CREDIT RISK MODELING WITH STOCHASTIC VOLATILITY, JUMPS AND STOCHASTIC INTEREST RATES

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

CREDIT RISK MODELING WITH STOCHASTIC VOLATILITY, JUMPS AND STOCHASTIC INTEREST RATES

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This thesis presents the modeling of credit risk by using structural approach. Three fundamental questions of credit risk literature are analyzed throughout the research: modeling single firm credit risk, modeling portfolio credit risk and credit risk pricing. First we analyze these questions under the assumptions that firm value follows a geometric Brownian motion and the interest rates are constant. We discuss the weaknesses of the geometric brownian motion assumption in explaining empirical properties of real data. Then we propose a new extended model in which asset value, volatility and interest rates follow affine jump diffusion processes. In our extended model volatility is stochastic, asset value and volatility has correlated jumps and interest rates are stochastic and have jumps. Finally, we analyze the modeling of single firm credit risk and credit risk pricing by using our extended model and show how our model can be used as a solution for the problems we encounter with simple models.

Keywords: Credit risk, Affine jump diffusion models, Stochastic volatility, Jump processes.

ÖZ

KREDİ RİSKİNİN STOKASTİK VOLATİLİTE, SIÇRAMA SÜREÇLERİ VE STOKASTİK FAİZ ORANLARI İLE MODELLENMESİ

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Bu çalışma, kredi riskinin yapısal yaklaşım aracılığıyla modellenmesini ortaya koymaktadır. Çalışmada kredi riskine ilişkin literatürde ele alınan üç temel problem incelenmiştir: münferit bir firmanın kredi riskinin modellenmesi, portföy kredi riskinin modellenmesi ve kredi riskinin fiyatlanması. İlk olarak, bahsi geçen üç problem, firma değerinin geometrik Brownian hareketi izlediği ve faiz oranlarının sabit olduğu varsayımları altında incelenmiştir. Geometrik brownian hareketi varsayımının, gerçek verilerin ampirik özelliklerini açıklamadaki zayıflıkları tartışılmıştır. Daha sonra, firma varlıklarının, volatilitenin ve faiz oranlarının afin sıçrama-yayınma süreci izlediği yeni bir genişletilmiş model ortaya konulmuştur. Genişletilmiş bu modelde volatilite stokastiktir, firma varlıkları ve volatilite korele sıçrama süreçlerine sahiptir ve faiz oranları stokastik ve sıçrama süreçlerine sahiptir. Son olarak münferit bir firmanın kredi riskinin modellenmesi ve kredi riskinin fiyatlanması problemleri genişletilmiş model aracılığıyla analiz edilmiş ve yeni modelin, daha basit modellerde ortaya çıkan problemlerin çözümünde nasıl kullanılabileceği gösterilmiştir.

Anahtar Kelimeler: Kredi riski, Afin sıçrama-yayınma modelleri, Stokastik volatilite, Sıçrama süreçleri.

To my lovely wife and my dear family

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INTRODUCTION

1.1 Definition of Credit Risk

Credit risk is one of the most prominent risks that financial institutions (especially a bank) are exposed to. Credit risk is defined as the risk of loss caused by a credit-related event. Credit-related events, for instance, include default of a counterparty or rating changes.

For classifying credit risk, we can use different classifications using different criteria. For example, by looking at the party who creates credit risk, we can classify credit risk into three broad categories. The first category includes *unilateral credit risk*. This simplest type of credit risk arises if a financial institution has a claim from a counterparty. The typical example is a bank loan. In the second category, we have *bilateral credit risk*, which generally arises in derivative exposures such as an interest rate swap. In this case, depending on the market factors affecting derivative value, at any time, either party to the contract can have a claim against each other. This is because some derivatives may have positive or negative value to a counterparty depending on the level of market factors (e.g. interest rates, exchange rates). In the third category we have a somewhat different type of credit risk, which is called *reference credit risk*. This type of credit risk may arise for exposures in securitisation or asset-backed securities. In these cases, even if the creditwothiness of the counterparty dose not change, a price fall occurs if the creditworthiness of reference obligors deteriorate. Another important classification is made based on the definition of loss. If we define the credit losses those caused only if a default occurs, we call this approach a default-mode (DM) approach. Alternatively, beside defaults, we can include losses stemming from rating downgrades and changes in credit spreads. The latter approach is called mark-to-market (MtM) approach.

1.2 Sources of Credit Risk

In this section, sources of credit risk (i.e. credit risk factors) are introduced. In order to determine the factors that affects credit risk, we first define credit loss. As mentioned in the previous section, credit risk can be defined in a DM or a MtM environment. First we will analyze credit risk factors in a DM environment.

Let assume a credit card exposure of a bank. In this contract, bank grants a credit line or limit to its client (i.e. credit card holder). By using credit card, the customer can borrow and repay any amount of money up to this limit. After setting the limit by the bank, the exact amount of money borrowed is in the discretion of the borrower. Assume that we want to asses the riskiness of a credit card exposure at time t=0. Our risk horizon is 1 year. The credit card limit is \$ 100, and only \$ 40 is used currently. Possible scenarios are given in Figure 1.1.

At the end of the risk horizon, i.e. at t=1, the drawn part can change. In our example the drawn part increases to \$ 70. The increase in the drawn amount increases credit risk. Therefore the first risk factor we should consider is the possible increase in the exposure amount. This is captured by *exposure at default (EAD)*. In our example, EAD amount is \$ 70, not \$ 40. Generally EAD is expressed as a ratio which will be applied to the current exposure. For example in our case EAD ratio is 70 / 40 = 1.75 .

At this time there are two possible events. In the first scenario we may have no default and the bank collects all its money (i.e. \$ 70). In this case there is no

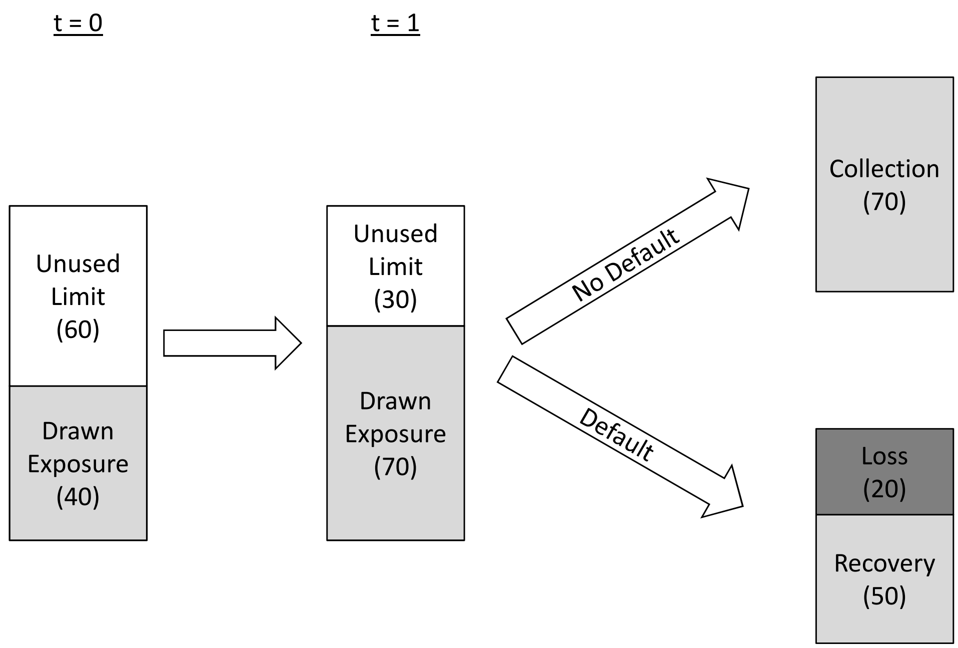

Figure 1.1: Default Mode Scenarios

credit loss. In the second scenario, our obligor defaults on his credit card claim. The possibility of default is a source of risk for our claim. This risk is captured by *probability of default (PD)*. Our credit risk increases with PD.

In the default case, generally banks may not collect all their money back. Rather a portion of the EAD amount can be recovered. In our example, the bank can only recovers \$ 50, and therefore incurs a credit loss of \$ 20. The final loss incurred on EAD amount is called *loss given default (LDG)*. In our example LGD amount is \$ 20. In general LGD is expressed as a ratio which will be applied to the EAD amount. In our case LGD ratio is 20 / 70 = 0.29 .

Therefore, in DM environment we have three different risk factors: PD, LGD and EAD. If we model these risk factors appropriately, we can model credit risk. There are two additional important issues in modeling credit risk. They are inter- and intra- dependencies for risk factors. The first term refers to the dependencies of PDs or LGDs or EADs for different obligors. For instance two firms may have dependent PDs since they operate in the same sector. Or two loans may have dependent LGDs because they have a same kind of collateral. Or two swaps may have dependent EADs because they have the same underlying interest rate. The intradependencies among risk factors is defined as the dependencies between PD and LGD, or PD and EAD, or LGD and EAD. For instance if there exists common factors that both affect PD and LGD of a firm, we should model the dependence structure between these two risk factors.

In a MtM environment, we should have more risk factors, since credit loss is caused not only by default, but also by other credit-related events. For example assume a bank that has an investment in a corporate bond. The current value for the bond is found, using arbitrage principles, by discounting all cash flows by the corresponding discounting factor. The discounting factors should include both risk-free rates for the corresponding maturities and credit spreads for the corresponding maturities and credit quality (i.e. credit rating). The example is schematized in Figure 1.2.

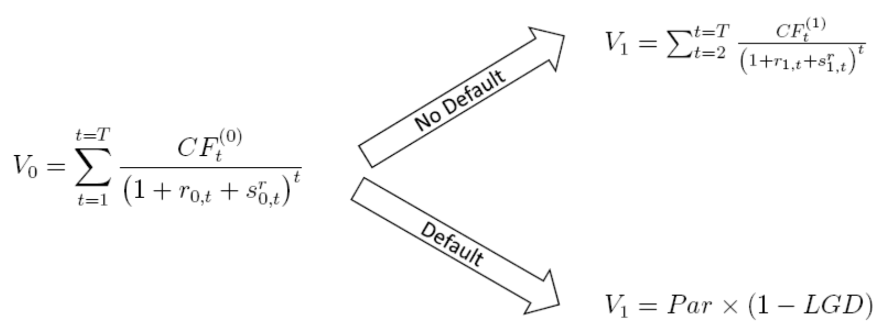

Figure 1.2: Mark-to-Market Scenarios

We can find the current value of the bond by using current term structure for risk-free yields and credit spreads. At the end of our risk horizon, i.e. t=1, the first thing that can happen is the change in the expected cash flows. For example this is more pronounced for interest rate swaps. This risk is captured by EAD.

Then we have two possible scenarios: default or no-default. Similar to the DM environment, the risk of default is captured by PD. If we have a default case, we may recover only a portion of the par value and may incur some loss. This risk is also captured by LGD.

Otherwise, if we have a no-default case, we apply the same procedure for pricing and discount all the expected cash flows by using market values of risk-free yields and credit spreads available at time t=1. Therefore, even if we do not have a default, the value of our credit-risky position may deteriorate if there is an adverse movement in risk-free yields or credit spreads, or if the creditworthiness of the obligor deteriorates (i.e. its rating is downgraded). Hence we have three additional sources of risk. The first one is caused by the risk-free term structure. However this source of risk is purely related to market interest rates and, rather than credit risk, it is generally classified within the *market risk*. The second source of risk is the credit spreads. This *credit spread risk* is an important element of credit risk in a MtM environment. For instance, if the market participants become more risk averse and credit spreads widens, the values of all credit-risky securities will decrease. Indeed credit spread risk is at the intersection of market and credit risks, and sometimes classified under market risk. The final risk factor is the *rating downgrade risk*. Even if the credit spreads do not change, a rating downgrade of an obligor will cause loss because the appropriate credit spreads are now different (i.e. higher).

To sum up we have five types of credit risk factors in a MtM environment. These are PD, LGD, EAD, credit spreads and rating downgrade probabilities. Similar to the DM environment, in modeling credit risk, we should also consider inter and intra- dependencies for these risk factors.

1.3 Main Problems Dealt within Credit Risk Literature

We have a huge literature on credit risk, in which different papers attempt to answer different questions related to credit risk. However, we can classify the main problems dealt within the whole credit risk literature into three main simple questions:

1. What is the riskiness of a single claim?
2. What is the riskiness of a portfolio of claims?
3. How can we price a single claim or a portfolio of claims?

The first question is related to estimating individual credit risk factors for a single claim (or firm). For example estimating PD of a single firm, or estimating potential recoveries from a certain collateral type (e.g. residential mortgages), or estimating EAD for an interest rate swap are all research fields related to the first question. In this category we see a numerous research on credit scoring / rating. Altman's Z-score [Alt68] is a popular example in this category. There are many papers that attempt to estimate PDs from bond prices. Although not as popular as PD estimation, estimation of LGDs and EADs are also an active area of research. For example Moody's LossCalc methodology [GS02] and [GS05] attempts to model LGDs using regression analysis.

The second main question is related to the credit risk of a portfolio. Even if we estimate all the credit risk factors of each individual claims in the portfolio, we should additionally model the dependencies to estimate a more realistic portfolio risk. This is because of *diversification effects*. Although we generally have positive correlation between credit risk factors for different obligors, there exists diversification benefits for a portfolio of claims. The main additional gradient for the portfolio modeling is the *default correlations* which are difficult to estimate. There are many different models that attempt to capture the portfolio risk. For example popular commercial models include Credit Metrics [GFB97], Credit Risk Plus [CS97] and Credit Portfolio View ([Wil97a] and [Wil97b]). Also the new Basel Accord, called Basel-II [BCBS06], which sets the regulatory rules for bank capital, also assumes a portfolio model for its internal ratings-based approaches. Estimating portfolio risk can help financial institutions to increase efficiency in taking investment decisions and allocation capital among business lines.

The final question is related to pricing. Pricing is somewhat a different problem than assessing riskiness. Because in pricing problems we use different probability spaces than we use in risk measurement. The first ground-breaking paper for credit pricing is the famous paper of Merton [Mer74]. In this paper, Merton treats credit-risky securities as contingent claims against firm's assets and uses option pricing theory to value these securities. This approach used by Merton is called *structural approach* to credit risk. After Merton's paper, many additional papers were published, each attempt to generalize the simplistic assumptions used by Merton. Beside structural approach, there are also papers that do not assume a structural relation between default and firm fundamentals, and treat default as a surprise event. This approach is called *reduced-form approach*. The literature on credit pricing deals with pricing bonds (e.g. corporate bonds) and loans (e.g. LIBOR market) as well as other more complex instruments such as asset-backed securities (e.g. mortgage-backed bonds) and credit derivatives (e.g. credit default swaps, credit linked notes, collateralized debt obligations).

1.4 Credit Risk Modeling Approaches

As mentioned in the previous section, there are two different approaches for credit risk modeling: structural approach and reduced-form approach.

In structural modeling, the default of a firm is directly related to its market value of assets. In order to understand this relation, first, let analyze the capital structure of a simple firm. The firm holds assets (A) and the available funds are liabilities (L) and capital (C). Therefore we have,

$$A = L + C$$

This equation is true for two different cases. In the first case we measure the assets, liabilities and capital by using accounting principles. Although these principles do not always yield economically reasonable values for these items, the equation always holds because the accounting principles are designed to do so. Additionally this equation also holds if we measure each item by using market values. Assume, for example, that the firm has issued only one traded zero coupon bond as a liability, invests in a traded zero coupon bond and the shares of the firm are also traded in the secondary market. Then the above equation holds for the capital structure. However, in real life, we generally do not observe the market values for all items. For example there are sometimes illiquid assets, or non-traded debt. However the unobservability of these values do not breach the equality.

Now assume that the market value of assets decreases to the level of liabilities, i.e. A=L. Then the market value of capital is zero, which means that the firm bankrupts, and hence defaults. Therefore we can simply define the default event as the equivalence of assets to liabilities. If we are able to model the evolution of assets and liabilities, then we can model the default event.

The structural approach uses this idea to model default events. The first structural model is the famous Merton's model [Mer74]. In this model the firm value is assumed to follow a geometric Brownian motion, and the value of liabilities are fixed. Then we can easily find the probability of default by finding the probability of the event that the asset value is below the liabilities at maturity.

Indeed the Merton model is simply an application of Black-Scholes option pricing model [BS73] to the firm's equity and debt. Because we can see firm's equity and debt as contingent claims written against firm's assets. To understand this, let analyze the payoffs to equity and debt. These payoffs are illustrated in Figure 1.3.

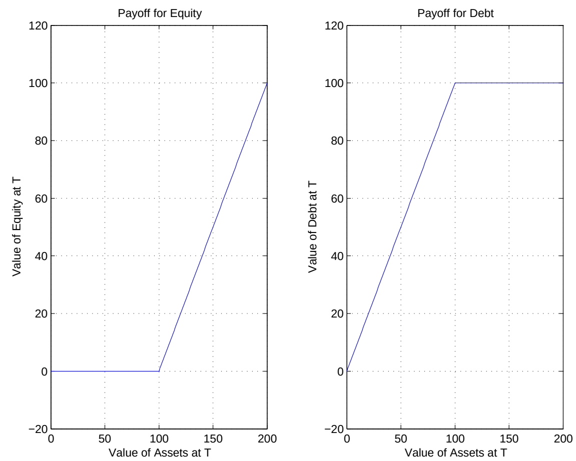

Figure 1.3: Payoffs for Debt and Equity

The equity holders have the potential to gain from extreme returns, but have limited liability which is bounded by their investment. Therefore if the asset values are below the liabilities, the payoff to the equity holders is always zero. But they take all the upside returns if the asset value is over the liabilities. For debt holders, the picture is the reverse. They take no upside gain if the firm has extreme profits (i.e. $A \geq L$). The only amount they can obtain is the interest and principal amount they lent. However, they may incur loss if the asset values can not meet liability requirements (i.e. $A \leq L$).

The payoffs we analyze are actually similar to the option payoffs. Indeed, the equity payoff is exactly the same with the payoff of a call option. And the payoff for debt is the same with the payoff of a portfolio with a long position in a risk-free bond and a short position in a put option. Therefore we can use standard option pricing framework for pricing credit-related instruments. Actually the isolated credit risk component of a bond, which is the difference between a risk-free and a risky bond, is equivalent to a put option on firm assets.

Although the structural approach has a well-grounded explanation for default event, there are many weaknesses of the simple Merton model. And these yield extensions to the original work. For example one of the important extensions to the original model was about the timing of default. In the original Merton model the default can only be happen at the maturity. But this is not a realistic assumption. [BC76] introduces the first-passage time models, in which the default occurs at the first hitting time of the asset value to the default barrier. There are also extensions which deals with the default barrier. In the simple model the default barrier is equal to the liabilities and it is constant. But we can assume stochastic default barriers and barrier may be determined endogenously or exogenously. Studies assuming different barrier specifications include [Le94], [LT96] and [BV97].

In the Merton model, the firm's asset value follows a geometric Brownian motion which produces gaussian returns and under this setting the default events become predictable. In order to remove this unrealistic assumption, [Zhou97] introduces jumps in the asset values which creates discontinuities. With the jumps, the default event becomes unpredictable. Additionally [FSS06] and [FWZ06] introduce stochastic volatility in the asset values which captures an important phenomena observed in financial asset prices, that is volatility clustering.

In pricing credit-risky securities, we also use risk-free interest rates as an input to the model. Merton model assumes constant interest rates. However, further studies examine the effects of stochastic interest rates including [STD93], [KRS93], [LS95], [BV97] and [Zhou97].

Contrary to the structural approach, the reduced-form approach do not assume any explicit relation between the default event and the firm fundamentals. Rather it takes the default event as an unpredictable (surprise) event. In reduced-form models, the default event is governed by an intensity-based or hazard-rate process. In these models the default time is an inaccessible stopping time. In this approach we model directly the distribution of the default time.

[JT95] assumes an exponentially distributed default time with constant hazard rate. Under this specification the default arrival has a Poisson distribution. [JLT97] extends the simple exponential default time model by assuming a continuous time Markov chain for the default event. In their model, the default occurs when the K-state Markov chain hits the absorbing state (which is default) with Markov probabilities are understood to be rating transition probabilities. [DS98] uses credit risk-adjusted short rates for valuing credit-risky bonds. In their model the risky short rate includes both risk-free short rate and risk premiums associated with credit and liquidity risk.

For a well-structured review of different modeling approaches see [Bohn00] and for a detailed mathematical text see [BR02].

1.5 Structure of the Thesis

This thesis aims at analyzing mathematical aspects of structural credit risk modeling with two distinct assumptions. The thesis has three main parts. In the first part, we will present the modeling approaches under the assumption that the firm value follows a geometric Brownian motion. Models for the three important credit risk-related questions are presented in chapters 2 to 4. Chapter 2 present modeling of single firm credit risk. Chapter 3 deals with portfolio credit risk. And chapter 4 is devoted to the credit pricing. We have chapter 5 in the second part, which deals with the weaknesses of geometric Brownian motion assumption. Also in this chapter, possible extensions for the geometric Brownian motion is discussed. In the third part of the thesis, we introduce stochastic volatility and jumps in the asset value process and make risk-free interest rates stochastic, and again analyze the credit risk-related questions. Chapter 6 introduces models for single firm credit risk with our extended assumptions. Chapter 7 deals with credit pricing with the same assumptions, and chapter 8 concludes.

Chapter 2

MODELING SINGLE FIRM CREDIT RISK

In this chapter, we analyze the modeling of single firm credit risk with a structural approach under the assumption of geometric Brownian motion. To serve our purpose, we first analyze the asset values and its returns. There are different sections devoted to the analysis of pathwise and distributional properties of returns, algorithms for simulation of returns as well as scaling and addition properties of returns. Then we will introduce the credit risk factors, PD, LGD and EAD, as well as correlation among them. And at the end we have the loss distribution for a single claim and the common risk measures.

2.1 Asset Value Process and Returns

Assume that value of the firm's assets follows a geometric Brownian motion. Then we can express asset value by using the following stochastic differential equation:

$$\frac{dV_t}{V_t} = \mu d_t + \sigma dW_t \tag{2.1}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and W_t is a standard Brownian motion.

Solution of asset value process can be found by using Ito Calculus. Let $Y_t = f(V_t)$ with $f(x) = \ln(x)$. Then we have $f'(x) = 1/x$ and $f''(x) = -1/x^2$. Then by using Ito Lemma:

$$\begin{aligned} dY_t &= f'(V_t)\,dV_t + \frac{1}{2} f''(V_t) \underbrace{d < V_t, V_t >}_{= {V_t}^2 \sigma^2 dt} \\ &= \frac{1}{V_t} dV_t + \frac{1}{2}\left(\frac{-1}{{V_t}^2}\right) {V_t}^2 \sigma^2 dt \\ &= \frac{1}{V_t}\left(\mu V_t dt + \sigma V_t dW_t\right) - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \end{aligned}$$

Therefore:

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_0^t \sigma dW_t \\ &= Y_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \end{aligned}$$

Since $Y_t = \ln(V_t)$:

$$\begin{aligned} \ln(V_t) &= \ln(V_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \\ V_t &= V_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \end{aligned} \tag{2.2}$$

Since the standard Brownian motion has a normal distribution, the asset value therefore follows a log-normal distribution. Simulated examples for asset values with different drift and volatility terms are given in Figure 2.1.

The continuously compounding (logarithmic) returns between time 0 and t is defined as:

$$R_{0,t} := \ln(V_t/V_0) = \ln(V_t) - \ln(V_0)$$

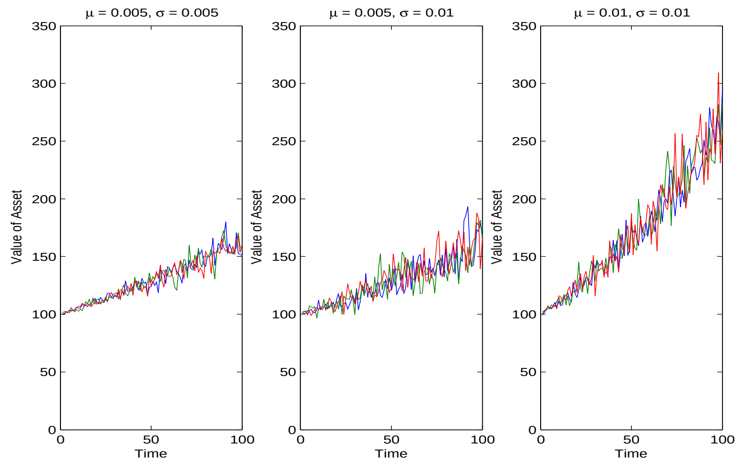

Figure 2.1: Simulated Asset Values

Therefore:

$$\begin{aligned} R_{0,t} &= \ln(V_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \ln(V_0) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \end{aligned} \tag{2.3}$$

And the return between time $(t - \Delta t)$ and t is:

$$\begin{aligned} R_{t-\Delta t,t} &:= \ln(V_t) - \ln(V_{t-\Delta t}) \\ &= \ln(V_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \ln(V_0) - \left(\mu - \frac{1}{2}\right)(t - \Delta t) - \sigma W_{t-\Delta t} \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\left(W_t - W_{t-\Delta t}\right) \end{aligned}$$

For R_t, the corresponding stochastic differential equation is:

$$dR_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \tag{2.4}$$

And the detrended returns are defined as:

$$R_t^D := R_t - \left(\mu - \frac{1}{2}\sigma^2\right) t = \sigma W_t \tag{2.5}$$

2.2 Pathwise Properties of Returns

In this section we analyze the pathwise properties of returns.

2.2.1 Continuity

Since Brownian motion is, by definition, continuous, the asset value process and return process are continuous.

2.2.2 Independence

For time intervals $(t - 2\Delta t, t - \Delta t)$ and $(t - \Delta t, t)$, we have:

$$\begin{aligned} R_{t-2\Delta t,t-\Delta t} &= \left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma \left(W_{t-\Delta t} - W_{t-2\Delta t}\right) \\ R_{t-\Delta t,t} &= \left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma \left(W_t - W_{t-\Delta t}\right) \end{aligned}$$

Since the two increments of Brownian motion, i.e. $(W_{t-\Delta t} - W_{t-2\Delta t})$ and $(W_t - W_{t-\Delta t})$, are, by definition, independent, the two return $R_{t-2\Delta t,t-\Delta t}$ and $R_{t-\Delta t,t}$, are also independent.

2.2.3 Stationarity

Since $(W_{t-\Delta t} - W_{t-2\Delta t})$ has the same distribution as $(W_t - W_{t-\Delta t})$, $R_{t-2\Delta t,t-\Delta t}$ has the same distribution as $R_{t-\Delta t,t}$, which means that the returns are stationary.

2.2.4 Martingale Property

The detrended or unanticipated part of returns has martingale property:

$$\begin{aligned}
R_t^D &= R_t - \left(\mu - \frac{1}{2}\sigma^2\right) t = \sigma W_t \\
E\left[R_t^D | F_s\right] &= E\left[\sigma W_t | F_s\right] = E\left[\sigma\left(W_t - W_s\right) + \underbrace{\sigma W_s}_{\in F_s} | F_s\right] \\
&= E\left[\underbrace{\sigma\left(W_t - W_s\right)}_{\text{Independent of } F_s} | F_s\right] + \sigma W_s \\
&= 0 + \sigma W_s \\
&= \sigma W_s \\
&= R_s^D
\end{aligned}$$

2.2.5 Markov Property

The detrended part of returns has Markov property. To show this, we should have, for each non-negative Borel-measurable function f, there is another Borel-measurable function g such that:

$$\begin{aligned}
E\left[f\left(R_t^D\right) | F_s\right] &= g\left(R_t^D\right) \\
&= E\left[f\left(\sigma W_t\right) | F_s\right] \\
&= E\left[f\left(\sigma\left(W_t - W_s\right) + \sigma W_s\right) | F_s\right]
\end{aligned}$$

Since $(W_t - W_s)$ is independent of F_s and $W_s \in F_s$, from Independence Lemma, we have:

$$E\left[f\left(\sigma\left(W_t - W_s\right) + \sigma W_s\right) | F_s\right] = g\left(W_s\right)$$

where $g(x) = E\left[f\left(\sigma\left(W_t - W_s\right) + \sigma x\right)\right]$.

Since $\sigma\left(W_t - W_s\right) \sim N\left(0, \sigma^2\left(t-s\right)\right)$,

$$g\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2\left(t-s\right)}} \int_{-\infty}^{\infty} f\left(\omega + \sigma x\right) e^{\frac{-\omega^2}{2\sigma^2(t-s)}} d\omega$$

We may change the variable $\tau = t - s$ and $y = \omega + \sigma x$:

$$g\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} f\left(y\right) e^{\frac{-(y-\sigma x)^2}{2\sigma^2\tau}} dy$$

And define the transition density $h\left(\tau, x, y, \sigma\right)$ as:

$$h\left(\tau, x, y, \sigma\right) := \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{\frac{-(y-\sigma x)^2}{2\sigma^2\tau}}$$

Therefore:

$$g\left(x\right) = \int_{-\infty}^{\infty} f\left(y\right) h\left(\tau, x, y, \sigma\right) dy$$

At the end, we obtain:

$$E\left[f\left(\sigma W_t\right) | F_s\right] = g\left(W_s\right) = \int_{-\infty}^{\infty} f\left(y\right) h\left(\tau, \omega_s, y, \sigma\right) dy$$

Therefore the detrended part of returns has Markov property.

2.2.6 Quadratic Variation

We analyze the quadratic variation of detrended returns. For any partition of time interval, $[0, T]$, $\pi = \{0 = t_o, t_1, \cdots, t_n = T\}$,with $\|\pi\| = \max_j\left(t_{j+1} - t_j\right)$, the sampled quadratic variation is defined as:

$$\begin{aligned} QV_\pi &= \sum_{j=0}^{n-1} \left(R^D_{t_{j+1}} - R^D_{t_j}\right)^2 \\ &= \sum_{j=0}^{n-1} \sigma^2 \left(W_{t_{j+1}} - W_{t_j}\right) \\ &= \sigma^2 \sum_{j=0}^{n-1} \left(W_{t_{j+1}} - W_{t_j}\right)^2 \end{aligned}$$

We can find $E\left[QV_\pi\right]$ as follows:

$$
\begin{aligned}
E\left[\left(W_{t_{j+1}}-W_{t_j}\right)^2\right] &= Var\left[W_{t_{j+1}}-W_{t_j}\right]=t_{j+1}-t_j \\
\Rightarrow E\left[QV_\pi\right] &= \sum_{j=0}^{n-1}\sigma^2 E\left[\left(W_{t_{j+1}}-W_{t_j}\right)^2\right] \\
&= \sigma^2\sum_{j=0}^{n-1}\left(t_{j+1}-t_j\right)=\sigma^2\tau
\end{aligned}
$$

We can find Var $\left[QV_\pi\right]$ as follows:

$$
\begin{aligned}
Var\left[\left(W_{t_{j+1}}-W_{t_j}\right)^2\right] &= E\left[\left\{\left(W_{t_{j+1}}-W_{t_j}\right)^2-\left(t_{j+1}-t_j\right)\right\}^2\right] \\
&= E\left[\left(W_{t_{j+1}}-W_{t_j}\right)^4-2\left(t_{j+1}-t_j\right)E\left[\left(W_{t_{j+1}}-W_{t_j}\right)^2\right]\right. \\
&\quad \left.+(t_{j+1}-t_j)^2\right] \\
&= 3(t_{j+1}-t_j)^2-2(t_{j+1}-t_j)^2+(t_{j+1}-t_j)^2 \\
&= 2(t_{j+1}-t_j)^2 \\
\Rightarrow Var\left[QV_\pi\right] &= \sum_{j=0}^{n-1}Var\left[\left(W_{t_{j+1}}-W_{t_j}\right)^2\right]=\sum_{j=0}^{n-1}2(t_{j+1}-t_j)^2
\end{aligned}
$$

Additionally, we have:

$$
Var\left[QV_\pi\right]=\sum_{j=0}^{n-1}2(t_{j+1}-t_j)^2\leq\sum_{j=0}^{n-1}2\|\pi\|\left(t_{j+1}-t_j\right)=2\|\pi\|\tau
$$

The quadratic variation for detrended returns is defined as:

$$
QV=\lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}\left(R^D_{t_{j+1}}-R^D_{t_j}\right)^2
$$

In the limit we have:

$$E\left[QV\right] = \lim_{\|\pi\|\to 0} E\left[\sum_{j=0}^{n-1}\left(R^D_{t_{j+1}} - R^D_{t_j}\right)^2\right] = \sigma^2\tau$$

Additionally since $Var\left[QV_\pi\right] = \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2$ is non-negative, $Var\left[QV_\pi\right] \leq 2\|\pi\|\tau$, and $\lim_{\|\pi\|\to 0} 2\|\pi\|\tau = 0$, we have:

$$\begin{aligned} Var\left[QV\right] &= \lim_{\|\pi\|\to 0} Var\left[\sum_{j=0}^{n-1}\left(R^D_{t_{j+1}} - R^D_{t_j}\right)^2\right] \\ &= \lim_{\|\pi\|\to 0} Var\left[QV_\pi\right] = 0 \end{aligned}$$

Therefore:

$$QV = \sigma^2\tau$$

2.3 Distributional Properties of Returns

In this section we analyze the density, moment generating function and moments of returns.

2.3.1 Density

For R_{t_1,t_2}, the cumulative distribution function is:

$$\begin{aligned} F\left(x\right) &= Pr\left\{R_{t_1,t_2} < x\right\} \\ &= Pr\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\left(t_2 - t_1\right) + \sigma\left(W_{t_2} - W_{t_1}\right) < x\right\} \\ &= Pr\left\{W_{t_2} - W_{t_1} < \frac{x - \left(\mu - \frac{1}{2}\sigma^2\right)\left(t_2 - t_1\right)}{\sigma}\right\} \end{aligned}$$

Since $(W_{t_2} - W_{t_1}) \sim N(0, t_2 - t_1)$, we have:

$$F(x) = \Phi\left[\frac{x - \left(\mu - \frac{1}{2}\sigma^2\right)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}}\right]$$

where Φ is the cumulative density function for standardized normal distribution.

For $\tau = t_2 - t_1$,

$$F(x) = \Phi\left[\frac{x - \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right]$$

And the density is:

$$f(x) = \frac{dF(x)}{dx} = \phi\left[\frac{x - \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right]$$

where ϕ is the density for standard normal distribution.

Therefore:

$$R_{t_1,t_2} \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right) \tag{2.6}$$

Examples of return distributions are given in Figure 2.2.

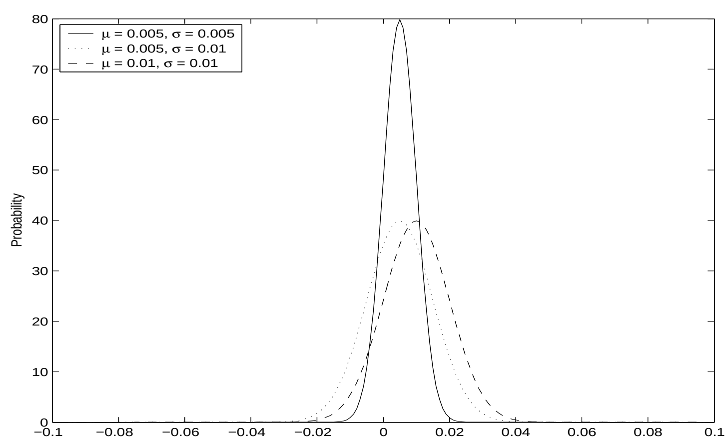

Figure 2.2: Examples of Return Distributions

2.3.2 Moment Generating Function

The moment generating function (MGF) of a random variable X is defined as:

$$MGF_X(u) = E\left[e^{uX}\right]$$

Then we can define the MGF of returns as follows:

$$\begin{aligned} MGF_{R_{t_1,t_2}}(u) &= E\left[e^{uR_{t_1,t_2}}\right] = E\left[e^{u\left(\mu-\frac{1}{2}\sigma^2\right)\tau+u\sigma\left(W_{t_2}-W_{t_1}\right)}\right] \\ &= e^{u\left(\mu-\frac{1}{2}\sigma^2\right)\tau}E\left[e^{u\sigma\left(Wt_2-W_{t_1}\right)}\right] \end{aligned}$$

Since $W_{t_2} - W_{t_1}$ has the same distribution with W_τ and using MGF of normal distribution:

$$\begin{aligned} MGF_{R_{t_1,t_2}}(u) &= e^{u\left(\mu-\frac{1}{2}\sigma^2\right)\tau}e^{\frac{1}{2}\sigma^2u^2\tau} \\ &= exp\left\{u\mu\tau - \frac{1}{2}u\sigma^2\tau + \frac{1}{2}u^2\sigma^2\tau\right\} \end{aligned}$$

2.3.3 Moments

We can find the moments by using the moment generating function. For any random variable X, the general relation with k^{th} moment, m^k, and moment generating function is as follows:

$$m^k := E\left[X^k\right] = \frac{d^k}{du^k}M_X(0)$$

For the returns, mean, variance, skewness and excess kurtosis is given by:

$$\text{Mean} = E\left[R_{t_1,t_2}\right] = \left(\mu - \frac{1}{2}\sigma^2\right)\tau$$

$$\text{Variance} = E\left[\left(R_{t_1,t_2} - E\left[R_{t_1,t_2}\right]\right)^2\right] = \sigma^2\tau$$

$$\text{Skewness} = \frac{E\left[\left(R_{t_1,t_2} - E\left[R_{t_1,t_2}\right]\right)^3\right]}{\left(E\left[\left(R_{t_1,t_2} - E\left[R_{t_1,t_2}\right]\right)^2\right]\right)^{\frac{3}{2}}} = 0$$

$$\text{Excess Kurtosis} = \frac{E\left[\left(R_{t_1,t_2} - E\left[R_{t_1,t_2}\right]\right)^4\right]}{\left(\left[\left(R_{t_1,t_2} - E\left[R_{t_1,t_2}\right]\right)^2\right]\right)^2} - 3 = 0$$

Therefore the distribution of returns is symmetric and mesokurtic.

2.4 Simulation of Returns

The model for asset value returns is a continuous model. In order to simulate it, we should use discretized version of the model. For this, we divide the time interval $[0, T]$ into discrete intervals, $\{0 = t_0, t_1, t_2 \cdots, t_N = T\}$. The discrete intervals need not be equally spaced. Then, by using *Euler Scheme*, the discretized model is given below.

$$R_{t_1,t_2} = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z$$

where $\Delta t = t_2 - t_1$ and $Z \sim N(0, 1)$.

There are many different techniques used for the simulation of normal variables (see for example [Rub81] or [Ross97]). The most popular ones are:

- The inverse transform method
- The Box-Muller method
- The rejection method

In all these three methods, first we need to generate random variables from uniform distribution. There are different algorithms for generating uniformly distributed random numbers, called low discrepancy numbers. In the next section we give one method for generating them.

2.4.1 Generating Low Discrepancy Numbers

One method that can be used for generating low discrepancy numbers is the Halton sequence. Haltons low discrepancy sequences are based on a simple idea [Bre02]:

- Representing an integer number n in a base b, where b is a prime number:

$$n = (d_m \cdots d_4 d_3 d_2 d_1 d_0)_b = \sum_{k=0}^{m} d_k b^k$$

- Reflecting the digits and adding a radix point to obtain a number, the nth number in the Halton sequence, within the unit interval [0, 1]:

$$h\left(n, b\right) = (0.d_0 d_1 d_2 d_3 d_4 \cdots d_m)_b = \sum_{k=0}^{m} d_k b^{-(k+1)}$$

2.4.2 The Inverse Transform Method

In this method, we use the following proposition.

Proposition: Let U be a uniform $(0, 1)$ random variable. For any continuous distribution function F, the random variable X defined by $X = F^{-1}\left(U\right)$ has distribution F [Ross97].

Then the algorithm for simulating random numbers from normal distribution is:

- Step 1: Simulate U ∼Uniform$(0, 1)$
- Step 2: $X = \Phi^{-1}\left(U\right)$ where Φ is the distribution function for standard normal distribution. Then X has a standard normal distribution.

2.4.3 The Box-Muller Method

The method uses the fact that the coordinates of a point (x, y) can be expressed as:

$$(x, y) = \left(cos\theta\sqrt{x^2 + y^2}, sin\theta\sqrt{x^2 + y^2}\right)$$

where θ is the angle between x-axis and the line (0,0)-(x,y).

