PRICING US CORPORATE BONDS BY JARROW/TURNBULL (1995) MODEL

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ABSTRACT

PRICING US CORPORATE BONDS BY JARROW/TURNBULL (1995) MODEL

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In this study Jarrow Turnbull (1995) Model, which is a reduced form approach for credit risk models, is employed to estimate the default intensity of US corporate bonds conditionally based on a fixed recovery rate. The estimations are performed with respect to the ratings of the bonds and the results were consistent with the ratings. US Treasury Bills are also used to since zero coupon default free prices, modeled by Svensson (1994) are necessary for pricing the default risky coupon bonds.

Key Words: Credit Risk, Corporate Bond, Reduced form Models, Default Intensity, Jarrow-Turnbull (1995) Model

ÖZ

AMERİKAN ÖZEL ŞİRKET TAHVİLLERİNİN JARROW/TURNBULL (1995) MODELİ İLE FIYATLANMASI

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Bu çalışmada, ABD özel şirket tahvillerinin fiyatlanması için reduced form modellerden birisi olan Jarrow-Turnbull(1995) kullanılmış ve recovery rate sabit tutularak, default intensity tahmini yapılmıştır. Jarrow-Turnbull (1995) modeline göre, default risky kuponlu tahvilin fiyatlanması için default free zero coupon bond fiyatı gerektiği için, Amerikan Hazine bonoları da Svensson (1994) modeli ile fiyatlandırılmıştır. Default intensity tahminleri yapılırken, bonolar derecelendirmelerine göre değerlendirilmiş ve çıkan sonuçların bu derecelendirmeler ile uyumlu oldukları gözlenmiştir.

Anahtar Kelimeler: Kredi Riski, Özel Şirker Tahvilleri, Reduced Form Models, Default Intensity, Jarrow-Turnbull (1995) Modeli

To my family

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CHAPTER 1

INTRODUCTION

Credit risk occurs when the issuer of a bond fails to meet its contractual obligations, repaying the principal and the interests on time. Credit risk is also defined as "The degree of value fluctuations in debt instruments and derivatives due to changes in the underlying credit quality of borrowers and counterparties" (Lopez and Saidenberg, 2000). Default risk, which relates the possibility that the counterparty will not fulfill its contractual commitments, is one of the issues of growing importance both in academia and in industry. Numerous credit risk models have been developed by the academicians and by the industry in order to model the default risk during the last decade.

Extensions have been made to the early Merton model (1974) on the "structural" modeling side, to refine the model while alternative variations for the intensity process are proposed on the "reduced-form" side, in modeling. However, a consensus on a model could not be reached. Controversial results yielded from different empirical studies for the validation of the theories is one of the main reasons for this deficit.

On the industry side, numerous developments are observed in the field of credit risk modeling, as evidenced by the public release of such models by a number of financial institutions such as J.P. Morgan (1998) and Credit Suisse Financial Products (1997).

Credit risk models are expected to be used to formally determine risk-adjusted, regulatory capital requirements by the International Swap Dealers Association

(ISDA) and the Institute of International Finance Working Group on Capital Adequacy (IIF) (Lopez and Saidenberg, 2000).

Although the lending institutions developed the risk management tools, default frequency has been observed to increase for the past decades. Thus, mathematical models have been implemented to quantify and view variations in the risk profile of the creditor and losses related to these changes to correctly price and manage the credit risk since correct measurement and valuation of credit risk and especially the likelihood of default and the loss given default is of great importance for the lending institutions.

In this thesis, the ratings of the corporate bonds selected are accepted as the main indicators of their risk. However, it is seen that these ratings given by S&P and Moody’s are judged during the recent crisis.

There are only two corporate bonds issued in Turkey. The main reason is the high yields provided by the treasury bills and therefore the impossibility of the competitiveness. Consequently US corporate bonds are selected for the estimation of the Jarrow Turnbull (1995) model. However, US corporate bond data are not publicly available, thus the data employed in this study is scarce. Nevertheless, this thesis will provide a basis for the pricing of the corporate bonds in Turkey, if corporate issue bonds.

The goal of this thesis is the estimation of the default intensity in the Jarrow/Turnbull (1995) model. Estimations of the default intensity based on default-free term structure data are performed. The thesis consists of 6 chapters including introduction. The rest of the paper evolves as follows: Chapter 1 presents introduction, and followed by the literature review chapter. Chapter 3 presents the preliminaries that are necessary for the theory to be well understood. Chapter 4 gives the details of the original Jarrow-Turnbull (J/T) (1995) model. In Chapter 5 the estimation technique of the corporate bond pricing with J/T (1995) is presented with the technique of the Svensson method that is utilized for the pricing of the default free bonds. Data issues and estimation results are also explained in chapter 5. Finally, in chapter 6 the main results are summarized in the conclusion part.

CHAPTER 2

LITERATURE REVIEW

There are two well-known approaches to the modeling of credit risk. First model is the "structural-form approach" which can be investigated as first generation and second generation models. Structural form model is originated with Black and Scholes (1973), proposed by Merton (1974) using the principles of option pricing. They endogenize the bankruptcy process by explicitly modeling the asset and liability structure of the company (Merton 1974). This approach views equity shares and debts as derivatives on the firm's assets. Default risk of a firm is explicitly connected to the variation in the asset value of the firm which conducts the default process. Merton Model proposes that default takes place when the liabilities of the firm is greater than the market value of the firm (Altman et al, 2004). This model derives the price of default risk by modeling the value of the firm relative to the firm's debt. The face value of the bond, which is thought as the strike price, minus a put option on the value of the firm yields the pay-off at maturity, equal to the maturity of the bond, to the bondholder (Altman et al, 2004). Merton used this approach to derive an explicit formula for risky bonds by which probability of default and the yield differential between a risky bond and a default-free bond can be estimated. Merton model states that the probability of default and the recovery rate , which are the functions of structural features of the firm, are related inversely. Black and Cox (1976), Geske (1977) and Vasicek (1984) can also be included in the first generation structural form models where Black and Cox (1976) presents the possibility of more complex capital structures, with subordinated debt; Geske (1977) suggests interest-paying debt and Vasicek (1984) establishes the distinction between short- and long-term liabilities in order to improve the basic Merton Model.

In practice structural form approach has some shortcomings. First, under Merton's model, default occurs only at maturity of the debt which contradicts with the real life. Second, the priority/seniority structures of various debts have to be specified in case of complex capital structures. Moreover, debts are assumed to be paid with respect to their seniority orders in this model. However, Franks and Torous (1994) empirically proved that the absolute-priority rules are often violated. Basic Merton model utilizes the lognormal distribution which tends to amplifies recovery of rate in the event of default (Altman, et al, 2004). Second generation structural form models tried to remove the unrealistic assumptions by adopting the basic Merton model. For instance default is allowed to occur at any time between the issuance and maturity of the debt when the value of the firm's assets reaches a lower threshold level. These models include Kim et al. (1993), Hull and White (1995), Nielsen et al. (1993), Longstaff and Schwartz (1995) and others. Recovery rate at default is taken as exogenous and is assumed to be independent from the firm's asset value. Being generally defined as a fixed ratio of the outstanding debt, Recovery rate is independent from the probability of default. Longstaff and Schwartz (1995) argue that, the recovery rate can be estimated utilizing the history of defaults and the recovery ratios for various classes of debt of comparable firms. They allow for a stochastic term structure of interest rates and they let defaults correlate with the interest rates and conclude that this correlation plays a significant role on the credit spread. Nonetheless three main drawbacks, explaining the relatively poor empirical performance of the structural models, remain unsolved by the second generation models. First problem is the need of estimates of the firm's asset value which is not observable. Secondly the credit rating changes are not incorporated into structural form models which are observed to decrease before the firms' default. Besides, these models postulates that the value of the firm is continuous and thus default time can be anticipated just before the default takes place (Altman et al, 2005). Therefore Duffie and Lando (2001) argue that this continuity assumption implies that there are no sudden surprises.

Structural models assume complete knowledge of a very detailed information set generated by continuous observations of both the firm's asset value and the default barrier, which means that modeler has continuous and detailed information about all of the firm's assets and liabilities. This information set is similar to that held by the firm's managers and regulators. Generally, complete information assumption means that the default time can be predicted which does not hold when the firm's asset value follows a continuous time jump diffusion process (Jarrow and Protter, 2004).

To overcome these shortcomings reduced form models are employed by Litterman and Iben (1991), Jarrow and Turnbull (1992), Madan and Unal (1998), Jarrow and Turnbull(1995), Duffie and Singleton (1997), Jarrow, Lando and Turnbull (1997), Lando (1998), Duffie and Singleton (1999). Reduced form models takes market into account as the only source of information and therefore default probabilities and credit risk dependencies are inferred by means of market prices of the firms' defaultable instruments (such as bonds or credit default swaps) (Abel, 2006). Reduced-form models do not condition default on the value of the firm, and parameters related to the firm's value need not be estimated to implement them unlike structural-form models. Moreover, reduced-form models present separate explicit assumptions on probability of default and recovery rate dynamics which are modeled independently from the structural features of the firm such as its asset volatility and leverage. Recovery rate at default is generally supposed to be exogenous and thereby independent from the probability of default. Reduced form models take the term structure of default-free interest rates, the recovery rate of defaultable bonds at default and stochastic process for default intensity as primitives (Altman et al., 2005).

Jarrow and Turnbull (1995) study the simplest case in which the default was driven by a Poisson process with constant intensity and a known payoff at default. Stochastic interest rates are incorporated, but they specify the processes for bankruptcy and the payoff on the risky debt conditional on default exogenously.

They make use of foreign currency analogy and payoffs to the risky security are made in nominal terms in a risky currency, 'XYZ-dollars'. Default risky bond price is formulated in terms of default free bond and a exchange rate implying the analogy of foreign currency by no-arbitrage conditions. Bankruptcy is modeled as a jump process in the continuous time case (Cooper and Martin, 1996). This model is built on arbitrary Heath et al. (1992) term structure model, remaining analytically tractable. The Jarrow/Turnbull model, assuming that the stochastic process driving the default-free term structure and the default process are independent, is especially functional when data is scarce. This assumption gives the opportunity to study term structure issues and default issues separately. Explicit pricing formulas for risky bonds and for options on interest rate sensitive stocks which facilitate implementation and calibration are also derived in this model (Frühwirth and Sögner, 2006). Empirical study of Houweling and Vorst (2003) showed that, in spite of its simplicity, the Jarrow and Turnbull model proves to work well in some situations.

Duffie and Singleton (1997) used a similar method. “They assume a multi-factor square-root process for the riskless interest rate and a Poisson process for default with state dependent values for the hazard rate and the loss in default” (Cooper and Martin, 1996). Valuation under the martingale probability measure is done by discounting the default-free payoff on the debt by a discount rate that is edited for the default process parameters. The valuation procedure becomes same for riskless claims, with an adjustment to the interest rate for the effect of default risk by this discounting procedure. This argument gave inspiration to those that simply assume a process for the spread and then use this in a way similar to that derived by Duffie and Singleton (Cooper and Martin, 1996).

The parameterization of the recovery rate forms the base on which the reduced-form models differ. Jarrow and Turnbull (1995) assumed that a the market value of a bond at default is equal to an exogenously given fraction of an equivalent default-free bond whereas Duffie and Singleton (1999) allow for closed-form solutions for the term-structure of credit spreads when market value at default, recovery rate, is exogenously specified. This model also allows for a random recovery rate depending on the pre-default value of the bond. This model lets default hazard-rate process and recovery rate correlate and assumes that expected loss at default process is exogenous implying that the recovery rate does not depend on the defaultable claim value. The correlation is modeled by combining independent Poisson processes by pre-specifying the sign of the correlation which is a handicap of the model.

Coupon level or maturity is taken as irrespective with the recovery rate and thus bond holder gets a fixed payment, at default according to Duffie (1998. This amount is the same fraction of face value as any other bond of the same seniority since the model assumes that bonds of the same issuer, seniority, and face value have the same recovery rate at default, regardless of the remaining maturity. Recovery parameters based on statistics and withdrawn from defaults are provided by rating agencies such as Moody's and they can be utilized with this assumption. Jarrow, Lando and Turnbull (1997) mainly concentrates on the migrations between credit rating classes and make use of transition matrices (historical probabilities of credit rating changes) to price defaultable bonds allowing for different debt seniorities to translate into different recovery rates for a given firm.

Empirical work on reduced form is rather limited (see e.g. Duffie and Singleton (1997), Duffee (1999), Tauren (1999), Düllmann and Windfuhr (2000), Bühler et al. (2001), Bakshi et al. (2001), Houweling and Vorst (2003) or Duffie et al. (2003)). The articles using intensity-based models generally build the cross-sections and specify them exogenously, frequently based on credit ratings, to derive the estimates. Daily data is used to extract the parameters based on the pre-specified cross-sections (Frühwirth, 2004). Duffee (1999) finds that these models have some problems in explaining the observed term structure of credit spreads across firms of different credit risk qualities using the Duffie and Singleton (1999) approach.

