# COMPLETION OF A LEVY MARKET MODEL AND PORTFOLIO OPTIMIZATION

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# ABSTRACT

# COMPLETION OF A LEVY MARKET MODEL AND PORTFOLIO OPTIMIZATION


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In this study, general geometric Levy market models are considered. Since these models are, in general, incomplete, that is, all contingent claims cannot be replicated by a self-financing portfolio consisting of investments in a risk-free bond and in the stock, it is suggested that the market should be enlarged by artificial assets based on the power-jump processes of the underlying Levy process. Then it is shown that the enlarged market is complete and the explicit hedging portfolios for claims whose payoff function depends on the prices of the stock and the artificial assets at maturity are derived. Furthermore, the portfolio optimization problem is considered in the enlarged market. The problem consists of choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained. It is shown that for particular choices of the equivalent martingale measure in the market, the optimal portfolio only consists of bonds and stocks. This corresponds to completing the market with additional assets in such a way that they are superfluous in the sense that the terminal expected utility is not improved by including these assets in the portfolio.

Keywords: Levy processes, Power-jump processes, Complete markets, Martingale Representation Property, Hedging portfolio, Portfolio optimization, Martingale method

# ÖZ

# LEVY PİYASASI TAMLAMASI VE PORTFÖY OPTİMİZASYONU

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Bu çalışmada, genel geometrik Levy piyasa modelleri incelenmiştir. Bu modeller, genellikle, tam değillerdir, yani, tüm şarta bağlı alacak hakları, tahvil ve hisse senetlerine yatırım yapılarak kendi kendini finanse eden portföy tarafından yinelenemezler. Bu sebepten piyasanın, söz konusu Levy süreçlerinin kuvvet-sıçrama süreçlerine dayalı yapay varlıklar tarafından genişletilmesi önerilmiştir. Bu durumda piyasanın tam olduğu gösterilmiş ve alacak hakkına ait ödeme fonksiyonunun hisse senedi ve yapay varlıkların vade sonu değerlerine bağlı riskten korunma portföyü açık olarak ifade edilmiştir. Ayrıca, genişletilen piyasada portföy optimizasyon problemi incelenmiştir. Problem, optimal portföyün, nihai servete ait beklenen faydasının maksimum olacak şekilde, seçiminden ibarettir. Piyasadaki denk martingale ölçüsünün özel seçimleri için, optimal portföyün sadece tahvil ve hisse senetlerinden oluştuğu gösterilmiştir. Bu durum piyasanın yeni varlıkları gereksiz kılacak şekilde tamlanmasına karşılık gelmektedir.

Anahtar Kelimeler: Levy süreçleri, Kuvvet-sıçrama süreçleri, Tam piyasalar, Martingale Temsili Özelliği, Riskten korunma portföyü, Portföy optimizasyonu, Martingale metodu

To my family

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# CHAPTER 1

# INTRODUCTION

In recent years more and more attention has been given to stochastic models of financial markets which depart from the famous Black-Scholes model [9]. Some of the most popular and still tractable models are the Lévy models. These models are able to take into account different important stylized features of financial time series. An accessible introduction, together with theoretical motivations to Lévy markets, can be found in Geman (2002) [20], as well as [24]. For an overview of the theory and the applications of Lévy processes in finance see [32] and [12].

It is well-known that the famous Black-Scholes model is complete, that is, all contingent claims can be replicated by a self-financing portfolio consisting of investments in a risk-free bond and in the stock. However, when the sources of randomness are more than the number of assets available for investment the incompleteness arises. In incomplete markets a perfect replication of a claim is, in general, not possible and most Lévy market models are incomplete. There are different approaches to hedging in incomplete markets, see Cont and Tankov (2004) [12].

A market model is said to be complete if for every integrable contingent claim there exists an admissible self-financing strategy replicating the claim. The question of market completeness is linked with the Predictable Representation Property (PRP) of a martingale. A martingale M is said to have the PRP if, for any square-integrable random variable $\mathrm{X} \in \mathcal{F}_{\mathrm{T}}$, we have $\mathrm{X} = \mathrm{E}(\mathrm{X}) + \int_0^{\mathrm{T}} \mathrm{h_s} \mathrm{dM_s}$, for some predictable process $\mathrm{h} = \{\mathrm{h_s}, 0 \leq \mathrm{s} \leq \mathrm{T}\}$ (see [32] (p.18)). If we have such a representation, the predictable process h gives us the admissible self-financing strategy replicating the claim. Unfortunately, this kind of PRP is a rather delicate and exceptional property, which is only possessed by a few martingales. Examples include Brownian motion, the compensated Poisson process and the Azéma martingale, see Dritschel and Protter (1999) [18]. The PRP for Brownian motion states that every square integrable random variable adapted to the filtration generated by a Brownian motion can be represented as a sum of its mean and a stochastic integral with respect to the Brownian motion, where the integrand is a predictable process. The PRP of Brownian motion implies the completeness of the Black-Scholes model [9] and gives the admissible self-financing strategy replicating a contingent claim whose price only depends on the time to maturity and the current stock price.

When the underlying asset is driven by a Lévy process, perfect hedging using only a risk-free bond (or a bank account) and the underlying asset is, in general, not possible and the market is said to be incomplete. However, further developments are possible. Nualart and Schoutens (2000) [26] proved the PRP for Lévy processes which satisfy some exponential moment conditions, see also [25]. This PRP states that every square integrable random variable adapted to the filtration generated by a Lévy process can be represented as an infinite sum of iterated stochastic integrals with respect to the orthogonalized compensated power-jump processes of the underlying Lévy process. In the light of [26] and [25], Corcuera et al. (2005) [14] suggested that the market should be enlarged by a series of very special assets (power-jump assets) so that perfect hedging can be achieved. Corcuera et al. (2006) [13] used this completeness to solve the portfolio optimization problem using the martingale method.

In this study, we work under a market which consists of one riskless asset (the bond) and one non-dividend paying risky asset (the stock) with price process formulated by a geometric Lévy model. Since general geometric Lévy market models are incomplete, except for the geometric Brownian and the geometric Poissonian models, the market is completed by following the approach suggested by [14], that is, the market is equipped with certain additional assets so that any final wealth is actually attainable by trading in the complete market. First the market is enlarged by artificial assets based on the power-jump processes of the underlying Lévy process [26, 25]. For pure jump processes the power-jump process of order two is the quadratic variation process and is related with the realized variance, see Barndorff-Nielsen and Shephard (2002), (2003) [5, 6]. Higher order power-jump processes can be related with realized skewness and realized kurtosis. These new assets can be related with options on the stock Balland (2002) [4] and with contracts on realized variance Carr and Madan (1998) [10], Demeterfi et al. (1999) [17] that are traded in OTC markets regulary. These new assets are strongly related to the realized higher moments and in a discrete time framework, they mainly coincide Corcuera et al. (2005a) [15], see also Schoutens (2005) [33]. These assets give you protection against different kinds of market shocks. Completeness of the enlarged market is shown by the Martingale Representation Property [26, 25]. The notion of completeness used is equivalent to the notion of approximate completeness of Björk et al. (1997) [8]. Also by giving the explicit hedging portfolios for claims whose payoff function depends on the prices of the stock and the artificial assets at maturity, the portfolio optimization problem is considered in the enlarged market [13]. The problem consists of choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained. A class of utility functions, including HARA, logarithmic and exponential utilities as special cases, is considered. Then, the optimal portfolio which maximizes the terminal expected utility is obtained by the martingale method: First, the optimal wealth is found and then the hedging portfolio replicating this wealth is obtained [22]. It is shown that for particular choices of the equivalent martingale measure in the market, the optimal portfolio only consists of bonds and stocks [21, 31]. This corresponds to completing the market with additional assets in such a way that they are superfluous in the sense that the terminal expected utility is not improved by including these assets in the portfolio. This in turn provides the solution to the problem of utility maximization in the real market, consisting only of the bond and the stock.

The organization of this study is as follows. In Chapter 2, basic definitions and concepts related to Lévy processes are given. In Chapter 3, the geometric Lévy market model is introduced. In Chapter 4, the power-jump processes are introduced and the Lévy market model is completed by artificial assets constructed from them. In Chapter 5, the hedging portfolio for the claims whose payoff function depends on the prices of the stock and the new assets at maturity is given. In Chapter 6, the portfolio optimization problem in the complete Lévy market is considered. And finally, in Chapter 7, the conclusion follows.

# CHAPTER 2

# PRELIMINARIES

Definitions and theorems given in this part are mainly taken from [12, 27, 28].

Assume that we are given a filtered, complete probability space $(\Omega, \mathcal{F}, \mathrm{F}, \mathbb{P})$, where $\mathrm{F} = (\mathcal{F}_t)_{0 \leq t \leq \infty}$, satisfying the usual hypotheses, that is,

(i) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$;

(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, for all t, $0 \leq t < \infty$; i.e. the filtration F is right continuous.

## 2.1 Basic Tools

**Definition 2.1.1** Two stochastic processes X and Y are modifications if $X_t = Y_t$ almost surely (a.s.) for each t.

**Definition 2.1.2** A function $f : [0,T] \to \mathbb{R}$ is said to be càdlàg if it is right continuous with left limits.

Of course, any continuous function is càdlàg but càdlàg functions can have discontinuities. If t is a discontinuity point we denote by $\Delta f(t) = f(t) - f(t-)$ the "jump" of f at t. However, càdlàg functions cannot jump aroud too wildly. A càdlàg function f can have at most a countable number of discontinuities: $\{t \in [0,T],\ f(t) \neq f(t-)\}$ is finite or countable. Also, for any $\varepsilon > 0$, the number of discontinuities ("jumps") on [0,T] larger than ε should be finite. So a càdlàg function on [0,T] has a finite number of "large jumps" (larger than ε) and posssibly infinite, but countable number of small jumps.

**Definition 2.1.3** A stochastic process X is said to be càdlàg if it a.s. has sample paths which are right continuous, with left limits.

**Definition 2.1.4** A family of random variables $(U_\alpha)_{\alpha \in A}$ is uniformly integrable if $\lim_{n\to\infty} \sup_\alpha \int_{\{|U_\alpha| \geq n\}} |U_\alpha| d\mathbb{P} = 0$.

**Theorem 2.1.1** Let X be a martingale. Then $(X_t)_{t\geq 0}$ is uniformly integrable if and only if $Y = \lim_{t\to\infty} X_t$ exists a.s., $E\{|Y|\} < \infty$, and $(X_t)_{0\leq t\leq\infty}$ is a martingale, where $X_\infty = Y$.

If X is a uniformly integrable martingale, then $X_t$ converges to $X_\infty = Y$ in $L^1$ as well as a.s..

## 2.2 Lévy Processes

**Definition 2.2.1** An adapted process $Z = (Z_t)_{t\geq 0}$ with $Z_0 = 0$ a.s. is a Lévy process if

(i) Z has increments independent of the past; that is, $Z_t - Z_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$; and

(ii) Z has stationary increments; that is, $Z_t - Z_s$ has the same distribution as $Z_{t-s}$, $0 \leq s < t < \infty$; and

(iii) $Z_t$ is continuous in probability; that is, $\lim_{t\to s} Z_t = Z_s$, where the limit is taken in probability; i.e. $\forall t \geq 0 \ \forall \varepsilon > 0$, $\lim_{s\to t} \mathbb{P}(|Z_s - Z_t| > \varepsilon) = 0$.

The simplest Lévy process is the linear drift, a deterministic process. Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths. Other examples of Lévy processes are the Poisson and compound Poisson processes.

Notice that the sum of a linear drift, a Brownian motion and a compound Poisson process is again a Lévy process; it is often called a "Lévy jump-diffusion" process.

**Theorem 2.2.1** Let Z be a Lévy process. There exists a unique modification Y of Z which is càdlàg and which is also a Lévy process.

We will henceforth always assume that we are using the (unique) càdlàg version of any given Lévy process. Lévy processes provide us with examples of filtrations that satisfy the 'usual hypotheses', as the next theorem shows.

**Theorem 2.2.2** Let Z be a Lévy process and let $\mathcal{G}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where $(\mathcal{F}_t^0)_{0\le t<\infty}$ is the natural filtration of Z, and $\mathcal{N}$ are the $\mathbb{P}$-null sets of $\mathcal{F}$. Then $(\mathcal{G}_t)_{0\le t<\infty}$ is right continuous.

There is a strong interplay between Lévy processes and infinitely divisible distributions. We first define infinitely divisible distributions and give some examples, and then describe their relationship to Lévy processes.

**Definition 2.2.2** The law $\mathbb{P}_X$ of a random variable X is infinitely divisible, if for all n $\in \mathbb{N}$ there exist i.i.d. random variables $X_1^{(1/n)}, \dots, X_n^{(1/n)}$ such that

$$X =^d X_1^{(1/n)} + \cdots + X_n^{(1/n)}.$$

Alternatively, we can characterize an infinitely divisible random variable X using its characteristic function $\varphi_X$. The law of a random variable X is infinitely divisible, if for all $n \in \mathbb{N}$, there exists a random variable $X^{(1/n)}$, such that

$$\varphi_X(u) = \left(\varphi_{X^{(1/n)}}(u)\right)^n.$$

Some examples of infinitely divisible distributions are the Normal distribution, the Poisson distribution, the compound Poisson distribution, the exponential, the Γ-distribution, the geometric, the negative binomial, the Cauchy distributions and the strictly stable distribution. On the other hand, the uniform and the binomial distributions are not infinitely divisible.

The next theorem provides a complete characterization of random variables with infinitely divisible distributions via their characteristic functions; this is the celebrated Lévy-Khintchine formula.

**Theorem 2.2.3** (Lévy-Khintchine Formula) The law $\mathbb{P}_X$ of a random variable X is infinitely divisible if and only if there exists a triplet $(\alpha, \sigma^2, \nu)$, with $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $\nu$ is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2)\, \nu(dx) < \infty$, such that

$$E\left[e^{iuX}\right] = \exp\left[iu\alpha - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|<1\}}\right)\nu(dx)\right],$$

where $u \in \mathbb{R}$.

The triplet $(\alpha, \sigma^2, \nu)$ is called the Lévy or characteristic triplet and

$$\psi(u) = iu\alpha - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|<1\}}\right)\nu(dx)$$

is called the Lévy or characteristic exponent. Moreover, $\alpha \in \mathbb{R}$ is called the drift term, $\sigma^2$ is the Gaussian or diffusion coefficient and $\nu$ is the Lévy measure.

**Theorem 2.2.4** For every Lévy process $(Z_t)_{t\geq 0}$, we have that

$$E\left[e^{iuZ_t}\right] = e^{t\psi(u)} = \exp\left[t\left(iu\alpha - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|<1\}}\right)\nu(dx)\right)\right],$$

where $\psi(u)$ is the characteristic exponent of $Z_1$, a random variable with an infinitely divisible distribution.

Therefore, any Lévy process can be associated with the law of an infinitely divisible distribution. The opposite, i.e., given any random variable X, whose law is infinitely divisible, we can construct a Lévy process $(Z_t)_{t\geq 0}$ such that $\mathfrak{L}(Z_1) \coloneqq \mathfrak{L}(X)$, where

$\mathfrak{L}(X)$ denotes the law of X, is also true. This will be the subject of the Lévy-Itô decomposition. We prepare this result with an analysis of the jumps of a Lévy process and the introduction of Poisson random measures.

The jump process $\Delta Z = (\Delta Z_t)_{t\geq 0}$ associated to the Lévy process Z is defined, for each $t \geq 0$, via $\Delta Z_t = Z_t - Z_{t-}$, where $Z_{t-} = \lim_{s\uparrow t} Z_s$, the left limit at t. The condition of stochastic continuity of a Lévy process yields immediately that for any Lévy process Z and any fixed $t > 0$, $\Delta Z_t = 0$ a.s.; hence, a Lévy process has no fixed times of discontinuity.

A convenient tool for analyzing the jumps of a Lévy process is the random measure of jumps of the process. Consider a set $\Lambda \in \mathfrak{B}(\mathbb{R}\backslash\{0\})$ such that $0 \notin \overline{\Lambda}$ and let $0 \leq t \leq T$, where $T \in [0,\infty]$; define the random measure of the jumps of the process Z by

$$\mathcal{J}^Z(\omega; t, \Lambda) = \#\{0 \leq s \leq t;\ \Delta Z_s(\omega) \in \Lambda\} = \sum_{s\leq t} 1_\Lambda\big(\Delta Z_s(\omega)\big)\ ;$$

hence, the measure $\mathcal{J}^Z(\omega; t, \Lambda)$ counts the jumps of the process Z of size in $\Lambda$ up to time t. $\mathcal{J}^Z(\cdot, \Lambda)$ is a Poisson process and $\mathcal{J}^Z$ is a Poisson random measure. The intensity of this Poisson process is $\nu(\Lambda) = E[\mathcal{J}^Z(1, \Lambda)]$.

**Theorem 2.2.5** The set function $\Lambda \rightarrow \mathcal{J}^Z(\omega; t, \Lambda)$ defines a σ-finite measure on $\mathbb{R}\backslash\{0\}$ for each (ω,t). The set function $\nu(\Lambda) = E[\mathcal{J}^Z(1, \Lambda)]$ defines a σ-finite measure on $\mathbb{R}\backslash\{0\}$.

**Definition 2.2.3** The measure ν defined by

$$\nu(\Lambda) = E[\mathcal{J}^Z(1, \Lambda)] = E\left[\sum_{0<s\leq 1} 1_\Lambda\big(\Delta Z_s(\omega)\big)\right]$$

is the Lévy measure of the Lévy process Z.

The Lévy measure ν is a measure on $\mathbb{R}$ that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\, \nu(dx) < \infty$. The Lévy measure describes the expected number of jumps of a certain height in a time interval of length 1. The Lévy measure has no mass at the origin, while singularities (i.e. infinitely many jumps) can occur around the origin (i.e. small jumps). Moreover, the mass away from the origin is bounded (i.e. only a finite number of big jumps can occur).

Now, using that $\mathcal{J}^{Z}(t,\Lambda)$ is a counting measure we can define an integral with respect to the Poisson random measure $\mathcal{J}^{Z}$. Consider a set $\Lambda \in \mathfrak{B}(\mathbb{R}\backslash\{0\})$ such that $0 \notin \overline{\Lambda}$ and a function $f : \mathbb{R} \to \mathbb{R}$, Borel measurable and finite on $\Lambda$. Then, the integral with respect to a Poisson random measure is defined as follows:

$$\int_{\Lambda} f(x)\mathcal{J}^{Z}(\omega;t,dx) = \sum_{s\le t} f(\Delta Z_s)\, 1_{\Lambda}\big(\Delta Z_s(\omega)\big).$$

Note that each $\int_{\Lambda} f(x)\mathcal{J}^{Z}(t,dx)$ is a real-valued random variable and generates a càdlàg stochastic process. The stochastic process

$$\int_0^{\cdot}\int_{\Lambda} f(x)\mathcal{J}^{Z}(ds,dx) = \left(\int_0^{t}\int_{\Lambda} f(x)\mathcal{J}^{Z}(ds,dx)\right)_{0\le t\le T}$$

is a compound Poisson process.