The algorithm is:

- Step 1: Generate two independent uniformly distributed random numbers: U_1 ~Uniform$(0, 1)$, U_2 ~Uniform$(0, 1)$.
- Step 2: Set $R^2 = -2\ln(U_1)$ and $\theta = 2\pi U_2$
- Step 3: Set $Z_1 = Rcos\theta$ and $Z_2 = Rsin\theta$. Then, Z_1 and Z_2 are independent and identically distributed (iid) standardised normal random variables.

2.4.4 The Rejection Method

The method uses the following theorem.

Theorem: Let X be a random variate distributed with the density function $f_x(x)$, which is represented as:

$$f_x(x) = C\, g(x)\; h(x)$$

where $C \geq 1, 0 \leq g(x) \leq 1$ and $h(x)$ is also a density. Let U and Y be distributed uniformly $(0, 1)$ and $h(x)$ respectively. Then:

$$f_y(x \mid U \leq g(Y)) = f_x(x)$$

For proof see [Rub81], theorem 3.4.1.

The algorithm is:

- Step 1: Generate $U \sim Uniform(0, 1)$

- Step 2: Generate Y from the density h(x)
- Step 3: If $U \leq g(Y)$ deliver Y as the variate generated from $f_x(x)$

2.5 Scaling and Addition Properties

In order to analyze the scaling and addition properties, we should determine the characteristic function for returns.

Define $\mu_R = \left(\mu - \frac{1}{2}\sigma^2\right)$ and $\sigma_R = \sigma$ such that:

$$R_{0,t} = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t = \mu_R t + \sigma_R W_t$$

Then we can define the characteristic function.

$$\begin{aligned}
\varphi_{R_{0,t}}(u) &= E\left[exp\left\{iuR_{0,t}\right\}\right] \\
&= E\left[exp\left\{iu\mu_R t + iu\sigma_R W_t\right\}\right] \\
&= exp\left\{iu\mu_R t\right\} E\left[exp\left\{iu\sigma_R W_t\right\}\right] \\
&= exp\left\{iu\mu_R t\right\} \varphi_{(\sigma_R W_t)}(u) \\
&= exp\left\{iu\mu_R t\right\} exp\left\{-\frac{1}{2}u^2\sigma_R^2 t\right\} \\
&= exp\left\{t\left(iu\mu_R - \frac{1}{2}u^2\sigma_R^2\right)\right\}
\end{aligned}$$

2.5.1 Scaling Property

We analyze the properties of $c\,R_{0,t}$ where $c \in \mathbb{R}$, by using the scaling property of characteristic functions, i.e.:

$$\varphi_{cR_{0,t}}(u) = \varphi_{R_{0,t}}(cu) = exp\left\{t\left(icu\mu_R - \frac{1}{2}u^2c^2\sigma_R^2\right)\right\}$$

As can be seen from the above equality, if $R_{0,t}$ follows a geometric Brownian motion with drift μ_R and volatility σ_R, then $c\,R_{0,t}$ follows a Geometric Brownian Motion with drift $c.\mu_R$ and volatility $c.\sigma_R$.

2.5.2 Addition Property

We want to analyze the properties of sum of two independent returns, i.e. $R_{0,t}^{(1)} + R_{0,t}^{(2)}$.

By using the addition property of characteristic functions, if $R_{0,t}^{(1)} \sim GBM(\mu_1, \sigma_1)$ and $R_{0,t}^{(2)} \sim GBM(\mu_2, \sigma_2)$, then we have:

$$\begin{aligned}
\varphi_{R_{0,t}^{(1)}+R_{0,t}^{(2)}}(u) &= \varphi_{R_{0,t}^{(1)}}(u)\,\varphi_{R_{0,t}^{(2)}}(u) \\
&= exp\left\{t\left(iu\mu_1 - \frac{1}{2}u^2\sigma_1^2\right)\right\} exp\left\{t\left(iu\mu_2 - \frac{1}{2}u^2\sigma_2^2\right)\right\} \\
&= exp\left\{t\left(iu(\mu_1+\mu_2) - \frac{1}{2}u^2\left(\sigma_1^2+\sigma_2^2\right)\right)\right\}
\end{aligned}$$

Therefore, if $R_{0,t}^{(1)} \sim GBM(\mu_1, \sigma_1)$ and $R_{0,t}^{(2)} \sim GBM(\mu_2, \sigma_2)$, then we have $R_{0,t}^{(1)} + R_{0,t}^{(2)} \sim GBM\left(\mu_1+\mu_2, \sqrt{\sigma_1^2+\sigma_2^2}\right)$.

2.6 Probability of Default

In this section, we determine the probability that the firm will go into default within a time horizon. In structural models, the default is assumed to happen, if the value of firm's assets is below the firm's liabilities. Let DP denote the default point, which is assumed to be the face value of firm's liabilities. Then the probability of default is defined as:

$$\begin{aligned}
PD = Pr\{V_T \leq DP\} &= P\{\ln(V_t) \leq \ln(DP)\} \\
&= Pr\{\ln(V_t/V_0) \leq \ln(DP/V_0)\} \\
&= Pr\{R_{0,T} \leq \ln(DP/V_0)\} \\
&= Pr\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T \leq \ln(DP/V_0)\right\} \\
&= Pr\left\{W_T \leq \frac{\ln(DP/V_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma}\right\}
\end{aligned}$$

Since $W_T \sim N(0,T)$, we have:

$$PD = \Phi\left[\frac{\ln(DP/V_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right] \tag{2.7}$$

Therefore the PD depends on the *level of leverage*, i.e. DP/V_0, parameters of asset value process, i.e. μ and σ, and the time horizon, i.e. T. In Figure 2.3, we see the PDs sketched with respect to leverage, μ and σ, assuming $T = 1$.

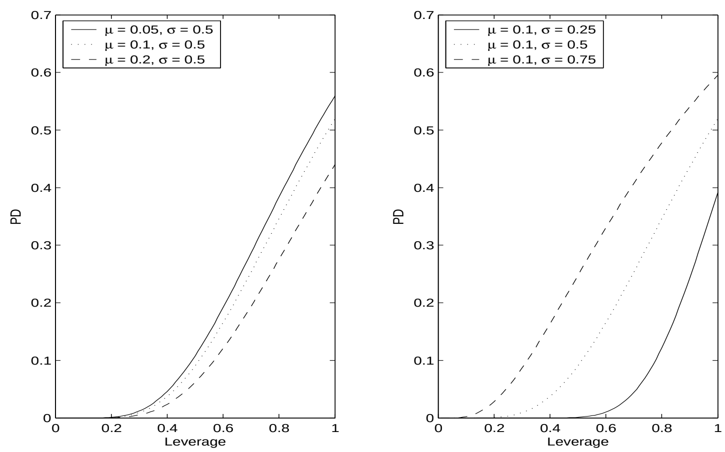

Figure 2.3: Probability of Default

As seen from the graphs, the PD increases with the volatility of asset values. This is because the more volatility the more the chance of hitting DP. The opposite is true for the asset drift. Because a high drift increases the chance of an upward sloping asset trajectory, which decreases the PD. And finally PD increases with leverage since high leverage means that the initial asset value is already so close to the default point.

Sometimes, a measure called *distance-to-default (DD)* is defined as:

$$DD := \frac{\ln(V_0/DP) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{2.8}$$

It measures the distance of the expected value of firm's assets from the default point in terms of its standard deviation. Then we have:

$$PD = \Phi\left(-DD\right) \;\Rightarrow\; DD = -\Phi^{-1}\left(PD\right)$$

There is also another approach called *first-passage time approach* to the definition of PD. Contrary to the classical approach where default can happen only at maturity, in first-passage time approach, the default happens at the first time asset value hits DP.

The default time, τ, and the PD in first-passage time approach are defined as follows.

$$\begin{aligned} \tau &:= inf\left\{\tau \in [0,T] : V_\tau \leq DP\right\} \\ PD^{FPT} &:= P\left\{\tau \leq T\right\} = P\left\{\inf_{0\leq t\leq T} V_t \leq DP\right\} \\ &= P\left\{\ln\left(\inf_{0\leq t\leq T} V_t\right) \leq \ln\left(DP\right)\right\} \end{aligned}$$

Since $\ln(.)$ is a monotonic function:

$$\begin{aligned} PD^{FPT} &= P\left\{\inf_{0\leq t\leq T} \ln\left(V_t\right) \leq \ln DP\right\} \\ &= P\left\{\inf_{0\leq t\leq T} \left[\ln\left(V_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \leq \ln DP\right\} \\ &= P\left\{\inf_{0\leq t\leq T} \left[\ln\left(V_0/DP\right) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \leq 0\right\} \end{aligned}$$

If we define:

$$\begin{aligned} \theta_1 &:= \ln\left(V_0/DP\right) \\ \theta_2 &:= \left(\mu - \frac{1}{2}\sigma^2\right) \\ \theta_3 &:= \sigma \end{aligned}$$

then:

$$PD^{FPT} = P\left\{\inf_{0\leq t\leq T} \theta_1 + \theta_2 t + \theta_3 W_t \leq 0\right\}$$

To find this probability, we use the following Lemma.

Lemma: Let Y be given by $Y_t = y_0 + \vartheta t + \sigma W_t$,where $\vartheta \in \mathbb{R}$, $\sigma > 0$ and W_t is a standard Brownian motion under $\mathbb{P}$. Then the random time $\tau := \inf\{t \geq 0, Y_t \leq 0\}$ has an inverse Gaussian probability distribution under $\mathbb{P}$. More specifically, for any $0 < t < \infty$, we have:

$$Pr\{\tau \leq t\} = \Phi(h_1(t)) + e^{-2y_0\vartheta/\sigma^2}\Phi(h_2(t))$$

where:

$$\begin{aligned} h_1(t) &= \frac{-y_0 - \vartheta t}{\sigma\sqrt{t}} \\ h_2(t) &= \frac{-y_0 + \vartheta t}{\sigma\sqrt{t}} \end{aligned}$$

For proof see [MR05].

When we apply this Lemma, we have:

$$\begin{aligned} PD^{FPT} &= \Phi\left(\frac{-\ln(V_0/DP) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &\quad + e^{-2\left(\mu-\frac{1}{2}\sigma^2\right)\ln(V_0/DP)/\sigma^2}\Phi\left(\frac{-\ln(V_0/DP) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &= \Phi\left(\frac{\ln(DP/V_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &\quad + \left(\frac{DP}{V_0}\right)^{2\left(\mu-\frac{1}{2}\sigma^2\right)/\sigma^2}\Phi\left(\frac{\ln(DP/V_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{2.9}$$

Note that the first term is the PD under the classical approach. Since both DP and V_0 are non-negative and the last term is a probability (i.e. non-negative), PD^{FPT} is always greater than the PD in the classical approach. Throughout the thesis, we use classical approach unless stated otherwise. The sensitivity of PD^{FPT} with respect to different parameters are shown in Figure 2.4.

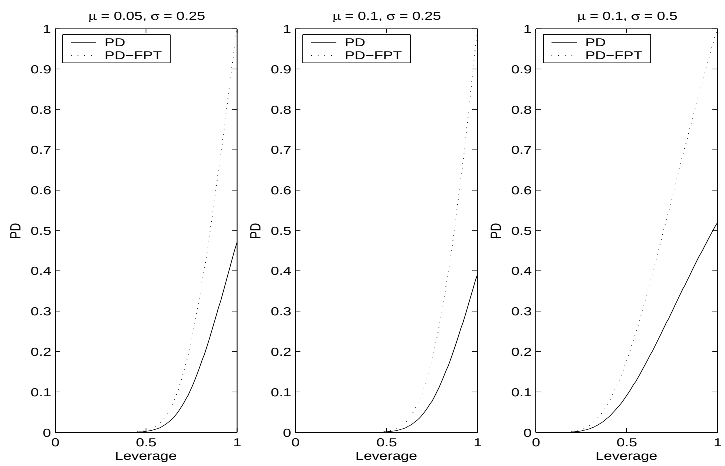

Figure 2.4: Probability of Default in First-Passage Approach

2.7 Loss Given Default

In Merton model, the recovery amount is equal to asset value in the event of default, i.e. V_T. Therefore in the model LGD is assumed to be stochastic. In structural modeling, sometimes LGD is assumed to be deterministic. However, there are substantial empirical findings showing that the LGD ratios are highly volatile (for example [AK96], [MIS02]). Therefore modeling LGD as a random variable is a reasonable assumption.

In Merton model, the recovery amount is V_T and the recovery ratio, i.e. $RR = 1 - LGD$, is $\frac{V_T}{F}$ where F is the face value of debt. We already know that V_T follows a lognormal distribution. Then we can express the distribution of LGD as a truncated lognormal distribution since we can only have a loss if a default occurs. Therefore:

$$RR = \frac{1}{F} \times V_T \text{ given } V_T \leq F \tag{2.10}$$

Then the expected RR is given by:

$$E[RR] = E\left[\frac{V_T}{F}|V_T \leq F\right] = \frac{1}{F} \times E[V_T|V_T \leq F]$$

The expectation of the truncated function is given by [ARS02] and for a formal proof see [LLKM97]:

$$E[V_T|V_T \leq F] = e^{\mu_* + \frac{\sigma_*^2}{2}} \times \frac{\Phi\left(\frac{ln(F)-\mu_*}{\sigma_*} - \sigma_*\right)}{\Phi\left(\frac{ln(F)-\mu_*}{\sigma_*}\right)}$$

where $\mu_* = ln(V_t) + \left(\mu - \frac{\sigma^2}{2}\right)\tau$ and $\sigma_*^2 = \sigma^2\tau$ are the mean and variance of $ln(V_t)$. When we put these into the equation:

$$\begin{aligned} E[V_T|V_T \leq F] &= e^{ln(V_t)+\mu\tau} \frac{\Phi\left(-\frac{ln\left(\frac{V_t}{F}\right)+\left(\mu+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)}{\Phi\left(-\frac{ln\left(\frac{V_t}{F}\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)} \\ &= V_t e^{\mu\tau} \frac{\Phi(-d_1)}{\Phi(-d_2)} \\ &= E[V_T] \frac{\Phi(-d_1)}{\Phi(-d_2)} \end{aligned}$$

where:

$$\begin{aligned} d_1 &= \frac{ln\left(\frac{V_t}{F}\right) + \left(\mu + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= \frac{ln\left(\frac{V_t}{F}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

Therefore the expected RR is:

$$\begin{aligned} E[RR] &= E\left[\frac{V_T}{F}|V_T \leq F\right] \\ &= \frac{1}{F}V_t e^{\mu\tau}\frac{\Phi(-d_1)}{\Phi(-d_2)} \\ &= E\left[\frac{V_T}{F}\right]\frac{\Phi(-d_1)}{\Phi(-d_2)} \end{aligned}$$

Therefore, for LGD ratio we have:

$$E[LGD] = 1 - \frac{V_t e^{\mu\tau}}{F}\frac{\Phi(-d_1)}{\Phi(-d_2)} \quad (2.11)$$

The sensitivity of expected LGD ratios to the leverage, drift and volatility is given in Figure 2.5. The expected LGD increases with volatility and leverage, and decreases with drift.

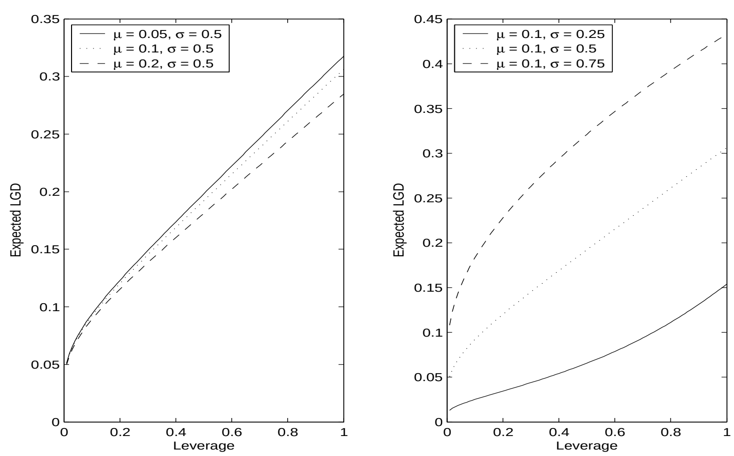

Figure 2.5: Expected Loss Given Default

Modeling recovery rates is an active area of research today. Apart from the initial setting of Merton model, there are different statistical specifications used in the literature. For example a widely used specification for recovery rates is the beta distribution. The beta distribution has the nice property that its domain is [0,1] and it has only two parameters. Therefore it is easy to calibrate the distribution if we have data on mean and standard deviation of recovery rates. Beta distribution is used in Credit Metrics [GFB97], Moody's LossCalc Model [GS02] and [GS05] and models of MoodysKMV [CB03]. Other specifications include a normal distribution [Frye00a] and [Frye00b], lognormal distribution [Pyk03] and a logit-normal specification such that:

$$RR = \frac{e^Z}{1 + e^Z}$$

where Z is a standard normal distribution.

In the literature, there are three different definitions used for the measurement of historical LGD ratios. These are:

1. Measuring recovery value as a percentage of face value of debt.
2. Measuring recovery value as a percentage of the debt value just before the default.
3. Measuring recovery value as a percentage of the value of a risk-free debt which has similar contract specifications (i.e. maturity, coupon, etc.).

In Merton model, both the occurance of default event and the LGD amount depends on the final value of assets. Therefore PD and expected LGD have a positive correlation. This can also be seen from Figure 2.6.

2.8 Exposure at Default

The last risk factor we analyze is the exposure at default (EAD). As mentioned in the introduction, for some certain types of claims the exposure amounts just before the default may be uncertain. For example in a revolving loan facility, the EAD amount can not be known in advance. Or for an interest rate swap, the swap value may change during the life of the contract, and therefore we can not exactly know the amount just before the default.

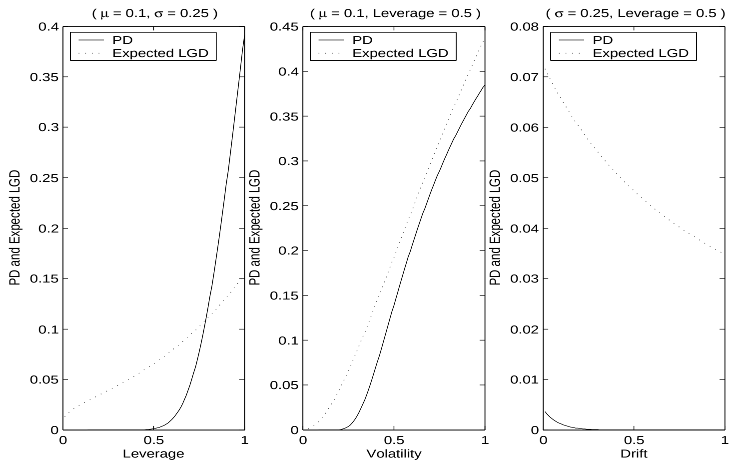

Figure 2.6: PD- LGD Correlation in Merton Model

For claims with stochastic exposures, EAD amount is defined as the amount just before the default and EAD ratio is defined as the ratio of EAD amount to the current exposure. In the simple Merton model, we assume a constant EAD amount which is equal to the face value of debt. However we can model stochastic exposures using the following generic model:

$$EADAmount = CE + PFE \tag{2.12}$$

where CE is the non-negative current exposure and PFE is the potential future exposure which represents the fact that our exposure may increase in the future. For example, for a revolving line of credit, a generic model can be:

$$EAD = Dr_0 + (1 - Dr_0) \times \delta \tag{2.13}$$

where Dr_0 is the initial draw-down rate and δ is the stochastic variable representing stochastic future draw-down rate.

For examples of modeling stochastic exposures, see [GFB97].

2.9 Loss Distribution of a Single Claim

If we assume a constant LGD, the loss distribution for a single claim from the firm has a discrete distribution with only two outcomes. If the firm is not in default, the holder of the claim receives the promised payment, let say F. This event has probability 1-PD. Otherwise, if the firm is in default, which means that the asset value is less than the claim, the holder of the claim receives only the remaining part of the asset, i.e. V_T. And this event has probability PD.

Therefore the payoff and the loss distribution is given by:

$$\text{Payoff} = \begin{cases} F & \text{with 1-PD} \\ V_T & \text{with PD} \end{cases}$$

Loss is defined as difference between promised payment and the actual payoff.

$$L = \begin{cases} 0 & \text{with 1-PD} \\ F - V_T & \text{with PD} \end{cases}$$

The loss distribution is given in Figure 2.7. The amount $F - V_T$ is the loss given default in dollar terms. In general it is expressed as a percentage, i.e.

$$LGD = \frac{F - V_T}{F}$$

2.10 Risk Measures

We can define different risk measures for the loss distribution. The most commonly used ones are expected loss, standard deviation, value-at-risk (VaR) and expected shortfall (ES).

For a single claim, the definitions for four risk measures are given below:

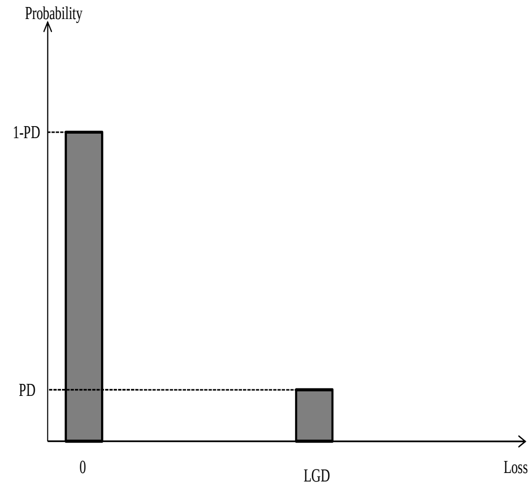

Figure 2.7: Loss Distribution of a Single Firm

1. Expected Loss

Expected loss is defined as the mean of loss distribution.

$$\mu_L = E[L] = \int_{-\infty}^{\infty} (L - E[L])^2 \, dP(L < x) \tag{2.14}$$

2. Standard Deviation

Standard deviation of loss is defined as the square root of the variance of loss distribution.

$$\sigma_L = \sqrt{Var(L)} = \int_{-\infty}^{\infty} (L - E[L])^2 dP(L < x) \tag{2.15}$$

3. Value-at-Risk (VaR)

One of the most common risk measures is the value-at-risk which is defined as the α percentile of the loss distribution.

$$VaR_{\alpha} = \inf\{\ell : Pr\{L \leq \ell\} \geq \alpha\} = F_L^{-1}(\alpha) \tag{2.16}$$

4. Expected Shortfall (ES)

Although VaR is a widely used measure of risk, it has an important deficiency. According to [ADEH99], an efficient measure of risk, ρ, should be a coherent one. And coherent measures of risk must satisfy the following four properties:

1. Monotonicity: For all X and Y, if $X \leq Y$, then $\rho(X) \leq \rho(Y)$.
2. Translation Invariance: For all X and for all $\alpha \in \mathbb{R}$, $\rho(X + \alpha) = \rho(X) - \alpha$.
3. Positive Homogeneity: For all $\lambda \geq 0$ and for all X, $\rho(\lambda X) = \lambda \rho(X)$.
4. Sub additivity: For all X and Y, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Although VaR satisfies the first three properties, the fourth property is not always satisfied. Therefore we need another risk measure which is coherent. A candidate is the expected shortfall. For a random variable X, ES is defined as:

$$ES_{\alpha}(X) = -\frac{1}{\alpha} \times \left\{ E\left[X \times 1_{(X \leq q^{\alpha}(X))}\right] - q^{\alpha}(X) \times \left(P\left\{X \leq q^{\alpha}(X) - \alpha\right\}\right)\right\} \tag{2.17}$$

where q represents the quantile.

Throughout the thesis we consider only expected loss, standard deviation and VaR.

Now we turn to the explicit formulas for risk measures. We analyze two cases: constant and stochastic LGDs.

(i) Constant LGD Case

Under the constant LGD case , we can take LGD out of expectation operator. The risk measures formulated under this assumption is given below.

$$
\begin{aligned}
\mu_L &= 0 \times (1 - PD) + LGD \times PD = PD \times LGD && (2.18)\\
\sigma_L &= \sqrt{(1 - PD)\,(0 - PD \times LGD)^2 + PD\,(LGD - PD \times LGD)^2} \\
&= LGD\sqrt{PD \times (1 - PD)} && (2.19)\\
VaR_\alpha &= \begin{cases} 0 & \text{if } \alpha > \text{PD} \\ LGD & \text{if } \alpha \leq \text{PD} \end{cases} && (2.20)
\end{aligned}
$$

(ii) Stochastic LGD Case

Under the stochastic LGD case, we should also consider LGD volatility. We assume zero correlation between PD and LGD. The risk measures formulated under this assumption is given below, where $L\bar{G}D$ represents the mean of LGD distribution.

$$
\begin{aligned}
\mu_L &= E\left[1_D \times LGD\right] = E\left[1_D\right] \times E\left[LGD\right] = PD \times L\bar{G}D && (2.21)\\
\sigma_L^2 &= \sqrt{E\left[\left(1_D \times LGD - PD \times L\bar{G}D\right)^2\right]} \\
&= \sqrt{E\left[(1_D \times LGD)^2\right] + PD^2 \times L\bar{G}D^2 - 2 \times PD \times L\bar{G}D \times E\left[1_D \times LGD\right]} \\
&= \sqrt{E\left[1_D^2\right] \times E\left[LGD^2\right] + PD^2 \times L\bar{G}D^2 - 2 \times PD^2 \times L\bar{G}D^2} \\
&= \sqrt{(\sigma_D^2 + PD^2) \times (\sigma_{LGD}^2 + LGD^2) - PD^2 \times L\bar{G}D^2} \\
&= \sqrt{(PD \times (1 - PD) + PD^2) \times (\sigma_{LGD}^2 + LGD^2) - PD^2 \times L\bar{G}D^2} \\
&= \sqrt{L\bar{G}D^2 \times PD \times (1 - PD) + PD \times \sigma_{LGD}^2} && (2.22)
\end{aligned}
$$

In the above formula for standard deviation of losses, the first part in the square root is equal to the standard deviation in the constant LGD case. Therefore stochastic LGD increases the standard deviation of loss distribution. For VaR, since we do not know the distribution of LGD, we do not have explicit formulas.

Chapter 3

MODELING PORTFOLIO CREDIT RISK

In general, the financial firms (for example banks) that are exposed to credit risk have many claims from different counterparties. Therefore, beside modeling single firm credit risk, modeling of credit risk caused by a portfolio of claims is also very important. Since, in a portfolio, there will be diversification effects caused by correlations among claims, modeling portfolio credit risk requires additional assumptions and techniques.

In a generic portfolio model, we do the following steps:

1. Define a factor model for the asset value process of the borrowers. This model may be a one-factor or a multi-factor model. The following steps for one and multi-factor models both rely on the same principles.

2. Determine the formula for unconditional default probability. For this we link the default point to the default probability.

3. Determine the formula for default probability conditional on the realization of factor(s) which are common to all obligors.

4. Determine the loss distribution conditional on the realization of common factor(s).

5. Using the distribution of common factor(s), determine the unconditional loss distribution.

For banks, credit portfolio models are also important for regulatory purposes. Because an *asymptotic single risk factor model* is used in Basel-II [BCBS06], the new international regime for bank capital requirements, to determine regulatory capital requirements against loans.

One factor structural modeling of portfolio credit risk is extensively analyzed by [Vas87] and [Vas91] and [Gor03], which assume a single systematic factor affecting all defaults. [Weh03] analyzed loss distributions for a heterogeneous portfolio with a one-factor model. [Frye00a] and [Frye00b] extended this framework to assume one-factor models for both PD and LGD. And finally, [Kup07] extended the framework by assuming one factor models for PD, LGD and EAD.

3.1 One Factor Modeling

In one-factor models, the asset value process is assumed to be driven by one systemic and one idiosyncratic (i.e. firm-specific) factor. The systemic factor represents the global factor that affects all firms simultaneously. It can be thought as an abstract factor that represents the general health of the economy or credit cycle.

3.1.1 Asset Value Process and Returns

In one-factor models, asset value process for firm i follows the following stochastic differential equation:

$$\frac{dV_t^i}{V_t^i} = \mu^i dt + \sigma^i \left(\sqrt{w^i} dX_t + \sqrt{1 - w^i} dZ_t^i \right) \tag{3.1}$$

where $\mu^i \in \mathbb{R}$, $\sigma^i \geq 0$, $0 \leq w^i \leq 1$, X_t is a standard Brownian motion that represents the systemic factor (same for all firms) and Z_t^i is a standard Brownian motion that represents the idiosyncratic factor. We assume that X_t and Z_t^i are independent.

By applying Ito Lemma to $Y_t = f(V_t)$ with $f(x) = \ln(x)$, we have (we remove superscript i from μ, σ, w to ease notation):

$$V_t = V_0\, exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\left(\sqrt{w}X_t + \sqrt{1-w}Z_t^i\right)\right\} \tag{3.2}$$

In general, the linear trend in log-prices, i.e $\left(\mu - \frac{1}{2}\sigma^2\right)t$, is assumed to be zero. This does not effect the model in a critical manner, since it can be justified as having a default point with a linear time trend. Under this assumption the model becomes:

$$V_t = V_0\, exp\left\{\sigma\left(\sqrt{w}X_t + \sqrt{1-w}Z_t^i\right)\right\}$$

Throughout the thesis we do not ignore the drift term. This will give us a more general formula. The continuously compounding returns and detrended returns are:

$$\begin{aligned} R_{0,t} &= \ln(V_t/V_0) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\left(\sqrt{w}X_t + \sqrt{1-w}Z_t^i\right) && (3.3) \\ R_{t-\Delta t,t} &= \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{w}\left(X_t - X_{t-\Delta t}\right) + \sigma\sqrt{1-w}\left(Z_t^i - Z_{t-\Delta t}^i\right) \\ R_t^D &= \sigma\left(\sqrt{w}X_t + \sqrt{1-w}Z_t^i\right) \end{aligned}$$

And the corresponding stochastic differential equation for R_t is:

$$dR_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\left(\sqrt{w}dX_t + \sqrt{1-w}dZ_t^i\right) \tag{3.4}$$

Indeed this one factor representation of asset returns is closely related to the one factor equity modeling, i.e. well-known Capital Asset Pricing Model (CAPM). The CAPM equation for the required return is:

$$r_i - r_f = \alpha_i^0 + \beta_i \times (r_m - r_f) + \epsilon_i$$

where r_i is the required return for asset i, r_f is the return of risk-free asset, r_m is the return of market index, α_i^0 and β_i are constants and ϵ_i represents the idiosyncratic risk. Then, by rearranging the equation, we can express the asset return as follows:

$$\begin{aligned} r_i &= \underbrace{\left[\alpha_i^0 + r_f \times (1-\beta_i)\right]}_{\alpha_i} + \beta_i \times r_m + \epsilon_i \\ &= \alpha_i + \beta_i \times r_m + \epsilon_i \end{aligned}$$

The errors of the expected values for the equation are:

$$\begin{aligned} Error &= r_i - \alpha_i - \beta_i \times E\left[r_m\right] \\ &= (\alpha_i + \beta_i \times r_m + \epsilon_i) - \alpha_i - \beta_i \times E\left[r_m\right] \\ &= \beta_i \times (r_m - E\left[r_m\right]) + \epsilon_i \end{aligned}$$

When we normalize the errors:

$$\begin{aligned} NormalizedError &= \frac{r_i - \alpha_i - \beta \times E\left[r_m\right]}{\sigma_i} \\ &= \beta_i \frac{r_m - E\left[r_m\right]}{\sigma_i} + \frac{\epsilon_i}{\sigma_i} \\ &= \left(\beta_i \frac{\sigma_m}{\sigma_i}\right) \frac{r_m - E\left[r_m\right]}{\sigma_m} + \left(\frac{\sigma_{\epsilon_i}}{\sigma_i}\right) \frac{\epsilon_i}{\sigma_{\epsilon_i}} \end{aligned}$$

Note that $y = \frac{r_i - \alpha_i - \beta \times E[r_m]}{\sigma_i}$, $x = \frac{r_m - E[r_m]}{\sigma_m}$ and $e = \frac{\epsilon_i}{\sigma_{\epsilon_i}}$ are normalized random variables, and hence have unit variances. Therefore:

$$\begin{aligned} 1 &= \left(\beta_i \frac{\sigma_m}{\sigma_i}\right)^2 + \left(\frac{\sigma_{\epsilon_i}}{\sigma_i}\right)^2 \\ \left(\frac{\sigma_{\epsilon_i}}{\sigma_i}\right)^2 &= 1 - \left(\beta_i \frac{\sigma_m}{\sigma_i}\right)^2 \end{aligned}$$

If we define $\rho_V := \left(\beta_i \frac{\sigma_m}{\sigma_i}\right)^2$ then we have:

$$y = \sqrt{\rho_V} x + \sqrt{1 - \rho_V} e$$

and the unanticipated changes in returns becomes:

$$Error = \sigma_i \left[\sqrt{\rho_V} x + \sqrt{1-\rho_V} e \right]$$

Therefore the sensitivity factor w in our one factor credit model, corresponds to the sensitivity factor in a CAPM world:

$$w = \left(\beta_i \frac{\sigma_m}{\sigma_i} \right)^2$$

For a detailed discussion see [TW04].

Indeed factor models are widely used in equity modeling. For stock returns, we may specify different factors. One alternative is using statistical factor models using principal component analysis. Or we may specify factor models using macroeconomic variables such as growth rate, unemployment rate, etc. or using stock indices like CAPM. We can also use firms' financial ratios as factors. For a discussion on the specification of factors, see [HLR03] .

3.1.2 Unconditional Probability of Default

The unconditional default probability, (PD^{uc}) is given as:

$$\begin{aligned} PD^{uc} &= Pr\left\{V_T \leq DP\right\} = Pr\left\{\ln\left(V_T\right) \leq \ln(DP)\right\} \\ &= Pr\left\{\ln(V_0) + (\mu - \frac{1}{2}\sigma^2)T + \sigma(\sqrt{w}X_T + \sqrt{1-w}Z_T^i) \leq \ln(DP)\right\} \\ &= Pr\left\{\sqrt{w}X_T + \sqrt{1-w}Z_T^i \leq \frac{\ln(DP/V_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma}\right\} \end{aligned}$$

Since $X_T \sim N(0,T)$ and $Z_T^i \sim N(0,T)$, we have:

$$(\sqrt{w}X_T + \sqrt{1-w}Z_T^i) \sim N(0,T)$$

Therefore:

$$PD^{uc} = \Phi\left(\frac{\ln(DP/V_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{3.5}$$

We can define the unconditional distance-to-default measure:

$$DD := -\Phi^{-1}(PD^{uc}) = \left(\frac{\ln(V_0/DP) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{3.6}$$

Note that the unconditional PD and the distance to default are equal to what we obtain in the previous chapter. In general, we assume T=1 which represents a one-year horizon.

3.1.3 Default Correlation

Before analyzing default correlation in a one-factor model, let us first derive the general formula.

Let A and B represents to firms and 1_A and 1_B are indicator functions for firm defaults defined as:

$$1_A := \begin{cases} 1 & \text{if A defaults} \\ 0 & \text{if A does not default} \end{cases}$$

$$1_B := \begin{cases} 1 & \text{if B defaults} \\ 0 & \text{if B does not default} \end{cases}$$

And let JPD represents the probability that both firms default. Then we have the following probabilities for joint events:

$$\begin{aligned} Pr\{1_A = 1, 1_B = 1\} &= JPD \\ Pr\{1_A = 1, 1_B = 0\} &= PD_A - JPD \\ Pr\{1_A = 0, 1_B = 1\} &= PD_B - JPD \\ Pr\{1_A = 0, 1_B = 0\} &= 1 - PD_A - PD_B + JPD \end{aligned}$$

where PD_A and PD_B represents the individual default probabilities of A and B.

The default correlation, $\rho_{A,B}$, is defined as the correlation of two default events:

$$\begin{aligned}
\rho_{A,B} &= Correl(1_A, 1_B) = \frac{Cov(1_A, 1_B)}{\sqrt{Var(1_A)Var(1_B)}} \\
&= \frac{\left\{\begin{array}{c} (1-PD_A)(1-PD_B)\text{JPD} \\ +(1-PD_A)(0-PD_B)(PD_A-JPD) \\ +(0-PD_A)(1-PD_B)(PD_B-JPD) \\ +(0-PD_A)(0-PD_B)(1-PD_A-PD_B+JPD) \end{array}\right\}}{\sqrt{PD_A(1-PD_A)PD_B(1-PD_B)}} \\
&= \frac{JPD - PD_A PD_B}{\sqrt{PD_A(1-PD_A)PD_B(1-PD_B)}} \qquad (3.7)
\end{aligned}$$

Therefore:

$$JPD = PD_A PD_B + \rho_{A,B}\sqrt{PD_A(1-PD_A)PD_B(1-PD_B)} \qquad (3.8)$$

Additionally, we can define default probabilities for a firm conditional on the default of the second firm.

$$\begin{aligned}
PD_{A|B} &:= \frac{JPD}{PD_B} = PD_A + \rho_{A,B}\sqrt{\frac{PD_A}{PD_B}(1-PD_A)(1-PD_B)} \\
PD_{B|A} &:= \frac{JPD}{PD_A} = PD_B + \rho_{A,B}\sqrt{\frac{PD_B}{PD_A}(1-PD_A)(1-PD_B)}
\end{aligned}$$

where $PD_{A|B}$ and $PD_{B|A}$ are conditional PD's.

Now, let us assume that the asset value processes of two firms follows a one-factor model, with same w:

$$V_t^A = V_0^A exp\left\{(\mu_A - \frac{1}{2}\sigma_A^2)t + \sigma_A\left(\sqrt{w}X_t + \sqrt{1-w}Z_t^A\right)\right\}$$

$$V_t^B = V_0^B exp\left\{(\mu_B - \frac{1}{2}\sigma_B^2)t + \sigma_B\left(\sqrt{w}X_t + \sqrt{1-w}Z_t^B\right)\right\}$$

Then JPD is given by:

$$\begin{aligned} JPD &= Pr\left\{1_A = 1, 1_B = 1\right\} = Pr\left\{V_T^A \le DP^A, V_T^B \le DP^B\right\} \\ &= Pr\left\{\frac{\sqrt{w}X_T + \sqrt{1-w}Z_T^A}{\sqrt{T}} \le \frac{\ln(DP^A/V_0^A) - (\mu_A - \frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}}\right., \\ &\quad \left.\frac{\sqrt{w}X_T + \sqrt{1-w}Z_T^B}{\sqrt{T}} \le \frac{\ln(DP^B/V_0^B) - (\mu_B - \frac{1}{2}\sigma_B^2)T}{\sigma_B\sqrt{T}}\right\} \\ &= Pr\left\{\frac{\sqrt{w}X_T + \sqrt{1-w}Z_T^A}{\sqrt{T}} \le \Phi^{-1}(PD^A)\right., \\ &\quad \left.\frac{\sqrt{w}X_T + \sqrt{1-w}Z_T^B}{\sqrt{T}} \le \Phi^{-1}(PD^B)\right\} \end{aligned}$$

Since $(\frac{\sqrt{w}X_T+\sqrt{1-w}Z_T^A}{\sqrt{T}})$ and $(\frac{\sqrt{w}X_T+\sqrt{1-w}Z_T^B}{\sqrt{T}})$ have a joint distribution which is a standard bivariate normal distribution with correlation w, we have:

$$JPD = \Phi_2(\Phi^{-1}(PD^A), \Phi^{-1}(PD^B); w) \tag{3.9}$$

where Φ_2 is the standard bivariate normal distribution. And default correlation is:

$$\rho_{A,B} = \frac{\Phi_2(\Phi^{-1}(PD^A), \Phi^{-1}(PD^B); w) - PD^A PD^B}{\sqrt{PD^A(1-PD^A)PD^B(1-PD^B)}} \tag{3.10}$$

Therefore the default correlation depends on firms' PDs and the dependence of their asset values to systematic factor, i.e. w.

3.1.4 Conditional Probability of Default

Let us think a portfolio of n claims from different firms. For all firms, X_t is the common factor effecting their asset value process. Also they each have their firm specific factors, i.e. $Z_t^i, i = 1, \cdots, n$. Since X_t and $(Z_t^i)_{i=1}^n$ are all independent, conditional on the realization of X_t, the asset value processes of all firms become independent. And this will give us extremely important advantages in modeling. This property of factor models is called *conditional independence property*.