The relationship between the probability of default and recovery rate has been the subject of the research for recent years (Frye (2000a, 2000b), Jokivuolle and Peura (2003), and Jarrow (2001)). Frye uses the conditional approach suggested by Finger (1999) and Gordy (2000). The state of the economy drives the default in these models. The same economic conditions are supposed to bring about the probability of default to rise and recovery rates to fall. Therefore the correlation between recovery rate and default is caused by the common dependence on the state variable. Frye found that default rates and recovery rates are negatively correlated. Frye's (2000) concluded that bond recoveries might decline 20-25 percentage points from their normal-year average whereas loan recoveries may decline by a similar amount, but from a higher level in a severe economic downturn.

Jarrow (2001) proposed a new methodology for the estimation of recovery rates and probability of default benefiting from the debt and equity prices which are ignored by the reduced form models and connected reduced form models with the structural models. The equity prices are incorporated in the estimation procedure allowing the separate identification of the recovery rates and probability of default which are assumed to be correlated and depend on the state of the macroeconomy. He incorporated the liquidity premium and price bubble effects in his method.

Jokivuolle and Peura (2003) proposed a model for bank loans in which collateral value is correlated with the probability of default. In their method the borrowing firm's total asset value determines the event of default. However, the recovery rate is not determined by the firm's asset value, but determined by the collateral value which is stochastic. Bakshi et al. (2001) allowed for a flexible correlation between the risk-free rate, the default probability and the recovery rate and empirically showed that recovery rates are negatively associated with default probability through the analysis of a sample of BBB-rated corporate bonds (Altman et al.,2005).

Zhou (2001) tried to associate the structural-form model and reduced-form model to benefit from the clear economic mechanism behind the default process in structural models and surprise of default in reduced-form models. He modeled the evolution of the firm as a jump-diffusion process linking the recovery rate to the firm value at default so that the variation in recovery rates is endogenously generated and the correlation between recovery rates and credit ratings reported in Altman (1989) and Gupton, Gates and Carty (2000) is verified (Altman et al., 2005).

CHAPTER 3

PRELIMINARIES

Before explaining the details of the reduced-form credit risk model of Jarrow and Turnbull (1995), the following definitions are provided as useful reminders:
A complete probability space (Ω, F, P) and a filtration $\{Gt : t \geq 0\}$ of sub-σ-algebras of F, which represents the investors' information set, are fixed. Let P denote the physical probability measure observed in the financial markets and Q denotes the risk neutral (pseudo) probability. These parameters are used to define a market model which is free of arbitrage.

<u>Definition 3.1:</u> A risk-neutral probability is a probability measure under which the discounted expected value of tomorrow's asset price is equal to today's asset price in an arbitrage-free world.

The significance of calculating the arbitrage-free prices with the risk neutral probabilities is that these prices are applicable for all investors regardless of their attitudes towards risk.

<u>Definition 3.2:</u> Let (Ω, F, P) be a probability space with $F = P(\Omega)$ and for all ω in Ω, $P(\{\omega\}) > 0$ with a filtration $(F_n)_{0 \leq n \leq N}$. Let M_n be an adapted sequence with $0 \leq n \leq N$ of real random variables. Then M_n is martingale if $E\,(M_{n+1}|F_n) = M_n$ for all $n \leq N - 1$.

Within the finance context, if the price of an asset A_n is martingale with $0 \leq n \leq N$, then the best estimate of A_{n+1}obtained by the least squares estimation at each n is A_n. That is, the best forecast of the future value of an asset is its last observed value.

Definition 3.3: Arbitrage is the possibility of earning a positive monetary return with zero equity investment and with a probability of 1.0. An arbitrage profit is a riskless profit. The assumption of no arbitrage is necessary to calculate a unique risk neutral price for financial assets since the risk neutral probabilities only exist in the absence of arbitrage. The existence of arbitrage is referred to as "mispricing" in the market.

Definition 3.4: The market is said to be "viable" if there is no arbitrage opportunity.

3.1 Bond Market

The aim of this study is pricing the default risky corporate bonds employing Jarrow-Turnbull (1995) model. For the pricing issues a relative pricing approach is needed. Zero coupon bonds are the basic instruments that are used in the following sections for developing a relative pricing approach since the underlying structure of Jarrow Turnbull (1995) model is the Heath Jarrow and Morton (1992) Model which has a zero coupon bond with maturity T.

Definition 3.1.1: A zero coupon bond with maturity T is a contract guaranteeing to pay 1 dollar to the investor at time T. The time t price of a zero coupon bond with maturity T is denoted by $p(t, T)$.

Definition 3.1.2: A coupon bond with maturity T delivers payments in $[0, T]$ and provides the holder of the bond with a deterministic cash flow.

After defining the zero-coupon and coupon bonds, the following assumptions are made in order to assure the existence of a bond market that is sufficiently deep:

1- There exists a frictionless market for all bonds of all maturities.

2- $p(t, t) = 1$ holds in order to avoid arbitrage.

3- $p(t,T)$ is differentiable with respect to T for all fixed t (Björk, 1998).

The first assumption guarantees that the market contains all possible bonds. The price of the bond is strictly positive for all t and the price process is adapted. The price of the bond $p(t,T)$ is a stochastic object with two variables t and T. If t is fixed, $p(t,T)$ gives the prices of the bonds with different maturities at a fixed time. The graph of this function is the term structure of the bond and it is smooth, meaning that $p(t,T)$ is differentiable with respect to T for all t. If the maturity T is fixed, $p(t,T)$ becomes a scalar stochastic process and its trajectory becomes irregular. Next, some interest rates are defined based on the above market conditions. While a simple interest notation is used in the market, the continuously-compounded interest notation is used in theoretical contexts. These two representations are logically equivalent. The interest rates are constructed according to the following scenario:

Assume that $t < S < T$.

Table 3.1.1: Scenario for the Construction of Interest Rates

Time	t	S	T
	Sell S bonds Buy $p(t,S)/p(t,T)$ T bonds	Pay out 1	Receive $p(t,S)/$ $p(t,T)$
Net investment	0	-1	$+p(t,S)/p(t,T)$

The above transactions can be summarized as follows: A deal is made at time t, to make an investment of one unit of money at time S, that is guaranteeing a yield of $p(t,S)/p(t,T)$ at time$T$. Therefore, Deal is made at riskless rate at time t, which is valid on the future period $[S,T]$. This rate is called as a forward rate.

The following definitions are implied by the above construction.

Definition 3.1.3: The simple spot rate for $[S, T]$ is defined as:

$$L(S,T) = -\frac{p(S,T) - 1}{(T-S)p(S,T)} \tag{3.1.1}$$

Definition 3.1.4: The simple forward rate for $[S, T]$ contracted at t is defined as:

$$L(t;S,T) = -\frac{\big(p(t,T) - p(t,S)\big)}{(\mathrm{T}-\mathrm{S})\mathrm{p}(\mathrm{S},\mathrm{T})} \tag{3.1.2}$$

Definition 3.1.5: The continuously compounded spot rate for $[S, T]$ is defined as:

$$R(S,T) = -\frac{logp(S,T)}{T-S} \tag{3.1.3}$$

Definition 3.1.6: The continuously compounded forward rate for $[S, T]$ contracted at t is defined as:

$$R(t;S,T) = -\frac{logp(t,T) - logp(t,S)}{T-S} \tag{3.1.4}$$

Definition 3.1.7: The instantaneous forward rate with maturity T, contracted at t is defined as:

$$f(t,T) = -\frac{\partial logp(t,T)}{\partial T} \tag{3.1.5}$$

Definition 3.1.8: The instantaneous short rate at time t is defined as:

$$r(t) = f(t,t) \tag{3.1.6}$$

If the limit of the continuously compounded forward rate when S goes to T is taken, the instantaneous forward rate is obtained. This rate can be interpreted as the riskless rate of interest, contracted at t over the infinitesimal interval $[T, T + dT]$ (Björk, 1998)

Definition 3.1.9: The money market account process is defined as:

$$B_t = \exp\{\int_0^t r(s)ds\} \tag{3.1.7}$$

Money market account can be interpreted as a strategy of instantaneously reinvesting at the current short rate (Kiesel and Bingham, 1998).

Definitions 3.1.3 through 3.1.9 imply the following lemma:

Lemma 3.1.1: For $t \leq s \leq T$, the following are true:

$$p(t,T) = p(t,s).\exp\{-\int_s^T f(t,u)du\} \tag{3.1.8}$$

and

$$p(t,T) = \exp\{-\int_t^T f(t,s)ds\} \tag{3.1.9}$$

Proof: Continuously compounded forward rate $R(t; S, T)$ is the solution of the following equation and therefore by solving this equation equation 3.1.11 is obtained:

$$e^{R(T-S)} = \frac{p(t,s)}{p(t,T)} \tag{3.1.10}$$

Thus,

$$p(t,T) = p(t,s)\exp(-R(T-S)) \tag{3.1.11}$$

can be written.

Replacing R with definition 3.1.6, the following equation is obtained:

$$p(t,T) = p(t,s).\exp\left(\frac{\log p(t,T) - \log(t,s)}{T-S}.(T-S)\right) \qquad (3.1.12)$$

Using the definition of the instantaneous forward rate, the following result is obtained:

$$= p(t,s).\exp\{-\int_s^T f(t,u)du\} \qquad (3.1.13)$$

The lemma puts forward the relationship between the forward rate and the price of the bond.

The bond market can be modeled by specifying either the dynamics of the short rate, or the dynamics of all possible bonds or the dynamics of all possible forward rates. According to the above formulation, in order to model a bond market, specifying the dynamics of all possible forward rates and then using Lemma 3.1.1 to obtain the price of the bond is sufficient. Thus, once the forward rate dynamics are specified, then the price of the bond can be determined. All of the approaches used to model a bond market are related to each other; hence their relationship is of importance. The short rate, bond price and forward rate dynamics will be stated and then proposition 3.2.1. which provides an equation for the derivative of the bond price process under the assumption of the forward rate dynamics will be given.

3.2. Relationship between the forward rate, bond price and short rate dynamics $\boldsymbol{df(t,T), dp(t,T)}$ *and* $\boldsymbol{dr(t)}$:

Short Rate Dynamics : $dr(t) = a(t)dt + b(t)dW(t)$

Bond Price Dynamics : $dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t)$

Forward Rate Dynamics: $df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$

Proposition 3.2.1:

a) If $p(t,T)$ satisfies $dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t)$ then for the forward rate dynamics the following equation holds:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are given by

$$\alpha(t,T) = v_T(t,T)v(t,T) - m_T(t,T)$$
$$\sigma(t,T) = -v_T(t,T)$$

and they denote the drift and the volatility parameters.

b) If $f(t,T)$ satisfies $df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$ then the short rate satisfies

$$dr(t) = a(t)dt + b(t)dW(t)$$

where $a(t)$ and $b(t)$ are given by

$$a(t) = f_T(t,t) + \alpha(t,t)$$
$$b(t) = \sigma(t,t)$$

where $f_T(t,t)$ is the derivative of forward rate process with respect to T.

c) If $f(t,T)$ satisfies $df(t,T) = a(t,T)dt + \sigma(t,T)dW(t)$ then $p(t,T)$ satisfies

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2} \parallel S(t,T \parallel^2\right\}dt + p(t,T)S(t,T)dW(t)$$

where $\|.\|$ denotes the Euclidean norm and

$$A(t,T) = -\int_t^T \alpha(t,s)ds$$

$$S(t,T) = -\int_t^T \sigma(t,s)ds$$

<u>Proof:</u>

By lemma 3.1.1 it is known that $p(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right)$ and $r(t) = f(t,t)$.

Since $f(t,T)$ satisfies $df(t,T) = a(t,T)dt + \sigma(t,T)dW(t)$ $f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW_u$ can be written by taking the integral of $df(t,T)$.

To facilitate the notation let $X_t = -\int_t^T f(t,s)\,ds$ then, $p(t,T) = e^{X_t}$

Applying Ito formula to X_t for $g(x) = e^x$, equation 3.1.1.14 is obtained;

$$p(t,T) = e^{X_t} = e^{X_0} + \int_0^t e^{X_s}\,dX_s + \frac{1}{2}\int_0^t e^{X_s}\,d < X,X >_s \qquad (3.2.1)$$

Derivative of equation 3.2.1 is taken since the proposition asserts the relationship between the derivative of the bond price process and the dynamics of the forward rate process.

$$dp(t,T) = e^{X_t} dX_t + \frac{1}{2} e^{X_t}\, d < X, X >_t \tag{3.2.2}$$

Thus in order to reach the required equation, dX_t should be found out. For the derivation of dX_t following trick will be used.

Let us write $X_t = \int_t^T \big(-f(s,s) + f(s,s) - f(t,s)\big) ds$

$$= -\int_t^T f(s,s)ds - \int_t^T f(0,s)ds - \int_t^T \int_0^t \alpha(u,s)duds - \int_t^T \int_0^t \sigma(u,s)dW_u\, ds$$

$$+ \int_t^T f(0,s)ds + \int_t^T \int_0^s \alpha(u,s)du\, ds + \int_t^T \int_0^s \sigma(u,s)dW_u ds$$

Then, by Fubini's theorem, an interchange of integrals yields the following equation;

$$= -\int_t^T f(s,s)ds + \int_t^T \int_u^T \alpha(u,s)dsdu + \int_t^T \int_u^T \sigma(u,s)\, dsdW_u$$

$X_0 = -\int_0^T f(s,s)ds + \int_0^T \int_u^T \alpha(u,s)dsdu + \int_0^T \int_u^T \sigma(u,s)\, dsdW_u$.