**Theorem 2.2.6** Let $\Lambda$ be a Borel set of $\mathbb{R}$, $0 \notin \overline{\Lambda}$. Let $\nu$ be the Lévy measure of Z.

(i) If $f1_{\Lambda} \in L^1(d\nu)$, then

$$E\left[\int_0^{t}\int_{\Lambda} f(x)\mathcal{J}^{Z}(ds,dx)\right] = t\int_{\Lambda} f(x)\nu(dx).$$

(ii) If $f1_{\Lambda} \in L^2(d\nu)$, then

$$E\left[\left(\int_0^{t}\int_{\Lambda} f(x)\mathcal{J}^{Z}(ds,dx) - t\int_{\Lambda} f(x)\nu(dx)\right)^2\right] = t\int_{\Lambda}\big(f(x)\big)^2\nu(dx).$$

**Corollary 2.2.1** Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and vanish in a neighborhood of 0. Then

$$E\left[\sum_{0<s\le t} f(\Delta Z_s)\right] = t\int_{-\infty}^{\infty} f(x)\nu(dx).$$

**Theorem 2.2.7** (Lévy- Itô Decomposition) Consider a triplet $(\alpha, \sigma^2, \nu)$, where $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $\nu$ is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\,\nu(dx) < \infty$. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which four independent Lévy processes $Z^{(1)}, Z^{(2)}, Z^{(3)}$ and $Z^{(4)}$ exist, where $Z^{(1)}$ is a constant drift, $Z^{(2)}$ is a Brownian motion, $Z^{(3)}$ is a compound Poisson process and $Z^{(4)}$ is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Taking $Z = Z^{(1)} + Z^{(2)} + Z^{(3)} + Z^{(4)}$, we have that there exists a probability space on which a Lévy process $Z = (Z_t)_{0 \le t \le T}$ with characteristic exponent

$$\psi(u) = iu\alpha - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|<1\}}\right)\nu(dx)$$

for all $u \in \mathbb{R}$, is defined.

We can decompose any Lévy processes Z into these four independent Lévy processes $Z = Z^{(1)} + Z^{(2)} + Z^{(3)} + Z^{(4)}$, as follows:

$$Z_t = \alpha t + \sigma W_t + \int_0^t \int_{\{|x|\ge 1\}} x\mathcal{J}^Z(ds, dx) + \int_0^t \int_{\{|x|<1\}} x(\mathcal{J}^Z(ds, dx) - \nu(dx)ds).$$

Here $Z^{(1)}$ is a constant drift, $Z^{(2)}$ is a Brownian motion, $Z^{(3)}$ is a compound Poisson process and $Z^{(4)}$ is a pure jump martingale.

The Lévy measure is responsible for the richness of the class of Lévy processes and carries useful information about the structure of the process. Path properties can be read from the Lévy measure. For example, the compound Poisson process has a finite number of jumps on every time interval, while the NIG and α-stable processes have an infinite one; we then speak of an infinite activity Lévy process.

**Proposition 2.2.1** Let Z be a Lévy process with triplet $(\alpha, \sigma^2, \nu)$.

(i) If $\nu(\mathbb{R}) < \infty$, then almost all paths of Z have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.

(ii) If $\nu(\mathbb{R}) = \infty$, then almost all paths of Z have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.

Whether a Lévy process has finite variation or not also depends on the Lévy measure (and on the presence or absence of a Brownian part).

**Proposition 2.2.2** Let Z be a Lévy process with triplet $(\alpha, \sigma^2, \nu)$.

(i) If $\sigma^2 = 0$ and $\int_{\{|x|\leq 1\}} |x| \nu(dx) < \infty$, then almost all paths of Z have finite variation.

(ii) If $\sigma^2 \neq 0$ or $\int_{\{|x|\leq 1\}} |x| \nu(dx) = \infty$, then almost all paths of Z have infinite variation.

The compound Poisson process has finite measure, hence it has finite variation as well; on the contrary, the NIG Lévy process has an infinite measure and has infinite variation. In addition, the CGMY Lévy process for $0 < Y < 1$ has infinite activity, but the paths have finite variation.

The Lévy measure also carries information about the finiteness of the moments of a Lévy process. The finiteness of the moments of a Lévy process is related to the finiteness of an integral over the Lévy measure (more precisely, the restriction of the Lévy measure to jumps larger than 1 in absolute value, i.e. big jumps).

**Proposition 2.2.3** Let Z be a Lévy process with triplet $(\alpha, \sigma^2, \nu)$.

(i) $Z_t$ has finite p-th moment for $p \in \mathbb{R}^+$ ( $E|Z_t|^p < \infty$ ) if and only if $\int_{\{|x|\geq 1\}} |x|^p \, \nu(dx) < \infty$.

(ii) $Z_t$ has finite p-th exponential moment for $p \in \mathbb{R}$ ( $E[e^{pZ_t}] < \infty$ ) if and only if $\int_{\{|x|\geq 1\}} e^{px} \, \nu(dx) < \infty$.

Actually, the conclusion of this proposition holds for a general class of submultiplicative functions, which contains $e^{px}$ and $|x|^p \vee 1$ as special cases (see Theorem 25.3 in [30]).

Note that the variation of a Lévy process depends on the small jumps (and the Brownian motion), the moment properties depend on the big jumps, while the activity of a Lévy process depends on all the jumps of the process.

Basic reference texts on Lévy processes are [3, 7, 23, 28] and [30]. For applications in finance see [12] and [32].

## 2.3 Elements from Semimartingale Theory

**Definition 2.3.1** A semimartingale is a stochastic process $X = (X_t)_{0 \le t \le T}$, which admits the decomposition

$$X_t = X_0 + \mathcal{M}_t + \mathcal{A}_t, \tag{2.1}$$

where $X_0$ is finite and $\mathcal{F}_0$-measurable, $\mathcal{M}$ is a local martingale with $\mathcal{M}_0 = 0$ and $\mathcal{A}$ is a finite variation process with $\mathcal{A}_0 = 0$.

**Definition 2.3.2** An adapted, càdlàg process Y is a classical semimartingale if there exist processes $\mathcal{M}$, $\mathcal{A}$ with $\mathcal{M}_0 = \mathcal{A}_0 = 0$ such that $Y_t = Y_0 + \mathcal{M}_t + \mathcal{A}_t$, where $\mathcal{M}$ is a local martingale and $\mathcal{A}$ is a finite variation process.

**Theorem 2.3.1** A classical semimartingale is a semimartingale.

**Definition 2.3.3** Let X be a semimartingale. If X has a decomposition $X_t = X_0 + \mathcal{M}_t + \mathcal{A}_t$, with $\mathcal{M}_0 = \mathcal{A}_0 = 0$, $\mathcal{M}$ a local martingale, $\mathcal{A}$ a finite variation process and with $\mathcal{A}$ predictable, then X is said to be a special semimartingale.

**Theorem 2.3.2** If X is a special semimartingale, then its decomposition $X = \mathcal{M} + \mathcal{A}$, with $\mathcal{A}$ predictable, is unique (it is assumed that $X_0 = 0$).

Every Lévy process is also a semimartingale; this follows easily from (2.1) and Lévy–Itô decomposition of a Lévy process. Every Lévy process with finite first moment is also a special semimartingale; conversely, every Lévy process that is a special semimartingale, has a finite first moment. This is the subject of the next result.

**Lemma 2.3.1** Let Z be a Lévy process with triplet $(\alpha, \sigma^2, \nu)$. The following conditions are equivalent:

(i) Z is a special semimartingale,

(ii) $\int_{\mathbb{R}} (|x| \wedge |x|^2)\, \nu(dx) < \infty$,

(iii) $\int_{\mathbb{R}} |x| 1_{\{|x| \geq 1\}}\, \nu(dx) < \infty$.

**Definition 2.3.4** Let X, Y be semimartingales. The quadratic variation process of X, denoted by $[X, X] = ([X, X]_t)_{t \geq 0}$, is defined by

$$[X, X] = X^2 - 2 \int X_- dX$$

where $X_{0-} = 0$. The quadratic covariation of X and Y is defined by

$$[X, Y] = XY - \int X_- dY - \int Y_- dX.$$

**Definition 2.3.5** For a semimartingale X, the process $[X, X]^c$ denotes the path-by-path continuous part of $[X, X]$.

We can then write

$$[X, X]_t = [X, X]_t^c + X_0^2 + \sum_{0 < s \leq t} (\Delta X_s)^2 = [X, X]_t^c + \sum_{0 \leq s \leq t} (\Delta X_s)^2.$$

Analogously, $[X, Y]^c$ denotes the path-by-path continuous part of $[X, Y]$, where Y is also a semimartingale.

For every finite variation process X, we have $[X, X]_t = \sum_{0 \leq s \leq t} (\Delta X_s)^2$.

**Example 2.3.1** (Quadratic variation of a Lévy process) If Z is a Lévy process with characteristic triplet $(\alpha, \sigma^2, \nu)$, its quadratic variation process is given by

$$[Z, Z]_t = \sigma^2 t + \int_0^t \int_{\mathbb{R}} x^2 \mathcal{J}^Z(ds, dx).$$

**Example 2.3.2** (Quadratic variation of a Poisson integral) Consider a Poisson random measure $\mathcal{N}$ on $[0, T] \times \mathbb{R}^d$ with intensity $\mu(ds \times dy)$ and a simple predictable random function $\psi: [0, T] \times \mathbb{R}^d \to \mathbb{R}$ . If

$$X_t = \int_0^t \int_{\mathbb{R}^d} \psi(s, y)\, \mathcal{N}(ds, dy),$$

then the quadratic variation of X is given by

$$[X, X]_t = \int_0^t \int_{\mathbb{R}^d} (\psi(s, y))^2\, \mathcal{N}(ds, dy).$$

**Example 2.3.3** Let $\mathcal{N}$ be a Poisson random measure on $[0, T] \times \mathbb{R}^d$ and $(W_t)_{t \in [0,T]}$ be a Wiener process, independent from $\mathcal{N}$. If

$$X_t^i = X_0^i + \int_0^t \phi_s^i dW_s + \int_0^t \int_{\mathbb{R}^d} \psi^i(s, y) \mathcal{N}(ds, dy), \qquad i = 1,2,$$

then the quadratic covariation $[X^1, X^2]$ is given by

$$[X^1, X^2]_t = \int_0^t \phi_s^1 \phi_s^2 ds + \int_0^t \int_{\mathbb{R}^d} \psi^1(s, y) \psi^2(s, y) \mathcal{N}(ds, dy).$$

**Definition 2.3.6** A semimartingale X is called quadratic pure jump if $[X, X]^c = 0$.

If X is quadratic pure jump, then $[X, X]_t = X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2$. Note that the trivial continuous process $X_t = t$ is a quadratic pure jump since $[X, X]_t^c = [X, X]_t = 0$. The Poisson process is an obvious example of a quadratic pure jump semimartingale.

More generally, if Z is a Lévy process with a Lévy decomposition $Z_t = B_t + X_t$, where B is a Brownian motion and

$$X_t = \alpha t + \int_0^t \int_{\{|x|\geq 1\}} x\mathcal{J}^Z(ds, dx) + \int_0^t \int_{\{|x|<1\}} x(\mathcal{J}^Z(ds, dx) - \nu(dx)ds),$$

then X is a quadratic pure jump semimartingale.

**Theorem 2.3.3** (Itô's Formula) Let $X = (X^1, \dots, X^n)$ be an n-tuple of semimartingales, and let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous second order partial derivatives. Then f(X) is a semimartingale and the following formula holds:

$$f(X_t) - f(X_0) = \sum_{i=1}^n \int_{0^+}^t \frac{\partial f}{\partial x_i}(X_{s-})dX_s^i + \frac{1}{2}\sum_{1\leq i,j\leq n} \int_{0^+}^t \frac{\partial^2 f}{\partial x_i\, \partial x_j}(X_{s-})d[X^i, X^j]_s^c$$
$$+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i \right\}.$$

# CHAPTER 3

# THE LÉVY MARKET MODEL

## 3.1 The Model

We will consider a market model consisting of one riskless asset (the bond) and one risky asset (the stock). In this market model, denoted by $\mathfrak{M}$, the value of the bond $B = \{B_t,\ t \geq 0\}$ is given by

$$B_t = \exp(rt), \tag{3.1}$$

where the risk-free interest rate r is constant; and the stock price process $S = \{S_t, t \geq 0\}$ follows a geometric Lévy process

$$\frac{dS_t}{S_{t-}} = bdt + dZ_t, \qquad S_0 > 0, \tag{3.2}$$

where b is a constant. Here $Z = \{Z_t,\ t \geq 0\}$ is a Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{F}, F, \mathbb{P})$, where $F = \{\mathcal{F}_t, t \geq 0\}$, $\mathcal{F}_t = \sigma(S_u: 0 \leq u \leq t)$, is the natural filtration generated by the stock price process completed with the $\mathbb{P}$-null sets. Since any Lévy process Z has a càdlàg modification, we will always assume that we are dealing with the càdlàg version.

If the process Z has the Lévy triplet $(\alpha, \sigma^2, \nu)$, where $\alpha \in \mathbb{R}$ , $\sigma \geq 0$ and $\nu$ on $\mathbb{R}\backslash\{0\}$ with $\int_{-\infty}^{\infty}(1 \wedge x^2)\, \nu(dx) < \infty$ is the Lévy measure of Z, then Z satisfies the following Lévy- Itô decomposition:

$$Z_t = \sigma W_t + X_t, \qquad t \geq 0, \tag{3.3}$$

where $W = \{W_t,\ t \geq 0\}$ is a standard Brownian motion, $X = \{X_t,\ t \geq 0\}$ is a pure jump Lévy process and W is independent of X. Moreover,

$$X_t = \int_{\{|x|<1\}} x\big(N\big((0,t], dx\big) - t\nu(dx)\big) + \int_{\{|x|\geq 1\}} xN\big((0,t], dx\big) + \alpha t, \tag{3.4}$$

where $N(dt, dx)$ is a Poisson random measure on $(0,+\infty)\times\mathbb{R}\backslash\{0\}$ with intensity $dt\times\nu$, dt denotes the Lebesgue measure and $\alpha = E\left(Z_1 - \int_{\{|x|\geq 1\}} xN\left((0,1], dx\right)\right)$.

In this model, it is required that the Lévy measure satisfies, for some $\varepsilon > 0$ and $\lambda > 0$,

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\, \nu(dx) < \infty. \tag{3.5}$$

This will ensure the existence of the predictable representation property, see [26] and [25], which will be used later. In particular, this assumption implies that

$$\int_{-\infty}^{\infty} |x|^i \nu\,(dx) < \infty\,, \qquad i \geq 2,$$

and there exist $0 < h_1, h_2 \leq \infty$ such that

$$E(\exp\,(-hZ_1)) < \infty, \quad \text{for all } h \in (-\,h_1, h_2). \tag{3.6}$$

Hence, all moments of $Z_t$ and $X_t$ exist (see Theorem 25.3 of [30]). Furthermore, $X_t$, given by (3.4), can be written as (see [28] (p.27))

$$X_t = \int_{-\infty}^{\infty} x(N\big((0,t],dx\big) - t\nu(dx)) + \left(\int_{\{|x|\geq 1\}} x\nu(dx) + \alpha\right)t, \tag{3.7}$$

where

$$\alpha = E(X_1) - \int_{\{|x|\geq 1\}} x\nu(dx). \tag{3.8}$$

Note that

$$M(dt,dx) \coloneqq N(dt,dx) - dt\nu(dx) \tag{3.9}$$

is the compensated Poisson random measure on $(0,+\infty)\times\mathbb{R}\backslash\{0\}$. Therefore, the Doob decomposition of X, in terms of a martingale part and a predictable process of finite variation, is given by

$$X_t = L_t + at, \tag{3.10}$$

where $L = \{L_t,\ t \geq 0\}$ defined by

$$L_t = \int_{-\infty}^{\infty} xM\big((0,t],dx\big) \tag{3.11}$$

is a martingale and

$$a = E(X_1). \tag{3.12}$$

Notice that $E(X_t) = at$.

Consequently, by (3.3) and (3.10), Z has the decomposition

$$Z_t = \sigma W_t + L_t + at = \sigma W_t + at + \int_{-\infty}^{\infty} xM\big((0,t],dx\big). \tag{3.13}$$

## 3.2 The Stock Price Formula

We will use Itô's formula for semimartingales to obtain the solution of (3.2). By (3.2) and (3.13), the stock price process has dynamics

$$dS_t = S_{t-}\big((a+b)dt + \sigma dW_t + dL_t\big)$$

$$dS_t = S_{t-}\left((a+b)dt + \sigma dW_t + \int_{-\infty}^{\infty} xM(dt,dx)\right). \qquad (3.14)$$

Apply Itô's formula to $f(S_t) = \log S_t$ :

$$f(S_t) = f(S_0) + \int_0^t f'(S_{s-})\, dS_s + \frac{1}{2}\int_0^t f''(S_{s-})\, d[S,S]_s^c$$
$$+ \sum\nolimits_{0<s\le t} (f(S_s) - f(S_{s-}) - f'(S_{s-})\Delta S_s). \qquad (3.15)$$

Note that $d[S,S]_s^c = S_{s-}^2 \sigma^2 ds$ and $\Delta S_s = S_{s-}\Delta L_s$, where $\Delta S_s = S_s - S_{s-}$. Hence, $S_s = S_{s-}(1+\Delta L_s)$ and $f(S_s) - f(S_{s-}) = \log(1+\Delta L_s)$. Therefore, (3.2) has the explicit solution

$$S_t = S_0 \exp\left(\sigma W_t + L_t + \left(a + b - \frac{\sigma^2}{2}\right)t\right)$$
$$\times \prod\nolimits_{0<s\le t} (1+\Delta L_s)\exp(-\Delta L_s). \qquad (3.16)$$

We must ensure that $S_t > 0$ for all $t \ge 0$ almost surely, and hence it is required that $\Delta L_t > -1$ for all t. Therefore, it is assumed that the Lévy measure ν is supported on $[\delta, +\infty)$ with $\delta > -1$.

Note that, by using (3.11), the stock price process can also be written as

$$S_t = S_0 \exp\left(\sigma W_t + \int_{-\infty}^{\infty} xM\big((0,t],dx\big) + \left(a + b - \frac{\sigma^2}{2}\right)t\right.$$
$$\left. + \int_{-\infty}^{\infty} (\log(1+x) - x)N\big((0,t],dx\big)\right),$$

$$S_t = S_0 \exp\left(\sigma W_t + \int_{-\infty}^{\infty} \log(1+x)\, M\big((0,t],dx\big)\right.$$
$$\left. + \left(a + b - \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (\log(1+x) - x)\nu(dx)\right)t\right). \qquad (3.17)$$

Therefore, the stock price process can also be represented as an usual exponential

$$S_t = S_0 \exp \mathcal{Z}_t, \tag{3.18}$$

where

$$\mathcal{Z}_t := \sigma W_t + \int_{-\infty}^{\infty} \log(1+x)\, M\big((0,t], dx\big) + \left(a + b - \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (\log(1+x) - x)\nu(dx)\right) t \tag{3.19}$$

is also a Lévy process.