Therefore, determining PD's conditional on the realization of X_t is very important.

$$\begin{aligned}
PD_i^c(x) &= Pr\left\{V_T^i \leq DP^i | X_T = x\right\} \\
&= Pr\left\{\ln(V_T^i) \leq \ln(DP^i) | X_T = x\right\} \\
&= Pr\left\{\ln(V_0^i) + (\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i(\sqrt{w_i}x + \sqrt{1-w_i}Z_T^i) \leq \ln(DP^i)\right\} \\
&= Pr\left\{Z_T^i \leq \frac{\frac{\ln(DP^i/V_0^i) - (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i} - \sqrt{w_i}x}{\sqrt{1-w_i}}\right\}
\end{aligned}$$

and for T=1,

$$\begin{aligned}
PD_i^c(x) &= Pr\left\{Z_t^i \leq \frac{\Phi^{-1}(PD_i^{uc}) - \sqrt{w_i}x}{\sqrt{1-w_i}}\right\} \\
&= \Phi\left(\frac{\Phi^{-1}(PD_i^{uc}) - \sqrt{w_i}x}{\sqrt{1-w_i}}\right) \qquad (3.11)
\end{aligned}$$

Sensitivity of PD^c with respect to the realization of systematic factor X for different values of PD^{uc} and w is shown in Figure 3.1.

3.1.5 Loss Given Default

In one factor modeling of asset values, we model the unexpected movements in asset value with systematic and idiosyncratic parts. A similar approach is used in [Frye00a] and [Frye00b] for modeling recovery rates. In these papers, RR is modeled as:

$$RR_i = \mu_i + \sigma_i\left(\sqrt{q}X + \sqrt{1-q}Z_i\right) \qquad (3.12)$$

where RR_i is the recovery rate for the claim i, X is the systematic factor, Z_i is the idiosyncratic factor, q is the sensitivity factor and μ and σ are the mean and standard deviation of RR.

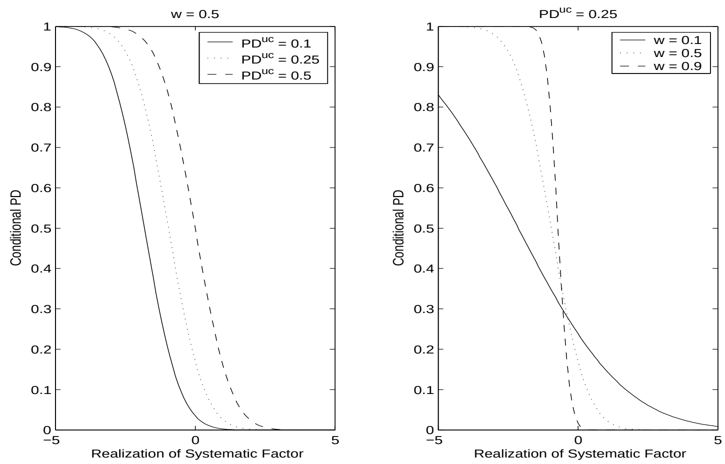

Figure 3.1: Conditional Probability of Default

Assuming that the systematic factor is common to all obligors and X and $(Z_i)_{i=1}^N$ are independent, the correlation between recovery rates of two obligors is:

$$\begin{aligned} Corr\,(RR_i, RR_j) &= Corr\left(\mu_i + \sigma_i\left(\sqrt{q_i}X + \sqrt{1-q_i}Z_i\right),\right. \\ &\quad \left.\mu_j + \sigma_j\left(\sqrt{q_j}X + \sqrt{1-q_j}Z_j\right)\right) \\ &= Corr\left(\sigma_i\sqrt{q_i}X, \sigma_j\sqrt{q_j}X\right) \\ &= q_i q_j \end{aligned} \tag{3.13}$$

With this setting, we can express the distribution of unconditional and conditional RR as follows:

$$\begin{aligned} F_{RR}\,(r) &= Pr\left\{RR^{uc} \leq r\right\} \\ &= Pr\left\{\mu_i + \sigma_i\left(\sqrt{q_i}X + \sqrt{1-q_i}Z_i\right) \leq r\right\} \\ &= Pr\left\{\sqrt{q_i}X + \sqrt{1-q_i}Z_i \leq \frac{r-\mu_i}{\sigma_i}\right\} \\ &= \Phi\left(\frac{r-\mu_i}{\sigma_i}\right) \end{aligned} \tag{3.14}$$

and

$$
\begin{aligned}
F_{RR}^{c}(r) &= Pr\left\{RR^{c} \leq r\right\} \\
&= Pr\left\{\mu_i + \sigma_i\left(\sqrt{q_i}x + \sqrt{1-q_i}Z_i\right) \leq r\right\} \\
&= Pr\left\{Z_i \leq \frac{r - \mu_i - \sigma_i\sqrt{q_i}x}{\sigma_i\sqrt{1-q_i}}\right\} \\
&= \Phi\left[\frac{\frac{r-\mu_i}{\sigma_i} - \sqrt{q}x}{\sqrt{1-q_i}}\right] \\
&= \Phi\left[\frac{\Phi^{-1}\left(RR^{uc}\right) - \sqrt{q}x}{\sqrt{1-q_i}}\right] \qquad (3.15)
\end{aligned}
$$

If both PD and RR are modeled using the same systematic factor, we will implicitly assume a negative correlation between PD and RR (i.e. a positive correlation between PD and LGD).

3.1.6 Exposure at Default

[Kup07] generalizes one factor modeling to include correlated stochastic exposures. The new model can accommodate any distribution and correlation assumption for the LGD and EAD rates and will produce a closed-form approximation for an asymptotic portfolio's loss rate.

In the model EAD for a revolving credit is modeled as follows. Assume that a revolving credit account, i, has a maximum line amount M_i and the initial drawn amount is equal to $M_i Dr_{i,0}$ where $Dr_{i,0}$ is the initial draw-down rate. Then we can express the end of period account exposure, by using a random variable U such that:

$$EAD = M_i U = M_i\left(Dr_{i,0} + \left(1 - Dr_{i,0}\right)\delta_i\right) \qquad (3.16)$$

where $\delta_i \in [0,1]$ is a random variable representing the draw rate. The equation means that the end of period account exposure has two parts: the initial drawn amount and the amount that will be drawn within the period.

[Kup07], using a grid approximation for distribution functions, proposes a one factor model for the draw rate:

$$\delta = F_{\delta}^{-1}\left[\Phi\left(\sqrt{q}X + \sqrt{1-q}Z_i\right)\right] \tag{3.17}$$

where F is the unconditional distribution function for δ. F can be found by fitting a grid approximation to the empirical distribution of draw rates or can be assumed to be a known distribution like beta distribution.

3.1.7 Conditional Loss Distribution

For each claim, if we assume a constant conditional LGD, LGD^c, the conditional loss distribution is a two-state discrete distribution:

$$L_i(x) = \begin{cases} 0 & \text{with probability } 1 - PD^c(x) \\ LGD^c(x) & \text{with probability } PD^c(x) \end{cases}$$

For determining conditional loss distribution, we may have different cases. For example our portfolio may have finite or infinite number of claims, which may be homogeneous or heterogeneous.

Case 1: Homogeneous Portfolio

Assume that we have a portfolio of n claims (loans). Each loan has a size of \$ 1/n , so the total portfolio value is 1\$. Also each borrower have same $LGD^c, \mu, \sigma, w, DP$ and V_0. This means that each has the same $PD^c(x)$.

Since, conditional on the realization of X, the firm defaults are independent, the number of defaults follows a binomial distribution:

$$Pr\left\{\text{k over n defaults}\right\} = \binom{\text{n}}{\text{k}} (PD^c(x))^k (1 - PD^c(x))^{n-k} \tag{3.18}$$

Since all loans have the same exposure size and LGD^c, we have the conditional loss distribution given by:

$$Pr\left\{L^c(x) = k\frac{\overline{LGD}}{n}\right\} = \begin{pmatrix} \text{n} \\ \text{k} \end{pmatrix} (PD^c(x))^k(1 - PD^c(x))^{n-k}$$

where L^c is the total conditional loss of the portfolio and $\overline{LGD}$ is the common LGD^c.

By using the moments of binomial distribution, the mean and the variance of the conditional loss distribution are:

$$E\left[L^c(x)\right] = PD^c(x)\overline{LGD} \tag{3.19}$$

$$Var\left[L^c(x)\right] = \frac{1}{n}\overline{LGD}^2 PD^c(x)(1 - PD^c(x)) \tag{3.20}$$

Case 2: Large Homogeneous Portfolio (LHP)

Assume that we have a homogeneous portfolio with infinite number of loans, i.e. $n = \infty$. We want to derive the conditional loss distribution of this portfolio.

By law of large numbers, conditional loss converges to its expectation, i.e. $PD^c(x)\overline{LGD}$, as $n \to \infty$. This can be seen from limiting mean and variance of $L^c(x)$.

$$\lim_{n\to\infty} E\left[L^c\right] = \lim_{n\to\infty} PD^c(x)\overline{LGD} = PD^c(x)\overline{LGD} \tag{3.21}$$

$$\lim_{n\to\infty} Var\left[L^c(x)\right] = \lim_{n\to\infty} \frac{1}{n}\overline{LGD}^2 PD^c(x)(1 - PD^c(x)) = 0 \tag{3.22}$$

Therefore for an *infinitely granular portfolio*, conditional on the realization of X, there is no uncertainty for the loss.

Indeed, the LHP approximation, sometimes called as the Asymptotic Single Risk Factor (ASRF) model, is used in Basel-II for deriving risk weights (see [BCBS06], [BCBS05], [Vas87] and [Vas91] and [Gor03]).

Case 3: Moderately Heterogeneous Portfolio (MHP)

A purely homogeneous portfolio is actually an unrealistic assumption. In practice we generally encounter heterogeneous portfolios with finite number of obligors. Therefore analyzing heterogeneous portfolio loss distributions are also important. Loss distribution of heterogeneous portfolios are analyzed in [Weh03].

In analyzing heterogeneous portfolios, we generally partition the portfolio into homogeneous sub portfolios. A moderately heterogeneous portfolio H is defined as the union of homogeneous sub-portfolios, H_j, j=1,...h:

$$H = U_{j=1}^{h} H_j$$

where each sub-portfolio H_j contains only identical clients with common conditional loss given default $\overline{LGD_j}$, and parameters $w_j, \sigma_j, \mu_j, DP^j, V_o^j$ (therefore common PD_j^{uc} and $PD_j^c(x)$).

Then for each portfolio, the conditional loss distribution is given by, $\forall j, j = 1, ...h,$:

$$Pr\left\{L_j^c(x) = k\frac{\overline{LGD_j}}{n_j}\right\} = \begin{pmatrix} n_j \\ \text{k} \end{pmatrix} (PD_j^c(x))^k (1 - PD_j^c(x))^{n_j - k} \tag{3.23}$$

where n_j is the number of exposures in sub-portfolio j. Since the conditional defaults are independent:

$$\begin{aligned} p\left(k_1, \cdots, k_n\right) &:= Pr\left\{L_j^c(x) = k_1\frac{\overline{LGD_1}}{n_1}, ..., L_h^c(x) = k_h\frac{\overline{LGD_h}}{n_h}\right\} \\ &= \prod_{j=1}^{h} \begin{pmatrix} n_j \\ k_j \end{pmatrix} (PD_j^c(x))^{k_j} (1 - PD_j^c(x))^{n_j - k_j} \end{aligned} \tag{3.24}$$

Case 4: Large Moderately Heterogeneous Portfolio (LMHP)

In a MHP, if we let, $\forall j, j = 1, ...h, \lim_{n_j \to \infty}$, we have infinitely granular sub-portfolios. Since for each sub-portfolio:

$$\lim_{n_j \to \infty} E\left[L^c_j(x)\right] = PD^c_j(x)\overline{LGD_j} \quad (3.25)$$

$$\lim_{n_j \to \infty} Var\left[L^c_j(x)\right] = 0 \quad (3.26)$$

the conditional loss distribution of each infinitely granular sub-portfolio converges to its mean. Therefore total conditional loss distribution becomes:

$$L^c(x) = \sum_{j=1}^{h} PD^c_j(x)\overline{LGD_j} \quad (3.27)$$

as $n_j \to \infty, j = 1, ...h$.

3.1.8 Granularity Adjustment

There is another important concept in credit risk modeling called *granularity adjustment*. This adjustment is used to allow one to use ASRF framework for a portfolio which is not infinitely granular. The main idea is to adjust ASRF loss distribution by using an adjustment factor which reflects the granularity (i.e. concentration/diversification) of the portfolio.

There are different methodologies proposed in the literature for granularity adjustment. The granularity adjustments are beyond the scope of this thesis. Therefore interested readers are directed to [ET03], [Gor04], [MW02] and [Wil01].

3.1.9 Unconditional Loss Distribution

Now we want to determine the unconditional loss distribution. From elementary probability we know:

$$Pr(A) = E\left[1_A\right] = E\left[E\left[1_A | B\right]\right]$$

Therefore, the unconditional loss probabilities are given as:

$$Pr(L = \ell) = \int_{-\infty}^{\infty} Pr(L^c(x) = \ell)\, dPr(x) \tag{3.28}$$

where x is the value of systematic factor X.

The unconditional loss distributions for different cases are analyzed below.

Case 1: Homogeneous Portfolio

$$\begin{aligned} Pr\left\{L = k\frac{\overline{LGD}}{n}\right\} &= \int_{-\infty}^{\infty} Pr\left\{L(x)^c = k\frac{\overline{LGD}}{n}\right\} d\Phi(x) \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} (PD^c(x))^k (1 - PD^c(x))^{n-k} d\Phi(x) \end{aligned} \tag{3.29}$$

where:

$$PD^c(x) = \Phi\left(\frac{\Phi^{-1}(PD^{uc}) - \sqrt{w}x}{\sqrt{1-w}}\right)$$

Case 2: Large Homogeneous Portfolio

Since $L^c(x) = PD^c(x)\overline{LGD}$, we have:

$$\begin{aligned} F_L(\ell \times \overline{LGD}) &= Pr\left\{L \leq \ell \times \overline{LGD}\right\} \\ &= Pr\left\{PD^c(x) \times \overline{LGD} \leq \ell \times \overline{LGD}\right\} \\ &= Pr\left\{PD^c(x) \leq \ell\right\} \\ &= Pr\left\{x \geq (PD^c)^{-1}(\ell)\right\} = \Phi\left(-(PD^c)^{-1}(\ell)\right) \\ &= \Phi\left(\frac{\sqrt{1-w}\Phi^{-1}(\ell) - \Phi^{-1}(PD^{uc})}{\sqrt{w}}\right) \end{aligned} \tag{3.30}$$

The last formula is used in Basel-II. Therefore it is important to analyze its properties (we assume $\overline{LGD} = 1$). For a more extended discussion see [Vas87] and [Vas91].

1. $E[L] = PD^{uc}$

2. $Var\left[L\right] = \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w) - (PD^{uc})^2$

3. $L_\alpha := F_L^{-1}(\alpha; PD^{uc}, w) = F_L(\alpha; 1 - PD^{uc}, 1 - w)$

4. (Symmetry Property) $F_L(\ell; PD^{uc}, w) = 1 - F(1 - \ell; 1 - PD^{uc}, w)$

5. The density is $\begin{cases} \text{unimodal if } w \leq 0,5 \\ \text{monotone if } w = 0,5 \\ \text{U-shaped if } w \geq 0,5 \end{cases}$

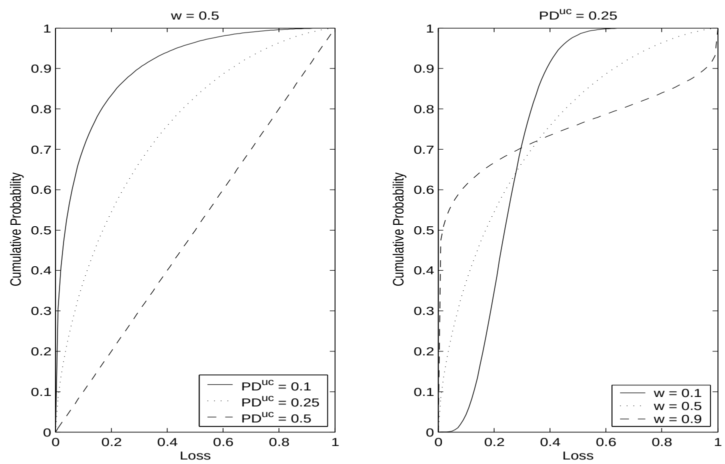

Figure 3.2: Cumulative Loss Distribution in Asymptotic Single Risk Factor Model

Case 3: Moderately Heterogeneous Portfolio

For each sub portfolio j, we have:

$$Pr\left\{L_j = k\frac{\overline{LGD_j}}{n_j}\right\} = \int_{-\infty}^{\infty} \begin{pmatrix} n_j \\ k \end{pmatrix} (PD_j^c(x))^k (1 - PD_j^c(x))^{n-k} d\Phi(x)$$

and

$$
\begin{aligned}
g(k_1, \cdots, k_h) &= Pr\left\{L_1 = k_1 \frac{\overline{LGD_1}}{n_1}, \cdots, L_h = k_h \frac{\overline{LGD_h}}{n_h}\right\} \\
&= \int_{-\infty}^{\infty} \prod_{j=1}^{h} \begin{pmatrix} n_j \\ k_j \end{pmatrix} (PD_j^c(x))^{k_j} (1 - PD_j^c(x))^{n_j - k_j} d\Phi(x)
\end{aligned}
$$

Case 4: Large Moderately Heterogeneous Portfolio

Since $L_j^c(x) = PD_j^c(x)\overline{LGD_j}$ and $L^c(x) = \sum_{j=1}^{h} PD_j^c(x)\overline{LGD_j}$, we have:

$$
\begin{aligned}
F_{L_j}(\ell \times \overline{LGD_j}) &= Pr\left\{L_j \leq \ell \times \overline{LGD_j}\right\} = Pr\left\{PD_j^c(x) \times \overline{LGD_j} \leq \ell \times \overline{LGD_j}\right\} \\
&= Pr\left\{PD_j^c(x) \leq \ell\right\} = Pr\left\{x \leq (PD_j^c)^{-1}(\ell)\right\} \\
&= \Phi((PD_j^c)^{-1}(\ell)) \\
&= \Phi\left(\frac{\sqrt{1 - w_j}\Phi^{-1}(\ell) - \Phi^{-1}(PD_j^{uc})}{\sqrt{w_j}}\right) \qquad (3.31)
\end{aligned}
$$

3.1.10 Risk Measures

In this section,we analyze the risk measures for different cases. We assume $\overline{LGD} = 1$

Case 1: Homogeneous Portfolio

We can find expected loss by using expected conditional loss.

$$
\mu_L = E\left[E\left[L^c(x)|X\right]\right] = E\left[PD^c(x)\right] = PD^{uc} \qquad (3.32)
$$

For any random variable, we can express its unconditional variance as a sum of two parts: variance of conditional mean and mean of conditional variance. Therefore:

$$
\sigma_L^2 = Var(PD^c(x)) + E\left[\frac{1}{n} PD^c(x)(1 - PD^c(x))\right]
$$

$$
\begin{aligned}
&= E\left[(PD^c(x) - E\left[PD^c(x)\right])^2\right] + \frac{1}{n}E\left[PD^c(x) - (PD^c(x))^2\right] \\
&= E\left[(PD^c(x))^2\right] - (E\left[PD^c(x)\right])^2 + \frac{1}{n}\left\{PD^{uc} - E\left[(PD^c(x))^2\right]\right\} \\
&= \int_{-\infty}^{\infty}\left[\Phi\left(\frac{\Phi^{-1}(PD^{uc}) - \sqrt{w}x}{\sqrt{1-w}}\right)\right]^2 d\Phi(x) \\
&\quad -(PD^{uc})^2 + \frac{1}{n}\left\{PD^{uc} - \int_{-\infty}^{\infty}\left[\Phi\left(\frac{\Phi^{-1}(PD^{uc}) - \sqrt{w}x}{\sqrt{1-w}}\right)\right]^2 d\Phi(x)\right\} \\
&= \left\{\Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w) - (PD^{uc})^2\right\} \\
&\quad + \frac{1}{n}\left\{PD^{uc} - \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w)\right\} \qquad (3.33)
\end{aligned}
$$

The first term in the formula represents the variance caused by systemic factor, i.e. systemic variance. And the second term is the idiosyncratic variance.

Since general conditional loss distribution is not monotone with respect to systemic factor, we can not derive analytical formulas for VaR_α.

Case 2: Large Homogeneous Portfolio

We have the same mean:

$$
\mu_L = E\left[E\left[L^c(x)|X\right]\right] = E\left[PD^c(x)\right] = PD^{uc} \qquad (3.34)
$$

Since, in LHP, we have $n \to \infty$, we can derive the formula by using the variance formula of homogeneous portfolio.

$$
\begin{aligned}
\sigma_L^2 &= \lim_{n\to\infty} \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w) - (PD^{uc})^2 \\
&\quad + \frac{1}{n}\left\{PD^{uc} - \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w)\right\} \\
&= \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w) - (PD^{uc})^2 \qquad (3.35)
\end{aligned}
$$

Therefore in the infinitely granular portfolio, the unsystematic variance is completely removed and we have only systematic variance. Since conditional loss is a monotonically decreasing function of X, we have:

$$\begin{aligned} VaR_\alpha &= F_L^{-1}(\alpha) = L^c(F_x^{-1}(1-\alpha)) \\ &= L^c(\Phi^{-1}(1-\alpha)) = PD^c(-\Phi^{-1}(\alpha)) \\ &= \Phi\left(\frac{\Phi^{-1}(PD^{uc}) + \sqrt{w}\Phi^{-1}(\alpha)}{\sqrt{1-w}}\right) \end{aligned} \tag{3.36}$$

This VaR figure is used in Basel-II to define the level of required capital for banks.

Case 3: Moderately Heterogeneous Portfolio

The expected loss is:

$$\mu_L = E\left[E\left[L^c(x)|X\right]\right] = E\left[\sum_{j=1}^{h} n_j PD_j^c(x)|X\right] = \sum_{j=1}^{h} n_j PD_j^{uc} \tag{3.37}$$

We can use the same variance decomposition that we use in previous section. Since, conditional on X, the sub portfolios are independent:

$$\sigma^2 = Var\left[\sum_{j=1}^{h} n_j PD_j^c(x)\right] + E\left[\sum_{j=1}^{h} \frac{1}{n_j} PD_j^c(x)(1 - PD_j^c(x))\right] \tag{3.38}$$

For VaR_α, we can not derive analytical formulas.

Case 4:Large Moderately Heterogeneous Portfolio

The expected loss is same as the expected loss of the finite portfolio.

$$\mu_L = E\left[E\left[L^c(x)|X\right]\right] = E\left[\sum_{j=1}^{h} n_j PD_J^c(x)|X\right] = \sum_{j=1}^{h} n_j PD_j^{uc} \tag{3.39}$$

And we can find the variance by limiting the finite portfolio variance:

$$\begin{aligned}\sigma^2 &= \lim_{\substack{n_j \to \infty \\ j=1,2,\cdots,h}} Var\left[\sum_{j=1}^{h} n_j PD_j^c(x)\right] + E\left[\sum_{j=1}^{h} \frac{1}{n_j} PD_j^c(x)(1-PD_j^c(x))\right] \\ &= Var\left[\sum_{j=1}^{h} n_j PD_j^c(x)\right] \qquad (3.40)\end{aligned}$$

Since conditional loss distribution is monotonically decreasing with respect to X, we have:

$$\begin{aligned}VaR_\alpha &= F_L^{-1}(\alpha) = L^c(F_x^{-1}(1-\alpha)) = L^c(\Phi^{-1}(1-\alpha)) \\ &= L^c(-\Phi^{-1}(\alpha)) = \sum_{j=1}^{h} \overline{LGD_j} PD_j^c(-\Phi^{-1}(\alpha)) \\ &= \sum_{j=1}^{h} \overline{LGD_j}\, \Phi\left(\frac{\Phi^{-1}(PD_j^{uc}) + \sqrt{w_j}\Phi^{-1}(\alpha)}{\sqrt{1-w_j}}\right) \qquad (3.41)\end{aligned}$$

3.1.11 Risk Decomposition

Because of diversification effects, the portfolio risk is generally less than the sum of risk of each loan (also called stand-alone risk) in the portfolio. Therefore we need a different metric (than stand-alone risk) for each firm's contribution to total risk. This process is called *risk decomposition* and the metric we obtain is called the *marginal risk*.

Assume that we have a portfolio of n assets, A_i $i = 1, 2, \cdots, n$ with weights w_i:

$$P = \sum_i w_i A_i$$

Then we can find the risk contribution of each asset by using the homogeneity property of our risk measure. We give the generic form for VaR below. Since all the risk measures we analyze (expected loss, standard deviation, VaR, ES) satisfy the positive homogeneity property defined in page 38, the following steps are also valid for other risk measures.

We have VaR for our portfolio:

$$VaR\,(P) = VaR\,(w_1 A_1, w_2 A_2, \cdots, w_n A_n)$$

From positive homogeneity, if we increase our investment by a factor of $\lambda > 0$, we have:

$$VaR\,(\lambda w_1 A_1, \lambda w_2 A_2, \cdots, \lambda w_n A_n) = \lambda VaR\,(w_1 A_1, w_2 A_2, \cdots, w_n A_n)$$

When we take derivatives of each side with respect to λ, we have:

$$\frac{\partial}{\partial \lambda} VaR\,(\lambda w_1 A_1, \lambda w_2 A_2, \cdots, \lambda w_n A_n) = VaR\,(w_1 A_1, w_2 A_2, \cdots, w_n A_n)$$

By using the chain rule of differentiation:

$$\begin{aligned}\frac{\partial}{\partial \lambda} VaR\,(\lambda w_1 A_1, \cdots, \lambda w_n A_n) &= \sum_i \frac{\partial}{\partial \lambda w_i} \frac{\partial \lambda w_i}{\partial \lambda} VaR\,(\lambda w_1 A_1, \cdots, \lambda w_n A_n) \\ &= \sum_i w_i \frac{\partial}{\partial \lambda w_i} VaR\,(\lambda w_1 A_1, \cdots, \lambda w_n A_n)\end{aligned}$$

Therefore:

$$\sum_i w_i \frac{\partial}{\partial \lambda w_i} VaR\,(\lambda w_1 A_1, \lambda w_2 A_2, \cdots, \lambda w_n A_n) = VaR\,(w_1 A_1, w_2 A_2, \cdots, w_n A_n)$$

Finally for $\lambda = 1$, we have:

$$\sum_i w_i \frac{\partial}{\partial w_i} VaR = VaR$$

The derivative of VaR with respect to asset allocation, i.e. $\frac{\partial VaR}{\partial w_i}$, is the marginal risk of i^{th} asset. An important property of marginal risks is that sum of all marginal risks weighted by position weights is always equal to total VaR.

Marginal risk statistics are important metrics since they show the contribution of each asset to the total portfolio risk and are used in economic capital allocation and risk-adjusted performance measurement.

3.1.12 Estimation

In order to fit a model to portfolio credit loss, we should estimate the model parameters from historical observations of portfolio losses. In general, we can use two methods: *maximum likelihood method (MLM)* and *moment matching method (MMM)*.

Maximum Likelihood Method

We can use this method if we know the density function of portfolio losses for a model. Let $f(\ell|\theta)$ be the density function for portfolio losses with a set of parameters. Then assume that we historically observe n values of portfolio losses. Additionally if the losses for each time point is iid, we can write their joint density as:

$$f(\ell_1, \ell_2, \cdots, \ell_n|\theta) = \prod_{i=1}^{n} f(\ell_i|\theta) \tag{3.42}$$

In MLM, we search for best values of θ that maximizes this joint density. Therefore, let L define the likelihood function as:

$$L(\theta|\ell) := \prod_{i=1}^{n} f(\ell_i|\theta)$$

and LL define the log-likelihood function as:

$$LL(\theta|\ell) := \sum_{i=1}^{n} \ln(f(\ell_i|\theta))$$

Then the best values that maximizes LL can be found by solving the following system:

$$\begin{aligned}\frac{\partial LL(\theta|\ell)}{\partial \theta} &= 0 \\ \frac{\partial^2 LL(\theta|\ell)}{\partial \theta^2} &\leq 0\end{aligned}$$

For example, for constant LGD, we know the loss distribution for a large homogeneous portfolio. Its distribution function is:

$$F_L(\ell|PD^{uc}, w) = \Phi\left(\frac{\sqrt{1-w}\Phi^{-1}(\ell) - \Phi^{-1}(PD^{uc})}{\sqrt{w}}\right)$$

and the density is:

$$f_L(\ell|PD^{uc}, w) = \varphi\left(\frac{\sqrt{1-w}\Phi^{-1}(\ell) - \Phi^{-1}(PD^{uc})}{\sqrt{w}}\right)$$

Then, for n observations of portfolio loss, we can write log-likelihood function:

$$LL(PD^{uc}, w|\ell) = \sum_{i=1}^{n} \ln \varphi\left(\frac{\sqrt{1-w}\Phi^{-1}(\ell_i) - \Phi^{-1}(PD^{uc})}{\sqrt{w}}\right) \tag{3.43}$$

The $\hat{PD}^{uc}$ and $\hat{w}$ that maximizes this two-parameter LL function are the maximum likelihood estimators for the model.

$$(\hat{PD}^{uc}, \hat{w}) = \arg\max LL(PD^{uc}, w|\ell) \tag{3.44}$$

Moment Matching Method

In this method, we match the moments of true model with sample moments. If we want to fit a model with k parameters to the data, we first determine the first k moments of the true model:

$$\mu_r^{'} := E\left[X^r\right], r = 1, 2, ..., k$$

Then we find the first k moments of the sample:

$$\mu_r^{'(Sample)} := \frac{1}{n}\sum_{i=1}^{n}(\ell_i)^r, r = 1, 2, ..., k$$

where n is the number of observations and $\{\ell_i\}_{i=1}^n$ are the historical observations for portfolio losses.

Then we solve the following system with k equations, for k parameters:

$$\mu_r^{'} = \mu_r^{'(Sample)}, r = 1, 2, ..., k \tag{3.45}$$

For example, for constant LGD, the mean and variance of portfolio loss for a large homogeneous portfolio are:

$$\begin{aligned} E\left[L\right] &= PD^{uc} \\ Var\left[L\right] &= \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w) - (PD^{uc})^2 \end{aligned}$$

Therefore the first two raw moments are:

$$\begin{aligned} \mu_1^{'} &= E\left[L\right] = PD^{uc} \\ \mu_2^{'} &= E\left[L^2\right] = Var\left[L\right] + (E\left[L\right])^2 = \Phi_2(\Phi^{-1}(PD^{uc}), \Phi^{-1}(PD^{uc}); w) \end{aligned}$$

Let assume that we have n observations for portfolio loss and calculate the sample moments:

$$\mu_1^{'(Sample)} = \frac{1}{n}\sum_{i=1}^{n}\ell_i \tag{3.46}$$

$$\mu_2^{'(Sample)} = \frac{1}{n}\sum_{i=1}^{n}\ell_i^2 \tag{3.47}$$

Then we can find the estimates $\hat{PD}^{uc}$ and $\hat{w}$ by solving the following two equations:

$$\hat{PD}^{uc} = \mu_1^{'(Sample)} \quad (3.48)$$

$$\Phi_2(\Phi^{-1}(\hat{PD}^{uc}), \Phi^{-1}(\hat{PD}^{uc}); \hat{w}) = \mu_2^{'(Sample)} \quad (3.49)$$

3.2 Multi-Factor Modeling

Although one factor modeling enables us parsimonious closed-form formulas, generally it has less explanatory power when compared to a multi-factor model. Representing all the systematic effects with a single random variable is a critical simplification. Therefore, just like in equity modeling, multi factor models are extremely used in credit risk modeling. The most popular examples of multi factor credit modeling are Credit Metrics and KMV models.

In multi factor models, asset values are driven by more than one systematic factors as well as a single systematic factor for each obligor. For a K-factor model, the asset value dynamics are:

$$V_t^i = V_0^i exp\left\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i\left(\sum_{k=1}^{K}\sqrt{w_{ik}}X_t^k + \sqrt{1-\sum_{k=1}^{K}w_{ik}}Z_t^i\right)\right\} \quad (3.50)$$

where $\left(X^1, \cdots X^K\right)$ and $(Z_i)_{i=1}^N$ are all independent and have standard normal distribution, and w_{ik} represents the factor loading of claim i to the k-th factor.

Then the unconditional PD is:

$$PD^{uc} = Pr\left\{V_t^i \leq DP_i\right\} = Pr\left\{ln\left(V_t^i\right) \leq ln\left(DP_i\right)\right\}$$

$$
\begin{aligned}
&= Pr\left\{ln\left(V_0^i\right) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t \right. \\
&\quad \left. +\sigma_i\left(\sum_{k=1}^{K}\sqrt{w_{ik}}X_t^k + \sqrt{1-\sum_{k=1}^{K}w_{ik}}Z_t^i\right) \leq ln\left(DP_i\right)\right\} \\
&= Pr\left\{\left(\sum_{k=1}^{K}\sqrt{w_{ik}}X_t^k + \sqrt{1-\sum_{k=1}^{K}w_{ik}}Z_t^i\right) \leq \frac{ln\left(\frac{DP_i}{V_0^i}\right) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i}\right\} \\
&= \Phi\left[\frac{ln\left(DP_i/V_0^i\right) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i}\right] \qquad (3.51)
\end{aligned}
$$

Given the realization of systematic factors, $\left(X^1, \cdots X^K\right) = \left(x^1, \cdots x^K\right)$, the conditional PD of the obligor is:

$$
\begin{aligned}
PD^c\left(x^1, \cdots x^K\right) &= Pr\left\{V_t^i \leq DP_i \middle| \left(X^1, \cdots X^K\right) = \left(x^1, \cdots x^K\right)\right\} \\
&= Pr\left\{ln\left(V_t^i\right) \leq ln\left(DP_i\right) \middle| \left(X^1, \cdots X^K\right) = \left(x^1, \cdots x^K\right)\right\} \\
&= Pr\left\{ln\left(V_0^i\right) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t \right. \\
&\quad \left. +\sigma_i\left(\sum_{k=1}^{K}\sqrt{w_{ik}}x_t^k + \sqrt{1-\sum_{k=1}^{K}w_{ik}}Z_t^i\right) \leq ln\left(DP_i\right)\right\} \\
&= Pr\left\{Z_t^i \leq \frac{ln\left(\frac{DP_i}{V_0^i}\right) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t - \sigma_i\left(\sum_{k=1}^{K}\sqrt{w_{ik}}x_t^k\right)}{\sigma_i\sqrt{1-\sum_{k=1}^{K}w_{ik}}}\right\} \\
&= \Phi\left[\frac{\Phi^{-1}\left(PD^{uc}\right) - \left(\sum_{k=1}^{K}\sqrt{w_{ik}}x_t^k\right)}{\sqrt{1-\sum_{k=1}^{K}w_{ik}}}\right] \qquad (3.52)
\end{aligned}
$$

The conditional and unconditional loss distributions can be found by using similar techniques with one factor modeling.

Chapter 4

PRICING CREDIT RISK

In this chapter, we will analyze pricing approaches and derive pricing formulas for different contingent claims. The instruments analyzed include bonds, stocks and credit default swaps.

4.1 Pricing Methods

For pricing credit-related contingent claims, we use two different approaches: *equivalent martingale measure (EMM) approach* and *partial differential equations (PDE) approach*. The connection between two approaches is given by Feynman Kac formula.

4.1.1 Equivalent Martingale Measure Approach

In EMM approach, we define a new probability measure, for which discounted asset values are martingales. And then we price the contingent claims using conditional expectations under this new probability measure.

Let the asset value of the firm satisfies:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t,$$

and the bank account (or discounting process), B_t, satisfies:

$$dB_t = -rB_t d_t$$

where r is the constant interest rate.

Under actual probability measure, P, the discounted asset value process, defined as $\tilde{V_t} := V_t D_t$ satisfies:

$$\begin{aligned} d\tilde{V_t} &= B_t dV_t + V_t dB_t + dB_t dV_t \\ &= B_t(\mu V_t d_t + \sigma V_t dW_t) - r_t B_t V_t d_t + 0 \\ &= \tilde{V_t}((\mu - r_t)d_t + \sigma dW_t) \end{aligned} \tag{4.1}$$

which is not a martingale.

Now, define a new probability measure, $\tilde{P}$, called the EMM such as:

$$\tilde{P}(A) := \int_A Z_T(\omega) dP(\omega),\ \forall A \in F \tag{4.2}$$

where Z is the Radon-Nikodym derivative defined as:

$$Z_t = exp\left\{ -\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 d_s \right\} \tag{4.3}$$

with $\theta_t = \frac{\mu - r}{\sigma}$. Then from Girsanov Theorem, we have (for proofs see [Shr04]):

1. $\tilde{W_t} = W_t + \int_0^t \theta_s ds$ is a standard Brownian motion under $\tilde{P}$
2. $\tilde{E}[X] = E[XZ_T]$ where the expectations are taken with respect to $\tilde{P}$ and P, respectively. And equivalently, $E[X] = \tilde{E}[X/Z_T]$.
3. The Radon-Nikodym derivate process is a martingale, i.e.

$$E[Z_t|F_s] = Z_s$$

4. For $0 \leq s \leq t, Y \in F_t$, we have

$$\tilde{E}\left[Y|F_s\right] = \frac{1}{Z_s} E\left[Y Z_t|F_s\right]$$

Additionally, under $\tilde{P}$, the discounted asset values are martingale:

$$\begin{aligned}
\frac{d\tilde{V}_t}{\tilde{V}_t} &= (\mu - r)d_t + \sigma dW_t \\
&= (\mu - r)d_t + \sigma(d\tilde{W}_t - \frac{\mu - r}{\sigma} d_t) \\
&= \sigma d\tilde{W}_t
\end{aligned}$$

Since we have one source of randomness, $\tilde{W}_t$, and one asset other than bank account, the model is complete (See [Bjo04]).

Therefore, we can use $\tilde{P}$ to price contingent claims written on V_t. For example, let C_t denote the price of a contingent claim written on V_t and have maturity T. Since the market is complete and arbitrage-free, there is a unique self-financing replicating portfolio, π, whose value is always equal to C_t.

Since, under $\tilde{P}$, discounted C_t is martingale:

$$B_t C_t = \tilde{E}\left[B_T C_T|F_t\right] = \tilde{E}\left[B_T \pi_T|F_t\right] = B_t \pi_t$$

Therefore:

$$C_t = \frac{1}{B_t} \tilde{E}\left[B_T \pi_T|F_t\right] \tag{4.4}$$

4.1.2 Partial Differential Equation Approach

The second approach we can use is the partial differential equation approach. In this approach we form a hedged, i.e. riskless, portfolio and equate its return to risk-free yield.

Assume that the asset value of the firm satisfies:

$$\frac{dV_t}{V_t} = \mu dt + \sigma W_t$$

And assume we have a contingent claim, Y_t, written on V_t. This claim can be a bond, a common stock, an option on common stock or a credit derivative.

Then we can apply Ito Lemma to $Y_t = f(V_t, t)$

$$\begin{aligned} dY &= f_V dV + \frac{1}{2} f_{VV} dV dV + f_t dt \\ &= f_V(\mu V dt + \sigma V dW) + \frac{1}{2} f_{VV} \sigma^2 V^2 dt + f_t dt \\ &= \underbrace{\left[\mu f_V V + \frac{1}{2} f_{VV} \sigma^2 V^2 + f_t\right]}_{:=\mu_y Y} dt + \underbrace{\sigma f_V V}_{:=\sigma_y Y} dW \\ &= \mu_y Y dt + \sigma_y Y dW_t \end{aligned}$$

Assume that we form a portfolio from firm's assets, V_t, contingent claim, Y_t, and riskless debt. The weights are $w_1, w_2, w_3 = -w_1 - w_2$, respectively. Then the return on portfolio, π satisfies:

$$\begin{aligned} d\pi_t &= w_1 \frac{dV_t}{V_t} + w_2 \frac{dY_t}{Y_t} + w_3 r dt \\ &= w_1(\mu dt + \sigma dW_t) + w_2(\mu_y dt + \sigma_y dWt) + (-w_1 - w_2) r dt \\ &= [w_1(\mu - r) + w_2(\mu_y - r)]\, dt + [w_1 \sigma + w_2 \sigma_y]\, dW_t \end{aligned}$$

Suppose that we apply a trading strategy such that $(w_1\sigma + w_2\sigma_y)$ is always zero. Since the market is complete, we can find such a strategy. Then our portfolio becomes riskless since there is no uncertainty caused by W_t. And since our portfolio requires zero net investment, i.e. $w_1 + w_2 + w_3 = 0$, the return on portfolio must be zero to avoid arbitrage.

