-order differential equation with real and unequal roots, our solution is given by,

$$f(\tau) = \beta_0 + \beta_1 e^{(-\tau/T_1)} + \beta_2 e^{(-\tau/T_2)} \quad (4.2.1)$$

where T_1 and T_2 are time constants associated with the equation, and β_0 , β_1 and β_2 are determined by initial conditions. We can get many forward rate curves by using this equation and forward rate curves take on monotonic, humped, or S shapes depending on the values of β_0 , β_1 and β_2 . The yield to maturity is the average of the forward rates, and given by,

$$R(\tau) = \frac{1}{\tau} \int_0^{\tau} r(x) dx \quad (4.2.2)$$

However, Nelson-Siegel concluded that the model is over-parametrized and decided to use equal root solution. Hence, instantaneous implied forward rate function becomes,

$$f(\tau) = \beta_0 + \beta_1 e^{(-\tau/T)} + \beta_2 [(\tau/T) e^{(-\tau/T)}] \quad (4.2.3)$$

To obtain yield as a function of maturity for the equal roots, model (4.2.3) is integrated from zero to τ and divided by τ . The resulting function is,

$$R(\tau) = \beta_0 + (\beta_1 + \beta_2) [1 - e^{(-\tau/T)}] / (\tau/T) - \beta_2 e^{(-\tau/T)} \quad (4.2.4)$$

If we take the limit of model as τ approaches to zero and infinity, we can understand features of parameters. The limiting value of $R(\tau)$ as τ approaches to infinity, is β_0 and, as τ approaches to zero, is $\beta_0 + \beta_1$. In other words, the contribution of the long term component is β_0 and the contribution of the short term component is $\beta_0 + \beta_1$.

Let's go over the algorithm of Nelson-Siegel model. In this model, from equation (4.2.4) we can write the theoretical price formula as follows.

$$P_i = e^{-\tau_i R_i} \quad (4.2.5)$$

where τ_i is the time to maturity of the i th bond and R_i is the yield of the i th bond. After that, in Matlab by using `fmincon` command, we can find the parameters of yield function (equation 4.2.4) which provide minimum deviation to theoretical prices from the observed prices, in other words, we calculate parameters β_0 , β_1 , β_2 and τ which are the solution of following problem.

$$\begin{aligned}
& \min((WP_{\text{theoretical}}) - (WP_{\text{observed}})) \\
& \text{subject to} \\
& \beta_0 > 0 \\
& \beta_0 + \beta_1 > 0 \\
& T > 0 \\
& \beta_2
\end{aligned} \tag{4.2.6}$$

This optimization problem is nonlinear problem, to ease our problem we fix T to some real number which is greater than zero, and so our problem becomes linear problem.

Our first step in the problem is to give some starting value to our parameters, and then the `fmincon` command finds the parameters, which minimize the our residual. After finding the parameters, we can calculate our theoretical yields for the zero-coupon bonds up to longest maturity we observed. Algorithm of the Nelson-Siegel model is as follows,

Step 1: Give the initial values to the parameters,

$$\beta_0, \beta_1, \beta_2 \text{ and } T$$

Step 2: Fix T to some real number greater than zero. Then use `fmincon` command in Matlab and solve the following problem,

$$\begin{aligned}
& \min((WP_{\text{theoretical}}) - (WP_{\text{observed}})) \\
& \text{subject to} \\
& \beta_0 > 0 \\
& \beta_0 + \beta_1 > 0 \\
& T > 0 \\
& \beta_2
\end{aligned}$$

We find the parameters of yield function (equation 4.2.4) which provide minimum deviation to theoretical prices from the observed prices.

Step 3: After finding parameters in step (2), calculate the theoretical yields $R(\tau)$.

Step 4: Finally draw the yield curve with respect to these theoretical yields.

4.3 COMPARISON OF THE MODELS

We compare the relative performances of two methods with respect to root mean squared yield error (RMSYE) and mean absolute yield error (MAYE). For each sample we evaluate RMSYE and MAYE then since for each month we have two sample, we evaluate the RMSYE and MAYE of the month by taking averages of RMSYE and MAYE of samples which are observed in the relating month. The RMSYE and MAYE are calculated by the following formulas [4].

$$\text{RMSYE} = \sqrt{\frac{1}{N} \sum_{i=1}^N \varepsilon_i^2} \quad (4.3.1)$$

$$\text{MAYE} = \frac{1}{N} \sum_{i=1}^N |\varepsilon_i| \quad (4.3.2)$$

Where N = number of bonds in the sample

ε_i = the error made in the i th yield, which is calculated by the model

By using these error measures, we can also compare the performances of the models at the different time intervals. In our thesis we evaluate the RMSYE of each sample at each models in the time intervals 0-90 days, 90-180 days, 180-270 days and 270-... days. Thus, we compare the ability of the models in fitting the yields for the long and short periods. We do our analysis as an in-sample analysis. Results of our application is presented in the table 4.3.1 and table 4.3.2 . Table 4.3.1 gives errors made in yield by McCulloch model and table 4.3.2 gives errors made in yield by Nelson-Siegel model. Our results show that MAYE and RMSYE of the McCulloch model is less than that of Nelson-Siegel model, thus performance of McCulloch model in fitting yield is better than that of Nelson-Siegel model. Prediction ability of Nelson-Siegel is better than McCulloch model

in the time interval 0-90 days but in the other time intervals McCulloch model shows better performances than Nelson-Siegel model. The results and figures of these applications as follows,

Date	MAYE	RMSYE	0-90	90-180	180-270	270-
03.01.2005	0,000452	0,000602	0,001844	0,000882	0,000643	0,000358
31.01.2005	0,000617	0,000903	0,001241	0,003097	0,001336	0,000384
Average	0,000535	0,000753	0,001543	0,001990	0,000990	0,000371
07.02.2005	0,000631	0,000987	0,002558	0,002265	0,000060	0,000139
21.02.2005	0,000555	0,000673	0,000342	0,001593	0,002060	0,000899
Average	0,000593	0,00083	0,00145	0,001929	0,00106	0,000519
15.03.2005	0,000141	0,000172	0,000002	0,000538	0,000198	0,000342
29.03.2005	0,000670	0,000955	0,000735	0,002219	0,002698	0,000137
Average	0,000406	0,000564	0,000369	0,0013785	0,001448	0,0002395
15.04.2005	0,000332	0,000450	0,000383	0,001469	0,000924	0,000296
29.04.2005	0,000742	0,000942	0,002319	0,002699	0,001970	0,000549
Average	0,000537	0,000696	0,001351	0,002084	0,001447	0,0004225
13.05.2005	0,000697	0,000887	0,002237	0,001942	0,002052	0,000630
31.05.2005	0,000371	0,000537	0,001298	0,001763	0,000200	0,000277
Average	0,000534	0,000712	0,001768	0,0018525	0,001126	0,0004535
14.06.2005	0,000468	0,000622	0,000171	0,002324	0,000519	0,000705
29.06.2005	0,000369	0,000491	0,000547	0,001749	0,000390	0,000591
Average	0,000419	0,000557	0,000359	0,0020365	0,0004545	0,000648

Table 4.1: Monthly Averages of in sample Yield Errors (McCulloch)

Date	MAYE	RMSYE	0-90	90-180	180-270	270-
03.01.2005	0,000709	0,000898	0,001801	0,001522	0,001504	0,001632
31.01.2005	0,000694	0,001009	0,000987	0,003249	0,001857	0,001150
Average	0,000702	0,000954	0,001394	0,002385	0,001680	0,001391
07.02.2005	0,000730	0,001003	0,002388	0,002404	0,000492	0,000591
21.02.2005	0,000615	0,000769	0,000114	0,001703	0,002300	0,0013640
Average	0,000672	0,000886	0,001251	0,002053	0,001396	0,000978
15.03.2005	0,000201	0,000280	0,000183	0,000970	0,000260	0,000368
29.03.2005	0,000892	0,001078	0,000060	0,002061	0,002809	0,002029
Average	0,000547	0,000679	0,000121	0,001515	0,001534	0,001199
15.04.2005	0,000508	0,000645	0,001711	0,001558	0,001050	0,000443
29.04.2005	0,001039	0,001241	0,001712	0,004129	0,002037	0,002262
Average	0,000774	0,000943	0,001712	0,002844	0,001544	0,001353
13.05.2005	0,001042	0,001242	0,001478	0,003305	0,003220	0,001655
31.05.2005	0,000653	0,000952	0,001139	0,003513	0,000844	0,001019
Average	0,000848	0,001097	0,001309	0,003409	0,002032	0,001337
14.06.2005	0,000556	0,000718	0,000053	0,002375	0,000438	0,001550
29.06.2005	0,000518	0,000630	0,000471	0,002057	0,000816	0,001106
Average	0,000537	0,000674	0,000262	0,002216	0,000627	0,001328

Table 4.2: Monthly Averages of in sample Yield Errors (Nelson-Siegel)