**Proposition 3.2.1** Let F(x) and f(x) be Borel functions satisfying the following assumptions:

(i) $F(x) > 0$ for all x in support of the Lévy measure ν and there are constants $\mu, \eta > 0$ such that $0 < \mu \le F(x)$ for all $x \in (-\eta, \eta)$.

(ii) $\int_{-\infty}^{\infty} \big|F(x) - 1 - f(x)\big| \nu(dx) < \infty$.

(iii) $\int_{-\infty}^{\infty} |f(x)|^2 \nu(dx) < \infty$.

(iv) There is an $\varepsilon > 0$ such that $\int_{-\varepsilon}^{\varepsilon} \big|F(x) - 1\big|^2 \nu(dx) < \infty$.

Then, the process $\mathcal{M}$ defined by

$$\mathcal{M}_t = \exp\left(\int_{-\infty}^{\infty} f(x) M\big((0,t], dx\big) - t \int_{-\infty}^{\infty} \big(F(x) - 1 - f(x)\big)\nu(dx)\right) \times \prod_{0<s\le t} F(\Delta X_s) \exp(-f(\Delta X_s)) \tag{3.20}$$

does not depend on f(x) and it is a local martingale.

**Proof:** First consider the process

$$\mathcal{R}_t := \int_{-\infty}^{\infty} f(x) M\big((0,t], dx\big) - t \int_{-\infty}^{\infty} \big(F(x) - 1 - f(x)\big)\nu(dx). \tag{3.21}$$

Note that the integral $\int_{-\infty}^{\infty}\big(F(x) - 1 - f(x)\big)\nu(dx)$ is well defined. Also note that, by definition, the compensated Poisson random measure $M\big((0,t],\Lambda\big)$ is a martingale, where $\Lambda$ is a Borel set in $\mathbb{R}$. Thus, by assumption (iii), $\int_{-\infty}^{\infty} f(x)M\big((0,t],dx\big)$ is a martingale. Therefore, $\mathcal{R}_t$ is a semimartingale.

Now consider the process

$$\mathcal{S}_t := \prod_{0<s\le t} F(\Delta X_s)\exp(-f(\Delta X_s)), \tag{3.22}$$

which has càdlàg paths and is adapted. By càdlàg property, the set $\{s: |\Delta X_s| \ge \varepsilon\}$ is finite, where we choose $\varepsilon > 0$ such that $\varepsilon < \eta$ and the condition (iv) is satisfied. Therefore, in order to show that $\mathcal{S}_t$ is a semimartingale, it is enough to show that

$$\mathcal{A}_t := \prod_{0<s\le t\,:\,|\Delta X_s|<\varepsilon} F(\Delta X_s)\exp(-f(\Delta X_s))$$

has paths of finite variation. To do this, consider the process

$$\log\mathcal{A}_t = \sum_{0<s\le t\,:\,|\Delta X_s|<\varepsilon} (\log(F(\Delta X_s)) - f(\Delta X_s)).$$

Then,

$$\mathrm{Var}(\log\mathcal{A}_t) \le \sum_{0<s\le t\,:\,|\Delta X_s|<\varepsilon} |\log(F(\Delta X_s)) - f(\Delta X_s)|.$$

Note that by assumption (i) and using the fact that $\log x \le x-1$, for $x > 0$, we have

$$\int_{-\varepsilon}^{\varepsilon} |\log(F(x)) - f(x)|^2\nu(dx) \le c\int_{-\varepsilon}^{\varepsilon} |F(x) - 1|^2\nu(dx) + 2\int_{-\varepsilon}^{\varepsilon} |f(x)|^2\nu(dx),$$

where c is a constant.

Thus, by assumptions (iii) and (iv), $\int_{-\varepsilon}^{\varepsilon}\big|\log\big(F(x)\big) - f(x)\big|^2\nu(dx) < \infty$, and hence $\log\mathcal{A}_t$ is a process with paths of finite variation. Therefore, $\mathcal{A}_t$ has paths of finite variation and thus, $\mathcal{S}_t$ is a semimartingale.

Consequently, we can apply Itô's formula for semimartingales to $\mathcal{M}_t := \hbar(\mathcal{R}_t, \mathcal{S}_t)$, where $\hbar(x, y) = e^x y$. Thus, we have

$$\mathcal{M}_t = 1 + \int_0^t \int_{-\infty}^{\infty} \mathcal{M}_{s-} f(x)\, M(ds, dx) - \int_0^t \int_{-\infty}^{\infty} \mathcal{M}_{s-} \big(F(x) - 1 - f(x)\big) \nu(dx) ds$$
$$+ \sum_{0<s\leq t} (\mathcal{M}_s - \mathcal{M}_{s-} - \mathcal{M}_{s-} f(\Delta L_s)), \tag{3.23}$$

since $d[\mathcal{R}, \mathcal{R}]_s^c = d[\mathcal{R}, \mathcal{S}]_s^c = d[\mathcal{S}, \mathcal{S}]_s^c = d\mathcal{S}_s^c = 0$ and $\Delta\mathcal{R}_s = f(\Delta L_s)$. Moreover, by using the facts that $\mathcal{M}_s = \mathcal{M}_{s-} F(\Delta L_s)$ and $M(ds, dx) = N(ds, dx) - ds\nu(dx)$, the equation (3.23) becomes

$$\mathcal{M}_t = 1 + \int_0^t \int_{-\infty}^{\infty} \mathcal{M}_{s-} f(x)\, M(ds, dx) + \int_0^t \int_{-\infty}^{\infty} \mathcal{M}_{s-} \big(F(x) - 1 - f(x)\big)\, M(ds, dx)$$
$$= 1 + \int_0^t \int_{-\infty}^{\infty} \mathcal{M}_{s-} (F(x) - 1)\, M(ds, dx).$$

Since the compensated Poisson random measure $M\big((0, t], \Lambda\big)$ is a martingale, where $\Lambda$ is a Borel set in $\mathbb{R}$, by assumption (iv) $\mathcal{M}$ is a local martingale.
Q.E.D.

## 3.3 Equivalent Martingale Measures

In this section we will characterize all structure-preserving $\mathbb{P}$-equivalent martingale measures $\mathbb{Q}$ under which Z remains a Lévy process and the discounted stock price process $\check{S} = \big\{\check{S}_t = S_t / B_t,\ \ 0 \leq t \leq T\big\}$ is an $\{\mathcal{F}_t\}$-martingale.

We have the following well-known result (see Theorem 33.1 and 33.2 in [30]).

**Theorem 3.3.1** Let $Z = \{Z_t,\ 0 \leq t \leq T\}$ be a Lévy process with Lévy triplet $(\alpha, \sigma^2, \nu)$ under some probability measure $\mathbb{P}$.

1) Then the following two conditions are equivalent:

(a) There is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that Z is a $\mathbb{Q}$-Lévy process with triplet $(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\nu})$.

(b) The triplet $(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\nu})$ satisfies

(i) $\tilde{\nu}(dx) = H(x)\nu(dx)$ for some Borel function $H: \mathbb{R} \to (0,\infty)$. (3.24)

(ii) $\tilde{\alpha} = \alpha + \int_{\{|x|<1\}} x\big(H(x) - 1\big)\nu(dx) + G\sigma$, for some $G \in \mathbb{R}$ . (3.25)

(iii) $\tilde{\sigma} = \sigma$. (3.26)

(iv) $\int_{-\infty}^{\infty} \left(1 - \sqrt{H(x)}\right)^2 \nu(dx) < \infty$. (3.27)

2) Suppose that any of the equivalent conditions above is satisfied. Then, the density process $\{d\mathbb{Q}_t/d\mathbb{P}_t = \xi_t,\ 0 \leq t \leq T\}$ is given by

$$\xi_t = \exp\Bigg(GW_t - \frac{1}{2}G^2 t + \lim_{\varepsilon\to 0}\left(\int_{\{|x|>\varepsilon\}} \log H(x) N\big((0,t], dx\big) - t\int_{\{|x|>\varepsilon\}} (H(x) - 1)\nu(dx)\right)\Bigg), \tag{3.28}$$

with $E(\xi_t) = 1$, for every $t \in [0,T]$ and the convergence is uniform in $t$ on any bounded interval, $\mathbb{P}$-a.s.

Moreover, the process $J = \{J_t,\ 0 \leq t \leq T\}$ given by

$$J_t = GW_t - \frac{1}{2}G^2 t + \lim_{\varepsilon\to 0}\left(\int_{\{|x|>\varepsilon\}} \log H(x) N\big((0,t], dx\big) - t\int_{\{|x|>\varepsilon\}} (H(x) - 1)\nu(dx)\right),$$

is a $\mathbb{P}$-Lévy process with triplet $\left(\alpha_J, \sigma_J^2, \nu_J\right)$ given by

$\alpha_J = -\frac{1}{2}G^2 - \int_{\mathbb{R}}\big(e^y - 1 - y1_{\{|y|\leq 1\}}\big)(\nu\vartheta^{-1})(dy)$,

$\sigma_J^2 = G^2$,

$\nu_J = [\nu\vartheta^{-1}]_{\mathbb{R}\setminus\{0\}}$ ,

where $\vartheta(x) \coloneqq \log H(x)$.

**Remark 3.3.1** Assume that the equivalent conditions in the previous theorem holds. If Z has Lévy triplet $(\tilde{\alpha}, \sigma^2, \tilde{\nu})$ under $\mathbb{Q}$, then we have the following:

1) $N(dt, dx)$ is a Poisson random measure on $(0,+\infty)\times\mathbb{R}\backslash\{0\}$ with intensity $dt \times \tilde{\nu}(dx)$ under $\mathbb{Q}$ and $\widetilde{M}(dt, dx) := N(dt, dx) - dt\tilde{\nu}(dx)$ is the compensated Poisson random measure.

2) By Lévy-Itô decomposition, we can write

$$Z_t = \sigma\widetilde{W}_t + \widetilde{X}_t, \qquad t \geq 0,$$

where $\sigma\widetilde{W} = Z - \widetilde{X}$ is a $\mathbb{Q}$-Brownian motion with coefficient $\sigma^2$ and $\widetilde{X}$ is defined by

$$\widetilde{X}_t = \int_{\{|x|<1\}} x(N\big((0,t], dx\big) - t\,\tilde{\nu}(dx)) + \int_{\{|x|\geq 1\}} xN\big((0,t], dx\big) + \tilde{\alpha}t,$$

where $\tilde{\alpha} = E_{\mathbb{Q}}\left(Z_1 - \int_{\{|x|\geq 1\}} xN\left((0,1], dx\right)\right)$.

Moreover, by using (3.3) and (3.4), we have

$$Z_t - \widetilde{X}_t = \sigma W_t + X_t - \widetilde{X}_t = \sigma(W_t - Gt),$$

which means that $\mathbb{Q}$-Brownian motion is defined by

$$\sigma\widetilde{W}_t = \sigma(W_t - Gt).$$

3) Moreover, if $\tilde{\nu}$ verifies the condition (3.5), then

$$\widetilde{X}_t = \int_{-\infty}^{\infty} x(N\big((0,t], dx\big) - t\,\tilde{\nu}(dx)) + \left(\int_{\{|x|\geq 1\}} x\,\tilde{\nu}(dx) + \tilde{\alpha}\right)t,$$

where $\tilde{\alpha} = E_{\mathbb{Q}}\big(\widetilde{X}_1\big) - \int_{\{|x|\geq 1\}} x\,\tilde{\nu}(dx)$.

Thus, the Doob-Meyer decomposition of $\widetilde{X}$ is given by

$$\widetilde{X}_t = \tilde{L}_t + \tilde{a}t,$$

where $\tilde{L} = \{\tilde{L}_t,\ t \geq 0\}$ defined by

$$\tilde{L}_t = \int_{-\infty}^{\infty} x\widetilde{M}\big((0,t], dx\big)$$

is a $\mathbb{Q}$-martingale and $\tilde{a} = \int_{\{|x|\geq 1\}} x\,\tilde{\nu}(dx) + \tilde{\alpha}$.

Using the above remark, we see that the equivalent conditions in the previous theorem imply that the process $\widetilde{W} = \{\widetilde{W}_t, 0 \leq t \leq T\}$ defined by

$$\widetilde{W}_t = W_t - Gt \tag{3.29}$$

is a standard Brownian motion under $\mathbb{Q}$.

Moreover, if $\nu$ and $\tilde{\nu}$ verify the condition (3.5), the process X is a quadratic pure jump Lévy process with Doob-Meyer decomposition (with respect to $\mathbb{Q}$)

$$X_t = \tilde{L}_t + \left(a + \int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx)\right)t, \tag{3.30}$$

where $\tilde{L} = \{\tilde{L}_t,\ 0 \leq t \leq T\}$ is a $\mathbb{Q}$-martingale and

$$\tilde{L}_t = L_t - t\int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx), \tag{3.31}$$

and the new Lévy measure is given by

$$\tilde{\nu}(dx) = H(x)\nu(dx). \qquad (\text{eqn. } (3.24))$$

This implies that the compensated Poisson random measure (with respect to $\mathbb{Q}$) on $(0,+\infty)\times\mathbb{R}\backslash\{0\}$ is given by

$$\widetilde{M}(dt, dx) = N(dt, dx) - \tilde{\nu}(dx)dt = M(dt, dx) - (H(x) - 1)\nu(dx)dt. \tag{3.32}$$

Now we want to find an equivalent martingale measure $\mathbb{Q}$ under which the discounted stock price process $\check{S}$ is a martingale. Using (3.16), (3.29) and (3.31), discounted price process can be written as

$$\check{S}_t = S_0 \exp\left(\sigma\widetilde{W}_t + \tilde{L}_t + \left(a + b - r + \sigma G - \frac{1}{2}\sigma^2\right)t\right)$$
$$\times \exp\left(t\int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx)\right)\prod_{0<s\leq t}\left(1 + \Delta\tilde{L}_s\right)\exp\left(-\Delta\tilde{L}_s\right). \tag{3.33}$$

Note that, by Proposition 3.2.1, the process

$$\exp\left(\sigma\widetilde{W}_t + \tilde{L}_t - \frac{1}{2}\sigma^2 t\right)\prod_{0<s\leq t}\left(1 + \Delta\tilde{L}_s\right)\exp\left(-\Delta\tilde{L}_s\right) \tag{3.34}$$

is a martingale. Hence, a necessary and sufficient condition for $\check{S}$ to be a $\mathbb{Q}$-martingale is the existence of $G$ and $H(x)$, with

$$\int_{-\infty}^{\infty}\left(1 - \sqrt{H(x)}\right)^2 \nu(dx) < \infty, \qquad \text{(eqn. (3.27))}$$

for which the process $\xi$ is a positive martingale, and such that

$$a + b - r + \sigma G + \int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx) = 0. \tag{3.35}$$

Thus, by (3.3), (3.29), (3.30) and (3.35), we have

$$Z_t = \sigma\widetilde{W}_t + \tilde{L}_t + (r - b)t, \tag{3.36}$$

where $\widetilde{W}$ is a $\mathbb{Q}$-Brownian motion and $\tilde{L}$ is a $\mathbb{Q}$-martingale. Therefore, the process $\tilde{Z} = \{\tilde{Z}_t, \quad 0 \leq t \leq T\}$, where

$$\tilde{Z}_t = Z_t + (b - r)t, \tag{3.37}$$

is a $\mathbb{Q}$-martingale.

Note that

$$\tilde{Z}_t = \sigma\widetilde{W}_t + \tilde{L}_t \tag{3.38}$$

and $E_{\mathbb{Q}}(\tilde{Z}_t) = 0$. Moreover, the dynamics of $\check{S}$ under $\mathbb{Q}$ is given by

$$d\check{S}_t = \check{S}_{t-}\left(\sigma d\widetilde{W}_t + d\tilde{L}_t\right) = \check{S}_{t-} d\tilde{Z}_t, \tag{3.39}$$

or,

$$\check{S}_t = S_0 \exp\left(\sigma\widetilde{W}_t + \tilde{L}_t - \frac{1}{2}\sigma^2 t\right) \prod_{0<s\leq t} \left(1 + \Delta\tilde{L}_s\right) \exp\left(-\Delta\tilde{L}_s\right). \tag{3.40}$$

Note that the dynamics of S under $\mathbb{Q}$ is given by

$$dS_t = S_{t-}\left(rdt + \sigma d\widetilde{W}_t + d\tilde{L}_t\right) = S_{t-}\left(rdt + d\tilde{Z}_t\right), \tag{3.41}$$

or,

$$S_t = S_0 \exp\left(\sigma\widetilde{W}_t + \tilde{L}_t + \left(r - \frac{\sigma^2}{2}\right) t\right) \prod_{0<s\leq t} \left(1 + \Delta\tilde{L}_s\right) \exp\left(-\Delta\tilde{L}_s\right). \tag{3.42}$$

**Remark 3.3.2** If there exists a (non-structure preserving) equivalent martingale measure $\mathbb{Q}_1$ under which Z is not a Lévy process, there always exists a (structure preserving) equivalent martingale measure $\mathbb{Q}_2$ under which Z is a Lévy process (see Eberlein and Jacod (1997) [19]).

# CHAPTER 4

# COMPLETION OF THE LÉVY MARKET MODEL

## 4.1 Power-Jump Processes

The following processes, introduced in Nualart and Schoutens (2000) [26], are considered:

$$Z_t^{(i)} = \sum\nolimits_{0<s\le t} (\Delta Z_s)^i, \quad i \ge 2, \tag{4.1}$$

and for convenience we put $Z_t^{(1)} = Z_t$ ,where $\Delta Z_s = Z_s - Z_{s-}$. Note that not necessarily $Z_t = \sum_{0<s\le t} \Delta Z_s$ holds; it is only true in the bounded variation case with $\sigma^2 = 0$.

If we define $X_t^{(1)} = X_t$ and

$$X_t^{(i)} = \sum\nolimits_{0<s\le t} (\Delta X_s)^i, \quad i \ge 2, \tag{4.2}$$

then we have

$$X_t^{(i)} = Z_t^{(i)}, \quad i \ge 2. \tag{4.3}$$

Notice that $[X, X]_t = X_t^{(2)}$.

