STATIC HEDGING STRATEGIES FOR BARRIER OPTIONS AND THEIR ROBUSTNESS TO MODEL RISK

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STATIC HEDGING STRATEGIES FOR BARRIER OPTIONS AND THEIR ROBUSTNESS TO MODEL RISK

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ABSTRACT

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With the rapid increase in the usage of barrier options on the OTC markets, pricing and especially hedging of these exotic instruments became an important field of research. This paper aims to explain, apply and compare current methods used for pricing and hedging barrier options with a simulation approach. An overview of most popular methods for pricing and hedging is presented in the first part, followed by application of these pricing methods and comparing the performances of different dynamic and static hedging techniques in Black-Scholes environment by simulation in the second part. In the third part different models such as ARCH type and Stochastic Volatility are used with different jump terms to relax the assumptions of the Black-Scholes and examine the effects of these incomplete models on both pricing and performance of different hedging techniques. In the fourth part diffusion models such as Constant Variance Elasticity, Heston Stochastic Volatility and Merton Jump Diffusion are used to complete the picture.

Keywords: Barrier Options,Static Hedging Strategies

ÖZ

BARİYER OPSİYONLARI İÇİN STATİK HEDGİNG STRATEJİLERİ VE BUNLARIN MODEL RİSKİNE GÖRE SAĞLAMLIĞI

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Bariyer Opsiyonlarının kullanımının OTC marketlerinde hızlı artışı ile birlikte, bunların fiyatlandırılması özeliklede hedgingi önemli bir araştırma alanı haline gelmiştir. Bu tez, bariyer opsiyonlarının güncel fiyalandırılma ve hedge methodlarının simulasyon yöntemi ile açıklaması, uygulaması ve karşılaştırmasını amaçlamaktadır. İlk bölümdeki güncel fiyatlandırma ve hedging methodlarının gözden geçirilmesini, ikinci bölümde Black- Scholes ortamında bu fiyatlandırma ve hedging stratejilerinin performaslarının simulasyon ile incelenmesi takip etmektedir. Üçüncü bölümde fiyatlandırma ve hedging çalışmalarında Black Scholes varsayımlarının rahatlatılması amacı ile ARCH tipi ve Stokastik Volatilite modellerinin değişik sıçrama terimleri simulasyon uygulamaları için kullanılmıştır. Son bölümde Sabit Varyans Elastik, Heston Stokastik Volatilite ve Merton Sıçrama Difuzyon modeli gibi difuzyon modelleri kullanılmıştır.

Anathtar Kelimeler: Bariyer Opsiyonları, Statik Hedging Stratejileri

To My Family

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BARRIER OPTIONS

1.1 Introductıon

Derivative markets have become an essential part of the global financial system. Growth of these markets appears in every kind of instrument and trading activity but the major increment happen in the over-the-counter (OTC) markets. The primary reason of the success of the OTC markets has been their ability to offer “exotic” derivatives customized to customer needs. Banks and financial institutions have been producing exotic options to match corporate end users and investor requirements: either to reduce premium expenditure or to better fit the risk profile of the client

Exotic options appear to be relatively modern phenomenon but in reality they have a very long history such that some of them have been available since 1967. However the trading volumes were not very high in the previous days and it was not until the end of 70’s and the beginning of 80’s that exotic options gain more interest. In these days the trading volumes are high and users have a diversified profile from large financial institutions to corporations, from fund managers to private bankers.

One of the most significant interests of exotic options has been on the barrier options. Barrier options are vanilla options with an additional feature which allows investors to control their positions at one or more critical levels. These instruments are attractive for clients as they are cheaper than the plain-vanilla counterparts. Barrier options fulfill many different needs and widely used in foreign-exchange and fixed-income derivatives markets. The uses of barrier options have been discussed by Steinherr (1998). Especially investors from developing markets who suffer from high volatilities prefer barrier options because these options allow market participants to tailor their trading strategies to their specific markets views.

1.2 Definitions

Barrier options are part of the class called path dependent options. Unlike the plain vanilla options which depend just on the magnitude of the underlying asset price at maturity and the strike price of the option, barrier options also depend on how the settlement price is reached. Depending on how the settlement price is reached there are two types: “in” or “knock in” and “out” or “knock out” barrier options.

1.2.1 Vanilla Barrier Options

As mentioned earlier barrier options are simple plain vanilla’s if the barrier is not hit during the life of the option. But if the barrier is hit they become worthless. Using this fact vanilla barrier options can be categorized according to the level of the barrier and direction that the barrier is hit.

The simplest barrier options are knock outs which start their lives active and become worthless if a certain barrier level is reached during the life of the option and knock ins which start their lives inactive and become alive if a certain barrier level is reached during the life of the option. Also the level of the barrier is determinative for a barrier option. If the barrier is hit below then the option is called either “up knock-out” or “up knock-in”. If the barrier is hit above then the option is called either “down knock-in” or “down knock-out". Moreover one has to add the put and call distinction as well. To sum up plain vanilla barrier options can be categorized in 8 different categories.

Table 1.1:Summary of types of barrier options

	Up	Down
Knock out	Up knock out call Up knock out put	Down knock out call Down knock out put
Knock in	Up knock in call Up knock in put	Down knock in call Down knock in put

In addition to the plain vanilla barrier options there are other vanillas such as double barrier options. These options knock in or out at the first hitting time of either a lower or upper barrier. In the case of single asset, one dimensional option, there can be only two barriers one is above and the other is below the initial spot price.

1.2.2 Other Types of Barrier Options

Although plain vanilla barrier options are the most traded ones, of course they are not the only ones that exist in the market. There are others which are called as exotic or second generation barrier options. These options have a big market share as well. Construction of these generally depends on the use of plain vanilla barriers.

One touch option is an option that gives the investor a payout when the price of the underlying asset hits a predetermined barrier during the life of the option. In contrast to the plain vanilla barrier options, one touch does not give the owner to buy or sell the underlying at a predefined price. Instead it gives the buyer a fixed amount of money in the condition that the barrier is hit. No touch options are similar to the one touch. Unlike one touch, no touch options pay the owner fixed amount of money if the barrier is not hit during the life of the option. One can also add multiple barriers to both of touch options. Double no touch is the most common example to this type. It gives the owner to receive a fixed amount if the both of the barriers are not touched during the life of the option. On contrary, owner of a double one touch receives a fixed amount if either of the barrier is reached during the life of the option.

Corridor options or so called dual barrier options are options with double barriers. They end worthless if knock out happens or become active if knock in occurs during the life option. Unlike plain vanilla, meaning of the in or out is different. For example it makes no difference whether the upper or the lower barrier hit. So it can be said that dual barrier option is an agreement to give the purchaser of the option the right to exchange a known quantity of one security for a known quantity of a specified currency - subject to the behavior of the price of the underlying security with respect to two price barriers. Such a security is usually used to lower the cost of acquiring a similar option without such barriers. For this reason these options are generally cheaper than plain vanilla barrier options. One last feature is that corridor options barriers do not have to be fixed.

One type that has to be mentioned is Forward-start barrier options. For this type of options the barrier is active only over the latter period of options life. They give the owner to postpone the moment that the barrier becomes effective. The forward starts either as down barrier or up barrier option when the starting time becomes valid, depending on whether the underling asset price is below or above the predefined barrier level. A forced forward start barrier option is a special case of forward start option which combines the properties of vanilla and forward starts barrier options as it can be guaranteed before to be down or up option before.

For the limited barrier option case, barrier is effective within one or more periods which are decided before during the life of the option. These predefined periods are called windows.

The fluctuation in the underlying asset price can be high which in turn affects the payoff of the barrier option. In order to solve this problem the combination of Asian options and barrier options is introduced. The option called Asian barrier option offer the possibility to trade barrier options based on the averages of the asset price during the life of the option. The average can be arithmetic or geometric for the Asian barrier options.

Outside barrier options are options in which the underlying asset and the trigger asset are different variables. Whether or not an outside barrier options is knock-out depends on whether the price of the measurement asset touches a pre specified barrier within the life of the option.

Rolling options are issued with a sequence of barriers which are all either above or below the initial spot price. For the below case it is called roll-down call and the above case roll-up put. When the stock price reaches the barrier, the strike of the option is lowered for the call case and increased for the up case. The option ends worthless if the last barrier is hit during the life of the option.

Ratchet options are a special case of rolling options. Each time the barrier is hit the strike price set as that barrier. If the last barrier is reached instead of ending worthless the ratchet option strike is ratchet for the last time to that barrier level.

Lookback option prices depend on the maximum or minimum price level that is reached during the lookback period. Lookback period can start before or after the time that option starts its life but it has to finish before options maturity.

The categorization mentioned is the most traded barrier options. Of course there are many more and structured barrier options traded on the market as well.

1.3 Pricing Barrier Options

There is a huge literature on pricing barrier options and for the most important one's solutions have been found analytically and numerically. One can start counting from down and out call in the Black-Scholes model by

Merton (1973) and go on till now. Although closed form solutions have been found for all plain vanilla barriers in the Black-Scholes environment there are no analytical solutions developed for other models in which volatility assumed to be not constant and change in time. Of course there are some suggestions but neither of them are as apprehensible as Black-Scholes model or have less assumptions than that. Following is a small summary of the analytical prices for the Black-Scholes model for barrier options.

1.3.1 Analytical Prices

As mentioned earlier analytical solutions for pricing barrier options are as old as Black-Scholes formula. Down and out call formula is given by Merton (1973) which was followed by a more detailed paper by Reiner&Rubenstein (1991) that provides the formulas for all 8 types of barriers. Moreover Haug (1998) gives a generalization of the set of formulas provided by Reiner&Rubinstein.

The following derivation is a shorter one given at Poulsen (2006) for Black-Scholes Model (BSM). Derivation depends on the reflection theorem which says that it is possible to find sets that resemble the class of all sets. The name "reflection principle" comes from the fact that properties of the universe of all sets are "reflected" down to a smaller set.

In order to use reflection principle stock price dynamics in 1.3.1 is considered under risk-neutral measure Q.

$$dS(t) = (r - d)S(t)dt + \sigma S(t)dW(t) \qquad 1.3.1$$

where;

$S(t)$: Stock Price at time t

r : risk-free interest rate

d : dividend yield

σ : volatility of the stock prices r, d, σ are known constants and W(t) is a Q- Brownian motion then consider a claim with payoff at maturity $g(S_T, T)$. For B>0 define a new function g^*, called image function as in 1.3.2.

$$g^*(x) = (x/B)^p . g(B^2/x)$$
$$p = 1 - \frac{2(r-d)}{\sigma^2} \qquad 1.3.2$$

The next theorem reveals that $g(S_T, T)$ g^* are related.

Reflection theorem: Let the set up for S_t is like given above and define the arbitrage-free price at time t of this claim;

$$\Pi^g(t) = e^{-r(T-t)} E_t^Q\left[g(S_T, T)\right] = e^{-r(T-t)} f(S(t), t)$$

Then if we consider a simple claim with pay-off function g^* the arbitrage free time-t value of this g^*-claim is;

$$\Pi^{g^*}(t) = e^{-r(T-t)} \left(\frac{S_t}{B}\right)^p f\left(\frac{B^2}{S_t}, t\right)$$

Proof: Define the process Z_t as: $Z_t = \left(\frac{S_t}{B}\right)^p$.Using the Ito's lemma, it follows that;

$$dZ_t = \left[p(r-d)Z_t + \frac{1}{2}p(p-1)\sigma^2 Z_t\right]dt + p\sigma Z_t dW_t$$

Since $p = 1 - \frac{2(r-d)}{\sigma^2}$, drift term cancels and;

$$dZ_t = p\sigma Z_t dW_t \; § \; \frac{Z_t}{Z_0} \text{ is a Q martingale.}$$

(At this point one has to mention that exact form of p is needed here and the result will not hold if σ is time-dependent or stochastic.)

From that it follows

$$\frac{dQ^Z}{dQ} = \frac{Z(T)}{Z(0)}$$

which defines a probability measure $Q^Z \sim Q$. From Girsanov theorem it is known that

$$dW_t^{Q^Z} = dW^Q(t) - p\sigma dt$$

defines a Q^Z Brownian motion.

Value of g^* is then;

$$\Pi^{g^*}(t) = e^{-r(T-t)} E_t^Q\left[g^*(S_T, T)\right] = e^{-r(T-t)} \left(\frac{S_t}{B}\right)^p E_t^{Q^Z}\left[g\left(\frac{B^2}{S_T}, T\right)\right]$$

Define a new process $Y_t = \dfrac{B^2}{S(t)}$. Then Ito formula and definition of W^{Q^Z} gives;

$$dY(t) = (r - d)dtY(t) + \sigma(-dW_t^{Q^Z})Y(t)$$

which means the law of Y under Q^Z is the same as the law of S under Q. Therefore;

$$E_t^{Q^Z}(g(Y(T))) = f(Y(t), t) = f\left(\frac{B^2}{S(t)}, t\right)$$

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Reflection principle has a very long history in Physics, partial differential equations and Stochastic processes and can be used to price barrier options. Additionally it gives insight on to the method of static hedging for barrier options. The weakness of the theorem is that it holds under the BSM assumptions. With volatility of the form $\sigma(S(t), t)$, even in simple case of time dependent volatility or drift unfortunately it breaks down.

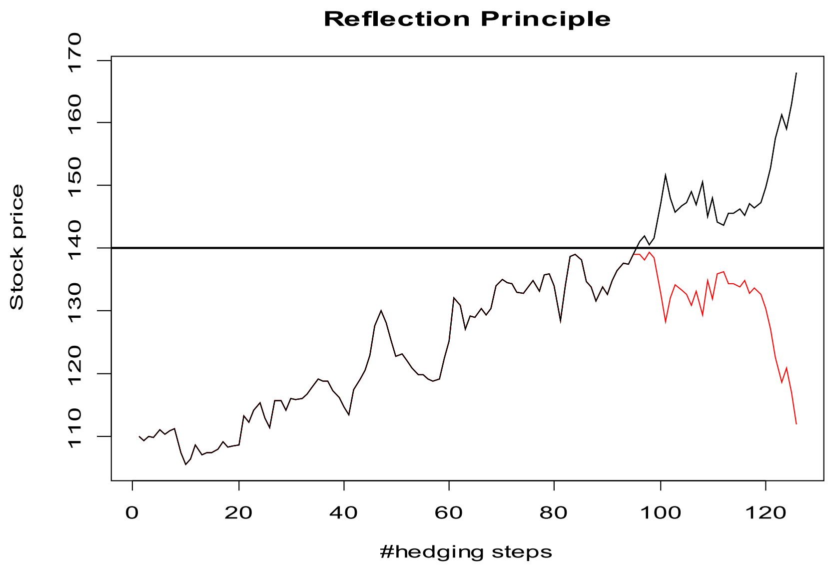


Figure 1.1: Reflection principle

1.3.1.1 Payoff Function & Price for Down and Out Call Option

There are two cases to be distinguished when considering the price and the payoff function for a down and out barrier option. For a regular down and out call option where the Barrier B is below the strike price K, payoff function $f(S(T))$ at maturity is given by 1.3.3.

$$f(S(T)) = \begin{cases} 0 & S(t) \leq B \\ S(T) - K & S(T) > K \,\&\, S(T) > B \end{cases} \qquad 1.3.3$$

Calculation of price of down and out call is given through 1.3.4-1.3.8.

$$d1 = \frac{\ln(S_0 / K) + (r - d - \sigma^2 / 2)(T)}{\sigma \sqrt{T}} \qquad 1.3.4$$

$$d2 = d1 - \sigma \sqrt{T} \qquad 1.3.5$$

$$h1 = \frac{\ln(\frac{B^2}{S_0 K}) + (r - d - \sigma^2/2)T}{\sigma\sqrt{T}} \quad 1.3.6$$

$$h2 = h1 - \sigma\sqrt{T} \quad 1.3.7$$

$$C^{DO}(S_0, \sigma, r, q, K, B, T) = e^{-dT} S_0 \left[N(d1) - (S_0/B)^{p-2} N(h1)\right] - e^{-rT} K\left[N(d2) - (S_0/B)^{p} N(h2)\right] \quad 1.3.8$$

Equation 1.3.8 shows that price of the down and out call option is the price of the European call minus a correction factor. This correction term is due to the barrier and although it is a bit more complicated than the European call, the results are still similar.

1.3.1.2 Payoff & Price for Up and Out Call Option

Two cases have to be distinguished when considering the price and payoff function for an up and out barrier option as well. For a reverse up an out call option where the Barrier is above the strike price the payoff function $f(S(T))$ at maturity is given by 1.3.9.

$$f(S(T)) = \begin{cases} 0 & S(t) > B \\ S(T) - K & S(T) > K \,\&\, S(T) \leq B \end{cases} \quad 1.3.9$$

Calculation of price for up and out call is given through 1.3.10-1.3.14.

$$x1 = \frac{\ln(S_0/B) + (r - d - \sigma^2/2)(T)}{\sigma\sqrt{T}} \quad 1.3.10$$

$$x2 = x1 - \sigma\sqrt{T} \quad 1.3.11$$

$$y1 = \frac{\ln(\frac{B}{S_0}) + (r - d - \sigma^2/2)T}{\sigma\sqrt{T}} \qquad 1.3.12$$

$$y2 = y1 - \sigma\sqrt{T} \qquad 1.3.13$$

$$\begin{aligned} C^{UO}(S_0,\sigma,r,q,K,B,T) &= e^{-dT}S_0\left[N(d1) - N(x1)\right] \\ &- e^{-rT}K\left[N(d2) - N(x2)\right] - B(S_0/B)^p e^{-dT}\left[N(h1) - N(y1)\right] \\ &+ e^{-rT}K(S_0/B)^{p+1}\left[N(h1) - N(y2)\right] \end{aligned} \qquad 1.3.14$$

The case for up and out call is more complicated than the down and out case. They have low premium which is restricted by the knock out feature. Moreover as option approaches the barrier, the value of the option decreases although the plain vanillas call option price increases. This kink is due to the discontinuity of the up and out call at the barrier level.

1.3.2 In-Out Parity

If at time t, one has a portfolio consisting of a down and out version of ϕ_{LO} as well as down and in version of ϕ_{LI} then it is obvious that price at maturity will be the same as plain vanilla call. So this result leads the in-out parity given by 1.3.17.

$$\phi_{LO}(x) = \begin{cases} \phi(x) & x > B \\ 0 & x \leq B \end{cases} \quad \text{Down and out payoff} \qquad 1.3.15$$

$$\phi_{LI}(x) = \begin{cases} \phi(x) & x < B \\ 0 & x \geq B \end{cases} \quad \text{Down and in payoff} \qquad 1.3.16$$

$$F(t,s;\phi) = F(t,s;\phi_{LO}) + F(t,s;\phi_{LI}) \qquad 1.3.17$$

In-out parity is of course valid for the up and out and up and in case as well. So using in-out parity; analytical results can be extended to the down and in and up and in versions of the barrier options easily.