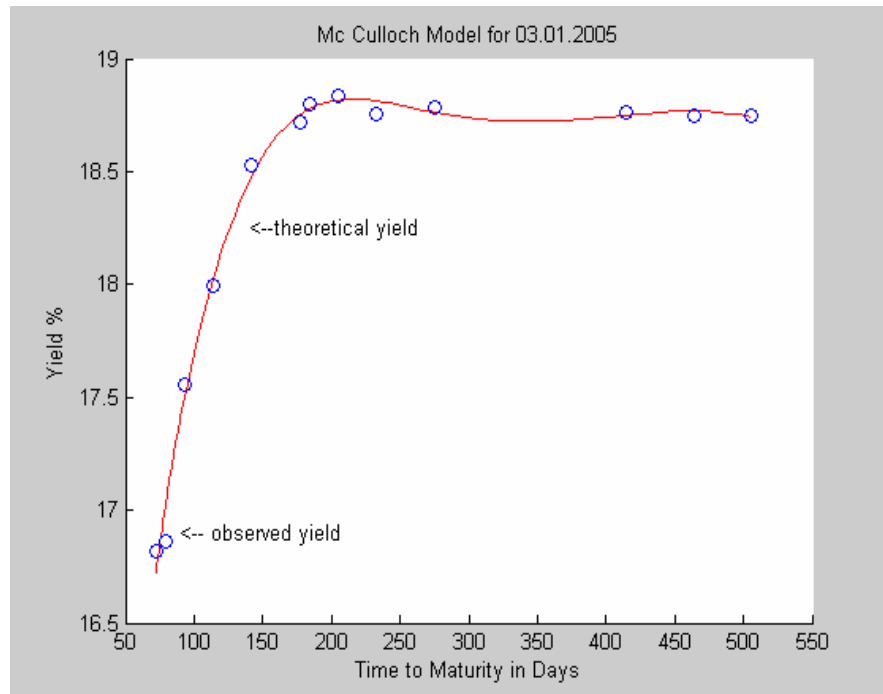


Figure 4.1: Mc Culloch Model for 03.01.2005

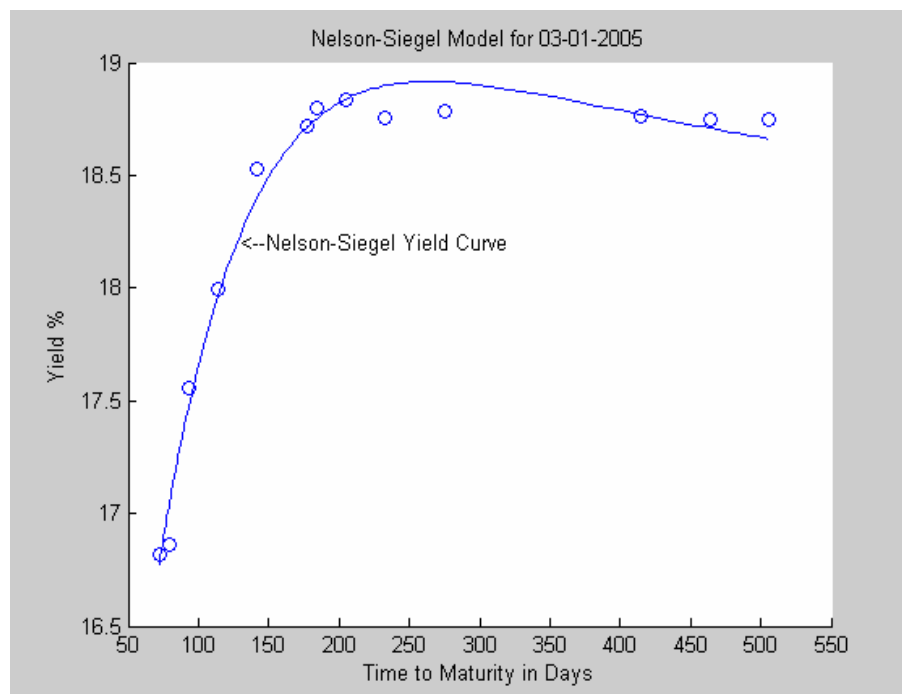


Figure 4.2: Nelson-Siegel Model for 03.01.2005

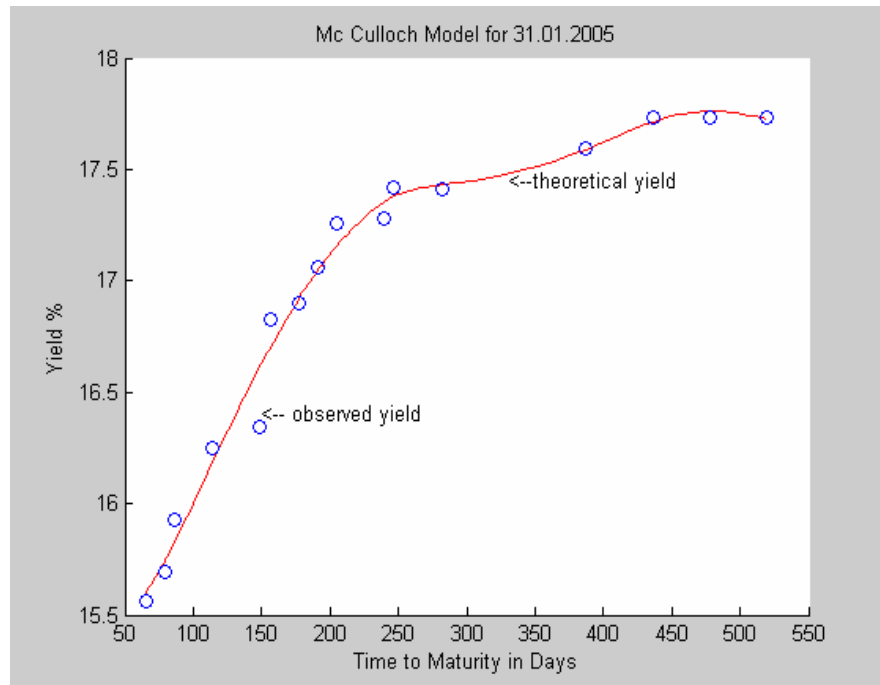


Figure 4.3: Mc Culloch Model for 31.01.2005

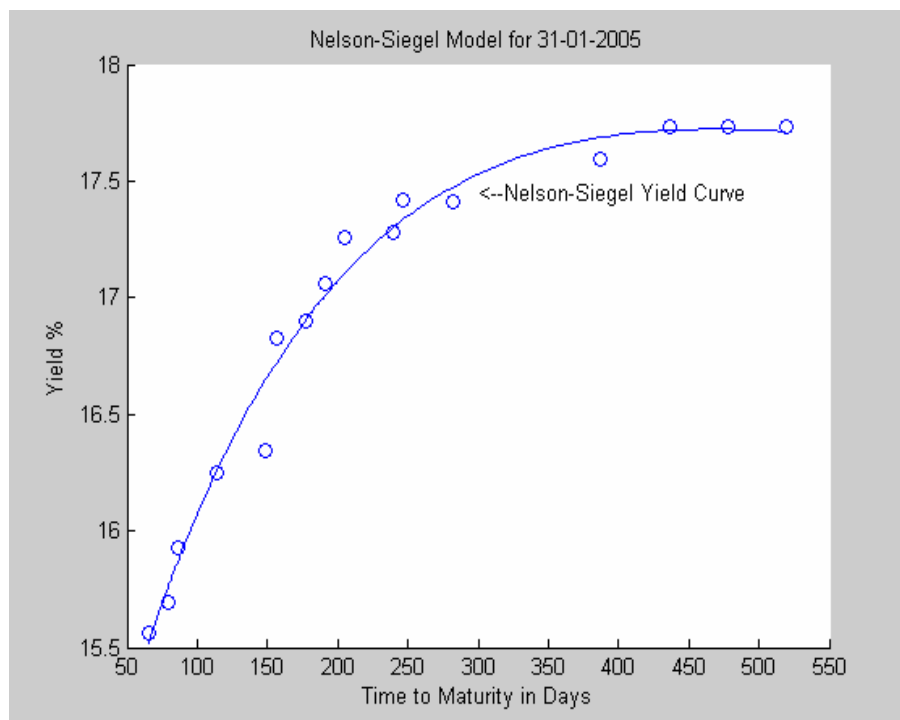


Figure 4.4: Nelson-Siegel Model for 31.01.2005

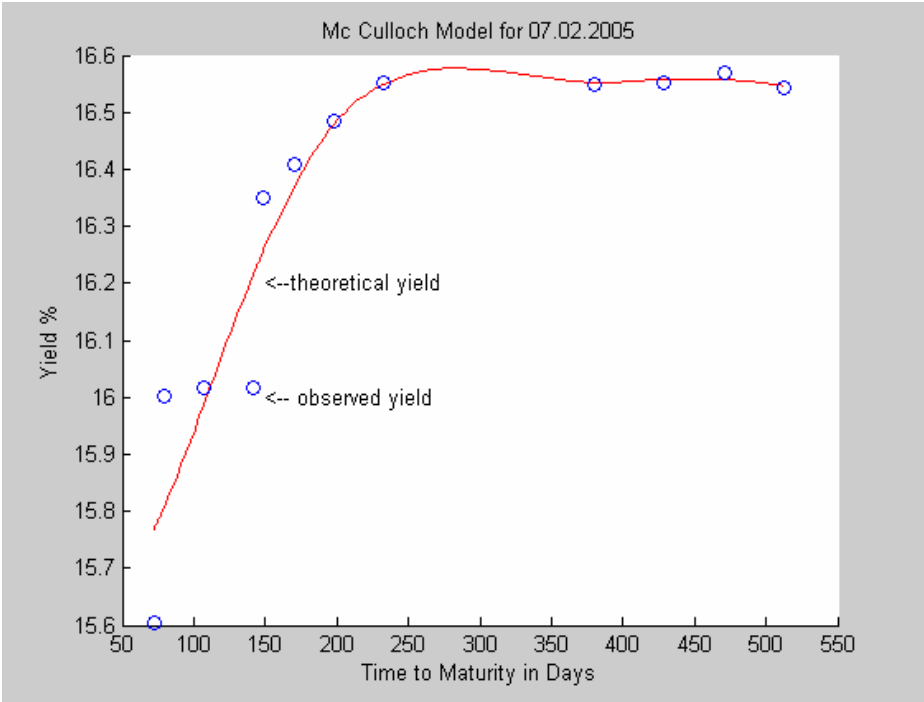


Figure 4.5: Mc Culloch Model for 07.02.2005

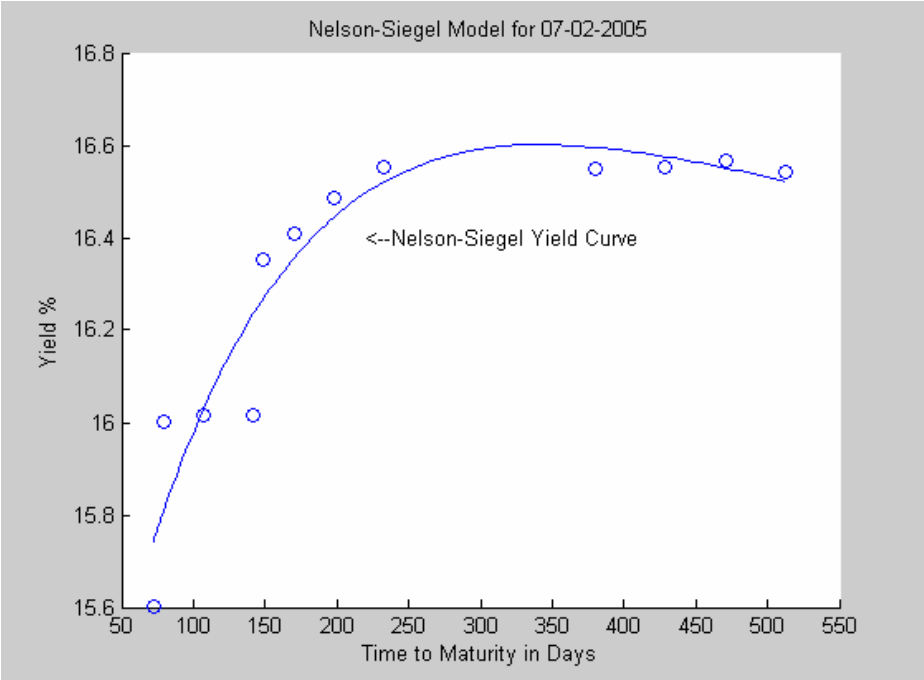


Figure 4.6: Nelson-Siegel Model for 07.02.2005

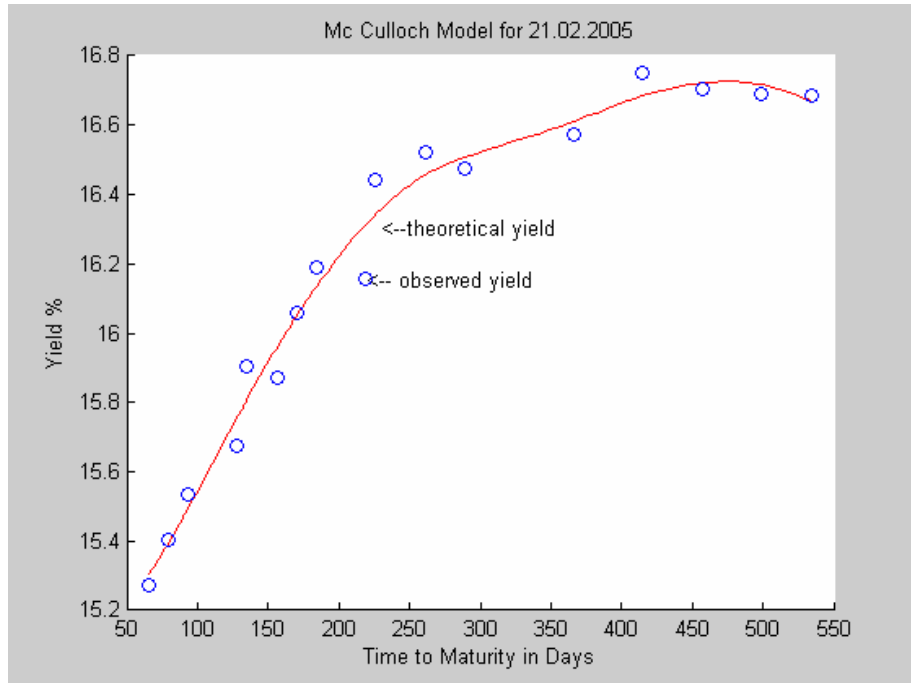


Figure 4.7: Mc Culloch Model for 21.02.2005

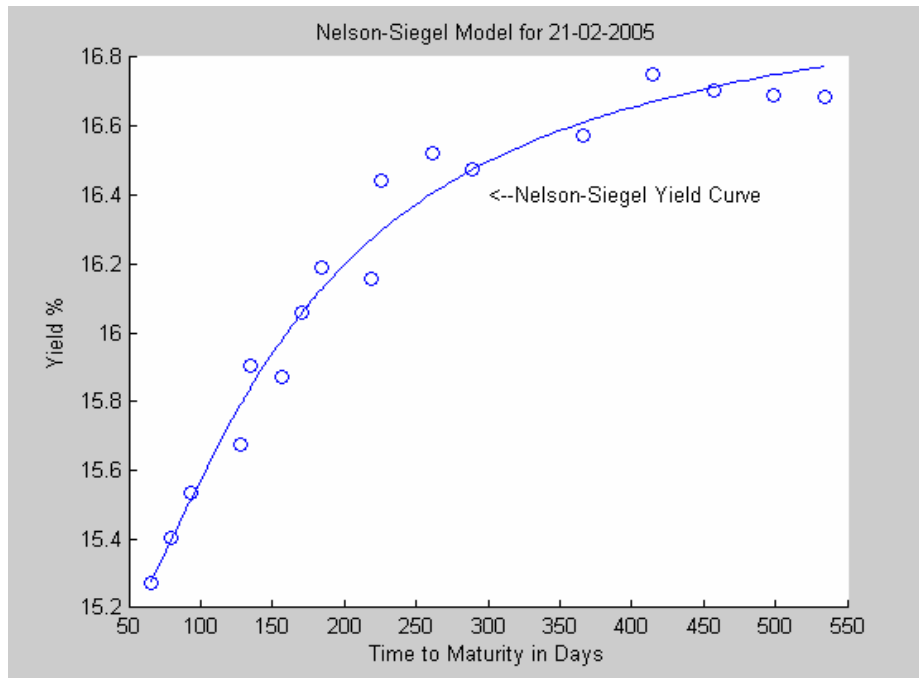


Figure 4.8: Nelson-Siegel Model for 21.02.2005

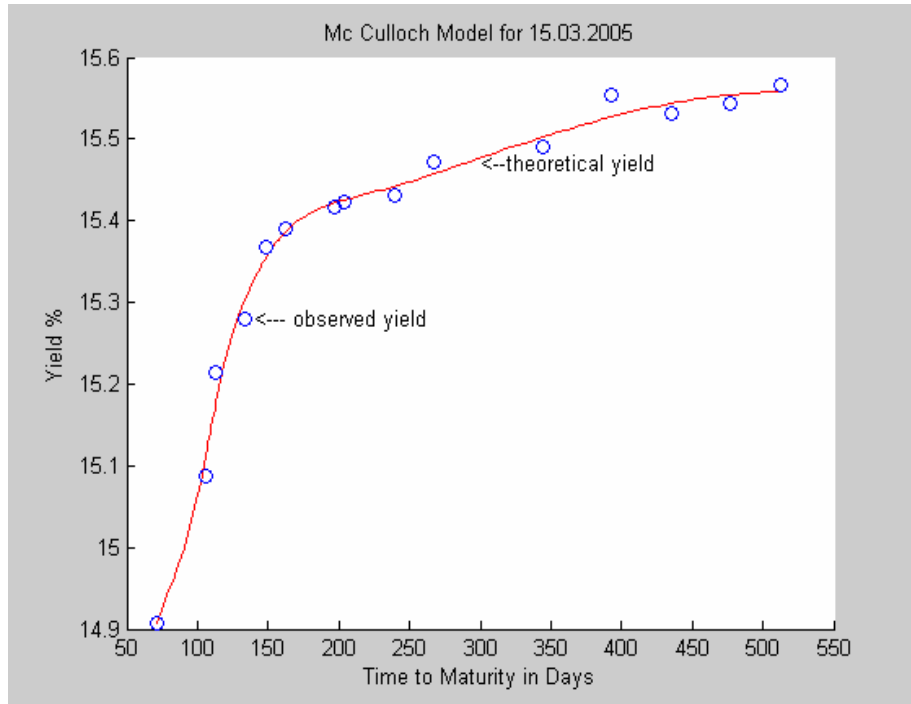


Figure 4.9: Mc Culloch Model for 15.03.2005

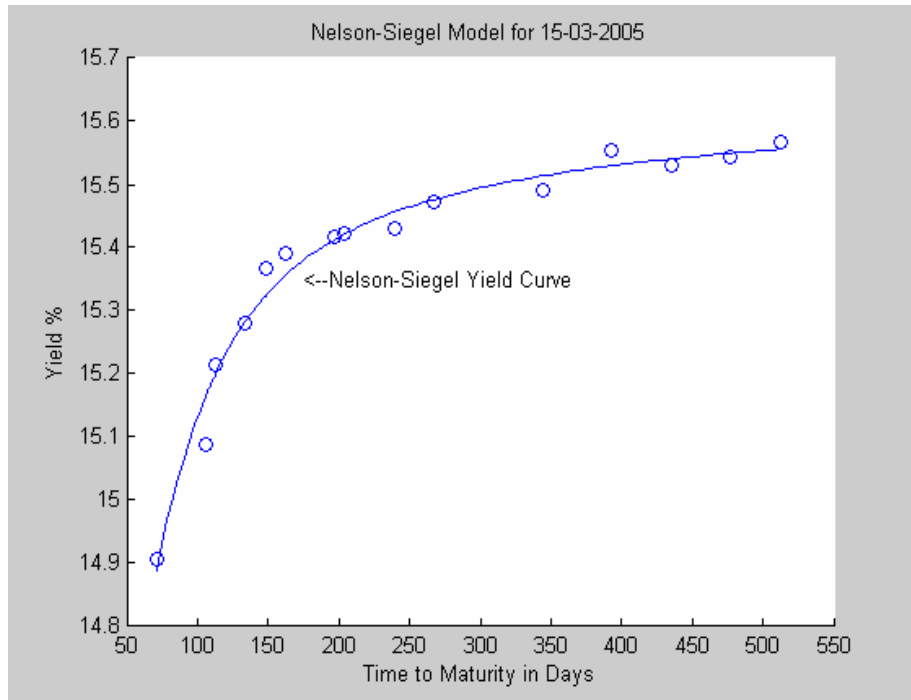


Figure 4.10: Nelson-Siegel Model for 15.03.2005

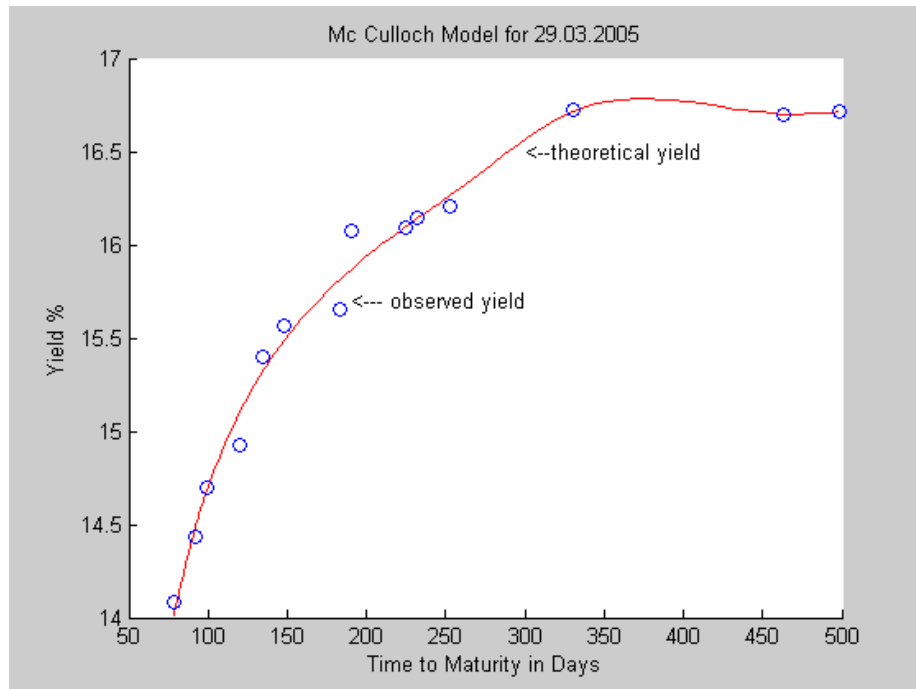


Figure 4.11: Mc Culloch Model for 29.03.2005

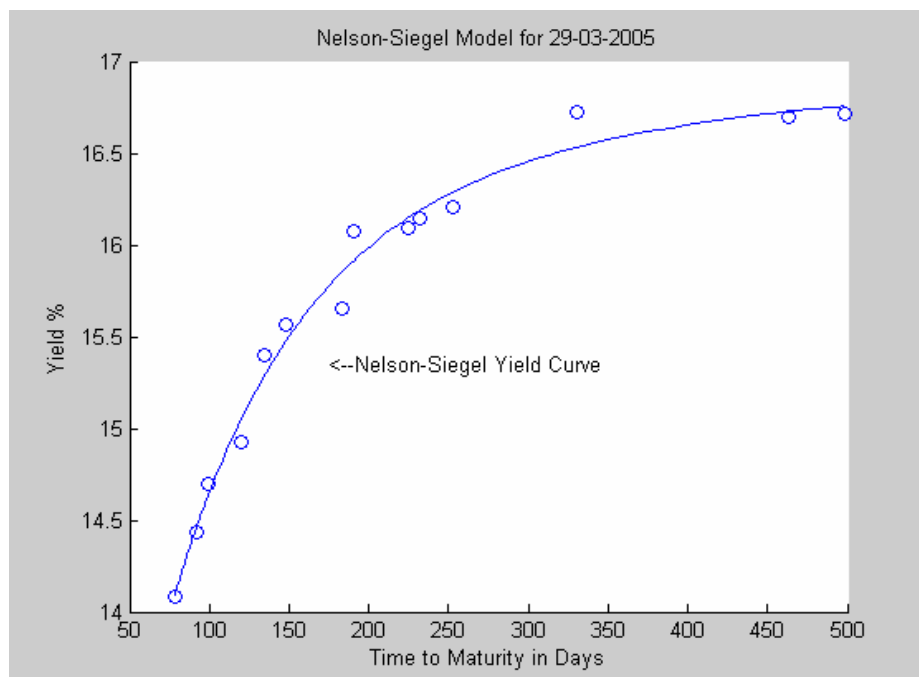


Figure 4.12: Nelson-Siegel Model for 29.03.2005

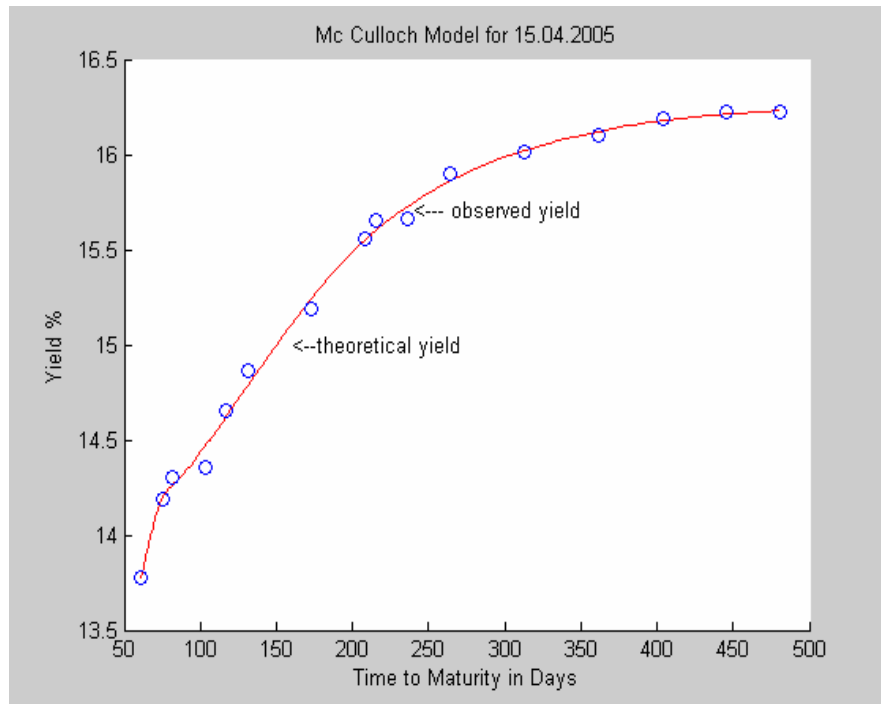


Figure 4.13: McCulloch Model for 15.04.2005

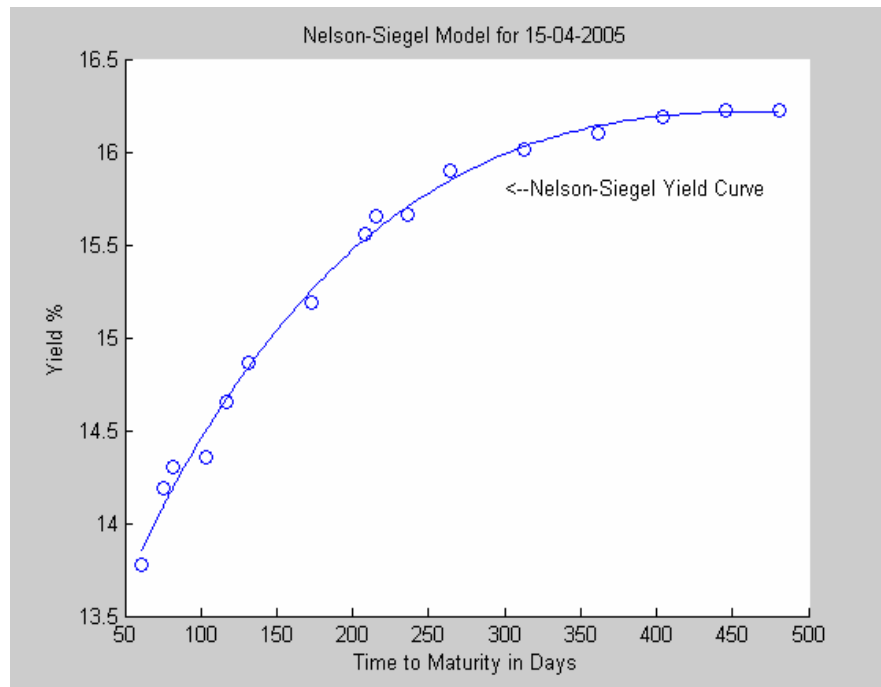


