COOPERATIVE INTERVAL GAMES

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ABSTRACT

COOPERATIVE INTERVAL GAMES

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Interval uncertainty affects our decision making activities on a daily basis making the data structure of intervals of real numbers more and more popular in theoretical models and related software applications. Natural questions for people or businesses that face interval uncertainty in their data when dealing with cooperation are how to form the coalitions and how to distribute the collective gains or costs. The theory of cooperative interval games is a suitable tool for answering these questions. In this thesis, the classical theory of cooperative games is extended to cooperative interval games. First, basic notions and facts from classical cooperative game theory and interval calculus are given. Then, the model of cooperative interval games is introduced and basic definitions are given. Solution concepts of selection-type and interval-type for cooperative interval games are intensively studied. Further, special classes of cooperative interval games like convex interval games and big boss interval games are introduced and various characterizations are given. Some economic and Operations Research situations such as airport, bankruptcy and sequencing with interval data and related interval games have been also studied. Finally, some algorithmic aspects related with the interval Shapley value and the interval core are considered.

Keywords: cooperative games, interval data, interval cores, uncertainty, convex games, big boss games, sequencing situations, bankruptcy rules, airport games, Operations Research

ÖZ

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Aralık belirsizliği, günlük bazda reel sayı aralıklarının veri yapılarını oluştururken teorik modellerde ve alakalı yazılım uygulamalarında gitgide popülerleşerek karar alma aktivitelerimizi etkilemektedir. İşbirliği ile ilgilenirken verileri aralık belirsizliğine dayanan şahıslar ve şirketler için doğal sorular, koalisyonların nasıl oluşacağı ve müşterek kazanç veya masrafların nasıl dağıtılacağıdır. İşbirliğine ait aralık oyunlarının teorisi bu soruları cevaplamak için uygun bir araçtır. Bu tezde, klasik işbirliğine ait oyun teorisi işbirliğine ait aralık oyunlarına genişletilmiştir. Önce klasik işbirliğine ait oyun teorisinin temel kavram ile unsurları ve aralık hesapları verilmiştir. Sonra işbirliğine ait aralık oyunlarının modeli tanıtılmış ve temel tanımları verilmiştir. İşbirliğine ait aralık oyunları için seçme tipli ve aralık tipli çözüm yöntemleri üzerinde yoğun olarak çalışılmıştır. Ayrıca, işbirliğine ait aralık oyunlarının konveks aralık oyunları ve büyük patron aralık oyunları gibi özel sınıfları tanıtılmış ve çeşitli nitelendirmeleri verilmiştir. Bunlara ek olarak, aralık verili havaalanı, iflas ve sıralama gibi bazı ekonomik ve işletme (yöneylem) araştırması durumları ve alakalı aralık oyunları çalışılmıştır. Son olarak, aralık Shapley değeri ve aralık çekirdeği ile ilgili bazı algoritmik bakış açıları ele alınmıştır.

Anahtar Kelimeler: İşbirliğine Ait Oyunlar, Aralık Verileri, Aralık Çekirdekleri, Belirsizlik, Konveks Oyunlar, Büyük Patron Oyunları, Sıralama Durumları, İflas Kuralları, Havaalanı Oyunları, İşletme (Yöneylem) Araştırması

In memory of my dear mother Selma Alparslan

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PREFACE

The answer to the question *can Mathematics or Operations Research model the complexity of nature and environment under the limitations of modern technology and in the presence of various societal problems?*, seems likely to be “yes”, but in the margins of our developing understanding only, in this sense: approximately, dynamically and, newly, as being in a game.