The processes $X^{(i)} = \left\{X_t^{(i)},\ t \geq 0\right\}, i \geq 2$, are again Lévy processes and are called the ith-power-jump processes. They have jumps at the same points as the original Lévy process, but the jump sizes are equal to the ith power of the jump sizes of the original Lévy process. We have

$$E(X_t) = E(X_t^{(1)}) = at \coloneqq m_1 t < \infty, \tag{4.4}$$

and (see [28] (p.29))

$$E(X_t^{(i)}) = E\left(\sum\nolimits_{0<s\leq t} (\Delta X_s)^i\right) = t\int_{-\infty}^{\infty} x^i \nu\,(dx) \coloneqq m_i t < \infty\,, \quad i \geq 2. \tag{4.5}$$

Therefore, for every $i \geq 1$, the compensated ith-power-jump processes $Y^{(i)} = \left\{Y_t^{(i)}, t \geq 0\right\}$ can be denoted by

$$Y_t^{(i)} = Z_t^{(i)} - E(Z_t^{(i)}) = Z_t^{(i)} - m_i t, \qquad i \geq 1. \tag{4.6}$$

$Y^{(i)}$ is also called as Teugels martingale of order i. Moreover, a set of pairwise strongly orthonormal martingales $\left\{\overline{Y}^{(i)}, i \geq 1\right\}$ can be constructed such that

$$\overline{Y}^{(i)} = c_{i,i}Y^{(i)} + c_{i,i-1}Y^{(i-1)} + \cdots + c_{i,1}Y^{(1)}, \quad i \geq 1. \tag{4.7}$$

$\overline{Y}^{(i)} = \left\{\overline{Y}_t^{(i)},\ t \geq 0\right\}$ are called the orthonormalized ith-power-jump processes. It was shown in [26] that the constants $c_{i,j}$ correspond to the coefficients of the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure

$$\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx).$$

Hence, we consider the orthogonalization with respect to the scalar product

$$< P(x), Q(x) > = \int_{-\infty}^{\infty} P(x)Q(x)x^2\nu(dx) + \sigma^2 P(0)Q(0),$$

where P(x) and Q(x) are real polynomials on the positive real line.

Notice that in the case of a Brownian motion, all power-jump processes of order $i > 1$ are zero. In the case of a Poisson process, all power-jump processes are equal to the original Poisson process and all compensated power-jump processes are equal to the compensated Poisson process.

## 4.2 Enlarging the Lévy Market

In this section, we fix a time interval $[0,T]$. Suppose we have an equivalent martingale measure $\mathbb{Q}$ under which Z remains a Lévy process on $[0,T]$ with triplet $(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\nu})$. We know that under this measure $\mathbb{Q}$, the discounted stock price process $\check{S} = \left\{\check{S}_t = S_t/B_t,\ 0 \le t \le T\right\}$ is a martingale. Moreover, the process $\tilde{Z} = \{\tilde{Z}_t,\ 0 \le t \le T\}$ defined by

$$\tilde{Z}_t = Z_t + (b - r)t, \quad \text{(eqn. (3.37))}$$

is a $\mathbb{Q}$-Lévy process with Lévy measure $\tilde{\nu}$ and a $\mathbb{Q}$-martingale, by (3.38).

Now consider the ith-power-jump processes based on $\tilde{Z} = \{\tilde{Z}_t,\ 0 \le t \le T\}$. Clearly, we have $\Delta\tilde{Z}_t = \Delta Z_t$ and $\tilde{Z}_t^{(i)} = Z_t^{(i)},\ i \ge 2$. Under $\mathbb{Q}$, we construct the compensated ith-power-jump processes $Y^{(i)} = \left\{Y_t^{(i)},\ 0 \le t \le T\right\}$ and their orthonormalized version $\overline{Y}^{(i)} = \left\{\overline{Y}_t^{(i)},\ 0 \le t \le T\right\}$ based on $\tilde{Z}$, that is, the compensators are

$$m_i t := tE_{\mathbb{Q}}\left(\tilde{Z}_1^{(i)}\right), \quad i \ge 1, \tag{4.8}$$

and the orthonormalization procedure is performed under $\mathbb{Q}$. Note that

$$m_i = \int_{-\infty}^{\infty} x^i\, \tilde{\nu}(dx), \quad i \ge 2, \tag{4.9}$$

where $\tilde{\nu}(dx)$ is the Lévy measure of Z (and $\tilde{Z}$ ) under $\mathbb{Q}$ and it is required that $\tilde{\nu}$ verifies (3.5). Notice that

$$Y_t^{(1)} = \tilde{Z}_t - tE_{\mathbb{Q}}(\tilde{Z}_1) = \tilde{Z}_t, \tag{4.10}$$

$$Y_t^{(i)} = Z_t^{(i)} - t\,E_{\mathbb{Q}}\left(Z_1^{(i)}\right) = Z_t^{(i)} - t\int_{-\infty}^{\infty} x^i\,\tilde{\nu}(dx), \qquad i \geq 2. \tag{4.11}$$

The Lévy market $\mathfrak{M}$ is enlarged with a series of artificial assets based on the above processes. Actually, in the enlarged market, the trade in assets with price processes $H^{(i)} = \left\{H_t^{(i)}, t \geq 0\right\}$, where

$$H_t^{(i)} = \exp(rt)\,Y_t^{(i)}, \qquad i \geq 2, \tag{4.12}$$

is allowed. Although $H^{(i)}, i \geq 2,$ are the price processes of new assets, for simplicity, they are called the ith-power-jump assets. The orthonormalized version of these assets $\overline{H}^{(i)} = \left\{\overline{H}_t^{(i)}, t \geq 0\right\}$ are defined by

$$\overline{H}_t^{(i)} = \exp(rt)\,\overline{Y}_t^{(i)}, \qquad i \geq 2. \tag{4.13}$$

These new assets give you protection against different kinds of market shocks. For example, the 2nd-power-jump asset, in some sense, measures the volatility of the stock and thus, it can be useful to cover possible losses due to the changes in the volatility regime. Similary, to protect against a wrongly estimated skewness or kurtosis power-jump-assets of higher order can be useful.

Notice that when the original Lévy market $\mathfrak{M}$ is enlarged for different, structure-preserving equivalent martingale measures $\mathbb{Q}$, different Lévy markets $\mathfrak{M}_{\mathbb{Q}}$ are obtained, in the sense that the new assets available in each $\mathfrak{M}_{\mathbb{Q}}$ are different for each $\mathbb{Q}$.

Clearly, by construction, the discounted versions of the power-jump assets $H^{(i)}$and the orthonormalized power-jump assets $\overline{H}^{(i)}$ are $\mathbb{Q}$-martingales:

$$E_{\mathbb{Q}}\left[\exp(-rt)\,H_t^{(i)} | \mathcal{F}_s\right] = E_{\mathbb{Q}}\left[Y_t^{(i)} | \mathcal{F}_s\right] = Y_s^{(i)}, \qquad 0 \leq s \leq t \leq T \tag{4.14}$$

and

$$E_{\mathbb{Q}}\left[\exp(-rt)\,\overline{H}_t^{(i)}|\mathcal{F}_s\right] = E_{\mathbb{Q}}\left[\overline{Y}_t^{(i)}|\mathcal{F}_s\right] = \overline{Y}_s^{(i)}, \qquad 0 \le s \le t \le T. \tag{4.15}$$

Therefore, the enlarged Lévy market $\mathfrak{M}_{\mathbb{Q}}$, allowing trade in the bond, the stock and the power-jump assets, remains arbitrage-free.

**Remark 4.2.1** Assume that the original Lévy market $\mathfrak{M}$ is enlarged with the ith-power-jump assets with price processes $H_t^{(i)} = \exp(rt)\,Y_t^{(i)} = \exp(rt)\,(X_t^{(i)} - n_i t)$, $i \ge 2$. The question is whether this enlargement leads to arbitrage or not. A sufficient condition to guarantee that the enlarged market is free of arbitrage is the existence of an equivalent martingale measure $\mathbb{Q}$ under which all the discounted prices of the traded assets are martingales [16]. We have seen that if $\mathbb{Q}$ is structure-preserving $\mathbb{P}$-equivalent martingale measure, by Theorem 3.3.1, the condition that the discounted stock price must be a martingale simplifies to the existence of $G$ and $H(x) > 0$ with

$$\int_{-\infty}^{\infty}\left(1 - \sqrt{H(x)}\right)^2 \nu(dx) < \infty, \qquad (\text{eqn. } (3.27))$$

such that

$$a + b - r + \sigma G + \int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx) = 0 \qquad \left(\text{eqn. } (3.35)\right)$$

holds. Moreover, the condition that the discounted $H^{(i)}, i \ge 2$, must be a martingale simplifies to the condition

$$\int_{-\infty}^{\infty} x^i H(x)\nu\,(dx) = n_i, \qquad i \ge 2. \tag{4.16}$$

The question is whether there exist $G$ and $H(x)$ such that (3.35) and (4.16) hold simultaneously. This is related with the moment problem: given a series of numbers $\{\mu_n\}$, find a necessary and sufficient condition for the existence of a measure with $\mu_n$ as the nth moment. Uniqueness of such a measure is the another point, see [1] and [34].

## 4.3 Market Completeness

In this section, it will be shown that the market enlarged with the ith-power-jump assets is complete in the sense that for every square-integrable contingent claim $\mathcal{X}$ (i.e. a non-negative square-integrable $\mathcal{F}_T$-measurable random variable) one can construct a sequence of self-financing portfolios whose values, at time T, converge in $L^2(\mathbb{Q})$ to $\mathcal{X}$. These portfolios will consist of a finite number of bonds, stocks and ith-power-jump assets. It will be said, for short, that $\mathcal{X}$ can be replicated. This notion of completeness is equivalent to the notion of approximate completeness of Björk et al. (1997) [8].

**Definition 4.3.1** A portfolio $\pi = \{\pi^n\}$ is a sequence of finite-dimensional predictable processes

$$\left\{\pi_t^n = \left(\alpha_t^n, \beta_t^n, \beta_t^{(2),n}, \beta_t^{(3),n}, \dots, \beta_t^{(k_n),n}\right), \qquad 0 \le t \le T, \qquad n \ge 2\right\}, \tag{4.17}$$

where $\alpha_t^n$ represents the number of bonds at time t, $\beta_t^n$ represents the number of stocks at time t, $\beta_t^{(i),n}$ represents the number of ith-power-jump assets $H^{(i)}$ at time t and $k_n$ is an integer which depends on n.

A portfolio $\pi = \{\pi^n\}$ is self-financing if each $\pi^n$ is self-financing.

**Definition 4.3.2** Fix $p \ge 1$. A contingent claim $\mathcal{X} \in L^p(\Omega, \mathcal{F}_T, \mathbb{Q})$ is called replicable in $L^p(\mathbb{Q})$ if there exists a self-financing portfolio whose values, at time T, converge to $\mathcal{X}$ in $L^p(\mathbb{Q})$.

**Definition 4.3.3** A contingent claim $\mathcal{X} \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ is called strongly replicable in $L^1(\mathbb{Q})$ if it is replicable in $L^1(\mathbb{Q})$ by a portfolio $\pi = \{\pi^n\}$ of the form

$$\left\{\pi_t^n = \left(\alpha_t, \beta_t, \beta_t^{(2)}, \beta_t^{(3)}, \dots, \beta_t^{(n)}\right), \qquad 0 \le t \le T, \qquad n \ge 2\right\}, \tag{4.18}$$

where the number of assets $\alpha_t, \beta_t, \beta_t^{(2)}, \ldots$ do not depend on n and if the series

$$\int_0^t \alpha_s dB_s + \int_0^t \beta_s dS_s + \sum_{i=2}^{\infty} \int_0^t \beta_s^{(i)} dH_s^{(i)} \tag{4.19}$$

converges absolutely in $L^1(\mathbb{Q})$ for each $t \in [0,T]$.

In order to show that the enlarged market is complete we need the following theorem, see Nualart and Schoutens (2000), (2001) [26, 25].

**Theorem 4.3.1** (Martingale Representation Property) Every square integrable $\mathbb{Q}$-martingale $\mathcal{M} = \{\mathcal{M}_t,\ 0 \le t \le T\}$ has a representation in the form

$$\mathcal{M}_t = \mathcal{M}_0 + \sum_{i=1}^{\infty} \int_0^t h_s^{(i)} d\overline{Y}_s^{(i)},$$

where $h_s^{(i)}, i \ge 1$, are predictable processes such that $E_{\mathbb{Q}}\left(\int_0^t \sum_{i=1}^{\infty} |h_s^{(i)}|^2\, ds\right) < \infty$.

The martingale representation property (MRP) allows the representation of any square integrable $\mathbb{Q}$-martingale as an orthogonal sum of stochastic integrals with respect to the orthonormalized power-jump processes $\{\overline{Y}^{(i)},\ i \ge 1\}$. In other words, any square-integrable $\mathbb{Q}$-martingale $\mathcal{M} = \{\mathcal{M}_t,\ 0 \le t \le T\}$ can be represented as follows:

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t h_s\, d\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} d\overline{Y}_s^{(i)}, \tag{4.20}$$

where $h_s$ and $h_s^{(i)}, i \ge 2$, are predictable processes such that $E_{\mathbb{Q}}\left(\int_0^t |h_s|^2 ds\right) < \infty$ and $E_{\mathbb{Q}}\left(\int_0^t \sum_{i=2}^{\infty} |h_s^{(i)}|^2\, ds\right) < \infty$. Remember that $\overline{Y}^{(i)},\ i \ge 1$, are the orthonormalized versions of $Y^{(i)}$, where

$$Y_t^{(1)} = \tilde{Z}_t - tE_{\mathbb{Q}}(\tilde{Z}_1) = \tilde{Z}_t, \quad (\text{eqn. } (4.10))$$

$$Y_t^{(i)} = Z_t^{(i)} - t\, E_{\mathbb{Q}}\left(Z_1^{(i)}\right) = Z_t^{(i)} - t\int_{-\infty}^{\infty} x^i\, \tilde{\nu}(dx), \qquad i \ge 2, \quad (\text{eqn. } (4.11))$$

with

$$\tilde{Z}_t = Z_t + (b - r)t \quad \big(\text{eqn.}\,(3.37)\big)$$

being a $\mathbb{Q}$-martingale. Also remember that the dynamics of $\check{S}$ under $\mathbb{Q}$ is given by

$$d\check{S}_t = \check{S}_{t-} d\tilde{Z}_t \,. \qquad (\text{eqn.}\,(3.39))$$

Therefore, the MRP implies that the enlarged market is complete. In fact, we have the following theorem.

**Theorem 4.3.2** The Lévy market model $\mathfrak{M}_{\mathbb{Q}}$, enlarged with the ith-power-jump assets, is complete in the sense that any square-integrable contingent claim $\mathcal{X} \in L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$ can be replicated in $L^2(\mathbb{Q})$.

**Proof:** Consider a square-integrable (with respect to $\mathbb{Q}$) contingent claim $\mathcal{X}$ with maturity T and let $\mathcal{M}_t = E_{\mathbb{Q}}(\exp(-rT)\,\mathcal{X}|\mathcal{F}_t)$. By the MRP, we have

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t h_s\, d\tilde{Z}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} d\overline{Y}_s^{(i)}. \qquad (\text{eqn.}\,(4.20))$$

If we define

$$\mathcal{M}_t^N \coloneqq \mathcal{M}_0 + \int_0^t h_s\, d\tilde{Z}_s + \sum_{i=2}^{N} \int_0^t h_s^{(i)} d\overline{Y}_s^{(i)}, \tag{4.21}$$

we have $\lim_{N\to\infty} \mathcal{M}_t^N = \mathcal{M}_t$ in $L^2(\mathbb{Q})$.

Define the sequence of portfolios (in terms of the orthonormalized ith-power-jump assets)

$$\phi^N = \left\{\phi_t^N = \left(\alpha_t^N, \beta_t, \beta_t^{(2)}, \beta_t^{(3)}, \dots, \beta_t^{(N)}\right), \qquad t \geq 0\right\}, \qquad N \geq 2,$$

by

$$\alpha_t^N = \mathcal{M}_{t-}^N - \beta_t S_{t-} e^{-rt} - e^{-rt} \sum_{i=2}^{N} \beta_t^{(i)} \overline{H}_{t-}^{(i)}, \tag{4.22}$$

$$\beta_t = e^{rt} h_t S_{t-}^{-1}, \tag{4.23}$$

$$\beta_t^{(i)} = h_t^{(i)}, \qquad i = 2,3, \dots N. \tag{4.24}$$

Here $\alpha_t^N$ represents the number of bonds, $\beta_t$ represents the number of stocks and $\beta_t^{(i)}$ represents the number of orthonormalized ith-power-jump assets at time t. Then, the value $V_t^N$ of the portfolio $\phi^N$ at time t is given by

$$V_t^N = \alpha_t^N e^{rt} + \beta_t S_t + \sum_{i=2}^{N} \beta_t^{(i)} \overline{H}_t^{(i)} = e^{rt} \mathcal{M}_t^N, \tag{4.25}$$

which implies that the sequence of portfolios $\{\phi^N,\ N \ \geq\ 2\}$ replicates the claim $\mathcal{X}$.

Thus, to complete the proof, it is enough to show that the portfolio $\phi^N$ is self-financing. That is,

$$G_t^N + \mathcal{M}_0 = e^{rt} \mathcal{M}_t^N, \tag{4.26}$$

where $G_t^N$ is the gain process corresponding to $\phi^N$ at time t, given by

$$G_t^N = r \int_0^t \alpha_s^N\, e^{rs} ds + \int_0^t \beta_s dS_s + \sum_{i=2}^{N} \int_0^t \beta_s^{(i)} d\overline{H}_s^{(i)}. \tag{4.27}$$

By (4.22), (4.23) and (4.24), we can write (4.27) as

$$G_t^N = r \int_0^t \mathcal{M}_{s-}^N\, e^{rs} ds - r \int_0^t h_s e^{rs} ds - r \sum_{i=2}^{N} \int_0^t h_s^{(i)} \overline{H}_{s-}^{(i)}\, ds + \int_0^t h_s e^{rs} S_{s-}^{-1} dS_s$$
$$+ \sum_{i=2}^{N} \int_0^t h_s^{(i)} d\overline{H}_s^{(i)}. \tag{4.28}$$

Note that integration by parts follows

$$r \int_0^t \mathcal{M}_s^N\, e^{rs} ds = e^{rt} \mathcal{M}_t^N - \mathcal{M}_0 - \int_0^t h_s e^{rs}\, d\tilde{Z}_s - \sum_{i=2}^{N} \int_0^t h_s^{(i)} e^{rs} d\overline{Y}_s^{(i)}. \tag{4.29}$$

Moreover, using

$$\overline{H}_t^{(i)} = \exp(rt)\, \overline{Y}_t^{(i)}, \qquad i \geq 2, \qquad \big(\text{eqn. } (4.13)\big)$$

and

$$dS_t = S_{t-}\big(r dt + d\tilde{Z}_t\big), \qquad \big(\text{eqn.}\,(3.41)\big)$$

we can write (4.29) as

$$r\int_0^t \mathcal{M}_s^N\, e^{rs} ds = e^{rt}\mathcal{M}_t^N - \mathcal{M}_0 - \int_0^t h_s e^{rs}\, S_{s-}^{-1} dS_s + r\int_0^t h_s e^{rs} ds$$
$$- \sum_{i=2}^N \int_0^t h_s^{(i)} d\overline{H}_s^{(i)} \; + r\sum_{i=2}^N \int_0^t h_s^{(i)} \overline{H}_{s-}^{(i)}\, ds. \tag{4.30}$$

Thus, by substituting (4.30) into (4.28), we obtain

$$G_t^N = e^{rt}\mathcal{M}_t^N - \mathcal{M}_0.$$

Therefore, $\{\phi^N,\ N \geq 2\}$ is the sequence of self-financing portfolios replicating the claim $\mathcal{X}$.