Therefore we have two equations.The first one guarantees a riskless portfolio:

$$w_1\sigma + w_2\sigma_y = 0$$

and the second one guarantees no arbitrage condition:

$$w_1(\mu - r) + w_2(\mu_y - r) = 0$$

Therefore we have:

$$\frac{w_1}{w_2} = \frac{-\sigma_y}{\sigma} = \frac{-(\mu_y - r)}{\mu - r} \Rightarrow \frac{\mu_y - r}{\sigma_y} = \frac{\mu - r}{\sigma}$$

When we substitute the equations for μ_y and σ_y:

$$\begin{aligned}
\frac{\mu f_V V + \frac{1}{2} f_{VV}\sigma^2 V^2 + f_t - rf}{\sigma f_V V} &= \frac{\mu - r}{\sigma} \\
\Rightarrow \mu f_V V + \frac{1}{2} f_{VV}\sigma^2 V^2 + f_t - rf &= \mu f_V V - r f_V V \\
\Rightarrow \frac{1}{2} f_{VV}\sigma^2 V^2 + r f_V V - rf + f_t &= 0 \qquad (4.5)
\end{aligned}$$

This equation is the fundamental PDE for pricing contingent claims. Any contingent claim should satisfy this equation. For each contingent claim, we have boundary conditions and initial values. The boundary conditions distinguish contingent claims from each other.

4.1.3 On the Equivalence of Two Methods

The link between PDE and EMM approaches can be developed by using Feynmann Kac Theorem. The discounted version of the theorem is given below.

(Discounted Feynmann-Kac Theorem) Consider the following stochastic differential equation.

$$dX_t = \alpha(t, X_t)dt + \gamma(t, X_t)dW_t$$

Let h(y) be a Borel-measurable function and let r be constant. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the new function:

$$f(t, x) = E^{t,x}\left[e^{-r(T-t)}h(X_T)\right]$$

where $E^{t,x}\left[h(X_T)\right] < \infty$ for all t and x. Then $f(t, x)$ satisfies the following partial differential equation:

$$f_t(t, x) + \alpha(t, x)f_x(t, x) + \frac{1}{2}\gamma^2(t, x)f_{xx}(t, x) = rf(t, x)$$

and the terminal condition is $f(T, x) = h(x)$ for all x. For proof see [LL96].

We can apply the theorem to our contingent claim pricing problem. In previous sections, we show that the price of contingent claim satisfies:

$$B_t C_t = \tilde{E}\left[B_T C_T | F_t\right] \Rightarrow e^{-rt}C_t = \tilde{E}\left[e^{-rT}C_T | F_t\right]$$

Let $C_t := f(t, V_t)$ where V_t is the asset value process.

Note that $e^{-rt}f(t, V_t)$ is a martingale:

$$\begin{aligned}\tilde{E}\left[e^{-rt}f(t, V_t)|F_s\right] &= \tilde{E}\left[\tilde{E}\left[e^{-rT}C_T|F_t\right]|F_s\right] \\ &= \tilde{E}\left[e^{-rT}C_T|F_s\right] \\ &= e^{-rs}f(s, V_s)\end{aligned}$$

Since it is a martingale, we should not have a drift term in its differential. We have, under $\tilde{P}$:

$$dV_t = rV_t dt + \sigma V_t d\tilde{W}_t$$

Then:

$$d(e^{-rt}f(t, V_t)) = -re^{-rt}f dt + e^{-rt}f_t dt + e^{-rt}f_V dV + e^{-rt}\frac{1}{2}f_{VV}dVdV$$

$$= e^{-rt}\underbrace{\left[-rf + f_t + rVf_V + \frac{1}{2}\sigma^2V^2f_{VV}\right]}_{\text{must be zero}}dt + e^{-rt}\sigma f_V dW$$

$$\Rightarrow 0 = \frac{1}{2}\sigma^2V^2f_{VV} + rVf_V - rf + f_t$$

This is exactly the same equation we found in PDE approach.

4.2 Risk-neutral Probability of Default and Market Price of Risk

In pricing applications we use risk-neutral PDs, rather than objective PDs. The risk-neutral PDs includes risk premiums over actual PDs. Let assume that, by using Girsanov Theorem, we define the risk-neutral probability measure $\tilde{P}$ with a new standard Brownian motion:

$$\begin{aligned} \tilde{W}_t &= W_t + \frac{\mu - r}{\sigma}t \\ d\tilde{W}_t &= dW_t + \frac{\mu - r}{\sigma}dt \end{aligned}$$

Then under $\tilde{P}$, the asset value process is:

$$\begin{aligned} \frac{dV_t}{V_t} &= \mu d_t + \sigma dW_t \\ &= \mu d_t + \sigma\left(d\tilde{W}_t - \frac{\mu - r}{\sigma}dt\right) \\ &= rd_t + \sigma d\tilde{W}_t \end{aligned}$$

Therefore at any time t, we can express the asset value as:

$$V_t = V_0 e^{\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{W}_t\right\}} \tag{4.6}$$

We can express the risk-neutral PD, $\tilde{PD}$, by using a similar approach with objective PD:

$$\begin{aligned}
\tilde{PD} = Pr\left\{V_T \leq DP\right\} &= P\left\{\ln\left(V_t\right) \leq \ln\left(DP\right)\right\} \\
&= Pr\left\{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\tilde{W}_T \leq \ln\left(DP/V_0\right)\right\} \\
&= Pr\left\{\tilde{W}_T \leq \frac{\ln\left(DP/V_0\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma}\right\}
\end{aligned}$$

Since $\tilde{W}_T \sim N\left(0, T\right)$, we have:

$$\tilde{PD} = \Phi\left[\frac{\ln\left(DP/V_0\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right] \tag{4.7}$$

The formula for $\tilde{PD}$ is similar to the formula for PD, except that the μ is replaced with r. More explicitly we have:

$$\tilde{PD} = \Phi\left[\Phi^{-1}\left(PD\right) + \frac{\mu - r}{\sigma\sqrt{T}}\right] \tag{4.8}$$

We can define explicit relations between actual and risk-neutral PDs, if we use an explicit pricing model for asset returns. For example assume a perfect CAPM market, where the expected return on any asset is given by:

$$\mu_i = r + \beta_i\left(\mu_M - r\right)$$

where $\beta_i = \rho_{iM}\frac{\sigma_i}{\sigma_M}$. Then we have:

$$\frac{\mu_i - r}{\sigma_i} = \rho_{iM}\frac{\mu_M - r}{\sigma_M} = \rho_{iM}\lambda$$

where $\lambda = \frac{\mu_M - r}{\sigma_M}$ is the *market price of risk*. Now we can express risk-neutral PD in terms of objective PD:

$$\tilde{PD} = \Phi\left[\Phi^{-1}\left(PD\right) + \lambda\,\rho_{iM}\,\sqrt{T}\right] \tag{4.9}$$

As seen from the formula, the objective probability of default is adjusted upwards to include a *risk premium*. The market price of risk is determined by the entire market and represents the reward per unit of market risk taken (i.e., an overall market Sharpe ratio). We use a one-factor model, i.e. CAPM, for asset returns. However different factor models yields different explicit linkages between PD and $\tilde{PD}$ [Bohn00].

4.3 Contingent Claims Pricing

In this section we derive pricing formulas for bonds and stocks using Gaussian assumption. We use EMM approach to price these claims.

4.3.1 Bonds

We derive the formulas for zero-coupon bonds. Coupon bonds can be thought as a portfolio of zero coupon bonds with different maturities.

The payoff for a zero coupon bond is:

$$\text{Payoff} = \begin{cases} L & \text{if no default} \\ V_T & \text{if default} \end{cases}$$

where L is the par value of bond and V_T is the firm's asset value at maturity T.

Then, under EMM, the discounted bond value is a martingale. Therefore:

$$B(t)D(t,T) = \tilde{E}\left[B(T)\left(L1_{\{\text{No Default}\}} + V_T 1_{\{\text{Default}\}}\right)|F_t\right]$$

where D(t,T) is the zero coupon bond price at t, which have maturity T and $B(t) = e^{-\int_0^t r_s ds}$ is the bank account, where r_t is the risk-free rate.

We derive pricing formulas for different cases below. See [BR02] for a more detailed discussion.

Case 1: Constant Interest Rate and Default Only at Maturity

Assume that r is constant. Then we can take B(T) outside the expectation

$$D(t,T) = \frac{B(T)}{B(t)} \tilde{E}\left[L 1_{\{\text{No Default}\}} + V_T 1_{\{\text{Default}\}} | F_t\right]$$

We know that, under EMM, V_t satisfies.

$$V_t = V_0 exp\left\{(r - \frac{1}{2}\sigma^2)t + \sigma \tilde{W}_t\right\}$$

and default event is given by:

$$1_{\{\text{Default}\}} = 1 - 1_{\{\text{No Default}\}} = \begin{cases} 1 & \text{if } V_T < L \\ 0 & \text{if } V_T \geq L \end{cases}$$

Then:

$$\begin{aligned} D(t,T) &= \frac{B(T)}{B(t)} \tilde{E}\left[L 1_{\{V_T \geq L\}} + V_T 1_{\{V_T < L\}} | F_t\right] \\ &= \frac{B(T)}{B(t)} \left\{L\, \tilde{P}\left\{V_T \geq L | F_t\right\} + \tilde{E}\left[V_T 1_{\{V_T < L\}} | F_t\right]\right\} \end{aligned}$$

For the first part:

$$\begin{aligned} \tilde{P}\left\{V_T \geq L | F_t\right\} &= \tilde{P}\left\{V_t\, exp\left\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)\right\} \geq L | F_t\right\} \\ &= \tilde{P}\left\{\tilde{W}_T - \tilde{W}_t \leq \frac{\ln(V_t/L) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma}\right\} \end{aligned}$$

Since, under $\tilde{P}$, $(\tilde{W}_T - \tilde{W}_t) \sim N(0, T-t)$, we have:

$$\begin{aligned} \tilde{P}\left\{V_T \geq L | F_t\right\} &= \Phi\left[\frac{\ln(V_t/L) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right] \\ &=: \Phi(d_2) \\ \text{with } d_2 &:= \frac{\ln(V_t/L) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

For the second part, define an auxiliary probability measure P^* with Radon-Nikodym density,

$$\frac{dP*}{d\tilde{P}} = exp\left\{\sigma\tilde{W}_T - \frac{1}{2}\sigma^2 T\right\} =: \eta_T$$

From Girsanov Theorem, $W_t^* = \tilde{W}_t - \sigma t$ is a standard Brownian motion under P^*. Therefore, under P^*, V_t satisfies:

$$\begin{aligned} dV_t &= V_t(rdt + \sigma d\tilde{W}_t) \\ &= V_t((r+\sigma)dt + \sigma dW_t^*) \\ \Rightarrow V_t &= V_0\, exp\left\{(r + \frac{1}{2}\sigma^2)t + \sigma W_t^*\right\} \end{aligned}$$

Therefore:

$$\begin{aligned} \tilde{E}\left[V_T 1_{[V_T<L]}|F_t\right] &= \tilde{E}\left[\underbrace{V_0\, e^{rT}}_{\text{constant}}\underbrace{e^{-\frac{1}{2}\sigma^2 T + \sigma\tilde{W}_T}}_{=\eta_T} 1_{\{V_T<L\}}|F_t\right] \\ &= V_0\, e^{rT}\tilde{E}\left[\eta_T 1_{\{V_T<L\}}|F_t\right] \\ &= V_0\, e^{rT}\eta_t E^*\left[1_{\{V_T<L\}}|F_t\right] \\ &= V_0\, e^{rT}\eta_t P^*\left\{V_T < L|F_t\right] \\ &= e^{r(T-t)}\underbrace{V_0\, exp\left\{(r - \frac{1}{2}\sigma^2)t + \sigma\tilde{W}_t\right\}}_{V_t} P^*\left\{V_T < L|F_t\right\} \\ &= e^{r(T-t)}V_t\, P^*\left\{V_T < L|F_t\right\} \end{aligned}$$

We can express the last probability as follows.

$$\begin{aligned} P^*\left\{V_T < L|F_t\right\} &= P^*\left\{V_t\, exp\left\{(r + \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^* - W_t^*)\right\} < L|F_t\right\} \\ &= P^*\left\{W_T^* - W_t^* < -\frac{\ln(V_t/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma}|F_t\right\} \end{aligned}$$

Since, under P^*, $(W_T^* - W_t^*) \sim N(0, T-t)$, we have:

$$\begin{aligned} P^* \{V_T < L | F_t\} &= \Phi\left[-\frac{\ln(V_t/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right] \\ &=: \Phi(-d_1) \\ \text{with } d_1 &:= \frac{\ln(V_t/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

To sum up, the price is given by:

$$D(t,T) = V_t\Phi(-d_1) + LB(t,T)\Phi(d_2) \tag{4.10}$$

where

$$d_1 = \frac{\ln(V_t/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(V_t/L) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and B(t,T) is the price of risk-free bond, defined as B(t,T):=$\frac{B(T)}{B(t)}$

In general, instead of price of a risky bond, we use its yield spread over risk-free yield to define its riskiness. The credit spread is defined as follows. For $B(t,T) = e^{-r(T-t)}$ where r is the risk-free yield and $D(t,T) = e^{-y(T-t)}$ where y is the risky yield,

$$S(t,T) := y - r = -\frac{\ln(D(t,T)/B(t,T))}{T-t}$$

Therefore, in our case:

$$S(t,T) = -\frac{\ln(D(t,T)/LB(t,T))}{T-t} = -\frac{\ln\left(\frac{V_t}{L}\Phi(-d_1) + \Phi(d_2)\right)}{T-t} \tag{4.11}$$

Sensitivity of credit spreads with respect to different parameters are shown in Figure 4.1.

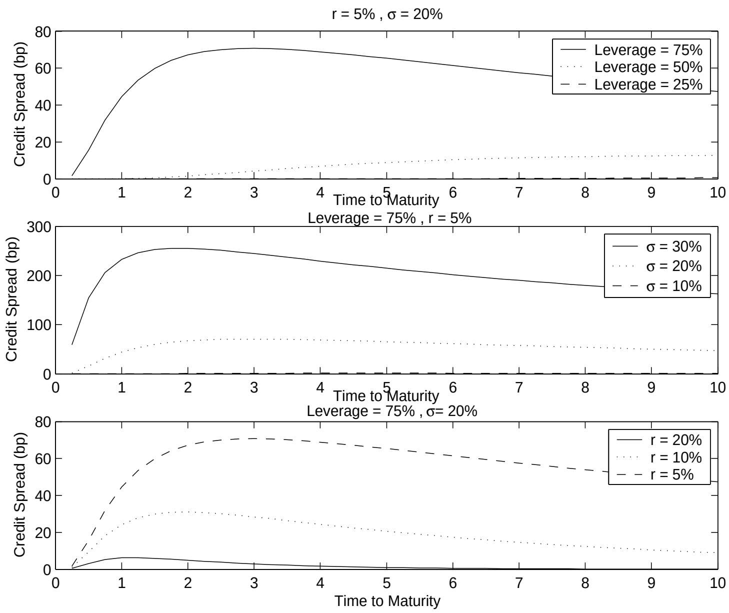

Figure 4.1: Credit Spreads

Case 2: Constant Interest Rate and First-Passage Approach

Assume that the default time is defined as:

$$\tau := inf\left\{t \geq 0 : V_t \leq L\right\}$$

Additionally, if default occurs before T, the bond holder receives a constant fraction of L, δ. Then the price is:

$$\begin{aligned} D^{FPT}(t,T) &= B(t,T)\tilde{E}\left[L1_{\{\tau>T\}} + L\delta 1_{\{\tau\leq T\}}|F_t\right] \\ &= B(t,T)L\left[\tilde{P}\left\{\tau > T|F_t\right\} + \delta\tilde{P}\left\{\tau \leq T|F_t\right\}\right] \\ &= B(t,T)L\left[1-(1-\delta)\tilde{P}\left\{\tau \leq T|F_t\right\}\right] \end{aligned}$$

From section 1.6, we know the formula for $P\left\{\tau \leq T|F_t\right\}$. But now we are working under $\tilde{P}$. Therefore let define $Y_t := \ln(V_t/L)$ where $dV_t = rV_tdt+\sigma V_t d\tilde{W}_t$

under $\tilde{P}$. Then:

$$\begin{aligned} dY_t &= \frac{1}{V_t}dV_t - \frac{1}{2}\frac{1}{V_t^2}dV_t dV_t = (rdt + \sigma d\tilde{W}_t) - \frac{1}{2}\sigma^2 dt \\ &= (r - \frac{1}{2}\sigma^2)dt + \sigma d\tilde{W}_t \\ \Rightarrow\ Y_t &= \ln(V_0/L)exp\left\{(r - \frac{1}{2}\sigma^2)t + \sigma\tilde{W}_t\right\} \\ \Rightarrow\ \tilde{P}\{\tau \le T|F_t\} &= \tilde{P}\left\{\inf_{t\le s\le T} Y_s \le 0|F_t\right\} \end{aligned}$$

Using the same Lemma of section 1.6, we have:

$$\begin{aligned} \tilde{P}\{\tau \le T|F_t\} &= \Phi\left(\frac{\ln(L/V_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad + \left(\frac{L}{V_t}\right)^{2\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}} \Phi\left(\frac{\ln(L/V_t) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

Therefore, the pricing formula is:

$$\begin{aligned} D^{FPT}(t,T) &= B(t,T)L\left[1 - (1-\delta)\left\{\Phi\left(\frac{\ln(L/V_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right.\right. \\ &\quad \left.\left. + \left(\frac{L}{V_t}\right)^{2\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}} \Phi\left(\frac{\ln(L/V_t) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right\}\right] \end{aligned} \tag{4.12}$$

And the spread is:

$$S(t,T) = -\frac{\ln(D^{FPT}(t,T)/LB(t,T))}{T-t} \tag{4.13}$$

Case 3: Stochastic Interest Rates

In the final case we analyze bond pricing with stochastic interest rates. For this we will use the technique called *change of numeraire* and define two new probability measures. Let P, $\tilde{P}$,$\tilde{P}^T$ and $\tilde{P}^V$ denote objective probability measure, risk-neutral probability measure, and newly defined *T-forward* and *V-forward* probability measures respectively. The new measures are defined as:

$$\begin{aligned}\tilde{P}^T(A) &:= \int_A Z_T^T(\omega)dP(\omega),\ \forall A \in F \\ \tilde{P}^V(A) &:= \int_A Z_T^V(\omega)dP(\omega),\ \forall A \in F\end{aligned}$$

where:

$$\begin{aligned}Z^T(t) &:= \frac{D(t)\,B(t,T)}{B(0,T)} \\ Z^V(t) &:= \frac{D(t)\,V(t)}{V(0)}\end{aligned}$$

Under $\tilde{P}^T$, $\frac{V(t)}{B(t,T)}$ is martingale. Similarly under $\tilde{P}^V$, $\frac{B(t,T)}{V(t)}$ is martingale (For proofs see [Shr04]). And we have the following relations:

$$\begin{aligned}Z^T(0) &= \frac{D(0)\,B(0,T)}{B(0,T)} = 1 \\ Z^T(T) &= \frac{D(T)\,B(T,T)}{B(0,T)} = \frac{D(T)}{B(0,T)} \\ Z^V(0) &= \frac{D(0)\,V(0)}{V(0)} = 1 \\ Z^V(T) &= \frac{D(T)\,V(T)}{V(0)} \\ \tilde{E}^T[X|F_t] &= \frac{1}{Z^T(t)}\tilde{E}\left[XZ^T(T)\,|F_t\right] \\ \tilde{E}^V[X|F_t] &= \frac{1}{Z^V(t)}\tilde{E}\left[XZ^V(T)\,|F_t\right]\end{aligned}$$

We know that, under risk-neutral measure $\tilde{P}$, discounted prices for both firm's asset value and the risk-free zero coupon bond are martingale:

$$\begin{aligned}
\frac{dV(t)}{V(t)} &= rdt + \sigma d\tilde{W}_t \\
\frac{dD(t)V(t)}{D(t)V(t)} &= \sigma d\tilde{W}_t \\
\frac{dB(t,T)}{B(t,T)} &= rdt + \sigma_B d\tilde{W}_t \\
\frac{dD(t)B(t,T)}{D(t)B(t,T)} &= \sigma_B d\tilde{W}_t
\end{aligned}$$

Additionally, by using martingale property, we have, for $\bar{\sigma} = \|\sigma - \sigma_B\| = \sqrt{|\sigma^2 - \sigma_B^2|}$,

$$\begin{aligned}
d\frac{V(t)}{B(t,T)} &= \frac{V(t)}{B(t,T)}\bar{\sigma}d\tilde{W}_t^T \\
\Rightarrow \frac{V(T)}{B(T,T)} &= \frac{V(0)}{B(0,T)}exp\left\{-\frac{1}{2}\bar{\sigma}^2 T + \bar{\sigma}\tilde{W}_T^T\right\} \\
d\frac{B(t,T)}{V(t)} &= \frac{B(t,T)}{V(t)}\bar{\sigma}d\tilde{W}_t^V \\
\Rightarrow \frac{B(T,T)}{V(T)} &= \frac{B(0,T)}{V(0)}exp\left\{-\frac{1}{2}\bar{\sigma}^2 T + \bar{\sigma}\tilde{W}_T^V\right\}
\end{aligned}$$

where $\tilde{W}^T$ and $\tilde{W}^V$ are standard Brownian motions under $\tilde{P}^T$ and $\tilde{P}^V$ respectively (see [Shr04]).

After defining these new probability measures, we can know start formulating our bond pricing formula. We know that the bond price is:

$$\begin{aligned}
D(t,T) &= \frac{1}{D(t)}\tilde{E}\left[D(T)\left\{L1_{\{V_T \geq L\}} + V(T)1_{\{V_T < L\}}\right\}|F_t\right] \\
&= \frac{1}{D(t)}\tilde{E}\left[D(T)L1_{\{V_T \geq L\}}|F_t\right] + \frac{1}{D(t)}\tilde{E}\left[D(T)V(T)1_{\{V_T < L\}}|F_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{B(t,T)}{D(t)}\tilde{E}\left[\frac{D(T)}{B(t,T)}L1_{\{V_T\geq L\}}|F_t\right] + \frac{V(0)}{D(t)}\tilde{E}\left[\frac{D(T)V(T)}{V(0)}1_{\{V_T<L\}}|F_t\right] \\
&= \frac{B(t,T)}{D(t)}L\frac{D(t)B(t,T)}{B(0,T)}\tilde{E}^T\left[1_{\{V_T\geq L\}}|F_t\right] \\
&\quad +\frac{V(0)}{D(t)}\frac{D(t)V(t)}{V(0)}\tilde{E}^V\left[1_{\{V_T<L\}}|F_t\right] \\
&= LB(t,T)\tilde{P}^T\{V_T\geq L|F_t\} + V(t)\tilde{P}^V\{V_T<L|F_t\} \\
&= LB(t,T)\tilde{P}^T\left\{\frac{V_T}{B(T,T)}\geq L|F_t\right\} + V(t)\tilde{P}^V\left\{\frac{V_T}{B(T,T)}>\frac{1}{L}|F_t\right\}
\end{aligned}
$$

We can easily find the probabilities:

$$
\begin{aligned}
\tilde{P}^T\left\{\frac{V_T}{B(T,T)}\geq L|F_t\right\} &= \tilde{P}^T\left\{ln\left(\frac{V(t)}{B(t,T)}\right)-\frac{\bar{\sigma}^2\tau}{2}\right. \\
&\quad \left.+\bar{\sigma}\left(\tilde{W}_T^T-\tilde{W}_t^T\right)\leq ln(L)|F_t\right\} \\
&= \tilde{P}^T\left\{\left(\tilde{W}_T^T-\tilde{W}_t^T\right)\geq\frac{ln\left(\frac{V(t)}{LB(t,T)}\right)-\frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}}|F_t\right\} \\
&= \Phi\left[\frac{ln\left(\frac{V(t)}{LB(t,T)}\right)-\frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}\sqrt{\tau}}\right] \\
&=: \Phi(d_1)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{P}^V\left\{\frac{B(T,T)}{V_T}>\frac{1}{L}|F_t\right\} &= \tilde{P}^V\left\{ln\left(\frac{B(t,T)}{V(t)}\right)-\frac{\bar{\sigma}^2\tau}{2}\right. \\
&\quad \left.+\bar{\sigma}\left(\tilde{W}_T^V-\tilde{W}_t^V\right)>ln\left(\frac{1}{L}\right)|F_t\right\} \\
&= \tilde{P}^V\left\{\left(\tilde{W}_T^V-\tilde{W}_t^V\right)<\frac{ln\left(\frac{LB(t,T)}{V(t)}\right)-\frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}}|F_t\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \tilde{P}^V \left\{ \left(\tilde{W}_T^V - \tilde{W}_t^V \right) < \frac{ln\left(\frac{V(t)}{LB(t,T)}\right) + \frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}} | F_t \right\} \\
&= \Phi \left[\frac{ln\left(\frac{V(t)}{LB(t,T)}\right) + \frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}\sqrt{\tau}} \right] \\
&=: \Phi\left(d_2\right)
\end{aligned}
$$

Therefore the price is:

$$
D\left(t,T\right) = LB\left(t,T\right)\Phi\left(d_1\right) + V\left(t\right)\Phi\left(d_2\right) \tag{4.14}
$$

where:

$$
\begin{aligned}
d_1 &= \frac{ln\left(\frac{V(t)}{LB(t,T)}\right) - \frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}\sqrt{\tau}} \\
d_2 &= \frac{ln\left(\frac{V(t)}{LB(t,T)}\right) + \frac{1}{2}\bar{\sigma}^2\tau}{\bar{\sigma}\sqrt{\tau}}
\end{aligned}
$$

4.3.2 Stocks

In structural modeling, the common stocks are treated as call options on firms assets, since their payoff is:

$$
\text{Payoff} = \begin{cases} V_T - L & \text{if no default} \\ 0 & \text{if default} \end{cases}
$$

Therefore, the corresponding pricing formula is:

$$
\begin{aligned}
E_t &= B\left(t,T\right)\tilde{E}\left[\left(V_T - L\right)\times 1_{\{\text{No Default}\}} + 0 \times 1_{\{\text{Default}\}} | F_t\right] \\
&= B\left(t,T\right)\tilde{E}\left[V_T \times 1_{\{\text{No Default}\}} | F_t\right] - B\left(t,T\right)L \times \tilde{P}\left\{\text{No Default}\right\}
\end{aligned}
$$

To derive the pricing formula, we can use similar techniques that we used for bonds. But there is also an easy way. Since at any time t, the firm's assets are equal to its liabilities, and if we assume that the firm's liabilities are one zero coupon bond and a common stock, we can derive the price of stock from the following (put-call parity) equation:

$$V_t = D(t,T) + E_t \Rightarrow E_t = V_t - D(t,T)$$

By using the formulas for $D(t,T)$ we obtained in previous section, we can find E_t.

$$E_t = V_t\Phi(d_1) + LB(t,T)\Phi(d_2) \tag{4.15}$$

where

$$d_1 = \frac{\ln(V_t/L) + (r + \frac{1}{2}\sigma^2)(T_t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(V_t/L) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and B(t,T) is the price of risk-free bond.

4.3.3 Credit Default Swaps

Credit default swaps (CDS) are the simplest type of credit derivatives, but they are also the basic building block for more complex credit derivatives. In order to price a CDS, we use a reverse engineering and decompose the cash flows of it. Under a simple CDS contract on a prespecified underlying such as a bond, the protection buyer agrees to pay periodic cash flows to the protection seller until a default occurs before the maturity of CDS. On the other side, the protection seller agrees to pay a contingent payment if a default occurs before the CDS maturity. The periodic payment done by the protection buyer is called *CDS spread* and is generally a fixed percentage of the notional amount of the underlying. If a default occurs before CDS maturity, the protection seller compensates the loss of the protection buyer, i.e. the amount paid to the protection buyer is equal to the loss given default (or 1 - recovery rate).

Therefore the cash flows for the protection buyer is:

$$CF = -\sum_{i=1}^{n} s1_{\{\text{No default until } t_i\}} + (1 - RR)\, 1_{\{\text{Default before } t_n\}}$$

where $t_i, i = 1, 2, \cdots, t_n$ are the times that corresponds to periodic payments and t_n is the maturity of CDS, s is the CDS spread, RR is the recovery rate. Note that buyer pays spread at each t_i but receives contingent payment at the time of default, say τ.

The value of CDS is calculated as the risk-neutral expected discounted cash flows. Assume that the interest rates are constant. Then:

$$\begin{aligned} V^{CDS} &= E^Q\left[-\sum_{i=1}^{n} e^{-rt_i} s1_{\{\text{No default until } t_i\}} + e^{-r\tau}(1 - RR)\, 1_{\{\text{Default before } t_n\}}\right] \\ &= -s\sum_{i=1}^{n} e^{-rt_i} E^Q\left[1_{\{\tau > t_i\}}\right] + (1 - RR)\, E^Q\left[e^{-r\tau} 1_{\{\tau < t_n\}}\right] \\ &= -s\sum_{i=1}^{n} \bar{B}(0, t_i) + (1 - RR)\, E(0, t_n) \end{aligned} \tag{4.16}$$

where $\bar{B}(0, t_i)$ is the value of a zero coupon risky bond with zero recovery rate, and $E(0, t_i)$ is the price of a unit contingent payment if a default occurs before t_i where the payment is made at the time of default. Formally:

$$\begin{aligned} \bar{B}(0, t_i) &= e^{-rt_i} E^Q\left[1_{\{\tau > t_i\}}\right] \\ E(0, t_i) &= E^Q\left[e^{-r\tau} 1_{\{\tau < t_n\}}\right] \end{aligned}$$

By using the first-passage default probabilities, we can define $\bar{B}(0, t_i)$ and $E(0, t_i)$.

$$\begin{aligned} \bar{B}(0, t_i) &= e^{-rt_i} E^Q\left[1_{\{\tau > t_i\}}\right] \\ &= e^{-rt_i}\left[1 - PD^{FPT}(0, t_i)\right] \\ &= B(0, t_i)\left[1 - PD^{FPT}(0, t_i)\right] \end{aligned}$$

where $B(0, t_i)$ is the risk-free zero coupon bond price.

$$\begin{aligned} E(0,t_i) &= E^Q\left[e^{-r\tau}1_{\{\tau<t_n\}}\right] \\ &= E^Q\left[\sum_{i=1}^{n} e^{-rt_i}1_{\{t_{i-1}<\tau<t_i\}}\right] \\ &= \sum_{i=1}^{n} e^{-rt_i}\underbrace{\left[1-PD^{FPT}(0,t_{i-1})\right]}_{SurvivalProbability}\underbrace{PD^{FPT}(t_{i-1},t_i)}_{DefaultProbability} \\ &= \sum_{i=1}^{n} B(0,t_i)\left[1-PD^{FPT}(0,t_{i-1})\right]PD^{FPT}(t_{i-1},t_i) \end{aligned}$$

where the last conditional probability can be easily derived (by a recursive approach) from the term structure of PDs by using the following relation:

$$PD^{FPT}(0,t_n) = 1-\prod_{i=1}^{n}\left[1-PD^{FPT}(t_{i-1},t_i)\right]$$

Thus we can totally define the value of a CDS by using zero coupon risk-free bond prices and first passage default probabilities:

$$\begin{aligned} V^{CDS} &= +(1-RR)\sum_{i=1}^{n} B(0,t_i)\left[1-PD^{FPT}(0,t_{i-1})\right]PD^{FPT}(t_{i-1},t_i) \\ &\quad -s\sum_{i=1}^{n} B(0,t_i)\left[1-PD^{FPT}(0,t_i)\right] \end{aligned} \tag{4.17}$$

4.4 Model Calibration

Model calibration refers to the *inverse problem* in asset pricing. In calibration problem, we have a model for an asset price, we observe prices of some assets (e.g. liquid ones) and want to price other assets on the same underlying (e.g illiquid, exotic or over-the-counter ones). Therefore we call the problem an inverse problem. For instance, in structural models defined in this chapter, the debt price is a function of five variables: the firm value, its volatility, time to maturity, risk-free rate and default point:

$$D(t,T) = f(V_t, \sigma, \tau, DP, r)$$

Therefore the generic problem is solved with a three step procedure:

- First obtain the known/observed values of input parameters. For example we can easily obtain the risk-free rate from government bond or LIBOR market. Or we can know in advance the time to maturity of a bond.

- Second, we determine some *benchmark* assets (bonds, CDSs, etc) which are liquid. Then we find the unobserved parameters which leads to the observed prices for these benchmark assets. Indeed this is an optimization problem. For this step we generally perform a least-square minimization. For the unknown (θ^u) and known (θ^k) parameter sets, we perform the following minimization:

$$\hat{\theta}^u = argmin \sum_{i=1}^{N} w_i |P_i\left(\theta^u; \theta^k\right) - P_i^{Market}|^2 \tag{4.18}$$

  where $i = 1, 2 \cdots N$ are the benchmark assets, $P_i\left(\theta^u; \theta^k\right)$ is the model price for i^{th} asset and D_i^{Market} is the market price of it and w_i are the weights used in the optimization. At the end we obtain *market-implied* parameters for the model.

- Then we can price any other asset on the same underlying by using the market-implied parameters.

$$P_j = P_j\left(\hat{\theta}^u; \theta^k\right) \tag{4.19}$$

The above procedure is a generic one that can be used in any asset pricing problem. However, for credit risk modeling we have an additional advantage that we can obtain parameters from equity markets. Since for firms with traded debt or CDS, we generally have more liquid markets for stocks, we can *calibrate* our model using the stock prices, and then by using stock market-implied parameters, we can price bonds, CDSs, etc. Procedures for determining model inputs are explained below.

4.4.1 Risk-free Rate

Theoretically the risk-free rates should be obtained from the Treasury bond and bill markets. However, if available, LIBOR swap rates are generally used to estimate risk-free term structures. Because Treasury securities are often assumed to contain a convenience yield, because they can be posted as collateral and allow to borrow at special repo rates ([Eck07]).

4.4.2 Time to Maturity of Debt

In structural models, a single number for time to maturity of debt is used. This is a consequence of the simplifying assumption that the firm has a single (zero-coupon) debt. However, this is an unrealistic assumption. Therefore we can use a weighted-average duration for all long-term liabilities ([Eck07]).

4.4.3 Default Point

In the original Merton model, the default point is equal to the face value of debt. This means that the firm will be in default if its assets are below the face value of debt at maturity. However, we should distinct short-term (i.e. shorter than the risk horizon) and long-term debt. Because if a firm's assets can meet firm's short-term liabilities but below its long-term debt, there is no reason for a default. The firm has the chance to increase its assets to meet longer-term debt until when they become due. Thus, one approach is to choose a value for default point between short-term debt and total debt ([Sun 01], [Eli05]). For example, MoodysKMV, by analyzing its empirical database for defaults, uses a default point which is equal to the short-term debt plus one half of the long term debt ([CB03]).

4.4.4 Market Value of Assets and Its Volatility

Asset value and its volatility are the most critical parameters in structural credit pricing. However these are also the parameters that are not generally observed. If all claims of a firm (i.e. bonds and stock) are traded publicly, we can easily infer the asset value, by using the identity,

$$\text{Market Value of Assets} = \text{Market Value of Debt} + \text{Market Value of Equity}$$

But this is generally not the case. A practical solution, if not all the claims are traded, is to apply a proxy of book-to-market value ratio obtained from traded debt to the non-traded one. This gives us the total market value of firm's assets.

If we can obtain asset values by using methods explained above, we calculate it volatility using historical time series of asset values. But this method has an implicit assumption. The asset volatility used in structural models is the volatility that corresponds to the future time period up to the risk horizon. Therefore, even if effectively estimate the asset value, the volatility calculated form historical values represents only the historical (i.e. realized) volatility, which may not be a good estimator for the future risk-neutral volatility.

There is a second approach used for estimating asset values and volatilities which is a maximum likelihood procedure based on the relationship between the standard deviation of the return to the firm and the equity (see for example [RV86], [ER05]).

Let firm assets (V) and equity (E) follow the following stochastic dynamics under risk neutral probability measure:

$$\begin{aligned} dV_t &= rV_t dt + \sigma V_t d\tilde{W}_t \\ dE_t &= rE_t dt + \sigma_E E_t d\tilde{W}_t \end{aligned}$$

Since the value of the equity is a function of time and of the value of the assets, $E_t = f(V_t, t)$, we can apply Ito Lemma to get:

$$dE_t = \left[\frac{\partial f(V_t,t)}{\partial t} + \frac{\partial f(V_t,t)}{\partial V_t} rV_t + \frac{1}{2}\frac{\partial f^2(V_t,t)}{\partial V_t^2}(V_t\sigma)^2\right] dt + \frac{\partial f(V_t,t)}{\partial V_t}\sigma V_t d\tilde{W}_t$$

Therefore:

$$\sigma_E E_t = \frac{\partial f(V_t,t)}{\partial V_t}\sigma V_t$$

Since:

$$\begin{aligned}
\frac{\partial f(V_t,t)}{\partial V_t} &= \Phi\left(\frac{\ln(V_t/L) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \Phi(d_1)
\end{aligned}$$

we have:

$$\sigma_E = \frac{V_t}{E_t}\Phi(d_1)\sigma$$

Thus, in order to obtain the asset value and its volatility, we should solve the following two-equation non-linear system in two unknowns (i.e. V_t and σ):

$$E_t^{Obs} = V_t\Phi(d_1) + LB(t,T)\Phi(d_2) \tag{4.20}$$

$$\sigma_E^{Obs} = \frac{V_t}{E_t}\Phi(d_1)\sigma \tag{4.21}$$

where E_t^{Obs} and σ_E^{Obs} observed stock value and its volatility respectively.

Chapter 5

PROBLEMS WITH GAUSSIAN MODEL AND POSSIBLE EXTENSIONS

In this chapter, we discuss the problems that the Gaussian model creates and potential extensions. Some of the problems caused by Gaussian model is directly related to the credit risk and some of them are more general problems that are common to all asset price modeling.

5.1 Problems with Gaussian Model

5.1.1 Pathwise and Distributional Properties of Asset Values and Returns

Finance literature is quite rich with regard to papers investigating the empirical properties of financial asset prices and returns in different markets such as stocks, commodities, exchange rates, interest rates and financial derivatives derivatives. Different markets, time periods and asset classes may have different empirical characteristics that are unique to that market. However, there is a set of certain properties observed by different studies that are common across many instruments, markets and time periods. These properties, called *stylized* *facts* show that seemingly random behavior of different asset prices do share quite non-trivial statistical properties.

In structural modeling of credit risk, the asset value of the firm is the main process that should be modeled. In general, the asset value can not be observed directly. It is simply the sum of market values of all assets that the firm owns. However, firms generally have some assets that are not traded in a liquid market. Because of this unobservability, we do not have prior information about the qualitative or statistical properties of firms' asset values. Since the firm's asset value is the sum of values of different assets, it seems reasonable to require that the firm's asset value should also satisfy the stylized statistical properties for different financial asset values.

[Cont01] presents a set of stylized facts emerging from the statistical analysis of price variations in various types of financial markets and lists the stylized statistical properties of asset returns as follows.

1. **Absence of autocorrelations:** (linear) autocorrelations of asset returns are often insignificant, except for very small intraday time scales (around 20 minutes) for which microstructure effects come into play.

2. **Heavy tails:** the (unconditional) distribution of returns seems to display a power-law or Pareto-like tail, with a tail index which is finite, higher than two and less than five for most data sets studied. In particular this excludes stable laws with infinite variance and the normal distribution. However the precise form of the tails is difficult to determine.

3. **Gain/loss asymmetry:** one observes large draw downs in stock prices and stock index values but not equally large upward movements. This property is not true for exchange rates where there is a higher symmetry in up/down moves.