Figure 4.14: Nelson-Siegel Model for 15.04.2005

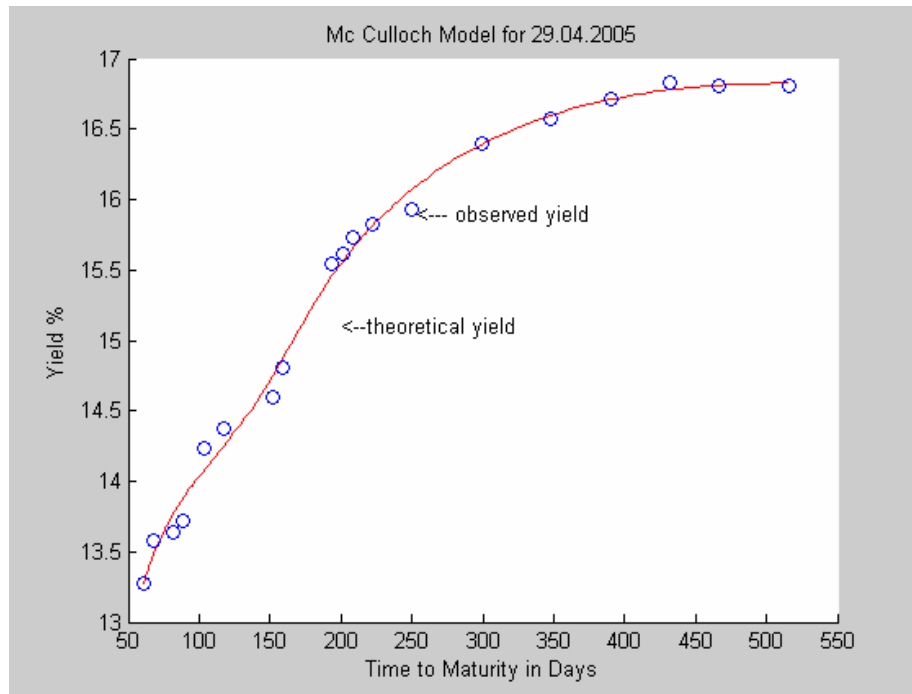


Figure 4.15: McCulloch Model for 29.04.2005

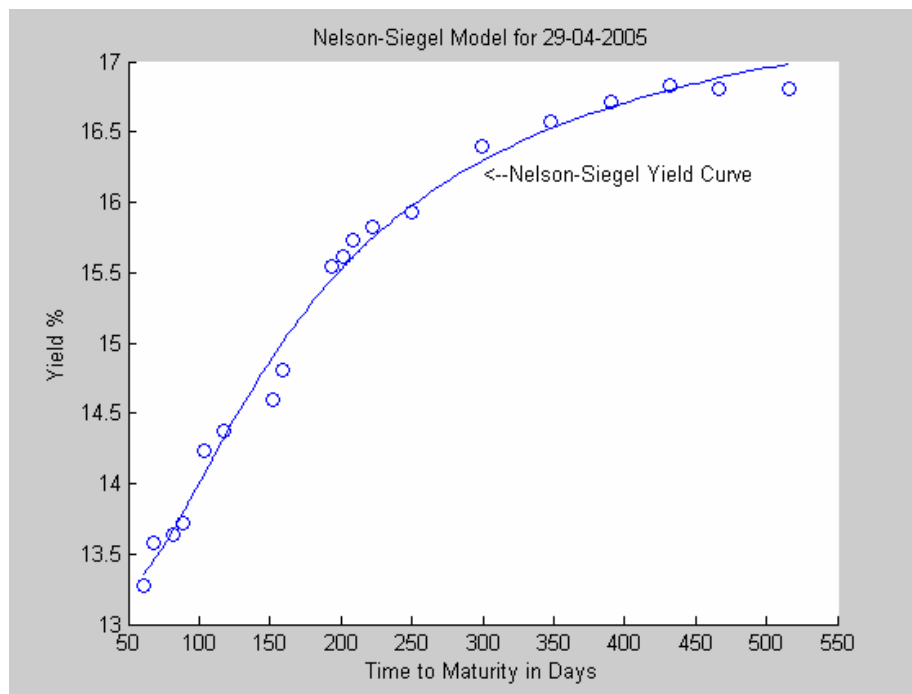


Figure 4.16: Nelson-Siegel Model for 29.04.2005

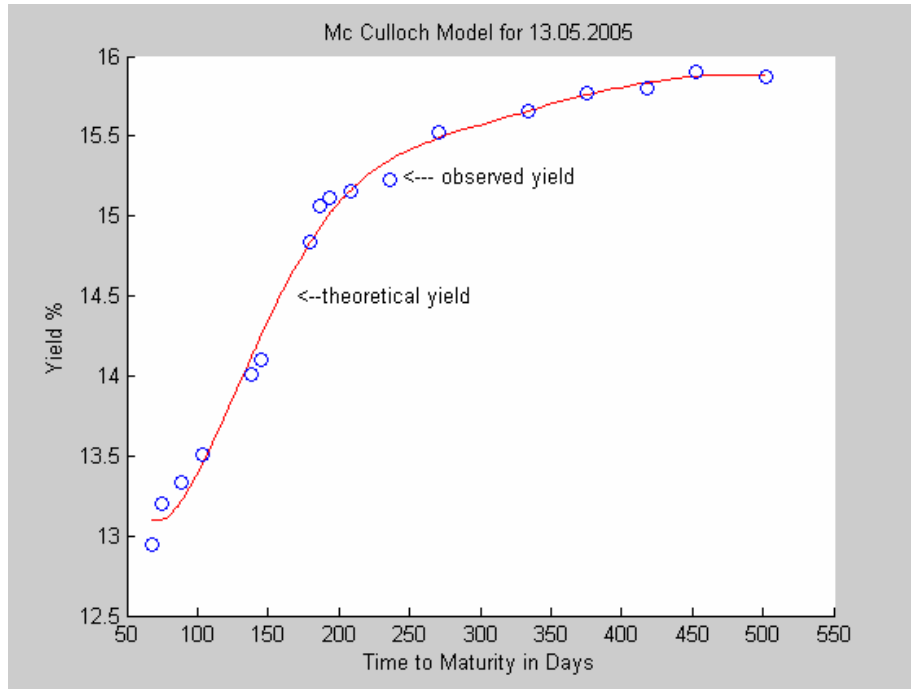


Figure 4.17: McCulloch Model for 13.05.2005

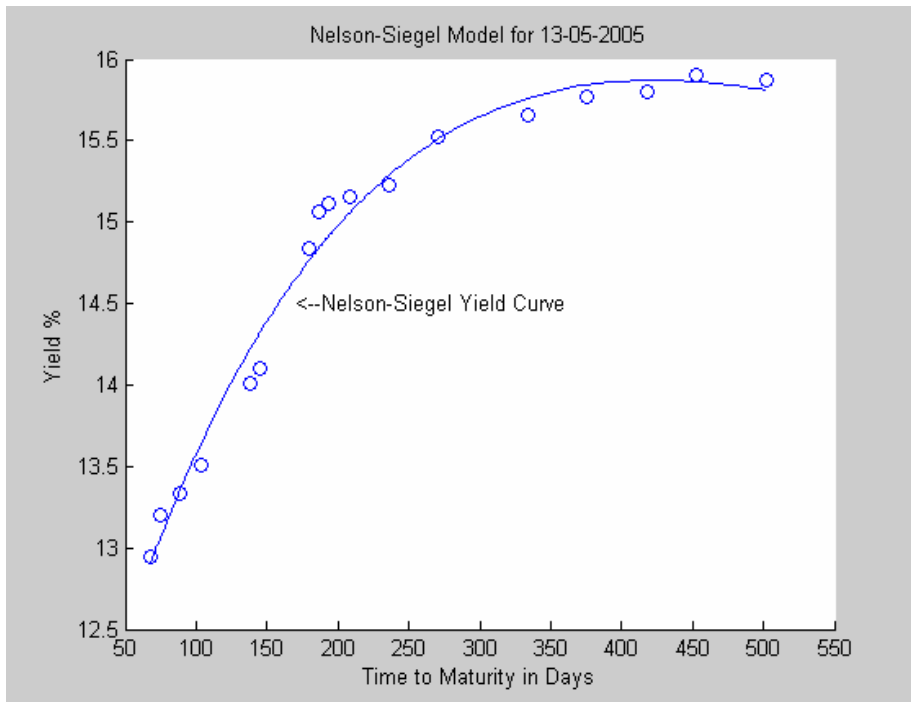


Figure 4.18: Nelson-Siegel Model for 13.05.2005

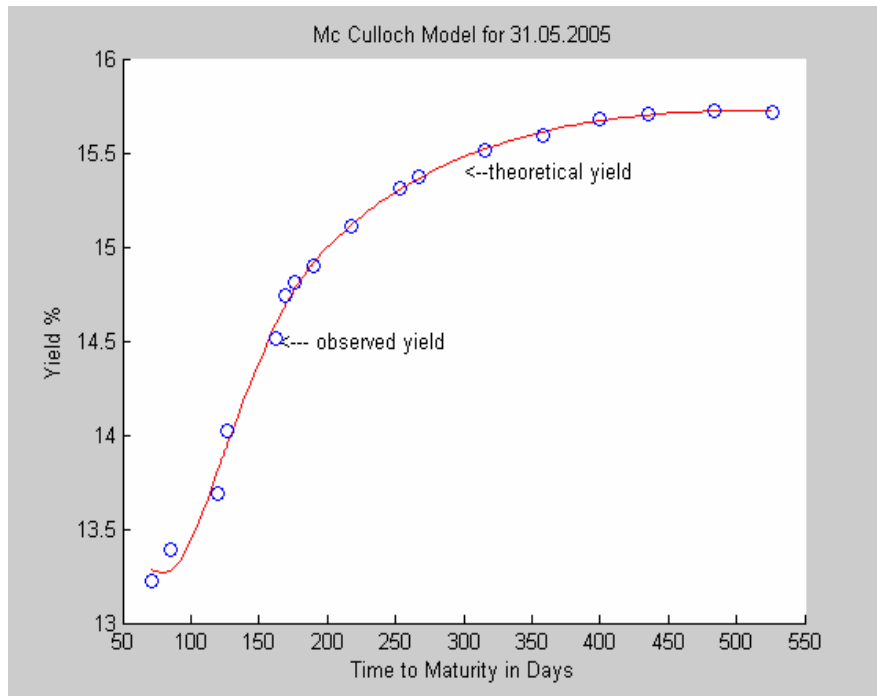


Figure 4.19: McCulloch Model for 31.05.2005

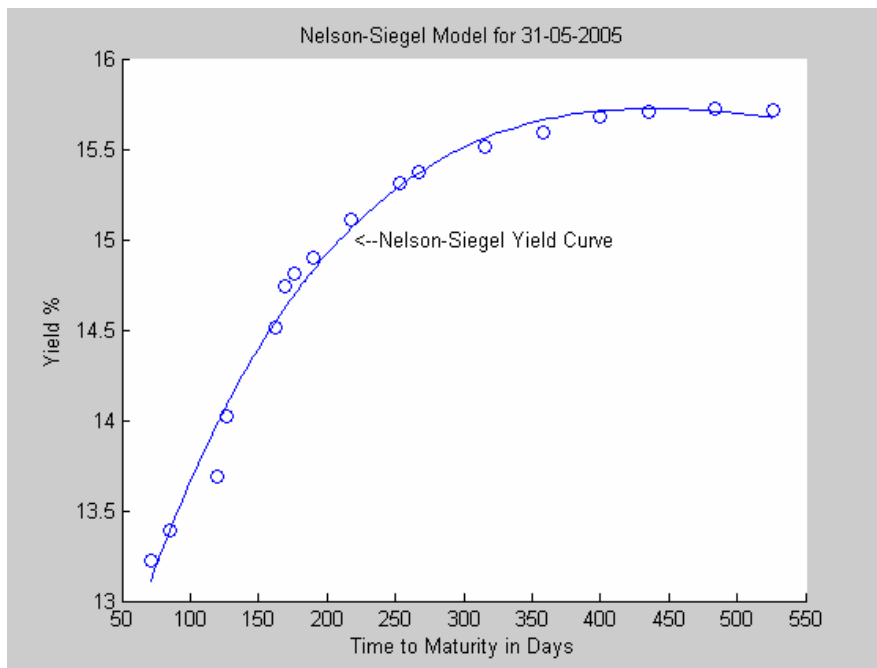


Figure 4.20: Nelson-Siegel Model for 31.05.2005

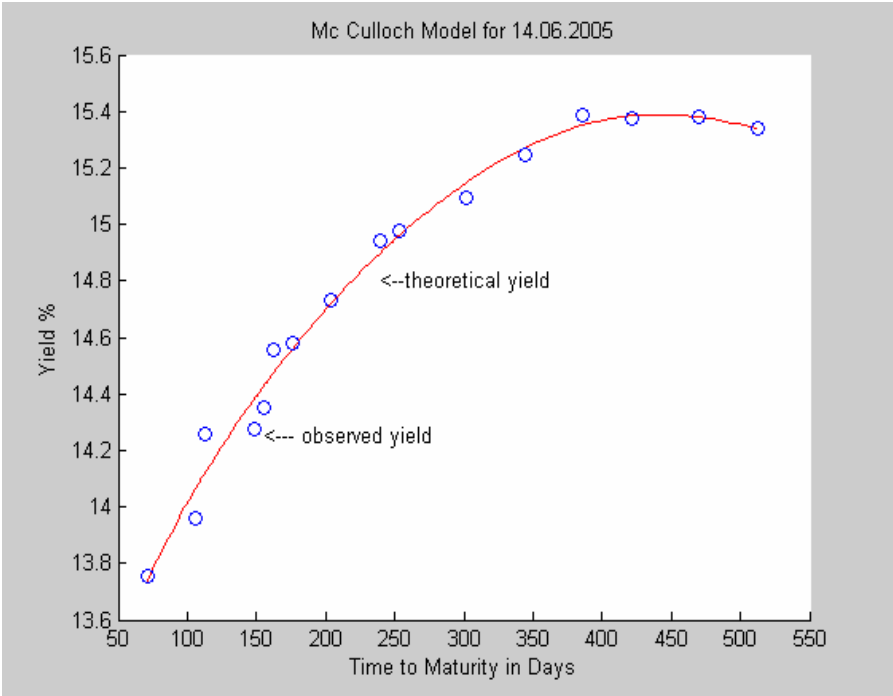


Figure 4.21: McCulloch Model for 14.06.2005

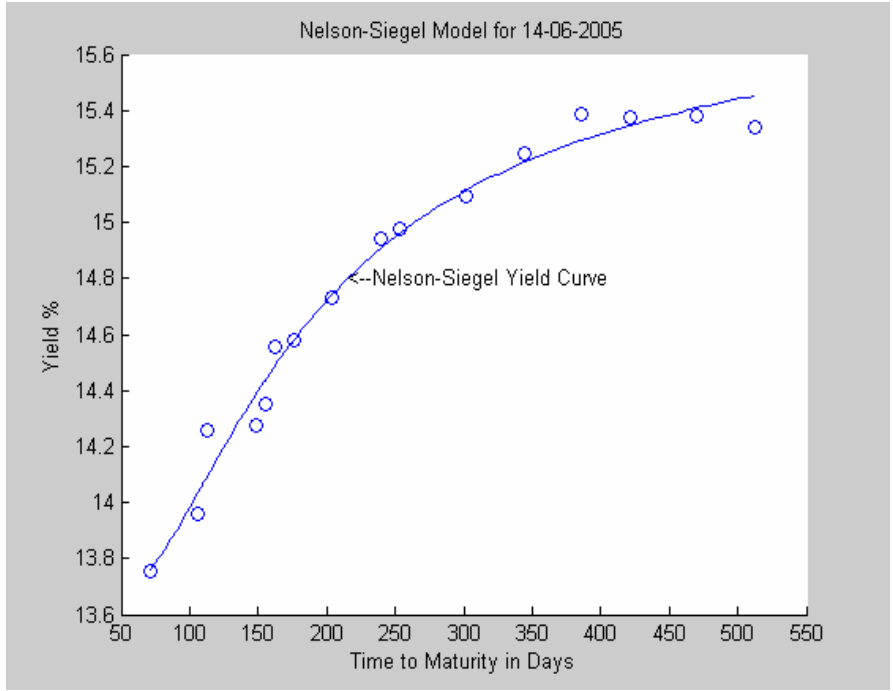


Figure 4.22: Nelson-Siegel Model for 14.06.2005

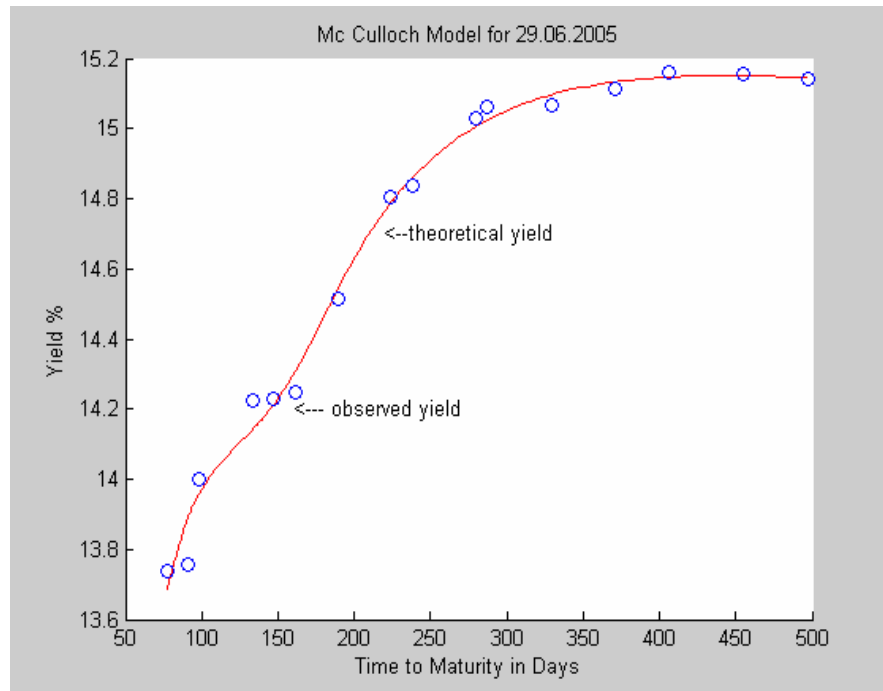


Figure 4.23: McCulloch Model for 29.06.2005

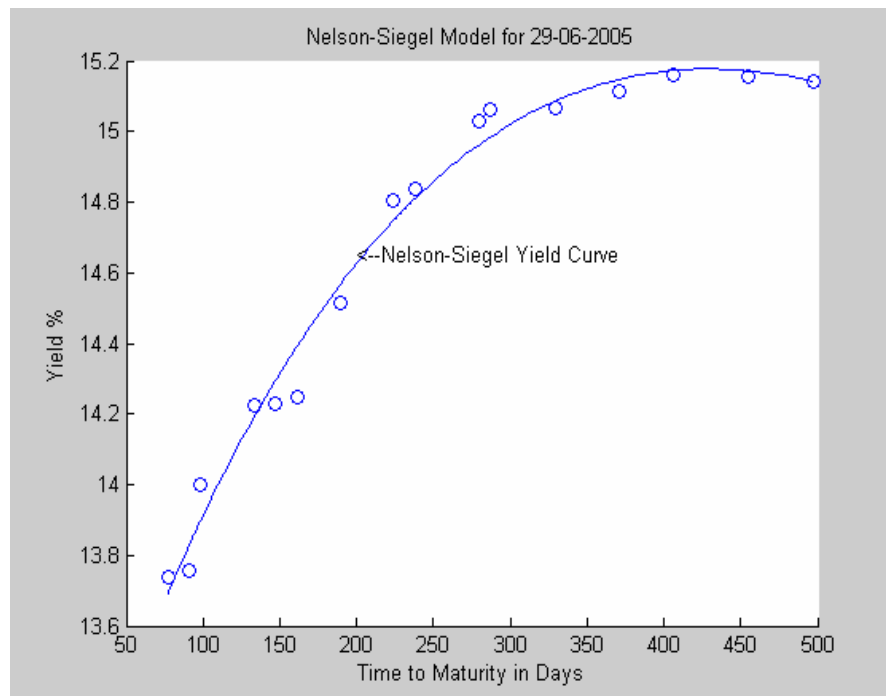


Figure 4.24: Nelson-Siegel Model for 29.06.2005

4.4 THE DATA

The data that is used in this study is the Turkish secondary government zero-coupon bond data. we use data from January 2005 through June 2005. In each month, we randomly select two days, and in application these data is used as a