Q.E.D.

# CHAPTER 5

# HEDGING PORTFOLIOS

## 5.1 Pricing Formula

Consider a contingent claim $\mathcal{X}$ whose payoff is only a function of the value, at maturity, of the stock price S, that is, the payoff is a function of the form $\phi(S_T)$. The value at time $t$ of the contingent claim with payoff $\mathcal{X} = \phi(S_T)$ is given by

$$F(t, S_t) = e^{-r(T-t)} E_{\mathbb{Q}}[\mathcal{X} | \mathcal{F}_t] = e^{-r(T-t)} E_{\mathbb{Q}}[\phi(S_T) | \mathcal{F}_t]. \tag{5.1}$$

Remember that the dynamics of S under $\mathbb{Q}$ is given by

$$dS_t = S_{t-}\big(rdt + \sigma d\widetilde{W}_t + d\tilde{L}_t\big). \qquad (\text{eqn.}\,(3.41))$$

By Itô's formula, it can easily be shown that

$$S_T = S_t \exp\left(\sigma\big(\widetilde{W}_T - \widetilde{W}_t\big) + (\tilde{L}_T + \tilde{L}_t) + \left(r - \frac{\sigma^2}{2}\right)(T - t)\right) \\ \times \prod_{t<s\leq T}\big(1 + \Delta\tilde{L}_s\big)\exp\big(-\Delta\tilde{L}_s\big). \tag{5.2}$$

Thus, the price function is given by

$$F(t, x) = e^{-r(T-t)} E_{\mathbb{Q}}\left[\phi\left(x\exp\left(\sigma\big(\widetilde{W}_T - \widetilde{W}_t\big) + (\tilde{L}_T + \tilde{L}_t) + \left(r - \frac{\sigma^2}{2}\right)(T - t)\right) \\ \times \prod_{t<s\leq T}\big(1 + \Delta\tilde{L}_s\big)\exp\big(-\Delta\tilde{L}_s\big)\right)\right]. \tag{5.3}$$

$$F(t,x) = e^{-r(T-t)} E_{\mathbb{Q}} \left[ \phi \left( x\exp \left( \sigma \widetilde{W}_{T-t} + \tilde{L}_{T-t} + \left( r - \frac{\sigma^2}{2} \right) (T-t) \right) \right. \right.$$

$$\left. \left. \times \prod_{0<s\leq T-t} \left(1 + \Delta\tilde{L}_s\right) \exp\left(-\Delta\tilde{L}_s\right) \right) \right]. \tag{5.4}$$

Remember that in the Black-Scholes model the price of the option with volatility σ is given by

$$F_{BS}(t,x) = e^{-r(T-t)} E_{\mathbb{Q}} \left[ \phi \left( x\exp \left( \sigma \widetilde{W}_{T-t} + \left( r - \frac{\sigma^2}{2} \right) (T-t) \right) \right) \right]. \tag{5.5}$$

# 5.2 Hedging Portfolios

In this section, we will obtain the hedging portfolio of a contingent claim $\mathcal{X}$ whose payoff is a function of the value, at maturity, of the stock price S and a pure jump process $K = \{K_t,\ 0 \leq t \leq T\}$. The jump process is defined by

$$K_t = \int_{-\infty}^{\infty} g(x)\, \widetilde{M}\big((0,t], dx\big), \tag{5.6}$$

where $g \in C^{\infty}, g(0) = g'(0) = 0$ such that $\int_{-\infty}^{\infty} |g(x)| \tilde{\nu}(dx) < \infty$ and

$$\widetilde{M}(dt, dx) = N(dt, dx) - \tilde{\nu}(dx)dt \qquad (\text{eqn. } (3.32))$$

is the compensated Poisson random measure. Thus, the payoff is a function of the form $\phi(S_T, K_T)$. Note that the jump process K will enable us to consider the portfolio optimization problem, which is to be discussed later.

The value of the contingent claim with payoff $\mathcal{X} = \phi(S_T, K_T)$ at time t is given by

$$F(t, S_t, K_t) = e^{-r(T-t)} E_{\mathbb{Q}}[\phi(S_T, K_T) | \mathcal{F}_t] = e^{-r(T-t)} E_{\mathbb{Q}} \left[ \phi \left( \frac{S_T}{S_t} S_t, K_T - K_t + K_t \right) | \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} E_{\mathbb{Q}} \left[ \phi \left( \frac{S_T}{S_t} x_1, K_T - K_t + x_2 \right) \right] \Big|_{\, x_1 = S_t,\, x_2 = K_t}.$$

Here we used the independence of $\frac{S_T}{S_t}$ and $K_T - K_t$ with respect to $\mathcal{F}_t$.

Now consider the following operators:

$$\mathcal{L}f(t,x) = D_0 f(t,x) + r x_1 D_1 f(t,x) - r f(t,x) + \frac{1}{2}\sigma^2 x_1^2 D_1^2 f(t,x)$$
$$- D_2 f(t,x) \int_{\mathbb{R}} g(y)\tilde{\nu}(dy), \qquad (5.7)$$

$$\mathcal{D}f(t,x) = \int_{-\infty}^{\infty} h(t,x,y)\tilde{\nu}(dy), \qquad (5.8)$$

where

$$x := (x_1, x_2),\ D_0 := \partial/\partial t,\ D_k := \partial/\partial x_k,\ D_1^i := \partial^i/\partial x_1^i \qquad (5.9)$$

and

$$h(t,x,y) := f\big(t, x_1(1+y), x_2 + g(y)\big) - f(t,x) - x_1 y D_1 f(t,x). \qquad (5.10)$$

We will show that the price function $F(t, x_1, x_2)$ satisfies a Partial Differential Integral Equation (PDIE). In order to do this, we need the following lemma.

**Lemma 5.2.1** Consider a real function $h(s,x,y): \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ which is analytic in the y variable and such that $h(s,x,0) = 0$ and $(\partial h/\partial y)(s,x,0) = 0$. Set

$$a_i(s,x) := \frac{1}{i!}\frac{\partial^i}{\partial y^i} h(s,x,0). \qquad (5.11)$$

Let $Y := \{Y_t,\ 0 \le t \le T\}$ be an adapted process with left continuous paths and with values on $\mathbb{R}^m$ and set

$$|m|_i := \int_{-\infty}^{\infty} |y|^i \tilde{\nu}(dy). \qquad (5.12)$$

If we assume that

$$\sum_{i=2}^{\infty} |m|_i \int_0^T E_{\mathbb{Q}}[|a_i(s, Y_s)|]\, ds < \infty, \qquad (5.13)$$

then

$$\sum_{t<s\le T} h(s, Y_s, \Delta X_s) = \sum_{i=2}^{\infty} \int_t^T a_i(s, Y_s) dY_s^{(i)} + \int_t^T \int_{-\infty}^{\infty} h(s, Y_s, y)\tilde{\nu}(dy)ds \tag{5.14}$$

a.s. and in the $L^1(\mathbb{Q})$-sense.

**Proof**: Since the function h(s,x,y) is analytic in the y variable, it can be expanded as

$$h(s, x, y) = \sum_{i=2}^{\infty} a_i(s, x)\, y^i.$$

Then we have

$$\sum_{t<s\le T} h(s, Y_s, \Delta X_s) = \sum_{t<s\le T} \sum_{i=2}^{\infty} a_i(s, Y_s)(\Delta X_s)^i. \tag{5.15}$$

Now we will show that $\sum_{i=2}^{\infty} \sum_{t<s\le T} |a_i(s, Y_s)||\Delta X_s|^i < \infty$. Notice that, since $\sum_{i=2}^{\infty} \sum_{t<s\le T} |a_i(s, Y_s)||\Delta X_s|^i$ is a nonnegative random variable, if $E_{\mathbb{Q}}\left[ \sum_{i=2}^{\infty} \sum_{t<s\le T} |a_i(s, Y_s)||\Delta X_s|^i\right] < \infty$, then $\sum_{i=2}^{\infty} \sum_{t<s\le T} |a_i(s, Y_s)||\Delta X_s|^i < \infty$ a.s..

Given $\varepsilon > 0$, and denote $B_\varepsilon \coloneqq \mathbb{R}\backslash(-\varepsilon, \varepsilon)$. Notice that

$$\sum_{t<s\le T} |a_i(s, Y_s)||\Delta X_s|^i\, 1_{\{|\Delta X_s|>\varepsilon\}} = \int_t^T \int_{B_\varepsilon} |a_i(s, Y_s)||y|^i\, N(ds, dy).$$

It can be shown that

$$E_{\mathbb{Q}}\left[\int_t^T \int_{B_\varepsilon} |a_i(s, Y_s)||y|^i\, N(ds, dy)\right] = E_{\mathbb{Q}}\left[\int_t^T \int_{B_\varepsilon} |a_i(s, Y_s)||y|^i\, \tilde{\nu}(dy)ds\right]$$
$$\le |m|_i \int_0^T E_{\mathbb{Q}}[|a_i(s, Y_s)|]ds.$$

Thus, by monotone convergence, as $\varepsilon \to 0$ we have

$$E_{\mathbb{Q}}\left[\sum_{t<s\le T} |a_i(s, Y_s)||\Delta X_s|^i\right] \le |m|_i \int_0^T E_{\mathbb{Q}}[|a_i(s, Y_s)|]ds,$$

and hence, by assumption (5.13), we have

$$E_{\mathbb{Q}}\left[\sum_{i=2}^{\infty}\sum_{t<s\le T}|a_i(s,Y_s)||\Delta X_s|^i\right]\le\sum_{i=2}^{\infty}|m|_i\int_0^T E_{\mathbb{Q}}[|a_i(s,Y_s)|]ds<\infty.$$

Therefore, by the above argument, we have $\sum_{i=2}^{\infty}\sum_{t<s\le T}|a_i(s,Y_s)||\Delta X_s|^i<\infty$ a.s.

Consequently, by Fubini's Theorem and assumption (5.13), we can write (5.15) as

$$\begin{aligned}\sum_{t<s\le T}h(s,Y_s,\Delta X_s)&=\sum_{i=2}^{\infty}\sum_{t<s\le T}a_i(s,Y_s)(\Delta X_s)^i\\&=\sum_{i=2}^{\infty}\int_t^T a_i(s,Y_s)dY_s^{(i)}+\sum_{i=2}^{\infty}\int_t^T a_i(s,Y_s)m_i ds\\&=\sum_{i=2}^{\infty}\int_t^T a_i(s,Y_s)dY_s^{(i)}+\int_t^T\int_{-\infty}^{\infty}\sum_{i=2}^{\infty}a_i(s,Y_s)\,y^i\,\tilde{v}(dy)ds\\&=\sum_{i=2}^{\infty}\int_t^T a_i(s,Y_s)dY_s^{(i)}+\int_t^T\int_{-\infty}^{\infty}h(s,Y_s,y)\tilde{v}(dy)ds\end{aligned}$$

a.s. and in $L^1(\mathbb{Q})$. Here we used the fact that $Y_t^{(i)}=X_t^{(i)}-m_i t,\ \ i\ge 2$, where

$$m_i=\int_{-\infty}^{\infty}y^i\,\tilde{v}(dy),\qquad i\ge 2.\quad (\text{eqn. } (4.9))$$

Q.E.D.

**Theorem 5.2.1** Let $F(t,S_t,K_t)=e^{-r(T-t)}E_{\mathbb{Q}}[\mathcal{X}|\mathcal{F}_t]$ be the value of the contingent claim $\mathcal{X}=\phi(S_T,K_T)$ at time t, where $\mathcal{X}\in L^1(\Omega,\mathcal{F}_T,\mathbb{Q})$. Let $x:=(x_1,x_2)$ and assume that $F(t,x)\in C^{1,\infty,\infty}$. Set $a_i(s,x):=\frac{1}{i!}\frac{\partial^i}{\partial y^i}h(s,x,0)$ and $Y_t:=(S_t,K_t)$, where

$$K_t=\int_{-\infty}^{\infty}g(x)\,\widetilde{M}\big((0,t],dx\big).\ \ (\text{eqn. } (5.6))$$

Assume that

(i) The function h given by

$$h(t,x,y):=F\big(t,x_1(1+y),x_2+g(y)\big)-F(t,x)-x_1yD_1F(t,x)\qquad(5.16)$$

is analytic in y.

(ii) $\sum_{i=2}^{\infty}|m|_i\int_0^T E_{\mathbb{Q}}[|a_i(s,Y_s)|]\,ds<\infty.$ (5.17)

Then, $F(t, S_t, K_t)$ is the solution of the following PDIE:

$$\begin{cases} \mathcal{L}F(t,x) + \mathcal{D}F(t,x) = 0, \\ \qquad F(T,x) = \phi(x). \end{cases} \tag{5.18}$$

**Proof:** By assumption, the discounted price process $e^{-rt}F(t, Y_t)$ is a $\mathbb{Q}$-martingale. Hence, for any decomposition $e^{-rt}F(t, Y_t) = F(0, Y_0) + \mathcal{M}_t + \mathcal{A}_t$, where $\mathcal{M}$ is a local martingale and $\mathcal{A}$ is a finite variation process, we must have $\mathcal{A}_t \equiv 0$. In the following we derive such a representation. By applying Itô's formula for semimartingales to $f(t, Y_t) = e^{-rt}F(t, Y_t)$, where $Y_t = (S_t, K_t)$, we have

$$\begin{aligned} f(t, Y_t) = f(0, Y_0) &+ \int_0^t D_0 f(s, Y_{s-}) ds + \int_0^t D_1 f(s, Y_{s-})\, dS_s + \int_0^t D_2 f(s, Y_{s-})\, dK_s^c \\ &+ \frac{1}{2}\int_0^t D_1^2\, f(s, Y_{s-}) d[S,S]_s^c \\ &+ \sum\nolimits_{0<s\le t} (f(s, Y_s) - f(s, Y_{s-}) - D_1 f(s, Y_{s-})\Delta S_s), \end{aligned}$$

since $[S,K]_s^c = [K,K]_s^c = 0$. Thus,

$$\begin{aligned} e^{-rt}F(t, Y_t) = F(0, Y_0) &+ \int_0^t \left(-re^{-rs}F(s, Y_{s-}) + e^{-rs}D_0F(s, Y_{s-})\right) ds \\ &+ \int_0^t e^{-rs} D_1 F(s, Y_{s-})\, dS_s - \int_0^t e^{-rs} D_2 F(s, Y_{s-}) \int_{\mathbb{R}} g(y)\tilde{\nu}(dy)\, ds \\ &+ \frac{1}{2}\sigma^2 \int_0^t e^{-rs} S_{s-}^2 D_1^2\, F(s, Y_{s-}) ds \\ &+ \sum\nolimits_{0<s\le t} e^{-rs}(F(s, Y_s) - F(s, Y_{s-}) - D_1 F(s, Y_{s-})\Delta S_s). \end{aligned} \tag{5.19}$$

Note that $\Delta S_s = S_{s-}\Delta X_s$ and $\Delta K_s = g(\Delta X_s)$ imply that

$$Y_s = (S_s, K_s) = \left(S_{s-}(1 + \Delta X_s), K_{s-} + g(\Delta X_s)\right),$$

so we have

$$F(s, Y_s) - F(s, Y_{s-}) - D_1 F(s, Y_{s-})\Delta S_s = h(s, Y_{s-}, \Delta X_s). \tag{5.20}$$

Thus, by Lemma 5.2.1,

$$\sum\nolimits_{0<s\leq t} e^{-rs}(F(s,Y_s)-F(s,Y_{s-})-D_1F(s,Y_{s-})\Delta S_s)$$

$$=\sum\nolimits_{i=2}^{\infty}\int_0^t \frac{e^{-rs}}{i!}\frac{\partial^i}{\partial y^i}h(s,Y_{s-},0)dY_s^{(i)}$$

$$+\int_0^t\int_{-\infty}^{\infty} e^{-rs}h(s,Y_{s-},y)\tilde{\nu}(dy)ds. \tag{5.21}$$

Moreover, the dynamics of S under $\mathbb{Q}$ is given by

$$dS_t = S_{t-}\big(rdt+\sigma d\widetilde{W}_t + d\tilde{L}_t\big), \qquad (\text{eqn.}\,(3.41))$$

where $\widetilde{W}$ is a $\mathbb{Q}$-Brownian motion and $\tilde{L}$ is a $\mathbb{Q}$-martingale.

Therefore, by substituting (3.41) and (5.21) into (5.19) and making necessary arrangements, we obtain

$$e^{-rt}F(t,Y_t) = F(0,Y_0)+\mathcal{M}_t+\mathcal{A}_t,$$

where

$$\mathcal{M}_t := \sum\nolimits_{i=2}^{\infty}\int_0^t \frac{e^{-rs}}{i!}\frac{\partial^i}{\partial y^i}h(s,Y_{s-},0)dY_s^{(i)}+\sigma\int_0^t e^{-rs}S_{s-}D_1F(s,Y_{s-})d\widetilde{W}_s$$
$$+\int_0^t e^{-rs}S_{s-}D_1F(s,Y_{s-})d\tilde{L}_s$$

and

$$\mathcal{A}_t := \int_0^t e^{-rs}\Bigg(-rF(s,Y_{s-})+D_0F(s,Y_{s-})+\frac{1}{2}\sigma^2S_{s-}^2D_1^2F(s,Y_{s-})$$
$$-D_2F(s,Y_{s-})\int_{\mathbb{R}}g(y)\tilde{\nu}(dy)+rS_{s-}D_1F(s,Y_{s-})$$
$$+\int_{-\infty}^{\infty}h(s,Y_{s-},y)\tilde{\nu}(dy)\Bigg)ds.$$

The process $\mathcal{M}$ is a $\mathbb{Q}$-local martingale, since by (5.17) the series $\sum_{i=2}^{\infty}\int_0^t \frac{e^{-rs}}{i!}\frac{\partial^i}{\partial y^i}h(s,Y_{s-},0)dY_s^{(i)}$ converges in $L^1(\mathbb{Q})$ and the processes

$\int_0^t \frac{e^{-rs}}{i!}\frac{\partial^i}{\partial y^i}h(s,Y_{s-},0)dY_s^{(i)}$ are martingales. The process $\mathcal{A}$ is a predictable finite variation process. Thus, the condition $\mathcal{A}_t \equiv 0$ yields

$$D_0F(s,Y_{s-}) + rS_{s-}D_1F(s,Y_{s-}) - rF(s,Y_{s-}) + \frac{1}{2}\sigma^2 S_{s-}^2 D_1^2F(s,Y_{s-})$$
$$- D_2F(s,Y_{s-})\int_{\mathbb{R}} g(y)\tilde{\nu}(dy) + \int_{-\infty}^{\infty} h(s,Y_{s-},y)\tilde{\nu}(dy) = 0.$$

Therefore, by (5.7) and (5.8), $F(t,x)$ satisfies the PDIE given by (5.18).
Q.E.D.