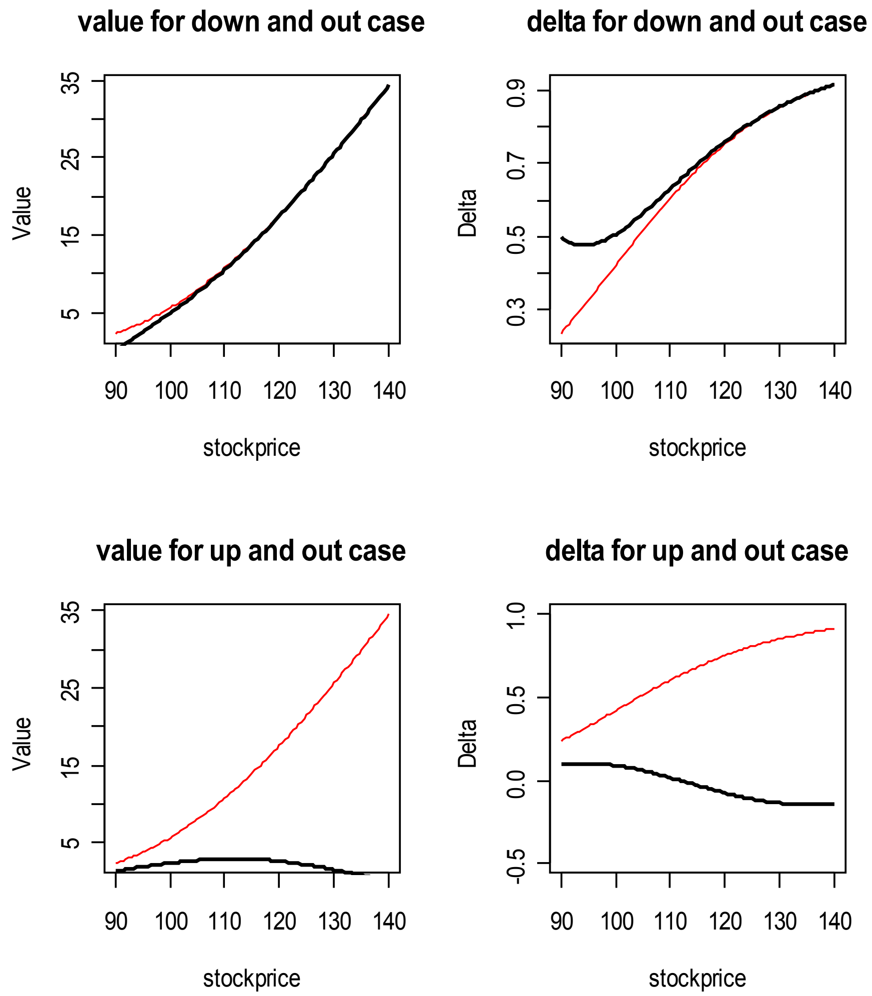


Figure1.2: Summary&Comparison of Delta and Prices for Barrier Options with Plain Vanillas

1.3.3 An Example

The following term sheet is an example of a real market structured product. The product is neither a barrier option nor a plain vanilla option completely.

It includes features of both types. Consequently it consist problems in pricing and/or finding the volatility or implied volatility driven by market for the product. The main question first to ask is whether the price of the product is reasonable or not?

As seen from the sheet that it is a simple put option with a simple premium if the trigger levels are not reached. However when trigger levels are reached, option behaves like one touch option in some sense. Ie the price of the option is 2% of the notional amount at the beginning, however (2+1.43) % of the notional amount after the first trigger level is reached. So the price of the option can be considered in three different levels given by 1.3.18.

With the given premiums, option is long in the put and short in the one touch. This makes sense because if the price of the underlying decreases then the put option ends in the money and one touch ends out of money and opposite as well.

Given S&P initial at 12 August 1998 at closing was 1281.43 USD (by Yahoo finance) then the strike price is 1217.3585 and #Contracts is 15608.

The price of this claim should be less than the price of the plain vanilla put option. For example if Black-Scholes model is used for pricing then price of a put option with expiry one year, strike price 1217.3585, initial price 1281.43 and volatility and interest assumed 0.2, 0.06 respectively is 43.74484. If we multiply this value with the number of contracts 15608 than the value is found as “682769” which is larger than 2% of notional amount “400000” as expected. So one can say that initial price of the claim is reasonable at the level of being less than plain vanilla put. Of course some other factors should be checked which I skip here.

The second question is finding a reasonable volatility for this option. For using an implied volatility, one needs the market price of the option. But for our case, market price is a three level constant. Consequently one has three different implied volatilities for three different intervals. For this reason it is not possible to find a well-defined implied volatility.

Table 1.2:Example

Over the Counter Option linked to the S&P 500 Index	
Option type	European put option, with contingent premium feature
Notional Amount	USD 20 MM
Trade Date	12 August 1998
Expiration date	11 August 1999
Underlying index	S&P 500
Settlement	Cash Settlement
Cash Settlement Date	5 Business days after the Expiration date
Cash Settlement Amount	Calculated as per the following formula: #Contracts*max[0, S&P strike- S&P final] where #Contracts=notional Amount/S&P initial This is same as a conventional put option: **S&P strike** will be equal to %95 of the closing price on the trade date **S&P final** will be the level of the Underlying Index at the valuation time on the Expiration date **S&P initial** is the level of the Underlying index at the time of execution
Initial Premium Amount	[2%] of Notional Amount
Initial Premium Payment Date	5 business days after Trade Date
Additional Premium Amounts	[1.43%] of Notional Amount per Trigger Level
Additional Premium Amounts Payment Dates	The additional premium amounts shall be due only if the underlying index at any time from and including the trade date and to and including the expiration date is equal to or greater than any of the trigger levels.
Trigger Levels	103%, 106%, 109% of **S&P initial**
Documentation	ISDA
Governing Law	New York

$$Price = \begin{cases} 2\%\, na & if \quad S\&P < 103\% \quad \text{of S\&P initial} \\ 3.43\%\, na & if \quad 103 \le S\&P < 106\% \quad \text{of S\&P initial} \\ 4.86\%\, na & if \quad 106 \le S\&P < 109\% \quad \text{of S\&P initial} \\ 6.29\%\, na & if \quad S\&P \ge 109\% \quad \text{of S\&P initial} \end{cases} \qquad 1.3.18$$

$na =$ notional amount

One approach for finding an implied volatility can be assuming that interest rate is constant and known moreover the probabilities (consequently the weights) of touching trigger levels are also constant and known so that a general constant price value (expected value) is found by 1.3.19.

$$\overline{P} = \sum_{i=1}^{4} w_i P_i \qquad 1.3.19$$

where w_i are the weights and P_i are the prices for trigger levels

After finding a constant price level one can use any model ie Black-Scholes formula for a simple implied volatility approach.

1.4 Hedging Barrier Options

Any investor who sells a derivative is faced with two problems:" How should I price the derivative" and "how should I deal with the risk about my position". In the previous part the answer to the first question is given. In this part the concern is the second one.

Although Barrier options are traded actively on the OTC markets, hedging of them is not that easy. The reason why they are more difficult is that they have unstable properties of the hedge parameters especially near the barrier. They combine the features of plain vanilla options with barriers as well. This additional feature makes barrier options sensitive to the price or volatility changes. So the Greeks of Barrier options exhibit slightly different behavior compared the plain vanilla counter parts. For example as stock price approaches to the barrier for the down and out call, the delta for the barrier option increases while the delta for the plain vanilla continues to decrease.

Moreover the gamma for the barrier options can be very large close to expiry. Due to the following reasons classical hedging techniques like dynamic hedging is not very practical for barrier options. On the other hand there have been many static hedging strategies that are developed since 90's which are more effective in some senses. But these static hedges in general have various restrictive assumptions as well. Some use BSM assumptions which leads constant volatility over time and it is a known fact that in real market volatility is not constant over time. Although some allows the volatility to be stochastic over time then they restrict the drift terms to be 0.

1.4.1 Dynamic versus Static Hedging

Hedging a derivative or a portfolio is an investment that is taken out specifically to reduce or cancel out the risk with a portfolio or derivative that is on the opposite direction of the existing one. Additionally a portfolio is considered to be dynamically hedged if the weights in the portfolio change dynamically with time. While static hedging releases its users from any re balancing needs by the certain combinations of vanilla options. The traditional method for valuing and hedging barrier option was using dynamic adjusted portfolios. But since mid-90'ies several static replication strategies have been developed and showed better performance then dynamic ones.

With the analytical solutions by Merton (1973) and Rubenstein&Reiner (1991), classical hedging techniques like delta hedging become possible for barrier options. An alternative approach was given by Bowie and Carr (1994) to the valuation and hedging of these options. Using the symmetry between puts and calls in the zero drift models, they have created portfolios of just a few options with fixed maturities to replicate barrier options. These results are extended to a symmetric volatility structure and to instruments like double and partial barrier by Carr,Ellis and Gupta (1996). Moreover Derman, Ergener and Kani(1995) created a static hedge portfolio with the use of plain vanilla options with different expiry dates.

Similarly Carr and Chou (1997) created a hedging strategy using vanilla options with same maturity and different strikes. Poulsen and Nalholm (2006) constructed many simulation studies to compare the success of static and dynamic hedge strategies on both Black-Scholes and Non Black-Scholes based models. Also Poulsen and Nalholm (2006) introduce a new method for static hedging of barrier options in the presence of leverage, correlated stochastic volatility, and jumps in the dynamics of the underlying.

Both the dynamic and static hedging strategies work perfectly in theory but the effect of real market dynamics is a serious problem for both. The main disadvantage of a perfect dynamic replication is that it requires continuous trading which can not be performed in real life because of huge transaction costs. The general solution to this problem is using delta of the Black and Scholes and doing periodical trading. This will reduce the error and the cost of replicating portfolio if the Gamma of the underlying is low. But barrier options often have high gammas which directly affect the delta hedging results. Also the volatility of the underlying asset has a direct effect on the dynamic hedge strategies but the high gammas often leads to high vegas as in the Barrier options. On the other hand replicating Exotic's like Barrier options with plain vanilla options are easier. Puts and calls can be traded on the market easily. They are much more liquid then OTC market options. Also Vanilla's contain more market information then just the underlying stock. Finally one needs more assets in incomplete markets and so on.

1.4.2 Dynamic Hedging Strategies for Barrier Options

Dynamic hedging is strategy that involves rebalancing hedge positions as market conditions change. The rebalancing can be daily, bidaily or more. But the shorter the time interval the better the hedging result is.

1.4.2.1 Pure Delta Hedging

The delta hedging method in the BSM is a standard method for dynamic replication. The delta represents the rate of change of the option price with respect to the change in the price of the underlying. By holding delta units of asset, investors offset the delta of the option position and thus, hold a delta neutral portfolio. Since the delta changes over time this action should be done periodically by the investor.

The delta of an option is defined by 1.4.1.

$$\Delta \equiv \frac{\partial F}{\partial S} \qquad 1.4.1$$

where F and S are the values of the option and the underlying, respectively.

In delta hedging, idea is immunizing the portfolio against small changes in the underlying asset price. Portfolio is constructed with adding a derivative to the underlying asset. Since the price of a derivative is perfectly correlated with the underlying asset price, one should be able to balance the derivative against the portfolio in such a way that the adjusted portfolio becomes delta neutral.

The delta value for the down and out call is defined by 1.4.2.

$$\Delta^{DO}(S_t,\sigma,r,q,K,B,T) = e^{-dT}\left[N(d1)\right]$$
$$- p\left(\frac{B}{S_t}\right)^{-p}\left[S_t e^{-rT}\left(\frac{B}{S_t}\right)^{2} N(h1) - Ke^{-qT}N(h2)\right] + \left(\frac{B}{S_t}\right)^{2-p} e^{-rt}N(h1) \qquad 1.4.2$$

Delta value for the up and out call is more complex and given by 1.4.3

$$\Delta^{UO}(S_t,\sigma,r,q,K,B,T) = e^{-dT}\left[N(d1)\right] - e^{-rT}\left[N(x1)\right] - e^{-rT}\left[n(x1)\right]\left(1-\frac{K}{B}\right)/\sigma\sqrt{\tau}$$
$$+ p\left(\frac{B}{S_t}\right)^{-p}\left[S_t e^{-rT}\left(\frac{B}{S_t}\right)^{2} N(-h1) - Ke^{-dT}N(-h2)\right] - \left(\frac{B}{S_t}\right)^{2-p} e^{-rt}N(h1)$$
$$- \left(\frac{B}{S_t}\right)^{-p}\left[S_t e^{-rT}\left(\frac{B}{S_t}\right)^{2} N(-y1) - Ke^{-dT}N(-y2)\right] - \left(\frac{B}{S_t}\right)^{2-p} e^{-rt}N(h1)$$

1.4.3

where $n(t)$ stands for the cumulative normal distribution

The hedging strategy described above depends on many restrictive assumptions. In practice, investors in this strategy are price takers or have very small influence compared to the market mass. So for a large change in prices can lead too large or too small deltas which are not acceptable because neither banks nor investors has unlimited amounts money that they can invest. They have limitations on the amount of borrowing and short selling. Also contrary what delta of an option suggests, securities are indivisible.

Moreover market has to be complete with no other expenses such as taxes or transaction costs. But in practice each delta adjustment implies a cost for the trader such as commissions and bid-ask spreads. Especially if gamma of the underlying is high it becomes much more difficult to construct delta hedge. Because high gammas lead high deltas which in turn mean more frequent transactions to re balance the position. Also continuous trading is not possible in real life. One has to construct discrete time intervals which in causes measurement errors.

Among the assumptions of the BSM, the strongest are the constant volatility and constant interest and dividend rate assumptions. It is shown by hundreds of evidence that assuming constant volatility over time can be very dangerous like in jumps for the underlying.

1.4.2.2 Mixed Delta Hedging

Greeks of the barrier options are different from the plain vanillas as mentioned earlier. This difference in Greeks cause additional problems other than mentioned above when it comes to delta hedging. In order to overcome these problems there are many suggestions. One of which can be called as mix delta hedging.

The general setup for the delta hedge was: a hedger sells/buys a barrier option on an underlying stock; he receives/pays the price of the option and sets up a hedge portfolio by buying/selling delta shares of the stock and investing the rest in the bank account. During time hedger adjusts the position continuously in order to maintain delta neutrality.

The construction of the mix hedge is quiet similar to the above construction. The idea is: since the large part of the error in the pure delta hedging is due to the kink in the payoff function off the plain vanilla call, this error can be eliminated by buying/selling one plain vanilla call at the strike price equal to the barrier option. So the delta of the strategy is as in 1.4.4.

$$\Delta(mixed) \equiv \Delta(barrier) - \Delta(call) \tag{1.4.4}$$

For example one sells a barrier option with strike price K and barrier B at time 0. Let this option costs X_1. Then he buys a call with strike price K for X_2. Moreover he buys the $\Delta(mixed) \equiv \Delta(barrier) - \Delta(call)$ units of the underlying stock which is equal to X_3. So the money that he needs to borrow from bank is $X_1 - X_2 - X_3$. And the delta which he will adjust during time is $\Delta(mixed) \equiv \Delta(barrier) - \Delta(call)$ which is much smaller than the delta of the plain vanilla. ie delta of the down and out and up and out call for mixed strategy is given through 1.4.5-1.4.6.

$$\Delta^{DO}(S_t,\sigma,r,q,K,B,T) = -p\left(\frac{B}{S_t}\right)^{-p}\left[S_t e^{-rT}\left(\frac{B}{S_t}\right)^2 N(h1) - Ke^{-qT}N(h2)\right] + \left(\frac{B}{S_t}\right)^{2-p} e^{-rt}N(h1) \tag{1.4.5}$$

$$\begin{aligned}\Delta^{UO}(S_t,\sigma,r,q,K,B,T) &= e^{-rT}\left[N(x1)\right] \\ &- e^{-rT}\left[n(x1)\right]\left(1-\frac{K}{B}\right)/\sigma\sqrt{\tau} - \left(\frac{B}{S_t}\right)^{2-p} e^{-rt}N(h1) \\ &+ p\left(\frac{B}{S_t}\right)^{-p}\left[S_t e^{-rT}\left(\frac{B}{S_t}\right)^2 N(-h1) - Ke^{-dT}N(-h2)\right] - \left(\frac{B}{S_t}\right)^{2-p} e^{-rt}N(h1) \\ &- \left(\frac{B}{S_t}\right)^{-p}\left[S_t e^{-rT}\left(\frac{B}{S_t}\right)^2 N(-y1) - Ke^{-dT}N(-y2)\right]\end{aligned} \tag{1.4.6}$$

1.4.3 Static Hedging Strategies for Barrier Options

Barrier options can be hedged "statically" using a portfolio of standard options or other products. Such a strategy avoids transaction costs from re-hedging and this is why it may be preferred especially in illiquid markets. With using static hedge portfolios the gamma and vega risk exposures is passed from the illiquid OTC markets to liquid plain vanilla options market. The idea is trying to construct a hedge such that the payoff function will be the same as barrier option if the barrier is hit or not hit.

1.4.3.1 Calendar Spread

One can create static hedges for barrier options with using plain vanilla puts and calls with different maturities. The calendar spread method of Derman, Ergener and Kani (1995) hedges the payoff of the barrier options along the barrier and at maturity using a portfolio of vanilla options. Here the idea is to replicate the behavior of the barrier option at any moment of its life with the use of plain vanillas. Consequently the more the plain vanilla options included in the portfolio, the better is the hedge.

For example if one wants to construct calendar spread for down and out case he has to start with buying a call with a strike price equal to the Barrier options strike price at time 0.

⟶Buy one call with strike K expiry T

⟶Sell one α_n strike B expiry T put

Such that;

$$\alpha_n * Put(B, t_{n-1} / B; T) + Call(B, t_{n-1} / K; T) = 0$$

This portfolio has zero value is the barrier is hit. If the barrier is not hit, the call option has the same pay off with the barrier option which in turn offset it.

Meanwhile more put options are sold with strike price equal to barrier with different maturities. So the value of the portfolio is forced to 0 along the barrier.

To continue;

⟶Sell α_{n-1} strike B expiry t_{n-1} puts such that;

$$\alpha_{n-1} * Put(B, t_{n-2} / B; t_{n-1}) + \alpha_n * Put(B, t_{n-2} / B; T) + Call(B, t_{n-2} / K; T) = 0$$

Various puts can be added to the portfolio such that;

$$\alpha_i * Put(B, t_{i-1} / B; t_i) + \sum_{j=i+1}^{n} \alpha_j * Put(B, t_{i-1} / B; t_j) + Call(B, t_{n-1} / K; T) = 0$$

This portfolio has the same characteristics with the previous one with one strike K call and strike B put since it has the same characteristic around barrier. If the barrier is hit portfolio has value 0, if it is not hit the puts end worthless and call option has the same payoff with the barrier option.