4. **Aggregational Gaussianity:** as one increases the time scale Δt over which returns are calculated, their distribution looks more and more like a

normal distribution. In particular, the shape of the distribution is not the same at different time scales.

5. **Intermittency:** returns display, at any time scale, a high degree of variability. This is quantified by the presence of irregular bursts in time series of a wide variety of volatility estimators.

6. **Volatility clustering:** different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time.

7. **Conditional heavy tails:** even after correcting returns for volatility clustering (e.g. via GARCH-type models), the residual time series still exhibit heavy tails. However, the tails are less heavy than in the unconditional distribution of returns.

8. **Slow decay of autocorrelation in absolute returns:** the autocorrelation function of absolute returns decays slowly as a function of the time lag, roughly as a power law with an exponent $\beta \in [[0.2, 0.4]]$. This is sometimes interpreted as a sign of long-range dependence.

9. **Leverage effect:** most measures of volatility of an asset are negatively correlated with the returns of that asset.

10. **Volume/volatility correlation:** trading volume is correlated with all measures of volatility.

11. **Asymmetry in time scales:** coarse-grained measures of volatility predict fine-scale volatility better than the other way round.

In addition to these properties, different empirical studies found significance evidence of discontinuities (i.e. jumps) in asset prices (e.g. [EJP02], [Lin06], [RS05], [Bates96]). These jumps corresponds to the sudden events generally caused by an important news or a crash in the market.

Therefore any successful model should satisfy the above mentioned properties as much as it can.

5.1.2 Default Predictability and Short-term Default Probability

The Gaussian model is a continuous model with a stationary distribution. The statistical properties of Gaussian distribution creates problems for default predictability and short-term default probabilities.

By definition of default, we know that the distance from asset value to default threshold determines how probable the default event is. With Gaussian model, if the asset value is far from the default threshold, it is very unlikely that the firm go into default in a short time period. Because the asset value process needs time to reach the default threshold. This makes default a predictable event, i.e. default does not come as a surprise ([Eli05]).

The consequence of default predictability in Gaussian model is that short-term default probabilities tend to go to zero. When we take the limit of PD for a short time period, we have:

$$\begin{aligned} \lim_{\tau \to 0} PD\left(t, t+\tau\right) &= \lim_{\tau \to 0} \; \Phi\left[\frac{\ln\left(DP/V_t\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right] \\ &= \begin{cases} 0 & \text{if } V_t > DP \\ 1 & \text{if } V_t < DP \end{cases} \end{aligned}$$

However default predictability and very low short-term PDs are unrealistic assumptions and we have many empirical counter examples. Sudden collapse of seemingly healthy firms, such as Enron, LTCM, WorldCom, within a relatively short period of time represents important counter-facts to prevailing approaches.

Therefore in a successful model, the default may include *predictable* and *surprise* events and short-term PDs may have non-zero values.

5.1.3 Credit Spreads and Implied Volatility Smiles

The empirical testing of Gaussian structural models in pricing credit risk in general has not been very successful (see [AS00], [EHH03] and [ER05]). One obvious reason for this poor performance is the unrealistic assumption of default predictability. This makes short term spreads zero.

$$\begin{aligned}\lim_{\tau \to 0} s\left(t, t+\tau\right) &= \lim_{\tau \to 0} -\frac{ln\left(\frac{V_t}{L}\Phi\left(-d1\right)+\Phi\left(d2\right)\right)}{\tau} \\ &= \begin{cases} 0 & \text{if } V_t > DP \\ \infty & \text{if } V_t < DP \end{cases}\end{aligned}$$

The difference between market and model prices and marked difference in model performance among different credit qualities suggest that we may encounter an implied volatility smile when we use Gaussian model. Additionally, since the Gaussian model has only two parameters, i.e. μ and σ, we do not have much flexibility for fitting the model to different shapes of credit spread curves that can be observed in the market. Therefore a successful model should have the capability to capture non-zero short-term spreads, can create different shapes for credit spread curves and can replicate implied volatility smiles with respect to the Gaussian model.

5.2 Possible Extensions

5.2.1 Non-constant Volatility Models

In the Gaussian model, volatility is constant. However, as explained in the previous section, observed volatility is not constant. Additionally it has clustering property and incorporate leverage effects. Therefore one obvious extension to the Gaussian model is making volatility non-constant.

The generic form for a non-constant volatility model is as follows:

$$dV_t = \mu V_t dt + \sigma_t V_t dW_t$$

where σ_t is non-constant. Different specifications for the function σ_t leads to different classes of volatility models. We can classify volatility models into two general classes: autoregressive conditional heteroskedasticity (ARCH) models and stochastic volatility models.

ARCH Models

ARCH models, including their generalized version (i.e. generalized ARCH or GARCH), models the conditional volatility. The generic form for a GARCH model is:

$$
\begin{aligned}
r_t &= \mu_t + \epsilon_t \\
\epsilon_t &= z_t \sigma_t \ , \ z_t \sim N(0,1) \\
\sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2
\end{aligned}
$$

where r_t is the return, μ_t is the conditional mean and σ_t is the conditional volatility. The above specification is for a GARCH(p,q) model where p and q determines the degree of model.

Simple GARCH(p,q) models with Gaussian distribution can not account for leverage effects. But with t-distribution or GED, GARCH models can properly account for volatility clustering and heavy tails. In practice EGARCH and GJR-GARCH models are heavily used due to their properties to allow for leverage effect and asymmetries. There exist so many extensions to GARCH models. For an excellent review of conditional volatility modeling, interested readers are directed to [PG03].

Note that in GARCH models $\sigma_{t+1} \in F_t$, i.e. tomorrow's variance is a deterministic function of price and variance history up to today. Therefore we have $Var\left[\sigma_{t+1}^2 | F_t\right] = 0$. This is a critical assumption which is the main difference between GARCH and stochastic volatility models.

Stochastic Volatility Models

Stochastic volatility models incorporate unpredictable conditional variances by allowing a stochastic term in the variance equation. The generic form of a stochastic volatility model is:

$$\begin{aligned} dV_t &= \mu dt + \sigma_t dW_t \\ \sigma_t &= f(Y_t) \\ dY_t &= \mu_Y(t, Y_t)\, dt + \sigma_Y(t, Y_t)\, dZ_t \end{aligned}$$

where f is a positive function, Y_t is an Ito process, W_t and Z_t are standard Brownian motions. Therefore volatility process has its own source of randomness caused by Z_t. The positivity of function f ensures that the volatility is non-zero. Additionally the generic form includes models with correlated Brownian motions, i.e. $d\langle W, Z\rangle_t = \rho dt$. The negative correlation captures the leverage effects.

For process, Y, the most common choices in the literature are:

- Lognormal (LN)

$$dY_t = \mu^Y Y_t dt + \sigma^Y Y_t dZ_t$$

- Ornstein-Uhlenbeck (OU)

$$dY_t = \beta(\alpha - Y_t)\, dt + \sigma^Y dZ_t$$

- Cox-Ingersoll-Ross (CIR)

$$dY_t = \beta(\alpha - Y_t)\, dt + \sigma^Y \sqrt{Y_t} dZ_t$$

Note that the OU and CIR processes have a mean reverting behavior but the LN does not. The LN and CIR process remain positive.

The following table (from [Tay04]) summarizes different models used by different authors.

Table 5.1: Stochastic Volatility Models

Authors	ρ	Y_t Process	f (y)
[HW88]	$\rho = 0$	Lognormal	$f(y) = \sqrt{y}$
[Sco87]	$\rho = 0$	Mean Reverting OU	$f(y) = y, e^y$
[Wig87]	$\rho \neq 0$	Mean Reverting	$f(y) = y$
[SS91]	$\rho = 0$	Mean Reverting OU	$f(y) = \|y\|$
[Hes93]	$\rho \neq 0$	CIR	$f(y) = \sqrt{y}$
[BR94]	$\rho = 0$	CIR	$f(y) = \sqrt{y}$

5.2.2 Jump Models

A second class of models that can be used for modeling financial asset prices are jump models. These models attempt to capture the discontinuities observed in asset prices. Jump processes can be classified as *finite activity* and *infinite activity* jump processes.

Finite Activity Jump Processes

These jump models have finite number of jumps for any finite time period. The simplest case is the *Poisson model*. The Poisson process, N_t, is a piecewise constant stochastic process which have finite jumps, and the inter arrival time for jumps have exponential distribution. Poisson process has a constant intensity which defines the average number of jumps expected within a unit period of time. The second example of a finite activity jump model is the *compound Poisson model*. In the compound Poisson process, Q_t, we have a Poisson process that determines the number and time of jumps, and a second process that drives the jump size. Formally, we have:

$$Q_t = \sum_{i=1}^{N_t} Y_i$$

where N_t is a Poisson process with intensity λ and Y_i is a (stochastic) process that drives jump sizes. The jumps in Q_t occur at the same times as the jumps in N_t, but whereas the jumps in N_t are always of size 1, the jumps in Q_t are of random size. Therefore compound Poisson process is also a piecewise constant stochastic process.

The piecewise constant models, such as Poisson and compound Poisson models, are alone not good candidates for modeling financial asset values. Thus, in general, jump models are combined with other stochastic processes. In a *jump-diffusion model*, we have a deterministic time trend, a stochastic part governed by a Brownian motion and a compound Poisson process. Formally,

$$dX_t = \mu (X_t, t)\, dt + \sigma (X_t, t)\, dW_t + \sum_{i=1}^{N_t} Y_i$$

where W_t is a standard Brownian motion, N_t is a Poisson process with intensity λ and $Y = (Y_i)_{i\geq 1}$ is an i.i.d. sequence of random variables with probability distribution F.

Infinite Activity Jump Processes

Second class of jump models are the infinite activity jump models in which we may have infinite number of small jumps within a finite time period. Contrary to the finite activity jump models, in infinite activity jump models, jumps are no longer rare events, the process moves essentially by jumps. Although we require diffusion parts to complement a finite activity jump process, we can directly model financial asset values with pure (infinite activity) jump models. For a complete resource for jump processes with financial applications refer to [CT04].

5.3 Approach Used in the Thesis

After searching for possible extensions, we opt to use a model with stochastic volatility and finite jumps. We call the model a *stocastic volatility correlated jumps with stochastic interest rates with jumps (SVCJ-SIJ) model*. The selected model attempts to solve the problems encountered with Gaussian case.

The selected model is a generalization of different models proposed in the literature and therefore by setting certain model parameters to zero, we can have different restricted models. The details of the model is explained in the next chapter.

Chapter 6

MODELING SINGLE FIRM CREDIT RISK: SVCJ-SIJ MODEL

6.1 Model

In this section we find the probability of default for a single firm which has an asset value process with stochastic volatility and jumps. First assume that the asset value follows the following stochastic differential equation.

$$\frac{dV_t}{V_t} = (\mu - \lambda_1 k)dt + \sqrt{h_t} dW_t^1 + (e^{Y^1} - 1)dN_t^1 \tag{6.1}$$

or equivalently, for $\nu_t := \ln V_t$

$$d\nu_t = (\mu - \lambda_1 k - \frac{1}{2}h_t)dt + \sqrt{h_t} dW_t^1 + Y^1 dN_t^1 \tag{6.2}$$

Thus the asset value process is governed by a diffusion process as well as a compensated jump process. The jumps are finite and jump sizes are normally distributed.

Additionally the volatility is also assumed as stochastic and follows a square-root (or Cox-Ingersoll-Ross - CIR) process with jumps. The jumps in the asset value and the volatility are caused by the same Poisson process. This means that the asset value and the volatility jumps simultaneously, but with different jump sizes. The jump size distribution for volatility process is exponential.

$$dh_t = \beta(\alpha - h_t)dt + \gamma\sqrt{h_t}dW_t^2 + Y^2 dN_t^1 \tag{6.3}$$

And finally, interest rate is also stochastic and follows a similar process, i.e. square-root plus jump, with volatility. But the Brownian motion and jump that drives interest rate is independent from asset value and volatility processes.

$$dr_t = \eta(\theta - r_t)dt + \delta\sqrt{r_t}dW_t^3 + Y^3 dN_t^3 \tag{6.4}$$

with the following additional assumptions:

- $\mu, \beta, \alpha, \eta, \theta \in \mathbb{R}$ and $\gamma, \delta \in \mathbb{R}^+$
- $k = E\left[e^{Y^1} - 1\right]$
- W^1, W^2, W^3 are standard Brownian Motions with the following variance-covariance matrix:

$$\sum = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- N_t^1 and N_t^2 are Poisson processes with intensities λ_1 and λ_2
- Y^2 and Y^3 are exponentially distributed with parameters μ_2 and μ_3 respectively. And the conditional distribution of Y^1 given Y^2 is normal.
  $(Y^1|Y^2 = y^2) \sim N(\mu_1 + \rho_j y_2, \sigma_1^2)$
  $Y^2 \sim exp(\mu_2)$
  $Y^3 \sim exp(\mu_3)$
- There is no interdependence between (W^1, W^2, W^3),(Y^1, Y^2, Y^3) and (N^1, N^2).

The model is called *stochastic volatility correlated jumps with stochastic interest rates with jumps (SVCJ-SIJ) model.* By setting certain parameters of the model, we can have restricted versions of the model such as jump diffusion (JD), stochastic volatility (SV), stochastic volatility jump diffusion (SVJD), etc.

6.2 Conditional Characteristic Function

Since under actual probability measure, the risk-free rate does not appear in the drift term of asset value (or return) process and r_t is independent of ν_t and h_t, we condition the characteristic function only to ν_t and h_t.

Therefore, the conditional characteristic function (CCF) is defined as

$$\begin{aligned} \varphi(u_1, u_2; \nu_T, h_T | \nu_t, h_t) &= \varphi^t(u_1, u_2; \nu_T, h_T) \\ &:= E\left[exp\left\{iu_1\nu_T + iu_2 h_T\right\} | \nu_t, h_t\right] \end{aligned} \tag{6.5}$$

φ^t is, by definition, a conditional expectation. And we also know that every conditional expectation is a martingale, i.e. for a filtration $F = \{F_t : t \geq 0\}$ and with $F_s \subset F_t$, we have:

$$E\left[E\left[X|F_t\right]|F_s\right] = E\left[X|F_s\right]$$

Thus φ^t satisfies all the properties of a martingale. Additionally we also know that a martingale has always constant expectation:

$$\begin{aligned} E\left[X_t|F_s\right] &= X_s \\ \Rightarrow E\left[X_t - X_s|F_s\right] &= 0 \\ \Rightarrow E\left[dX_s\right] &= 0 \end{aligned}$$

where $dX_s := X_t - X_s$. When we apply this property to φ^t, we have:

$$E\left[d\varphi^t\right] = 0 \tag{6.6}$$

To find $d\varphi^t$, we apply Ito to φ^t (we drop t to ease notation):

$$d\varphi = \varphi_t dt + \varphi_\nu d\nu + \varphi_h dh + \frac{1}{2}(\varphi_{\nu\nu} d\nu d\nu + 2\varphi_{\nu h} d\nu dh + \varphi_{hh} dh dh) \tag{6.7}$$

Since the jump parts are independent from the diffusion parts, we can express the change in φ in two parts:

$$\text{Total change} = \text{Change without jumps} + \text{Change because of jumps}$$

Thus:

$$\begin{aligned} E[d\varphi] &= E\left[\varphi_t dt + \varphi_\nu \left\{\left(\mu - \lambda_1 k - \frac{1}{2}h\right)dt + \sqrt{h}dW^1\right\}\right. \\ &\quad \left. + \varphi_h \left\{\beta(\alpha - h)dt + \gamma\sqrt{h}dW^2\right\} + \frac{1}{2}\varphi_{\nu\nu} h dt + \varphi_{\nu h}\gamma h \rho dt + \frac{1}{2}\varphi_{hh}\gamma^2 h dt\right] \\ &\quad + \lambda^1 E\left[\varphi\left(u_1, u_2; \nu_T, h_T | \nu_t + Y^1, h_t + Y^2\right) - \varphi\left(u_1, u_2; \nu_T, h_T | \nu_t, h_t\right)\right] dt \end{aligned}$$

Since W^1 and W^2 are martingale, we have $E[dW^1] = 0$ and $E[dW^2] = 0$. Thus, in order to obtain $E[d\varphi] = 0$, we should have:

$$\begin{aligned} 0 &= \varphi_t + \varphi_\nu\left(\mu - \lambda_1 k - \frac{1}{2}h\right) + \varphi_h \beta(\alpha - h)\frac{1}{2}\varphi_{\nu\nu} h + \varphi_{\nu h}\gamma h \rho + \frac{1}{2}\varphi_{hh}\gamma^2 h \\ &\quad + \lambda^1 E\left[\varphi\left(u_1, u_2; \nu_T, h_T | \nu_t + Y^1, h_t + Y^2\right) - \varphi\left(u_1, u_2; \nu_T, h_T | \nu_t, h_t\right)\right] \end{aligned}$$

Following [DPS00] and [RS05], we guess the general form of characteristic function as follows:

$$\varphi^t(u_1, u_2; \nu_T, h_T) = e^{iu_1\nu_t + A(\tau;u_1,u_2) + B(\tau;u_1,u_2)h_t + J^1(\tau;u_1,u_2)} \tag{6.8}$$

where $\tau = T - t$

Then the differentials of φ are:

$$\begin{aligned}
\varphi_t &= -\left[\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \tau} h_t + \frac{\partial J^1}{\partial \tau}\right]\varphi \\
\varphi_\nu &= iu_1\varphi \\
\varphi_h &= B\varphi \\
\varphi_{\nu\nu} &= i^2u_1^2\varphi \\
\varphi_{hh} &= B^2\varphi \\
\varphi_{\nu h} &= iu_1B\varphi
\end{aligned}$$

For the last term in $E\left[d\varphi\right]$, which is caused by jump events, define the jump transform:

$$\Theta^1(c_1, c_2) := E\left[e^{c_1Y^1 + c_2Y^2}|\nu_t, h_t\right]$$

where the transform is the joint conditional moment generating function of Y^1 and Y^2. Then, with A, B and J^1 as defined above:

$$\begin{aligned}
& E\left[\varphi(u_1, u_2; \nu_T, h_T|\nu_t + Y^1, h_t + Y^2) - \varphi(u_1, u_2; \nu_T, h_T|\nu_t, h_t)\right] \\
= \; & E_t\left[exp\left\{iu_1(\nu_t + Y^1) + A + B(h_t + Y^2) + J^1\right\}\right] \\
& -E_t\left[exp\left\{iu_1\nu_t + A + Bh_t + J^1\right\}\right] \\
= \; & exp\left\{iu_1\nu_t + A + Bh_t + J^1\right\} E_t\left[exp\left\{iu_1Y^1 + BY^2\right\} - 1\right] \\
= \; & \varphi_t(u_1, u_2; \nu_T, h_T)\left[\Theta^1(iu_1, B) - 1\right] \qquad (6.9)
\end{aligned}$$

When we input the differentials:

$$\left\{\begin{array}{l}
-\frac{\partial A}{\partial \tau} - \frac{\partial B}{\partial \tau}h - \frac{\partial J^1}{\partial \tau} + iu_1(\mu - \lambda_1 k - \frac{1}{2}h) + B\beta(\alpha - h) \\
\\
+\frac{1}{2}i^2u_1^2h + iu_1B\gamma h\rho + \frac{1}{2}B^2\gamma^2h + \lambda^1\left[\Theta^1(iu_1, B) - 1\right]
\end{array}\right\}\varphi = 0$$

When we group the terms with h, terms related to diffusion and jump parts, we have:

$$\left\{ \begin{array}{l} \left[-\frac{\partial A}{\partial \tau} + iu_1\mu + B\beta\alpha\right] \\ \\ +h\left[-\frac{\partial B}{\partial \tau} - \frac{1}{2}iu_1 - B\beta + \frac{1}{2}i^2u_1^2 + \frac{1}{2}B^2\gamma^2 + iu_1B\gamma\rho\right] \\ \\ +\left[-\frac{\partial J^1}{\partial \tau} - iu_1\lambda_1 k + \lambda_1\left[\Theta^1(iu_1, B) - 1\right]\right] \end{array} \right\} \varphi = 0$$

When we equate each group in brackets to zero, we obtain a system of complex-valued ordinary differential equations (ODE):

$$\begin{aligned} \frac{\partial A}{\partial \tau} &= iu_1\mu + B\beta\alpha && (6.10) \\ \frac{\partial B}{\partial \tau} &= \frac{1}{2}iu_1(iu_1 - 1) + B(iu_1\gamma\rho - \beta) + \frac{1}{2}B^2\gamma^2 && (6.11) \\ \frac{\partial J^1}{\partial \tau} &= -iu_1\lambda_1 k + \lambda_1\left[\Theta^1(iu_1, B) - 1\right] && (6.12) \end{aligned}$$

And the boundary conditions for the system are:

$$\begin{aligned} A(0; u_1, u_2) &= 0 && (6.13) \\ B(0; u_1, u_2) &= iu_2 && (6.14) \\ J^1(0; u_1, u_2) &= 0 && (6.15) \end{aligned}$$

so that

$$\varphi_T(u_1, u_2; \nu_T, h_T) = exp\left\{iu_1\nu_T + iu_2h_T\right\} \qquad (6.16)$$

In the next step, we derive the conditional characteristic function for returns. First observe that:

$$\begin{aligned} \varphi(w; \nu_T|\nu_t, h_t) &= E\left[e^{iw\nu_T}|\nu_t, h_t\right] \\ &= \varphi(w, 0; \nu_T, h_T|\nu_t, h_t) \\ &= exp\left\{iw\nu_t + A(\tau; w, 0) + B(\tau; w, 0)h_t + J^1(\tau; w, 0)\right\} \end{aligned}$$

And for logarithmic returns defined as $R_{t,T} := \nu_T - \nu_t$,

$$\begin{aligned}
\varphi(w; R_{t,T}|\nu_t, h_t) &= E\left[e^{iw(\nu_T-\nu_t)}|\nu_t, h_t\right] \\
&= E\left[e^{iw\nu_T}|\nu_t, h_t\right] e^{-iw\nu_t} \\
&= exp\left\{iw\nu_t + A(\tau; w, 0) + B(\tau; w, 0)h_t + J^1(\tau; w, 0) - iw\nu_t\right\} \\
&= exp\left\{A(\tau; w, 0) + B(\tau; w, 0)h_t + J^1(\tau; w, 0)\right\}
\end{aligned}$$

Since the functional form does not depend on ν_t, there is no need for conditioning on ν_t. Therefore we can express the conditional characteristic function of returns as $\varphi(w; R_{t,T}|h_t)$.

$$\varphi(w; R_{t,T}|h_t) = exp\left\{A(\tau; w, 0) + B(\tau; w, 0)h_t + J^1(\tau; w, 0)\right\} \tag{6.17}$$

6.3 Unconditional Characteristic Function

In the next step, we will derive the unconditional characteristic function (UCF) of returns, i.e. $\varphi(w; R_{t,T})$.

The UCF is simply the expectation of CCF. Therefore:

$$\begin{aligned}
\varphi(w; R_{t,T}) &= E\left[e^{iwR_{t,T}}\right] \\
&= E\left[E\left[e^{iwR_{t,T}}|h_t\right]\right] \\
&= E\left[\varphi(w; R_{t,T}|h_t)\right] \\
&= E\left[exp\left\{A(\tau; w, 0) + B(\tau; w, 0)h_t + J^1(\tau; w, 0)\right\}\right] \\
&= exp\left\{A(\tau; w, 0) + J^1(\tau; w, 0)\right\} E\left[exp\left\{B(\tau; w, 0)h_t\right\}\right]
\end{aligned}$$

The expectation can be derived from the characteristic function of h_t. Since, for h_t, the diffusion part which follows a square-root process (also called Cox-Ingersoll-Ross (CIR) process) and the jump part are independent, the characteristic function of h_t is simply the product of characteristic functions of these two parts (see [RS05]).

$$\varphi_t\left(w;h_T\right) := E\left[e^{iwh_T}|h_t\right] = \varphi_t^{CIR}\left(w\right)\varphi_t^{Jump}\left(w\right) \tag{6.18}$$

Since the distribution of the square-root process is gamma, the characteristic function of the diffusion part is (see, for example, appendix in [Jia02]):

$$\varphi_t^{CIR}(u) = \left(1 - \frac{iu\gamma^2}{2\beta}\right)^{-2\alpha\beta/\gamma^2}$$

Characteristic function of jump part is as follows. By using the independence property of Poisson events:

$$\begin{aligned} Jump &= \sum_{k=N_t}^{N_T} Y_k \sim \sum_{k=1}^{N_\tau} Y_k \\ \varphi_t^{Jump}(u) &= E\left[exp\left\{iu\sum_{k=1}^{N_\tau} Y_k\right\}\right] \\ &= E\left[E\left[exp\left\{iu\sum_{k=1}^{n} Y_k\right\}|N_\tau = n\right]\right] \end{aligned}$$

Since N_t is independent of $\{Y_k\}$,

$$\begin{aligned} \varphi_t^{Jump}(u) &= \sum_{n=0}^{\infty} E\left[exp\left\{iu\sum_{k=1}^{n} Y_k\right\}\right] P\left\{N_\tau = n\right\} \\ &= P\left\{N_\tau = 0\right\} + \sum_{n=1}^{\infty} E\left[\underbrace{\prod_{k=1}^{n} exp\left\{iuY_k\right\}}_{\{Y_k\}\ are\ independent}\right] \frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau} \end{aligned}$$

$$
\begin{aligned}
&= e^{-\lambda\tau} + \sum_{k=1}^{\infty} \prod_{k=1}^{n} \underbrace{E\left[exp\left\{iuY_k\right\}\right]}_{\text{Char. fnc. of } Y_k} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} \\
&= e^{-\lambda\tau} + \sum_{n=1}^{\infty} (\varphi_Y)^n \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} \\
&= e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(\varphi_Y \lambda\tau)^n}{n!} \\
&= exp\left\{\lambda\tau(\varphi_Y(u) - 1)\right\}
\end{aligned}
$$

where $\varphi_Y(u)$ is the characteristic function of Y.

Since the jump size distribution is exponential, we have $\varphi_Y(u) = \frac{1}{1-iu\mu}$ where μ is the scale parameter, and thus:

$$
\varphi_t^{Jump}(u) = exp\left\{\lambda\tau\left(\frac{1}{1-iu\mu} - 1\right)\right\} = exp\left\{\frac{\lambda\tau iu\mu}{1-iu\mu}\right\}
$$

Therefore, the characteristic function of h_t is:

$$
\varphi_t(u; h_T) = exp\left\{\frac{\lambda_1 \tau iu\mu_2}{1-iu\mu_2}\right\} \left(1 - \frac{iu\gamma^2}{2\beta}\right)^{-2\alpha\beta/\gamma^2} \tag{6.19}
$$

Note that in order to find the unconditional characteristic function for returns, we should find unconditional characteristic function for h_t. Therefore:

$$
\varphi\left(w; h_t\right) := E\left[e^{iwh_t}\right] = \left(1 - \frac{iu\gamma^2}{2\beta}\right)^{-2\alpha\beta/\gamma^2}
$$

for $\tau = 0$.

By using characteristic function of h_t we can express the last term in $\varphi\left(w, R_{t,T}\right)$:

$$
\begin{aligned}
E\left[exp\left\{B\left(\tau; w, 0\right) h_t\right\}\right] &= \varphi\left(\frac{B\left(\tau; w, 0\right)}{i}; h_t\right) \\
&= \left(1 - \frac{B\left(\tau; w, 0\right)\gamma^2}{2\beta}\right)^{-2\alpha\beta/\gamma^2}
\end{aligned}
$$

At the end, we can derive the formula for the UCF of returns:

$$\begin{aligned}\varphi(w; R_{t,T}) &= exp\left\{A(\tau; w, 0) + J^1(\tau; w, 0)\right\} E\left[exp\left\{B(\tau; w, 0)h_t\right\}\right] \\ &= exp\left\{A(\tau; w, 0) + J^1(\tau; w, 0)\right\} \left(1 - \frac{B(\tau; w, 0)\gamma^2}{2\beta}\right)^{-2\alpha\beta/\gamma^2} \qquad (6.20)\end{aligned}$$

6.4 Solutions

First,note that the jump transform, $\Theta^1(c_1, c_2)$, is:

$$\begin{aligned}\Theta^1(c_1, c_2) &= E\left[e^{c_1 Y^1 + c_2 Y^2} | \nu_t, h_t\right] \\ &= E\left[E\left[e^{c_1 Y^1 + c_2 Y^2} | Y^2 = \overline{y}\right]\right] \\ &= E\left[e^{c_2 Y^2} E\left[e^{c_1 Y^1} | Y^2 = \overline{y}\right]\right]\end{aligned}$$

Since $(Y^1 | Y^2 = \overline{y}) \sim N(\mu_1 + \rho_j \overline{y}, \sigma_1^2)$, by using the moment generating function of normal distribution we have:

$$\begin{aligned}\Theta^1(c_1, c_2) &= E\left[e^{c_2 Y^2} e^{(\mu_1 + \rho_j Y^2)c_1 + \frac{1}{2}c_1^2\sigma_1^2}\right] \\ &= e^{\mu_1 c_1 + \frac{1}{2}c_1^2\sigma_1^2} E\left[e^{(c_2 + \rho_j c_1)Y^2}\right]\end{aligned}$$

Since $Y^2 \sim exp(\mu_2)$, by using the moment generating function of exponential distribution we have:

$$\begin{aligned}\Theta^1(c_1, c_2) &= e^{\mu_1 c_1 + \frac{1}{2}c_1^2\sigma_1^2} \frac{1}{1 - (c_2 + \rho_j c_1)\mu_2} \\ &= \frac{exp\left\{\mu_1 c_1 + \frac{1}{2}c_1^2\sigma_1^2\right\}}{1 - \rho_j c_1 \mu_2 - c_2 \mu_2} \qquad (6.21)\end{aligned}$$

And for the jump compensator:

$$\begin{aligned} k &= E\left[e^{Y^1}-1\right] \\ &= E\left[E\left[e^{Y^1}|Y^2\right]\right]-1 \\ &= E\left[exp\left\{\mu_1+\rho_j Y^2+\frac{1}{2}\sigma_1^2\right\}\right]-1 \\ &= e^{\mu_1+\frac{1}{2}\sigma_1^2}E\left[e^{\rho_j Y^2}\right]-1 \\ &= \frac{exp\left\{\mu_1+\frac{1}{2}\sigma_1^2\right\}}{1-\rho_j\mu_2}-1 \\ &= \frac{exp\left\{\mu_1+\frac{1}{2}\sigma_1^2\right\}+\rho_j\mu_2-1}{1-\rho_j\mu_2} \end{aligned} \tag{6.22}$$

Now we can find the solution for the complex valued ODEs. The details of the solutions are given in the Appendix A.

$$A(\tau;u_1,u_2) = \left(iu_1\mu-\frac{\alpha\beta}{\gamma^2}g_1\right)\tau+\frac{\alpha\beta}{\gamma^2}ln\left(\frac{1+g_3^2}{1+g_4^2}\right) \tag{6.23}$$

$$B(\tau;u_1,u_2) = \frac{g_2g_3-g_1}{\gamma^2} \tag{6.24}$$

$$\begin{aligned} J^1(\tau;u_1,u_2) &= \left(g_8-\lambda_1+\frac{g_5g_7}{(g_5^2+g_6^2)}\right)\tau \\ &\quad +\frac{2g_6g_7}{g_2(g_5^2+g_6^2)}\,ln\left(\frac{(g_6g_4-g_5)\sqrt{1+g_3^2}}{(g_6g_3-g_5)\sqrt{1+g_4^2}}\right) \end{aligned} \tag{6.25}$$

where:

$$g_1(u_1,u_2) := iu_1\gamma\rho-\beta \tag{6.26}$$

$$g_2(u_1,u_2) := \sqrt{iu_1(iu_1-1)\gamma^2-g_1^2} \tag{6.27}$$

$$g_3(u_1,u_2) := tan\left[\frac{1}{2}g_2\tau+arctan(g_4)\right] \tag{6.28}$$

$$g_4(u_1,u_2) := \frac{g_1+iu_2\gamma^2}{g_2} \tag{6.29}$$

$$g_5(u_1,u_2) := 1-\rho_j i u_1\mu_2+\frac{\mu_2 g_1}{\gamma^2} \quad (6.30)$$

$$g_6(u_1,u_2) := \frac{\mu_2 g_2}{\gamma^2} \quad (6.31)$$

$$g_7(u_1,u_2) := \lambda_1 exp\left(\mu_1 i u_1-\frac{1}{2}u_1^2\sigma_1^2\right) \quad (6.32)$$

$$g_8(u_1,u_2) := -iu_1\lambda_1\frac{exp\left(\mu_1+\frac{1}{2}\sigma_1^2\right)+\rho_j\mu_2-1}{1-\rho_j\mu_2} \quad (6.33)$$

Since we found solutions for A, B and J^1, we can totally define the conditional and unconditional characteristic function for returns.

6.5 Simulation of Returns

We can simulate returns with the following algorithm. Note that in the algorithm we generate correlated Brownian motions, simultaneous jumps and conditional jump size for the log asset value.

6.5.1 Simulation Algorithm

1. Fix the time horizon T, and initial values ν_0 and h_0.
2. Establish an equally spaced time grid with $\Delta t := \frac{T}{M}$ where M is the number of grid points and $t_i = i\,\Delta t,\ i=0,1,\cdots,M$ are the time points.
3. First discretize the process for ν and h by using the Euler scheme:

$$\Delta\nu_t = \left(\mu-\lambda_1 k-\frac{1}{2}h_t\right)\Delta t+\sqrt{h_t}\Delta W_t^1+Y^1\Delta N_t^1$$

$$\Delta h_t = \beta(\alpha-h_t)\Delta t+\gamma\sqrt{h_t}\Delta W_t^2+Y^2\Delta N_t^1$$

4. Simulate two dimensional Brownian increments
   - First simulate $2\times M$ dimensional matrix of independent standard nor-

mal random numbers. Call this matrix as $Z^{(2)}$.

$$Z^{(2)}(i,j) \sim N(0,1) \ , i = 1, 2, \ j = 1, 2, \cdots, M$$

- Find Cholesky decomposition of the covariance matrix of Brownian increments. Call this matrix as C:

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \Rightarrow \ C = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix}$$

- Then the simulated (correlated)Brownian increments is a $2 \times M$ matrix, ΔW, given by:

$$\Delta W = \sqrt{\Delta t}\, C\, Z^{(2)}$$

5. The number of jumps, N, can be simulated using the Poisson distribution:

$$N \sim Poisson(\lambda_1 T)$$

6. Simulate N jump times. The inter arrival times of a Poisson distribution is exponentially distributed. However in order to increase efficiency of simulation algorithm, given the number of jumps, we can simulate jump times by using uniform distribution (see [CT04]). Therefore the jump times can be generated on $[0, M]$ as follows:

$$t_j^{Jump} \sim Uniform(0,T) \ , \ forj = 1, 2, \cdots, N$$

Call this $1 \times N$ vector as T^{Jump}.

7. Simulate jump sizes

- First we simulate N jump sizes for volatility. Call this $1 \times N$ vector as Y^2.

$$Y_j^2 \sim Exp(\mu_2) \ , \ forj = 1, 2, \cdots, N$$

- Then we simulate N jump sizes for log asset value, conditional on the jump size of volatility. For this, first we simulate N independent standard normal random numbers and call this $Z^{(1)}$:

$$Z^{(1)}(j) \sim N(0,1)\ ,\ for j = 1, 2, \cdots, N$$

Then the jump size vector Y^1 is found as:

$$Y^1 = \mu_1 Ones(1, N) + \rho_j Y^2 + \sigma_1 Z^{(1)}$$

where $Ones(i,j)$ is a $i \times j$ dimensional unit matrix and ρ_j is the correlation parameter for jump sizes.

8. Finally we can simulate variance and log asset value as follows:

$$\begin{aligned}
&For\ i = 1\ to\ M \\
&\qquad h_{i+1} = h_i + \beta(\alpha - h_i)\Delta t + \gamma\sqrt{h_i}\Delta W(2,i) + \sum_{j=1}^{N} Y^2(j)\, 1_{\left\{i\Delta t < t_j^{Jump} \le (i+1)\Delta t\right\}} \\
&\qquad \nu_{i+1} = \nu_i + \left(\mu - \lambda_1 k - \frac{1}{2}h_i\right)\Delta t + \sqrt{h_i}\Delta W(1,i) + \sum_{j=1}^{N} Y^1(j)\, 1_{\left\{i\Delta t < t_j^{Jump} \le (i+1)\Delta t\right\}} \\
&Loop
\end{aligned}$$

9. We can find log returns from log asset values:

$$R_i = \nu_{i+1} - \nu_i$$

6.5.2 Algorithms for Exponential and Poisson Random Numbers

In the above algorithm, we should generate random numbers from uniform, standard normal, exponential and Poisson distributions. Random number generation for uniform and standard normal distributions are given in Chapter 2. Now we give two additional algorithms for exponential and Poisson distributions.

In order to simulate a random number from an exponential distribution with parameter μ, we can use the following algorithm which depends on the inverse transform method [Rub81]:

- Step 1: Generate a random number from uniform distribution.

$$U \sim Uniform\,(0,1)$$

- Step 2: Find X such that:

$$X = -\mu\, ln\,(U)$$

- Step 3: X is a exponentially distributed random variable.

The following algorithm simulates random numbers from a Poisson distribution with intensity λ [Rub81]:

- Step 1: Generate a random number from uniform distribution.

$$U \sim Uniform\,(0,1)$$

- Step 2: Set $i = 0$, $p = e^{-\lambda}$ and $F = p$
- Step 3: If $U < F$, set $X = i$ and stop.
- Step 4: Set $p = \frac{\lambda p}{i+1}$, $F = F + p$ and $i = i + 1$.
- Step 5: Go to step 3.

6.6 Statistical Properties of Asset Values and Returns

In this section we analyze the pathwise and distributional properties of asset values and returns and show that our extended model can capture stylized facts of empirical data.

6.6.1 Pathwise Properties

The pathwise properties of the SVCJ-SIJ model is discussed in the following paragraphs. We simulate a path for the volatility and log asset value processes in Figure 6.1. In the figure both continuous and jump parts of the processes are shown. However, note that since we have a recursive relation, the total parts are not simply sum of continuous and jump parts.

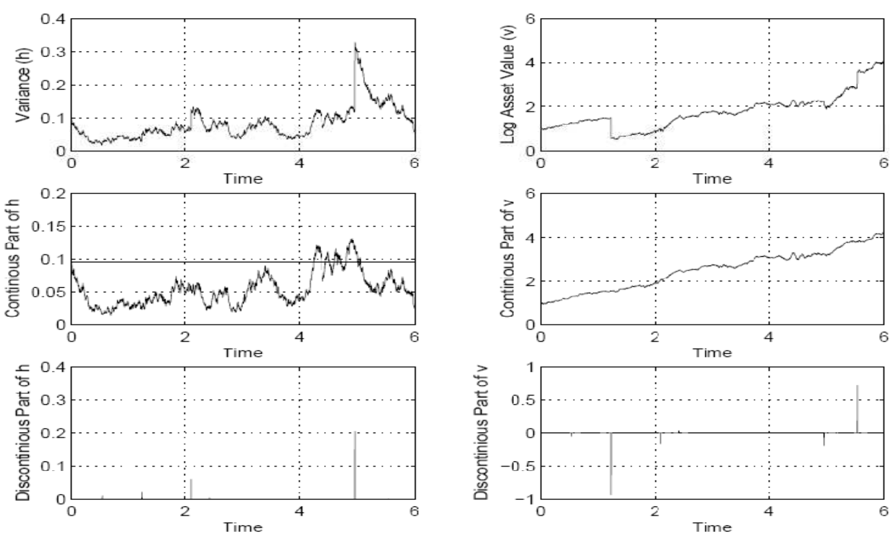

Figure 6.1: Simulation of SVCJ-SIJ Model

Model Property 1: *Non-constant volatility*

In SVCJ-SIJ model, the volatility is no longer constant. This is in line with the empirical findings since realized and implied volatilities are not constant in real life.