**Theorem 5.2.2** Consider the value $F(t,S_t,K_t)$, at time $t$, of a contingent claim $\mathcal{X} = \phi(S_T,K_T)$, satisfying the conditions of the previous theorem, where

$$K_t = \int_{-\infty}^{\infty} g(x)\,\widetilde{M}\big((0,t],dx\big). \qquad \text{(eqn. (5.6))}$$

Then, $\mathcal{X}$ is strongly replicable in $L^1(\mathbb{Q})$ and its replicating portfolio at time $t$ is given by

$$\alpha_t = \frac{1}{B_t}\left(F(t,S_{t-},K_{t-}) - S_{t-}D_1F(t,S_{t-},K_{t-}) - \sum_{i=2}^{\infty}\frac{1}{i!\,B_t}\frac{\partial^i}{\partial y^i}h(t,S_{t-},K_{t-},0)H_{t-}^{(i)}\right), \tag{5.22}$$

$$\beta_t = D_1F(t,S_{t-},K_{t-}), \tag{5.23}$$

$$\beta_t^{(i)} = \frac{1}{i!\,B_t}\frac{\partial^i}{\partial y^i}h(t,S_{t-},K_{t-},0), \quad i = 2,3,\dots, \tag{5.24}$$

where

$$h(t,x_1,x_2,y) := F\big(t,x_1(1+y),x_2+g(y)\big) - F(t,x_1,x_2) - x_1yD_1F(t,x_1,x_2),$$

$\alpha_t$ represents the number of bonds at time $t$, $\beta_t$ represents the number of stocks at time $t$ and $\beta_t^{(i)}$ represents the number of ith-power-jump assets $H^{(i)}$ at time $t$.

**Proof:** By applying Itô's formula for semimartingales to $F(t,Y_t)$, where $Y_t = (S_t,K_t)$, we have

$$F(T, Y_T) = F(t, Y_t) + \int_t^T D_0 F(s, Y_{s-}) ds + \int_t^T D_1 F(s, Y_{s-})\, dS_s$$
$$- \int_t^T D_2 F(s, Y_{s-}) \int_{\mathbb{R}} g(y)\tilde{\nu}(dy)\, ds + \frac{1}{2}\sigma^2 \int_t^T S_{s-}^2 D_1^2 F(s, Y_{s-}) ds$$
$$+ \sum_{t<s\leq T} \left(F(s, Y_s) - F(s, Y_{s-}) - D_1 F(s, Y_{s-}) \Delta S_s\right). \qquad (5.25)$$

As in the proof of the previous theorem, we have

$$F(s, Y_s) - F(s, Y_{s-}) - D_1 F(s, Y_{s-}) \Delta S_s = h(s, Y_{s-}, \Delta X_s). \qquad \text{(eqn. (5.20))}$$

Thus, by Lemma 5.2.1,

$$\sum_{t<s\leq T} \left(F(s, Y_s) - F(s, Y_{s-}) - D_1 F(s, Y_{s-}) \Delta S_s\right)$$
$$= \sum_{i=2}^{\infty} \int_t^T \frac{1}{i!} \frac{\partial^i}{\partial y^i} h(s, Y_{s-}, 0) dY_s^{(i)} + \int_t^T \int_{-\infty}^{\infty} h(s, Y_{s-}, y)\tilde{\nu}(dy) ds.$$
$$(5.26)$$

Therefore, by substituting (5.26) into (5.25) and using (5.7) and (5.8), we have

$$F(T, Y_T) = F(t, Y_t)$$
$$+ \int_t^T \left(\mathcal{L}F(s, Y_{s-}) + \mathcal{D}F(s, Y_{s-}) - rS_{s-} D_1 F(s, Y_{s-}) + rF(s, Y_{s-})\right) ds$$
$$+ \int_t^T D_1 F(s, Y_{s-})\, dS_s + \sum_{i=2}^{\infty} \int_t^T \frac{1}{i!} \frac{\partial^i}{\partial y^i} h(s, Y_{s-}, 0) dY_s^{(i)}. \qquad (5.27)$$

Note that by the previous theorem, F satisfies the PDIE (5.18). Also using

$$H_t^{(i)} = \exp(rt)\, Y_t^{(i)}, \qquad i \geq 2, \qquad \text{(eqn. (4.12))}$$

and making necessary arrangements, we write (5.27) as

$$F(T,Y_T) = F(t,Y_t)$$
$$+ \int_t^T \frac{1}{B_s}\Bigg(F(s,Y_{s-}) - S_{s-}D_1F(s,Y_{s-})$$
$$- \sum_{i=2}^{\infty} \frac{1}{i!\,B_s}\frac{\partial^i}{\partial y^i} h(s,Y_{s-},0)H_{s-}^{(i)}\Bigg) dB_s + \int_t^T D_1F(s,Y_{s-})dS_s$$
$$+ \sum_{i=2}^{\infty} \int_t^T \frac{1}{i!\,B_s}\frac{\partial^i}{\partial y^i} h(s,Y_{s-},0)dH_s^{(i)}. \qquad (5.28)$$

Therefore, we have

$$F(T,Y_T) = F(t,Y_t) + \int_t^T \alpha_s dB_s + \int_t^T \beta_s dS_s + \sum_{i=2}^{\infty} \int_t^T \beta_s^{(i)} dH_s^{(i)},$$

where $\alpha_s$ is the number of bonds, $\beta_s$ is the number of stocks and $\beta_s^{(i)}$ is the number of ith-power-jump assets, given by (5.22), (5.23) and (5.24), one should have in his portfolio at time s to hedge the contingent claim $\mathcal{X}$.

Q.E.D.

**Remark 5.2.1** If a contingent claim $\mathcal{X}$ with a payoff depending only on the value, at maturity, of the stock price is considered, i.e. $\mathcal{X} = \phi(S_T)$, then

$$F = F(t,S_t) \text{ and } h(t,x,y) = F\big(t,x(1+y)\big) - F(t,x) - xyD_1F(t,x).$$

The price function F(t,x) satisfies (see also [11] and [29])

$$D_0F(t,x) + rxD_1F(t,x) + \frac{1}{2}\sigma^2x^2D_1^2F(t,x) + \mathcal{D}F(t,x) = rF(t,x)$$

with $F(T,S_T) = \phi(S_T)$, where

$$\mathcal{D}F(t,x) = \int_{-\infty}^{\infty}\Big(F\big(t,x(1+y)\big) - F(t,x) - xyD_1F(t,x)\Big)\tilde{\nu}(dy)$$

and

$$D_0 := \partial/\partial t,\, D_1 := \partial/\partial x,\ D_1^n := \partial^n/\partial x^n.$$

Notice that

$$h(t,x,0) = 0, \qquad \frac{\partial}{\partial y} h(t,x,0) = 0, \qquad \frac{\partial^n}{\partial y^n} h(t,x,0) = x^n D_1^n F(t,x), \qquad n \geq 2,$$

and thus, by equation (5.28), we have

$$\begin{aligned} F(T,S_T) = F(t,S_t) & \\ & + \int_t^T \frac{1}{B_s}\Bigg( F(s,S_{s-}) - S_{s-} D_1 F(s,S_{s-}) \\ & - \sum_{i=2}^{\infty} \frac{S_{s-}^i}{i!\,B_s} D_1^i F(s,S_{s-})\, H_{s-}^{(i)} \Bigg) dB_s + \int_t^T D_1 F(s,S_{s-}) dS_s \\ & + \sum_{i=2}^{\infty} \int_t^T \frac{S_{s-}^i}{i!\,B_s} D_1^i F(s,S_{s-}) dH_s^{(i)}. \end{aligned} \tag{5.29}$$

That is, the hedging portfolio at time t is given by

$$\alpha_t = \frac{1}{B_t}\left( F(t,S_{t-}) - S_{t-} D_1 F(t,S_{t-}) - \sum_{i=2}^{\infty} \frac{S_{t-}^i}{i!\,B_t} D_1^i F(t,S_{t-}) H_{t-}^{(i)} \right),$$

$$\beta_t = D_1 F(t,S_{t-}),$$

$$\beta_t^{(i)} = \frac{S_{t-}^i}{i!\,B_t} D_1^i F(t,S_{t-}), \quad i = 2,3,\ldots.$$

**Remark 5.2.2** If the Black-Scholes model is considered, the risk-neutral dynamics of the stock price is given by

$$\frac{dS_t}{S_t} = r dt + \sigma d\widetilde{W}_t, \qquad S_0 > 0,$$

where $\widetilde{W} = \{\widetilde{W}_t,\ t \geq 0\}$ is a standard Brownian motion. Notice that the market is already complete and hence, an enlargement is not necessary, that is, all processes $H^{(i)} = \left\{H_t^{(i)}, t \geq 0\right\}$, $i \geq 1$, are zero. Therefore, by the above remark, the hedging portfolio at time t is given by

$\alpha_t = \frac{1}{B_t}\big(F(t,S_t) - S_t D_1 F(t,S_t)\big),$

$\beta_t = D_1 F(t,S_t),$

$\beta_t^{(i)} = 0, \quad i = 2,3, \ldots.$

**Remark 5.2.3** In the case of the geometric Poisson model, the risk-neutral dynamics of the stock price is given by

$$\frac{dS_t}{S_{t-}} = (r-\lambda)dt + d\mathcal{N}_t\,, \qquad S_0 > 0,$$

where $\mathcal{N} = \{\mathcal{N}_t,\ t \geq 0\}$ is a Poisson process with intensity parameter $\lambda > 0$. Notice that all the compensated power-jump processes are equal to the compensated Poisson process, that is, $Y_t^{(i)} = \mathcal{N}_t - \lambda t,\ i \geq 1$. Remember that

$$H_t^{(i)} = \exp(rt)\, Y_t^{(i)}, \qquad i \geq 2. \qquad (\text{eqn. } (4.12))$$

By Itô's formula, it can easily be shown that

$$S_t = S_0 \exp\big((r-\lambda)t\big)\, 2^{\mathcal{N}_t},$$

which implies that if $f(T,\mathcal{N}_T) = F(T,S_T)$ is set, we have $f(T,\mathcal{N}_T + 1) = F(T,2S_T)$. Thus, by using

$$\begin{aligned}\sum_{i=2}^{\infty} \frac{S_{s-}^i}{i!} D_1^i F(s,S_{s-}) &= F(s,2S_{s-}) - F(s,S_{s-}) - S_{s-} D_1 F(s,S_{s-}) \\ &= f(s,\mathcal{N}_{s-} + 1) - f(s,\mathcal{N}_{s-}) - S_{s-} D_1 F(s,S_{s-}),\end{aligned}$$

we can write (5.29) as

$$F(T,S_T) - F(t,S_t) = \int_t^T \alpha_s dB_s + \int_t^T \beta_s dS_s,$$

where

$\alpha_s = \frac{1}{B_s}\big(2f(s,\mathcal{N}_{s-}) - f(s,\mathcal{N}_{s-} + 1)\big),$

$$\beta_s = \frac{1}{S_{s-}}\left(f(s, \mathcal{N}_{s-} + 1) - f(s, \mathcal{N}_{s-})\right).$$

This means that $\beta_s^{(i)} = 0, \quad i = 2,3, \ldots$, and hence, an enlargement is not necessary.

# CHAPTER 6

# PORTFOLIO OPTIMIZATION

In this part, the portfolio optimization problem in the complete Lévy market, that is, the market enlarged with the power-jump-assets, is considered. The problem consists of choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained.

A class of utility functions, including HARA, logarithmic and exponential utilities as special cases, is considered. Then, the optimal portfolio which maximizes the terminal expected utility is obtained by the martingale method: First, the optimal wealth is found and then the hedging portfolio replicating this wealth is obtained.

It is shown that for particular choices of the equivalent martingale measure in the market, the optimal portfolio only consists of bonds and stocks. This corresponds to completing the market with new assets in such a way that they are superfluous in the sense that the terminal expected utility is not improved by including these assets in the portfolio.

## 6.1 The Optimal Wealth

Let us fix a structure-preserving $\mathbb{P}$-equivalent martingale measure $\mathbb{Q}$. The aim is to solve the portfolio optimization problem in the enlarged market $\mathfrak{M}_{\mathbb{Q}}$. Given an initial wealth $w_0 > 0$ and a utility function U, we want to find the optimal terminal wealth

$\mathcal{W}_T$, that is, the value of $\mathcal{W}_T$ that maximizes $E(U(\mathcal{W}_T))$ and which can be strongly replicated in $L^1(\mathbb{Q})$ by a portfolio with initial value $w_0$.

Let us begin with some basic definitons.

**Definition 6.1.1** A utility functon is a mapping $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ which is strictly increasing, continuous on $\{U > -\infty\}$, of class $C^\infty$, strictly concave on the interior of $\{U > -\infty\}$ and satisfies $U'(\infty) \coloneqq \lim_{x\to\infty} U'(x) = 0$, that is, marginal utility tends to zero when wealth tends to infinity.

Denoting the interior of $\{U > -\infty\}$ by $\text{dom}(U)$, only the following cases are considered:

Case 1. $\text{dom}(U) = (0,\infty)$, in which case U satisfies

$$U'(0) \coloneqq \lim_{x\to 0^+} U'(x) = \infty.$$

Case 2. $\text{dom}(U) = \mathbb{R}$, in which case U satisfies

$$U'(-\infty) \coloneqq \lim_{x\to -\infty} U'(x) = \infty.$$

The HARA utility functions $U(x) = \frac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}^+ \backslash \{0,1\}$ and the logarithmic utility function $U(x) = \log(x)$ are typical examples for Case 1, and the exponential utility function $U(x) = -\frac{1}{p} e^{-px}, p \in (0,\infty)$, is a typical example for Case 2.

**Definition 6.1.2** A self-financing portfolio $\pi = \{\pi^n\}$ of the form

$$\left\{\pi_t^n = \left(\alpha_t, \beta_t, \beta_t^{(2)}, \beta_t^{(3)}, \dots, \beta_t^{(n)}\right), \qquad 0 \le t \le T, \qquad n \ge 2\right\}$$

(eqn. (4.18))

is called admissible if its value process is bounded from below.

The set of all admissible portfolios is denoted by $\mathfrak{A}$.

**Definition 6.1.3** $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ is called an attainable wealth if it can be strongly replicated in $L^1(\mathbb{Q})$ by a portfolio in $\mathfrak{A}$.

**Proposition 6.1.1** For any $\pi \in \mathfrak{A}$, its discounted value process is a $\mathbb{Q}$-supermartingale.

**Proof:** The discounted value process of $\pi$ is a sum of bounded below stochastic integrals of predictable processes with respect to martingales. This process is a $\mathbb{Q}$-local martingale, see [2], and since it is bounded from below, by Fatou's lemma, it is a $\mathbb{Q}$-supermartingale.
Q.E.D.

If the initial wealth is $w_0$ and $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ is attainable, then we have $E_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T}\right) \leq w_0$, by the previous proposition. Thus, the following optimization problem is considered:

$$\max_{\mathcal{W}_T \in L^1(\mathbb{Q})} \left\{ E\big(U(\mathcal{W}_T)\big):\ E_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T}\right) \leq w_0 \right\},$$

which has the same solution as

$$\max_{\mathcal{W}_T \in L^1(\mathbb{Q})} \left\{ E\big(U(\mathcal{W}_T)\big):\ E_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T}\right) = w_0 \right\}, \tag{6.1}$$

since U is an increasing function. The Lagrangian for this optimization problem is given by

$$E(U(\mathcal{W}_T)) - \lambda_T E_{\mathbb{Q}}\left(\frac{\mathcal{W}_T}{B_T} - w_0\right) = E\left(U(\mathcal{W}_T) - \lambda_T\left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\frac{\mathcal{W}_T}{B_T} - w_0\right)\right).$$

**Definition 6.1.4** $\mathcal{W}_T$ is called the optimal terminal wealth if it is a solution to the optimization problem given by (6.1).
The optimal terminal wealth is given by

$$\mathcal{W}_T = (U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right), \tag{6.2}$$

where $\lambda_T$ is the solution of the equation

$$E_{\mathbb{Q}}\left(\frac{1}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right)\right) = w_0. \qquad (6.3)$$

**Remark 6.1.1** Suppose that $\mathbb{Q}$ is such that $H(x)$ is positive and of class $C^\infty$ on the support of the Lévy measure $\nu$. Moreover, assume that $H(0) = 1, H'(0) = {}^{G}/_{\sigma}$ , $\sigma > 0$ and that there are constants $\varepsilon > 0, \lambda > 0$ such that

1. $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}H(x)\nu(dx) < \infty$.
2. $\int_{(-\varepsilon,\varepsilon)^c}|\log H(x)|\nu(dx) < \infty$.
3. $\int_{(-\varepsilon,\varepsilon)^c}|\log H(x)|\tilde{\nu}(dx) = \int_{(-\varepsilon,\varepsilon)^c}|\log H(x)|H(x)\nu(dx) < \infty$.