For the portfolio constructed above if infinitely many puts are taken then the portfolio is perfectly hedged and the error is 0. But this is obviously impossible in practice and one has to choose the optimal number of puts that should be taken.

One last thing to mention is about the weights of the puts to be calculated. It is easy to calculate the weights of the puts because the system to be solved is a triangular system of equations as through 1.4.7 to 1.4.9.

$$X * \alpha + Y = 0 \qquad 1.4.7$$

$$X = \begin{pmatrix} 0 & \dots & 0 & Put(B,B,t_n,\tau_n) \\ 0 & \dots & Put(B,B,t_{n-1},\tau_{n-1}) & Put(B,B,t_{n-1},\tau_n) \\ \vdots & & \vdots & \vdots \\ 0 & \dots & Put(B,B,t_0,\tau_{n-1}) & Put(B,B,t_0,\tau_0) \end{pmatrix} \qquad 1.4.8$$

$$Y = \begin{pmatrix} Call(K,B,t_n,\tau_n) \\ Call(K,B,t_{n-1},\tau_n) \\ \vdots \\ Call(K,B,t_0,\tau_0) \end{pmatrix} \qquad 1.4.9$$

For an up and out call case, the construction of the portfolio is quiet similar. One has to buy calls instead of puts to hedge the barrier option.

1.4.3.2 Strike Spread

Another way of creating static hedges is with using strike prices of the options in the portfolio is constructed. Here the main idea is to force the payoff of the barrier option to the zero around the barrier. In order to do this one has to buy options with different strikes with a weight such that if the barrier is hit the value of the portfolio will be 0.

1.4.3.2.1 Strike Spread by Carr, Ellis and Gupta

The first method to be considered is the one by Carr, Ellis and Gupta (1998) (CEG). They use a terminology called put call symmetry (PCS), to develop a method for valuation and static hedging of exotic options. PCS can be viewed as both an extension and a restriction of widely known put call parity (PCP) CEG (1998). Extension is PCS allow the puts and calls to have different strikes. Meanwhile the restriction is zero drift and symmetric volatility structure. The symmetry that they found is the following;

European Put-Call Symmetry: Given frictionless markets, no arbitrage, zero drift, and the symmetry condition, following relation holds;

$$C(K)K^{-1/2} = P(B)B^{-1/2}$$

where the geometric mean of the call strike K and the put strike B is the forward price F:

$$(KB)^{1/2} = F$$

With the above symmetry one can construct strike spreads for various types of barrier options. Single barrier options, multiple barrier options, ratchet options, lookback options are some examples.

With the use of PCS the construction of static hedge for the down and call option where $B < K$ and when $F = B$ is as in 1.4.10.

$$C(K) = KH^{-1}P(B^2K^{-1}) \qquad 1.4.10$$

According to the equation above one writes KH^{-1} European puts struck at H^2K^{-1} to complete the hedge. To complete the replicating portfolio for a DOC one sells a down and out call with strike K and buys one plain call with strike K as well. Then sells KH^{-1} European puts struck at H^2K^{-1} to complete the hedge. If the barrier is not hit the call and the barrier option will have the same payoff. On the other hand if the barrier is hit, PCS guarantees that the proceeds from selling the call will be exactly offset by the cost of buying back the puts.

$$DOC(K,B) = C(K) - KB^{-1}P(B^2K^{-1}) \qquad 1.4.11$$

One other application of PCS is for up and out call. The construction of the hedge is a bit complex but it is still simple to understand.

$$UOC(K,B) = C(K) - UIP(K,B) - (B-K)UIB(B) \quad B > K,F \qquad 1.4.12$$

where UIP stands for up and in put and UIB stands for up and in bond. Writing the equation as above with UIP&UIB has an advantage that it can be used for continuous processes. But it has a disadvantage because UIP&UIB may not be traded on the market. Applying PCS $UIP(K,B)$ can be replicated with KH^{-1} European calls struck at H^2K^{-1}. And $UIB(B)$ can be replicated with binary calls BC struck at B and B^{-1} calls struck at B as;

$$UIB(B) = 2BC(B) + B^{-1}C(B) \qquad 1.4.13$$

To sum up the following information, $UOC(K,B)$ can be hedged with the use of following equation;

$$UOC(K,B) = C(K) - KB^{-1}C(B^2K^{-1}) - (B-K)\left[2BC(B) + B^{-1}C(B)\right] \qquad 1.4.14$$
$$B > K, F$$

Thus to hedge an up and out call one sells one up and out call with strike buys one vanilla call with strike K, sells KB^{-1} vanilla call strike B^2K^{-1} and sells (B-K) times two binary calls with strike B and B^{-1} call with strike B as well.

1.4.3.2.2 Strike Spread by Carr and Chou

Carr and Chou convert the problem of hedging a barrier option to a problem of hedging a European security with non-linear payoff function. The strike spread that they found, matches the payoff function of the barrier option above or below the barrier, depending on the type of the barrier option. For example if the barrier option is up and out call then the strikes and the matching points will be larger than the barrier. The success of the strike spread depends on increasing the matching points of the hedge. If the hedging is done for infinitely many points then the hedge will be perfect and the error will be 0. But of course this is again not possible in practice. The adjusted payoff function for down and out call can be written by 1.4.15.

$$h_{DO}(x) = \begin{cases} g(S) & S > B \\ -\left(\frac{S}{B}\right)^{p} g\left(\frac{B^2}{S}\right) & S \leq B \end{cases} \qquad 1.4.15$$

where $g(s) = (S - K)^{+}$ and $p = 1 - 2(r - d)/\sigma^{2}$

Similarly the adjusted payoff for up and put case is as in 1.4.16

$$h_{UO}(x) = \begin{cases} -\left(\frac{S}{B}\right)^{p} g\left(\frac{B^2}{S}\right) & S > B \\ v(S) & S \leq B \end{cases} \qquad 1.4.16$$

The construction of the strike spread for up and out call is done by using calls with different strikes. Since it is impossible to match the adjusted payoff at every point in real life, one has to choose some matching points and strikes above the barrier. Say the strike points K_i then matching points as X_i such that the 1.4.17 holds.

$$X_1 > K_1 > X_2 > K_2 \ldots X_n > K_n > B \qquad 1.4.17$$

moreover let α_i be the weights of the calls in the portfolio. Then unlike DEK's triangular system of equation one has in 1.4.18-1.4.20;

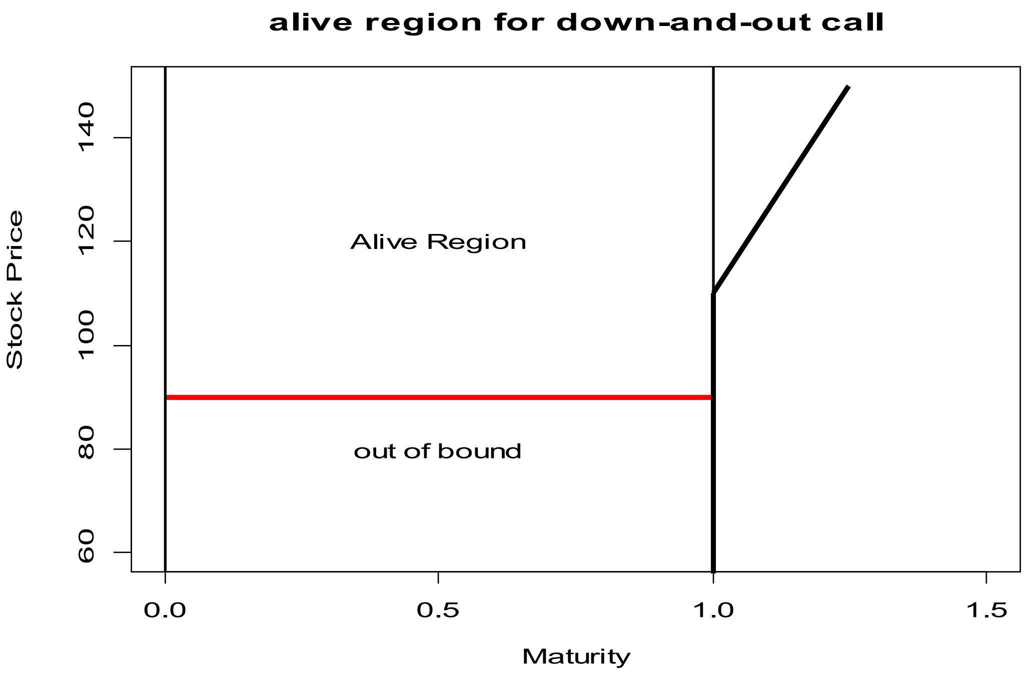
$$X * \alpha + Y = 0 \qquad 1.4.18$$

$$X = \begin{pmatrix} C(X_1, K_1, T) & 0 & & 0 \\ C(X_2, K_1, T) & C(X_2, K_2, T) & \dots & 0 \\ \vdots & & & \\ C(X_{n,}, K_1, T) & C(X_n, K_2, T) & \dots & C(X_n, K_n, T) \end{pmatrix} \qquad 1.4.19$$

$$Y = \begin{pmatrix} h_{UO}(X_1) - C(X_1, K, T) \\ h_{UO}(X_2) - C(X_2, K, T) \\ \vdots \\ h_{UO}(X_n) - C(X_n, K, T) \end{pmatrix} \qquad 1.4.20$$

The construction is the similar for down and out call. First difference is choosing puts instead of calls while constructing the spread. And the second difference is matching points and strikes are constructed as in 1.4.21;

$$X_1 < K_1 < X_2 < K_2 \dots X_n < K_n < B \qquad 1.4.21$$



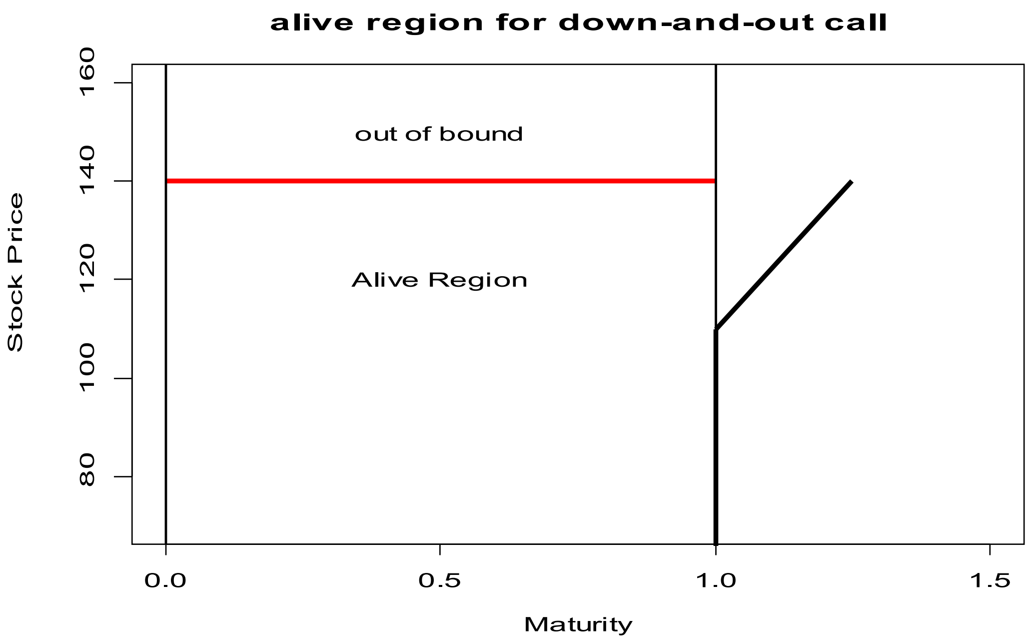


Figure1.3: Payoff function for down and out and up and out calls

CHAPTER 2

PRICING AND HEDGING IN BLACK-SCHOLES MODEL

2.1 Black-Scholes Model

The Black-Scholes model, for which Merton and Scholes received the 1997 Nobel Prize in Economics, is a tool for option pricing. Prior to its foundation there was no standardized way of pricing options. So one can say that Black-Scholes model marks the beginning of the modern era of pricing financial derivatives.

Like all successful models, its success is its ability to simplify the reality. But the weakness of the model is that it relies on many restrictive assumptions such as;

- The price of the underlying instrument (stock price for our case) follows a geometric Brownian motion. This means there are no jumps through the life of the option

$$S(t) = S(0)\exp\left(r - \frac{1}{2}\sigma^2 \right) t + \sigma W(t)$$

where;

$S(t), r, \sigma$ are as defined previosusly

$S(0)$ is the stock price at time 0

t is the time step

$W(t)$ is Geometric Brownian motion

- The volatility of the stock price σ is constant during the life of the option

- The interest rate r is constant during the life of the option

- There are no arbitrage opportunities in the market.

- It is possible to short sell the underlying stock

- It is possible to borrow or lend cash at a constant rate r

- There are no transaction costs such as taxes or bid ask spreads for neither the option nor the stock

- Trading in the stock is continuous

- All securities are divisible perfectly. Ie one can buy 1% of the share

- Investors can only exercise the option at the expiration

- The stock pays no dividends.

2.2 Pricing Barrier Options in BSM

Pricing of options is a big pool of literature and survey. Some of these literatures construct their results on analytical solutions like for plain vanilla barrier options while some others depends on simulation techniques.

2.2.1 Simulation Approach

In quantitative finance, problems like finding the arbitrage-free price of a derivative is reduced to computation of an integral. Most of the cases for these integrals, solutions are found analytically or computed with partial differential equations as in BSM. However when the number of dimensions in the problem is large, PDE'S and integrals became intractable. At this point simulation, which is an imitation of real things, state affairs or processes, becomes a good candidate for solving such kind of high dimension problems.

The fundamental option pricing theory is that the value of an option is simply the expected value of the discounted payoffs where the expectation is taken under the risk neutral measure. So finding the value of an option using simulation is just generating stock price paths for the given model and following the value of the derivative through the life of the stock. The steps for simulation are the following;

- Simulate a stock price path for the underlying asset with the model desired under the risk neural condition, over the given time horizon

- Discount the payoff of the option corresponding to the path at the risk free rate

- Repeat the procedure for large number of times

- Average the results which are discounted payoffs to obtain the desired option price.

The law of large numbers guarantees the convergence of these averages to the actual price of the option. Moreover central limit theorem guarantees that the standard error of the estimate tends to zero with a rate of convergence $\frac{1}{\sqrt{N}}$. If one thinks that the convergence rate is slow then further variance reduction techniques can be used. Moreover it is possible to have better rate of convergence with using quasi-random numbers to reduce the number of trials that is needed.

For example if BSM is used for the simulation of stock price paths the following can be used;

$$S(t) = S(0)\exp\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma W(\Delta t)$$

where;
all the parameters are same as above
Δ is the time interval between observations

2.2.2 Simulated Prices for Down and Out Call Options

Barrier options are good applications for simulation study since their payoff are more complicated than the plain vanilla counter parts. To check the reliability of the simulations, simulated plain vanilla prices are good controllers. So through the simulations the prices of the plain vanillas are also going to be found. Through simulation studies for the price of down and out barrier options the following steps are followed;

- Price paths for the stock are simulated using Geometric Brownian motion at risk free rate
- At each time step it is checked whether the barrier is hit or not
- If the barrier is hit the barrier option ends with zero value while plain vanilla counter part continues its life
- At the end of maturity the prices for the options are found
- The result is discounted back with the appropriate discount rate for both barrier option and plain vanilla
- The simulation study is done for10^4 times
- The mean of the results are taken for both cases and are counted as the simulated price of the option.

In order to simulate the stock price paths as defined above the R program is used and the parameters for the program are given in Table 2.1.

Deviations from analytical prices are also a concern for the following simulation study. For this reason one last thing to be mentioned before continuing to the simulation results is the comparison of the simulated prices with the analytical ones.

The error term in 2.2.1 can be used to have an idea of pricing deviation of the simulations for different barrier levels.

$$error = \frac{Abs(BSM\ \ price - simulatedprice)}{BSM\ \ price} * 100 \qquad 2.2.1$$

Table 2.1: Parameters and symbols for simulation study

Quantity	**Symbol**	**Value**
Initial Stock Price	S(0)	100
Interest rate	r	0.06
Dividend Yield	d	0.02
B-S Volatility	σ	0.2
Carr-Chou ρ	$p = 1 - 2(r - d)/\sigma^2$	-1
Strike of the underlying call	K	110
Expiry of the underlying call	T	1
Down and out Barrier	B^{DO}	90
Up and out Barrier	B^{UO}	140
Delta Hedging Time Step	Δt	1/126
No of Simulated Paths	N	10.000

Table 2.2:Results of simulation for different barrier levels

Barrier level	**Down&out call price in BS model**	**Simulated plain vanilla call price**	**Simulated down&out call p.**
95	3.326043	5.589557	3.810011
90	4.852342	5.498102	4.936848
85	5.387124	5.566095	5.42816
80	5.521628	5.501014	5.489589
75	5.544258	5.555913	5.555365

Table 2.3:Results of simulation with different confidence bands

Barrier	%95 confidence level Lower bound	%95 confidence level Upper bound	Error %
95	2.055146	5.564877	14.55088
90	3.042632	6.831063	1.741594
85	3.479294	7.377027	0.7617449
80	3.58351	7.395668	0.5802481
75	3.595096	7.515634	0.2003465

The vanilla price in the BS model with the parameters defined above is 5.5467. Through all the simulations the samples are chosen such that the error term for the plain vanilla is less than 1%.

The results show that as the barrier level lowers the error in the simulated prices gets smaller. This is not surprising because as the barrier level decreases the probability of hitting to the barrier decreases which means the barrier option more behave like plain vanilla option. This result is also verified by the analytical price of the barrier option. As the barrier gets lower then analytical prices for the plain vanilla and down&out call becomes closer. One surprising result happens when the barrier level moves from 90 to 95. The error in the simulated prices is nearly 10 times more than the previous barrier level.