raw data. The raw data have some elements, which have the maturity is less than two months, when we construct the data set for empirical applications, we exclude these elements. We also exclude the bonds which are traded at the early hours of the morning and traded very seldomly on the day. After the exclusion, we have approximately 20 elements for each sample. Moreover, bond prices are constructed by using daily closing prices of zero-coupon bonds. The one sample used in this study is summarized in the following table.

Issuing Date	Time to Maturity (Days)	Interest Rate (%)	Bond Price
21.02.2005	65	15,48	97,317
21.02.2005	79	15,66	96,721
21.02.2005	83	15,85	96,119
21.02.2005	128	16,11	94,651
21.02.2005	135	16,38	94,287
21.02.2005	156	16,42	93,442
21.02.2005	170	16,68	92,793
21.02.2005	184	16,87	92,163
21.02.2005	219	16,96	90,762
21.02.2005	226	17,31	90,320
21.02.2005	261	17,54	88,857
21.02.2005	289	17,60	87,772
21.02.2005	366	18,03	84,689
21.02.2005	415	18,45	82,660
21.02.2005	457	18,58	81,130
21.02.2005	499	18,75	79,600
21.02.2005	534	18,78	78,338

Table 4.3: Sample Turkish government bonds used for fitting term structure

CHAPTER 5

CONCLUSIONS