Then, the condition $\int_{-\infty}^{\infty}\left(1-\sqrt{H(x)}\right)^2 \nu(dx) < \infty$ of Theorem 3.3.1 is satisfied and the density process $\{d\mathbb{Q}_t/d\mathbb{P}_t = \xi_t,\ 0 \le t \le T\}$ given by

$$\xi_t = \exp\Bigg(GW_t - \frac{1}{2}G^2 t + \lim_{\varepsilon\to 0}\left(\int_{\{|x|>\varepsilon\}} \log H(x) N\big((0,t],dx\big) - t\int_{\{|x|>\varepsilon\}}(H(x)-1)\nu(dx)\right)\Bigg) \qquad \text{(eqn.(3.28))}$$

can be written simply as

$$\xi_t = \exp\left(GW_t - \frac{1}{2}G^2 t + \int_{-\infty}^{\infty}\log H(x) M\big((0,t],dx\big) - t\int_{-\infty}^{\infty}\big(H(x)-1-\log H(x)\big)\nu(dx)\right), \qquad (6.4)$$

where

$$M(dt,dx) = N(dt,dx) - dt\nu(dx). \qquad \big(\text{eqn. }(3.9)\big)$$

With the assumptions of the above remark, we have that

$$\xi_T = \frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = \exp\Bigg(GW_T - \frac{1}{2}G^2T + \int_{-\infty}^{\infty} \log H(x) M\big((0,T], dx\big)$$
$$- T\int_{-\infty}^{\infty}\big(H(x) - 1 - \log H(x)\big)\nu(dx)\Bigg). \quad (6.5)$$

Moreover, by (3.17), we have

$$\exp\,(GW_T) = S_T^{G/\sigma} S_0^{-G/\sigma} \exp\Bigg(-\frac{G}{\sigma}\int_{-\infty}^{\infty} \log(1+x)\, M\big((0,T], dx\big)$$
$$- \frac{TG}{\sigma}\Bigg(a + b - \frac{\sigma^2}{2} + \int_{-\infty}^{\infty}(\log(1+x) - x)\nu(dx)\Bigg)\Bigg). \quad (6.6)$$

Hence, by substituting (6.6) into (6.5) and making necessary arrangements we have

$$\xi_T = \frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = S_T^{G/\sigma} S_0^{-G/\sigma} \exp\Bigg(-\frac{1}{2}G^2T - \frac{G}{\sigma}\Bigg(a + b - \frac{\sigma^2}{2}\Bigg)T$$
$$+ T\int_{-\infty}^{\infty}\Bigg(\log H(x) - \frac{G}{\sigma}\log(1+x) - H(x) + 1 + \frac{G}{\sigma}x\Bigg)\nu(dx)$$
$$+ \int_{-\infty}^{\infty}\Bigg(\log H(x) - \frac{G}{\sigma}\log(1+x)\Bigg) M\big((0,T], dx\big)\Bigg). \quad (6.7)$$

Note that by equation (3.32) we have

$$M((0,t], dx) + t\nu(dx) = \widetilde{M}((0,t], dx) + tH(x)\nu(dx). \quad (6.8)$$

Thus, by using (6.7) and (6.8), we can write (6.2) as

$$\mathcal{W}_T = (U')^{-1}\Bigg(\frac{\lambda_T}{B_T} S_T^{G/\sigma} S_0^{-G/\sigma} \exp\Bigg(-\frac{1}{2}G^2T - \frac{G}{\sigma}\Bigg(a + b - \frac{\sigma^2}{2}\Bigg)T$$
$$+ T\int_{-\infty}^{\infty}\Bigg(\Bigg(\log H(x) - \frac{G}{\sigma}\log(1+x)\Bigg)H(x) - H(x) + 1 + \frac{G}{\sigma}x\Bigg)\nu(dx)$$
$$+ \int_{-\infty}^{\infty}\Bigg(\log H(x) - \frac{G}{\sigma}\log(1+x)\Bigg)\widetilde{M}\big((0,T], dx\big)\Bigg)\Bigg). \quad (6.9)$$

Therefore, the optimal terminal wealth is

$$\mathcal{W}_T = (U')^{-1}\left(\varpi(T) S_T^{G/\sigma} e^{K_T}\right), \tag{6.10}$$

where

$$\varpi(t) := \frac{\lambda_t}{B_t} S_0^{-G/\sigma} \exp\left(-\frac{1}{2}G^2 t - \frac{G}{\sigma}\left(a + b - \frac{\sigma^2}{2}\right)t \right. \\ \left. + t\int_{-\infty}^{\infty}\left(\left(\log H(x) - \frac{G}{\sigma}\log(1+x)\right)H(x) - H(x) + 1 + \frac{G}{\sigma}x\right)\nu(dx)\right) \tag{6.11}$$

and

$$K_t = \int_{-\infty}^{\infty} g(x)\, \widetilde{M}\big((0,t], dx\big), \tag{6.12}$$

with

$$g(x) := \log H(x) - \frac{G}{\sigma}\log(1+x). \tag{6.13}$$

Note that $H(0) = 1$ and $H'(0) = {}^{G}/_{\sigma}$ yields $g'(0) = 0$, since $g'(x) = \frac{H'(x)}{H(x)} - \frac{G}{\sigma}\frac{1}{1+x}$.

In order to replicate the optimal terminal wealth $\mathcal{W}_T$, we need to know its price process, and this depends on the utility function considered.

Now suppose that the utility function satisfies

$$(U')^{-1}(xy) = k_1(x)(U')^{-1}(y) + k_2(x), \tag{6.14}$$

for any $x, y \in (0,\infty)$, for certain $C^{\infty}$ functions $k_1(x),\ k_2(x)$. Then, the price function of $\mathcal{W}_T$ can be written as

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T | \mathcal{F}_t\right] = E_{\mathbb{Q}}\left[\frac{B_t}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right)|\mathcal{F}_t\right] \\ = \frac{B_t}{B_T}E_{\mathbb{Q}}\left(k_1\left(\frac{\lambda_{T,t}}{B_{T,t}}\frac{d\mathbb{Q}_{T,t}}{d\mathbb{P}_{T,t}}\right)\right)(U')^{-1}\left(\frac{\lambda_t}{B_t}\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right) \\ + \frac{B_t}{B_T}E_{\mathbb{Q}}\left(k_2\left(\frac{\lambda_{T,t}}{B_{T,t}}\frac{d\mathbb{Q}_{T,t}}{d\mathbb{P}_{T,t}}\right)\right).$$

Here we used the fact that $\{d\mathbb{Q}_t/d\mathbb{P}_t = \xi_t,\ 0 \le t \le T\}$ is a $\mathbb{P}$-exponential Lévy process, by Theorem 3.3.1 and equation (6.5), with $\frac{d\mathbb{Q}_{T,t}}{d\mathbb{P}_{T,t}} = \frac{d\mathbb{Q}_T/d\mathbb{P}_T}{d\mathbb{Q}_t/d\mathbb{P}_t}$ and $\frac{\lambda_{T,t}}{B_{T,t}} = \frac{\lambda_T/B_T}{\lambda_t/B_t}$. Thus, we have

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \varphi(t,T)\mathcal{W}_t + \psi(t,T), \tag{6.15}$$

where

$$\mathcal{W}_t = (U')^{-1}\left(\varpi(t)S_t^{G/\sigma}e^{K_t}\right), \tag{6.16}$$

with $\varpi(t)$ and $K_t$ are given by (6.11) and (6.12), respectively.

The following lemma shows the structure of the utility functions that satisfy (6.14).

**Lemma 6.1.1** $(U')^{-1}(xy) = k_1(x)(U')^{-1}(y) + k_2(x)$, for any $x,\ y \in (0,\infty)$, for certain $C^\infty$ functions $k_1(x)$, $k_2(x)$ if and only if $\frac{U'(x)}{U''(x)} = \mathcal{m}x + \mathcal{n}$, for any $x \in \mathrm{dom}(U)$, for some $\mathcal{m}, \mathcal{n} \in \mathbb{R}$.

**Proof:** First suppose that $(U')^{-1}(xy) = k_1(x)(U')^{-1}(y) + k_2(x)$. If we write $\hbar(x) = (U')^{-1}(x)$, then we have $\hbar(xy) = k_1(x)\hbar(y) + k_2(x)$. Thus, by differentiating with respect to x, we have that

$$y\hbar'(xy) = k_1'(x)\hbar(y) + k_2'(x). \tag{6.17}$$

Note that $\hbar'(x) = \frac{1}{U''((U')^{-1}(x))} = \frac{1}{U''(\hbar(x))}$.

Thus, by taking $y = \hbar^{-1}(z)$ and $x = 1$, the equation (6.17) becomes

$$\hbar^{-1}(z)\hbar'(\hbar^{-1}(z)) = \frac{U'(z)}{U''(z)} = k_1'(1)z + k_2'(1).$$

Now suppose that $\frac{U'(x)}{U''(x)} = \mathcal{m}x + \mathcal{n}$. Then, by integration of the differential equation, we have

$$U(x) = C_1 \log(x - n) + C_2, \qquad \text{if } m = -1, \tag{6.18}$$

$$U(x) = \frac{C_1}{m\left(1 + \frac{1}{m}\right)} (mx + n)^{1+\frac{1}{m}} + C_2, \qquad \text{if } m \notin \{-1,0\}, \tag{6.19}$$

$$U(x) = C_1 n e^{x/n} + C_2, \qquad \text{if } m = 0, \tag{6.20}$$

where $C_1$ and $C_2$ are integration constants.

Therefore, we have

$$(U')^{-1}(x) = cx^m - \frac{n}{m}, \qquad \text{if } m \neq 0, \tag{6.21}$$

implying that equation (6.14) holds with

$$k_1(x) = x^m \text{ and } k_2(x) = \frac{n}{m} x^m - \frac{n}{m}; \tag{6.22}$$

and

$$(U')^{-1}(x) = n \log x + c, \qquad \text{if } m = 0, \tag{6.23}$$

implying that equation (6.14) holds with

$$k_1(x) = 1 \text{ and } k_2(x) = n \log x, \tag{6.24}$$

where $c \in \mathbb{R}$.

Q.E.D.

From now on we only consider the class of utility funtions of the form

$$(U')^{-1}(xy) = k_1(x)(U')^{-1}(y) + k_2(x). \quad (\text{eqn. } (6.14))$$

This will ensure that the optimal portfolio consisting only of bonds and stocks can be constructed.

In order to solve the optimization problem in the complete market $\mathfrak{M}_{\mathbb{Q}}$, it is required that $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$. Thus, we need the following proposition.

**Proposition 6.1.2** $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ if and only if there is an $\varepsilon > 0$ such that

$$\int_{(-\varepsilon,\varepsilon)^c} \left|(U')^{-1}\big(H(x)\big)\right| \nu(dx) < \infty. \tag{6.25}$$

**Proof:** The optimal terminal wealth is given by

$$\mathcal{W}_T = (U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right). \qquad \text{(eqn. (6.2))}$$

Remember that we only consider the class of utility funtions of the form

$$(U')^{-1}(xy) = k_1(x)(U')^{-1}(y) + k_2(x). \quad \text{(eqn. (6.14))}$$

Thus, we can write $\mathcal{W}_T$ as

$$\mathcal{W}_T = k_1\left(\frac{\lambda_T}{B_T}\right)(U')^{-1}(\xi_T) + k_2\left(\frac{\lambda_T}{B_T}\right).$$

Since $\lambda_T/B_T$ is deterministic, to show $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$ it is enough to prove that $E_{\mathbb{Q}}[|(U')^{-1}(\xi_T)|] < \infty$.

Moreover, the utility functions we consider are such that (recall the proof of Lemma 6.1.1)

$$(U')^{-1}(x) = cx^{m} - \frac{n}{m}, \qquad \text{if } m \neq 0, \qquad \text{(eqn. (6.21))}$$

or

$$(U')^{-1}(x) = n\log x + c, \qquad \text{if } m = 0. \qquad \text{(eqn. (6.23))}$$

First consider the case where $(U')^{-1}(x) = cx^{m} - \frac{n}{m}$, for $m > 0$. We have

$$E_{\mathbb{Q}}[|(U')^{-1}(\xi_T)|] = E_{\mathbb{Q}}\left[\left|c(\xi_T)^{m} - \frac{n}{m}\right|\right].$$

By Theorem 3.3.1 and Remark 6.1.1, we have $\xi_t = \exp(J_t)$, where the process $J = \{J_t,\ 0 \leq t \leq T\}$ given by

$$J_t = GW_t - \frac{1}{2}G^2 t + \int_{-\infty}^{\infty} \log H(x) M\big((0,t], dx\big) - t\int_{-\infty}^{\infty} \big(H(x) - 1 - \log H(x)\big)\nu(dx)$$

is a $\mathbb{P}$-Lévy process with Lévy measure $\nu_J(du) = (\nu\vartheta^{-1})(du)$. Here, $u = \vartheta(x) \coloneqq \log H(x)$.

Thus, we have

$$E_{\mathbb{Q}}[|\xi_T|^{m}] = E_{\mathbb{Q}}[\exp(m J_T)].$$

Notice that $E_{\mathbb{Q}}[\exp(m J_T)] < \infty$ if and only if $E_{\mathbb{Q}}[\exp(m(J_T \vee 0))] < \infty$. Moreover, by Proposition 25.4 in [30], $E_{\mathbb{Q}}\left[\exp\left(m(J_T \vee 0)\right)\right] < \infty$ if and only if

$$\int_{\{|u|>1\}} \exp\left(m(u \vee 0)\right) \nu_J(du) < \infty,$$

which is equivalent to

$$\int_{(-\varepsilon,\varepsilon)^c} \left(H(x)\right)^{m} \nu(dx) < \infty$$

for some $\varepsilon > 0$, since $\log H(0) = 0$ and $H(x)$ is of class $C^{\infty}$.

The case where $(U')^{-1}(x) = cx^{m} - \frac{n}{m}$, for $m < 0$, can be treated analogously.

Now consider the second case where $(U')^{-1}(x) = n\log x + c$. We have

$$E_{\mathbb{Q}}[|(U')^{-1}(\xi_T)|] = E_{\mathbb{Q}}[|n\log\xi_T + c|]$$

and

$$E_{\mathbb{Q}}[|\log\xi_T|] = E_{\mathbb{Q}}[|J_T|].$$

Notice that $E_{\mathbb{Q}}[|J_T|] < \infty$ if and only if $E_{\mathbb{Q}}[|J_T| \vee 1] < \infty$. Moreover, again by Proposition 25.4 in [30], $E_{\mathbb{Q}}[|J_T| \vee 1] < \infty$ if and only if

$$\int_{\{|u|>1\}} (|u| \vee 1)\, \nu_J(du) < \infty,$$

which is equivalent to

$$\int_{(-\varepsilon,\varepsilon)^c} |\log H(x)|\, \nu(dx) < \infty,$$

for some $\varepsilon > 0$.

Q.E.D.

## 6.2 The Optimal Portfolio

Remember that, by (6.15) and (6.16), the price function of the optimal terminal wealth is given by

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = F(t, S_t, K_t)$$

with

$$F(t, x_1, x_2) := \varphi(t, T)(U')^{-1}\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right) + \psi(t, T). \tag{6.26}$$

Also, the function h defined in Theorem 5.2.2 is given by

$$h(t, x_1, x_2, y) := F\big(t, x_1(1+y), x_2 + g(y)\big) - F(t, x_1, x_2) - x_1 y D_1 F(t, x_1, x_2). \tag{6.27}$$

Note that

$$\begin{aligned} F\big(t, x_1(1+y), x_2 + g(y)\big) &= \varphi(t, T)(U')^{-1}\left(\varpi(t)(x_1(1+y))^{G/\sigma}e^{x_2+g(y)}\right) + \psi(t, T) \\ &= \varphi(t, T)(U')^{-1}\left(\varpi(t)x_1^{G/\sigma}e^{x_2}H(y)\right) + \psi(t, T) \\ &= \varphi(t, T)k_1\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right)(U')^{-1}\big(H(y)\big) + \varphi(t, T)k_2\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right) \\ &\quad + \psi(t, T). \end{aligned} \tag{6.28}$$

Here we used the fact that

$$g(x) := \log H(x) - \frac{G}{\sigma}\log(1+x) \quad \text{(eqn. (6.13))}$$

and

$$(U')^{-1}(xy) = k_1(x)(U')^{-1}(y) + k_2(x) \qquad \big(\text{eqn. (6.14)}\big)$$

with $x = \varpi(t)x_1^{G/\sigma}e^{x_2}$ and $y = H(y)$.

Also note that

$$D_1F(t, x_1, x_2) = \frac{G}{\sigma}\frac{\varphi(t, T)\varpi(t)x_1^{\frac{G}{\sigma}-1}e^{x_2}}{U''\left((U')^{-1}\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right)\right)}. \tag{6.29}$$

Here we used the fact that $((U')^{-1}(x))' = \frac{1}{U''\left((U')^{-1}(x)\right)}$.

Thus, by substituting (6.26), (6.28) and (6.29) into (6.27) and making necessary arrangements, we have that

$$h(t, x_1, x_2, y) = \varphi(t,T)k_1\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right)(U')^{-1}\big(H(y)\big) + \varphi(t,T)k_2\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right) - \varphi(t,T)(U')^{-1}\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right) - \frac{G}{\sigma}\frac{\varphi(t,T)\varpi(t)x_1^{G/\sigma}e^{x_2}}{U''\left((U')^{-1}\left(\varpi(t)x_1^{G/\sigma}e^{x_2}\right)\right)}y. \tag{6.30}$$

Therefore, by applying Theorem 5.2.2, the following result is obtained.

**Theorem 6.2.1** Let H be a positive function that satisfies the conditions of Remark 6.1.1 and let $G \in \mathbb{R}$ be a solution of

$$a + b - r + \sigma G + \int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx) = 0. \qquad \big(\text{eqn. } (3.35)\big)$$

Moreover, assume that

(i) $(U')^{-1}(H(y))$ is an analytic function.

(ii) $\sum_{i=2}^{\infty}\frac{|m|_i}{i!}\left|\frac{d^i}{dy^i}(U')^{-1}(H(y))\right|_{y=0} < \infty$, where $|m|_i := \int_{-\infty}^{\infty}|y|^i\tilde{\nu}(dy)$.

Then, the optimal terminal wealth $\mathcal{W}_T$ is strongly replicable in $L^1(\mathbb{Q})$ and the number of stocks and power-jump assets of the replicating portfolio is given by

$$\beta_t = \frac{G}{\sigma}\frac{\varphi(t,T)\varpi(t)S_{t-}^{\frac{G}{\sigma}-1}e^{K_{t-}}}{U''\left((U')^{-1}\left(\varpi(t)S_{t-}^{G/\sigma}e^{K_{t-}}\right)\right)} = \frac{G\varphi(t,T)(m\mathcal{W}_{t-} + n)}{\sigma S_{t-}} \tag{6.31}$$

and

$$\beta_t^{(i)} = \frac{1}{i!\,B_t}\varphi(t,T)k_1(U'(\mathcal{W}_{t-}))\frac{d^i}{dy^i}(U')^{-1}(H(y))|_{y=0}, \quad i = 2,3,\ldots, \tag{6.32}$$

respectively, where the constants $m$ and $n$ depend on the utility function (see Lemma 6.1.1).