Model Property 2: *Mean reversion in volatility*

In SVCJ-SIJ model, the volatility has a mean reversion, i.e. it has a tendency to go towards its long run (mean) value. The mean reversion is captured by the drift term, $\beta(\alpha - h_t)$, with the jump part, where β is the speed of mean reversion. The long run mean of the volatility is process is $\alpha + \frac{\lambda_1\mu_2}{\beta}$. This can be seen by integrating the SDE for the variance process, and by using the zero expectation property of integrals against Brownian motion and Fubini theorem:

$$
\begin{aligned}
E[h_t] &= E[h_0] + E\left[\int_0^t \beta(\alpha - h_s)\,ds\right] + E\left[\int_0^t \gamma\sqrt{h_s}dW_s^2\right] + E\left[\sum_{i=0}^{N_t^1} Y^2\right] \\
\bar{h} &= \bar{h} + \beta\left(\alpha - \bar{h}\right)t + E\left[\sum_{i=0}^{N_t^1} Y^2\right] \\
&= \bar{h} + \beta\left(\alpha - \bar{h}\right)t + \lambda_1\mu_2 t \\
&= \alpha + \frac{\lambda_1\mu_2}{\beta}
\end{aligned}
\tag{6.34}
$$

The volatility process has a reflecting barrier at zero so that we have non-negative values for the volatility. If the volatility reaches zero, the Gaussian part driven by the Brownian motion has zero volatility, and the drift part pulls volatility upwards and make it non-zero. The jumps in volatility makes it even more probable to have non-zero (and non-negative) values. The γ is called the *volatility of volatility* which captures the small symmetric movements in the volatility.

Model Property 3: *Volatility clustering*

In the model, we have volatility clustering, i.e. volatility process has positive autocorrelation. This can be seen by a visual inspection of the Figure 6.1. Actually, the variance process is a continuous time first-order autoregressive process with disturbances coming from normal and exponential distributions. To see this let us take the conditional expectation of the variance within a unit time period:

$$
\begin{aligned}
E_t\left[h_{t+1}\right] &= h_t + \alpha\beta\tau - \beta\int_t^{t+1} E_t\left[h_s\right] ds + \gamma E_t \underbrace{\left[\int_t^{t+1} \sqrt{h_s} dW_s^2\right]}_{Martingale} + E\left[\sum_{N_t^1}^{N_{t+1}^1} Y^2\right] \\
&= h_t + \beta\left[\left(\alpha + \frac{\lambda_1\mu_2}{\beta}\right)\tau - \int_t^{t+1} E_t\left[h_s\right] ds\right]
\end{aligned}
$$

which can be solved by solving the following ODE:

$$
f' = \alpha\beta + \lambda_1\mu_2 - \beta f
$$

The solution is:

$$
E_t\left[h_{t+1}\right] = h_t e^{-\beta\tau} + \left(\alpha + \frac{\lambda_1\mu_2}{\beta}\right)\left(1 - e^{-\beta\tau}\right)
$$

Therefore we can express h_{t+1} as follows:

$$
h_{t+1} = \left(\alpha + \frac{\lambda_1\mu_2}{\beta}\right)\left(1 - e^{-\beta\tau}\right) + e^{-\beta\tau} h_t + \text{Error} \tag{6.35}
$$

which is in the first-order autoregressive form.

Model Property 4: *Leverage effects*

The Brownian motions driving log asset value and volatility has correlation. This captures an important empirical phenomena called leverage effects. For a leveraged firm, when value of its assets decrease, the share of equity in total assets also decreases. Since the equity bears the full risk of the firm, the percentage volatility of equity rises. This dependence between asset returns and volatility is captured by correlation among Brownian motions.

Model Property 5: *Jumps in returns*

The model has jumps in the returns which can capture effects of important news. The jumps makes the asset value process discontinuous. The jump intensity is constant. This means that we can expect *on average* same number of jumps for two different time periods with same length. This is indeed a simplistic assumption since there may also be clustering in jump events, e.g. for times of market stress. The jump sizes are conditionally normally distributed. The symmetry property of normal distribution allows both positive and negative jumps. This means that the asset value may have a sudden decrease or increase. This feature is very intuitive since in real life asset prices can be affected by good or bad news and can have negative or positive jumps. Since the change in the log asset value does not depend on previous values of itself, the jumps in asset value has a non-persistent effect, i.e. it affects only the current period returns.

Model Property 6: *Jumps in volatility*

We have also jumps in volatility process which also captures a similar economic reasoning. Again the intensity is constant. But the jump size is exponentially distributed. Therefore volatility can only have positive jumps. This is also intuitive, since a sudden decrease in volatility is far less probable than a sudden increase in it. Since the change in volatility depends on the current level of volatility, the jumps in volatility has a persistent effect.

Model Property 7: *Correlated jumps in asset value and volatility*

The jump processes in asset value and volatility have a two-way dependence: the jump times are same and jump sizes are correlated. Having identical jump times captures characteristics of certain events, e.g. news can cause both a sudden change in asset value and a sudden increase in volatility. As an alternative the model can be changed by allowing independent jump times. But we opt not to include such specifications in our model.

In the model, the jump sizes are also correlated. This is also intuitive: the impact of news on asset values and volatility are correlated. This feature also adds to the leverage effects caused by diffusive parts. Additionally, if μ_1 and ρ_j have same sign, having a jump in volatility increases the mean of jump size in asset value in absolute value terms. As an example, if both μ_1 and ρ_j are negative, the jump component adds mass to the left tail of the return distribution.

6.6.2 Distributional Properties

Model Property 8: *Conditional non-symmetries and fat tails*

In SVCJ-SIJ model, for a small time period dt, the conditional change in log asset value ν, which is equal to the continuously compounded return, has 3 parts: a) a drift part, b) a Gaussian part caused by Brownian motion, which has zero drift and $h_t dt$ variance, c) a jump part which is a sum of conditional normals, but the number of random numbers that are summed is random and the means are different. Therefore, conditionally, we have a *mixture of normals* (not a sum of normals) for the return distribution. Normal mixture distributions have quite flexible structures that can accommodate skewness and kurtosis. The conditional non-symmetries and fat tails are caused by the jumps in the asset values. Stochastic volatility (with or without jumps) alone does not cause conditional non-normalities.

We derive the instantaneous conditional moments for the changes in log asset value and variance processes in the Appendix B. They are given below.

$$\lim_{dt\to 0}\frac{1}{dt}E_t\left[dh_t\right] = \beta(\alpha - h_t) + \lambda_1\mu_2 \tag{6.36}$$

$$\lim_{dt\to 0}\frac{1}{dt}Var_t\left[dh_t\right] = \gamma^2 h_t + 2\mu_2^2\lambda_1 \tag{6.37}$$

$$\lim_{dt\to 0}\frac{1}{dt}E_t\left[d\nu\right] = \mu - \frac{1}{2}h_t + \lambda_1\left(\mu_1 + \rho_j\mu_2 - k\right) \tag{6.38}$$

$$\lim_{dt\to 0}\frac{1}{dt}Var_t\left[d\nu\right] = h_t + \left(\mu_1^2 + 2\mu_1\mu_2\rho_j + 2\rho_j\mu_2^2 + \sigma_1^2\right)\lambda_1 \tag{6.39}$$

$$\begin{aligned}
\lim_{dt\to 0}\frac{1}{dt}E_t\left[\left(d\nu - E_t\left[d\nu\right]\right)^3\right] &= \left[\mu_1^3 + 6\mu_1\mu_2^2\rho_j^2 + 2\mu_1\mu_2\rho_j\right. \\
&\quad +3\mu_1\sigma_1^2 + \mu_1^2\mu_2\rho_j + 6\mu_2^2\rho_j^3 \\
&\quad \left.+3\rho_j\mu_2\sigma_1^2\right]\lambda_1 \qquad (6.40)
\end{aligned}$$

$$\begin{aligned}
\lim_{dt\to 0}\frac{1}{dt}E_t\left[\left(d\nu - E_t\left[d\nu\right]\right)^4\right] &= \left[6\mu_1^2\sigma_1^2 + 4\mu_1^3\mu_2\rho_j + 4\mu_1^2\mu_2^2\rho_j\right. \\
&\quad +4\rho_j\mu_2^2\sigma_1^2 + 24\mu_1\mu_2^3\rho_j^2 + 8\mu_1^2\mu_2^2\rho_j^2 \\
&\quad +3\sigma_1^4 + \mu_1^4 + 24\mu_2^4\rho_j^2 + 8\mu_2^2\rho_j^2\sigma_1^2 \\
&\quad \left.+11\mu_1\mu_2\rho_j\sigma_1^2\right]\lambda_1 \qquad (6.41)
\end{aligned}$$

Therefore, with suitable choice of parameters we can have conditional skewness (i.e. non-symmetries) and conditional kurtosis (i.e. fat tails) in SVCJ-SIJ model.

Model Property 9: *Unconditional non-symmetries and fat tails*

Since we know the unconditional characteristic function for returns, in principle, we can find the density of returns by inverting the characteristic function:

$$f_R\left(x\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixt}\varphi dt \qquad (6.42)$$

Additionally, the moments can also be found by taking the derivatives of characteristic function:

$$\mu_m := E\left[R^m\right] = \left(-i\right)^r \frac{d^r}{dw^r}\varphi\left(w\right)|_{w=0} \qquad (6.43)$$

where μ_m denotes the m-th central moment.

We give three simulation examples for the unconditional density of returns in Figure 6.2. Note that for small time periods the model can accommodate skewness and kurtosis. However as the time period increases, by central limit theorem, the return distribution approaches to normal distribution.

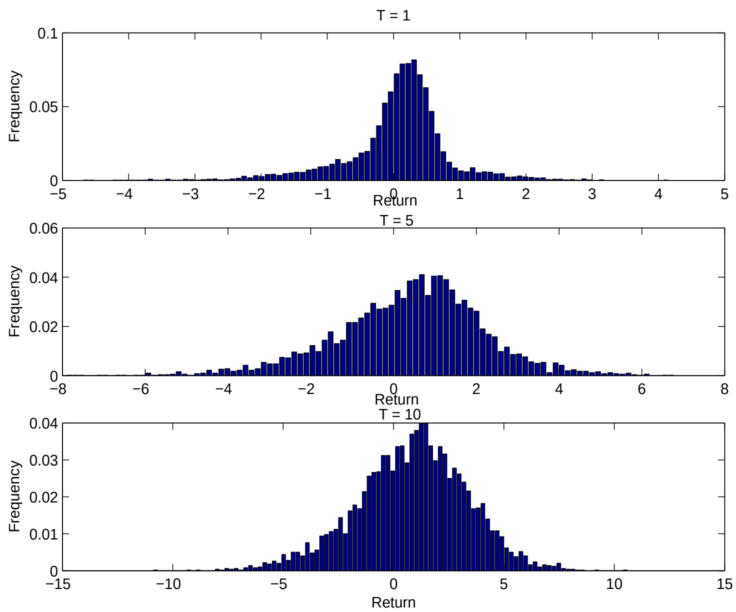

Figure 6.2: Examples for Unconditional Density of Returns

6.7 Probability of Default

The probability of default is defined as:

$$PD = Pr\left\{V_T \leq DP\right\} = Pr\left\{R_{t,T} \leq ln\left(DP\right) - \nu_t\right\}$$

Therefore PD can be found from the characteristic function by using the following inversion formula:

$$F_R\left(b\right) - F_R\left(a\right) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-iat} - e^{-ibt}}{it} \varphi\left(t\right) dt \tag{6.44}$$

Model Property 10: *Unpredictability of default events*

The main problem with the Gaussian model was the predictability of default because it has only drift and Gaussian parts. In SVCJ-SIJ model, the stochastic changes in asset value is driven by two different processes: a Gaussian part and a jump part. The Gaussian part evolves with small increments and captures *predicted* default events. But the jump part is composed of sudden changes and captures *unpredictable* default events. Because even if we have a high asset value relative to the default point (which means a low leverage), we may have a sudden default caused by the jump part.

Model Property 11: *Non-zero short-term default probabilities*

The natural result of default predictability in Gaussian model was very low (if not zero) short term PDs. In SVCJ-SIJ model, the existence of jumps avoids this problem. To gain an intuition, let us look at instantaneous conditional PD for a small time period, dt.

$$\begin{aligned} PD\left(t,t+dt\right) &= Pr\left\{ \left(\mu-\lambda_1 k-\frac{1}{2}h_t\right)dt+\sqrt{h_t}dW_t^1 \right. \\ &\quad \left. +Y^1 dN_t^1 < ln\left(DP\right)-\nu_t \right\} \end{aligned}$$

As $dt \to 0$, the drift term as well as Gaussian part gets smaller. Additionally for sufficiently small dt, having more than one jump has an ignorable probability. Therefore we can write the probability of having jumps in the time interval $(t, t+dt)$ as follows:

$$\begin{aligned} Pr\left\{\text{No jumps}\right\} &= 1-\lambda_1 dt+\mathcal{O}\left(dt\right) \\ Pr\left\{\text{One jump}\right\} &= \lambda_1 dt+\mathcal{O}\left(dt\right) \\ Pr\left\{\text{More than one jumps}\right\} &= \mathcal{O}\left(dt\right) \end{aligned}$$

where $\mathcal{O}\left(dt\right)$ is the asymptotic order symbol which is defined by $\psi\left(x\right) := \mathcal{O}\left(x\right)$, if $\lim_{x\to 0}\psi\left(x\right)/x = 0$.

And if we a jump in the small time interval $(t, t + dt)$, the change in the log asset value will be Y^1 which may take large values. Therefore, with suitable parameter specifications, even for very small dt, we may have probability of defaults that are not so small. This solves the zero short term PD problem that we encounter in Gaussian model.

Chapter 7

PRICING CREDIT RISK: SVCJ-SIJ MODEL

In this chapter we derive the pricing formula for a credit risky bond under our extended model. For this first we show that our model belongs to a wider class of stochastic models called *affine jump diffusion models*, and then by using the theorems on affine jump diffusion processes we derive pricing formula for the risky bond. The chapter concludes with a discussion of calibration problem.

7.1 Model

Since we can not observe the firm's asset value process, we can not simultaneously estimate/calibrate the model under actual and risk-neutral process. The only thing we can do is assuming a model under risk-neutral process. But, for justification only, we can assume certain functional forms for market prices of risk such that the asset value, variance and interest rate follow similar process under actual (P) and risk-neutral ($\tilde{P}$) processes.

For example, for our model under P:

$$\begin{aligned} d\nu_t &= (\mu - \lambda_1 k - \frac{1}{2}h_t)dt + \sqrt{h_t}dW_t^1 + Y^1 dN_t^1 \\ dh_t &= \beta(\alpha - h_t)dt + \gamma\sqrt{h_t}dW_t^2 + Y^2 dN_t^1 \\ dr_t &= \eta(\theta - r_t)dt + \delta\sqrt{r_t}dW_t^3 + Y^3 dN_t^2 \end{aligned}$$

the risk-neutral counterpart is:

$$\begin{aligned}
d\nu_t &= (\mu - \lambda_1 k - \frac{1}{2}h_t - \pi_\nu h_t - \pi_{J_1})dt + \sqrt{h_t}d\tilde{W}_t^1 + \tilde{Y^1}d\tilde{N}_t^1 \\
dh_t &= [\beta(\alpha - h_t) - \pi_h - \pi_{J_2}]\,dt + \gamma\sqrt{h_t}d\tilde{W}_t^2 + \tilde{Y^2}d\tilde{N}_t^2 \\
dr_t &= [\eta(\theta - r_t) - \pi_r - \pi_{J_3}]\,dt + \delta\sqrt{r_t}d\tilde{W}_t^3 + \tilde{Y^3}d\tilde{N}_t^3
\end{aligned}$$

where π's denote the market prices of risk:

π_ν : price premium

π_{J_1} : premium for jumps in log asset value

π_h : volatility premium

π_{J_2} : premium for jumps in volatility

π_r : interest rate premium

π_{J_3} : premium for jumps in interest rate

Now following [Bates96] and [Lin06], we assume certain functional forms for the market prices of risk:

$$\begin{aligned}
\pi_\nu &= \frac{\mu - r}{h} \\
\pi_{J_1} &= \tilde{\lambda_1}\tilde{k} - \lambda_1 k \\
\pi_h &= \pi_2 h \\
\pi_{J_2} &= \lambda_1 k^2 - \tilde{\lambda_1}\tilde{k^2} \\
\pi_r &= \pi_3 r \\
\pi_{J_3} &= \lambda_2 k^3 - \tilde{\lambda_2}\tilde{k^3}
\end{aligned}$$

where π_1, π_2, π_3 are constants, $\tilde{\lambda_1}, \tilde{\lambda_2}$ are risk neutral intensities and $\tilde{k}, \tilde{k^2}, \tilde{k^3}$ are risk neutral mean jump sizes. Then, under $\tilde{P}$, the model can be expressed as:

$$
\begin{aligned}
d\nu_t &= (\mu - \lambda_1 k - \frac{1}{2}h_t - \frac{\mu - r_t}{h_t}h_t + \lambda_1 k - \tilde{\lambda_1}\tilde{k})dt + \sqrt{h_t}d\tilde{W}_t^1 + \tilde{Y^1}d\tilde{N}_t^1 \\
&= (r_t - \tilde{\lambda_1}\tilde{k} - \frac{1}{2}h_t)dt + \sqrt{h_t}d\tilde{W}_t^1 + \tilde{Y^1}d\tilde{N}_t^1 \\
dh_t &= (\alpha\beta - \beta h_t - \pi_2 h_t - \lambda_1 k^2 + \tilde{\lambda_1}\tilde{k^2})dt + \gamma\sqrt{h_t}d\tilde{W}_t^2 + \tilde{Y^2}d\tilde{N}_t^2 \\
&= \tilde{\beta}(\tilde{\alpha} - h_t)dt + \gamma\sqrt{h_t}d\tilde{W}_t^2 + \tilde{Y^2}d\tilde{N}_t^2 \\
dr_t &= (\eta\theta - \eta r_t - \pi_3 r_t - \lambda_2 k^3 + \tilde{\lambda_2}\tilde{k^3})dt + \delta\sqrt{r_t}d\tilde{W}_t^3 + \tilde{Y^3}d\tilde{N}_t^3 \\
&= \tilde{\eta}(\tilde{\theta} - r_t)dt + \delta\sqrt{r_t}d\tilde{W}_t^3 + \tilde{Y^3}d\tilde{N}_t^3
\end{aligned}
$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\eta}$ and $\tilde{\theta}$ are newly-defined coefficients:

$$
\begin{aligned}
\tilde{\alpha} &:= \frac{\alpha\beta - \lambda_1 k^2 + \tilde{\lambda_1}\tilde{k^2}}{\tilde{\beta} + \pi_2} \\
\tilde{\beta} &:= \beta + \pi_2 \\
\tilde{\eta} &:= \eta + \pi_3 \\
\tilde{\theta} &:= \frac{\eta\theta + \tilde{\lambda_2}\tilde{k^3} - \lambda_2 k^3}{\eta + \pi_3}
\end{aligned}
$$

Therefore the models under P and $\tilde{P}$ have similar stochastic processes except that the drift term under P, μ, is replaced with r under $\tilde{P}$.

In order to ease notation, we drop ($\tilde{}$) in parameters. And under risk-neutral probability measure, our model is:

$$
\begin{aligned}
d\nu_t &= (r - \lambda_1 k - \frac{1}{2}h_t)dt + \sqrt{h_t}dW_t^1 + Y^1 dN_t^1 && (7.1) \\
dh_t &= \beta(\alpha - h_t)dt + \gamma\sqrt{h_t}dW_t^2 + Y^2 dN_t^1 && (7.2) \\
dr_t &= \eta(\theta - r_t)dt + \delta\sqrt{r_t}dW_t^3 + Y^3 dN_t^2 && (7.3)
\end{aligned}
$$

where:

- $\beta, \alpha, \eta, \theta \in \mathbb{R}$ and $\gamma, \delta \in \mathbb{R}^+$
- $k = E\left[e^{Y^1} - 1\right]$

- W^1, W^2, W^3 are standard Brownian Motions under $\tilde{P}$ with covariance matrix:

$$\sum = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- N_t^1 and N_t^2 are Poisson processes under $\tilde{P}$ with intensities λ_1 and λ_2

- The jump size distributions are:

$$\left.\begin{array}{c} (Y^1|Y^2 = y^2) \sim N(\mu_1 + \rho_j y_2, \sigma_1^2) \\ \mathrm{Y}^2 \sim exp(\mu_2) \\ \mathrm{Y}^3 \sim exp(\mu_3) \end{array}\right\} \textit{all under } \tilde{P}$$

- There is no interdependence between (W^1, W^2, W^3) and (Y^1, Y^2, Y^3) and (N^1, N^2)

Since, in our extended model, the number of sources of randomness is greater than the number of risky assets, the market is arbitrage free but not complete (see [Bjo04]). Therefore there is more than one pricing probability measure. However, in this chapter, we directly built our model under the risk-neutral probability measure, and we do not interested in choosing a risk neutral probability measure among others.

7.2 Affine Jump Diffusion Models

In this section we first define affine jump diffusion models (AJD) and show that our SVCJ-SIC model is a special case of AJD processes. And then we give an important theorem on Fourier-Stieltjes transforms of AJDs which we will use in pricing credit risky bonds. For a general discussion on AJDs and their application in finance see [DFS] and [DPS00].

7.2.1 Definition

First we fix a probability space (Ω, F, P) and an information filtration $(F_t) = \{F_t : t \geq 0\}$, suppose that X_t is a Markov process in some state space $D \in \mathbb{R}^n$, following the stochastic differential equation:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + dZ_t$$

where W_t is an (F_t)-standard Brownian motion in $\mathbb{R}^n$, $\mu(.) : D \to \mathbb{R}^n$ and $\sigma(.) : D \to \mathbb{R}^n$ are respectively the drift function and diffusion function, and Z is a pure jump process whose jumps has a fixed probability distribution on $\mathbb{R}^n$ and arrive with intensity $\{\lambda(X_t) : t \geq 0\}$ for some $\lambda(.) : D \to [0, \infty)$. X_t is called a continuous *jump diffusion (JD) process.* JD processes are widely used in finance literature to model the dynamics of asset prices, interest and exchange rates, etc. Intuitively, the drift term $\mu(.)$ represents an instantaneous deterministic time trend of the process, the diffusion term $\sigma(.)\,\sigma(.)^T$ represents an instantaneous volatility of the process when no jumps occur, and the jump term Z_t captures the discontinuous change of the sampling path with both random arrival of jumps and random jump sizes.

Now assume that drift and diffusion functions and the jump intensity have *affine structure*, i.e.:

$$\begin{aligned}
\mu(X_t) &= K_0 + K_1 X_t \\
\left[\sigma(X_t)\,\sigma(X_t)^T\right]_{ij} &= [H_0]_{ij} + [H_1]_{ij}\,X_t \\
\lambda(X_t) &= \ell_0 + \ell_1 X_t
\end{aligned}$$

where $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$, $H = (H_0, H_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}$ and $\ell = (\ell_0, \ell_1) \in \mathbb{R} \times \mathbb{R}^n$. Then the process X_t is called an *affine jump diffusion process.*

Note that our SVCJ-SIJ model is an example of AJD processes. In our model the state vector is (ν, h, r). And we can express our model as an affine structure on the state vector:

$$\begin{bmatrix} d\nu \\ dh \\ dr \end{bmatrix} = \left(\begin{bmatrix} -\lambda_1 k \\ \alpha\beta \\ \eta\theta \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ 0 & -\beta & 0 \\ 0 & 0 & -\eta \end{bmatrix} \begin{bmatrix} \nu \\ h \\ r \end{bmatrix} \right) \begin{bmatrix} dt \\ dt \\ dt \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{h} & 0 & 0 \\ 0 & \gamma\sqrt{h} & 0 \\ 0 & 0 & \delta\sqrt{r} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{bmatrix} + \begin{bmatrix} Y^1 dN_t^1 \\ Y^2 dN_t^1 \\ Y^3 dN_t^2 \end{bmatrix}$$

7.2.2 Theorems

[DPS00] showed that for the function $G\left(.;a,b,X_t,T\right):\mathbb{R}\rightarrow\mathbb{R}^+$, defined as:

$$G\left(y;a,b,X_t,T\right) := \tilde{E}\left[exp\left\{-\int_t^T r_u du\right\} e^{aX_T} 1_{\{bX_t \leq y\}} | X_t\right] \quad (7.4)$$

where X is the state vector which follows an AJD, r is an affine function of X, $a,b\in\mathbb{R}^n$ and $y\in\mathbb{R}$; we can define the Fourier-Stieltjes transform of G as:

$$\begin{aligned} \mathcal{G}(w;a,b,X_t,T) &= \int_{\mathbb{R}} e^{iwy} dG(y;a,b,X_t,T) \quad (7.5) \\ &= \tilde{E}\left[exp(-\int_t^T r_u du) e^{(a+iwb)X_T} | X_t\right] \\ &=: \psi(a+iwb, X_t, t, T) \end{aligned}$$

where:

$$\psi(a,X_t,t,T) := \tilde{E}\left[exp\left\{-\int_t^T r_u du\right\} e^{aX_t} | X_t\right] \quad (7.6)$$

Additionally we can find G by inverting $\mathcal{G}$:

$$\begin{aligned} G(y;a,b,X_t,T) &= \frac{\psi(a,X_t,t,T)}{2} \\ &\quad -\frac{1}{\pi}\int_0^\infty \frac{Im\left[\psi(a+iwb,X_t,t,T)e^{-iwy}\right]}{w} dw \quad (7.7) \end{aligned}$$

where $Im(c)$ denotes the imaginary part of $c \in \mathbb{C}$. For proof, see Appendix in [DPS00].

Indeed the function G has a flexible form that can be used in many applications. For instance,

- the function in its original form defines the price of a security that pays a conditional payoff e^{aX_T} if the event $\{bX_T \leq y\}$ occurs (e.g. options).
- with $b = y = 0$, the function defines the price of a security that pays an unconditional payoff e^{aX_T} (e.g. futures).
- with $a = b = y = 0$, the function defines the risk-free bond price.
- with $a = iw,\ b = y = r = 0$, the function defines the characteristic function of X.
- with $a = w,\ b = y = r = 0$, the function defines the moment generating function of X.

7.3 Fourier-Stieltjes Transform

First note that, for state vector $(\nu, h, r)^T$, the functional forms for G and $\mathcal{G}$ are:

$$G((y_1, y_2, y_3); (a_1, a_2, a_3), (b_1, b_2, b_3), (\nu_t, h_t, r_t), (T, T, T))$$

$$= \tilde{E}\left[exp(-\int_t^T r_u du) exp\left\{a_1 \nu_T + a_2 h_T + a_3 r_T\right\}\right.$$

$$\left. 1_{\{\{b_1 \nu_T \leq y_1\} \cap \{b_2 h_T \leq y_2\} \cap \{b_3 r_T \leq y_3\}\}} | \nu_t, h_t, r_t \right]$$

and

$$\mathcal{G}((w_1, w_2, w_3); (a_1, a_2, a_3), (b_1, b_2, b_3), (\nu_t, h_t, r_t), (T, T, T))$$

$$= \psi((a_1 + iw_1 b_1, a_2 + iw_2 b_2, a_3 + iw_3 b_3), (\nu_t, h_t, r_t), (t, t, t), (T, T, T))$$

$$= \tilde{E}\left[exp(-\int_t^T r_u du)\right.$$

$$\left. exp\left\{(a_1 + iw_1 b_1)\nu_T + (a_2 + iw_2 b_2)h_T + (a_3 + iw_3 b_3)r_T\right\} | \nu_t, h_t, r_t\right]$$

Then, similar with the previous chapter, we guess the functional form of ψ as:

$$\begin{aligned} \psi\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} \nu \\ h \\ r \end{bmatrix}, \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \begin{bmatrix} T \\ T \\ T \end{bmatrix}\right) &= exp\left\{iu_1\nu_t + A(\tau; u_1, u_2, u_3)\right. \\ &\quad +B(\tau; u_1, u_2, u_3)h_t + C(\tau; u_1, u_2, u_3)r_t \\ &\quad +J^1(\tau; u_1, u_2, u_3) \\ &\quad \left.+J^2(\tau; u_1, u_2, u_3)\right\} \end{aligned} \tag{7.8}$$

with $\tau = T - t$.

Since ψ is a conditional expectation, by using the martingale property of conditional expectations, we have E[$d\psi$]=0. When we apply Ito to ψ:

$$\begin{aligned} d\psi &= \psi_t dt + \psi_\nu d\nu + \psi_h dh + \psi_r dr + \frac{1}{2}(\psi_{\nu\nu} d\nu d\nu + \psi_{hh} dhdh + \psi_{rr} drdr) \\ &\quad +\psi_{\nu h} d\nu dh + \psi_{\nu r} d\nu dr + \psi_{hr} dhdr \end{aligned}$$

First note that since r_t is independent of ν_t and h_t we have:

$$d\nu dr = dhdr = 0$$

Additionally, since the jump parts are independent from the diffusion parts, we can express the change in ψ in two parts:

$$\text{Total change} = \text{Change without jumps} + \text{Change because of jumps}$$

When we input the differentials into the equation, we have:

$$\begin{aligned}
E\left[d\psi\right] &= E\Bigg[\psi_t dt + \psi_\nu \left\{\left(r - \lambda_1 k - \frac{1}{2}h\right)dt + \sqrt{h}dW^1\right\} \\
&\quad + \psi_h \left\{\beta(\alpha - h)dt + \gamma\sqrt{h}dW^2\right\} + \psi_r \left\{\eta(\theta - r)dt + \delta\sqrt{r}dW^3\right\} \\
&\quad + \frac{1}{2}\psi_{\nu\nu} hdt + \frac{1}{2}\psi_{hh}\gamma^2 hdt + \frac{1}{2}\psi_{rr}\delta^2 rdt + \psi_{\nu h}\gamma h\rho dt\Bigg] \\
&\quad + \lambda_1 E_t\left[\psi(\nu + Y^1, h + Y^2, r) - \psi(\nu, h, r)\right] \\
&\quad + \lambda_2 E_t\left[\psi(\nu, h, r + Y^3) - \psi(\nu, h, r)\right] \qquad (7.9)
\end{aligned}$$

Since W^1 and W^2 are martingale, we have $dW^1 = dW^2 = 0$. Thus, in order to obtain $E\left[d\psi\right] = 0$, we should have:

$$\begin{aligned}
0 &= \psi_t + \psi_\nu \left(r - \lambda_1 k - \frac{1}{2}h\right) + \psi_h \beta(\alpha - h) + \psi_r \eta(\theta - r) \\
&\quad + \frac{1}{2}\psi_{\nu\nu} h + \frac{1}{2}\psi_{hh}\gamma^2 h + \frac{1}{2}\psi_{rr}\delta^2 r + \psi_{\nu h}\gamma h\rho \\
&\quad + \lambda_1 E_t\left[\psi(\nu + Y^1, h + Y^2, r) - \psi(\nu, h, r)\right] \\
&\quad + \lambda_2 E_t\left[\psi(\nu, h, r + Y^3) - \psi(\nu, h, r)\right]
\end{aligned}$$

Additionally, by using the functional form of ψ, we can find the differentials :

$$
\begin{aligned}
\psi_t &= -\left[\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \tau}h + \frac{\partial C}{\partial \tau}r + \frac{\partial J^1}{\partial \tau} + \frac{\partial J^2}{\partial \tau}\right]\psi \\
\psi_\nu &= iu_1\psi \\
\psi_h &= B\psi \\
\psi_r &= C\psi \\
\psi_{\nu\nu} &= i^2u_1^2\psi \\
\psi_{hh} &= B^2\psi \\
\psi_{rr} &= C^2\psi \\
\psi_{\nu h} &= iu_1B\psi
\end{aligned}
$$

For the last two terms in $E\left[d\psi\right]$, define jump transforms Θ^1 and Θ^2 as:

$$
\Theta^1(c_1, c_2) := E\left[e^{c_1Y^1+c_2Y^2}|\nu_t, h_t, r_t\right] \tag{7.10}
$$

$$
\Theta^2(c) := E\left[e^{cY^3}|\nu_t, h_t, r_t\right] \tag{7.11}
$$

then:

$$
\begin{aligned}
E_t\left[\psi(\nu+Y^1, h+Y^2, r) - \psi(\nu, h, r)\right] &= E_t\left[exp\left\{iu_1(\nu+Y^1) + A\right.\right. \\
&\quad \left.+B(h+Y^2) + Cr + J^1 + J^2\right\} \\
&\quad -exp\left\{iu_1\nu + A + Bh + Cr\right. \\
&\quad \left.\left.+J^1 + J^2\right\}\right] \\
&= exp\left\{iu_1\nu + A + Bh + Cr + J^1 + J^2\right\} \\
&\quad E_t\left[exp\left\{iu_1Y^1 + BY^2\right\} - 1\right] \\
&= \psi(\nu, h, r)\left[\Theta^1(iu_1, B) - 1\right]
\end{aligned}
$$

and:

$$
\begin{aligned}
E_t\left[\psi(\nu,h,r+Y^3)-\psi(\nu,h,r)\right] &= E_t\left[exp\left\{iu_1\nu+A+Bh+C(r+Y^3)\right.\right.\\
&\quad \left.+J^1+J^2\right\}\\
&\quad \left.-exp\left\{iu_1\nu+A+Bh+Cr+J^1+J^2\right\}\right]\\
&= exp\left\{iu_1\nu+A+Bh+Cr+J^1+J^2\right\}\\
&\quad E_t\left[exp\left\{CY^3\right\}-1\right]\\
&= \psi(\nu,h,r)\left[\Theta^2(C)-1\right]
\end{aligned}
$$

From the previous chapter, we already know that:

$$
\begin{aligned}
\Theta^1(c_1,c_2) &= E\left[e^{c_1Y^1+c_2Y^2}|\nu_t,h_t,r_t\right]=E\left[e^{c_1Y^1+c_2Y^2}\right]\\
&= \frac{exp\left\{\mu_1c_1+\frac{1}{2}c_1^2\sigma_1^2\right\}}{1-\rho_jc_1\mu_2-c_2\mu_2}
\end{aligned}
$$

Additionally, by using the moment generating function of exponential distribution.

$$
\begin{aligned}
\Theta^2(c) &= E\left[e^{cY^3}|\nu_t,h_t,r_t\right]=E\left[e^{cY^3}\right]\\
&= \frac{1}{1-c\mu_3}
\end{aligned}
$$

We put the differentials:

$$
\left\{
\begin{array}{c}
-\frac{\partial A}{\partial\tau}-\frac{\partial B}{\partial\tau}h-\frac{\partial C}{\partial\tau}r-\frac{\partial J^1}{\partial\tau}-\frac{\partial J^2}{\partial\tau}+iu_1(r-\lambda_1k-\frac{1}{2}h)\\
\\
+B\beta(\alpha-h)+C\eta(\theta-r)+\frac{1}{2}i^2u_1^2h+\frac{1}{2}B^2\gamma^2h+\frac{1}{2}C^2\delta^2r\\
\\
+iu_1B\gamma h\rho+\lambda_1\left[\Theta^1(iu_1,B)-1\right]+\lambda_2\left[\Theta^2(C)-1\right]
\end{array}
\right\}\psi=0
$$

Then when we group the terms with h, r, and terms related to diffusion and jump parts, we have:

$$
\begin{aligned}
0 \;=\; & \left[-\frac{\partial A}{\partial \tau} + \alpha\beta B + \eta\theta C\right] \\
& + h\left[-\frac{\partial B}{\partial \tau} + \frac{1}{2} i u_1 (i u_1 - 1) + B(i u_1 \gamma \rho - \beta) + \frac{1}{2}\gamma^2 B^2\right] \\
& + r\left[-\frac{\partial C}{\partial \tau} + i u_1 - \eta C + \frac{1}{2}\delta^2 C^2\right] \\
& + \left[-\frac{\partial J^1}{\partial \tau} - i u_1 \lambda_1 k + \lambda_1 \left[\Theta^1(i u_1, B) - 1\right]\right] \\
& + \left[-\frac{\partial J^2}{\partial \tau} + \lambda_2 \left[\Theta^2(C) - 1\right]\right]
\end{aligned}
$$

When we equate each group in brackets to zero, we obtain a system of complex-valued ODEs:

$$\frac{\partial A}{\partial \tau} = \alpha\beta B + \eta\theta C \quad (7.12)$$

$$\frac{\partial B}{\partial \tau} = \frac{1}{2} i u_1 (i u_1 - 1) + B(i u_1 \gamma \rho - \beta) + \frac{1}{2}\gamma^2 B^2 \quad (7.13)$$

$$\frac{\partial C}{\partial \tau} = i u_1 - \eta C + \frac{1}{2}\delta^2 C^2 \quad (7.14)$$

$$\frac{\partial J^1}{\partial \tau} = -i u_1 \lambda_1 k + \lambda_1 \left[\Theta^1(i u_1, B) - 1\right] \quad (7.15)$$

$$\frac{\partial J^2}{\partial \tau} = \lambda_2 \left[\Theta^2(C) - 1\right] \quad (7.16)$$

The boundary conditions are:

$$A(0; u_1, u_2, u_3) = 0 \quad (7.17)$$

$$B(0; u_1, u_2, u_3) = i u_2 \quad (7.18)$$

$$C(0; u_1, u_2, u_3) = i u_3 \quad (7.19)$$

$$J^1(0; u_1, u_2, u_3) = 0 \quad (7.20)$$

$$J^2(0; u_1, u_2, u_3) = 0 \quad (7.21)$$

so that:

$$\begin{aligned}\psi(u,X,T,T) &= \tilde{E}\left[exp(-\int_T^T r_u du)e^{uX_T}|F_T\right] \\ &= e^{uX_T} = exp\left\{iu_1\nu_T + iu_2h_T + iu_3r_T\right\} \qquad (7.22)\end{aligned}$$

7.4 Solutions

The solutions for the complex valued ODEs are given below. The details of the solutions are given in the Appendix C.

$$A\left(\tau;u_1,u_2,u_3\right) = \left[-\frac{\alpha\beta g_1}{\gamma^2} - \frac{\eta\theta h_1}{\delta^2}\right]\tau \qquad (7.23)$$

$$+\frac{\alpha\beta}{\gamma^2}\, ln\left(\frac{1+g_3^2}{1+g_4^2}\right) + \frac{\eta\theta}{\delta^2}\, ln\left(\frac{1+h_3^2}{1+h_4^2}\right) \qquad (7.24)$$

$$B\left(\tau;u_1,u_2,u_3\right) = \frac{g_2g_3-g_1}{\gamma^2} \qquad (7.25)$$

$$C\left(\tau;u_1,u_2,u_3\right) = \frac{h_2h_3-h_1}{\delta^2} \qquad (7.26)$$

$$\begin{aligned}J^1\left(\tau;u_1,u_2,u_3\right) &= \left[g_8-\lambda_1+\frac{g_5g_7}{g_5^2+g_6^2}\right]\tau \\ &+\frac{2g_6g_7}{g_2\left(g_5^2+g_6^2\right)}\, ln\left(\frac{\left(g_6g_4-g_5\right)\sqrt{1+g_3^2}}{\left(g_6g_3-g_5\right)\sqrt{1+g_4^2}}\right) \qquad (7.27)\end{aligned}$$

$$\begin{aligned}J^2\left(\tau;u_1,u_2,u_3\right) &= \left[\frac{h_5h_7}{h_5^2+h_6^2}-h_7\right]\tau \\ &+\frac{2h_6h_7}{h_2\left(h_5^2+h_6^2\right)}\, ln\left(\frac{\left(h_4h_6-h_5\right)\sqrt{1+h_3^2}}{\left(h_3h_6-h_5\right)\sqrt{1+h_4^2}}\right) \qquad (7.28)\end{aligned}$$

where:

$$g_1(u_1,u_2,u_3) := iu_1\gamma\rho-\beta \quad (7.29)$$

$$g_2(u_1,u_2,u_3) := \sqrt{iu_1(iu_1-1)\gamma^2-g_1^2} \quad (7.30)$$

$$g_3(u_1,u_2,u_3) := tan\left[\frac{1}{2}g_2\tau+arctan(g_4)\right] \quad (7.31)$$

$$g_4(u_1,u_2,u_3) := \frac{g_1+iu_2\gamma^2}{g_2} \quad (7.32)$$

$$g_5(u_1,u_2,u_3) := 1-\rho_j iu_1\mu_2+\frac{\mu_2 g_1}{\gamma^2} \quad (7.33)$$

$$g_6(u_1,u_2,u_3) := \frac{\mu_2 g_2}{\gamma^2} \quad (7.34)$$

$$g_7(u_1,u_2,u_3) := \lambda_1 exp\left(\mu_1 iu_1-\frac{1}{2}u_1^2\sigma_1^2\right) \quad (7.35)$$

$$g_8(u_1,u_2,u_3) := -iu_1\lambda_1\frac{exp\left(\mu_1+\frac{1}{2}\sigma_1^2\right)+\rho_j\mu_2-1}{1-\rho_j\mu_2} \quad (7.36)$$

$$h_1(u_1,u_2,u_3) := -\eta \quad (7.37)$$

$$h_2(u_1,u_2,u_3) := \sqrt{2iu_1\delta^2-h_1^2} \quad (7.38)$$

$$h_3(u_1,u_2,u_3) := tan\left[\frac{1}{2}h_2\tau+arctan(h_4)\right] \quad (7.39)$$

$$h_4(u_1,u_2,u_3) := \frac{h_1+iu_3\delta^2}{h_2} \quad (7.40)$$

$$h_5(u_1,u_2,u_3) := 1+\frac{h_1\mu_3}{\delta^2} \quad (7.41)$$

$$h_6(u_1,u_2,u_3) := \frac{\mu_3 h_2}{\delta^2} \quad (7.42)$$

$$h_7(u_1,u_2,u_3) := \lambda_2 \quad (7.43)$$

7.5 Bond Pricing

The risky bond price is simply the expected payoff under $\tilde{P}$ discounted by stochastic risk-free rate process. Therefore:

$$
\begin{aligned}
D(t,T) &= \tilde{E}\left[exp(-\int_t^T r_u du)(L1_{\{NoDefault\}} + V_T 1_{\{Default\}})|F_t\right] \\
&= L\,\tilde{E}\left[exp(-\int_t^T r_u du)1_{\{\nu_T \geq \ln DP\}}|F_t\right] \\
&\quad +\tilde{E}\left[exp(-\int_t^T r_u du)e^{\nu_T}1_{\{\nu_T < \ln DP\}}|F_t\right]
\end{aligned}
$$

where $D(t,T)$ is the price of a zero-coupon risky bond, r_t is the short rate and L is the par value of risky bond.