Cooperative game theory in coalitional form is a popular research area with many new developments in the last few years. In classical cooperative game theory payoffs to coalitions of players are known with certainty. However, interval uncertainty affects our decision making activities on a daily basis making the data structure of intervals of real numbers more and more popular in theoretical models and related software applications. There are many real-life situations where people or businesses face interval uncertainty in decision making regarding cooperation, i.e., they only know the smallest and the biggest values for potential rewards/costs. In other words, the agents are uncertain about their coalition payoffs. Situations with uncertain payoffs in which the agents cannot await the realizations of their coalition payoffs cannot be modeled according to classical game theory. A suitable game theoretic model to support decision making under interval uncertainty of coalition values is that of cooperative interval games. Cooperative interval games are an extension of that of cooperative games in coalitional form in case the worth of coalitions is affected by interval uncertainty. The model of cooperative interval games, firstly introduced in Branzei, Dimitrov and Tijs (2003) to handle bankruptcy situations where the estate is known with certainty while claims belong to known intervals of real numbers, fits all the situations where participants consider cooperation and know with certainty only the lower and upper bounds of all potential revenues or costs generated via cooperation.

In this thesis we present our recent contributions to the theory of cooperative interval games and its applications. The thesis is organized as follows.

In Chapter 1, first we motivate the model of cooperative interval games. Second, basic notions and facts from classical cooperative games that are used for the extension of cooperative interval games are established. Finally, we recall basic notions from interval calculus. Chapter 2 presents formally the model of cooperative interval games and gives basic definitions. It includes selection-based solution concepts based on Alparslan Gök, Miquel and Tijs (2009) and interval solution concepts based on Alparslan Gök, Branzei and Tijs (2008a,b). Also, a basic guide for handling interval solution concepts is provided, which is based on Branzei, Tijs and Alparslan Gök (2008b). In Chapters 3 and 4 interesting classes of cooperative interval games, namely, $\mathcal{I}$-balanced interval games and size monotonic interval games are introduced and studied. In Chapter 5, the focus is on convex interval games and their characterizations, which is based on Alparslan Gök, Branzei and Tijs (2008b) and Branzei, Tijs and Alparslan Gök (2008a). Chapter 6 is based on Alparslan Gök, Branzei and Tijs (2008c) and Branzei, Tijs and Alparslan Gök (2008a). Here, we deal with another interesting class of cooperative interval games called big boss interval games. Chapter 7 discusses some applications of cooperative interval games in economic and Operations Research (OR) situations. It is composed of mainly three parts. Airport interval games and their Shapley value is based on Alparslan Gök, Branzei and Tijs (2008d); Bankruptcy problems with interval uncertainty is based on Branzei and Alparslan Gök (2008) and Sequencing interval situations and related games is based on Alparslan Gök et al. (2008). Chapter 8 is devoted to some algorithmic aspects related with cooperative interval games. Finally, in Chapter 9 we conclude and suggest some topics for further research.

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CHAPTER 1

INTRODUCTION

1.1 MOTIVATION

Classical cooperative game theory deals with coalitions that coordinate their actions and pool their winnings. Natural questions for individuals or businesses when dealing with cooperation are: Which coalitions should form? How to distribute the collective gains or costs among the members of the formed coalition? Generally, the situations here are considered from a deterministic point of view. However, in most economical situations potential rewards or costs are not known precisely, but often it is possible to estimate intervals to which they belong. The theory of cooperative interval games can be a suitable tool for answering these questions. This thesis deals with this model of cooperative games. The model of cooperative interval games is an extension of that of cooperative games in coalitional form in case the worth of coalitions is affected by interval uncertainty. Many real life situations can be modeled in a natural way as cooperative interval games or extensions of them. For example, Drechsel and Kimms (2008) modeled as a cooperative interval game a lot sizing problem with uncertain demand. As in the classical model, the decision regarding players' cooperation and the division of the interval-type joint reward are based on solution concepts.