**Proof:** Clearly, assumptions (i) and (ii) imply that (6.25) is satisfied, and hence $\mathcal{W}_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q})$, by Proposition 6.1.2. Note that in order to apply Theorem 5.2.2 to the function h given by (6.30), the following must be satisfied:

$$\sum_{i=2}^{\infty} |m|_i \int_0^T E_{\mathbb{Q}}[|a_i(s, Y_s)|]\, ds < \infty, \qquad (\text{eqn. } (5.17))$$

where $a_i(s, Y_s) = \frac{1}{i!}\frac{\partial^i}{\partial y^i} h(s, S_s, K_s, 0)$. That is, we should have

$$\sum_{i=2}^{\infty} \frac{|m|_i}{i!} \left| \frac{d^i}{dy^i}(U')^{-1}(H(y))\right|_{|y=0} \int_0^T |\varphi(t,T)| E_{\mathbb{Q}}\left[\left|k_1\left(\varpi(t) S_{t-}^{G/\sigma} e^{K_{t-}}\right)\right|\right] dt < \infty,$$

which means, by assumption (ii), that

$$\int_0^T |\varphi(t,T)| E_{\mathbb{Q}}\left[\left|k_1\left(\varpi(t) S_{t-}^{G/\sigma} e^{K_{t-}}\right)\right|\right] dt < \infty \qquad (6.33)$$

must hold. Note that in order to have a bounded price function

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T} \mathcal{W}_T | \mathcal{F}_t\right] = \varphi(t,T)\mathcal{W}_t + \psi(t,T), \qquad (\text{eqn. } (6.15))$$

$|\varphi(t,T)|$ must be finite for all t and bounded in [0,T]. Also note that $k_1(x)$ is a linear function of $(U')^{-1}(x)$, (see proof of Lemma 6.1.1), and $U'(\mathcal{W}_t) = \varpi(t) S_t^{G/\sigma} e^{K_t}$. Thus, we have that $k_1\left(\varpi(t) S_{t-}^{G/\sigma} e^{K_{t-}}\right)$ is a linear function of $\mathcal{W}_{t-}$ and $\mathcal{W}_{t-} \in L^1(\mathbb{Q})$ for all $t \in [0,T]$, and therefore $E_{\mathbb{Q}}\left[\left|k_1\left(\varpi(t) S_{t-}^{G/\sigma} e^{K_{t-}}\right)\right|\right]$ is bounded in [0,T]. Consequently, (6.33) holds and thus, with assumptions (i) and (ii), we can apply Theorem 5.2.2 to the function h given by (6.30).

Therefore, by Theorem 5.2.2, the optimal terminal wealth $\mathcal{W}_T$ is strongly replicable in $L^1(\mathbb{Q})$ and the number of stocks and power-jump assets of the replicating portfolio is given by

$$\beta_t = D_1 F(t, S_{t-}, K_{t-}), \quad (\text{eqn. } (5.23))$$

and

$$\beta_t^{(i)} = \frac{1}{i!\, B_t} \frac{\partial^i}{\partial y^i} h(t, S_{t-}, K_{t-}, 0), \qquad i = 2,3,\ldots, \qquad (\text{eqn. } (5.24))$$

respectively.

Note that, by using (6.29), we have that

$$\beta_t = \frac{G}{\sigma}\frac{\varphi(t,T)\varpi(t)S_{t-}^{\frac{G}{\sigma}-1}e^{K_{t-}}}{U''\left((U')^{-1}\left(\varpi(t)S_{t-}^{G/\sigma}e^{K_{t-}}\right)\right)} = \frac{G}{\sigma}\frac{\varphi(t,T)U'(\mathcal{W}_{t-})}{U''(\mathcal{W}_{t-})}S_{t-}^{-1}$$

$$= \frac{G}{\sigma}\frac{\varphi(t,T)(m\mathcal{W}_{t-}+n)}{S_{t-}}.$$

Here we used the fact that $U'(\mathcal{W}_{t-}) = \varpi(t)S_{t-}^{G/\sigma}e^{K_{t-}}$ and $\frac{U'(\mathcal{W}_{t-})}{U''(\mathcal{W}_{t-})} = m\mathcal{W}_{t-} + n$.

Also note that, by using (6.30), we have that

$$\beta_t^{(i)} = \frac{1}{i!\,B_t}\varphi(t,T)k_1(U'(\mathcal{W}_{t-}))\frac{d^i}{dy^i}(U')^{-1}(H(y))|_{y=0}, \quad i = 2,3, \ldots.$$

Q.E.D.

**Corollary 6.2.1** Let H be a positive function that satisfies the conditions of Remark 6.1.1 and let $G \in \mathbb{R}$ be a solution of

$$a + b - r + \sigma G + \int_{-\infty}^{\infty} x(H(x)-1)\nu(dx) = 0. \qquad \left(\text{eqn. } (3.35)\right)$$

Moreover, assume that

(i) $(U')^{-1}(H(y))$ is a polynomial function of degree n.

(ii) $\mathcal{W}_T \geq 0$.

Then, the optimal terminal wealth $\mathcal{W}_T$ is attainable and the number of stocks and power-jump assets of the replicating portfolio is given by

$$\beta_t = \frac{G}{\sigma}\frac{\varphi(t,T)\varpi(t)S_{t-}^{\frac{G}{\sigma}-1}e^{K_{t-}}}{U''\left((U')^{-1}\left(\varpi(t)S_{t-}^{G/\sigma}e^{K_{t-}}\right)\right)} = \frac{G\varphi(t,T)(m\mathcal{W}_{t-}+n)}{\sigma S_{t-}} \qquad (6.34)$$

and

$$\beta_t^{(i)} = \frac{1}{i!\,B_t}\varphi(t,T)k_1(U'(\mathcal{W}_{t-}))\frac{d^i}{dy^i}(U')^{-1}(H(y))|_{y=0}, \quad i = 2,3,\ldots,n, \qquad (6.35)$$

respectively, where the constants $m$ and $n$ depend on the utility function (see Lemma 6.1.1).

**Proof:** Clearly, the conditions of the previous theorem are satisfied and thus, the number of stocks and power-jump assets of the replicating portfolio is given by (6.31) and (6.32), respectively. Note that the replicating portfolio involves only a finite number of power-jump assets since, by assumption, $(\mathrm{U}')^{-1}(\mathrm{H}(\mathrm{y}))$ is a polynomial function of degree n. Also note that the replicating portfolio is admissible, since its value process is bounded from below. Therefore, the optimal terminal wealth $\mathcal{W}_\mathrm{T}$ is attainable with replicating portfolio given by (6.34) and (6.35).

Q.E.D.

Notice that we have the following result: If it is required that the optimal portfolio involves only stocks and bonds $\mathbb{Q}$ must be chosen so that

$$(\mathrm{U}')^{-1}(\mathrm{H}(\mathrm{y})) = \mathscr{p} + \mathscr{q}\mathrm{y},$$

where $\mathscr{p}, \mathscr{q} \in \mathbb{R}$.

It can be easily shown that $\mathrm{H}(0) = 1$ and $\mathrm{H}'(0) = {}^{\mathrm{G}}/_{\sigma}$ imply that

$$\mathrm{H}(\mathrm{y}) = \begin{cases} \left(1 + \dfrac{m\mathrm{G}}{\sigma}\mathrm{y}\right)^{1/m}, & \text{if } m \neq 0 \qquad (6.36) \\ \exp\left(\dfrac{\mathrm{G}}{\sigma}\mathrm{y}\right), & \text{if } m = 0 \qquad (6.37) \end{cases}$$

where the constant $m$ depends on the utility function (see Lemma 6.1.1). Remember that G must satisfy the equation (3.35). Moreover, if G is such that $\mathrm{H}(\mathrm{y}) > 0$ on the support of the Lévy measure $\nu$ and $\mathcal{W}_\mathrm{T} \geq 0$, the conditions of the above corollary are satisfied and $\beta_t^{(i)} = 0$, for all $i \geq 2$. Equivalently, if $\mathbb{Q}$ is chosen so that either (6.36) or (6.37) holds, then the optimal portfolio in the Lévy market $\mathfrak{M}_\mathbb{Q}$ involves only stocks and bonds.

## 6.3 Application

**Example 6.3.1** Consider the logarithmic utility function $U(x) = \log(x)$. Then $(U')^{-1}(x) = \frac{1}{x}$ and by solving

$$E_{\mathbb{Q}}\left(\frac{1}{B_T}(U')^{-1}\left(\frac{\lambda_T}{B_T}\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right)\right) = w_0, \qquad \text{(eqn. (6.3))}$$

we have

$$\mathcal{W}_T = w_0 B_T \frac{d\mathbb{P}_T}{d\mathbb{Q}_T} = \left(\varpi(T) S_T^{G/\sigma} e^{K_T}\right)^{-1}.$$

Moreover, the price function of $\mathcal{W}_T$ at time t is given by

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = w_0 B_t E_{\mathbb{Q}}\left[\frac{d\mathbb{P}_T}{d\mathbb{Q}_T}|\mathcal{F}_t\right] = w_0 B_t \frac{d\mathbb{P}_t}{d\mathbb{Q}_t} = \mathcal{W}_t,$$

which implies that $\varphi(t, T) = 1$ and $\psi(t, T) = 0$ in

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \varphi(t, T)\mathcal{W}_t + \psi(t, T). \qquad \text{(eqn. (6.15))}$$

It follows from the proof of Lemma 6.1.1 that $m = -1$, $n = 0$, $k_1(x) = x^{-1}$ and $k_2(x) = 0$. Note that $U'(x) = \frac{1}{x}$ and $U''(x) = -\frac{1}{x^2}$.

Therefore, it follows from Theorem 6.2.1 that the relative wealth invested in stocks, at time t, is constant and given by

$$\frac{\beta_t S_{t-}}{\mathcal{W}_{t-}} = -\frac{G}{\sigma};$$

and the number of power-jump assets in the optimal portfolio, at time t, is

$$\beta_t^{(i)} = \frac{\mathcal{W}_{t-}}{i!\, B_t}\frac{d^i}{dy^i}\frac{1}{H(y)}\bigg|_{y=0}, \quad i = 2,3, \ldots.$$

Moreover, if it is desired to have the optimal portfolio that consists of only stocks and bonds, then an equivalent martingale measure $\mathbb{Q}$ must be chosen so that (6.36) and (3.35) hold, that is

$$\mathrm{H}(\mathrm{y}) = \left(1 - \frac{\mathrm{G}}{\sigma}\mathrm{y}\right)^{-1},$$

where G satisfies the equation

$$\mathrm{a} + \mathrm{b} - \mathrm{r} + \sigma\mathrm{G} + \frac{\mathrm{G}}{\sigma}\int_{-\infty}^{\infty} \frac{\mathrm{x}^2}{1 - \frac{\mathrm{G}}{\sigma}\mathrm{x}} \nu(\mathrm{dx}) = 0.$$

The existence and the uniqueness of the solution is considered in the next example.

**Example 6.3.2** Consider the HARA utilities $\mathrm{U}(\mathrm{x}) = \frac{\mathrm{x}^{1-\mathrm{p}}}{1-\mathrm{p}}$ for $\mathrm{p} \in \mathbb{R}^+ \backslash \{0,1\}$. Then $(\mathrm{U}')^{-1}(\mathrm{x}) = \mathrm{x}^{-1/\mathrm{p}}$ and by solving

$$\mathrm{E}_{\mathbb{Q}}\left(\frac{1}{\mathrm{B}_\mathrm{T}}(\mathrm{U}')^{-1}\left(\frac{\lambda_\mathrm{T}}{\mathrm{B}_\mathrm{T}}\frac{\mathrm{d}\mathbb{Q}_\mathrm{T}}{\mathrm{d}\mathbb{P}_\mathrm{T}}\right)\right) = \mathcal{w}_0, \qquad (\text{eqn. } (6.3))$$

we have

$$\mathcal{W}_\mathrm{T} = \mathcal{w}_0 \mathrm{B}_\mathrm{T} \frac{(\mathrm{d}\mathbb{P}_\mathrm{T}/\mathrm{d}\mathbb{Q}_\mathrm{T})^{1/\mathrm{p}}}{\mathrm{E}_{\mathbb{Q}}((\mathrm{d}\mathbb{P}_\mathrm{T}/\mathrm{d}\mathbb{Q}_\mathrm{T})^{1/\mathrm{p}})} = \left(\varpi(\mathrm{T})\mathrm{S}_\mathrm{T}^{\mathrm{G}/\sigma}\mathrm{e}^{\mathrm{K}_\mathrm{T}}\right)^{-1/\mathrm{p}}.$$

Moreover, the price function of $\mathcal{W}_\mathrm{T}$ at time t is given by

$$\begin{aligned}
\mathrm{E}_{\mathbb{Q}}\left[\frac{\mathrm{B}_\mathrm{t}}{\mathrm{B}_\mathrm{T}}\mathcal{W}_\mathrm{T}|\mathcal{F}_\mathrm{t}\right] &= \mathcal{w}_0 \mathrm{B}_\mathrm{t} \frac{\mathrm{E}_{\mathbb{Q}}\left[(\mathrm{d}\mathbb{P}_\mathrm{T}/\mathrm{d}\mathbb{Q}_\mathrm{T})^{1/\mathrm{p}}|\mathcal{F}_\mathrm{t}\right]}{\mathrm{E}_{\mathbb{Q}}((\mathrm{d}\mathbb{P}_\mathrm{T}/\mathrm{d}\mathbb{Q}_\mathrm{T})^{1/\mathrm{p}})} \\
&= \mathcal{w}_0 \mathrm{B}_\mathrm{t} \frac{\mathrm{E}_{\mathbb{Q}}\left[\left(\mathrm{d}\mathbb{P}_{\mathrm{T,t}}/\mathrm{d}\mathbb{Q}_{\mathrm{T,t}}\right)^{1/\mathrm{p}}(\mathrm{d}\mathbb{P}_\mathrm{t}/\mathrm{d}\mathbb{Q}_\mathrm{t})^{1/\mathrm{p}}|\mathcal{F}_\mathrm{t}\right]}{\mathrm{E}_{\mathbb{Q}}\left(\left(\mathrm{d}\mathbb{P}_{\mathrm{T,t}}/\mathrm{d}\mathbb{Q}_{\mathrm{T,t}}\right)^{1/\mathrm{p}}(\mathrm{d}\mathbb{P}_\mathrm{t}/\mathrm{d}\mathbb{Q}_\mathrm{t})^{1/\mathrm{p}}\right)} \\
&= \mathcal{w}_0 \mathrm{B}_\mathrm{t} \frac{(\mathrm{d}\mathbb{P}_\mathrm{t}/\mathrm{d}\mathbb{Q}_\mathrm{t})^{1/\mathrm{p}}}{\mathrm{E}_{\mathbb{Q}}((\mathrm{d}\mathbb{P}_\mathrm{t}/\mathrm{d}\mathbb{Q}_\mathrm{t})^{1/\mathrm{p}})} = \mathcal{W}_\mathrm{t},
\end{aligned}$$

which implies that $\varphi(\mathrm{t},\mathrm{T}) = 1$ and $\psi(\mathrm{t},\mathrm{T}) = 0$ in

$$E_{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \varphi(t,T)\mathcal{W}_t + \psi(t,T). \quad \text{(eqn. (6.15))}$$

It follows from the proof of Lemma 6.1.1 that $m = -\frac{1}{p}, n = 0, k_1(x) = x^{-1/p}$ and $k_2(x) = 0$. Note that $U'(x) = x^{-p}$ and $U''(x) = -px^{-p-1}$.

Therefore, it follows from Theorem 6.2.1 that the relative wealth invested in stocks, at time t, is constant and given by

$$\frac{\beta_t S_{t-}}{\mathcal{W}_{t-}} = -\frac{G}{\sigma p};$$

and the number of power-jump assets in the optimal portfolio, at time t, is

$$\beta_t^{(i)} = \frac{\mathcal{W}_{t-}}{i!\,B_t}\frac{d^i}{dy^i}(H(y))^{-1/p}|_{\,y=0}\,, \qquad i = 2,3,\ldots.$$

Moreover, if it is desired to have the optimal portfolio that consists of only stocks and bonds, then an equivalent martingale measure $\mathbb{Q}$ must be chosen so that (6.36) and (3.35) hold, that is

$$H(y) = \left(1 - \frac{G}{\sigma p}y\right)^{-p},$$

where G satisfies the equation

$$a + b - r + \sigma G + \int_{-\infty}^{\infty} x\left(\left(1 - \frac{G}{\sigma p}x\right)^{-p} - 1\right)\nu(dx) = 0.$$

It is required that $H(y) > 0$ for all y on the support of the Lévy measure $\nu$ in order to obtain an equivalent measure $\mathbb{Q}$ (see Theorem 3.3.1). Remember that the support of the Lévy measure is given by $[\delta,\theta]$, where $\delta > -1$ and $\theta$ is a positive constant. Thus, we must have $1 - \frac{G}{\sigma p}y > 0$ for all $y \in [\delta,\theta]$, which implies that

$$\frac{\sigma p}{\delta} < G < \frac{\sigma p}{\theta}, \qquad \text{if } -1 < \delta < 0$$

and

$$G < \frac{\sigma p}{\theta}, \qquad \text{if} \quad \delta \geq 0.$$

Moreover, if G is a solution of the equation

$$a + b - r + \sigma G + \int_{-\infty}^{\infty} y\left(\left(1 - \frac{G}{\sigma p} y\right)^{-p} - 1\right) \nu(dy) = 0,$$

then the probability measure $\mathbb{Q}$ is an equivalent martingale measure.

Notice that the function

$$f(G) := \sigma G + \int_{-\infty}^{\infty} y\left(\left(1 - \frac{G}{\sigma p} y\right)^{-p} - 1\right) \nu(dy)$$

is strictly increasing. This means that we have at most one solution of the equation

$$a + b - r + f(G) = 0,$$

and this solution exists only if

$$\frac{\sigma^2 p}{\delta} + \int_{-\infty}^{\infty} y\left(\left(1 - \frac{y}{\delta}\right)^{-p} - 1\right) \nu(dy) < r - a - b$$

$$< \frac{\sigma^2 p}{\theta} + \int_{-\infty}^{\infty} y\left(\left(1 - \frac{y}{\theta}\right)^{-p} - 1\right) \nu(dy).$$

# CHAPTER 7

# CONCLUSION

In this study, general geometric Lévy market models are considered. Since these models are, in general, incomplete, that is, all contingent claims cannot be replicated by a self-financing portfolio consisting of investments in a risk-free bond and in the stock, it is suggested that the market should be enlarged by artificial assets based on the power-jump processes of the underlying Lévy process. These artificial assets can be related with options on the stock and contracts on realized variance that are traded in OTC markets regulary. By making use of the Predictable Representation Property for Lévy processes, it is shown that the enlarged market is complete. Then the explicit hedging portfolios for claims whose payoff function depends on the prices of the stock and the artificial assets at maturity are derived.

Moreover, the portfolio optimization problem is considered in the enlarged market. The problem consists of choosing an optimal portfolio in such a way that the largest expected utility of the terminal wealth is obtained. A class of utility functions, including HARA, logarithmic and exponential utilities as special cases, is considered. Then, the optimal portfolio which maximizes the terminal expected utility is obtained by the martingale method. It is shown that for particular choices of the equivalent martingale measure in the market, the optimal portfolio consists only of bonds and stocks. This corresponds to completing the market with additional assets in such a way that they are superfluous in the sense that the terminal expected utility is not improved by including these assets in the portfolio. This in turn provides a solution to the problem of utility maximization in the real market, consisting only of the bond and the stock.

The new assets, by which the market is completed, are not traded in the market and thus, considering the portfolio optimization in the enlarged market does not seem to be realistic. However, the replication formula for these artificial assets in terms of call options with the same maturity and with a continuum of strikes can be derived, which is the subject of another study.

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