Thus,

$$X_t = X_0 + \int_0^t f(s,s)ds - \int_0^t \int_u^T \alpha(u,s)dsdu - \int_0^t \int_u^T \sigma(u,s)\, dsdW_u$$

The above equation allows us writing X_t by the parameters of the forward rate process. Taking the derivative of both sides dX_t is obtained.

$$dX_t = f(t,t)dt + \int_t^T \alpha(t,s)dsdt - \int_t^T \sigma(t,s)\, dsdW_t$$

Thus, substituting dX_t in the relevant equation the following equation is obtained;

$$dp(t,T) = e^{X_t}\left(f(t,t)dt + \int_t^T \alpha(t,s)dsdt - \int_t^T \sigma(t,s)\,dsdW_t\right) + e^{X_t}\frac{1}{2}d < X,X >_t$$

where $d < X,X >_t = \frac{1}{2}(\int_t^T \sigma(t,s)ds)^2\, dt$

Finally; by writing $e^{X_t} = p(t,T)$ the equation below is reached:

$$dp(t,T) = p(t,T)\left(f(t,t)dt + \int_t^T \alpha(t,s)dsdt - \int_t^T \sigma(t,s)\,dsdW_t\right) + p(t,T)\frac{1}{2}(\int_t^T \sigma(t,s)ds)^2\, dt$$

Given, $A(t,T) = -\int_t^T \alpha(t,s)ds$

$S(t,T) = -\int_t^T \sigma(t,s)ds$

$r(t) = f(t,t)$,

the equation can be written as required.

Proposisiton 3.2.1. c asserts the relationship between the forward rate dynamics and the bond price dynamics. After constructing the relationship between the forward rates, bond prices and the short rate dynamics the Jarrow Turnbull (1995) model which accepts the Heath, Jarrow and Turnbull (1992) forward rate model imposing the exogenous stochastic structure upon the forward rates instead of zero coupon bond prices as a basis will be constructed.

CHAPTER 4

THE JARROW-TURNBULL (1995) MODEL

The Jarrow-Turnbull (JT) model introduces a reduced form approach to the pricing of derivatives that involve credit-risk. This model takes the stochastic term structure of interest rates and the stochastic maturity-specific credit-risk spread as given. Once these term structures are determined, the option-type securities are priced by a martingale measure under the no-arbitrage assumption. In this thesis, the JT model is used to price corporate bonds that are subject to default.

The JT model is built on the Heath-Jarrow-Morton (HJM) model which was introduced in a 1992 study. The HJM model is an improvement of short-term interest rate models. Working with short-term interest rate models has some advantages. First, these models specify the short-term rate as the solution of a stochastic differential equation. These stochastic equations allow the Markov process and partial differential equations to be used which further makes it possible to derive analytical formulas for the price of bonds. However, these models also have some drawbacks. First, the models attempt to explain the economy by means of only one variable (the short-term interest rate). Second, if the short-term interest rate model is made more realistic, then the inversion of the yield curve within the model becomes problematic since such an inversion would require the incorporation of all available information into the yield curve and would rely on the markets being dynamically complete. With these models, maturity preferences of investors are embedded into the observable term structure, and, thereby, arbitrage opportunities among bonds of different maturities are precluded (Kijima and Muromachi, 2000) .

4.1 The Heath- Jarrow-Morton (HJM) Model (1992)

Heath, Jarrow and Morton use the entire forward rate curve as the state variable in their model. The HJM model generalizes the Ho and Lee model (1986) which takes the initial bond price as a given and the bond price process as exogenous in a discrete trading economy. The Ho and Lee model is a single factor model causing the bonds of all maturities to become perfectly correlated. Also, during parameter estimation, as the step sizes get larger, the parameters become dependent on each other. The generalization of this model into continuous time eliminates this estimation difficulty.

The HJM model takes the initial forward rate curve and a family of potential stochastic processes for its subsequent movements as a given. Typically, in such models, zero-coupon bond prices with a fixed maturity and thereby a time-varying volatility are used to back out the yield curve. However, the HJM model imposes an exogenous stochastic structure upon forward rates by means of which forward rate volatilities become constant. These constant volatilities are consistent with a fixed value for a zero coupon bond as well (Heath et al., 1992). The model does not require an inversion of the term structure in order to back out the market prices of risk from contingent claim values since such a requirement would be highly demanding due to the nonlinearity of the bond pricing formulae.

The HJM model is based on an equivalent martingale measurement technique. In order to understand the framework of the model, initially all parameters are considered under an objective probability measure P. It is also necessary to make some assumptions:

Assumption 4.1.1:

For every fixed $T > 0$, the forward rate $f(.,T)$ where T denotes the maturity has a stochastic differential with respect to t which is given by the following under the objective measure P:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\bar{W}(t) \tag{4.1.1}$$

$$f(0,T) = f^{*}(0,T) \tag{4.1.2}$$

In this equation, $\bar{W}$ is a (d-dimensional) P-Wiener process, $\alpha(.,T)$ and $\sigma(.,T)$ are adapted processes.

The above equation is a stochastic differential in the t variable for each fixed choice of T which serves as a parameter. $\{f^{*}(0,T); T \geq 0\}$ is the initial condition which provides a perfect fit between the observed and theoretical bond prices at t=0.

Here the problem is characterizing $\alpha(t,T)$ and $\sigma(t,T)$ in such a way that there is no arbitrage opportunity. The following theorem makes this possible by means of the HJM drift condition:

Theorem 4.1.1: Assume that the family is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\bar{W}(t)$$

and the bond market is free of arbitrage. In that case, there exists a d-dimensional column -vector process $\lambda(t)=[\lambda_1(t),\ldots \lambda_d(t)]'$ with the property that for all $T \geq 0$ and for all $t \leq T$, there is

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'\,ds - \sigma(t,T)\lambda(t) \tag{4.1.3}$$

In this equation, the symbol prime (‘) denotes the transpose.

Proof: In proposition 3.2.1, it is proven that the bond dynamics satisfies the following equation if $f(t,T)$ satisfies $df(t,T) = a(t,T)dt + \sigma(t,T)d\overline{W}(t)$

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2} \parallel S(t,T) \parallel^2\right\}dt + p(t,T)S(t,T)d\overline{W}(t)$$

where

$$A(t,T) = -\int_t^T \alpha(t,s)ds$$

$$S(t,T) = -\int_t^T \sigma(t,s)ds$$

Thus the risk premium is given by

$$A(t,T) + \frac{1}{2} \parallel S(t,T) \parallel^2$$

In an arbitrage free market, there exists a market price of risk process, λ(t), which is common to all assets in the market and satisfying

$\alpha_{\Pi}(t) - r = \sigma_{\Pi}(t)\lambda(t)$ P a.s

when the price process satisfies

$$d\Pi(t) = \Pi(t)\alpha_{\Pi}(t)dt + \Pi(t)\,\sigma_{\Pi}(t)\,d\overline{W}(t)$$

Therefore d-dimensional column vector process λ exists such that

$$A(t,T) + \frac{1}{2} \parallel S(t,T) \parallel^2 = \sum_{i=1}^{d} S_i(t,T)\,\lambda_i(t)$$

Taking the derivative of the above equation with respect to T the following result is obtained.

$$-\alpha(t,T) - \sigma(t,T).S(t,T) = -\sigma(t,T)\lambda(t)$$

Rearranging the above equation the result is obtained.

Now for risk neutral modeling, forward rates will be assumed to be driven by the martingale measure Q as:

$$df(t,T) = a(t,T)dt + \sigma(t,T)dW(t)$$
$$f(0,T) = f^*(0,T)$$

where W is a d- dimensional Q Wiener process. Since we are studying with martingale probability measures, we no longer have to worry about the arbitrage as martingale probabilities directly provide the arbitrage free prices. However, we now have two equations for the bond prices one of which is related to the short rate process and the other one is related to the forward rate process.

$$p(0,T) = exp\,\{-\int_0^T f(0,s)ds\ \}$$
$$p(0,T) = E^Q[exp\,\{-\int_0^T r(s)ds\,\}]$$

where the short rate and the forward rate are related to each other by $r(t) = f(t,t)$. For the above formulae to hold simultaneously, HJM imposes a consistency relation between α and σ in the forward rate dynamics.

Proposition 4.1.1: Under the martingale measure Q, the processes α and σ must satisfy the following relation, for every t and every $T \geq t$.

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s\acute{)}\,ds$$

Proof: From proposition 3.2.1 it is known that

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2} \| S(t,T) \|^2\right\} dt + p(t,T)S(t,T)d\overline{W}(t)$$

Under martingale measure, the local rate of return should be equal the short rate r, leading the following equation.

$$r(t) + A(t,T) + \frac{1}{2} \| S(t,T) \|^2 = r(t)$$

Thus $A(t,T) + \frac{1}{2} \| S(t,T) \|^2 = \lambda = 0$

The result is obtained by theorem 4.1.1.

Proposition 4.1.1 implies that if forward rate dynamics can be specified, then the volatility structure can be specified. Then the drift parameters will be uniquely determined. Below is an algorithm to use the HJM model.

1- Specify, by your own choice, the volatilities

2- The drift parameters are now given by

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)' \, ds$$

3- Observe the today's forward rate structure form the market.

$$\{f^*(0,T); T \geq 0\}$$

4- Integrate in order to get the forward rates as

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T) \, dW(s)$$

5- Compute the bond prices using the formula

$$p(t,T) = \exp\{-\int_t^T f(t,s)ds\}$$

4.2 Description of the economy of JT(1995) Model

The economy is assumed to be frictionless. The trading horizon is set to be $[0,\tau]$ which can be either discrete or continuous. Default free zero coupon bonds of all maturities and ABC zero coupon bonds of all maturities are traded.

Let $p_0(t,T)$ be the time t dollar value of the default-free zero coupon bond paying a certain dollar at $T \geq t$.

$p_0(t,T) > 0$
$p_0(t,t) = 1$ (in order to be default free)

Let $v_1(t,T)$ be the time t value of the ABC zero coupon bond promising a dollar at T≥t.

$v_1(t,T) > 0$ (to avoid dividing by zeros)

Now $p_1(t,T)$ and $e_1(t)$ will be defined in order to be to used in the decomposition of $v_1(t,T)$ into two hypothetical quantities, namely a zero coupon bond denominated in a hypothetical currency, a promised ABC dollar and hereafter called as ABC and a price in dollars of ABC, which can be interpreted as spot exchange rate of dollar per ABC. Let the $p_1(t,T)$and $e_1(t)$ denote the following:

$p_1(t,T)$: is the time t value in units of ABCs of one ABC delivered at T.
$e_1(t)$: is the time t dollar value of one promised ABC dollar delivered at t.

$$e_1(t) \equiv \; v_1(t,t)$$

In order to decompose $v_1(t,T)$ the spot exchange rate analogy will be used. If ABC is not in default, the exchange rate will be 1, meaning that each promised ABC dollar will exactly be worth a dollar; but if ABC defaults then the exchange rate will be smaller than 1, meaning that each promised ABC dollar may be worth less than a dollar.

Therefore the ABC dollar paying zero coupon bond is constructed as:

$$p_1(t,T) \equiv \frac{v_1(t,T)}{e_1(t)}$$

By rearranging, the following equation is obtained:

$$v_1(t,T) = \; p_1(t,T).\, e_1(t)$$

The equation displays that dollar value of an ABC bond is the ABC dollar value of the bond times the spot exchange rate of dollars per ABC. In their paper, Jarrow and Turnbull prefers using the foreign currency analogy since the foreign currency option pricing techniques are well understood and they want to apply these techniques to price the derivatives involving credit risk.

Furthermore, by definition of $e_1(t)$; ABC dollar paying zero coupon bond is default free in ABC.

$$p_1(T,T) \equiv \frac{v_1(T,T)}{e_1(T)} \equiv \frac{v_1(T,T)}{v_1(T,T)} \equiv 1$$

$e_1(t)$ can be interpreted a pay off ratio. The decomposition will be used to characterize the term structure of ABC bond in terms of $p_1(t,T)$ and the pay off ratio $e_1(t)$ separately.

The following table presents a summary of the bond prices and their riskiness

Table 4.2.1: Summary of Bond Prices

$v_1(t,T)$	Promises a dollar	Risky
$p_1(t,T)$	Pays an ABC dollar	Default free in ABC
$p_0(t,T)$	Pays a certain dollar	Default free in dollar

4.3 Two- Period Discrete Trading Economy

After defining the relevant bond prices, the study will continue with the two period discrete trading economy to clarify the bond price dynamics. In this economy there are two time periods with trading dates $t \in \{0, 1, 2\}$. This section will include the term structure of the default free zero coupon bonds and the ABC zero coupon bonds.