Cooperative interval games are introduced in Branzei, Dimitrov and Tijs (2003) to handle bankruptcy situations where the estate is known with certainty while claims belong to known bounded intervals of real numbers. They defined two Shapley-like values each of which associating with each game with interval data a vector of intervals, and studied their interrelations using the arithmetic of intervals (Moore (1966)), and inspired by the work of Yager and Kreinovich (2000). Methods of interval arithmetic and analysis (Moore (1979)) have played a key role for new models of games based on interval uncertainty. Two-person zero-sum non-cooperative games with interval strategies and interval payoff function are studied by Shashikhin (2004). Interval matrix games arising from situations where the payoffs vary within closed intervals for fixed strategies are introduced in Collins and Hu (2005). Carpente et al. (2008) considered games in strategic form and constructed related cooperative interval games similarly with the procedure used by von Neumann (1928) and von Neumann and Morgenstern (1944). An interesting motivating example for the model of cooperative interval games can be found in Bauso and Timmer (2006): a joint replenishment situation where each retailer faces a demand bounded by a minimum and a maximum value. Pulido, Sánchez-Soriano and Llorca (2002) and Pulido et al. (2008) consider special interval cooperative games arising from bankruptcy-like situations. Throughout the foregoing literature motivation from different points of view for studying interval games is provided.

Alparslan Gök, Miquel and Tijs (2009) considered cooperative interval games and looked at selections of such games which are classical cooperative games. Based on classical solutions on the selections such as the core and the Shapley value then they define solutions for the interval cooperative games. Also, a bankruptcy situation where the claims are certain but the available estate can vary within a closed and bounded interval is used to illustrate cores for two-person interval games. In Alparslan Gök, Branzei and Tijs (2008a), another approach is taken, where solutions are described with the aid of tuples of intervals, the focus being on interval cores and stable sets. Other interval solution concepts like the Shapley value and the Weber set are introduced on a special class of cooperative interval games in Alparslan Gök, Branzei and Tijs (2008b). First, these solution concepts are suitable tools to support decision-making regarding cooperation in situations with interval data. Second, when the realization of the worth of the grand coalition is known with certainty, an interval payoff vector generated by such solution concepts is transformed into a traditional payoff vector. The essential issue of the usefulness of interval solutions depending on how the vectors of intervals can be handled when the uncertainty regarding joint gains/costs is removed is studied in Branzei, Tijs and Alparslan Gök (2008b).

Classical convex games have many applications in economic and real-life situations. It is well-known that classical public good situations (Moulin (1988)), sequencing situations (Curiel, Pederzoli and Tijs (1989)) and bankruptcy situations (O'Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1987)) lead to convex games. However, there are many real-life situations in which people or businesses are uncertain about their coalition payoffs. Situations with uncertain payoffs in which the agents cannot await the realizations of their coalition payoffs cannot be modeled according to classical game theory. Several models that are useful to handle uncertain payoffs exist in the game theory literature. We refer here to chance-constrained games (Charnes and Granot (1973)), cooperative games with stochastic payoffs (Suijs et al. (1999)), cooperative games with random payoffs (Timmer, Borm and Tijs (2005)). In all these models, probability theory plays an important role. The class of classical big boss games (Muto et al. (1988)) has received much attention in cooperative game theory and various situations were modeled using such games. We refer here to information market situations (Muto, Potters and Tijs (1989)), information collecting situations (Branzei, Tijs and Timmer (2001a,b), Tijs, Timmer and Branzei (2006)) and holding situations (Tijs, Meca and López (2005)). In case such situations are described in terms of interval data the corresponding cooperative games are under restricting conditions big boss interval games. Convex interval games and big boss interval games are introduced and studied in Alparslan Gök, Branzei and Tijs (2008b,c). In Branzei, Tijs and Alparslan Gök (2008a) characterizations of convex interval games using the notions of superadditivity and exactness are considered, and characterizations of big boss interval games in terms of subadditivity and exactness are derived.