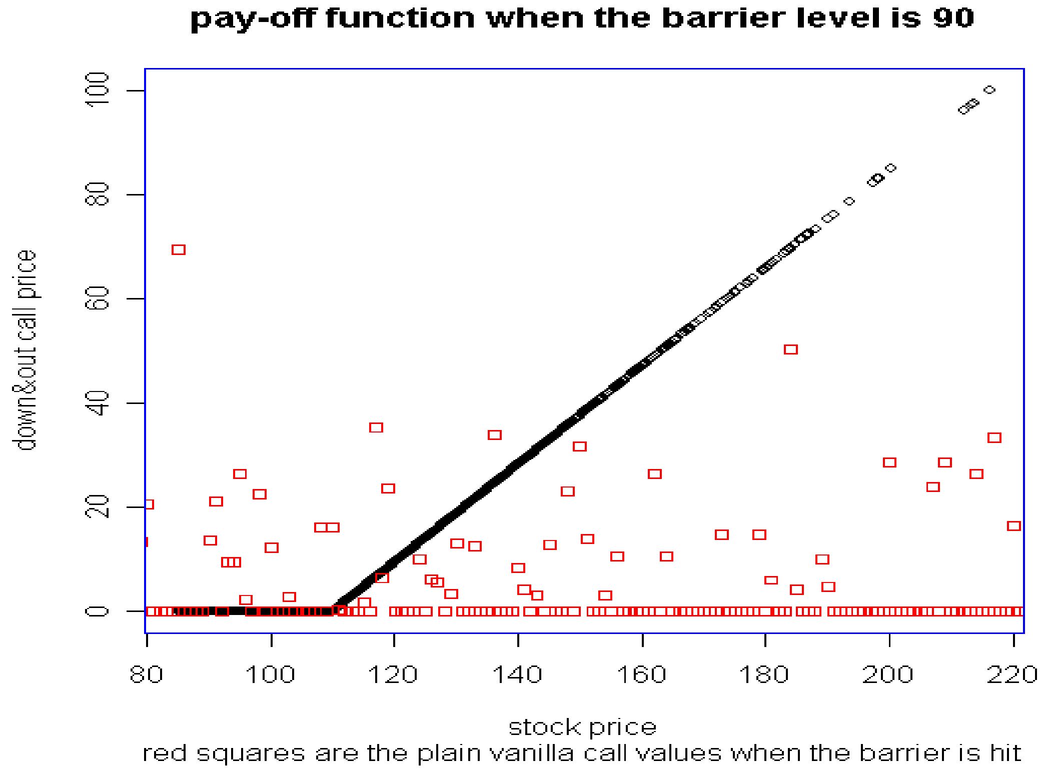


Figure 2.1:Payoff function for down and out call option

2.2.3 Simulated Prices for Up and Out Call Options

The simulation study for up and out call option is done with the same principle as in down&out call. Only difference is that the barrier level is now 140 instead of 90 so that the barrier is hit from below.

Table 2.4:Results for up&out call case

Barrier level	Barrier option call BS model	Simulated plain vanilla call price	Simulated up&out call price
150	3.512527	5.524443	3.692134
145	2.929400	5.517768	3.131986
140	2.276883	5.507764	2.469064
135	1.600082	5.552361	1.852425
130	0.9684055	5.494151	1.198186

Table 2.5:Results of simulation with different confidence bands

Barrier	%95 confidence level Lower bound	%95 confidence level Upper bound	Error %
150	2.391001	4.993268	5.113338
145	1.977027	4.286945	6.915616
140	1.501880	3.436248	8.440544
135	1.073829	2.631021	15.77063
130	0.6172578	1.779115	23.72776

Plain vanilla option price is of course not affected from the change in barrier level and it is still 5.5467 and the idea of including samples which have less then 1% error for plain vanillas are chosen again.

According to the simulated results it is seen that the error in pricing is much higher than the down&out case. This is due to the discontinuity of the up and out call at the barrier. Not surprisingly as the barrier level increases error of the simulation decreases. This is again the result of barrier options to behave like a plain vanilla as the barrier level deviates from the strike price.

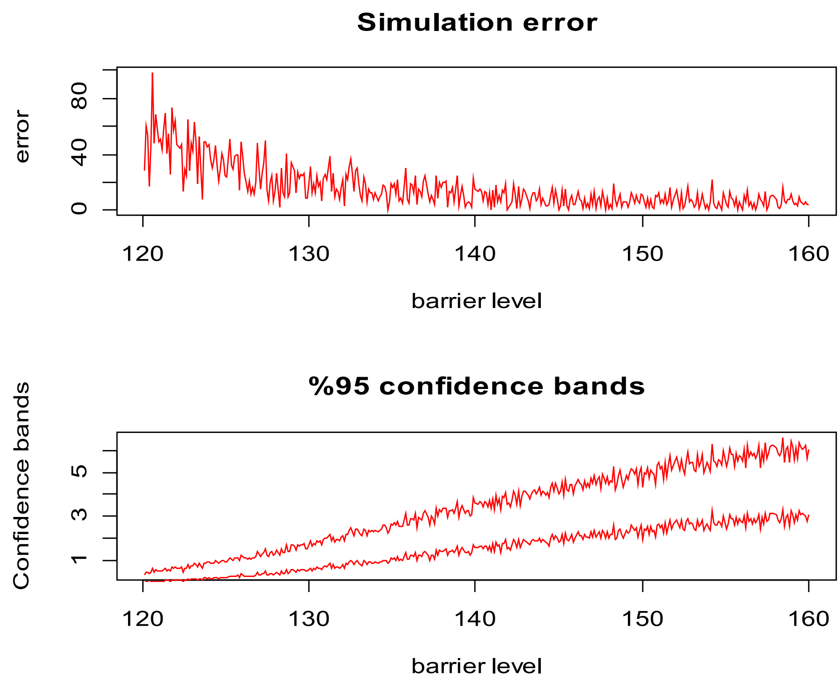


Figure 2.2:Simulation Errors

2.3 Hedging Barrier Options in BSM

2.3.1 Application of Simulation to Hedging Strategies

Simulation techniques can be very useful to compare different hedging strategies for barrier options. The idea is simulating stock price paths and then checking the success of the hedging technique used by looking to the simulation results. One can construct the simulation steps as the following;

- Specify the model to be used in the simulation study. For example use Geometric Brownian motion for BSM model at risk free rate.

- Suggest different hedging strategies like the ones defined in the previous chapter.

- Simulate paths for the stock price from the model chosen.

- The expiry dates and the strike prices of the options that are used in the simulation stay constant through the study. Moreover barrier level that is used is linear and does not change during the life of the option as well.

- Adjust the dynamic hedge strategies along the stock price path which means construct the required hedge portfolio every trading date using the delta. For the static hedging strategies construct the portfolio once at the beginning before starting trading. For dynamic hedging strategies keep the strategy self financing by borrowing or lending from the bank.

- At each time step check whether the barrier is hit or not.

- If the barrier is hit, barrier option value ends as a zero value option then liquidate the portfolio that is constructed.

- If the barrier is not hit; find the price at the end of maturity. The price of the barrier option will be the same with the plain vanilla option.

- Subtract the barrier option price from the portfolio value and save the results as error(Profit/Loss)

- Discount the errors back with the appropriate discount rate and hitting time for barrier option.

- Do the simulation for many times

- Find the moments such as mean and variance of the errors with respect to the real option price. And check the distributional characteristics of the errors.

During the construction of the simulation study that is described above the parameters that are used during pricing simulations are used.

2.3.2 Dynamic Hedging Strategies

2.3.2.1 Pure Delta Hedging

The pure delta hedging technique for barrier options is described in detail in the chapter 1. Following are some numerical results to compare the performance of this strategy with the other strategies.

2.3.2.1.1 Down and Out Case

The simulation study is done for different barrier levels and the following results are on table 2.6.

By looking the results it is seen that the mean values of the errors are close to 0 and the standard deviations are around 10% for the error terms.

These values do not decrease or increase depending on the level of error. This is an interesting result because one expects that the error term will decrease as the barrier level decreases.

The reason for this is the structure of the delta hedge for down and out case. The strategy is continuous at every point so at the hitting time there is no big difference between portfolio value and barrier option price. Moreover the delta used at this strategy depends on the barrier level as well. Ie if the barrier level is 90 then portfolio has a value 0 around 90 if the barrier is 95 again the portfolio has a value 0 around 95 because delta of the barrier option reflects barrier level as well.

Table 2.6:Results for down&out pure delta hedging

Barrier level	Real price	Mean of errors	SD of errors
95	3.326043	0.0003615876	0.1047099
90	4.852340	0.0009534759	0.1013058
85	5.387124	-0.001083549	0.1071984
80	5.521628	-0.001717777	0.1103815
75	5.544258	-0.001146284	0.1101915

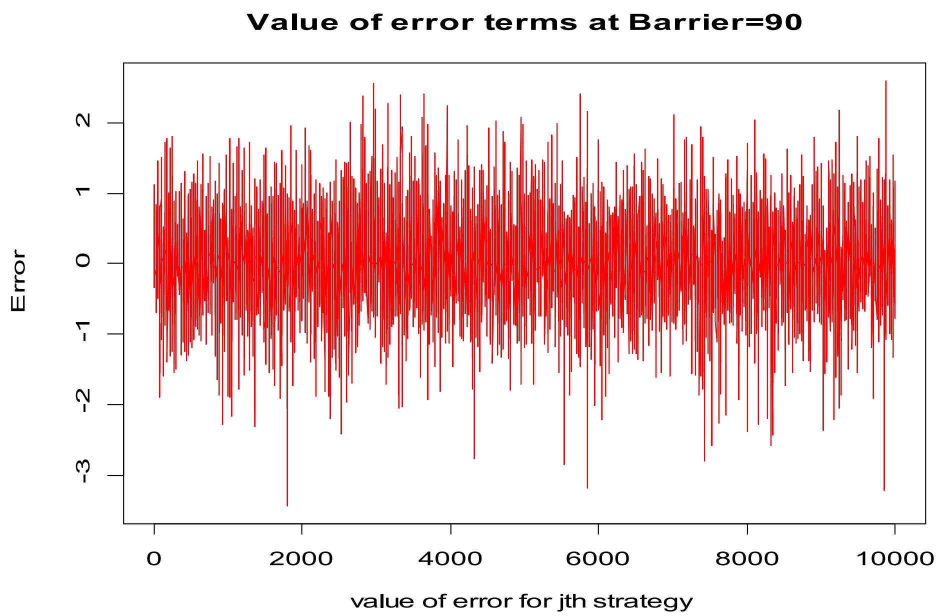


Figure 2.3:Error Terms

One more detail that can be mentioned is about the time step. Because it is known as the time step increases then the success of delta hedging strategy increases. And if the time step for hedging is assumed to be continuous then the error will be zero. It can be seen from table 2.7 that as time interval for the hedging strategy increases the success of the strategy increases as expected.

Table 2.7:Results for increasing time interval for pure delta hedge

Hedging time step (Barrier=90)	**SD of errors**
126(every two working day)	0.1013058
252(every working day)	0.07299965
504(two times every working day)	0.05175068
1008(four times every working day)	0.03620822
2016(eight times every working day)	0.02526977

2.3.2.1.2 Up and Out Case

The simulation study is constructed for different barrier levels and results on table 2.8 are obtained.

Table 2.8:Results for up&out pure delta hedging

Barrier	**Real price**	**Mean value of errors**	**Standard deviation of errors**
150	3.512527	-0.007121712	0.5651317
145	2.929400	0.002121180	0.7580842
140	2.276883	-0.01432327	0.8172131
135	1.600082	0.001035372	1.135409
130	0.9684055	0.006445856	1.395589

Results are, especially the standard error terms, are high for the up and out case. The reason having this much of error is due the discontinuity of the up and out call at the barrier level. ie consider a delta hedged portfolio that has a value around 29 at the stock price level 139 when the barrier is 140. At this point the barrier option price is around 29 as well. But at the time the barrier is hit, the portfolio still has a value around 30 on the other hand barrier option has a value of 0 which in turn cause extremely high error terms. This kind of behavior affects the success of the hedging such that as the barrier level decreases SD of error increases. More hitting cases with discontinuities occur around strike price.

Table 4 is the graphical representation of the results above. As it can be seen from the graphs where the barrier value is around 160 standard deviation and mean of the error terms are near zero. As barrier level gets closer to the strike price then the standard deviation appears to be around 10 which is totally not an acceptable number for a hedging strategy.

2.3.2.2 Mixed Delta Hedging

Since large part of the error in the pure delta hedging is due to the kink in the payoff function off the plain vanilla call, this error can be eliminated by buying one plain vanilla call at the strike price equal to the barrier option as mentioned in the first chapter. Consequently the delta of the strategy is give by 2.3.1.

$$\Delta(mixed) \equiv \Delta(barrier) - \Delta(call) \qquad 2.3.1$$

By using delta in 2.3.1, residuals are delta hedged with the same parameters that are used for the pure delta hedging.

2.3.2.2.1 Down and Out Case

Simulation study is done for the same barrier levels as in the pure delta case. Only difference is the change in delta that is used during the hedging. Results are given in table 2.9 and 2.10.

Table 2.9:Results for down&out mix delta hedging

Barrier	**Real price**	**Mean value of error**	**Standard deviation of error**
95	3.326043	0.0002084342	0.06477233
90	4.852340	-0.0006053655	0.02747667
85	5.387124	-0.000001008832	0.01071951
80	5.521628	-0.000001352101	0.003268369
75	5.544258	-0.000000710509	0.0006085129

Table 2.10:Results for increasing time interval for mix delta hedge

Hedging time step (Barrier=90)	**SD of errors**
126(every two working day)	0.02747667
252(every working day)	0.01900768
504(two times every working day)	0.01328525
1008(four times every working day)	0.009168254
2016(eight times every working day)	0.006607005

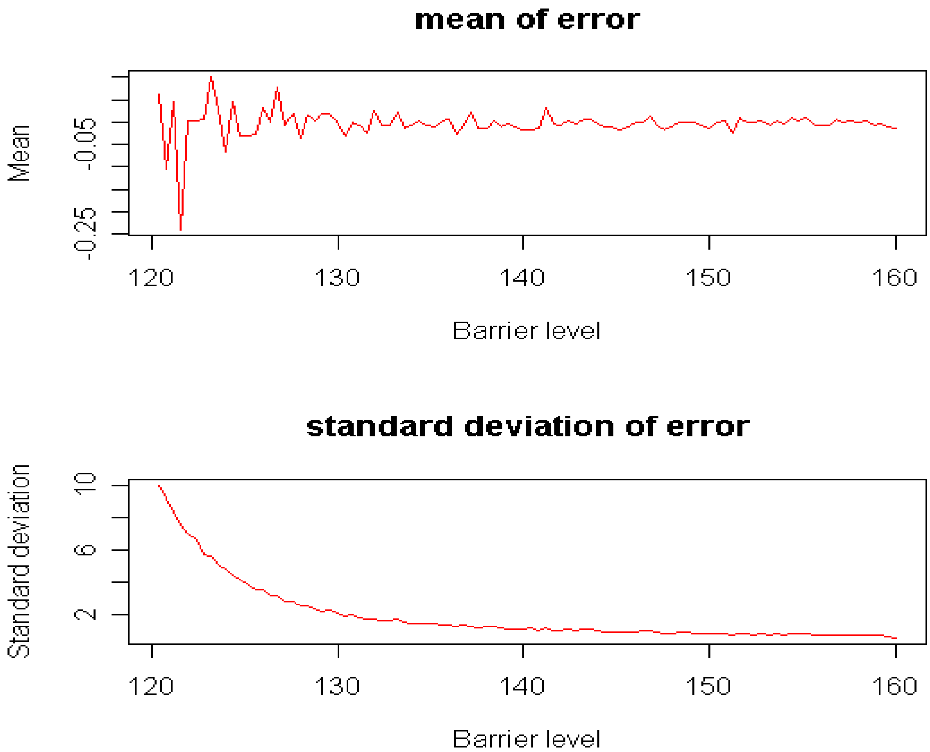


Figure 2.4:Mean and Deviation of Errors for Different Barriers

By looking to table 2.9 it is seen that the mean value is much smaller than the case in pure delta hedging. It is almost around zero. Moreover there is a high improvement in the standard errors of the estimates. Standard errors are more than 5 times less compared to the pure delta hedging. This proves that the kink in the delta of the barrier option is mainly from the plain vanilla part. Moreover by eliminating the delta of the plain vanilla I have received a decreasing standard error with barrier level.

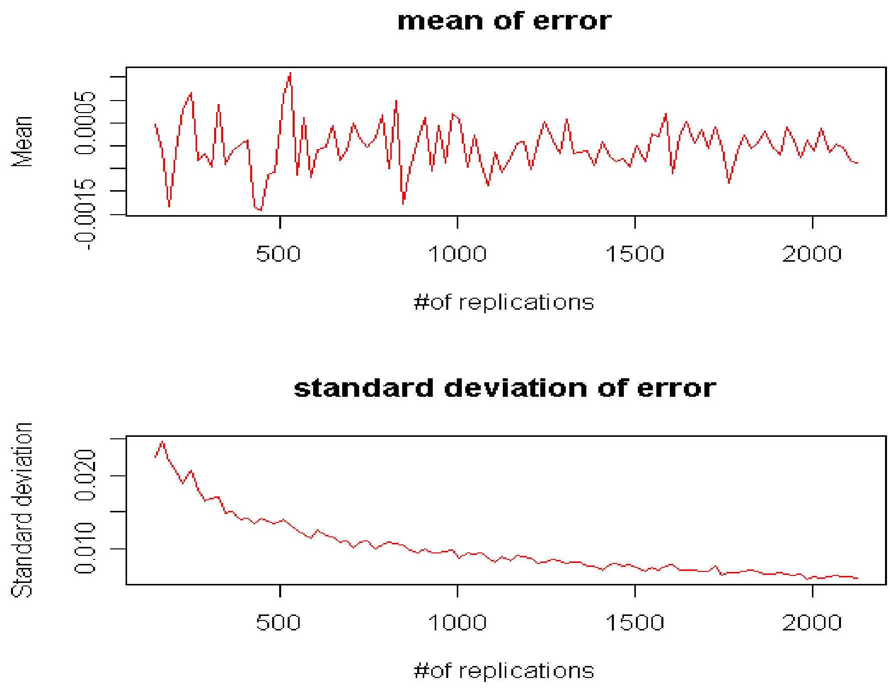


Figure 2.5:Mean and Standard Deviation for Errors for Increasing Time Intervals

Figure 2.5 illustrates that increase in time intervals decrease the standard deviation of the hedge error. Ie there is a SD around 2%when the time interval is 126 days, every two working day, while the error that is observed is almost zero as the number of replication goes to 2000 times, eight times every working day. Of course this result is not very surprising because it is a known fact for any kind of dynamic hedging that as the number of time intervals that is taken increased the performance of the hedge increases. This point is one of the most argued assumptions of the dynamic hedges as well because in real life it is not possible to make infinitely many time intervals.

2.3.2.2.2 Up and Out Case

Similar arguments can be found for up and out mix hedge as well. Table 2.11 is a short summary what I have obtained for different barrier levels.

Table 2.11:Results for up&out mix delta hedging

Barrier	**Real price**	**Mean value of errors**	**Standard deviation of errors**
150	3.512527	9.26751e-05	0.5629783
145	2.929400	0.01033533	0.7535478
140	2.276883	-0.01200034	0.8033402
135	1.600082	0.005614266	1.134652
130	0.9684055	0.02243552	1.59919

Again even the mix delta hedge does not remove the problems caused by the discontinuity of the up and out call option at the barrier level. The results are almost the same with the results of pure delta hedging.