The B-spline curve fitting technique is one of the most popular empirical methodologies for estimating the term structure of interest rates, due to its stability and reliability in practical applications. This thesis applies the B-spline technique for estimating the term structure and it compares McCulloch and Nelson-Siegel models' performance in-sample analysis with respect to our data and from the results we see that B-spline technique, which was used in McCulloch model in this thesis, has better performances. Moreover, with respect to the time intervals, our analysis shows that prediction ability of Nelson-Siegel is better than McCulloch model in the time interval 0-90 days but in the other time intervals McCulloch model shows better performances than Nelson-Siegel model.

In this study, we used zero-coupon bond data of Turkish government zero coupon bond data. In the further research we may use data, which include coupon payments, and by using these data we may compare relative performances of model with respect to coupon effects on the prices. In the paper of Bing-Hui Lin (2002), who wrote the paper about fitting term structure of interest rates using B-splines for Taiwanese government bonds, found the evidence that most of the bonds with coupons paying annually is almost consistently traded at lower yields than those with coupons paying semi-annually. We may do this analysis on Turkish government coupon bond data.

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APPENDICES

MATLAB CODES

A. MATLAB CODES FOR MC CULLOCH MODEL

A.1 Construction of B-Splines and Solution of the Optimization Problem

```
data=xlsread('data_zero_07-02-2005');
P=data(:,4);
D=data(:,2);
E=[-20 -10 0 72 222 308 600 650 700 750]';
tt=[72 222 308 600];
uu=[0 0 0 0 0];
FF=zeros(1,6);
x=(D./365)';
t=(E./365)';
n=length(t)-4;
for(i=1:length(x))
    for(k=1:n)
        if(x(i)>=t(k)&x(i)<t(k+1))
            B(i,k)=(x(i)-t(k))^3/((t(k+3)-t(k))*(t(k+1)-t(k))*(t(k+2)-t(k)));
        else if(x(i)>=t(k+1)&x(i)<t(k+2))
            B(i,k)=((x(i)-t(k))^2*(t(k+2)-x(i)))/((t(k+3)-t(k))*(t(k+2)-t(k))*(t(k+2)-t(k+1)))+...
                ((x(i)-t(k))*(t(k+3)-x(i))*(x(i)-t(k+1)))/((t(k+3)-t(k))*(t(k+3)-t(k+1))*(t(k+2)-t(k+1)))+...
                ((t(k+4)-x(i))*(x(i)-t(k+1))^2)/((t(k+4)-t(k+1))*(t(k+2)-t(k+1))*(t(k+3)-t(k+1)));
        else if(x(i)>=t(k+2)&x(i)<t(k+3))
            B(i,k)=((x(i)-t(k))*(t(k+3)-x(i))^2)/((t(k+3)-t(k))*(t(k+3)-t(k+1))*(t(k+3)-t(k+2)))+...