We can express the bond price using function G:

$$
D(t,T) = L \times G\left(\begin{bmatrix} -\ln(DP) \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \nu \\ h \\ r \end{bmatrix}, \begin{bmatrix} T \\ T \\ T \end{bmatrix}\right)
$$

$$
+G\left(\begin{bmatrix} \ln(DP) \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \nu \\ h \\ r \end{bmatrix}, \begin{bmatrix} T \\ T \\ T \end{bmatrix}\right)
$$

with Fourier-Stieltjes transforms:

$$
\mathcal{G}_1 = \psi\left(\begin{bmatrix} -iu_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \nu \\ h \\ r \end{bmatrix}, \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \begin{bmatrix} T \\ T \\ T \end{bmatrix}\right)
$$

$$
\mathcal{G}_2 = \psi\left(\begin{bmatrix} 1+iu_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \nu \\ h \\ r \end{bmatrix}, \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \begin{bmatrix} T \\ T \\ T \end{bmatrix}\right)
$$

Price can be found by using the Fourier inversion methods explained in the previous sections. In Figure 7.1 we give an example of a path for log asset value, variance and interest rate.

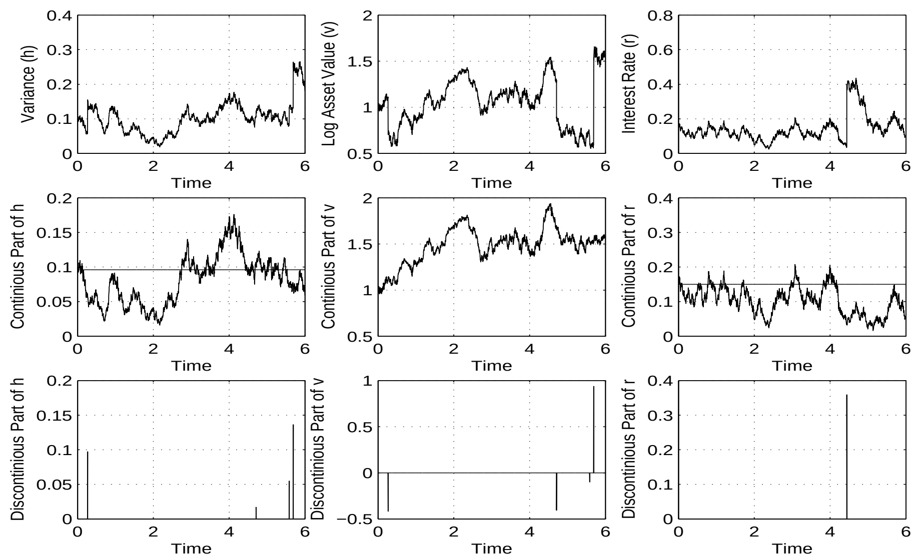

Figure 7.1: Simulation of SVCJ-SIJ Model under Risk Neutral Probability Measure

For SVCJ-SIJ model, we have two important property regarding bond prices.

Model Property 12: *Non-zero short-term credit spreads*

The unpredictability of default in SVCJ-SIJ model discussed in previous chapter results in non-zero short term PDs. And this yields non-zero credit spreads even for small maturities. This solves the problem of zero short term spreads in Gaussian model.

Model Property 13: *Implied volatility smiles*

SVCJ-SIJ model, with its large set of parameters can create quite flexible shapes for implied volatilities with respect to Gaussian model. This solves an important empirical problem. We give a simulation example. In the example, we first calculate risky bond prices by using SVCJ-SIJ model. And then we find the *implied* volatility that equates the price we found with the price calculated with Gaussian model. Given fixed value of parameters, the volatility smile is shown in Figure 7.2. The line in the figure represents a second-order polinomial fitted to the smile.

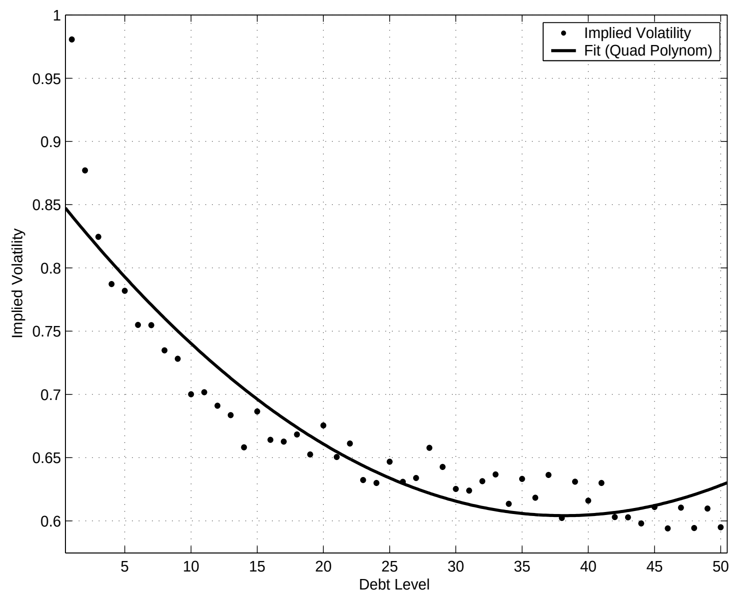

Figure 7.2: Implied Volatility Smile

Chapter 8

CONCLUSION

In this work we analyze the three fundamental research questions related to credit risk by using structural approach. These questions are modeling single firm credit risk, modeling portfolio credit risk and credit risk pricing. First we present the modeling framework for the three problem by using the assumption that firm value follows a geometric Brownian motion and interest rates are constant. Regarding the modeling of single firm credit risk, we present the asset value and return processes and analyze their pathwise and distributional properties. Additionally we discuss and derive the formulas for firm probability of default, expected loss given default, exposure at default, loss distribution and risk measures. For modeling portfolio credit risk, we extensively discuss the one-factor modeling and derive formulas for unconditional and conditional default probabilities, default correlation, conditional and unconditional portfolio loss distributions and risk measures. We also discuss two methods for estimation. Then we present approaches for pricing credit risky securities and discuss and derive formulas for pricing stocks, risky bonds and credit default swaps. We also discuss issues related to calibration.

Although the assumptions for Gaussian asset returns and constant interest rates yield tractable models, the assumptions are unrealistic. We discuss the potential problems caused by Gaussian model. The problems are classified as problems related to pathwise and distributional properties of asset values and returns, problems related to default predictability and short term default probabilities, and problems related to credit spreads and implied volatility smiles. Additionally we discuss the two main possible extensions to the basic model, namely non-constant volatility models and jump models.

After searching for an extended model, we conclude with a model called *stochastic volatility correlated jumps with stochastic interest rates with jumps (SVCJ-SIJ) model*. In the extended model, asset value, volatility and interest rates follow affine jump diffusion processes. In the model volatility is stochastic, asset value and volatility has correlated jumps and interest rates are stochastic and have jumps. Our extended model is a generalization of many popular models proposed in the literature, and by restricting certain parameters we may have restricted simpler models.

After introducing extended model, we analyze the modeling of single firm credit risk and credit risk pricing by using our extended model and show how our model can be used as a solution for the problems we encounter with simple models. We derive the conditional and unconditional characteristic function for returns which can be used in deriving density for returns as well as default probabilities. Additionally we derive formulas for instantaneous conditional moments for the model and propose simulation algorithms. We also discuss the statistical properties of asset values and returns as well as default probabilities. Regarding credit risk pricing, we derive the conditional Fourier-Stieltjes transforms for the bond pricing function, which can be inverted to find bond prices. Additionally we give a simulation example for the implied volatility smile created by the extended model. We discuss 13 important properties of the extended model, and conclude that our extended model can be a solution for the critical problems we encounter with the Gaussian case.

References

[AK96] Altman, E. and Kishore, V, Almost Everything You Wanted to Know about Recoveries on Defaulted Bonds, *Financial Analysts Journal*, November/December (1996), pp. 57-64.

[Alt68] Altman, E., Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy, *The Journal of Finance*, Vol. 23, No. 4 (Sep., 1968), pp. 589-609.

[ARS04] Altman, E., Resti, A. and Sironi, A., Default Recovery Rates in Credit Risk Modeling : a Review of the Literature and Empirical Evidence, *Economic Notes by Banca Monte dei Paschi di Siena SPA*, 33 (2) (2004), pp. 183-208.

[ARS02] Altman, E., Resti, A. and Sironi, A., The link between default and recovery rates : effects on the procyclicality of regulatory capital ratios, *BIS Working Papers*, 113 (2002).

[AS00] Anderson, R. and Sundaresan, S., A Comparative Study of Structural Models of Corporate Bond Yields, *Journal of Banking and Finance*, 24 (2000), pp. 255-269.

[ADEH99] Artzner P., Delbean F., Eber J.M. and Heath D., Coherent Measures of Risk, *Mathematical Finance*, 9, (1999), pp. 203-228.

[BR94] Ball, C. A. and Roma, A., Stochastic Volatility Option Pricing, *The Journal of Financial and Quantitative Analysis*, Vol 29, No. 4, (1994), pp 589-607.

[BCBS05] Basel Committee on Banking Supervision, An Explanatory Note on the Basel II IRB Risk Weight Functions, July (2005).

[BCBS06] Basel Committee on Banking Supervision, International Convergence of Capital Measurement and Capital Standards A Revised Framework Comprehensive Version, June (2006).

[Bates96] Bates, D. S., Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, *The Review of Financial Studies*, Vol. 9, No. 1., Spring (1996), pp. 69-107.

[BR02] Bielecki, R. and Rutkowski, M., *Credit Risk: Modeling, Valuation and Hedging*, Springer, (2002).

[Bjo04] Bjork, T., *Arbitrage Theory in Continuous Time*, Oxford, (2004).

[BC76] Black, F. and Cox, J.C., Valuing Corporate Securities: Some Effects of Bond Indenture Provisions, *Journal of Finance*, 31 (2), (1976), pp. 351-367.

[BS73] Black, F., and M. Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81 (1973), pp. 637-659.

[Bohn00] Bohn, J., A Survey of Contingent-Claims Approaches to Risky Debt Valuation, *The Journal of Risk Finance*, Spring (2000), pp. 53-69.

[Bre02] Brandimarte, p., *Numerical Methods in Finance: A Matlab-based Introduction*, Wiley and Sons, (2002).

[BV97] Briys, E. and de Varenne, F., Valuing Risky Fixed Rate Debt: An Extension, *Journal of Financial and Quantitative Analysis*, 32 (2) (1997), pp. 239-248.

[CT04] Cont, R and Tankov, C., *Financial Modelling With Jump Processes*, Chapman and Hall, (2004).

[Cont01] Cont, R., Empirical properties of asset returns: stylized facts and statistical issues , *Quantitative Finance*, 1 (2001), pp. 223-236.

[CS97] Credit Suisse, CreditRisk+: A Credit Risk Management Framework, Credit Suisse Financial Products (1997).

[CB03] Crosbie, P. and Bohn, J.R., Modeling Default Risk, Moody's KMV, December (2003).

[MIS02] Moody's Investor Service, Default & Recovery Rates of European Corporate Bond Issuers, 1985-2001, (2002).

[DS98] Duffie, D. and Singleton, K., Modeling Term Structures of Defaultable Bonds, Working Paper, Stanford University, 1998.

[DFS] Duffie, D., Filipovic, D. and Schachermayer, W., Affine Processes and Applications in Finance, Working Paper.

[DPS00] Duffie, D., Pan, J. and Singleton, K., Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica*, 68 (6) (2000), pp. 1343-1376.

[Eck07] Eckner, A., Computational Techniques for basic Affine Models of Portfolio Credit Risk, Working Paper, (2007).

[Eli05] Elizalde, A., Credit Risk Models II: Structural Models, Working Paper, November (2005).

[ET03] Emmer, S. and Tasche, D., Calculating credit risk capital charges with the one-factor model, Working Paper, September (2003).

[EHH03] Eom, Y. H., Helwege, J., and Huang, J. Z., Structural Models of Corporate Bond Pricing: An Empirical Analysis, *Review of Financial Studies*, 17 (2003), pp. 499-544.

[EJP02] Eraker, B., Johannes, M. and Polson, N., The Impact of Jumps in Volatility and Returns, Working Paper, May (2002).

[ER05] Ericsson, J. and Reneby, J., Estimating Structural Bond Pricing Models, *Journal of Business*, 78 (2005), pp. 707-736.

[FSS06] Fouque, J.P., Sircar, R. and Solna, K., Stochastic Volatility Effects on Defaultable Bonds, *Applied Mathematical Finance*, Vol 13, No:3, September, (2006), pp.215-244.

[FWZ06] Fouque, J.P., Wignall, B., and Zhou, X., Modeling Correlated Defaults: First Passage Model under Stochastic Volatility, Working Paper, July, (2006).

[Frye00a] Frye, J., Collateral Damage, *Risk*, April (2000), pp. 91-94.

[Frye00b] Frye, J., Depressing Recoveries, *Risk*, November (2000), pp. 108-111.

[Gor03] Gordy, M., A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules, *Journal of Financial Intermediation*, 12, July (2003), pp. 199-232.

[Gor04] Gordy, M., Granularity adjustment in portfolio credit risk measurement, *Risk Measures for the 21st Century* , John Wiley, (2004).

[GS05] Gupton, G. and Stein, R., LossCalc V2 : Dynamic Prediction of LGD, Moody's, (2005).

[GS02] Gupton, G. and Stein, R., LossCalc: Model for Predicting Loss Given Default (LGD), Moody's, (2002).

[GFB97] Gupton, G., Finger, C. and Bhatia, M., CreditMetrics - Technical Document, J.P.Morgan, (1997).

[HLR03] Hamerle, A., Liebig, T. and Rosch, D. Credit Risk Factor Modeling and the Basel II IRB Approach, Deutsche Bundesbank Discussion Paper, (2002).

[Hes93] Heston, S., A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies*, 6 (2) (1993), pp. 327-343.

[HW88] Hull, J. and White, A., The Pricing of Options on Assets with Stochastic Volatility, *Journal of Finance*, June(1988), pp.281-300.

[JT95] Jarrow, R.A. and Turnbull, S.M., Pricing Derivatives on Financial Securities Subject to Credit Risk, *Journal of Finance*, 50 (1) (1995), pp. 53-85.

[JLT97] Jarrow, R.A., Lando, D. and Turnbull, S., A Markov Model for the Term Structure of Credit Risk Spreads, *Review of Financial Studies*, 10 (2) (1997), pp. 481-523.

[JK00] Jiang, G. and Knight, J., Estimation of Continuous Time Processes via the Empirical Characteristic Function, Working Paper, August (2000).

[Jia02] Jiang, G., Testing Option Pricing Models with Stochastic Volatility, Random Jump and Stochastic Interest Rate , *International Review of Finance*, 3 (2002), pp. 233-272.

[KRS93] Kim, I.J., Ramaswamy, K. and Sundaresan, S., Does Default Risk in Coupons Affect the Valuation of Corporate Bonds?: A Contingent Claims Model, *Financial Management*, 1993, pp. 117-131.

[Kup07] Kupiec, P.H., A Generalized Single Common Factor Model of Portfolio Credit Risk, FDIC Center for Financial Research Working Paper Series, June, (2007).

[LL96] Lamberton, D. and Lapeyre, B., *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall, June (1996).

[LT96] Leland, H.E. and K.B. Toft, Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads, *Journal of Finance*, 51 (3) (1996), pp. 987-1019.

[Le94] Leland, H.E., Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance*, 49 (4) (1994), pp. 1213-1252.

[Lin06] Lin, Y.N., Pricing VIX Futures on Affine Stochastic Volatility Models with Simultaneous Sate-Dependent Jumps both in the S&P 100 Price and Variance Processes: Evidence from Integrated Physical and Risk-Neutral Probability Measures, Working Paper, December (2006).

[LLKM97] Liu, S., Lu, J. C. , Kolpin, D. W. and Meeker, W. Q., Analysis of Environmental Data With Censored Observations, *Environmental Science and Technology*, v. 31., (1997).

[LS95] Longstaff, F.A. and Schwartz, E.S., A Simple Approach to Valuing Risky Fixed and Floating Rate Debt, *Journal of Finance*, 50 (1995), pp. 789-820.

[MW02] Martin, R. and Wilde, T., Unsystematic credit risk, *Risk Magazine*, 15(11), (2002), pp 123-128.

[Mar04] Martin, R., *Credit Portfolio Modeling Handbook*, Credit Suisse First Buston, (2004).

[Mer74] Merton, R., On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance*, 29 (1974), p. 449-470.

[MR05] Musiela, M and Rutkowski, M., *Martingale Methods in Financial Modelling*, Springer, (2005).

[PG03] Poon, S.H. and Granger, C.W.J., Forecasting Volatility in Financial Markets: A Review, *Journal of Economic Literature*, Vol. 41, No. 2., June (2003), pp. 478-539.

[Pyk03] Pykhtin, M., Unexpected recovery risk, *Risk*, August,(2003), pp. 74-78.

[RS05] Rockinger, M. and Semenova, M., Estimation of Jump-Diffusion Processes via Empirical Characteristic Functions, Research Paper, FAME, June (2005).

[RV86] Ronn, E. I. and Verma, A. K., Pricing Risk-Adjusted Deposit Insurance: An Option Based Model, *Journal of Finance*, 41 (1986), pp. 871-895.

[Ross97] Ross, S., *Simulation*, Harcourt Academic Press, (1997).

[Rub81] Rubinstein, R., *Stimulation and the Monte Carlo Method*, John Wiley and Sons, (1981).

[Sco87] Scott, L. O., Option pricing when the variance changes randomly theory, estimation, and an application, *Journal of Financial and Quantitative Analysis*, (1987).

[STD93] Shimko, D., Tejima, N. and van Deventer, D., The Pricing of Risky Debt When Interest Rates are Stochastic, *Journal of Fixed Income*, 3, March 1993, pp. 58-65.

[Shr04] Shreve, S.E., *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, (2004).

[SS91] Stein, E. M. and Stein, J. C., Stock Price Distributions with Stochastic Volatility: An Analytic Approach, *The Review of Financial Studies*, Vol 4, No.4, (1991), pp. 727-752.

[Sun 01] Sundaram, R. K., The Merton/KMV Approach to Pricing Credit Risk, Working Paper, NYU, (2001).

[Tay04] Taylor, J., Review of Option Pricing under Stochastic Volatility and Levy Processes, ETH University of Zurich, (2004).

[TW04] Thomas, H. and Wang, Z., Interpreting the Internal Ratings-Based Capital Requirements in Basel II, Working Paper, September, (2004).

[Vas91] Vasicek, O., Limiting Loan Loss Probability Distribution, Moody's KMV, August (1991).

[Vas87] Vasicek, O., Probability of Loss on Loan Portfolio, Moody's KMV, February (1987).

[Weh03] Wehrspohn, U., Analytic Loss Distributions of Heterogeneous Portfolios in the Asset Value Credit Risk Model, Working Paper, (2003).

[Wig87] Wiggins, J. B., Option values under stochastic volatility, *Journal of Financial Economics*, 19 (1987), pp. 351-372.

[Wil03a] Wilde, T., IRB Approach Explained, Credit Suisse First Buston, (2003).

[Wil03b] Wilde, T., Probing Granularity, Credit Suisse First Buston, (2003).

[Wil01] Wilde, T., Probing granularity, *Risk Magazine*, 14(8), (2001), pp 103-106.

[Wil97a] Wilson, T., Portfolio credit risk I, *Risk*, 10 (9), September (1987).

[Wil97b] Wilson, T., Portfolio credit risk II, *Risk*, 10 (10), October (1987).

[Zhou97] Zhou, C, A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities, Federal Reserve Board, 1997.

Appendix A

Solutions for Characteristic Function

In this section we find the solution for the characteristic function for returns. The ODE for $B(\tau; u_1, u_2)$ is a Ricatti equation. First we solve this Ricatti equation. Then by using the solution for $B(\tau; u_1, u_2)$, we find $A(\tau; u_1, u_2)$ and $J^1(\tau; u_1, u_2)$.

Before going into the solutions, first define the following transforms:

$$
\begin{aligned}
g_1(u_1, u_2) &:= iu_1\gamma\rho - \beta \\
g_2(u_1, u_2) &:= \sqrt{iu_1(iu_1 - 1)\gamma^2 - g_1^2} \\
g_3(u_1, u_2) &:= \tan\left[\frac{1}{2}g_2\tau + \arctan(g_4)\right] \\
g_4(u_1, u_2) &:= \frac{g_1 + iu_2\gamma^2}{g_2} \\
g_5(u_1, u_2) &:= 1 - \rho_j iu_1\mu_2 + \frac{\mu_2 g_1}{\gamma^2} \\
g_6(u_1, u_2) &:= \frac{\mu_2 g_2}{\gamma^2} \\
g_7(u_1, u_2) &:= \lambda_1 exp\left(\mu_1 iu_1 - \frac{1}{2}u_1^2\sigma_1^2\right) \\
g_8(u_1, u_2) &:= -iu_1\lambda_1\frac{exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right) + \rho_j\mu_2 - 1}{1 - \rho_j\mu_2}
\end{aligned}
$$

A.1 Solution for $B(\tau; u_1, u_2)$

The ODE for B is:

$$\frac{\partial B}{\partial \tau} = Q_1 + BQ_2 + B^2 Q_3$$

where:

$$\begin{aligned} Q_1 &:= \frac{1}{2} i u_1 (i u_1 - 1) \\ Q_2 &:= i u_1 \gamma \rho - \beta \\ Q_3 &:= \frac{1}{2} \gamma^2 \end{aligned}$$

with the boundary condition:

$$B(0; u_1, u_2) = i u_2$$

$$\begin{aligned} \frac{dB}{d\tau} &= Q_1 + BQ_2 + B^2 Q_3 \\ d\tau &= \frac{dB}{Q_1 + BQ_2 + B^2 Q_3} \\ \int d\tau &= \tau = \int \frac{dB}{Q_1 + BQ_2 + B^2 Q_3} \end{aligned}$$

Note that, since we have more than one integral, it does not matter where we put a constant in the equality. Now define the following transforms:

$$\begin{aligned} K &:= \frac{\sqrt{4Q_1 Q_3 - Q_2^2}}{2\sqrt{Q_3}} \\ X &:= \frac{2Q_3 B + Q_2}{2\sqrt{Q_3}} \end{aligned}$$

Then we have:

$$\begin{aligned} K^2 + X^2 &= \frac{4Q_1Q_3 - Q_2^2}{4Q_3} + \frac{4Q_3^2B^2 + Q_2^2 + 4Q_2Q_3B}{4Q_3} \\ &= Q_1 - \frac{Q_2^2}{4Q_3} + Q_3B^2 + \frac{Q_2^2}{4Q_3} + Q_2B \\ &= Q_1 + Q_2B + Q_3B^2 \end{aligned}$$

Additionally we have:

$$\begin{aligned} dX &= d\left(\frac{2Q_3B + Q_2}{2\sqrt{Q_3}}\right) = \sqrt{Q_3}dB \\ dB &= \frac{dX}{\sqrt{Q_3}} \end{aligned}$$

Therefore we have:

$$\begin{aligned} \int \frac{dB}{Q_1 + BQ_2 + B^2Q_3} &= \int \frac{1}{\sqrt{Q_3}\,(K^2 + X^2)}dX \\ &= \frac{1}{\sqrt{Q_3}} \int \frac{1}{K^2 + X^2}dX = \tau \\ \int \frac{K}{K^2 + X^2}dX &= \tau\sqrt{Q_3}K \end{aligned}$$

Since we know $\int \frac{a}{a^2+x^2}dx = \arctan\left(\frac{x}{a}\right) - \bar{c}$, we have:

$$\arctan\left(\frac{X}{K}\right) - \bar{c} = \tau\sqrt{Q_3}K$$

When we solve for X:

$$X = K\ \tan\left(K\tau\sqrt{Q_3} + \bar{c}\right)$$

Using the definition of X and K:

$$\frac{2Q_3B+Q_2}{2\sqrt{Q_3}} = \frac{\sqrt{4Q_1Q_3-Q_2^2}}{2\sqrt{Q_3}}\, tan\left(\frac{1}{2}\tau\sqrt{4Q_1Q_3-Q_2^2}+\bar{c}\right)$$
$$B = \frac{\sqrt{4Q_1Q_3-Q_2^2}\ \tan\left(\frac{1}{2}\tau\sqrt{4Q_1Q_3-Q_2^2}+\bar{c}\right)-Q_2}{2Q_3}$$

Now we can use the boundary condition, $B(0;u_1,u_2)=iu_2$, to find $\bar{c}$.

$$iu_2 = \frac{\sqrt{4Q_1Q_3-Q_2^2}\ \tan(\bar{c})-Q_2}{2Q_3}$$
$$\tan(\bar{c}) = \frac{2iu_2Q_3+Q_2}{\sqrt{4Q_1Q_3-Q_2^2}}$$
$$\bar{c} = \arctan\left(\frac{2iu_2Q_3+Q_2}{\sqrt{4Q_1Q_3-Q_2^2}}\right)$$

Now we can totally define B:

$$B=\frac{\sqrt{4Q_1Q_3-Q_2^2}\ \tan\left(\frac{1}{2}\tau\sqrt{4Q_1Q_3-Q_2^2}+\arctan\left(\frac{2iu_2Q_3+Q_2}{\sqrt{4Q_1Q_3-Q_2^2}}\right)\right)-Q_2}{2Q_3}$$

By using previously defined g_1, g_2, g_3,

$$B(\tau;u_1,u_2)=\frac{g_2g_3-g_1}{\gamma^2} \tag{A.1}$$

A.2 Solution for $A(\tau;u_1,u_2)$

We have:

$$\frac{\partial A}{\partial \tau}=iu_1\mu+\alpha\beta B=iu_1\mu+\alpha\beta\frac{g_2g_3-g_1}{\gamma^2}$$

Note that g_1 and g_2 does not depend on τ.

$$\begin{aligned}
dA &= \left(iu_1\mu - \frac{\alpha\beta}{\gamma^2}g_1\right)d\tau + \frac{\alpha\beta}{\gamma^2}g_2g_3d\tau \\
\int dA &= \int\left(iu_1\mu - \frac{\alpha\beta}{\gamma^2}g_1\right)d\tau + \int\frac{\alpha\beta}{\gamma^2}g_2g_3d\tau \\
A &= \left(iu_1\mu - \frac{\alpha\beta}{\gamma^2}g_1\right)\tau + \frac{\alpha\beta}{\gamma^2}g_2\int g_3d\tau
\end{aligned}$$

Therefore we should find $\int g_3d\tau$. For $x := \frac{1}{2}g_2$ and $y := \arctan(g_4)$, we have:

$$\int g_3d\tau = \int \tan(x\tau + y)\,d\tau$$

We know that:

$$\begin{aligned}
\frac{d}{d\tau}\frac{1}{2x}ln\left(1+\tan^2(x\tau+y)\right) &= \frac{1}{2x}\frac{2\tan(x\tau+y)\,x/\cos^2(x\tau+y)}{1+\tan^2(x\tau+y)} \\
&= \frac{\tan(x\tau+y)/\cos^2(x\tau+y)}{1/\cos^2(x\tau+y)} \\
&= \tan(x\tau+y)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\int g_3d\tau &= \int \tan(x\tau+y)\,d\tau \\
&= \frac{1}{2x}\left[ln\left(1+\tan^2(x\tau+y)\right)+\bar{c}\right] \\
&= \frac{1}{2}\frac{2}{g_2}\left[ln\left(1+g_3^2\right)+\bar{c}\right] \\
&= \frac{1}{g_2}\left[ln\left(1+g_3^2\right)+\bar{c}\right]
\end{aligned}$$

When we input:

$$\begin{aligned}
A &= \left(iu_1\mu - \frac{\alpha\beta}{\gamma^2}g_1\right)\tau + \frac{\alpha\beta}{\gamma^2}g_2\frac{1}{g_2}\left[ln\left(1+g_3^2\right)+\bar{c}\right] \\
&= \left(iu_1\mu - \frac{\alpha\beta}{\gamma^2}g_1\right)\tau + \frac{\alpha\beta}{\gamma^2}\left[ln\left(1+g_3^2\right)+\bar{c}\right]
\end{aligned}$$

Now by using the boundary condition, i.e. $A(0; u_1, u_2) = 0$, we will find $\bar{c}$.

$$\begin{aligned} 0 &= 0 + \frac{\alpha\beta}{\gamma^2} ln\left[1 + \tan^2\left(\arctan\left(g_4\right)\right)\right] + \frac{\alpha\beta}{\gamma^2}\bar{c} \\ &= \frac{\alpha\beta}{\gamma^2} ln\left[1 + g_4^2\right] + \frac{\alpha\beta}{\gamma^2}\bar{c} \\ \bar{c} &= -ln\left[1 + g_4^2\right] \end{aligned}$$

Now we can totally define $A(\tau; u_1, u_2)$.

$$\begin{aligned} A(\tau; u_1, u_2) &= \left(iu_1\mu - \frac{\alpha\beta}{\gamma^2} g_1\right)\tau + \frac{\alpha\beta}{\gamma^2}\left[ln\left(1 + g_3^2\right) - ln\left(1 + g_4^2\right)\right] \\ &= \left(iu_1\mu - \frac{\alpha\beta}{\gamma^2} g_1\right)\tau + \frac{\alpha\beta}{\gamma^2} ln\left(\frac{1 + g_3^2}{1 + g_4^2}\right) \end{aligned} \tag{A.2}$$

A.3 Solution for $J^1(\tau; u_1, u_2)$

By using the solution for B and our previous transformations g_1 to g_8, we can express the ODE for J^1 as follows:

$$\begin{aligned} \frac{\partial J^1}{\partial \tau} &= g_8 - \lambda_1 + \frac{g_7}{g_5 - g_6 g_3} \\ \int dJ^1 &= \int \left(g_8 - \lambda_1\right) d\tau + g_7 \int \frac{1}{g_5 - g_6 g_3} d\tau \\ J^1 &= \left(g_8 - \lambda_1\right)\tau + g_7 \int \frac{1}{g_5 - g_6 g_3} d\tau \end{aligned}$$

Since $g_3 = \tan\left(\frac{1}{2} g_2 \tau + \arctan\left(g_4\right)\right)$, we have:

$$\frac{dg_3}{d\tau} = \frac{1}{2} g_2 \left(1 + g_3^2\right) \Rightarrow d\tau = \frac{2}{g_2\left(1 + g_3^2\right)} dg_3$$

Therefore we should find:

$$\begin{aligned} \int \frac{1}{g_5 - g_6 g_3} d\tau &= \frac{2}{g_2} \int \frac{1}{\left(g_5 - g_6 g_3\right)\left(1 + g_3^2\right)} dg_3 \\ &= -\frac{2}{g_6 g_2} \int \frac{1}{\left(g_3 - \frac{g_5}{g_6}\right)\left(1 + g_3^2\right)} dg_3 \end{aligned}$$

For the integrand, we have:

$$\frac{1}{\left(g_3 - \frac{g_5}{g_6}\right)\left(1+g_3^2\right)} = \frac{g_6^2}{\left(g_5^2+g_6^2\right)\left(g_3 - \frac{g_5}{g_6}\right)} - \frac{g_6\left(g_5+g_6 g_3\right)}{\left(g_5^2+g_6^2\right)\left(1+g_3^2\right)}$$

Thus:

$$\int \frac{1}{g_5 - g_6 g_3} d\tau = -\frac{2g_6}{g_2\left(g_5^2+g_6^2\right)} \int \frac{1}{g_3 - \frac{g_5}{g_6}} dg_3 + \frac{2}{g_2\left(g_5^2+g_6^2\right)} \int \frac{g_5+g_6 g_3}{1+g_3^2} dg_3$$

Additionally we have:

$$\begin{aligned}
\int \frac{1}{g_3 - \frac{g_5}{g_6}} dg_3 &= ln\left(g_3 - \frac{g_5}{g_6}\right) \\
\int \frac{g_5+g_6 g_3}{1+g_3^2} dg_3 &= g_5 \ \arctan\left(g_3\right) + \frac{1}{2} g_6 \ ln\left(1+g_3^2\right) \\
&= \frac{1}{2} g_5 g_2 \tau + g_5 \ \arctan\left(g_4\right) + \frac{1}{2} g_6 \ ln\left(1+g_3^2\right)
\end{aligned}$$

We put a constant term at this stage.

$$\begin{aligned}
\int \frac{1}{g_5 - g_6 g_3} d\tau &= -\frac{2g_6}{g_2\left(g_5^2+g_6^2\right)} \ ln\left(g_3 - \frac{g_5}{g_6}\right) + \frac{2}{g_2\left(g_5^2+g_6^2\right)} \frac{1}{2} g_5 g_2 \tau \\
&\quad + \frac{2}{g_2\left(g_5^2+g_6^2\right)} g_5 \ \arctan\left(g_4\right) + \frac{2}{g_2\left(g_5^2+g_6^2\right)} \frac{g_6}{2} \ ln\left(1+g_3^2\right) \\
&\quad \frac{2\bar{c}}{g_2\left(g_5^2+g_6^2\right)} \\
&= \frac{2}{g_2\left(g_5^2+g_6^2\right)} \\
&\quad \left\{ - g_6 \ ln\left(g_3 - \frac{g_5}{g_6}\right) + \frac{g_5 g_2}{2} \tau + g_5 \ \arctan\left(g_4\right) \right. \\
&\quad \left. + \frac{g_6}{2} \ ln\left(1+g_3^2\right) + \bar{c} \right\}
\end{aligned}$$

Note that:

$$
\begin{aligned}
\frac{1}{2}ln\left(1+g_3^2\right)-ln\left(g_3-\frac{g_5}{g_6}\right) &= ln\left(\frac{\sqrt{1+g_3^2}}{g_3-\frac{g_5}{g_6}}\right)\\
&= ln\left(\frac{\sqrt{1+\frac{\sin^2\left(\frac{1}{2}g_2+\arctan(g_4)\right)}{\cos^2\left(\frac{1}{2}g_2+\arctan(g_4)\right)}}}{\frac{\sin\left(\frac{1}{2}g_2+\arctan(g_4)\right)}{\cos\left(\frac{1}{2}g_2+\arctan(g_4)\right)}-\frac{g_5}{g_6}}\right)\\
&= -ln\left(\sin\left(\frac{1}{2}g_2+\arctan\left(g_4\right)\right)\right.\\
&\quad \left.-\frac{g_5}{g_6}\cos\left(\frac{1}{2}g_2+\arctan\left(g_4\right)\right)\right)
\end{aligned}
$$

By using the definition of g_3:

$$
\begin{aligned}
\sin\left(\frac{1}{2}g_2+\arctan\left(g_4\right)\right) &= \frac{g_3}{\sqrt{1+g_3^2}}\\
\cos\left(\frac{1}{2}g_2+\arctan\left(g_4\right)\right) &= \frac{1}{\sqrt{1+g_3^2}}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&= -g_6 ln\left(\frac{g_3-\frac{g_5}{g_6}}{\sqrt{1+g_3^2}}\right)\\
&= -g_6 ln\left(\frac{g_6 g_3-g_5}{g_6\sqrt{1+g_3^2}}\right)
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\int \frac{1}{g_5-g_6 g_3}d\tau &= \frac{2}{g_2\left(g_5^2+g_6^2\right)}\\
&\quad \left\{\frac{g_5 g_2}{2}\tau+g_5\ \arctan\left(g_4\right)-g_6\ ln\left(\frac{g_6 g_3-g_5}{g_6\sqrt{1+g_3^2}}\right)+\bar{c}\right\}
\end{aligned}
$$

At the end we obtain the expression for J^1:

$$\begin{aligned} J^1 &= (g_8 - \lambda_1)\,\tau \\ &+ \frac{2g_7}{g_2\left(g_5^2+g_6^2\right)}\left\{\frac{g_5 g_2}{2}\tau + g_5\ \arctan\left(g_4\right) - g_6\ ln\left(\frac{g_6 g_3 - g_5}{g_6\sqrt{1+g_3^2}}\right) + \bar{c}\right\} \end{aligned}$$

By using the boundary condition $J^1\left(0; u_1, u_2\right) = 0$ and noting that for $\tau = 0$, $g_3 = g_4$, we have:

$$\begin{aligned} 0 &= \frac{2g_5 g_7}{g_2\left(g_5^2+g_6^2\right)}\ \arctan\left(g_4\right) - \frac{2g_6 g_7}{g_2\left(g_5^2+g_6^2\right)}\ ln\left(\frac{g_6 g_4 - g_5}{g_6\sqrt{1+g_4^2}}\right) + \frac{2g_7}{g_2\left(g_5^2+g_6^2\right)}\bar{c} \\ \bar{c} &= g_6\ ln\left(\frac{g_6 g_4 - g_5}{g_6\sqrt{1+g_4^2}}\right) - g_5\ \arctan\left(g_4\right) \end{aligned}$$

When we input $\bar{c}$:

$$\begin{aligned} J^1\left(\tau; u_1, u_2\right) &= (g_8 - \lambda_1)\,\tau \\ &+ \frac{2g_7}{g_2\left(g_5^2+g_6^2\right)}\left\{\frac{g_5 g_2}{2}\tau + g_6\ ln\left(\frac{g_6 g_4 - g_5}{g_6\sqrt{1+g_4^2}}\frac{g_6\sqrt{1+g_3^2}}{g_6 g_3 - g_5}\right)\right\} \\ &= \left(g_8 - \lambda_1 + \frac{g_5 g_7}{\left(g_5^2+g_6^2\right)}\right)\tau \\ &+ \frac{2g_6 g_7}{g_2\left(g_5^2+g_6^2\right)}\ ln\left(\frac{\left(g_6 g_4 - g_5\right)\sqrt{1+g_3^2}}{\left(g_6 g_3 - g_5\right)\sqrt{1+g_4^2}}\right) \end{aligned} \tag{A.3}$$

Appendix B

Instantaneous Conditional Moments

For a random variable, the first central moment, i.e. mean, and the moments around the mean are given by:

$$\begin{aligned}
\mu &:= E[X] \\
\sigma^2 &:= E\left[(X-\mu)^2\right] = E\left[X^2\right] + (E[X])^2 \\
s &:= \frac{E\left[(X-\mu)^3\right]}{\sigma^3} = \frac{E[X^3] - 3E[X^2]E[X] + 2(E[X])^3}{\sigma^3} \\
\kappa &:= \frac{E\left[(X-\mu)^4\right]}{\sigma^4} = \frac{E[X^4] - 4E[X^3]E[X] + 6E[X^2](E[X])^2 - 3(E[X])^4}{\sigma^4}
\end{aligned}$$