2.3.3 Static Hedging Strategies

2.3.3.1 Calendar Spread

Through the simulation study below the calendar spread of DEK is used which has the idea to force the payoff of the barrier option to 0 around the barrier level with using different combinations of plain vanillas with different strike prices.

2.3.3.1.1 Down and Out Case

To construct the simulation for the down and out case it is assumed that a plain vanilla call option with strike price equals to the strike of down&out call option and expiration date which is again the same with the down&out call is sold. While doing this transaction 4 plain-vanilla puts with strike prices equal to the barrier level are bought. But unlike plain vanilla call these puts have different expirations which are 1, 0.75, 0.5, 0.25 times the maturity of the down&out call option respectively. The weights of the put options that are bought are constructed by using the algorithm explained in the first chapter. The value of constructing such a portfolio with the weight given in table 2.12 and it is 4.785633. The value of the barrier option at time 0 is 4.8523. So the error at the beginning is 0.066667.

Also one can see in Table 2.12 the weights and the expenses that are needed to construct DEK calendar spread. Figure 2.6 is an illustrative graph that shows how calendar spread works. The graph is constructed for 252 equal time intervals. So the maturities of the puts are 63,126, 189, and 252. The error that is obtained during the life of the option is the smallest at the beginning. But after the first option expiries its gets around 0.15 and fluctuates at that value till the end of the time that down and out call expiries.

Table 2.12: Results for down&out DEK calendar spread

Weight	Strike	Maturity	Price
1*Call	110	1	5.546741
-0.03420489*Put	90	1	-0.08487433
-0.1650233*Put	90	0.75	-0.3260472
-0.1860512*Put	90	0.5	-0.2515123
-0.1705399*Put	90	0.25	-0.09867438
Value of Portfolio			4.785633

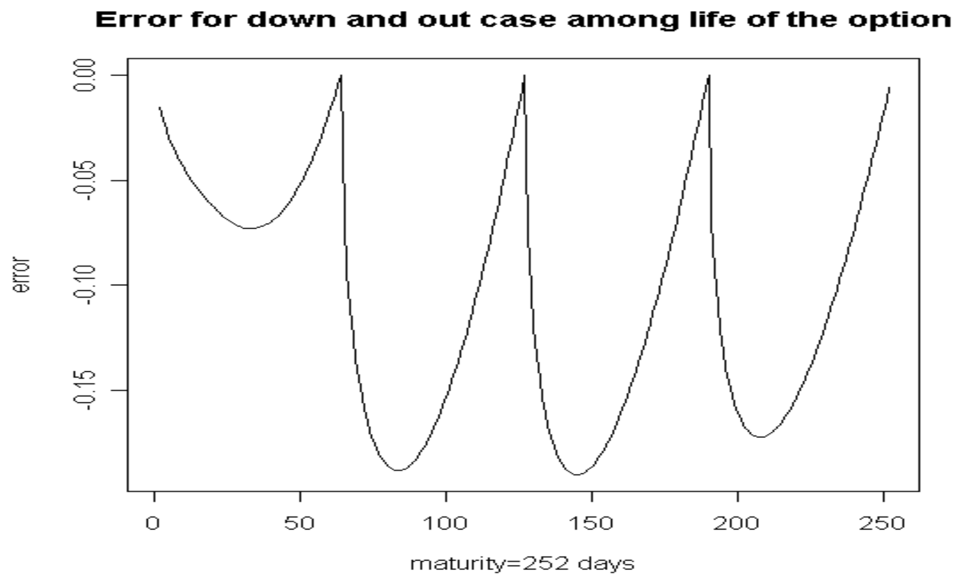


Figure 2.6:Calendar Spread

It was mentioned before that if one increases the plain vanillas taken, then the strategy becomes a perfect hedge. Taking more plain vanillas means decreasing the time distance between puts for DEK calendar spread. Table 2.13 shows how this relation works. For example the value of the portfolio is 4.7856 when 4 puts are used to hedge the call. As the number of puts used for the spread increases then the portfolio value approaches to the barrier option price. ie the value of the portfolio when 12 put taken is 4.8391 which is just 1% less than the real price. Also while the number of puts taken increases the standard deviation of the errors decrease. For example standard deviation of 12 puts is nearly 4 times less than the 4 puts case and is nearly zero. One last thing to mention is the mean values which also decreases and becomes less than 1 percent after 5 puts.

Table 2.13: Results for down&out DEK calendar spread

Number of options	Portfolio value	Mean value of errors	Standard deviation of errors
3	4.752915	-0.0208227	0.02488744
4	4.785633	-0.01339361	0.01687666
5	4.803933	-0.001758020	0.01656511
6	4.815303	0.003540346	0.01502267
7	4.822894	0.004519392	0.01236491
8	4.828238	0.004047468	0.009679059
9	4.832155	0.003713828	0.00802814
10	4.835121	0.003007378	0.006451549
11	4.837427	0.002502163	0.00545995
12	4.839153	0.002210231	0.00461457

2.3.3.1.2 Up and Out Case

Like the previous simulations up and out case is a bit problematic in this case as well. In order to construct the simulation of DEK approach for up and out case it is assumed that again a plain vanilla call option with strike price and expiration date, equal to the strike and expiration of up&out call is sold. Different from down&out case, 12 plain vanilla calls with strike prices equal to the barrier level are bought. The idea about the expiration of these calls and weights are the same like down and out case. The value of constructing such a portfolio with the weight given in table 2.14 and it is 2.481337. The value of the barrier option at time 0 is 2.277. So the error at the beginning is 0.204. Moreover it is seen in Table 2.14 that the calls which have less maturity has less weight on the portfolio and their contribution is very small compared to the ones with maturity equal to up&out call.

Figure 2.7 show the error obtained during the life of the option. It can be seen from the graph that even though very small time intervals are taken to hedge the call error explodes as the options gets close to the expiration. And it explodes in the end.

Table 2.14:Results for up&out DEK calendar spread

Weight	Strike	Maturity (months)	Price
1*Call	110	1	5.546741
-8.78366834*Call	140	1	-6.07033
3.86844096*Call	140	11/12	2.121116
1.31730262*Call	140	10/12	0.5523386
0.65089526*Call	140	9/12	0.198943
0.38597922*Call	140	8/12	0.08065678
0.25493789*Call	140	7/12	0.0333192
0.18080306*Call	140	6/12	0.01298086
0.13487280*Call	140	5/12	0.00434941
0.10448389*Call	140	4/12	0.001076084
0.08335004*Call	140	3/12	0.0001418492
0.06806551*Call	140	2/12	3.915823e-06
0.05665622*Call	140	1/12	2.507307e-10
Value of Portfolio			2.481337

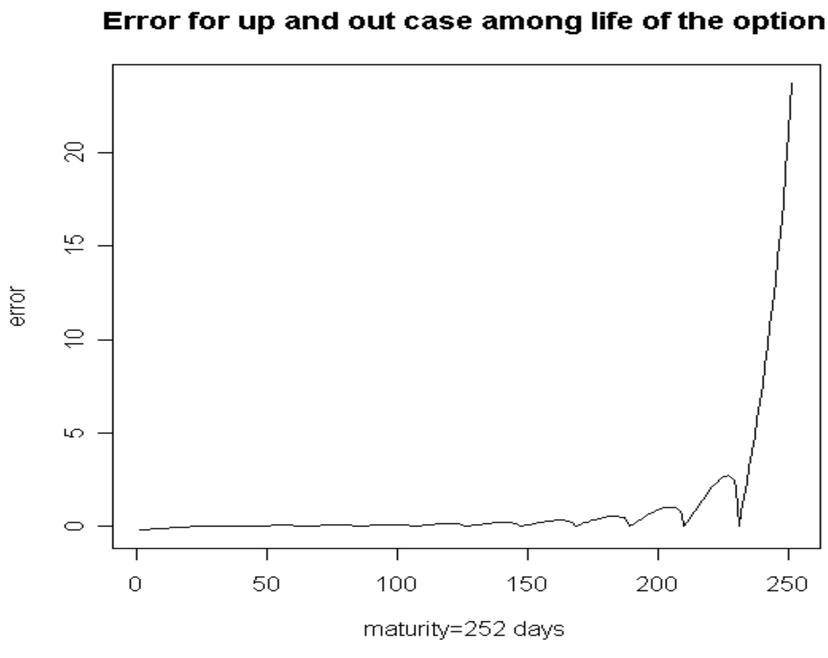


Figure 2.7:Error for Calendar Spread

One last thing to mention is the success of the strategy if the number of options taken increased. Table 2.15 is the summary of the means and standard deviations of errors respective to the number of options that is taken. It is seen that as the options taken increased then the Standard deviation gets smaller. However the change in SD is very small compared to the down&ut case.

Table 2.15:Results for up&out DEK calendar spread

Number of options	Portfolio value	Mean value of errors	Standard deviation of errors
4	2.828615	0.1976777	1.094790
5	2.73168	0.1832293	1.127013
6	2.663754	0.1580366	1.115838
7	2.613534	0.1300902	0.8874873
8	2.574899	0.1289557	1.033159
9	2.544254	0.1047604	0.9054844
10	2.519353	0.0990386	0.8858212
11	2.498717	0.1003617	0.7974267
12	2.481337	0.0912353	0.7878372

2.3.3.2 Strike Spread

The strike spread approach that is used in the following study is Carr&Chou approach. The idea is trying to force the value of the portfolio that is constructed to 0 around the barrier by buying/selling plain vanillas with different strikes.

2.3.3.2.1 Down and Out Case

For the down and out case the strategy can be constructed by buying one call at time 0 with maturity T and strike price equal to the strike of the underlying barrier option. At the same time puts with maturity T and strike prices lower then barrier are sold so that the price is forced to 0 among the levels defined(lower than barrier again). Finding the weights of the puts is more complex than the calendar spread but it is still easy and described in detail in chapter 1. The price of constructing such a portfolio is found to be 4.79 for down and out case.

Table 2.16 shows the weights and the corresponding prices for constructing strike spread. The strike prices are chosen such that they are around $\frac{B^2}{K} = 73.63$. The contributions of the puts are not very high because the match points for constructing the hedge are chosen as: minus one the strike of puts. Since in BSM there are no jumps the values can not go far lower than the barrier which means it can not reach the matching points any time through the life of the option. Moreover the strategy has a mean error of -1.2% and standard deviation error of 1.6%. Again increasing the number of puts taken with different strikes will increase the efficiency of the strategy. If infinitely many puts taken with infinitely many matching points then the hedge will be a perfect hedge

Table 2.16: Results for down&out C&C strike spread

Weight	Strike	Maturity	Price
1*Call	110	1	5.546741
-3.680654079*Put	72	1	-0.7662671
0.017560624*Put	73	1	0.004381155
0.011497392*Put	74	1	0.00341625
0.005912212*Put	75	1	0.002079772
Value of Portfolio			4.790351

2.3.3.2.2 Up and Out Case

For the up and out case the maturities and computation of the weights are the same with the down&out case. But there are two differences in the construction of the portfolio. Firstly plain vanilla calls are taken instead of plain vanilla puts, secondly the matching points are chosen plus one of the corresponding strike prices. CC strike spread costs 2.46 for up&out case.

Table 2.17 shows the weights and the corresponding prices for constructing strike spread. The strike prices are chosen such that they start just above the barrier. The contributions of the first two calls are the largest. The others are 100 times smaller when compared to the first two. The strategy has a mean error of 8% and standard deviation error of 47%.

Table 2.17:Results for up&out C&C strike spread

Weight	Strike	Maturity (months)	Price
1*Call	110	1	5.546741
-59.80136814*Call	140	1	-41.32829
59.96933021*Call	141	1	38.35004
-0.02973914*Call	142	1	-0.01759057
-0.02884226*Call	143	1	-0.01577299
-0.02797772*Call	144	1	-0.01414010
-0.02714416*Call	145	1	-0.01267352
-0.02634027*Call	146	1	-0.01135667
-0.02556481*Call	147	1	-0.01017458
-0.02481661 *Call	148	1	-0.009113714
-0.02409455*Call	149	1	-0.008161885
-0.02339756*Call	150	1	-0.00730809
Value of Portfolio			2.462200

2.3.4 Summary of the Hedging Results

Different hedging techniques are investigated in chapter 2 with BSM. Table 2.18 is the summary what is obtained for specific barrier levels for different hedging strategies. The results for are such that static hedging strategies perform better or at least with the same performance that of dynamic strategies

The accuracy when it’s passed from delta to mix delta is 5 times higher. This result shows that most of the error occurred in delta hedge is because of the kink in plain vanillas delta. Moreover taking only 4 plain vanilla options is enough to have better results than the mix hedge for static ones. Static hedges have accuracies around two times more than the mix hedge. If one consider the cost of constructing a static hedge portfolio in real market, it is definitely more acceptable than the dynamic ones in BSM.

For the up and out case static hedging does not remove the exploding Greeks problem. Still the error terms are too high and risky. The standard deviations vary from %50 to %100. This is because with taking plain vanillas the discontinuity problem of up and out call is not solved, it is just shifted from barrier to strike price K.

Table 2.18: Summary of results for different hedging techniques

BLACK SCHOLES MODEL

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8523	0.1%	10%	2.2772	-0.1%	105%
Mix Delta	4.8523	-0.1%	2.7%	2.2772	1%	80%
STR-Static	4.7903	-1.2%	1.6%	2.4622	8%	47%
CAL-Static	4.7856	-1.3%	1.6%	2.4813	9%	78%

CHAPTER 3

PRICING AND HEDGING IN ARCH TYPE AND SV MODELS

3.1 Introduction

Volatility is the most basic statistical risk measure. It can be used for any purpose from measuring the risk of a derivative or the risk of a portfolio of instruments including this derivative. While it can be expressed in many different ways, the most common one which is used in finance is simply the standard deviation.

The measurement of fluctuation and movement of a price series is said to be volatility of the series. Intuitively, a series that fluctuates a lot or has high volatility includes more risk. For this reason, volatility becomes a major market parameter to which key concepts in risk management and derivatives pricing are based upon.

However in the BSM, the volatility of the underlying is assumed to be constant which constructs the weakest part of the famous model. Hence, it is a very logical question how to devise a method where one can estimate and forecast a reasonable volatility.

The popular method of estimating volatility is deducing the implied volatility. Implied volatility of an option is simply the volatility implied by the real market price of the option based on the BSM. This method of estimating volatility has advantage on estimating volatility directly from historical price changes however implied volatility is found to give rise to a skew or smile depending on the market. Moreover implied volatility only gives the current market estimate of volatility. It gives no insight what is going to happen in future. Given that on of the most important component of option pricing theory, often practitioners and investors require more general picture of volatility.

It is known that volatility is higher in some periods while more stationary in others. And it is also known that high volatility periods tend to make clusters. For this reason one would expect to some extend there is a correlation between present and future volatilities. Moreover volatility tends to revert to some long run average known as mean reversion which ensures that the volatility processes are statistically stationary. These two reasons canalize researchers develop models to estimate volatility which both includes former persistency and the mean reversions at the same time.

The first idea comes from Engle (1982) which is called the autoregressive conditional heteroskedasticity (ARCH) model to estimate the volatility of UK inflation. The idea was developed by Bollerslev (1986) to a more generalized form of ARCH model which is called Generalized ARCH (GARCH). The structure of the parameters of GARCH model leads Bollerslev and Engle (1986) to develop another model called Integrated GARCH (IGARCH) which has a different mean reversion but still reasonable approach. One important contribution to the GARCH family volatility modeling is developed by Nelson (1991) with Exponential GARCH (EGARCH).

One alternative approach of ARCH-type volatility modeling is to allow volatility to depend not on the past observations but on some unobserved components or latent structure. Taylor (1986) developed the most famous one called Stochastic Volatility Models (SV). There are some extensions developed to generate more realistic SV models like the ones which include heavy tails, jumps as well. These kinds of extensions and comparisons are examined in

Kim, Shephard, Chib (1998) in detail. The statistical aspects of all the models that up till now are investigated by Shephard(1996) in detail.

3.2 Conditional Heteroscedastic Models

3.2.1 ARCH Model

3.2.1.1 Description of the Model

The ARCH by Engle (1982) is the first model that provides a systematic framework for volatility modeling. The model uses the log returns of the asset that is chosen. Two points to be mentioned about Arch Model is that;

- Mean corrected asset return X_t is serially uncorrelated but dependent.
- The dependence of X_t can be described by quadratic function of its lagged values.

So with the following information the ARCH (p) model is given in 3.2.1;

$$X_t = \log(S_t) - \log(S_{t-1})$$
$$X_t = \sigma_t \varepsilon_t \quad \text{where} \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \ldots + \alpha_p X_{t-p}^2, \varepsilon_t \overset{iid}{\sim} (0,1), \alpha_i > 0 \; for \, i > 0 \qquad 3.2.1$$

In general ε_t is assumed to be Standard Normal distributed but if one wants to include jumps in the model it is also possible to assume ε_t to be Student-t distributed as well. It can be seen from the model that under the ARCH frame work, large shocks tend to be followed by another large shock. This future of ARCH is similar to the volatility clustering of observed stock price returns.

3.2.1.2 ARCH (1)

Since the concern of the simulation study will be ARCH (1) it is enough to give some very general properties of this model. The model is given in 3.2.2;

$$X_t = \sigma_t \varepsilon_t \quad \text{where} \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2, \varepsilon_t \overset{iid}{:} (0,1), \alpha_{0,1} > 0 \qquad 3.2.2$$

For $\alpha_0 > 0 \, and \, 0 \leq \alpha_1 < 1$ the model is geometrically ergodic and has finite second order moments. The above is an important point because it makes it much easier to estimate the parameters $\hat{\alpha}_0, \hat{\alpha}_1$.

3.2.1.3 Weaknesses of ARCH Model

In addition to its properties mentioned above there are some negative effects of estimation in ARCH model such as;

- It is assumed in the model that both the positive and negative shocks have the same effect on volatility because of taking the square of the previous log stock price. In practice it is know that the response of stock price to positive and negative shocks is different.

- The model is likely to over predict the volatility because of its slow response to large shocks.

- It has some restrictive assumptions of the parameters that are estimated to have the higher order moments.

- Model just gives a general idea about what is happening to volatility estimates. It does not include any details like what or how much of something has an effect on the volatility that is estimated.

3.2.1.4 Pricing in ARCH (1)

One can use two different innovation terms for the simulation study for ARCH (1). First one is standard normal innovations. For standard normal innovations no additional parameters are needed. Second one is student-t innovations. In order to construct student-t innovations, one needs reasonable degrees of freedom (df). The smaller the df the larger the frequency of the jumps and the jump size that occur during the simulations.

All the parameters that are used in the simulation study are assumed to be under **risk- neutral measure** ¤ . Construction of the simulation for pricing is;

- Simulate stock price paths assuming the parameters are ¤ **parameters**.

- If the barrier is hit barrier option price ends as 0 while plain vanilla continues till the end of expiration.