4.3.1 Term Structure of the Default Free Zero Coupon Bonds

The default free zero coupon's bond price, $p_0(t,T)$, is assumed to depend only on the spot interest rate. The definitions and the assumptions will be asserted in the following items. These items will include the features related to the default free zero coupon bond price, the current, up state and down state spot interest rate, the risk neutral or the pseudo probability of a rise in the spot interest rate and the money market account.

a- The current one period spot interest rate is defined by;

$$r(0) = \frac{1}{p_0(0,1)}$$

where $p_0(0,1)$ is the time 0 price of a default free zero coupon bond with maturity 1.

b- If the interest rate rises at time 1, then the one period spot interest rate is defined by;

$$r(1)_u \equiv 1/p_0(1,2)_u$$

where $p_0(1,2)$ is the time 1 price of a default free zero coupon bond with maturity 2.

c- If the interest rate decreases at time 1, then the one period spot interest rate is defined by;

$$r(1)_d \equiv 1/p_0(1,2)_d$$

$p_0(1,2)_u < p_0(1,2)_d$ is assumed to hold without loss of generality.

d- The risk neutral probability of a rise of the spot interest rate is denoted by Π_0

e- It is assumed that the investor invests an amount of 1$, thus

i) $B(0) \equiv 1$. (Initial amount of money)

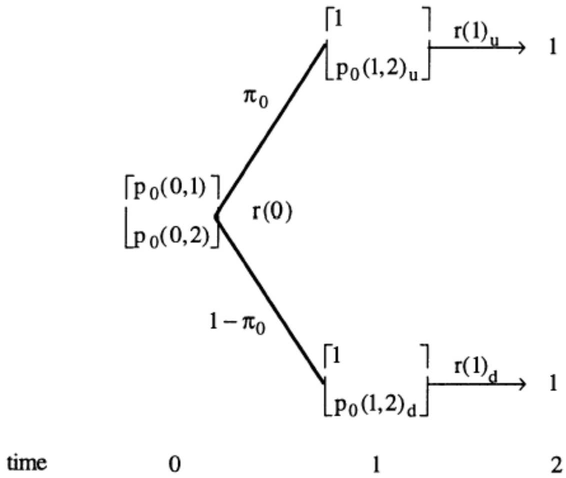
ii) $B(1) \equiv r(0)$. (Time t=1 amount of money invested at time t=0 since 1$ is invested with an interest rate of $r(0)$)

iii) $B(2)_u \equiv r(0)r(1)_u$

iv) $B(2)_d \equiv r(0)r(1)_d$

The following graph demonstrates the stochastic evolution of the spot interest rates and thereby the default free zero coupon bond price process at time t = {0, 1, 2}.

Graph 4.3.1.1: The Default Free Zero Coupon Bond Price Process For Two Period Economy



4.3.2 The Term Structure of ABC Bond

This section includes the stochastic evolution of the default-risky zero coupon bond. If the ABC bond has not defaulted, the payoff is the face value of the bond, meaning that the pay off ratio is equal to 1. However, if the default has occurred then the payoff is less than the face value of the bond yielding a pay off ratio smaller than 1.

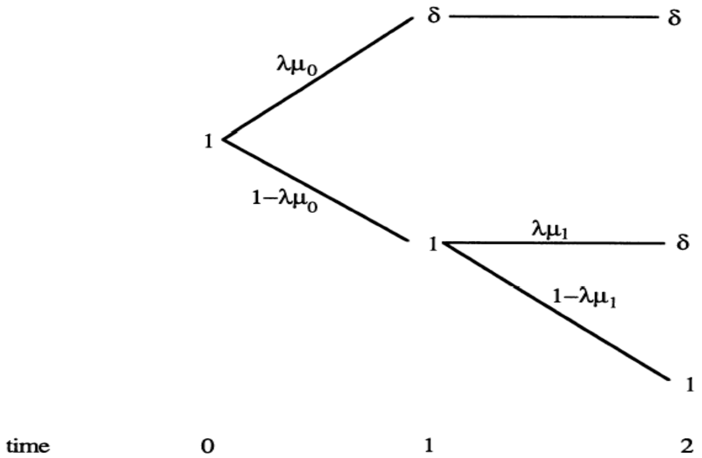
Since the absolute priority rule is often violated as Eberhart, Moore and Roenfeldt (1990) asserted and the payoff is affected by various factors, modeling the payoff at default becomes complicated. Consequently, Jarrow and Turnbull preferred to take the payoff ratio at default as an exogenously given constant as a first approximation and pay off per unit of face value is denoted by δ which is assumed to be same for all instruments in a given credit risk class.

The spot exchange rate has been defined, to utilize the foreign currency analogy, at time $t = 0$ as $e(0)$ and the equality is set to be $e(0) = 1$. Since the ABC bond is default risky, at $t = 1$ and $t = 2$, the spot exchange rate takes different values. Due to the relation between the binomial process and the Poisson process, J/T decided to choose the discrete time binomial process for the evolution of the spot exchange rate. Poisson random variable may be used to approximate the binomial random variable when the binomial parameter n is large and the p is small (Ross, 2007).

The pseudo- probability of default at $t = 1$ is denoted by $\lambda\mu_0$. Consequently, the pseudo-probability that the default does not take place at $t = 1$ is $(1 - \lambda\mu_0)$.
The model is constructed in a way that if the default occurs at $t = 1$ then the ABC bond maintains default at $t = 2$. Thus the pay off ratio at $t = 2$ is fixed at δ per unit of face value. If the default does not occur at $t = 1$, the pseudo probability of default at $t = 2$ is $\lambda\mu_1$.

The following graph demonstrates the stochastic evolution of the pay off ratio process for ABC debt in the two- period economy. The payoff ratio depends on the seniority of the debt.

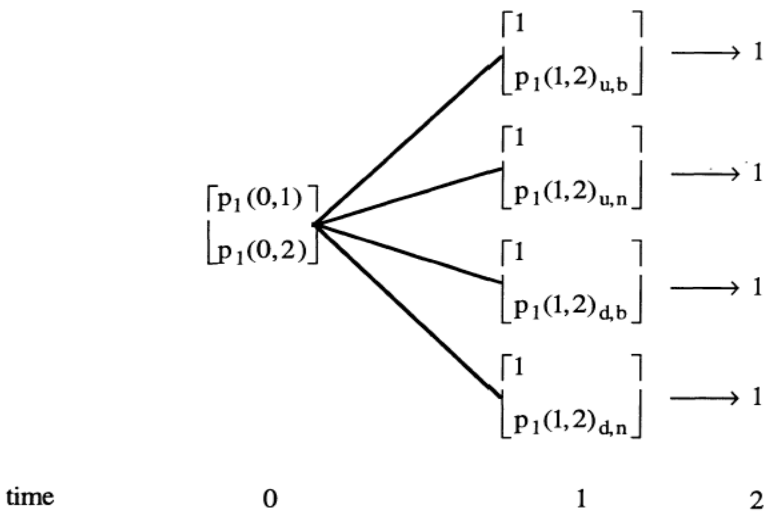
Graph 4.3.2.1: The Stochastic Evolution of the Pay off Ratio for ABC Zero Coupon Bond in Two-Period Economy.



ABC bond has been constructed in a hypothetical currency and it is default risky. Therefore the term structure of the ABC Bond will be determined by bankruptcy cases as well as the spot interest rate.

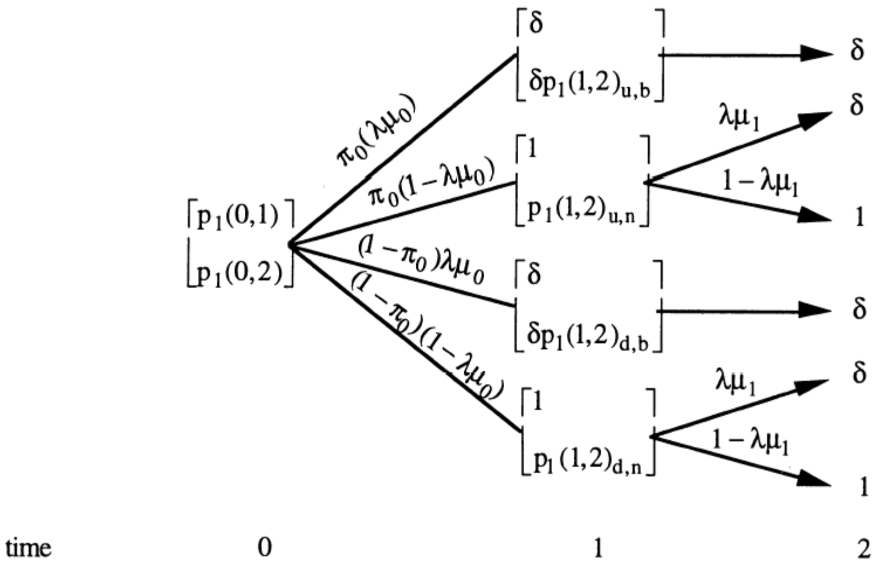
The following graph shows the stochastic evolution of the ABC zero- coupon bond in the hypothetical currency ABC. The similarity between the graph 4.3.1.1 and the graph 4.3.2.2 can be noticed. However, in the last graph it can be seen that the bankruptcy cases are added since the ABC bond is default risky in dollar. Nevertheless as the ABC bond is default free in the hypothetical currency ABC, at maturity the investor gets 1 ABC dollar in each case. If the ABC Bond defaults then 1ABC dollar is worth less than 1 dollar whereas if default does not take place then 1 ABC dollar worth is worth exactly 1 dollar.

Graph 4.3.2.2: The stochastic evolution of the ABC zero coupon bond price process for the two-period economy in terms of the hypothetical currency ABC dollar.



Jarrow and Turnbull (1995) assume that the spot interest rate process and the bankruptcy process are independent under the pseudo probabilities to simplify the analysis. If the market prices of risk are nonrandom in an economy, then the spot interest rate process and the bankruptcy process are independent under the true probabilities. Therefore the graph demonstrating ABC zero coupon bond price process for the two-period economy in dollars (not in ABC dollars since this type of bond is default free in ABC dollars) is constructed by multiplying the pseudo probabilities in Graph 4.3.2.2 and Graph 4.3.2.3.

Graph 4.3.2.3: The ABC zero coupon bond price process for the two period economy in dollars



4.3.3 Arbitrage free restrictions

Definition 4.3.3.1: The market is viable if there is no arbitrage opportunity.

Theorem 4.3.3.1: The market is viable if and only if there exists a probability measure Q equivalent to P such that the discounted prices of assets are Q martingales (Harrison and Pliska, 1981).

Propositon 4.3.3.1**:** Assume that the general binomial model is free of arbitrage, then it is complete.

Consequently, the theorem implies that the nonexistence of arbitrage opportunities is equivalent to the existence of pseudo probabilitiesΠ_0, $\lambda\mu_0$ and $\lambda\mu_1$ such that the discounted prices;

$\frac{p_0(t,1)}{B(t)}$, $\frac{p_0(t,2)}{B(t)}$, $\frac{v_1(t,1)}{B(t)}$, $\frac{v_2(t,1)}{B(t)}$ are Q martingales, meaning that the expected values are equal to the equal current values. That is, the absence of arbitrage opportunities is guaranteed by the existence of an equivalent martingale measure. As Harrison and Pliska asserted in 1981, the market is complete when the equivalent martingale measure is unique. In this case, all claims can be replicated by dynamic self financing trading strategies in the primary traded assets with unique prices determined by the cost of the replication strategy (Jarrow and Madan, 1995).

Therefore necessary and sufficient conditions for the absence of arbitrage and the existence of the complete market and thereby the existence of unique pseudo probabilities are given in this section.

To obtain the pseudo probabilities, Π_0, $\lambda\mu_0$ and $\lambda\mu_1$ the default free bond market and the default risky bond markets need to be investigated separately. Π_0 being the risk neutral probability of a rise of the spot interest rate is determined from the characteristics of the default-free bond market whereas the $\lambda\mu_0$ and $\lambda\mu_1$ are determined in the default risky bond market since they indicate the pseudo probability of default at $t = 1$ and $t = 2$ respectively.

Proposition 4.3.3.2: If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim is its discounted expected value calculated by the unique pseudo probability.

Graph 4.3.1.1 showing the default free zero coupon bond price process for two-period economy shows that;

$$p_0(0,2) = \frac{[\Pi_0 \ p_0(1,2)_u + (1 - \Pi_0) \ p_0(1,2)_d]}{r(0)} \qquad (4.3.3.1)$$

This equation demonstrates the fact that the time 0 long term zero coupon bond price is its time 1 discounted expected value calculated by pseudo probability Π_0.

If Π_0 is calculated from the above formula, following expression for Π_0 is obtained

$$\Pi_0 = \frac{[\ p_0(1,2)_d - r(0)p_0(0,2)]}{[\ p_0(1,2)_d - \ p_0(1,2)_u]} \qquad (4.3.3.2)$$

According to the equation Π_0 can exist and be unique and $0 < \Pi_0 < 1$ if and only if

$$p_0(1,2)_u < r(0)p_0(0,2) < \ p_0(1,2)_d \qquad (4.3.3.3)$$

This equation states that the long term zero coupon bond should not be dominated by the short term zero coupon bond.