Thus:

$$\begin{aligned}
E[X] &= \mu \\
E\left[X^2\right] &= \sigma^2 + \mu^2 \\
E\left[X^3\right] &= s\sigma^3 + 3\mu\sigma^2 + \mu^3 \\
E\left[X^4\right] &= \kappa\sigma^4 + 4s\mu\sigma^3 + 6\mu^2\sigma^2 - 5\mu^4
\end{aligned}$$

Additionally for two independent random variables X and Y, we have:

$$
\begin{aligned}
E[X+Y] &= E[X]+E[Y] := \bar{\mu} \\
E\left[(X+Y-\bar{\mu})^2\right] &= E\left[X^2\right]+E\left[Y^2\right]-(E[X])^2-(E[X])^2 \\
E\left[(X+Y-\bar{\mu})^3\right] &= E\left[X^3\right]+E\left[Y^3\right]-3E\left[X^2\right]E[X]-3E\left[Y^2\right]E[Y] \\
&\quad +2(E[X])^3+2(E[Y])^3 \\
E\left[(X+Y-\bar{\mu})^4\right] &= E\left[X^4\right]+E\left[Y^4\right]-3(E[X])^4-3(E[Y])^4 \\
&\quad +6E\left[X^2\right]E\left[Y^2\right]-6E\left[X^2\right](E[Y])^2+6E\left[X^2\right](E[X])^2 \\
&\quad -6E\left[Y^2\right](E[X])^2+6(E[X])^2(E[Y])^2 \\
&\quad +6E\left[Y^2\right](E[Y])^2-4E\left[X^3\right]E[X]-4E\left[Y^3\right]E[Y]
\end{aligned}
$$

B.1 Instantaneous Conditional Moments for Variance Process

We want to find the first two conditional moments of the variance process which are valid *instantaneously*. Formally we want to find:

$$
\begin{aligned}
\text{Instantaneous Conditional Mean} &:= \lim_{dt\to 0}\frac{1}{dt}E\left[dh_t|F_t\right] \\
\text{Instantaneous Conditional Variance} &:= \lim_{dt\to 0}\frac{1}{dt}Var\left[dh_t|F_t\right]
\end{aligned}
$$

The variance process h_t satisfies the following SDE:

$$
dh_t = \underbrace{\beta(\alpha-h_t)dt}_{=:P}+\underbrace{\gamma\sqrt{h_t}dW_t^2}_{=:Q}+\underbrace{Y^2dN_t^1}_{=:R}
$$

Note that we have:

$$
\begin{aligned}
dW_t &\sim N(0,dt) \\
dN_t^1 &\sim Pois(\lambda_1 dt)
\end{aligned}
$$

Then:

$$
\begin{aligned}
E_t\left[P\right] &= \beta(\alpha - h_t)dt \\
E_t\left[Q\right] &= 0 \\
E_t\left[Q^2\right] &= \gamma^2 h_t dt \\
E_t\left[R\right] &= E\left[E\left[\sum_{i=0}^{N} Y_i^2 | N_{dt}^1 = N\right]\right] = \lambda_1 \mu_2 dt \\
E_t\left[R^2\right] &= E\left[\left(\sum_{i=0}^{N_{dt}^1} Y_i^2\right)^2\right] \\
&= E\left[E\left[\left(\sum_{i=0}^{N} Y_i^2\right)^2 | N_{dt}^1 = N\right]\right] \\
&= E\left[NE\left[\left(Y^2\right)^2\right] + N\left(N-1\right)\left(E\left[Y^2\right]\right)^2\right] \\
&= \mu_2^2\left[2\lambda_1 dt + \left(\lambda_1 dt\right)^2\right]
\end{aligned}
$$

Therefore instantaneous conditional mean and variance of the variance process are:

$$
\begin{aligned}
E\left[dh_t|F_t\right] &= E_t\left[P\right] + E_t\left[Q\right] + E_t\left[R\right] \\
&= \beta(\alpha - h_t)dt + 0 + \lambda_1 \mu_2 dt
\end{aligned}
$$

$$
\lim_{dt \to 0} \frac{1}{dt} E\left[dh_t|F_t\right] = \beta(\alpha - h_t) + \lambda_1 \mu_2 \tag{B.1}
$$

$$
\begin{aligned}
Var\left[dh_t|F_t\right] &= 0 + \gamma^2 h_t dt + \mu_2^2\left[2\lambda_1 dt + \left(\lambda_1 dt\right)^2\right] - \left(\lambda_1 \mu_2 dt\right)^2 \\
&= \gamma^2 h_t dt + 2\mu_2^2 \lambda_1 dt
\end{aligned}
$$

$$
\lim_{dt \to 0} \frac{1}{dt} Var\left[dh_t|F_t\right] = \gamma^2 h_t + 2\mu_2^2 \lambda_1 \tag{B.2}
$$

B.2 Instantaneous Conditional Moments for Log Asset Value Process

Before going forward to the moments of log asset value process, note that the conditional moments of the jump process as well as the jump sizes are given as follows (W represents a standard normal random variable):

$$\begin{aligned}
E\left[Y^2\right] &= \mu_2 \\
E\left[\left(Y^2\right)^2\right] &= 2\mu_2^2 \\
E\left[\left(Y^2\right)^3\right] &= 6\mu_2^3 \\
E\left[\left(Y^2\right)^4\right] &= 24\mu_2^4 \\
E\left[N_{dt}^1\right] &= \lambda_1 dt \\
E\left[\left(N_{dt}^1\right)^2\right] &= \lambda_1 dt + (\lambda_1 dt)^2 \\
E\left[\left(N_{dt}^1\right)^3\right] &= (\lambda_1 dt)^3 + 3\left(\lambda_1 dt\right)^2 + \lambda_1 dt \\
E\left[\left(N_{dt}^1\right)^4\right] &= (\lambda_1 dt)^4 + 6\left(\lambda_1 dt\right)^3 + 7\left(\lambda_1 dt\right)^2 + \lambda_1 dt \\
E\left[W\right] &= 0 \\
E\left[W^2\right] &= 1 \\
E\left[W^3\right] &= 0 \\
E\left[W^4\right] &= 3
\end{aligned}$$

$$\begin{aligned}
E\left[Y^1\right] &= E\left[E\left[Y^1 | Y^2 = y^2\right]\right] \\
&= E\left[\mu_1 + \rho_j y^2 + \sigma_1 W\right] = \mu_1 + \rho_j \mu_2 \\
E\left[\left(Y^1\right)^2\right] &= E\left[E\left[\left(\mu_1 + \rho_j y^2 + \sigma_1 W\right)^2 | Y^2 = y^2\right]\right] \\
&= \mu_1^2 + 2\rho_j \mu_2^2 + 2\mu_1 \mu_2 \rho_j + \sigma_1^2
\end{aligned}$$

$$
\begin{aligned}
E\left[\left(Y^1\right)^3\right] &= E\left[E\left[\left(\mu_1+\rho_j y^2+\sigma_1 W\right)^3|Y^2=y^2\right]\right] \\
&= \mu_1^3+2\mu_1\mu_2^2\rho_j^2+2\mu_1\mu_2\rho_j+\mu_1\sigma_1^2 \\
&\quad +\mu_1^2\mu_2\rho_j+6\rho_j^3\mu_2^3+4\mu_1\mu_2^2\rho_j^2 \\
&\quad +\rho_j\mu_2\sigma_1^2+2\mu_1\sigma_1^2+2\mu_2\rho_j\sigma_1^2 \\
E\left[\left(Y^1\right)^4\right] &= E\left[E\left[\left(\mu_1+\rho_j y^2+\sigma_1 W\right)^4|Y^2=y^2\right]\right] \\
&= \mu_1^4+2\mu_1^2\mu_2^2\rho_j+2\mu_1^3\mu_2\rho_j+\mu_1^2\sigma_1^2 \\
&\quad +2\mu_1^2\mu_2^2\rho_j+24\mu_2^4\rho_j^2+12\mu_1\mu_2^3\rho_j^2 \\
&\quad +2\rho_j\mu_2^2\sigma_1^2+2\mu_1^3\mu_2\rho_j+12\mu_1\mu_2^3\rho_j^2 \\
&\quad +8\mu_1^2\mu_2^2\rho_j^2+\mu_1\mu_2\rho_j\sigma_1^2 \\
&\quad +\mu_1^2\sigma_1^2+2\mu_2^2\rho_j\sigma_1^2+2\mu_1\mu_2\rho_j\sigma_1^2 \\
&\quad +3\sigma_1^4+4\mu_1^2\sigma_1^2+4\mu_1\mu_2\rho_j\sigma_1^2 \\
&\quad +4\mu_1\mu_2\rho_j\sigma_1^2+8\mu_2^2\rho_j^2\sigma_1^2
\end{aligned}
$$

The SDE for the log asset value is:

$$
d\nu_t = \underbrace{(\mu-\lambda_1 k-\frac{1}{2}h_t)dt}_{=:P}+\underbrace{\sqrt{h_t}dW_t^1}_{=:Q}+\underbrace{Y^1 dN_t^1}_{=:R}
$$

The conditional moments for P, Q and R are:

$$
\begin{aligned}
E_t\left[P\right] &= (\mu-\lambda_1 k-\frac{1}{2}h_t)dt \\
E_t\left[Q\right] &= 0 \\
E_t\left[Q^2\right] &= h_t dt \\
E_t\left[Q^3\right] &= 0 \\
E_t\left[Q^4\right] &= 3h_t^2 dt^2
\end{aligned}
$$

$$
\begin{aligned}
E_t\left[R\right] &= E\left[E\left[E\left[\sum_{i=1}^{N}\mu_1+\rho_j y^2+\sigma_1 W|Y^2=y^2\right]|N_{dt}^1=N\right]\right] \\
&= \lambda_1\left(\mu_1+\rho_j\mu_2\right)dt \\
E_t\left[R^2\right] &= E\left[E\left[\left(Y_1^1+\cdots+Y_N^1\right)^2|N_{dt}^1=N\right]\right] \\
&= \left(\mu_1^2+2\mu_1\mu_2\rho_j+\rho_j^2\mu_2^2\right)E\left[\left(N^1\right)^2\right] \\
&\quad \left(2\mu_2^2\rho_j-\rho_j^2\mu_2^2+\sigma_1^2\right)E\left[N^1\right] \\
&= \left(\mu_1^2+2\mu_1\mu_2\rho_j+2\rho_j\mu_2^2+\sigma_1^2\right)\lambda_1 dt \\
&\quad +\left(\mu_1^2+\mu_2^2\rho_j^2+2\mu_1\mu_2\rho_j\right)\left(\lambda_1 dt\right)^2 \\
E_t\left[R^3\right] &= E\left[E\left[\left(Y_1^1+\cdots+Y_N^1\right)^3|N_{dt}^1=N\right]\right] \\
&= E\left[NE\left[\left(Y^1\right)^3\right]+3N\left(N-1\right)E\left[\left(Y^1\right)^2\right]E\left[Y^1\right]\right. \\
&\quad \left.N\left(N-1\right)\left(N-2\right)\left(E\left[Y^1\right]\right)^3\right] \\
&= \left[\mu_1^3+2\mu_1\mu_2^2\rho_j^2+2\mu_1\mu_2\rho_j+\mu_1\sigma_1^2\right. \\
&\quad +\mu_1^2\mu_2\rho_j+6\mu_2^2\rho_j^3+4\mu_1\mu_2^2\rho_j^2 \\
&\quad \left.+\rho_j\mu_2\sigma_1^2+2\mu_1\sigma_1^2+2\mu_2\rho_j\sigma_1^2\right]\lambda_1 dt \\
&\quad +\left[9\mu_1^3+27\mu_1^2\mu_2\rho_j+18\mu_1\mu_2^2\rho_j\right. \\
&\quad +18\mu_2^3\rho_j^2+18\mu_1\mu_2^2\rho_j^2+9\mu_1\sigma_1^2 \\
&\quad \left.+9\mu_2\sigma_1^2\rho_j\right]\left(\lambda_1 dt\right)^2 \\
&\quad +\left[4\mu_1^3+12\mu_1^2\rho_j\mu_2+6\mu_1\rho_j\mu_2^2\right. \\
&\quad +6\rho_j^2\mu_2^3+9\mu_1\mu_2^2\rho_j^2+3\mu_1\sigma_1^2 \\
&\quad \left.+3\mu_2\rho_j\sigma_1^2+\mu_2^3\rho_j^3\right]\left(\lambda_1 dt\right)^3
\end{aligned}
$$

$$
\begin{aligned}
E_t\left[R^4\right] &= E\left[E\left[\left(Y_1^1+\cdots+Y_N^1\right)^4|N_{dt}^1=N\right]\right] \\
&= E\left\{NE\left[\left(Y^1\right)^4\right]+4N\left(N-1\right)E\left[\left(Y^1\right)^3\right]E\left[Y^1\right]\right. \\
&\quad 3N\left(N-1\right)\left(E\left[\left(Y^1\right)^2\right]\right)^2 \\
&\quad +6N\left(N-1\right)\left(N-2\right)E\left[\left(Y^1\right)^2\right]\left(E\left[Y^1\right]\right)^2 \\
&\quad \left.N\left(N-1\right)\left(N-2\right)\left(N-3\right)\left(E\left[Y^1\right]\right)^4\right\} \\
&= \left[4\mu_1^3\mu_2\rho_j+4\mu_1^2\mu_2^2\rho_j+4\rho_j\mu_2^2\sigma_1^2\right. \\
&\quad +24\mu_1\mu_2^3\rho_j^2+8\mu_1^2\mu_2^2\rho_j^2+8\rho_j^2\mu_2^2\sigma_1^2 \\
&\quad +6\mu_1^2\sigma_1^2+11\mu_1\mu_2\rho_j\sigma_1^2+3\sigma_1^4 \\
&\quad \left.+24\mu_2^4\rho_j^2+\mu_1^4\right]\lambda_1 dt \\
&\quad +\left[8\mu_1^2\rho_j\mu_2+48\mu_1\mu_2^3\rho_j^3+40\mu_1^2\mu_2^2\rho_j^2\right. \\
&\quad +12\rho_j\mu_2^2\sigma_1^2+24\mu_1\mu_2^3\rho_j^2+12\mu_1^2\mu_2^2\rho_j \\
&\quad +12\rho_j^2\mu_2^2\sigma_1^2+8\mu_1\mu_2^2\rho_j^2+20\mu_1^3\mu_2\rho_j \\
&\quad +3\sigma_1^4+12\mu_2^4\rho_j^2+18\mu_1^2\sigma_1^2+24\rho_j^4\mu_2^4 \\
&\quad \left.+36\mu_1\mu_2\rho_j\sigma_1^2+7\mu_1^4\right]\left(\lambda_1 dt\right)^2 \\
&\quad +\left[12\mu_1\mu_2^3\rho_j^3+24\mu_1\mu_2^3\rho_j^2+12\mu_1^2\mu_2^2\rho_j\right. \\
&\quad +30\mu_1^2\mu_2^2\rho_j^2+24\mu_1^3\mu_2\rho_j+6\rho_j^2\mu_2^2\sigma_1^2 \\
&\quad +12\mu_1\mu_2\rho_j\sigma_1^2+12\rho_j^3\mu_2^4+6\mu_1^2\sigma_1^2 \\
&\quad \left.+6\mu_1^4\right]\left(\lambda_1 dt\right)^3 \\
&\quad +\left[4\mu_1\mu_2^3\rho_j^3+6\mu_1^2\mu_2^2\rho_j^2+4\mu_1^3\mu_2\rho_j\right. \\
&\quad \left.+\mu_1^4+\rho_j^4\mu_2^4\right]\left(\lambda_1 dt\right)^4
\end{aligned}
$$

Now we can find the conditional moments of the log asset value process. The instantaneous conditional mean and variance are:

$$
\begin{aligned}
E_t\left[d\nu\right] &= \left(\mu-\lambda_1 k-\frac{1}{2}h_t\right)dt \\
&\quad +\lambda_1\left(\mu_1+\rho_j\mu_2\right)dt \\
\lim_{dt\to 0}\frac{1}{dt}E_t\left[d\nu\right] &= \mu-\frac{1}{2}h_t+\lambda_1\left(\mu_1+\rho_j\mu_2-k\right) \qquad \text{(B.3)}
\end{aligned}
$$

$$
\begin{aligned}
Var_t[d\nu] &= h_t dt \\
&\quad + \left(\mu_1^2 + 2\mu_1\mu_2\rho_j + 2\rho_j\mu_2^2 + \sigma_1^2\right)\lambda_1 dt \\
&\quad + \left(\mu_1^2 + \mu_2^2\rho_j^2 + 2\mu_1\mu_2\rho_j\right)(\lambda_1 dt)^2 \\
&\quad -\lambda_1^2(\mu_1 + \rho_j\mu_2)^2 dt^2 \\
&= h_t dt + \left(\mu_1^2 + 2\mu_1\mu_2\rho_j + 2\rho_j\mu_2^2 + \sigma_1^2\right)\lambda_1 dt \\
\lim_{dt\to 0}\frac{1}{dt}Var_t[d\nu] &= h_t + \left(\mu_1^2 + 2\mu_1\mu_2\rho_j + 2\rho_j\mu_2^2 + \sigma_1^2\right)\lambda_1 \qquad \text{(B.4)}
\end{aligned}
$$

The third conditional moment around mean is:

$$
\begin{aligned}
E_t\left[(d\nu - E_t[d\nu])^3\right] &= \left[6\mu_1\mu_2^2\rho_j^2 + 3\rho_j\mu_2\sigma_1^2 + 9\mu_1^2\mu_2\rho_j\right. \\
&\quad +6\mu_1\mu_2^2\rho_j + 3\mu_1\sigma_1^2 + 6\mu_2^3\rho_j^2 \\
&\quad \left.+3\mu_1^3\right]\lambda_1^3 dt^3 \\
&\quad + \left[6\mu_1\sigma_1^2 + 12\mu_2^3\rho_j^2 + 18\mu_1^2\mu_2\rho_j\right. \\
&\quad +12\mu_1\mu_2^2\rho_j + 12\mu_1\mu_2^2\rho_j^2 + 6\mu_1^3 \\
&\quad \left.+6\rho_j\mu_2\sigma_1^2\right]\lambda_1^2 dt^2 \\
&\quad + \left[\mu_1^3 + 6\mu_1\mu_2^2\rho_j^2 + 2\mu_1\mu_2\rho_j\right. \\
&\quad +3\mu_1\sigma_1^2 + \mu_1^2\mu_2\rho_j + 6\mu_2^2\rho_j^3 \\
&\quad \left.+3\rho_j\mu_2\sigma_1^2\right]\lambda_1 dt
\end{aligned}
$$

Therefore instantaneous conditional third moment around mean is:

$$
\begin{aligned}
\lim_{dt\to 0}\frac{1}{dt}E_t\left[(d\nu - E_t[d\nu])^3\right] &= \left[\mu_1^3 + 6\mu_1\mu_2^2\rho_j^2 + 2\mu_1\mu_2\rho_j\right. \\
&\quad +3\mu_1\sigma_1^2 + \mu_1^2\mu_2\rho_j + 6\mu_2^2\rho_j^3 \\
&\quad \left.+3\rho_j\mu_2\sigma_1^2\right]\lambda_1 \qquad \text{(B.5)}
\end{aligned}
$$

The fourth conditional moment around mean is:

$$
\begin{aligned}
E_t\left[(d\nu - E_t[d\nu])^4\right] &= \left[6\mu_1^2\sigma_1^2 + 4\mu_1^3\mu_2\rho_j + 4\mu_1^2\mu_2^2\rho_j\right.\\
&+4\rho_j\mu_2^2\sigma_1^2 + 24\mu_1\mu_2^3\rho_j^2 + 8\mu_1^2\mu_2^2\rho_j^2\\
&+3\sigma_1^4 + \mu_1^4 + 24\mu_2^4\rho_j^2 + 8\mu_2^2\rho_j^2\sigma_1^2\\
&\left.+11\mu_1\mu_2\rho_j\sigma_1^2\right]\lambda_1 dt\\
&+\left[12\rho_j\mu_2^2 + 12\mu_1\mu_2\rho_j + 6\mu_1^2 + 6\sigma_1^2\right] h_t\lambda_1 dt^2\\
&+\left[12\mu_1\mu_2\rho_j\sigma_1^2 + 12\rho_j\mu_2^2\sigma_1^2 + 24\mu_1\mu_2^3\rho_j^2\right.\\
&+12\mu_1^2\mu_2^2\rho_j + 6\mu_1^2\sigma_1^2 + 12\mu_1^2\mu_2^2\rho_j^2\\
&+12\mu_1^3\mu_2\rho_j + 24\rho_j^4\mu_2^4 + 24\mu_1\mu_2^3\rho_j^3\\
&+3\sigma_1^4 + 3\mu_1^4 + 12\mu_2^4\rho_j^2 - 24\mu_2^2\rho_j^3\mu_1\\
&\left.-24\mu_2^3\rho_j^4\right]\lambda_1^2 dt^2 + 3h_t^2 dt^2\\
&+\left[-48\mu_1\mu_2\rho_j\sigma_1^2 - 24\mu_1^2\sigma_1^2\right.\\
&-24\mu_2^2\rho_j^2\sigma_1^2 - 24\mu_1^4 - 48\mu_1\mu_2^3\rho_j^3\\
&-96\mu_1\mu_2^3\rho_j^2 - 48\mu_1^2\mu_2^2\rho_j - 120\mu_1^2\mu_2^2\rho_j^2\\
&\left.-96\mu_1^3\mu_2\rho_j - 48\rho_j^3\mu_2^4\right]\lambda_1^3 dt^3\\
&+\left[-24\mu_1\mu_2^3\rho_j^3 - 60\mu_1^2\mu_2^2\rho_j^2\right.\\
&-48\mu_1^3\mu_2\rho_j - 12\mu_1^4 - 12\mu_1^2\sigma_1^2 - 48\mu_1\mu_2^3\rho_j^2\\
&-24\mu_1^2\mu_2^2\rho_j - 24\rho_j^3\mu_2^4 - 24\mu_1\mu_2\rho_j\sigma_1^2\\
&\left.-12\mu_2^2\rho_j^2\sigma_1^2\right]\lambda_1^4 dt^4
\end{aligned}
$$

Therefore instantaneous conditional fourth moment around mean is:

$$
\begin{aligned}
\lim_{dt\to 0}\frac{1}{dt}E_t\left[(d\nu - E_t[d\nu])^4\right] &= \left[6\mu_1^2\sigma_1^2 + 4\mu_1^3\mu_2\rho_j + 4\mu_1^2\mu_2^2\rho_j\right.\\
&+4\rho_j\mu_2^2\sigma_1^2 + 24\mu_1\mu_2^3\rho_j^2 + 8\mu_1^2\mu_2^2\rho_j^2\\
&+3\sigma_1^4 + \mu_1^4 + 24\mu_2^4\rho_j^2 + 8\mu_2^2\rho_j^2\sigma_1^2\\
&\left.+11\mu_1\mu_2\rho_j\sigma_1^2\right]\lambda_1 \qquad \text{(B.6)}
\end{aligned}
$$

Appendix C

Solutions for Fourier Transform

In this section we find the solution for the fourier transform that we used for bond pricing. For this, first we solve the Ricatti equations for $B(\tau; u_1, u_2, u_3)$ and $C(\tau; u_1, u_2, u_3)$, and then find $A(\tau; u_1, u_2, u_3)$,$J^1(\tau; u_1, u_2, u_3)$ and $J^2(\tau; u_1, u_2, u_3)$.

Before going into the solutions, in addition to the transforms g_1 to g_8 that we define in the previous section, define the following additional transforms:

$$\begin{aligned}
h_1(u_1, u_2, u_3) &:= -\eta \\
h_2(u_1, u_2, u_3) &:= \sqrt{2iu_1\delta^2 - h_1^2} \\
h_3(u_1, u_2, u_3) &:= \tan\left[\frac{1}{2}h_2\tau + \arctan(h_4)\right] \\
h_4(u_1, u_2, u_3) &:= \frac{h_1 + iu_3\delta^2}{h_2} \\
h_5(u_1, u_2, u_3) &:= 1 + \frac{h_1\mu_3}{\delta^2} \\
h_6(u_1, u_2, u_3) &:= \frac{\mu_3 h_2}{\delta^2} \\
h_7(u_1, u_2, u_3) &:= \lambda_2
\end{aligned}$$

C.1 Solution for $B(\tau; u_1, u_2, u_3)$

Since the ODE for B is same with the one we have in the previous section with the same boundary condition, we have the same solution:

$$B(\tau; u_1, u_2, u_3) = \frac{g_2 g_3 - g_1}{\gamma^2} \tag{C.1}$$

Note that g_1, g_2 and g_3, and hence B do not involve any term with u_3. This is because of the assumption that interest rate process is independent from asset value and variance processes.

C.2 Solution for $C(\tau; u_1, u_2, u_3)$

The ODE for C is:

$$\frac{\partial C}{\partial \tau} = R_1 + CR_2 + C^2 R_3$$

where:

$$\begin{aligned} R_1 &:= iu_1 \\ R_2 &:= -\eta \\ R_3 &:= \frac{1}{2}\delta^2 \end{aligned}$$

with the boundary condition:

$$C(0; u_1, u_2 - u_3) = iu_3$$

$$\begin{aligned} \frac{dC}{d\tau} &= R_1 + CR_2 + C^2 R_3 \\ d\tau &= \frac{dC}{R_1 + CR_2 + C^2 R_3} \\ \int d\tau &= \tau = \int \frac{dC}{R_1 + CR_2 + C^2 R_3} \end{aligned}$$

Note that, since we have more than one integral, it does not matter where we put a constant in the equality. Now define the following transforms:

$$\begin{aligned} K &:= \frac{\sqrt{4R_1R_3 - R_2^2}}{2\sqrt{R_3}} \\ X &:= \frac{2R_3C + R_2}{2\sqrt{R_3}} \end{aligned}$$

Then we have:

$$\begin{aligned} K^2 + X^2 &= \frac{4R_1R_3 - R_2^2}{4R_3} + \frac{4R_3^2C^2 + R_2^2 + 4R_2R_3C}{4R_3} \\ &= R_1 - \frac{R_2^2}{4R_3} + R_3C^2 + \frac{R_2^2}{4R_3} + R_2C \\ &= R_1 + R_2B + R_3C^2 \end{aligned}$$

Additionally we have:

$$\begin{aligned} dX &= d\left(\frac{2R_3C + R_2}{2\sqrt{R_3}}\right) = \sqrt{R_3}dC \\ dC &= \frac{dX}{\sqrt{R_3}} \end{aligned}$$

Therefore we have:

$$\begin{aligned} \int \frac{dC}{R_1 + CR_2 + C^2R_3} &= \int \frac{1}{\sqrt{R_3}\left(K^2 + X^2\right)}dX \\ &= \frac{1}{\sqrt{R_3}} \int \frac{1}{K^2 + X^2}dX = \tau \\ \int \frac{K}{K^2 + X^2}dX &= \tau\sqrt{R_3}K \end{aligned}$$

Since we know $\int \frac{a}{a^2+x^2}dx = \arctan\left(\frac{x}{a}\right) - \bar{c}$, we have:

$$\arctan\left(\frac{X}{K}\right) - \bar{c} = \tau\sqrt{R_3}K$$

When we solve for X:

$$X = K\ \tan\left(K\tau\sqrt{R_3} + \bar{c}\right)$$

Using the definition of X and K:

$$\begin{aligned}\frac{2R_3C + R_2}{2\sqrt{R_3}} &= \frac{\sqrt{4R_1R_3 - R_2^2}}{2\sqrt{R_3}}\ \tan\left(\frac{1}{2}\tau\sqrt{4R_1R_3 - R_2^2} + \bar{c}\right) \\ C &= \frac{\sqrt{4R_1R_3 - R_2^2}\ \tan\left(\frac{1}{2}\tau\sqrt{4R_1R_3 - R_2^2} + \bar{c}\right) - R_2}{2R_3}\end{aligned}$$

Now we can use the boundary condition, $C\left(0; u_1, u_2, u_3\right) = iu_3$, to find $\bar{c}$.

$$\begin{aligned}iu_3 &= \frac{\sqrt{4R_1R_3 - R_2^2}\ \tan\left(\bar{c}\right) - R_2}{2R_3} \\ \tan\left(\bar{c}\right) &= \frac{2iu_3R_3 + R_2}{\sqrt{4R_1R_3 - R_2^2}} \\ \bar{c} &= \arctan\left(\frac{2iu_3R_3 + R_2}{\sqrt{4R_1R_3 - R_2^2}}\right)\end{aligned}$$

Now we can totally define C:

$$C = \frac{\sqrt{4R_1R_3 - R_2^2}\ \tan\left(\frac{1}{2}\tau\sqrt{4R_1R_3 - R_2^2} + \arctan\left(\frac{2iu_3R_3 + R_2}{\sqrt{4R_1R_3 - R_2^2}}\right)\right) - R_2}{2R_3}$$

By using previously defined h_1, h_2, h_3,

$$C\left(\tau; u_1, u_2, u_3\right) = \frac{h_2h_3 - h_1}{\delta^2} \tag{C.2}$$

C.3 Solution for $A(\tau; u_1, u_2, u_3)$

We have:

$$\begin{aligned}\frac{\partial A}{\partial \tau} &= \alpha\beta B + \eta\theta C \\ &= \alpha\beta\frac{g_2 g_3 - g_1}{\gamma^2} + \eta\theta\frac{h_2 h_3 - h_1}{\delta^2}\end{aligned}$$

Note that g_1, g_2, h_1 and h_2 do not depend on τ. Therefore:

$$\begin{aligned}dA &= \left(\frac{\alpha\beta g_2}{\gamma^2}g_3 + \frac{\eta\theta h_2}{\delta^2}h_3 - \frac{\alpha\beta g_1}{\gamma^2} - \frac{\eta\theta h_1}{\delta^2}\right)d\tau \\ \int dA &= \left[-\frac{\alpha\beta g_1}{\gamma^2} - \frac{\eta\theta h_1}{\delta^2}\right]\tau + \frac{\alpha\beta g_2}{\gamma^2}\int g_3 d\tau + \frac{\eta\theta h_2}{\delta^2}\int h_3 d\tau\end{aligned}$$

Therefore we should find $\int g_3 d\tau$ and $\int h_3 d\tau$. In the previous section, we already found $\int g_3 d\tau$:

$$\int g_3 d\tau = \frac{1}{g_2}\left[ln\left(1+g_3^2\right) + \bar{c}\right]$$

In a similar manner, we have:

$$\int h_3 d\tau = \frac{1}{h_2}\left[ln\left(1+h_3^2\right) + \bar{c}\right]$$

When we input these results, we have the following equality. Note that it is unimportant where we put the constant term.

$$\begin{aligned}\int dA = A &= \left[-\frac{\alpha\beta g_1}{\gamma^2} - \frac{\eta\theta h_1}{\delta^2}\right]\tau \\ &\quad + \frac{\alpha\beta g_2}{\gamma^2}\frac{1}{g_2}\left[ln\left(1+g_3^2\right)+\bar{c}\right] + \frac{\eta\theta h_2}{\delta^2}\frac{1}{h_2}ln\left(1+h_3^2\right) \\ &= \left[-\frac{\alpha\beta g_1}{\gamma^2} - \frac{\eta\theta h_1}{\delta^2}\right]\tau \\ &\quad + \frac{\alpha\beta}{\gamma^2}\left[ln\left(1+g_3^2\right)+\bar{c}\right] + \frac{\eta\theta}{\delta^2}ln\left(1+h_3^2\right)\end{aligned}$$

Now, by using the boundary condition, $A(0;u_1,u_2,u_3)=0$, we will find $\bar{c}$.

$$\begin{aligned} 0 &= \frac{\alpha\beta}{\gamma^2}\left[ln\left(1+g_4^2\right)+\bar{c}\right]+\frac{\eta\theta}{\delta^2}ln\left(1+h_4^2\right) \\ \bar{c} &= -ln\left(1+g_4^2\right)-\frac{\eta\theta\gamma^2}{\alpha\beta\delta^2}ln\left(1+h_4^2\right) \end{aligned}$$

When we input $\bar{c}$:

$$\begin{aligned} A(\tau;u_1,u_2,u_3) &= \left[-\frac{\alpha\beta g_1}{\gamma^2}-\frac{\eta\theta h_1}{\delta^2}\right]\tau+\frac{\alpha\beta}{\gamma^2}\,ln\left(1+g_3^2\right)+\frac{\eta\theta}{\delta^2}\,ln\left(1+h_3^2\right) \\ &\quad -\frac{\alpha\beta}{\gamma^2}\,ln\left(1+g_4^2\right)-\frac{\eta\theta}{\delta^2}\,ln\left(1+h_4^2\right) \\ &= \left[-\frac{\alpha\beta g_1}{\gamma^2}-\frac{\eta\theta h_1}{\delta^2}\right]\tau \\ &\quad +\frac{\alpha\beta}{\gamma^2}\,ln\left(\frac{1+g_3^2}{1+g_4^2}\right)+\frac{\eta\theta}{\delta^2}\,ln\left(\frac{1+h_3^2}{1+h_4^2}\right) \end{aligned} \tag{C.3}$$

C.4 Solution for $J^1(\tau;u_1,u_2,u_3)$

Since the ODE for J^1 is same with the one we have in the previous section with the same boundary condition, we have the same solution:

$$\begin{aligned} J^1(\tau;u_1,u_2,u_3) &= \left[g_8-\lambda_1+\frac{g_5g_7}{g_5^2+g_6^2}\right]\tau \\ &\quad +\frac{2g_6g_7}{g_2\left(g_5^2+g_6^2\right)}\,ln\left(\frac{\left(g_6g_4-g_5\right)\sqrt{1+g_3^2}}{\left(g_6g_3-g_5\right)\sqrt{1+g_4^2}}\right) \end{aligned} \tag{C.4}$$

C.5 Solution for $J^2(\tau;u_1,u_2,u_3)$

With the transforms h_1 to h_8, we can express the ODE for J^2:

$$\begin{aligned}
\frac{\partial J^2}{\partial \tau} &= \frac{h_7}{h_5 - h_6 h_3} - h_7 \\
\int dJ^2 &= h_7 \int \frac{1}{h_5 - h_6 h_3} d\tau - \int h_7 d\tau \\
J^2 &= -h_7 \tau + h_7 \int \frac{1}{h_5 - h_6 h_3} d\tau \\
&= -h_7 \tau + \frac{h_7}{h_6} \int \frac{1}{h_3 - \frac{h_5}{h_6}} d\tau
\end{aligned}$$

Since:

$$h_3 = \tan\left(\frac{1}{2} h_2 \tau + \arctan\left(h_4\right)\right)$$

we have:

$$\frac{\partial h_3}{\partial \tau} = \frac{1}{2} h_2 \left(1 + h_3^2\right) \Rightarrow d\tau = \frac{2}{h_2\left(1 + h_3^2\right)} dh_3$$

Therefore we should find:

$$\int \frac{1}{h_3 - \frac{h_5}{h_6}} d\tau = \frac{2}{h_2} \int \frac{1}{\left(h_3 - \frac{h_5}{h_6}\right)\left(1 + h_3^2\right)} dh_3$$

For the integrand we have:

$$\frac{1}{\left(h_3 - \frac{h_5}{h_6}\right)\left(1 + h_3^2\right)} = \frac{h_6^2}{\left(h_5^2 + h_6^2\right)\left(h_3 - \frac{h_5}{h_6}\right)} - \frac{h_6\left(h_5 + h_6 h_3\right)}{\left(h_5^2 + h_6^2\right)\left(1 + h_3^2\right)}$$

Thus:

$$\int \frac{1}{h_3 - \frac{h_5}{h_6}} d\tau = \frac{2h_6^2}{h_2\left(h_5^2 + h_6^2\right)} \int \frac{1}{\left(h_3 - \frac{h_5}{h_6}\right)} dh_3 - \frac{2h_6}{h_2\left(h_5^2 + h_6^2\right)} \int \frac{h_5 + h_6 h_3}{1 + h_3^2} dh_3$$

Additionally we have:

$$\begin{aligned}
\int \frac{1}{h_3 - \frac{h_5}{h_6}} dh_3 &= ln\left(h_3 - \frac{h_5}{h_6}\right) \\
\int \frac{h_5 + h_6 h_3}{1 + h_3^2} dh_3 &= h_5 \ \arctan\left(h_3\right) + \frac{1}{2} h_6 \ ln\left(1 + h_3^2\right) \\
&= \frac{1}{2} h_2 h_5 \tau + h_5 \ \arctan\left(h_4\right) + \frac{1}{2} h_6 \ ln\left(1 + h_3^2\right)
\end{aligned}$$

We put a constant term at this stage.

$$\begin{aligned}
\frac{1}{h_6} \int \frac{1}{h_3 - \frac{h_5}{h_6}} d\tau &= \frac{2h_6}{h_2\left(h_5^2 + h_6^2\right)} \ ln\left(h_3 - \frac{h_5}{h_6}\right) - \frac{2}{h_2\left(h_5^2 + h_6^2\right)} \frac{1}{2} h_2 h_5 \tau \\
&\quad - \frac{2}{h_2\left(h_5^2 + h_6^2\right)} h_5 \ \arctan\left(h_4\right) - \frac{2}{h_2\left(h_5^2 + h_6^2\right)} \frac{1}{2} h_6 \ ln\left(1 + h_3^2\right) \\
&\quad - \frac{2}{h_2\left(h_5^2 + h_6^2\right)} \bar{c} \\
&= \frac{2}{h_2\left(h_5^2 + h_6^2\right)} \left\{ h_6 \ ln\left(\frac{h_3 - \frac{h_5}{h_6}}{\sqrt{1 + h_3^2}}\right) - \frac{h_2 h_5}{2} \tau \right. \\
&\quad \left. - h_5 \ \arctan\left(h_4\right) - \bar{c} \right\} \\
&= \frac{2}{h_2\left(h_5^2 + h_6^2\right)} \left\{ h_6 \ ln\left(\frac{h_3 h_6 - h_5}{h_6 \sqrt{1 + h_3^2}}\right) - \frac{h_2 h_5}{2} \tau \right. \\
&\quad \left. - h_5 \ \arctan\left(h_4\right) - \bar{c} \right\}
\end{aligned}$$

Now we can obtain J^2:

$$\begin{aligned}
J^2 &= -h_7 \tau \\
&\quad + \frac{2h_7}{h_2\left(h_5^2 + h_6^2\right)} \left\{ \frac{h_2 h_5}{2} \tau + h_5 \ \arctan\left(h_4\right) - h_6 \ ln\left(\frac{h_3 h_6 - h_5}{h_6 \sqrt{1 + h_3^2}}\right) + \bar{c} \right\}
\end{aligned}$$

By using the boundary condition, $J^2(0; u_1, u_2, u_3) = 0$, and noting that the for $\tau = 0$ we have $h_3 = h_4$:

$$\begin{aligned} 0 &= \frac{2h_7}{h_2(h_5^2+h_6^2)} \left\{ h_5 \ \arctan(h_4) - h_6 \ ln\left(\frac{h_4h_6 - h_5}{h_6\sqrt{1+h_4^2}}\right) + \bar{c} \right\} \\ \bar{c} &= h_6 ln\left(\frac{h_4h_6 - h_5}{h_6\sqrt{1+h_4^2}}\right) - h_5 \ \arctan(h_4) \end{aligned}$$

When we input $\bar{c}$:

$$\begin{aligned} J^2(\tau; u_1, u_2, u_3) &= -h_7\tau + \frac{2h_7}{h_2(h_5^2+h_6^2)} \frac{h_2h_5}{2}\tau \\ &\quad \frac{2h_6h_7}{h_2(h_5^2+h_6^2)} \left\{ -ln\left(\frac{h_3h_6 - h_5}{h_6\sqrt{1+h_3^2}}\right) + ln\left(\frac{h_4h_6 - h_5}{h_6\sqrt{1+h_4^2}}\right) \right\} \\ &= \left[\frac{h_5h_7}{h_5^2+h_6^2} - h_7\right]\tau \\ &\quad + \frac{2h_6h_7}{h_2(h_5^2+h_6^2)} \ ln\left(\frac{(h_4h_6 - h_5)\sqrt{1+h_3^2}}{(h_3h_6 - h_5)\sqrt{1+h_4^2}}\right) \end{aligned} \quad \text{(C.5)}$$