- At expiry calculate the price of both barrier option and plain vanilla. Keep in mind that even if the barrier is hit, plain vanilla continues its life.

- Do the simulation for many times say $10\,^\wedge 5$ times.

- Take the average of both barrier option and the plain vanilla prices and register as the price of the derivative.

Spot price, strike price, interest rate, barriers for both up and down, expiry and number of hedging steps are same as in BSM for pricing simulation. On the other hand number of replications for the simulation is constructed for 10^5 times and ARCH (1) model given in 3.2.3 is used to simulate stock prices which's parameters are assumed to be under risk-neutral measure ¤ . Also in order to have rare jumps with relatively small jump size, df is chosen to be 10 for the student-t innovations.

$$X_t = \sigma_t \varepsilon_t \quad \sigma_t^2 = 0.0002 + 0.45 X_{t-1}^2 \quad \text{where } \varepsilon_t \overset{iid}{\sim} \text{either normal or student-t} \tag{3.2.3}$$

The R code for pricing down and out call is the following;

```
source("Barrieroption.R")1
pricing<-function(spot,r,expiry,strike,barrier,nohedges,nofrep,sigmas,alfa)
{
st<-rep(spot,nofrep)
dt<-expiry/nohedges
hit<-rep(1<0,nofrep)
tau<-rep(0,nofrep)
stau<-st
yt<-0

for(i in 1:nohedges)
{
        sigmat<-sigmas+alfa*yt^2
        yt<-sqrt(sigmat)*rnorm(nofrep)
        ## for student-t innovations use  rt(nofrep,10) instead of
rnorm(nofrep)##
        st<-exp(yt)*st
        oldhit<-hit
        hit<-hit|st<barrier
        tau[!hit]<-i*dt
```

```
        stau[!hit]<-st[!hit]
        update<-oldhit==FALSE&hit==TRUE
        stau[update]<-st[update]
}
barrieroption<-exp(-r*(expiry))*pmax(stau-strike,0)
calloption<-exp(-r*(expiry))*pmax(st-strike,0)
print(mean(barrieroption))
print(mean(calloption))
}
```

Figure one shows two different simulations of ARCH (1) model. It can be seen from figure 3.1 that student-t returns has wider interval when compared with normal returns. It is four times wider than normal innovations in the positive side and three times in the negative side.

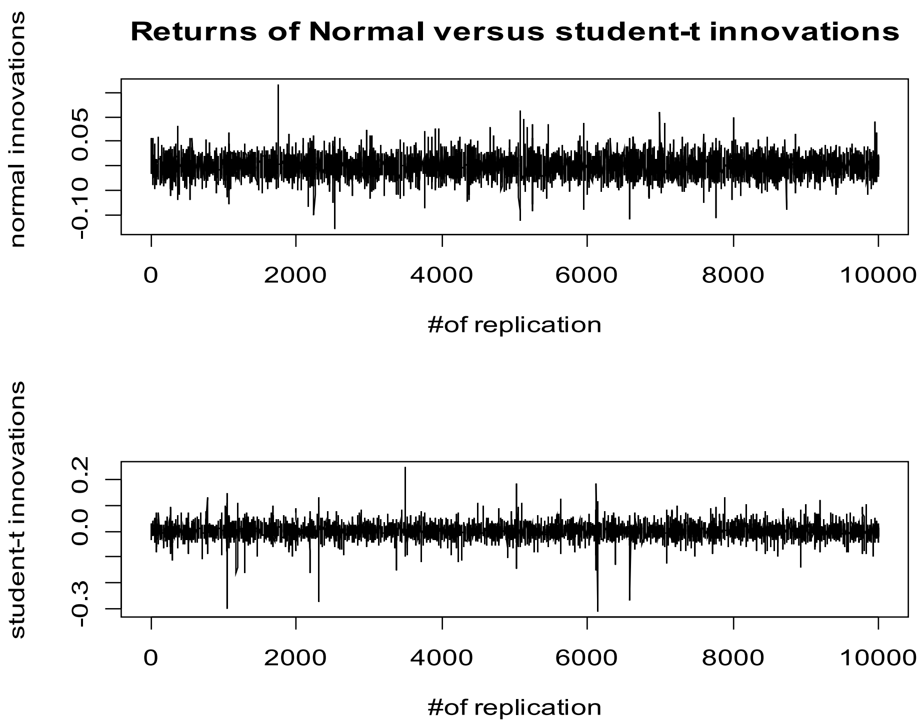


Figure 3.1:Normal versus Student-t Innovations

By looking table 3.1 it is seen that, error term(computed same as in BSM simulations) when compared with BSM prices is %2 for plain vanilla call, 3% for down and out case and 5% for up and out case for normal innovations. On the other hand for student-t innovations errors increase to 40%, 28%, and 15% respectively. Which means it is more difficult to find the price of options with jumps. Moreover if one includes jumps in the process, value of the option increases for the down and out case. However for up and out case it decreases. This is because if there is a big jump which happens above the barrier then the probability of option ending worthless increases on the other hand jumps below the strike does not have the same effect.

Table 3.1:Pricing Summary for ARCH (1)

	Innovations With	
	Normal	**Student-t**
Down &out call with B=90	4.692304	6.219885
Plain vanilla call	5.339224	7.768752
Up &out call with B=140	2.146543	1.926766

3.2.1.5 Hedging in ARCH (1)

For hedging study, all parameters are again assumed to be under risk- neutral measure¤ . Spot price, strike price, interest rate, dividend, barriers for both up and down, expiry, number of hedging steps and number of replications for the study are same as in BSM. However there is an additional problem about the dynamic hedges when it comes to models other than BSM. The problem is “how to find the delta” of the option in order to construct dynamic hedging. The easiest approach is using the delta of the BSM with employing an implied volatility. But as mentioned earlier there are some problems that can be caused by using implied volatility. So instead of implied volatility, volatility in 3.2.4 is plugged into the simulation which allows volatility to fluctuate with the log returns. As seen from the figure 3.2 both negative and positive shocks can be seen in the volatility.

$$\sigma_t^{\Delta} = \sigma^2 + \sigma_t \text{ where;} \quad 3.2.4$$
$\sigma^2 = 0.18$,
σ_t as defined in 3.2.3;
σ_t^{Δ} = Volatility of the Δ hedge

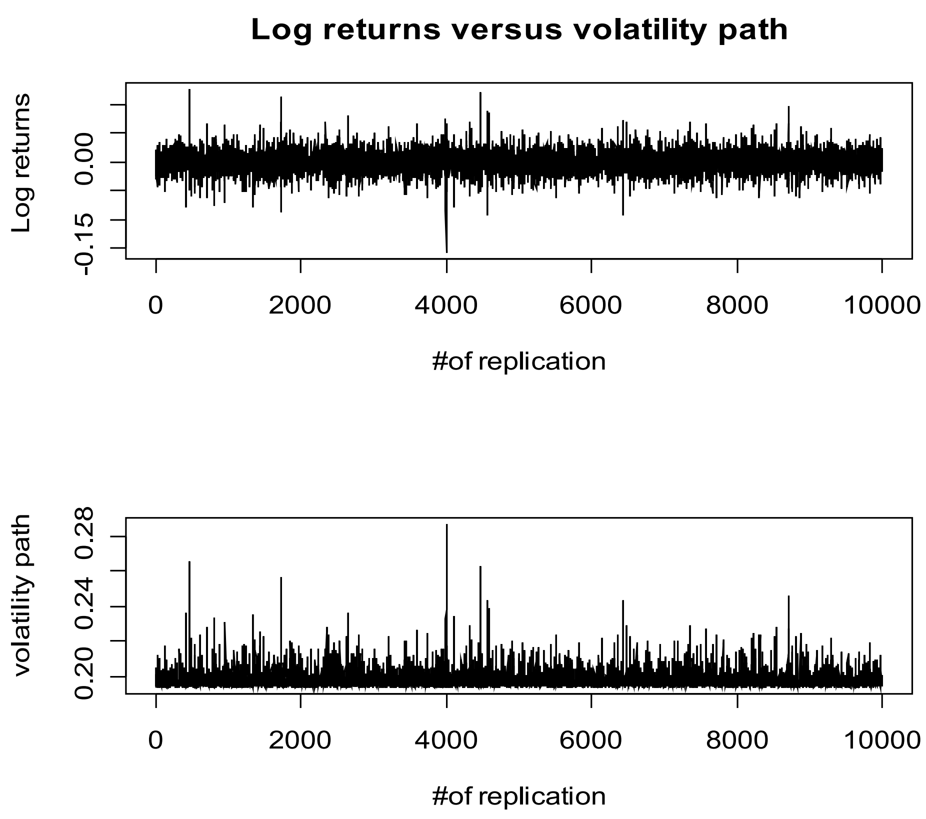


Figure 3.2:Log Returns and Volatility Paths

Example: Construction of the R code for simple delta hedging

Construction of codes for simple delta hedging strategy for down and out call can be categorized in three steps although these steps are connected in coding.

First step, which can be considered as transactions at time zero, is the one that the prices of the barrier options consequently the initial investments are found. Also first delta of the option is found and plugged at this stage. The first bank account and portfolio is constructed at this stage as well.

Second step is simulating the stock price and checking whether the barrier is hit or not also plugging again the relevant delta. Portfolio is adjusted dynamically at this stage.

Third step is finishing the stock price simulation, liquidating the portfolio and finding the price of the barrier option, the error of the hedge strategy, the mean and the variance of the hedge error relative to the price.

i. Construction of the code at time 0;

```
source("Barrieroption.R")
Deltaarch<-
function(spot,r,q,expiry,strike,barrier,sigma,nohedges,nofrep,sigmas,alfa)
{
dt<-expiry/nohedges
st<-rep(spot,nofrep)
sigmat<-rep(sigmas,nofrep)
alfa<-rep(alfa,nofrep)
initialinvestment<-StandardBarrier("cdo",spot,strike,barrier,0,expiry,r,r-q,
sigma+sqrt(sigmas))
deltabarrier<-
delta.StandardBarrier("cdo",st,strike,barrier,rebate=0,expiry,r,r-q,
sigma+sqrt(sigmas))
nofstock<-deltabarrier
portfolio<-rep(initialinvestment,nofrep)
bankaccount<-portfolio-nofstock*st
hit<-rep(1<0,nofrep)
tau<-rep(0,nofrep)
```

```
stau<-st
s.sigmat<-sigmat
yt<-0
```

The initial investment and the delta for the barrier options are obtained using BSM model. Volatility used in the BSM is the one as defined in 3.2.4. ie sigma is assumed to be 0.18 and volatility of volatilities assumed to be 0.0002. Consequently volatility of the BSM model is 0.1941 and initial investment, which is the price of barrier option, is found to be 4.7113.

ii. Construction of the code at time t;

```
for(i in 2:nohedges)
 {
sigmat<-sigmas+alfa*yt^2
yt<-sqrt(sigmat)*rnorm(nofrep)
### For student-t innovations use rt(nofrep,10) instead of rnorm(nofrep)###
st<-exp(yt)*st
oldhit<-hit
hit<-hit|st<barrier
tau[!hit]<-i*dt
stau[!hit]<-st[!hit]
s.sigmat[!hit]<-sigmat[!hit]
update<-oldhit==FALSE&hit==TRUE
stau[update]<-st[update]
portfolio<-nofstock*stau*exp(q*dt)+bankaccount*exp(r*dt)
deltabarrier[!hit]<-
delta.StandardBarrier("cdo",stau[!hit],strike,barrier,rebate=0,expiry-dt*(i-
1),r,r-q, sigma+sqrt(s.sigmat[!hit]))
nofstock[!hit]<-deltabarrier[!hit]
bankaccount[!hit]<-portfolio[!hit]-nofstock[!hit]*st[!hit]
}
```

In this part, price paths are simulated and whether the stock price hits the barrier or not is checked. The delta of the option is found using BSM model again. The volatility that is used in BSM calculations is not constant and assumed to follow the pattern in 3.2.4. The portfolio is dynamically adjusted using BSM delta if the barrier is not hit. If the barrier is hit process stops for that path.

iii. Construction of the code either at T or when the barrier is hit;

```
sigmat<-sigmas+alfa*yt^2
yt<-sqrt(sigmat)*rnorm(nofrep)
## For student-t innovations use rt(nofrep,10) instead of rnorm(nofrep)##
st<-exp(yt)*st
oldhit<-hit
hit<-hit|st<barrier
stau[!hit]<-st[!hit]
s.sigmat[!hit]<-sigmat[!hit]
update<-oldhit==FALSE&hit==TRUE
stau[update]<-st[update]
portfolio<-nofstock*stau*exp(q*pmin(expiry-
tau,dt))+bankaccount*exp(r*dt)
barrieroption<-StandardBarrier("cdo",stau,strike,barrier,0,expiry-tau,r,r-q,
sigma+sqrt(s.sigmat))
error<-(portfolio-barrieroption)*exp(-r*tau)
print(initialinvestment)
print(mean(error/initialinvestment))
print(sqrt(var(error/initialinvestment)))
plot(error,type="l",xlab="#of
errors",ylab="Error",main="Barrier=90",col="red")
}
```

At the beginning of the code one last stock price path is simulated since the simulation in part two is started from 2. This is done for the cases where barrier is not hit till maturity. The prices of the barrier options are found simply by BSM model. The volatility of the model is inserted as in 3.2.4. At the end of the process, portfolios are liquidated and the error term is found by simply subtracting the portfolio value from barrier option price, and discounting the value with the appropriate rate. The mean and the variance of the error are found with respect to the initial investment found.

A short description of R codes for other types of hedging strategies:

For the *mix delta hedge*, an additional plain vanilla call price and delta is needed. These values of call price and delta can be found using BSM model with employing the volatility as in 3.2.4. All the assumptions and the parameters are same with simple delta hedge for this strategy as well.

For the strike spread, portfolio construction in the first step is the same as in BSM with the same strike prices and same maturities. Only difference is the volatility that is used in the model which is assumed to be 0.1941. This difference leads small changes in the weights of the puts and/or calls taken for the hedge. No dynamic adjustment is needed for this case so there is no such problem as delta. At time T BSM is used with volatility given in 3.2.4 to find the price of plain vanilla and barrier options.

For calendar spread the maturities are the same with the ones used in BSM model. Only difference again is the volatility employed at the beginning. Moreover no delta is needed for this strategy again. At time T BSM is used with volatility given in 3.2.4 to find the price of plain vanilla and barrier options.

By looking the results of Table 3.2 and Table 3.3 it is seen that the static hedging strategies perform better than the dynamic ones.

For the model with normal innovations, SD of the down and out call with delta hedge is 23.5. Passing from delta to mix hedge one has 4 times less SD. Moreover the accuracy of the static hedges is nearly two times more when compared to the dynamic ones. When it comes to up out call the strike spread performs the best with the lowest accuracy which is three times better than the delta hedge

When moved from normal to the student-t innovations, static hedges still perform better. Again strike spread performs the best for up and out case while calendar spread performs best for down and out. Largest increase in the mean value happens in mix delta hedging strategy when it is passed form down to up case. Standard deviations increase 150% when compared to normal innovations for up and out case and 200% for down and out case for all hedges.

Table 3.2:Hedging Summary for ARCH (1) with normal innovations

ARCH(1)Model with standard normal innovations

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.7113	8.8%	23.5%	2.3328	9.3%	112%
Mix Delta	4.7113	2.8%	6.8%	2.3328	38%	131%
STR-Static	4.6730	-1.9%	3.7%	2.5232	10%	48%
CAL-Static	4.6488	-0.6%	2.6%	2.5285	6.2%	96%

3.2.2 GARCH Model

3.2.2.1 Description of the Model

Although ARCH model is a very simple model, it requires many lags to adequately describe the process. As a result of many lags one needs to estimate many parameters as well. This complexity of ARCH modeling is reduced with the introduction of GARCH model by Bollerslev (1986). GARCH (p,q) model is formulized in 3.2.5.

As used in ARCH model different ε_t's like, Standard Normal or Student-t distributed can be used in GARCH. In addition to simple GARCH model there are some extensions of the model which is discussed in more detail later.

Table 3.3:Hedging Summary for ARCH (1) with student-t innovations

ARCH(1)Model with student-t innovations with df=10

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.7113	-28%	45.6%	2.3328	32%	184%
Mix Delta	4.7113	8.6%	13.9%	2.3328	141%	231%
STR-Static	4.6730	3.2%	8%	2.5232	21%	67%
CAL-Static	4.6488	-0.5%	5.4%	2.5285	2.7%	159%

$X_t = \sigma_t \varepsilon_t$ where X_t is the same as in ARCH

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \ldots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_q \sigma_{t-q}^2$$

$$= \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_p \sigma_{t-i}^2 \qquad 3.2.5$$

$$\varepsilon_t \overset{iid}{:} (0,1), \alpha_0 > 0, \alpha_i, \beta_j > 0 \; for\, i, j > 0, \sum_{i=1}^{\max(p,q)} \alpha_i + \beta_i < 1$$

3.2.2.2 GARCH (1, 1)

GARCH (1, 1) is usually enough to describe the behavior of a volatility series. So in the simulation GARH (1, 1) is employed with the structure in 3.2.6.

$$X_t = \sigma_t \varepsilon_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \qquad 3.2.6$$
$$0 < \alpha_1 + \beta_1 < 1$$

The statistical estimation of parameters of a GARCH process is uncomplicated as one can use Gaussian quasi-maximum likelihood. Moreover by looking the model it can be concluded;

- A large X_{t-1} tends to cause X_t which is the well known behavior of volatility clustering
- The tail distribution of the process is heavier than that of normal distribution

- Model provides a simple parametric function that can be used to describe the volatility evolution.

3.2.2.3 Weaknesses of GARCH Model

GARCH model has the same weaknesses as in ARCH model. For example the positive and negative shocks still have the same effect on the model. Moreover, empirical studies of high frequency financial time series data reveals that the tail behavior of GARCH models remains too short even with standardized Student- t innovations.

3.2.2.4 Pricing in GARCH (1, 1)

Simulation study for GARCH (1, 1) model is constructed with same innovation terms, assumptions and parameters used in ARCH (1) model. The only difference is GARCH model its self. GARCH (1, 1) model with the parameters given in 3.2.7 is used to simulate stock price paths. Study is again constructed for standard normal and student–t (with d.f=10) innovations. Also parameters are assumed to be under risk neutral measure ¤ .

$$X_t = \sigma_t \varepsilon_t \quad \text{where } \varepsilon_t \overset{iid}{:} \text{ either normal or student-t}$$
$$GARCH \text{ § } \sigma_t^2 = 0.000075 + 0.05 X_{t-1}^2 + 0.75 \sigma_{t-1}^2 \qquad 3.2.7$$

By looking table 3.4 it is seen that, error term(found like in BSM simulations) when compared with BSM prices is 3.6% for plain vanilla call, 2.3% for down and out call and 4.7% for up and out call for normal innovations. On the other hand for student-t innovations, errors increase to 25%, 19%, and 14% respectively. Differences in simulated prices and BSM prices are high when jumps are included in the model. Moreover when compared to ARCH (1) model, simulated prices are very close for normal innovations. For example GARCH (1, 1) price for plain vanilla is 5.341446 while ARCH (1) is 5.339224. However for student-t innovations, effect of jumps is smaller in GARCH (1, 1) model when compared to ARCH (1). For example plain vanilla price in GARCH (1, 1) model is 6.937883 while in ARCH (1) price is 7.768752.