The conditions for the existence and uniqueness of $\lambda\mu_0$ and $\lambda\mu_1$ will be determined investigating the default risky bond market. Therefore Graph 4.3.2.3 should be investigated.

If the spot interest rates rise and bankruptcy occurs at time 1, $e(t)$ turns out to be δ by Graph 4.3.2.1 and thus $v_1(1,2)_{u,b} = \delta p_1(1,2)_{u,b}$

$$v_1(1,2)_{u,b} = \delta p_1(1,2)_{u,b} = \frac{\delta}{r(1)_u} \qquad (4.3.3.4)$$

If the spot interest rates rise but the bankruptcy does not occur at time 1, then the pay off ratio becomes 1. Thus;

$$v_1(1,2)_{u,n} = p_1(1,2)_{u,n} = \frac{[\lambda\mu_1\delta + (1-\lambda\mu_1)]}{r(1)_u} \qquad (4.3.3.5)$$

If the spot interest rates decrease and the bankruptcy takes place at time 1, the pay off ratio becomes δ and thus;

$$v_1(1,2)_{d,b} = \delta p_1(1,2)_{d,b} = \frac{\delta}{r(1)_d} \quad (4.3.3.6)$$

If both the spot interest rates decrease and the bankruptcy occurs then the payoff ratio becomes 1 and thus;

$$v_1(1,2)_{u,n} = p_1(1,2)_{u,n} = \frac{[\lambda\mu_1\delta + (1-\lambda\mu_1)]}{r(1)_d} \quad (4.3.3.7)$$

Equation 4.3.3.7 indicates that, the time 1 long term ABC bond price is its time 2 discounted expected value calculated by the pseudo-probabilities. From equations 4.1.3.3.5 and 4.1.3.3.7, $\lambda\mu_1$ is derived and it is;

$$\lambda\mu_1 = \frac{[1-p_1(1,2)_{u,n}r(1)_u]}{[1-\delta]} = \frac{[1-p_1(1,2)_{d,n}r(1)_d]}{[1-\delta]} \quad (4.3.3.8)$$

Thus, $\lambda\mu_1$ exists, is unique and satisfies $0 < \lambda\mu_1 < 1$ if and only if

$$p_1(1,2)_{u,b} = \frac{1}{r(1)_u} \quad \text{(from equation 4.1.3.3.4)} \quad (4.3.3.9)$$

$$p_1(1,2)_{d,b} = \frac{1}{r(1)_d} \quad \text{(from equation 4.1.3.3.6)} \quad (4.3.3.10)$$

In order that $0 < \lambda\mu_1 < 1$ equation 4.3.3.8 implies that

$$\frac{\delta}{r(1)_u} < p_1(1,2)_{u,n} < \frac{1}{r(1)_u} \quad (4.3.3.11)$$

$$\frac{\delta}{r(1)_d} < p_1(1,2)_{d,n} < \frac{1}{r(1)_d} \quad (4.3.3.12)$$

For $\lambda\mu_1$ to be unique, two sides of the equations should be equal implying the following equation:

$$r(1)_u p_1(1,2)_{u,n} = r(1)_d p_1(1,2)_{d,n} \qquad (4.3.3.13)$$

Equation 4.3.3.9 and 4.3.3.10 can be interpreted as the equality of the price of a default free bond in units of dollars and the ABC denominated ABC bonds if the bankruptcy occurs. This equality is because of the absence of the uncertainty after the default. However, if the bankruptcy does not occur at $t = 1$ then the dollar value of the ABC zero coupon bond is less than the dollar value of a default free zero coupon bond price and is greater than a claim paying δ dollars for sure which is asserted by equation 4.3.3.11 and equation 4.3.3.12.

The independence of the pseudo probability $\lambda\mu_1$ from the spot interest rate process is guaranteed by Equation 4.3.3.13.

The default risky bond market will also be analyzed so as to determine $\lambda\mu_0$ by using time 0 default risky bond market since $\lambda\mu_0$ is the pseudo- probability of default at $t = 1$. Graph 4.3.2.3 is consulted again.

$$v_1(0,1) = p_1(0,1) = \frac{[\lambda\mu_0\delta + (1-\lambda\mu_0)]}{r(0)} \qquad (4.3.3.14)$$

$$v_1(0,2) = p_1(0,2)$$
$$= \frac{\left[\Pi_0(\lambda\mu_0)\delta p_1(1,2)_{u,b} + \Pi_0(1-\lambda\mu_0)p_1(1,2)_{u,n} + (1-\Pi_0)\lambda\mu_0\delta p_1(1,2)_{d,b} + (1-\Pi_0)(1-\lambda\mu_0)p_1(1,2)_{d,n}\right]}{r(0)}$$

(4.3.3.15)

These conditions ensure that time 0 prices are their time 1 discounted expected values calculated by the pseudo probabilities. Substitution of equation 4.3.3.2 and equation 4.3.3.8 into equation 4.3.3.15 and then simplification gives us the equation 4.3.3.16.

$$v_1(0,2) = p_1(0,2) = p_0(0,2)[\lambda\mu_0\delta + (1-\lambda\mu_0)r(1)_d p_1(1,2)_{d,n}] \qquad (4.3.3.16)$$

By equations 4.3.3.14, 4.3.3.15 and 4.3.3.16 $\lambda\mu_0$ is obtained as follows:

$$\lambda\mu_0 = \frac{[1-r(0)p_1(0,1)]}{[1-\delta]}$$

$$= \frac{\left[r(1)_d p_1(1,2)_{d,n} - \frac{p_1(0,2)}{p_0(0,2)}\right]}{\left[r(1)_d p_1(1,2)_{d,n} - \delta\right]} \qquad (4.3.3.17)$$

Thus, $\lambda\mu_0$ exists, is unique and satisfies 0< $\lambda\mu_0 < 1$ if and only if;

$$\frac{\delta}{r(0)} < p_1(0,1) < 1/r(0) \qquad (4.3.3.18)$$

$$\delta p_0(0,2) < p_1(0,1) < p_0(0,2)r(1)_d p_1(1,2)_{d,n} \qquad (4.3.3.19)$$

$$\frac{\left[r(1)_d p_1(1,2)_{d,n} - \frac{p_1(0,2)}{p_0(0,2)}\right]}{\left[r(1)_d p_1(1,2)_{d,n} - \delta\right]} = \frac{[1-r(0)p_1(0,1)]}{[1-\delta]} \qquad (4.3.3.20)$$

Equation 4.3.3.18 asserts that the dollar value of the ABC zero coupon bond maturing at $t = 1$ must be worth less than receiving a dollar for sure and greater than receiving δ dollars for sure. Equation 4.3.3.19 states that the ABC zero coupon bond maturing at $t = 2$ must be worth more than receiving δ dollars for sure and less than receiving $r(1)_d p_1(1,2)_{d,n}$ dollars for sure at $t = 2$. Equation 4.3.3.20 guarantees that under the pseudo probabilities, the bankruptcy process is independent of the default free spot interest rate process which is imposed for analytical convenience.

From now on, Jarrow and Turnbull assume that the conditions for the existence and uniqueness conditions for the pseudo probabilitiesΠ_0, $\lambda\mu_0$ and $\lambda\mu_1$ hold. That is the market is assumed to be arbitrage free and complete.

4.3.4 ABC Zero Coupon Bonds

With the assumption of the existence and uniqueness of the pseudo probabilities and thereby the market completeness and absence of arbitrage; ABC zero coupon bond prices can be stated in an equivalent form by means of the discounted expected values.

Expected pay off ratios at future dates can be calculated by time t conditional expected value under the pseudo probabilities. Namely, $\tilde{E}_t(.)$ denotes the time t conditional expected value under the pseudo probability.

Thus, the expected pay off ratios at future dates can be calculated by means of Graph 4.3.2.1 as follows:

$$\tilde{E}_1\big(e_1(2)\big) = \begin{cases} \delta \; if \; bankrupt \; at \; t = 1 \\ \lambda\mu_1\delta + (1 - \lambda\mu_1) if \; not \; bankrupt \; at \; t = 1 \end{cases} \tag{4.3.4.1}$$

Equation 4.3.4.1 expresses that the expected pay off ratio of $t = 2$ at $t = 1$, is δ if the bankruptcy occurs since it was assumed that if the bankruptcy occurs, it will remain until maturity; $\lambda\mu_1\delta + (1 - \lambda\mu_1)$ if the bankruptcy does not take place at $t = 1$.

$$\tilde{E}_0\big(e_1(2)\big) = \lambda\mu_0\delta + (1 - \lambda\mu_0)[\lambda\mu_1\delta + (1 - \lambda\mu_1)] \tag{4.3.4.2}$$

Equation 4.3.4.2 signifies that the expected pay off ratio at $t = 2$ as viewed from $t = 0$. Tracking Graph 4.3.2.1, it can be seen that the two states of the world, bankruptcy and nonbankruptcy are taken into account and the expected value is calculated.

$$\tilde{E}_0\big(e_1(1)\big) = \lambda\mu_0\delta + (1 - \lambda\mu_0) \qquad (4.3.4.3)$$

Equation 4.3.4.3 specifies the expected pay off ratio at $t = 1$ when looked at $t = 0$. What must be noticed in these equations is that all of them are less than 1 since all of them are related to bankruptcy in any state of the world.

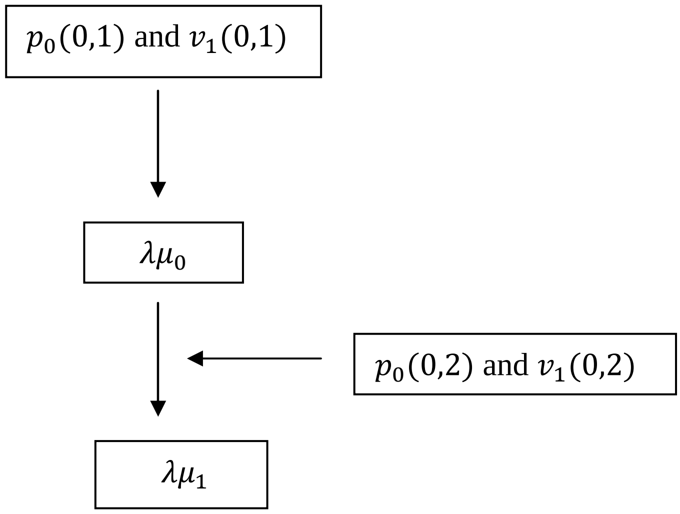
The equations referred imply equation 4.3.4.4

$$v_1(t,T) = p_0(t,T)\ \tilde{E}_t\big(e_1(T)\big) \qquad (4.3.4.4)$$

The decomposition can be imposed because of the independence assumption under the pseudo probabilities. In an arbitrage free market, the ABC zero coupon bond price is its discounted expected payoff at time T under pseudo probabilities. Equation 4.3.4.4 makes evident that the discount factor is the default free zero coupon bond price.

By Equation 4.3.4.4 expected pay off ratio at $t = T$ can be estimated if $v_1(t,T)$and $p_0(t,T)$ are known. On the other hand, if someone is given the estimation of the pay off ratio δ, one can estimate the pseudo probabilities recursively. Namely, $\lambda\mu_0$ can be estimated if the default free zero coupon bond prices, $p_0(0,1)$ and the time 0 value of the ABC zero coupon bond promising a dollar at $T = 1$, $v_1(0,1)$ are given by equation 4.3.4.4 and equation 4.3.4.3. Then, having $p_0(0,2)$ and $v_1(0,2)$, $\lambda\mu_1$ can be estimated by equation 4.3.4.4 and equation 4.3.4.2. The following graph shows the estimation procedure of the pseudo probabilities:

Graph 4.3.4.1: Estimation procedure of the pseudo probabilities



Equation 4.3.3.4 shows that the ABC zero coupon bond is strictly less valuable than a default free zero coupon bond of equal maturity since $\tilde{E}_t\left(e_1(T)\right)$ is strictly less than 1. Consequently, the credit spread must be positive in a bankruptcy.

4.3.5 ABC Coupon Bonds

In this section the coupon payments will be included in the computations. Jarrow and Turnbull assume that ABC coupon bearing bond promises k_1 dollars at $t = 1$ and k_2 dollars at $t = 2$ where k_2 coupon includes the principal payment too. The coupon bearing bond can be considered as a portfolio consisting of k_1 zero coupon bonds with maturity 1 and k_2 zero coupon bonds with maturity 2.

Let $D(t)$ denote the time t dollar value of the ABC coupon bond. As stated, the price of ABC coupon bond is the discounted expected payoff due to the risk neutral valuation using the pseudo probabilities. Consequently the following result is obtained:

$$D(0) = \tilde{E}_0\left(\frac{k_1 e_1(1)}{B(1)} + \frac{k_2 e_1(2)}{B(2)}\right) \quad (4.3.5.1)$$

$$= D(0) = k_1 \tilde{E}_0\left(\frac{e_1(1)}{B(1)}\right) + k_2 \tilde{E}_0\left(\frac{e_1(2)}{B(2)}\right)$$

$$= D(0) = k_1 v_1(0,1) + k_2 v_1(0,2) \text{ (by equation 4.3.4.4)} \quad (4.3.5.2)$$

Equation 4.3.5.2 offers the advantage of deducing the prices of the ABC zero coupon bonds, $v_1(t,T)$ from the traded prices of only a few issues of ABC coupon bearing bonds, where $B(t)$ is the money market account.