Cooperative interval games are a useful tool for modeling various economic and OR situations where payoffs for people or businesses are affected by interval uncertainty. For example, sealed bid second price auctions and flow situations with interval uncertainty are modeled by interval peer group games in Branzei, Mallozzi and Tijs (2008). In such situations decisions regarding cooperation as well as estimations of potential shares of achieved collective gains have to be made ex-ante, i.e., by taking into account all possible realizations which belong to intervals whose lower and upper bounds are known with certainty. We mention here minimum spanning tree networks (Montemanni (2006), Moretti et al. (2008)), management applications such as funds' allocation of firms among their divisions, cost allocation and/or surplus sharing in joint projects, sequencing situations, conflict resolution and bankruptcy situations, assignment of taxes, when there is interval uncertainty regarding the homogeneous good at stake. Other interesting applications for the model of cooperative interval games can be also found in literature.

Before closing this section we notice that in this thesis the rewards/costs taken into account are not random variables, but just closed and bounded intervals of real numbers with no probability distribution attached.

1.2 CLASSICAL COOPERATIVE GAME THEORY

In this section, we give some definitions and results concerning classical cooperative game theory needed in the thesis. For an extensive description of classical game cooperative theory see Tijs (2003) and Branzei, Dimitrov and Tijs (2005, 2008).

A *cooperative game* in *coalitional form* is an ordered pair $< N, v >$, where $N := \{1, 2, ..., n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. Often, we also refer to such a game as a *TU (transferable utility) game* and identify a game $< N, v >$ with its *characteristic function* v. In some situations, costs are considered instead of rewards. A cost game $< N, c >$ is a cooperative game, where N is the set of players, and $c : 2^N \to \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a real number, $c(S)$, which is the cost of the coalition S with $c(\emptyset) = 0$. The set G^N of coalitional games with player set N, equipped with the usual operators of addition and scalar multiplication of functions, forms a $(2^{|N|} - 1)$-dimensional linear space. A basis of this space is supplied by the unanimity games u_T (or $< N, u_T >$), $T \in 2^N \setminus \{\emptyset\}$, which are defined by

$$u_T(S) := \begin{cases} 1, & \text{if } T \subset S \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that for each $v \in G^N$ we have $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T$ with
$c_T = \sum_{S : S \subset T} (-1)^{|T|-|S|} v(S)$.

The interpretation of the unanimity game u_T is that a gain (or cost savings) of 1 can be obtained if and only if all players in coalition T are involved in cooperation.

The *dual T-unanimity game* u_T^* is defined by

$$u_T^*(S) := \begin{cases} 1, & T \cap S \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

A *multi-solution* is a multi-function $\mathcal{F} : G^N \twoheadrightarrow \mathbb{R}^n$ and a *one-point solution* is a map $f : G^N \to \mathbb{R}^n$.

A payoff vector $x \in \mathbb{R}^n$ is called an *imputation* for the game $< N, v >$ if

(i) x is *individually rational*, i.e., $x_i \geq v(\{i\})$ for all $i \in N$,

(ii) x is *efficient (Pareto optimal)*, i.e., $\sum_{i=1}^n x_i = v(N)$.

The set of imputations of $< N, v >$ is denoted by $I(v)$. Note that $I(v) = \emptyset$ if and only if $v(N) < \sum_{i \in N} v(\{i\})$.

The core (Gillies (1959)) of a game $< N, v >$ is the set

$$C(v) := \left\{ x \in I(v) | \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

Note that the core is a convex set. If $x \in C(v)$, then no coalition $S \neq N$ has any incentives to split off from the grand coalition if x is the proposed reward allocation in N, because the total amount $\sum_{i \in S} x_i$ allocated to S is not smaller than the amount $v(S)$ which the players can obtain by forming the subcoalition. If $C(v) \neq \emptyset$, then elements of $C(v)$ can easily be obtained, because the core is defined with the aid of a finite system of linear inequalities. The core is a polytope [1]. For a two-person game $< N, v >$, $I(v) = C(v)$.

A map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ is called a *balanced map* if $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N$. Here, e^S is the *characteristic vector* for coaliton S with

$$e_i^S := \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \in N \setminus S. \end{cases}$$

An n-person game $< N, v >$ is called a *balanced game* if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) v(S) \leq v(N)$.