Table 3.4:Pricing Summary for GARCH (1, 1)

	GARCH(1,1)	
	Normal	**Student-t**
Down &out call with B=90	4.696828	5.774296
Plain vanilla call	5.341446	6.937883
Up &out call with B=140	2.168628	1.957688

3.2.2.5 Hedging in GARCH (1, 1)

For hedging study, all the dynamic and static hedging strategy principles are same as in ARCH (1) model, also parameters are again assumed to be under risk- neutral measure ¤ . Spot price, strike price, interest rate, dividend, barriers for both up and down, expiry, number of hedging steps and number of replications for the study are same as in BSM. However volatility (σ^2) is numerically different from the previous model. For GARCH (1, 1), σ^2 is 0.19 instead of 0.18. This volatility is used because constant term or the variance of variances for GARCH (1, 1) model is smaller this time. GARCH (1, 1) parameters specified in the model described in section 3.2.7 are employed again for simulating stock price paths. Moreover the volatility approach for calculation of delta which was used for ARCH (1) is used again.

$$\sigma_t^{\Delta} = \sigma^2 + \sigma_t \quad \text{where} \quad \sigma^2 = 0.19 \,\&\, \sigma_t^2 = 0.000075 + 0.05X_{t-1}^2 + 0.75\sigma_{t-1}^2 \quad 3.2.7$$

From table 3.5&3.6 it is seen that static hedges performs better for both models. When compared with the BSM model, Standard Deviations of dynamic hedges in GARCH (1, 1) are a bit higher.

For the model with normal innovation term, simple delta hedging has the lowest accuracy with 13% when compared with all the other strategies. Moreover static hedges have more accuracy then the mixed delta hedge if one looks to the SD's. Calendar spread performs best for the down and out case with lowest accuracy 1.3%. For the up and out case strike spread has the lowest SD with 42% and is the best.

When it comes to student-t innovations, the accuracy of dynamic models lowered nearly twice. The means of the static hedges remains relatively same when compared with the dynamic hedges for the down and out case. Largest loose in accuracy is in Delta hedging. If one looks to the up and out case there is a dramatic increase in the mean of mix delta hedge.

Strike spread is the best with SD 52% for up and out case. Calendar spread looses it is accuracy when it is passed from down& out to up& out case with an increase in SD from 1.7% to 109%.

Table 3.5:Hedging Summary for GARCH (1,1) with normal innovations

GARCH(1, 1)Model with standard normal innovations

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8205	-7.8%	13%	2.2899	7.5%	99%
Mix Delta	4.8205	3.1%	3.6%	2.2899	32%	95%
STR-Static	4.7643	-3%	3.4%	2.4845	10%	42%
CAL-Static	4.7547	0.01%	1.3%	2.4924	4.3%	91%

Table 3.6:Hedging Summary for GARCH (1, 1) with student-t innovations

GARCH(1, 1)Model with student-t innovations with df=10

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8205	-21%	25%	2.2899	24%	124%
Mix Delta	4.8205	6.7%	5.6%	2.2899	102%	136%
STR-Static	4.7643	-3.6%	3.7%	2.4845	15.9%	52%
CAL-Static	4.7547	0.4%	1.7%	2.4924	5.4%	109%

3.2.3 Extensions of ARCH Type Models

3.2.3.1 IGARCH

All volatility models are expected to deal with real life data. However when GARCH model is employed for this purpose it often results with a strange outcome as it is often observed that parameters tend to sum up very close to one.

$$\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} \beta_j \approx 1$$

This fact of GARCH model is formulized by Engle and Bollerslev (1986) as IGARCH which has the following property;

$$\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} \beta_j = 1$$

Although this model makes sense in practical applications it consists some difficulties about statistical properties since the process has infinite variance.

IGARCH model is interesting for pricing and hedging studies because in this model shocks are very persistent. For example a major positive shock followed by another one is important for up and out call hedging as dynamic strategies are effected from these kind of behavior negatively. Simulation studies can be constructed easily with using estimated parameters sum up to one for all the hedging studies constructed before.

3.2.3.2 EGARCH

To overcome some weaknesses of GARCH model Nelson (1991) introduced a new model called EGARCH. With this model returns are no more has to be symmetric. It responds non-symmetrical to shocks. The model is the one given by 3.2.8.

The difference of the model from the previous ones is that conditional variance is modeled as a function of variables not just depending on the squares the observations. The asymmetry of information is useful especially for dynamic hedging strategies because it allows the variance to respond more rapidly to falls in the market than the corresponding rises. Although model looks complicated it has nice statistical properties which are easy to find

$$X_t = \sigma_t \varepsilon_t, \ln\left(\sigma_t^2\right) = \alpha_0 + \frac{1+\beta_1 B+\ldots+\beta_s B^s}{1+\alpha_1 B+\ldots+\alpha_m B^m} g\left(\varepsilon_{t-1}\right)$$

where;

$$g\left(\varepsilon_t\right) = \begin{cases} \left(\left(\theta+\gamma\right)\varepsilon_t - \gamma E\left(\left|\varepsilon_t\right|\right) & if\ \varepsilon_t \geq 0 \\ \left(\left(\theta-\gamma\right)\varepsilon_t - \gamma E\left(\left|\varepsilon_t\right|\right) & if\ \varepsilon_t < 0\right) \end{cases} \qquad 3.2.8$$

3.3 Stochastic Volatility Models

There are many ways to model variance changes. One way of modeling was to let the conditional variance be a function of squares of previous observations and/or past variances which is called the Conditional Heteroscedastic models.

The basic alternative to ARCH-type modeling is to allow σ_t^2 depend on not on the past observations but on some unobserved components or latent structure. These types of models are called Stochastic Volatility (SV) models which have gradually emerged as a useful way of modeling time-varying volatility with significant potential for applications, especially in finance

Definition: Given an iid sequence ε_t of random variables with zero mean and unit variance one can say that y_t is a stochastic volatility model if

$$yt = \sigma_t \varepsilon_t$$

where σ_t is a stochastic process independent of ε_t. The volatility σ_t usually is a Markov chain. One has to note that ARCH processes do not fit in to this definition as σ_t depends on y_{t-1} and therefore ε_{t-1}

3.3.1 Log Normal SV Model

3.3.1.1 Description of the Model

The simplest formulation of SV model is the log normal model as introduced in Taylor (1986) which is by 3.3.1.

$$
\begin{aligned}
y_t &= \beta e^{\frac{h_t}{2}} \varepsilon_t, t > 0 \\
h_t &= \mu + \Phi \left(h_{t-1} - \mu \right) + \sigma_n \eta_{t-1} \\
h_1 &: N\left(\mu, \frac{\sigma^2}{1 - \Phi^2} \right)
\end{aligned}
\qquad 3.3.1
$$

y_t is the mean corrected return on holding the asset at time t, h_t is log volatility at time t which is assumed to be stationary for(|Φ1>|) with h_1 is normally distributed with mean μ, and variance $\frac{\sigma^2}{1-\Phi^2}$. For the simulation study the constant scaling factor β is chosen 1 so that the parameter μ is unrestricted where β=exp (μ/2). $\varepsilon_t \& \eta_t$ are uncorrelated standard normal white noise parameters. Φ is the persistence in volatility and σ_n is the volatility of lognormal volatility.

The model above is heavily analyzed in literature for most of the statistical properties. Many Monte Carlo Markov Chain (MCMC) algorithms are developed to estimate and find the distribution of this model. A good example of these MCMC algorithms is given by Kim (1998)

3.3.1.2 Pricing in SV Models

In order to construct the simulation study for SV model, there is no need to add any parameters other than the model parameters its self which is given in 3.3.2. Again a student-t distribution can be used to model a more tailed return.

To sum up; all parameters are assumed to be under **risk neutral measure**¤ and simulation study is constructed with same innovation terms and parameters that are used in ARCH (1) model other than the model parameters of SV model. In the SV context, Student-t error-based models were used by Harvey, Ruiz and Shephard (1994).

$$h_t = -8 + 0.95\left(h_{t-1} + 8\right) + 0.140\eta_{t-1}$$
$$h_1 : N\left(-8, \frac{0.04}{1 - 0.9025}\right) \quad 3.3.2$$

By looking table 3.7 it is seen that SV model prices are approximately same with the simulated prices in the previous models. Again the price for up and out case with student-t innovations is less than the normal innovation up and out prices. Moreover opposite relation of simulated plain vanilla and down and out case prices still remain for the down and out case.

Table 3.7:Pricing Summary for SV

	Innovations With	
	Normal	**Student-t**
Down &out call with B=90	4.798011	5.881475
Plain vanilla call	5.437155	6.805012
Up &out call with B=140	2.104128	1.976563

3.3.1.3 Hedging in SV Models

For hedging study, all the dynamic and static hedging strategy principles are same as in ARCH (1) model except; for simplicity in the construction of the code at time=0 BSM volatility is used. Spot price, strike price, interest rate, volatility, dividend, barriers for both up and down, expiry, number of hedging steps and number of replications for the study are same as in BSM.

However for hedging in SV models the problem of finding a suitable delta at time t for dynamic hedges still remains. Contrary to ARCH type models this time a different volatility is plugged to the find delta of BSM model. The calculation of the delta is given in 3.3.3.

$$\begin{aligned} &h_t^{\Delta} = \sigma^2 * \exp\left(\sqrt{h_t}\right) \quad \text{where;} \\ &\sigma^2 = 0.2, \\ &h_t \text{ as defined in 3.3.2,} \\ &h_t^{\Delta} = \text{Volatility of the} \Delta \text{ hedge.} \end{aligned} \qquad 3.3.3$$

The summary of table 3.8&3.9 is that static hedging strategies perform better for both cases.

For the model with normal innovations calendar spread performs the best for the down and out case with a SD of 1.5%. Strike spread has roughly the same result with mix delta hedge which are around 5-6%. Delta hedge has a very low accuracy with both having the highest mean and variance at the same time. However for the up and out case strike spread defeats the performance calendar spread with having the lowest SD 44%. Moreover mix hedging results has lower performance than delta hedge results like in the previous models. Calendar spread has SD around 70% which is more than strike spread this time.

When student-t innovations are checked it is seen that problematic results are generally means. The SD's remains roughly the same with the model with normal innovations. The most dramatic change in the means is again at the mix delta hedge. The best static hedging results for up and out case is strike spread with SD 54%. Then calendar spread follows with and SD of 84%. For down and out case calendar spread is best with SD 1.7%. There is a big difference between calendar spread and other models even the strike spread is 3 has three times less accuracy.

Table 3.8:Hedging Summary for SV with normal innovations

SV Model with standard normal innovations

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8523	-7%	21%	2.2768	9.6%	111%
Mix Delta	4.8523	1.6%	5.9%	2.2768	30%	120%
STR-Static	4.7903	-4.6%	5.4%	2.4622	8.6%	44%
CAL-Static	4.7856	-0.8%	1.5%	2.4813	5.9%	70%

Table 3.9:Hedging Summary for SV with student-t innovations

SV Model with student-t innovations with df=10

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8523	-17%	28%	2.2768	20%	132%
Mix Delta	4.8523	4.5%	7.2%	2.2768	84%	158%
STR-Static	4.7903	-5.2%	5.8%	2.4622	12%	54%
CAL-Static	4.7856	0.1%	1.7%	2.4813	6.9%	84%

3.3.2 Log Normal SV Model with Jumps

3.3.2.1 Description of the Model

Jump models are quite popular in continuous time models of financial asset pricing. Keeping this in mind another model is used for simulation study which is log normal SV model that contains a jump component in the observation equation to allow for large, transient movements. This model, which is called SV model with jumps model is as given by 3.3.4.

In the model given by 3.3.4 q_t is a Bernoulli random variable that takes the value one with unknown probability k and the value zero with probability $1-k$. Moreover time-varying variable k_t represents the size of the jump when a jump occurs. Taken together $k_t q_t$ can be viewed as a discretization of a finite activity Levy process. Recent econometric work on SV models with jumps for parameter estimation, model checking, volatility estimation via filtering and model comparisons can be found on the paper by Nielsen and Shephard (2001).

$$
\begin{aligned}
& y_t = \beta e^{\frac{h_t}{2}} \varepsilon_t + q_t k_t, \ t > 0 \\
& h_t = \mu + \Phi\left(h_{t-1} - \mu\right) + \sigma_n \eta_{t-1} \\
& h_1 : N\left(\mu, \frac{\sigma^2}{1-\Phi^2}\right) \\
& q_t : Bernoulli(k) \\
& \log\left(1+k_t\right) : N\left(-0.5\delta^2, \delta^2\right)
\end{aligned}
\qquad 3.3.4
$$

Figure 3.3 shows that returns of SV model with jumps has four times wider interval on both at positive and negative side of the graph then the model without jumps. Although rarely happening, these wider intervals are the result of the jumps that occurs during the process. These jumps have direct effect on both pricing and hedging results. For example if one remembers one of the requirements of simple delta hedging which assumes that the change of the price change called gamma should be low then it is understood better how important the effect of jumps for a process. Moreover for the strike spread that is constructed there was some matching points that the stock price is assumed to reach. But for a jump model these prices can be surpassed easily.

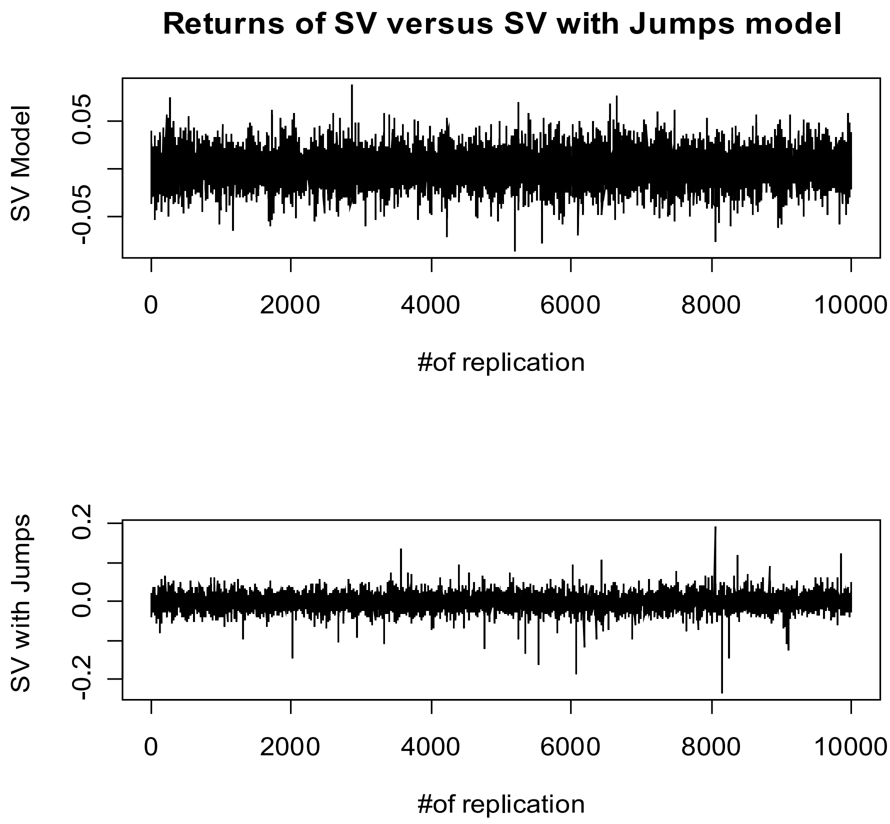


Figure 3.3:Returns of SV Models

3.3.2.2 Pricing in SV Model with Jumps

In order to price the barrier options in SV model with jumps; model in 3.3.5 is used. All parameters are assumed to be under risk neutral measure ¤ and simulation study is constructed with same innovation terms and parameters that are used in SV model other than the model parameters of SV model with jumps.

$$\begin{aligned} & h_t = -8 + 0.95(h_{t-1} + 8) + 0.14\eta_{t-1} \\ & h_1 : N\left(-8, \frac{0.04}{1-0.95^2}\right) \\ & q_t : Bernoulli(0.004) \\ & \log(1+k_t) : N\left(-0.5(0.1)^2, (0.1)^2\right) \end{aligned} \qquad 3.3.5$$

Table 3.10 is the summary of the pricing results in SV model with jumps. The prices that are found are higher than the model without jumps for both normal and student-t errors for down and out case. This is because if a jump occurs on the inactive side of the option ($S_t < K$), this movement has no effect on the value since option is already worthless. Contrary if a jump occurs when $S_t > K$, the value becomes even higher when compared to the model without jumps. So up and out prices are less when jump effect is added. One can also include the innovation effect like in the previous models as well.

Table 3.10:Pricing Summary for SV with jumps

	Innovations With	
	Normal	**Student-t**
Down &out call with B=90	4.950269	5.864632
Plain vanilla call	5.913776	7.072211
Up &out call with B=140	2.044065	1.791158

3.3.2.3 Hedging in SV with Jumps

There is no difference with the model without jumps in either hedging strategies or parameters and assumptions of the strategies. The only difference is the model that is used. From table 3.11&3.12 it is seen that jump model with both innovations has higher results than the model without jumps.

For the case of standard normal innovations the SD's of dynamic hedges are approximately one and half times the SV model without jumps. There are increases in the static hedges as well but it is small compared to the dynamic ones. Especially strike spread seems to be not effected by the jumps at the model at all with a change form 4.8 to 4.9. For the up and out case again the changes in the Static hedges are less when compared to the model without jumps. Again strike spread performs much better than all the strategies with a SD of 40%.

The results of student-t innovations are more or less the same. The accuracy of all hedges decreases when compared with the model without jumps and the model with jumps with normal innovations. Especially there are large increases in the mean values. Strike spread performs best for both down&out and up&out cases.