4.4. The Continuous Trading Economy

4.4.1 Derivation of the stochastic processes for the default free zero coupon bond, the ABC bond and the money market account

So far, the two period discrete time economy has been described so as to extend it to its multi period, continuous time limit. Under the pseudo probabilities, the bankruptcy process for ABC bond and the default free term structure are independent.

The trading is assumed to take place over the time interval [0,τ]. Let;

τ_1^*: the time of the bankruptcy for ABC firm. It is assumed to be exponentially distributed over [0,∞) with parameter λ_1 with the fact that alternative distributions may be utilized. In this section the basic definitions of forward rate, spot rate and the money market account will be used The indices 0 and 1 will differentiate the default free and the default risky bonds. Namely, $f_0(t,T), r_0(t,T) and\ B_0(t,T)$ will characterize the default free bond whereas $f_1(t,T), r_1(t,T) and\ B_1(t,T)$ will characterize the default risky bond.

The exogenous stochastic structure will be imposed by the proceeding assumptions on the forward rates $f_0(t,T), f_1(t,T)$ and the payoff ratio $e_1(t)$ to be consistent with HJM.

Assumption 4.4.1.1: Default free forward rates

$$df_0(t,T) = \alpha_0(t,T)dt + \sigma(t,T)dW_1(t)$$

where $W_1(t)$ refers to a Brownian motion, which is a real valued continuous process with independent and stationary increments. $\sigma(t,T)$ is a random shock with volatility and it is deterministic. $\alpha_0(t,T)$ is the drift. $(\alpha_0(t,T), \sigma(t,T))$ satisfy the following smoothness and boundedness conditions:

1- $\alpha_0(t,T), \sigma^2(t,T)$ are L^1 in t

2- $\alpha_0(t,T)$ is in $L^{(1,1)}$ in (t,T)

3- $\sigma(t,T)$ is $L^{(2,1)}$

The assumption on the stochastic movement of the default free forward rates, imply that the default free forward rates change is equal to a drift plus a random shock with volatility. The volatility function is assumed to be deterministic in order to facilitate the derivation of the closed form solution.

The second assumption is for the ABC bonds.

Assumption 4.4.1.2: ABC Forward rates

$$df_1(t,T) = \begin{cases} [\alpha_1(t,T) - \theta_1(t,T)\lambda_1]dt + \sigma(t,T)dW_1(t) & if\ t < \tau_1^* \\ [\alpha_1(t,T) - \theta_1(t,T)\lambda_1]dt + \sigma(t,T)dW_1(t) + \theta_1(t,T) & if\ t = \tau_1^* \\ \alpha_1(t,T)dt + \sigma(t,T)dW_1(t) & if\ t > \tau_1^* \end{cases}$$

where $\alpha_1(t,T)$ and $\theta_1(t,T)$satisfy the following measurability and integrability conditions.

1- $\alpha_0(t,T), \sigma^2(t,T)$ and $\theta_1(t,T)\lambda_1$ are L^1 in t

2- $\alpha_0(t,T)$ and $\theta_1(t,T)\lambda_1$ in $L^{(1,1)}$ in (t,T)

As seen from the equation, the process for the ABC forward rates is very similar to the default free forward rates. Before the bankruptcy, the drift is adjusted downward to reflect the expected change $\theta_1(t,T)\lambda_1$. This term is added to the equation at the bankruptcy. After the bankruptcy the forward rate process is the same as the default free forward rate process except the subscripts.

Assumption 4.4.1.3: The ABC payoff ratio

$$e_1(t) = \begin{cases} 1 & if\ t < \tau_1^* \\ \delta_1 & if\ t \geq \tau_1^* \end{cases}$$

where $0<\delta_1<1$.

As described before, the payoff ratio before the bankruptcy is 1 and it is equal to δ_1 at the time of the bankruptcy. This equation is the continuous time limit of the bankruptcy process in Graph 4.3.2.1. The payoff ratio is determined by the seniority of the debt likewise in the two period discrete time economy. For the simplicity of the estimation and the computation the payoff ratio is assumed to be constant, though this assumption can be relaxed and the payoff ratio can be random and dependent on an additional Brownian motion representing the randomness generating the value of the firm.

Once given the forward rate and payoff ratio processes the stochastic processes of the default free bond price, default risky ABC bond price and the money market account can be derived. These are the continuous time limits of the graphs 4.3.1.1 and 4.3.2.3

4.4.1.1. Derivation of the stochastic process for $p_0(t,T)$

Proposition 4.1.1.c asserts that, if $f_0(t,T)$ satisfies

$df_0(t,T) = \alpha_0(t,T)dt + \sigma(t,T)dW_1(t)$ then $p_0(t,T)$ satisfies

$$dp_0(t,T) = p_0(t,T)\left\{r_0(t) + A(t,T) + \frac{1}{2} \parallel S(t,T \parallel^2\right\}dt + p_0(t,T)S(t,T)dW_1(t)$$

where $\|.\|$ denotes the Euclidean norm and

$$A(t,T) = -\int_t^T \alpha_0(t,s)ds$$

$$S(t,T) = -\int_t^T \sigma(t,s)ds$$

Dividing by $p_0(t,T)$, the following result is obtained;

$$\frac{dp_0(t,T)}{p_0(t,T)} = \left\{r(t) + A(t,T) + \frac{1}{2} \parallel S(t,T \parallel^2\right\}dt + S(t,T)dW_1(t)$$

The equation can be simplified by denoting;

$$\beta_0(t,T) = -\int_t^T \alpha_0(t,s)ds + \frac{1}{2} \parallel S(t,T) \parallel^2$$

Thus the equation turns out to be;

$$\frac{dp_0(t,T)}{p_0(t,T)} = \{r(t) + \beta_0(t,T)\}dt + S(t,T)dW_1(t) \qquad (4.4.1.1.1)$$

Eqn.4.4.1.1.1 is the return process followed by the default free zero coupon bond. The return equals to the default free interest rate plus a risk premium plus a random shock with volatility $S(t,T)$. The volatility function goes to zero as t approaches to T, i.e as the bond matures.

4.4.1.2. Derivation of the stochastic process for $\boldsymbol{v_1(t,T)}$

The return process of the ABC zero coupon bond imitates the return process of the default free zero coupon bond. Before the bankruptcy, the return equals to a drift adjusted for a change at the time of the bankruptcy plus a random shock with volatility $S(t,T)$. At the bankruptcy, the return varies by ($\delta_1 e^{\Theta_1(t,T)}$-1).

Assumption 4.4.1.2 imposes the following process for the forward rate before the bankruptcy.

$$df_1(t,T) = [\alpha_1(t,T) - \theta_1(t,T)\lambda_1]dt + \sigma(t,T)dW_1(t) \qquad if\ t < \tau_1^*$$

$$p_1(t,T) = \exp\left(-\int_t^T f_1(t,s)ds\right)$$

$$f_1(t,T) = f_1(0,T) + \int_0^t [\alpha_1(u,T) - \theta_1(u,T)\lambda_1]du + \int_0^t \sigma(u,T)dW_1(u)$$

Let X_t be $\int_t^T -f_1(s,s) + f_1(s,s) - f_1(t,s)\, ds$

where $p_1(t,T) = \exp(X_t)$

Let $g(x) = e^x$ then, $g'(x) = e^x$ and $g''(x) = e^x$

$p_1(t,T) = \exp(X_t) = \exp(X_0) + \int_0^t e^{X_s}\, dX_s + 1/2 \int_0^t e^{X_s} d < X,X >_s$

$$dp_1(t,T) = e^{X_t} dX_t + \frac{1}{2} e^{X_t} d < X,X >_t$$

Then,

$$X_t = -\int_t^T f_1(s,s)ds - \int_t^T \int_0^t \alpha_1(u,s)\, duds$$
$$- \int_t^T \int_0^t \theta_1(u,s)\lambda_1 duds - \int_t^T \int_0^t \sigma(u,s)\, dW_1(u)ds$$
$$+ \int_t^T \int_0^s \alpha_1(u,s)duds$$
$$+ \int_t^T \int_0^s \theta_1(u,s)\lambda_1 duds + \int_t^T \int_0^s \sigma(u,s)\, dW_1(u)ds$$

By Fubini Theorem and interchanging the integrals, the following equation is obtained.

$$X_t = -\int_t^T f_1(s,s)ds + \int_t^T \int_u^T \alpha_1(u,s)\, dsdu$$
$$+ \int_t^T \int_u^T \theta_1(u,s)\lambda_1 dsdu + \int_t^T \int_u^T \sigma(u,s)\, dsdW_1(u)$$

$$X_0 = -\int_0^T f_1(s,s)ds + \int_0^T \int_u^T \alpha_1(u,s)\, dsdu$$
$$+ \int_0^T \int_u^T \theta_1(u,s)\lambda_1 dsdu + \int_0^T \int_u^T \sigma(u,s)\, dsdW_1(u)$$

$$X_t = X_0 + \int_0^t f_1(s,s)ds - \int_0^t \int_u^T \alpha_1(u,s)\, dsdu$$
$$- \int_0^t \int_u^T \theta_1(u,s)\lambda_1 dsdu + \int_0^t \int_u^T \sigma(u,s)\, dsdW_1(u)$$

$$dX_t = f_1(t,t)dt - \int_t^T \alpha_1(t,s)\, dsdt - \int_t^T \theta_1(t,s)\lambda_1 dsdt + \int_t^T \sigma(t,s)\, dsdW_1(t)$$

$$dp_1(t,T) = p_1(t,T)(f_1(t,t)dt - \int_t^T \alpha_1(t,s)\, dsdt - \int_t^T \theta_1(t,s)\lambda_1 dsdt$$
$$+ \int_t^T \sigma(t,s)\, dsdW_1(t)) + \frac{1}{2} p_1(t,T)(\int_t^T \sigma(t,s)ds)^2\, dt$$

since $d < X, X >_t = \frac{1}{2}(\int_t^T \sigma(t,s)ds)^2\, dt$.

Let,

$$\beta_1(t,T) = -\int_t^T \alpha_1(t,u)\, du + \frac{1}{2} S(t,T)^2$$

$$\Theta_1(t,T) = -\int_t^T \theta_1(t,u) du$$

Then $dp_1(t,T)/p_1(t,T)$ can be written as follows;

$$\frac{dp_1(t,T)}{p_1(t,T)} = [r_1(t) + \beta_1(t,T) - \Theta_1(t,T)\lambda_1]dt + S(t,T)dW_1(t)$$

Since $e_1(t)$ is 1 and since there is no jump making $v_1(t,T) = v_1(t\_,T)$ before bankruptcy, the equation can be written as follows;

$$\frac{dv_1(t,T)}{v_1(t\_,T)} = [r_1(t) + \beta_1(t,T) - \Theta_1(t,T)\lambda_1]dt + S(t,T)dW_1(t)$$

Since the bankruptcy process follows a Poisson process, the jump process should be taken into account. Therefore, Ito-Doeblin formula is applied to $h(x) = e^x$.

To derive the equation for $v_1(t,T)$ at the bankruptcy $p_1(t,T) = \exp\,(-\int_t^T f_1\,(t,s)ds)$ is set. Then

$$= -log p_1(t,T) = \int_t^T f_1(\text{t,s})\text{ds}$$

$$= \int_t^T f(0,s)ds + \int_t^T \int_0^t \alpha_1(t,s)duds + \int_t^T \int_0^t \theta_1(t,s)duds$$

$X = X^c + \int \Theta_1(t,T)\, dN(t) = X^c + \Theta_1(t,T)N(t)$ where $X^c$is the continuous part.