The importance of this notion becomes clear by the following theorem proved by Bondareva (1963) and Shapley (1967). This theorem characterizes games with a non-empty core.

Theorem 1.2.1 *Let $< N, v >$ be an n-person game. Then the following two assertions are equivalent:*

(i) $C(v) \neq \emptyset$,

(ii) $< N, v >$ *is a balanced game.*

Now, we recall other subsets of imputations which are solution concepts for coalitional games: the dominance core (D-core) and stable sets. They are based on the dominance relation over vectors in $\mathbb{R}^n$.

Let $v \in G^N$, $x, y \in I(v)$, and $S \in 2^N \setminus \{\emptyset\}$. We say that x *dominates* y *via coalition* S, and denote it by $x \,\mathrm{dom}_S\, y$, if

[1] For details on polytope structure and convexity see Rockafellar (1970).

(i) $x_i > y_i$ for all $i \in S$,

(ii) $\sum_{i \in S} x_i \leq v(S)$.

Note that if (i) holds, then the payoff x is better than the payoff y for all members of S; condition (ii) guarantees that the payoff x is reachable for S.

Let $v \in G^N$, $x, y \in I(v)$. We say that x *dominates* y, and denote it by x dom y, if there is an $S \in 2^N \setminus \{\emptyset\}$ such that $x \operatorname{dom}_S y$. For $S \in 2^N \setminus \{\emptyset\}$ we denote by $D(S)$ the set of imputations which are dominated via S; note that players in S can successfully protest against any imputation in $D(S)$. An imputation x is called *undominated* if there does not exist y and a coalition S such that $y \operatorname{dom}_S x$. The *dominance core (D-core)* $DC(v)$ of a game $v \in G^N$ consists of all undominated elements in $I(v)$, i.e., it is the set $I(v) \setminus \cup_{S \in 2^N \setminus \{\emptyset\}} D(S)$.

For $v \in G^N$ and $A \subset I(v)$ we denote by $\operatorname{dom}(A)$ the *set consisting of all imputations that are dominated by some element in A*. Note that $DC(v) = I(v) \setminus \operatorname{dom}(I(v))$.

For $v \in G^N$ a subset K of $I(v)$ is called a *stable set* if the following conditions hold:

(i) (*Internal stability*) $K \cap \operatorname{dom}(K) = \emptyset$.

(ii) (*External stability*) $I(v) \setminus K \subset \operatorname{dom}(K)$.

Let $v \in G^N$. For each $i \in N$ and for each $S \in 2^N$ with $i \in S$, the *marginal contribution* of player i to the coalition S is $M_i(S, v) := v(S) - v(S \setminus \{i\})$.

Let $\Pi(N)$ be the set of all permutations $\sigma : N \to N$ of N.

The set $P^\sigma(i) := \left\{ r \in N | \sigma^{-1}(r) < \sigma^{-1}(i) \right\}$ consists of all predecessors of i with respect to the permutation σ.

Let $v \in G^N$ and $\sigma \in \Pi(N)$. The *marginal contribution vector* $m^\sigma(v) \in \mathbb{R}^n$ with respect to σ and v has the i-th coordinate the value $m_i^\sigma(v) := v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ for each $i \in N$.

The *Shapley value* (Shapley (1953)) $\phi(v)$ of a game $v \in G^N$ is the average of the marginal vectors of the game, i.e., $\phi(v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v)$.

This value associates to each n-person game one (payoff) vector in $\mathbb{R}^n$. It is proved that the Shapley value is the unique solution satisfying the properties of additivity, efficiency, anonymity and the dummy player property (see Theorem 61 in Branzei, Dimitrov and Tijs (2008)).

A player i is called a *dummy* in the game $< N, v >$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \in 2^{N \setminus \{i\}}$.

A *dummy player* is a player whose marginal contribution to any coalition is always equal to the worth of his/her own coalition. His/her Shapley value equals his/her own worth. For details about the properties of one-point solution concepts for cooperative games, we refer to Driessen (1988).