```



```

MAYE_yield=(1/12)*norm((theo_yield-obs_yield),1);
MAPE_price=(1/12)*norm((PP-P),1);
RMSYE_yield
RMSPE_price
MAYE_yield
MAPE_price
D=data(:,2);
for(s=1:12)
if(D(s)<90)
theo_yield_0_90(s)=theo_yield(s);
obs_yield_0_90(s)=obs_yield(s);
PP_0_90(s)=PP(s);
P_0_90(s)=P(s);
else if(D(s)>=90&D(s)<180)
theo_yield_90_180(s)=theo_yield(s);
obs_yield_90_180(s)=obs_yield(s);
PP_90_180(s)=PP(s);
P_90_180(s)=P(s);
else if(D(s)>=180&D(s)<270)
theo_yield_180_270(s)=theo_yield(s);
obs_yield_180_270(s)=obs_yield(s);
PP_180_270(s)=PP(s);
P_180_270(s)=P(s);
else
theo_yield_270(s)=theo_yield(s);
obs_yield_270(s)=obs_yield(s);
PP_270(s)=PP(s);
P_270(s)=P(s);
end
end
end
end
error_yield_0_90=norm((theo_yield_0_90-obs_yield_0_90),2);
error_yield_90_180=norm((theo_yield_90_180-obs_yield_90_180),2);
error_yield_180_270=norm((theo_yield_180_270-obs_yield_180_270),2);
error_yield_270=norm((theo_yield_270-obs_yield_270),2);
error_price_0_90=norm((PP_0_90-P_0_90),2);
error_price_90_180=norm((PP_90_180-P_90_180),2);
error_price_180_270=norm((PP_180_270-P_180_270),2);
error_price_270=norm((PP_270-P_270),2);
error_yield_0_90
error_yield_90_180
error_yield_180_270