Then $f(X_t) =$

$$f(X_0) + \int_0^t f'\,(X_s)dX_s^c + \frac{1}{2}\int_0^t f''\,(X_s)d < X^c, X^c > + \sum_{0<s<t} f(X_s) - f(X_{s\_})$$

$$= f(X_0) + \int_0^t e^{X_s}\{r(s) + \beta_0(s,T)\}dt + \ S(s,T)dW_1(s) - \lambda_1\Theta_1(s,T)ds$$

$$+\frac{1}{2}\int_0^t e^{X_s}S(s,T)^2\, ds + \sum_{0<s<t} e^{X(s)} - \mathrm{e}^{X(s\_)}$$

$$e^{X(s)} - \mathrm{e}^{X(s\_)} = \exp\big(X^c + \Theta_1(s,T)N(s)\big) - \exp\big(X^c + \Theta_1(s,T)N(s\_)\big)$$

Let $S(u)$ denote $e^{X(s)}$. Then $e^{X(s\_)} = K(u\_)$

At bankruptcy, a single jump is observed. Thus $K(u\_)$ is multiplied by δ_1. Moreover, since N(s) is 1 at jumps, $K(u) = K(u\_)(e^{\Theta_1(t,T)}\delta_1)$. Then,

$$K(u) - K(u\_) = \begin{cases} K(u\_)\left(e^{\Theta_1(u,T)}\delta_1 - 1\right) \text{ at jumps} \\ 0 \quad \text{if there is no jump} \end{cases}$$

Thus, $K(u) - K(u\_)$ can be written as $K(u\_)\left(e^{\Theta_1(u,T)}\delta_1 - 1\right)\Delta N(u)$, therefore,

$$\sum_{0<s<t} e^{X(s)} - e^{X(s\_)} = e^{X(t)}\left(e^{\Theta_1(s,T)}\delta_1 - 1\right)$$

Finally, for the ABC zeros,

$$dv_1(t,T)/v_1(t-,T)$$
$$= \begin{cases} [r_1(t) + \beta_1(t,T) - \Theta_1(t,T)\lambda_1]dt + S(t,T)dW_1(t) & if\ t < \tau_1^* \\ [r_1(t) + \beta_1(t,T) - \Theta_1(t,T)\lambda_1]dt + S(t,T)dW_1(t) + \left(e^{\Theta_1(t,T)}\delta_1 - 1\right) & if\ t = \tau_1^* \\ [r_1(t) + \beta_1(t,T)]dt + S(t,T)dW_1(t) & if\ t > \tau_1^* \end{cases}$$

(4.4.1.2.1)

Equation 4.4.1.2.1 expresses the return process of the ABC zeros. As seen before the bankruptcy, the return has a drift which is adjusted for the change at bankruptcy and a random shock with volatility $S(t,T)$. At the time of bankruptcy, the return process has an additional term $\left(e^{\Theta_1(t,T)}\delta_1 - 1\right)$ due to the jump process occurring at bankruptcy.

4.4.2 Arbitrage- Free Restrictions

In discrete time economy, the absence of the arbitrage is guaranteed by means of the existence of the unique equivalent pseudo probabilities under which the relative prices $\frac{p_0(t,1)}{B(t)}$, $\frac{p_0(t,2)}{B(t)}$, $\frac{v_1(t,1)}{B(t)}$, $\frac{v_2(t,1)}{B(t)}$ are Q martingales which means that the expected values are equal to the equal current values since the absence of arbitrage opportunities is guaranteed by the existence of an equivalent martingale measure. As Harrison and Pliska asserted in 1981, the market is complete when the equivalent martingale measure is unique. In a similar way, to ensure that there is no arbitrage opportunity, the relative prices; $\frac{p_0(t,T)}{B(t)}, \frac{v_1(t,T)}{B(t)}$ and $\frac{B_1(t)e_1(t)}{B(t)}$ should be proved to be Q martingales. These conditions are analogous to the conditions in discrete time case. In order to get these conditions, Assumption 4 is imposed.

Assumption 4.4.2.1: $\left(e^{\Theta_1(t,T)}\delta_1 - 1\right) \neq 0$ for all $t \leq \tau_1^*$ and $T \in [0, \tau]$

Equation 4.4.1.2.1 states that the bankruptcy process has an impact on the return process by $\left(e^{\Theta_1(t,T)}\delta_1 - 1\right)$. Therefore this coefficient should be different from zero.

Under assumption 4.4.2.1, the system of equations below has a unique solution $(\Upsilon_1(t), \mu_1(t))$.

$$\beta_0(t,T) + \Upsilon_1(t)S(t,T) = 0 \qquad (4.4.2.1)$$

$$r_1(t) - r_0(t) + \beta_1(t,T) + \Upsilon_1(t)S(t,T) - \Theta_1(t,T)\lambda_1 + \left(e^{\Theta_1(t,T)}\delta_1 - 1\right)\lambda_1\mu_1(t) = 0 \qquad if\ t < \tau_1^* \qquad (4.4.2.2)$$

$$r_1(t) - r_0(t) + \beta_1(t,T) + \Upsilon_1(t)S(t,T) = 0 \qquad if\ t \geq \tau_1^* \qquad (4.4.2.3)$$

$$r_1(t) = r_0(t) + (1 - \delta_1)\lambda_1\mu_1(t) \qquad if\ t < \tau_1^* \qquad (4.4.2.4)$$

$$r_1(t) = r_0(t) \qquad\qquad if\ t \geq \tau_1^* \qquad\qquad (4.4.2.5)$$

The unique solution of the above equations; $(\Upsilon_1(t), \mu_1(t))$ denote the market prices of risk. $\beta_0(t,T)$ denotes the excess expected return on the T maturity default free zero coupon bond and the equation 4.4.2.3 indicates that it is proportional to its volatility $S(t,T)$ by the risk premium $\Upsilon_1(t)$.

If equation 4.4.2.2 is rewritten, equation 4.4.2.6 is obtained.

$$\beta_1(t,T) - \Theta_1(t,T)\lambda_1 = \beta_0(t,T) - \delta_1\left(e^{\Theta_1(t,T)} - 1\right)\lambda_1\mu_1(t) \qquad if\ t < \tau_1^* \qquad (4.4.2.6)$$

Before the bankruptcy, the excess expected return on the ABC zero coupon bond; $\beta_1(t,T) - \Theta_1(t,T)\lambda_1$ is equal to the excess expected return on the default free zero coupon bond $\beta_0(t,T)$ plus an adjustment for the default risk which is proportional to the bankruptcy shock $\delta_1\left(e^{\Theta_1(t,T)} - 1\right)$ by the risk premium $\lambda_1\mu_1(t)$.

In order to analyze the case after the bankruptcy, the equations 4.4.2.1,3 and 5 are combined and the following equation is obtained.

$$\beta_1(t,T) = \beta_0(t,T) \qquad\qquad if\ t \geq \tau_1^* \qquad\qquad (4.4.2.7)$$

Since after the bankruptcy, the default risk vanishes, the excess expected return on ABC zeros and the default free zeros are identical by equation 4.4.2.7. Thus after the bankruptcy takes place the term structures of the default free and the ABC zeros are identical like in the discrete time setting, indicating that

$$p_1(t,T) = p_0(t,T)$$

and

$v_1(t,T) = \delta_1 p_0(t,T)$

For simplicity Assumption 4.4.2.2 which implies the statistical independence of the bankruptcy process under the martingale probabilities is imposed. This assumption makes the time of the bankruptcy process an exponential distribution under the martingale probabilities with parameter $\lambda_1 \mu_1$.

<u>Assumption 4.4.2.2:</u>

$\mu_1(t) \equiv \mu_1 > 0$ is a positive constant

4.4.3 The ABC Bonds

Assumptions 4.4.1.1, 4.4.1.2, 4.4.1.3, 4.4.2.1 and 4.4.2.2are used to simplify $v_1(t,T)$ to

$$v_1(t,T) = \tilde{E}_t\left(\frac{e_1(T)}{B(T)}\right)B(t) = \tilde{E}_1(e_1(T))p_0(t,T)$$

$$= \begin{cases} e^{-\lambda_1 \mu_{1(T-t)+}}\delta_1\left(1 - e^{-\lambda_1 \mu_{1(T-t)}} p_0(t,T)\right) & if\ t < \tau_1^* \\ \delta_1 p_0(t,T) & if\ t \geq \tau_1^* \end{cases}$$

(4.4.3.1)

Equation 4.4.3.1 reveals the fact that, $\lambda_1 \mu_1$ and δ_1 are the only parameters that are needed to compute the stochastic process of $v_1(t,T)$. It is also noticed that $\Theta_1(t,T)$ does not included in the equation since the martingale restrictions under assumption 4.4.2.2 specify $\Theta_1(t,T)$ in terms of the bankruptcy parameters. This case provides easiness for the empirical estimations. Lemma 1 puts forward the relationship of $\Theta_1(t,T)$ with the default risk parameters.

Lemma 4.4.3.1:

$\delta_1\left(e^{\Theta_1(t,T)} - 1\right) = (e^{-\lambda_1 \mu_1 (T-t)}(\delta_1 - 1))/(e^{-\lambda_1 \mu_1 (T-t)} + \delta_1(1 - e^{-\lambda_1 \mu_1 (T-t)}))$
for $t < \tau_1^*$

CHAPTER 5

ESTIMATION METHODOLOGY

In order to implement the Jarrow Turnbull Model, the estimates of default intensity, λ called by Frühwirth and Sögner (2006) corresponding to $\lambda_1\mu_1$ in the original equation and the recovery rate, δ, parameters need to be estimated. Moreover as equation (4.1.4.3.1) demonstrates that default-free term structure, default intensity and recovery rate are necessary and sufficient to compute default risky bond price. The term structure dynamics $\alpha(t,T)$, drift parameter, and $\sigma(t,T)$, volatility parameter, are not needed to be estimated for the default risky bond price $(v(t,T))$ since their effect is restricted to the default free bond price $p_0(t,T)$. This is because of the independence assumption of the stochastic process driving the default-free term structure and the default process and it is an advantage of the model in implementation issue. As a result, to implement Jarrow Turnbull model, $p_0(t,T)$ and the estimates of default risk parameters default intensity λ and recovery rate δ are required. Default intensity λ is determined by the issuer's long term senior unsecured credit rating while recovery rate depends on the seniority of the bond (Gupton *et al* (1997), Lando(1994), Altman and Kishore (1996)).

5.1 Estimation of the parameters

5.1.1 Estimation of the default-free zero coupon bond prices

Default free bond price process is assumed to follow the Heath-Jarrow-Morton Model. However, as the default risky bond price depends only on the default free bond price, the estimation of the term structure dynamics is decided to be waived and Svensson (1994) model is decided to be used. The default free zero coupon bond prices are the estimated firstly of since for the estimation of the default risky bond prices, the default free zero coupon bond prices are required.

Default-free zero coupon bond prices are inferred through default-free spot rates which are computed by means of Svensson parameters and Svensson Function. Therefore $\alpha(t,T)$ and $\sigma(t,T)$ are not required for $p_0(t,T)$.

For maturities smaller than 12 months spot rates are available in the form of rates as Treasury bills which are zero-coupon bonds. However, for longer maturities, zero coupon bonds are usually not available for sufficiently many maturities and in sufficiently large issues to be sufficiently liquid. Therefore spot rates will have to be estimated from yields on coupon bonds (Svensson, 1994).

Let $r(t,T)$ be continuously compounded spot interest rate for a zero coupon bond traded at time t with a maturity $T > t$. Denote $T - t$ with m, which is interpreted as the time to maturity. Let $d(t,T)$ be the price of a zero coupon bond at time t that pays 1$ at maturity T. Then $d(t,T)$is called as the discount function. The following formula gives the relation between the spot rate $r(t,T)$ and discount function $d(t,T)$.

$$d(t,T) \ = \ exp(-r(t,T)\ /100.(T-t)) \qquad (5.1.1.1)$$

Let us consider a coupon bond with a coupon rate % c per year with a face value R at maturity T. Then the time t price of the bond with time to maturity m is;

$$p(t,t+m) = \sum_{k=1}^{m} cd(t,t+k) + Rd(t,t+m) \qquad (5.1.1.2)$$

Since yield to maturity is the internal rate of return for the coupon bond, $y(t, t+m)$ satisfies the following equation.

$$p(t, t+m) = \sum_{k=1}^{m} cexp(\frac{(-y(t, t+m)}{100})k) + Rexp((-\frac{y(t, t+m)}{100})m) \quad (5.1.1.3)$$

Continuously compounded forward rate is

$$f(t; S, T) \ = \ [(T-t)r(t, T) - (S-t)r(t, S)]/(T-S) \quad (5.1.1.4)$$

where t is the trade date, S is the settlement date and T is the maturity. This equation gives us the relation between the spot rate $r(t, T)$ and the continuously compounded forward rate $f(t; S, T)$. The instantaneous forward rate which can be interpreted as the riskless rate of interest is defined as the limit of the continuously compounded forward rate $f(t; S, T)$ as T goes to S. It is the forward rate for a forward contract with an infinitesimal investment period after the settlement date.

$$f(t, S\ ) = \lim_{T \to S} f(t; S, T) \quad (5.1.1.5)$$

The spot rate $r(t, T)$ at t with maturity T is identical to the average of the instantaneous forward rates with settlement dates between t and maturity T.

$$r(t, T) = \int_{\tau=t}^{T} f(t, \tau)\, d\tau \ /(T-t) \quad (5.1.1.6)$$

By multiplying $r(t, T)$ with $T-t$ and taking the derivatives of both sides with respect to T, equation 5.1.1.7. is obtained;

$$f(t,T) = r(t,T) + (T-t)\partial r(t,T)/\partial T \qquad (5.1.1.7)$$

The above equation is the relation between the spot rates and the instantaneous forward rates.

Nelson and Siegel assume that instantaneous forward rate is the solution to a second order differential equation with to equal roots. Svensson simplified the notation by replacing $f(t,t+m)$ with $f(m)$ and added new terms to the Nelson Siegel's forward rate and obtained the following equation.

$$f(m;b) =$$
$$\beta_0 + \beta_1 exp(-m/\tau_1) + \beta_2 (m/\tau_1)\, exp(-m/\tau_1) + \beta_3(m\,\tau_2)\, exp(-m/\tau_2)$$

where $b = [\beta_0, \beta_1, \beta_2, \tau_1, \beta_3, \tau_2]$ (5.1.1.8)

Here;

β_0 : constant

$\beta_1 \exp(-m/\tau_1)$: is monotonically decreasing for $\beta_1 > 0$ or increasing $\beta_1<0$ towards 0 as a function of time to settlement.