Table 3.11:Hedging Summary for SV with jumps and normal innovations

SV Model with jumps and standard normal innovations

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.7438	-11.7%	28.5%	2.3203	14%	140%
Mix Delta	4.7113	3%	9.1%	2.3203	58%	158%
STR-Static	4.7005	-4%	4.8%	2.5008	9.8%	40%
CAL-Static	4.6804	-0.7%	2.9%	2.5181	6.5%	80%

Table 3.12:Hedging Summary for SV with jumps and student-t innovations

SV Model with jumps and student-t innovations with df=10

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.7438	-21%	36%	2.3203	25%	157%
Mix Delta	4.7113	5.8%	9.5%	2.3203	108%	185%
STR-Static	4.7005	-4.4%	4.9%	2.5008	10%	42%
CAL-Static	4.6804	-0.9%	3.0%	2.5181	5.2%	114%

3.3.3 Extensions of SV Model

The study constructed for SV models can be extended in some other directions as well. The most remarkable one is the SV models in which the parameters are allowed to switch among a given number of states according to a Hidden Markov process. The Markov model can be formulized as in 3.3.6.

$$y_t = \sigma_t \varepsilon_t, \varepsilon_t \overset{iid}{:} (0,1) \textit{ independent of } \sigma_t$$

σ_t can take r values $\sigma_1, \ldots, \sigma_r$ according to finite state markov chain $x_t$on $E\{1, \ldots, r\}$
The distribution of the one step transition of the markov chain x_t is given by the transtion matix;

$$p = \begin{pmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & \ddots & \vdots \\ p_{r1} & \cdots & p_{rr} \end{pmatrix} \text{ where;}$$

$$p_{ij} = P(x_t = j \mid x_{t-1} = i) \qquad 3.3.6$$

The process is interesting and important for pricing and hedging studies of barrier options because with this model it is possible to model regime switching in a time series. For example regime changes is expected to have a direct affect on the prices or the persistence's in volatilities.

3.4 Summary of Chapter

Through discrete time study frequently used models are taken as starting points. Some extensions to these models are discussed and the benefits of using these extended models are described briefly. The prices and the hedging results of barrier options for frequently used methods are found.

Pricing study is constructed for normal and student-t innovations for all models. For down and out case prices increase one move from standard normal to student-t distribution. Conversely for up and out case when one moves from standard normal to student-t innovations prices decrease. Some prices have values which were close to the ones in BSM however all these experiments depend on the parameters chosen. So no real comparison can be made at this point.

For all models static hedges perform better than dynamic ones. Either strike spread or calendar spread has lower SD's than both delta and mix delta hedges. Again in all simulation results classic delta hedging performs the worst for down and out case. On the other hand mix delta hedge was the worst for the up and out case. The strike spread in general was the best for up and out case while calendar spread was the best down and out case for most of the models.

CHAPTER 4

PRICING AND HEDGING IN DIFFUSION MODELS

4.1 Introduction

It is necessary to remind the readers that the prices of the options that are path dependent, such as barrier options, generally much more sensitive to the specification of the underlying price process than the plain vanilla options. Keeping this in mind without discussing the diffusion models, picture of pricing and hedging of barrier options is not going to be complete. These models can be considered as alternatives or extensions of the BSM model. For the models that are of interest in this study, stock prices have Markov property and follow a geometric Brownian motion.

These diffusion models can be categorized into two groups. First one is one factor models which consider only one random parameter. The Constant Elasticity of Variance Model (CEV) is a part of this group. The second group is the multi-factor models like stochastic volatility or jump models. Both of these models have more than one random variable as their parameters.

One can start with CEV which was originally proposed by Cox and Ross (1976). In this model volatility of the process depends on the stock price and the size of the elasticity parameter chosen which makes sense if one considers of stocks as options on firm value.

Another model to be investigated for hedging strategies is the Stochastic Volatility model of Heston (1993). The aim of the model is to capture the price change variances which display variation over time.

One last model that is taken into consideration is Merton jump diffusion model (1976). In this model returns of the stock are hit by Poisson arrivals of lognormal jumps.

Before passing on the results I have to mention that the results for diffusion models are similar to the study from Poulsen and Nalholm (2006). In this paper one can find more jump models and various extensions of the static hedging strategies described before.

4.2 Constant Elasticity of Variance Model

A company with improvement which is enjoying high profits expects to have higher stock prices which lead a decreasing volatility. On the other hand a company which suffers from financial troubles should expect decreasing stock prices which in turn increase the volatility. With these in mind CEV model assumes that the volatility at time t of the underlying asset depends on its price level. The stock prices are assumed to follow 4.1 under the risk neutral probability measure Q.

$$dS_t = S_t\left(r - d\right)dt + S_t\sigma\, S_t^{\alpha-1}dWt \text{ where;} \qquad 4.1$$

Wt is a standard Brownian motion, $\sigma \,\&\, \alpha$ are constant parameters and elasticity factor α has to be in the interval $\left[0,1\right)$.

If one assumes that $\alpha = 1$ then CEV model becomes a BSM model. When $\alpha < 1$ volatility increases as stock price falls. Moreover it can be shown that the variance of elasticity is constant.

Hedging study for CEV model can be constructed with the same parameters for BSM. α assumed to be 0.5 for the model and $\sigma_{CEV} = 2.0472$.

One can create different implied volatilities for finding delta in this model. For simulation study BSM volatility, which is equal to 0.2, is used for simplicity.

The simulated price for down and out call is 4.9063 which is slightly more than BSM price. On the other hand price for up and out call is 2.9066 which is again just a bit higher than BSM price.

Table 4.1 is the summary of hedging results constructed for CEV model. For the down and out case pure delta hedging has again the lowest accuracy with 10%. Contrary to previous models this time strike spread has the second lowest SD. With a SD equals to 1.8% calendar spread performs double in accuracy when compared to mix delta hedge.

For up and out case the accuracies are again very low compared to the down and out case. Mix delta hedging does not form any improvement over pure delta hedge with a SD 129%. The best strategy for up and out model is the strike spread with an accuracy of 37%. This time calendar spread ha a low accuracy compared to strike spread with a SD of 75%.

Table 4.1:Hedging Summary for CEV model

CEV Model

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8523	1%	10%	2.2772	-14%	125%
Mix Delta	4.8523	0.8%	2.9%	2.2772	2%	129%
STR-Static	4.7903	-17%	14%	2.4622	9%	37%
CAL-Static	4.7856	-1.5%	1.8%	2.4813	20%	75%

Poulsen and Nalholm (2006) construct the hedging study for CEV model in many different ways. For example they give two different approaches which are BSM and CEV based hedging methods for dynamic strategies. Moreover they have constructed three different Strike spreads which are BSM based, uniformly scaled and Smile scaled strike spreads. Finally they include two different Calendar spreads which are again BSM and CEV based.

4.3 Stochastic Volatility Model

Heston (1993) stochastic volatility model become a classical reference and a very good starting point in the research related to stochastic modeling. The main assumption is that the volatility of the underlying asset price is stochastic. In this model volatility process is chosen to be mean reverting square root variance process.

Heston model allows for stochastic interest rates and can be used for other exotic options as well. In this model stock price follows the usual diffusion process while variance is supposed to fluctuate according to well-known square root process by Cox, Ingersoll and Ross (1985). The model is given in 4.2.

$$\begin{aligned} dS_t &= (r - d) S_t dt + \sqrt{v_t} S_t dW_1 t \\ dv_t &= \kappa (\theta_{SV} - v_t) dt + \eta \sqrt{v_t} \left(\rho \, dW_1 t + \sqrt{1 - \rho^2} dW_2 t \right) \end{aligned} \qquad 4.2$$

The skewness of spot returns can be explained by the correlation between asset prices and volatility. If there is a positive correlation than the model will provide higher prices for out of the money options and lower prices for in the money ones compared to BSM. The reason for kurtosis and fat tails is the volatility of the volatility.

The simulation study for Heston SV model is constructed with the parameters on 4.3. The volatility that is used for the simulation of dynamic hedges is simply the square root of the volatility in the model which is $\sqrt{v_t}$.

Mean reversion of variance	κ	1.301
Long-term variance level	θ_{SV}	0.044
Volatility of variance	η	0.105
Correlation(stock,variance)	ρ	-0.608

4.3

The price of the down and out call is found to be 4.8860 for down and out case in Heston SV model. This price is very close to the BSM price which was 4.8523. Moreover the price for up and out call is simulated to be 2.8415. This price is higher compared to the BSM model.

Table 4.2 is the summary of hedging results constructed for CEV model. For the down and out case pure delta hedging has again the lowest accuracy with 15%. Mix delta hedging performs better than strike spread with a SD of 4.6. Best hedging strategy for this model is calendar spread with a SD of 3.3

For Up and out case strike spread performs the best with an accuracy of 52%.Calendar spread does not perform any improvement over the dynamic hedges with a SD of 90% which is equal to the SD of pure delta hedge and a bit less than mix delta hedge which is 98%.

Again in the paper by Poulsen and Nalholm (2006) one can find some extensions of hedging strategies for Heston model. For example they use Fink (2003) approach for calendar spread. Moreover they have included calendar spread with conditional volatility in the spirit of Dupire (1994).

Table 4.2:Hedging Summary for Heston SV model

Heston SV Model

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8524	2.3%	15%	2.2768	-14.7%	90%
Mix Delta	4.8524	1%	4.6%	2.2828	-19%	98%
STR-Static	4.7903	-4.2%	6.3%	2.4663	4.7%	52%
CAL-Static	4.7856	0.7%	3.3%	2.4863	16.7%	90%

4.4 Merton Jump Diffusion Model

A jump model which allows a fatter tail process is given by Merton (1976). The model has the form in 4.4.

$$dS_t = \left(r - d - \lambda_k\right) S_t dt + \sigma_{JD} S_t dWt + \left(J_t - 1\right) S_t dq_t \ where;$$
$$\ln\left(J_t\right) : N\left(\gamma, \delta\right), dq_t : Poisson\left(\lambda\right) \ and \ k = E^Q\left[J_t - 1\right] \qquad 4.4$$

Thus we can say that Merton jump diffusion model is capable to contain skewness and excess kurtosis caused by random jumps in the underlying asset returns. The first element for the random jumps is the jump size which determines the amount of jump that occurs during the process. Other component is the frequency of the jumps which determines how of then the jumps occur during the process. One can also add that as maturity approaches these jumps effects cancel each other in time.

The parameters of the simulation study are given on 4.5 for Merton jump diffusion model. The volatility employed is simply the BSM volatility for dynamic hedging strategies.

Jump intensity	λ	1.158	
Mean jump size	γ	-0.135	
Jump size variance	δ	$4.7 * 10^{-6}$	4.5
Volatility of diffusion part	σ_{JD}	0.148	

Table 4.3 is the summary of the results for Merton model. Pure delta hedging has the lowest accuracy with 32% for the down and out case. Mix delta hedging again performs better than strike spread with a SD of 6.With lowest SD 2.9% best hedging strategy for this model is calendar spread.

For Up and out case strike spread performs the best with an accuracy of 32%. It is much lower than other dynamic strategies which show again in jump models strike spread is a good candidate. Calendar spread is also powerful tool when compared to dynamic one with an SD of %57 less tan half of dynamic strategies. Dynamic hedges with a SD of around 130-150% have very low accuracies for Merton jump model.

If one want to increase the jump models for hedging he can include the variance gamma model from Madan, Carr and Chang (1998) to use a pure a jump process with small jumps but infinite arrival time as investigated in Poulsen and Nalholm (2006). Moreover the strike spread extensions which are uniformly scaled and smile scaled, can also provide improvements over the traditional static hedges. One can also check the papers by Leisen (1998) and Metwlly&Atiya(2002) if interested the research methods to reduce convergence time and quantitative errors, on jump processes for barrier options.

Table 4.3:Hedging Summary for Merton Jump Diffusion Model

Merton Jump Diffusion Model

	Down and out call B=90			Up and out call B=140		
HEDGE M	**Price**	**Mean**	**SD**	**Price**	**Mean**	**SD**
Delta	4.8564	3%	32%	2.2752	-14.7%	129%
Mix Delta	4.8564	-2%	6%	2.2752	-19%	147%
STR-Static	4.1378	-11%	13%	2.9942	4.9%	32%
CAL-Static	4.7777	-1%	2.9%	3.0145	6.1%	57%

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APPENDIX

R Codes for calculating delta and price for plain vanilla and barrier options

```
################# B/S
BlackScholesFormula <- function (spot,timetomat,strike,r,q=0,sigma,
opttype=1,greektype=1)

{

d1<-(log(spot/strike)+((r-q)+0.5*sigma^2)*timetomat)
/(sigma*sqrt(timetomat))

d2<-d1-sigma*sqrt(timetomat)
if (opttype==1 && greektype==1) result<-spot*exp(-
q*timetomat)*pnorm(d1)-strike*exp(-r*timetomat)*pnorm(d2)

if (opttype==2 && greektype==1) result<-spot*exp(-
q*timetomat)*pnorm(d1)-strike*exp(-r*timetomat)*pnorm(d2)-spot*exp(-
q*timetomat)+strike*exp(-r*timetomat)

if (opttype==1 && greektype==2) result<-exp(-q*timetomat)*pnorm(d1)

if (opttype==2 && greektype==2) result<-exp(-q*timetomat)*(pnorm(d1)-1)

BlackScholesFormula<-result
```

```
}

StandardBarrier <- function (TypeFlag,S, X, H, K,  time2mat, r, b, v){

mu <-(b - v ^ 2 / 2) / v ^ 2

lambda <- sqrt(mu ^ 2 + 2 * r / v ^ 2)

X1 <- log(S / X) / (v * sqrt(time2mat)) + (1 + mu) * v * sqrt(time2mat)

X2 <- log(S / H) / (v * sqrt(time2mat)) + (1 + mu) * v * sqrt(time2mat)

y1 <- log(H ^ 2 / (S * X)) / (v * sqrt(time2mat)) + (1 + mu) * v *
sqrt(time2mat)

y2 <- log(H / S) / (v * sqrt(time2mat)) + (1 + mu) * v * sqrt(time2mat)

Z <- log(H / S) / (v * sqrt(time2mat)) + lambda * v * sqrt(time2mat)

 if (TypeFlag=="cdi" | TypeFlag=="cdo"){

eta <- 1

phi <- 1
 }

 if (TypeFlag=="cui" |  TypeFlag=="cuo"){

eta <- -1

phi <- 1
```

```
}
if (TypeFlag=="pdi" |  TypeFlag=="pdo"){

eta <- 1

 phi <- -1
}
if (TypeFlag=="pui" |  TypeFlag=="puo"){

eta <- -1

phi <- -1
}

####### code is formally correct, but breaks down for low vols due to eval of Inf*0

#     f1 <- phi * S * exp((b - r) * time2mat) * pnorm(phi * X1) - phi * X * exp(-r * time2mat) * pnorm(phi * X1 - phi * v * sqrt(time2mat))

#     f2 <- phi * S * exp((b - r) * time2mat) * pnorm(phi * X2) - phi * X * exp(-r * time2mat) * pnorm(phi * X2 - phi * v * sqrt(time2mat))

#     f3 <- phi * S * exp((b - r) * time2mat) * (H / S) ^ (2 * (mu + 1)) * pnorm(eta * y1) - phi * X * exp(-r * time2mat) * (H / S) ^ (2 * mu) * pnorm(eta * y1 - eta * v * sqrt(time2mat))

#     f4 <- phi * S * exp((b - r) * time2mat) * (H / S) ^ (2 * (mu + 1)) * pnorm(eta * y2) - phi * X * exp(-r * time2mat) * (H / S) ^ (2 * mu) * pnorm(eta * y2 - eta * v * sqrt(time2mat))
```

```
#       f5 <- K * exp(-r * time2mat) * (pnorm(eta * X2 - eta * v * sqrt(time2mat)) - (H / S) ^ (2 * mu) * pnorm(eta * y2 - eta * v * sqrt(time2mat)))

#    f6 <- K * ((H / S) ^ (mu + lambda) * pnorm(eta * Z) + (H / S) ^ (mu - lambda) * pnorm(eta * Z - 2 * eta * lambda * v * sqrt(time2mat)))

if (X > H){

if(TypeFlag=="cdi") StandardBarrier <- f3 + f5

if(TypeFlag=="cui") StandardBarrier <- f1 + f5

if(TypeFlag=="pdi") StandardBarrier <- f2 - f3 + f4 + f5

if(TypeFlag=="pui") StandardBarrier <- f1 - f2 + f4 + f5

if(TypeFlag=="cdo") StandardBarrier <- f1 - f3 + f6

if(TypeFlag=="cuo") StandardBarrier <- f6

if(TypeFlag=="pdo") StandardBarrier <- f1 - f2 + f3 - f4 + f6

if(TypeFlag=="puo") StandardBarrier <- f2 - f4 + f6

}
if (X <= H){

if(TypeFlag=="cdi") StandardBarrier <- f1 - f2 + f4 + f5

if(TypeFlag=="cui") StandardBarrier <- f2 - f3 + f4 + f5
```

```
if(TypeFlag=="pdi") StandardBarrier <- f1 + f5

if(TypeFlag=="pui") StandardBarrier <- f3 + f5

if(TypeFlag=="cdo") StandardBarrier <- f2 + f6 - f4

if(TypeFlag=="cuo") StandardBarrier <- f1 - f2 + f3 - f4 + f6

if(TypeFlag=="pdo") StandardBarrier <- f6

if(TypeFlag=="puo") StandardBarrier <- f1 - f3 + f6
}
StandardBarrier
}
delta.StandardBarrier <-function(TypeFlag, spot, strike, barrier, rebate,
time2mat, r, b, v)
{
eps<-0.001

eps
(StandardBarrier(TypeFlag,spot+eps,strike,barrier,rebate,time2mat,r,b,v) -
StandardBarrier(TypeFlag,spot-eps,strike,barrier,rebate,time2mat,r,b,v))/
(2*eps)
}

DOprob.BS<- function (spot,timetomat,barrier,r,q,sigma)
{
y<-log(barrier/spot)
nu<-(r-q-0.5*sigma^2)
if(y<=0)
```

```
{
Doprob.BS<-pnorm((y+nu*timetomat)/(sigma*sqrt(timetomat)))
exp(2*y*nu/sigma^2)* pnorm((y + nu*timetomat)/(sigma*sqrt(timetomat)))

}
if(y>0) {DOprob.BS<- 1}

DOprob.BS
}
UOprob.BS<- function (spot,timetomat,barrier,r,q,sigma)
{
y<-log(barrier/spot)
nu<-(r-q-0.5*sigma^2)
if(y>=0)
{

Uoprob.BS<-pnorm((y+-nu*timetomat)/(sigma*sqrt(timetomat)))
exp(2*y*nu/sigma^2)* pnorm((-y - nu*timetomat)/(sigma*sqrt(timetomat)))
}
if(y<0) {UOprob.BS<- 1}
1-UOprob.BS
}
fhat<- function(type,x,strike,barrier,p)
{
fhat<-1:length(x)

if(type=="cdo")
{
fHIGH<-pmax(x-strike,0)

fLOW<--(x/barrier)^p*pmax(barrier^2/x-strike,0)
```

```
}
if(type=="cuo")
{

fLOW<-pmax(x-strike,0)

fHIGH<--(x/barrier)^p*pmax(barrier^2/x-strike,0)

}
HIGH<-(x>=barrier)

fhat<-c(fLOW[!HIGH],fHIGH[HIGH])
}
```