```

```

error_yield_270
error_price_0_90
error_price_90_180
error_price_180_270
error_price_270

```

A.3 Interpolation and Fitting Yield Curve

```

function [p]=tSpline_discmdl_07_02_2005(k,f)
discoun_mdl_07_02_2005;
step=10;
A=xlsread('data_zero_07-02-2005');
k=A(:,5);
f=theo_yield;
obs_yield=A(:,6);
N=length(k);
for(i=2:N)
    h(i-1)=k(i)-k(i-1);
    s(i-1)=(f(i)-f(i-1))/h(i-1);
end
for(i=1:N)
    if(i==1)
        g(i)=0;
        d(i)=0;
    else if(i==N)
        g(i)=1;
        d(i)=0;
    else
        g(i)=h(i)/(h(i-1)+h(i));
        d(i)=6*(s(i)-s(i-1))/(h(i-1)+h(i));
    end
end
end
V=spalloc(N,N,3*N);
V(1,1:2)=[2 g(1)];
for (i=2:N-1)
    V(i,i-1:i+1)=[1-g(i) 2 g(i)];
end
V(N,N-1:N)=[1-g(N) 2];
m=V\d';
for (i=1:N-1)
    x=linspace(k(i),k(i+1),step)
    x=x(1:step);
    X1((1+(i-1)*(step)):i*(step))=x';

```

```

p(((1+(i-1)*(step))^i*(step))=((m(i)*((k(i+1)-x).^3)))/(6*h(i))+...
((m(i+1)*((x-k(i)).^3)))/(6*h(i))+...
((f(i)-((m(i)*(h(i)^2))/6))*(k(i+1)-x))/h(i)+...
((f(i+1)-((m(i+1)*(h(i)^2))/6))*(x-k(i)))/h(i);
end
X1;
F=p';
end
F
X2=X1.*365;
k1=k.*365;
hold on
plot(X2,F,'r'),xlabel('Time to Maturity in Days'),ylabel('Yield
%'),text(150,16.0,'<-- observed yield'),text(150,16.2,'<--theoretical yield')
title('Mc Culloch Model for 07.02.2005')
plot(k1,obs_yield,'o')
hold off

```

B. MATLAB CODES FOR NELSON-SIEGEL MODEL

B.1 Fmincon and Solution of Optimization Problem

```
AA=xlsread('data_zero_07-02-2005');
global m;
global obs_price;
global theo_price;
m=AA(:,5);
A=[-1 0 0;-1 -1 0;0 0 0];
B=[0.001 0.001 0.001]';
beta_16=[0.05 0.05 0.05]';
beta(4)=0.4;
[beta,FVAL]=fmincon(@residual_07february,beta_16,A,B)
```

B.2 Second Part of Optimization Problem

```
function residual_07february=ns(beta_16)
AA=xlsread('data_zero_07-02-2005');
weight_07february;
global m;
global obs_price;
global theo_price;
m=AA(:,5);
obs_price=AA(:,4);
weight_obs_price=W*obs_price;
beta_16(4)=0.4;
for(j=1:12)
    theo_yield(j)=beta_16(1)+(beta_16(2)+beta_16(3))*(1-exp(-
m(j)/beta_16(4)))/(m(j)/beta_16(4))-beta_16(3)*exp(-m(j)/beta_16(4));
    theo_price(j)=exp(-theo_yield(j)*m(j))*100;
end
weight_theo_price=W*(theo_price)';
residual_07february=norm((weight_obs_price-weight_theo_price),2);
theo_yield;
```

B.3 Weight Matrix

```
AA=xlsread('data_zero_07-02-2005');
kk=AA(:,5);
for(i=1:12)
    vv(i)=1/kk(i);
end
```

```

total=sum(vv);
for(j=1:12)
vvv(j)=vv(j)/total;
W(j,j)=vvv(j);
end
W;

```

B.4 Error Measures

```

beta_16=[0.15741 -0.0096151 0.04248 0.4];
AA=xlsread('data_zero_07-02-2005');
P=(AA(:,4))';
obs_yield=((AA(:,6)))/100;
for(j=1:12)
    theo_yield(j)=beta_16(1)+(beta_16(2)+beta_16(3))*(1-exp(-
m(j)/beta_16(4)))/(m(j)/beta_16(4))-beta_16(3)*exp(-m(j)/beta_16(4));
    PP(j)=exp(-theo_yield(j)*m(j))*100;
end
RMSYE_yield=(1/sqrt(12))*norm((theo_yield-obs_yield),2);
RMSPE_price=(1/sqrt(12))*norm((PP-P),2);
MAYE_yield=(1/12)*norm((theo_yield-obs_yield),1);
MAPE_price=(1/12)*norm((PP-P),1);
RMSYE_yield
RMSPE_price
MAYE_yield
MAPE_price
D=data(:,2);
for(s=1:12)
if(D(s)<90)
theo_yield_0_90(s)=theo_yield(s);
obs_yield_0_90(s)=obs_yield(s);
PP_0_90(s)=PP(s);
P_0_90(s)=P(s);
else if(D(s)>=90&D(s)<180)
theo_yield_90_180(s)=theo_yield(s);
obs_yield_90_180(s)=obs_yield(s);
PP_90_180(s)=PP(s);
P_90_180(s)=P(s);
else if(D(s)>=180&D(s)<270)
theo_yield_180_270(s)=theo_yield(s);
obs_yield_180_270(s)=obs_yield(s);
PP_180_270(s)=PP(s);
P_180_270(s)=P(s);
else

```

```

theo_yield_270(s)=theo_yield(s);
obs_yield_270(s)=obs_yield(s);
PP_270(s)=PP(s);
P_270(s)=P(s);
end
end
end
end
error_yield_0_90=norm((theo_yield_0_90-obs_yield_0_90),2);
error_yield_90_180=norm((theo_yield_90_180-obs_yield_90_180),2);
error_yield_180_270=norm((theo_yield_180_270-obs_yield_180_270),2);
error_yield_270=norm((theo_yield_270-obs_yield_270),2);
error_price_0_90=norm((PP_0_90-P_0_90),2);
error_price_90_180=norm((PP_90_180-P_90_180),2);
error_price_180_270=norm((PP_180_270-P_180_270),2);
error_price_270=norm((PP_270-P_270),2);
error_yield_0_90
error_yield_90_180
error_yield_180_270
error_yield_270
error_price_0_90
error_price_90_180
error_price_180_270
error_price_270

```

B.5 Fitting Yield Curve

```

fmincon_07february;
AA=xlsread('data_zero_07-02-2005');
obs_yield=AA(:,6);
beta(4)=0.4;
m=AA(:,5);
N=length(m);
step=10;
for(i=1:N-1)
    xx=linspace(m(i),m(i+1),step);
    xx=xx(1:step);
    XX((1+(i-1)*(step)):i*(step))=(xx)';
end
NN=length(XX);
for(j=1:NN)
    theo_yield(j)=beta(1)+(beta(2)+beta(3))*(1-exp(-XX(j)/beta(4)))/(XX(j)/beta(4))-
    beta(3)*exp(-XX(j)/beta(4));
end

```

```
theo_yield1=100.*theo_yield;
XXX=XX.*365;
md=m.*365;
hold on
plot(XXX,theo_yield1),xlabel('Time to Maturity in Days'),ylabel('Yield
%'),text(220,16.4,'--Nelson-Siegel Yield Curve')
title('Nelson-Siegel Model for 07-02-2005')
plot(md,obs_yield,'o')
hold off
```