$\beta_2 (m/\tau_1) \exp(-m/\tau_1)$: is a function of time to settlement and it generates the hump- shape when $\beta_2 > 0$ and U- shape when $\beta_2 < 0$

$\beta_3(m/\tau_2) \exp(-m/\tau_2)$: second hump-shape or U-shape parameter with $\tau_2 > 0$

When τ goes to ∞, $f(m,b)$ approaches to β_0 and when τ goes to 0, $f(m,b)$ approaches to $\beta_0 + \beta_1$

As indicated before the spot rate can be derived by integrating the forward rate. Therefore equation 5.1.1.9 is obtained;

$$r(m;b) = \beta_0 + \beta_1\left(\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}\right) + \beta_2(\left(\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}\right) -$$

$$\exp\left(-\frac{m}{\tau_1}\right) + \beta_3(\left(\frac{1-\exp\left(\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}}\right) - \exp\left(\frac{m}{\tau_2}\right)) \qquad (5.1.1.9)$$

The discount function is given by

$$d(m;b) = \exp\left(\left(-\frac{r(t,T)}{100}\right)m\right) \qquad (5.1.1.10)$$

The following algorithm is used to estimate the spot rates for n coupon bonds (c_j, m_j, y_j, p_j) where c_j is the coupon rate, m_j is the time to maturity, y_j is the observed yield to maturity and p_j is the observed price of the bond j with a face value R \$, $j = 1,2,\ldots,n$.

1- A trade date is fixed.

2- A vector of starting parameters $b = [\beta_0, \beta_1, \beta_2, \tau_1, \beta_3, \tau_2]$ is selected.

3- The discount function is determined by means of this parameter vector and (5.1.1.10)

4- The discount function is used to find the starting model price of each bond by the following formula.

$$P_j(b) = \sum_{k=1}^{K_i} c_j\, d(\tau_{ik};b) + Rd(\tau_{ik};b) \qquad j = 1,2,\ldots,n \qquad (5.1.1.11)$$

$$\tau_{jk} = m_j - [m_j] + k - 1$$

$$K_j = [m_j] + 1$$

$[m_j]$ denotes the largest integer that is strictly smaller than $[m_j]$

5- Numerical optimization procedures are performed to estimate the set of parameters that minimizes the sum of squared yield errors. The estimated yield to maturity for bond j, $Y_j(b)$, is computed from the estimated bond price $P_j(b)$ by 5.1.1.3 The observed yield to maturity is allowed to deviate from the estimated yield to maturity.

$$y_j = Y_j(b) + e_j \qquad (5.1.1.12)$$

6-Finally the estimated set of parameters are used to determine the spot rate function $r(m;b)$ given by (5.1.1.9).

5.1.2. Estimation of the Default Risk Parameters

5.1.2.1 Non-linear Least Squares Estimation of the Parameters

The main objective of this part is evaluating the Jarrow Turnbull model for default risky bond pricing purposes. Implementing the model necessitates estimating the required parameters, default intensity and the recovery rate which determine the bond price. Default risk parameters can be derived by means of either separate estimation or joint estimation. Frühwirth and Sögner built the joint estimation procedure and used the following non linear least squares estimation:

$$\left(\hat{\lambda}_{I,t}, \hat{\delta}_{I,t}\right) = \underset{\lambda,\delta}{\operatorname{argmin}} f(\lambda, \delta) \qquad (5.1.2.1.1)$$

where

$$f(\lambda, \delta) := \sum_{i \in I} \sum_{t=1}^{T} [B_i^{obs}(t, T_i) - B_i(t, T_i)]^2 \qquad (5.1.2.1.2)$$

and

$$B(t,T) = \sum_{u=t}^{T} v_1(t,u) C_B\,(u) \qquad (5.1.2.1.3)$$

The usual method to estimate non-linear regression models is to minimize the squared sum of residuals. The above equation computes the pricing error for all bonds $i \in I$ between the observed price and the Jarrow- Turnbull model price. Then the equation computes the sum of squared pricing errors, therefore the above equation defines a contrast function. After computing the contrast function, the parameters are estimated by minimizing the contrast function with different λ and δ which are restricted to be positive.

A non-linear regression model must be identified to get unique parameter estimates. However, Frühwirth and Sögner showed that joint estimation is numerically unstable and poorly identified which means that the Hessian matrix of the contrast function is nearly singular by means of a simulation study.

They therefore employ separate estimation which takes one parameter as fixed in (5.1.2.1.1) and estimate the other parameter conditionally on the fixed parameter. To decide which parameter is going to be fixed they searched the literature and they decided on the recovery rate parameter since they are already recovery rate estimates drawn from actual defaults provided in Moody's (1992), Altman and Kishore (1996), Standard & Poor's (2000) and Moody's (2000). These studies propose a recovery rate of (approximately) 50% for unsecured senior bonds. Moreover default intensity estimates are scarce especially under the equivalent martingale measure Q.

Following Frühwirth and Sögner (2005), recovery rate will be fixed and default intensity λ will be estimated. The estimate of the default intensity will be denoted by $\hat{\lambda}_{I,t}$. The corresponding non linear estimation will be as follows:

$$\left(\hat{\lambda}_{I,t}\right) = \underset{\lambda}{\operatorname{argmin}}\, f_{\delta}(\lambda) \qquad (5.1.2.1.4)$$

where

$$f_{\delta}(\lambda) = \sum_{i\in I}[B_i^{obs}(t,T_i) - B_i(t,T_i)]^2 \qquad (5.1.2.1.5)$$

Checking for the poor identification and stability of the estimation is again required. For poor identification problem, the Hessian of $f_{\delta}(\lambda)$ should be computed The Hessian of the contrast function is as follows:

$$h_{11} = \sum_{i\in I}\sum_{t}^{T} 2[\frac{\partial B_i(t,T_i)}{\partial\lambda}]^2 \qquad (5.1.2.1.6)$$

First of the results is that h_{11} is stictly positive and the number of bonds. Poor identification can be met only at the maturity since $\exp\left(-\lambda(T-t)\right)$ is close to 1 for λ sufficiently small which makes sense since the probability of default is zero at maturity for any default intensity.

As a second step whether the minimization procedure is numerically stable should be checked. Therefore values are assigned to δ and λ initially and these values are called true values. Then $\widetilde{\lambda}_{i,t}$ the noisy default intensity is modeled as follows:

$$\widetilde{\lambda}_{i,t} = \lambda\exp\left(-0{,}5\mathrm{c}_{\lambda}^2\right)\exp\left(\mathrm{c}_{\lambda}\xi_{\lambda,\mathrm{i,t}}\right) \qquad (5.1.2.1.7)$$

where $\mathrm{c}_{\lambda} > 0$ is the distortion factor between the noisy default intensity and the Jarrow Turnbull default intensity and $\xi_{\lambda,\mathrm{i,t}}$ is standard normally distributed noise term. $\exp\left(-0{,}5\mathrm{c}_{\lambda}^2\right)$ is used to guarantee that the expectation of $\widetilde{\lambda}_{i,t}$ is λ.

$$E(\widetilde{\lambda}_{i,t}) = E(\lambda \exp(-0{,}5c_\lambda^2)\exp(c_\lambda \xi_{\lambda,i,t})) = \lambda \exp(-0{,}5c_\lambda^2)\, e\,\mathrm{xp}(-0{,}5c_\lambda^2) = \lambda \tag{5.1.2.1.8}$$

since $E(\lambda \exp(c_\lambda \xi_{\lambda,i,t})) = \exp(-0{,}5c_\lambda^2)$ (5.1.2.1.9)

when $\xi_{\lambda,i,t} \sim N(0,1)$

The simulated coupon bonds are used in the estimation procedure for fixed recovery rates and various default intensity starting valuesλ^{start} to obtain an estimate for the default intensity. Frühwirth and Sögner (2006) investigated the impact of the starting values of the default intensities on the default intensity estimate and concluded that this impact is minor, and indicated that the estimation procedure based on a fixed recovery rate is numerically stable.

5.2 Data

The data set used in this study consists of two parts one of which is the default free zero coupon bond prices and the other part is the default risky bond prices.

Default free bond prices comprise the daily closing prices over the period 19.06.2008 through 21.11.2008 consisting of 3365 observations and obtained from www.wsj.com.

Default risky bond data comprises 34 dollar denominated fixed-rate senior unsecured bonds without sinking fund provisions or embedded options. Issuers are banks and non-bank corporates from the rating classes AA, A and B. Thus the data is grouped into six clusters as AA bank bonds: i = 1,…,5, AA non bank bonds: i=6,…,11, A bank bonds : i=12,…,18, A non bank bonds: i=19,…,26 bank bonds: i=27,…,30, and B non bank bonds: i=31,…,34, with different maturities. The default risky data is obtained from Reuters Information System Database.

5.3 Results of the Default Intensity Estimation

The purpose of this part of the thesis is estimation of default intensity from empirical data by 5.1.1.16.

The default intensity parameter is estimated from the whole data by MATLAB according by making a cross section with respect to their ratings assuming that these ratings are correct.

The algorithm for the estimation of default intensity can be described as follows:

1- A certain type of a bond class with the same rating and type of corporation (with the discrimination of bank or non bank) is selected.
2- A trade date is fixed
3- For a bond $i \in I$, for the fixed date t, default free zero coupon bond price is computed by Svensson (1994) method.
4- The recovery rate is taken as 0,5 in consistence with Moody's report (2000).
5- The equation 5.1.2.1.5 computes the pricing error between the observed price and model price for observation date and for all bonds in the same rating and corporation class.
6- Finally equation 5.1.2.1.4 yields the default intensity estimate which makes the pricing error minimum.

The estimation procedure yielded the following results:

Table 5.3.1: results of the default intensity estimation

Rating and type	Estimated default intensity
AA rated bank bonds	$\hat{\lambda}_{AA,b}$=0,0047
AA rated non- bank bonds	$\hat{\lambda}_{AA,nb} = 0{,}0052$
A rated bank bonds	$\hat{\lambda}_{A,b}$=0,0057
A rated non-bank bonds	$\hat{\lambda}_{A,nb} = 0{,}0064$
B rated bank bonds	$\hat{\lambda}_{B,b} = 0{,}0136$
B rated non-bank bonds	$\hat{\lambda}_{B,nb} = 0{,}0289$

The results presented in Table 5.3.1 are consistent with respect to the rating classes, since as the rating falls, the default intensity is expected to increase. However, these results involve estimation errors due to the fixed recovery rate assumption.

CHAPTER 6

CONCLUSION

This thesis presents the Jarrow Turnbull (1995) model and the estimation methodology employed in the Frühwirth and Sögner's paper which used German data. In the first chapter, introduction is given which is followed by a literature survey in which structural and reduced form models are compared. In the third chapter, preliminaries for the corporate bond pricing issues are explained with the underlying Heath, Jarrow and Morton (1992) Model. Chapter 4 gave the details of the Jarrow Turnbull (1995) Model starting from the discrete time model extending to the continuous time model utilizing the analogy of the foreign exchange rate. In chapter 5, estimation methods are explained and performed based on the Svensson (1994) term structure procedure for the default-free term structure since for maturities smaller than 12 months spot rates are available in the form of rates as Treasury bills which are zero-coupon bonds while for longer maturities, zero coupon bonds are usually not available for sufficiently many maturities and in sufficiently large issues to be sufficiently liquid. Therefore spot rates are estimated from yields on coupon bonds (Svensson, 1994).

In order to find out the default risky bond price, it is shown that default free zero coupon bond price, default intensity and the recovery rate are sufficient. Thus the aim of this thesis is the estimation of the default free bond price and default intensity since Frühwirth and Sögner proposed analytically, by means of a simulation study and using empirical data, that it is not possible to jointly estimate the default intensity and the recovery rate implicitly and thus suggested estimating the default intensity conditionally on a fixed recovery rate drawn from empirical studies in literature. The joint estimation of the default risk parameters, default intensity and the recovery rate is proven to be numerically unstable and purely identified. Therefore the default intensity is estimated based on recovery rate and thus recovery rate is assumed to be fixed. The recovery rate estimates are available drawn from actual defaults provided in Moody's (1992), Altman and Kishore (1996), Standard & Poor's (2000) and Moody's (2000). These studies propose a recovery rate of (approximately) 50% for unsecured senior bonds. Estimated default intensities depend on the default-free term structure estimation since the term structure of default risky bonds involve default free zero coupon bond price.

Due to lack of Turkish corporate bond data, US corporate bonds, namely bank and non bank bonds form different rating classes and thereby US Treasury bill data are employed in the estimation procedure by means of MATLAB.

The estimated default intensities are proposed in Table 5.3.1 and they are consistent with the rating classes, thereby their risk classes, of the bonds employed in the estimation procedure.

For further study, pooling of data, which is shown to yield better estimates of the parameters with term structure models by De Munnik and Schotman (1994), can be investigated whether it can improve the estimates also with credit risk models.

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