The Shapley value $\phi(u_T^*)$ of the *dual T-unanimity game* u_T^* is defined by

$$\phi_i(u_T^*) := \begin{cases} 1/|T|, & i \in T \\ 0, & i \in N \setminus T. \end{cases}$$

The *marginal vectors* of a two-person game $< N, v >$ are $m^{(12)}(v) = (v(\{1\}), v(\{1,2\}) - v(\{1\}))$ and $m^{(21)}(v) = (v(\{1,2\}) - v(\{2\}), v(\{2\}))$.

For a two-person game $< N, v >$ we have

$$\phi_i(v) = v(\{i\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2}, \; i \in \{1,2\}.$$

Note that for a two-person game $< N, v >$, the Shapley value is the standard solution which is in the middle of the core and the marginal vectors are the extreme (or extremal) points of the core whose average gives the Shapley value.

A game $v \in G^N$ is refered to as *additive* if $v(S \cup T) = v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$. An additive game $v \in G^N$ is determined by the vector $a = (v(\{1\}), \ldots, v(N)) \in \mathbb{R}^n$ since $v(S) = \sum_{i \in S} a_i$ for all $S \in 2^N$. Let $v_1, v_2 \in G^N$. The game v_2 is strategically equivalent to the game v_1 if there exist $k > 0$ and an additive game a such that $v_2(S) = kv_1(S) + \sum_{i \in S} a_i$ for all $S \in 2^N \setminus \{\emptyset\}$. The core is entitled *relative invariant with respect to strategic equivalence*: if $v_2 \in G^N$ is strategic equivalent to $v_1 \in G^N$, say $v_2 = kv_1 + a$, then $C(v_2) = kC(v_1) + a$. A game $< N, v >$ is called *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$; it is called *subadditive* if $v(S \cup T) \leq v(S) + v(T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$. In a superadditive game, it is advantageous for the players to cooperate. A two-person cooperative game $< N, v >$ is superadditive if and only if $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$ holds. Note that a two-person cooperative game $< N, v >$ is superadditive if and only if the game is balanced.

A game $v \in G^N$ is called *convex (or supermodular)* if and only if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for each $S, T \in 2^N$; it is called *concave (or submodular)* if and only if $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$ for all $S, T \in 2^N$. The family of all convex games with player set N is denoted by CG^N. Each convex (concave) game is also superadditive (subadditive). In the following, we give characterizations of classical convex games.

Theorem 1.2.2 *(Theorem 4.9 in Branzei, Dimitrov and Tijs (2005)) Let $v \in G^N$. The following five assertions are equivalent:*

(i) $< N, v >$ *is convex.*

(ii) *For all $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$ we have*

$$v(S_1 \cup U) - v(S_1) \leq v(S_2 \cup U) - v(S_2).$$

(iii) *For all $S_1, S_2 \in 2^N$ and $i \in N$ such that $S_1 \subset S_2 \subset N \setminus \{i\}$ we have*

$$v(S_1 \cup \{i\}) - v(S_1) \leq v(S_2 \cup \{i\}) - v(S_2).$$

(iv) *Each marginal vector $m^\sigma(v)$ of the game v with respect to the permutation σ belongs to the core $C(v)$.*

(v) *$W(v) = C(v)$, where $W(v)$ is the Weber set (Weber (1988)) of v which is defined as the convex hull of the marginal vectors of v.*

Convex games are balanced games. Notice that Theorem 1.2.2 implies that convex games have a nonempty core. On the class of convex games, solution concepts have nice properties. We recall that the Shapley value of a convex (concave) game belongs to the core of the game and the core is the unique stable set of the game. Also, the core is an additive map on the class of convex games (Dragan, Potters and Tijs (1989)).
For a game $v \in G^N$ and a coalition $T \in 2^N \setminus \{\emptyset\}$, the *subgame* with player set T, (T, v_T), is the game v_T defined by $v_T(S) := v(S)$ for all $S \in 2^T$. In the sequel, we denote such subgames by $< T, v >$. For $T \subset N$, the *marginal* game of v based on T is defined by $v^T(S) := v(S \cup T) - v(T)$ for each $S \subset N \setminus T$. A game $< N, v >$ is called *exact* if for each $S \in 2^N \setminus \{\emptyset\}$ there is an $x \in C(v)$ with $\sum_{i \in S} x_i = v(S)$. It is well-known that:

(i) subgames of convex games are also convex (and subgames of concave games are also concave);

(ii) convex games are (total) exact games and total exact games (i.e., games whose all subgames are also exact) are convex (Biswas et al. (1999), Azrieli and Lehrer (2007));

(iii) games whose marginal games are all superadditive are convex (Branzei, Dimitrov and Tijs (2004), Martinez-Legaz (1997, 2006)).

For a traditional cooperative game $< N, v >$, Biswas et al. (1999) proved that the game is convex if and only if each subgame $< S, v >$, with $S \subset N$, is an exact game. In the sequel, we prove in Theorem 5.2.10 that a similar characterization holds true in the interval data setting.
A game $v \in G^N$ is called *totally balanced* if all its subgames are balanced. Equivalently, the game v is totally balanced if $C(v_T) \neq \emptyset$ for all $T \in 2^N \setminus \{\emptyset\}$. The class of totally balanced games includes the class of games with a *population monotonic allocation scheme (pmas)* (Sprumont (1990)).
Let $v \in G^N$. A scheme $a = (a_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ of real numbers is a pmas of v if

(i) $\sum_{i \in S} a_{iS} = v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $a_{iS} \leq a_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

It is known that for $v \in CG^N$ the (total) Shapley value and (total) Dutta-Ray solution (Dutta and Ray (1989)) generate population monotonic allocation schemes.
Let $v \in G^N$. An imputation $b \in I(v)$ is *pmas extendable* if there exists a pmas $a = (a_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ such that $a_{iN} = b_i$ for each $i \in N$.
A game $< N, v >$ is called *a big boss game with n as a big boss* (Muto et al. (1988), Tijs (1990)) if the following conditions are satisfied:

(i) $v \in G^N$ is monotonic, i.e., $v(S) \leq v(T)$ if for each $S, T \in 2^N$ with $S \subset T$.

(ii) $v(S) = 0$ if $n \notin S$.

(iii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\}))$ for all S, T with $n \in S$.

Definition 1.2.1 *Let $v \in G^N$ and $n \in N$. Then, this game is a total big boss game with n as a big boss, if the following conditions are satisfied:*

(i) *$v \in G^N$ is monotonic, i.e., $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subset T$;*

(ii) *$v(S) = 0$ if $n \notin S$;*

(iii) *$v(T) - v(S) \geq \sum_{i \in T \setminus S} (v(T) - v(T \setminus \{i\}))$ for all S, T with $n \in S \subset T$.*

Note that big boss games form a cone in G^N. Further, a game $< N, v >$ is a total big boss game with big boss n if and only if $< T, v >$ is a big boss game for each $T \in 2^N$ with $n \in T$.

In this thesis, we only consider total big boss games and call them shortly big boss games. We denote by P_n the set $\{S \subset N | n \in S\}$ of all coalitions containing the big boss.

Let $v \in G^N$ be a big boss game with n as a big boss. We call a scheme $a = (a_{iS})_{i \in S, S \in P_n}$ an *allocation scheme* for v if $(a_{iS})_{i \in S}$ is a core element of the subgame $< S, v >$ for each coalition $S \in P_n$. Such an allocation scheme $a = (a_{iS})_{i \in S, S \in P_n}$ is called a *bi-monotonic allocation scheme (bi-mas)* (Branzei, Tijs and Timmer (2001b)) for v if for all $S, T \in P_n$ with $S \subset T$ we have $a_{iS} \geq a_{iT}$ for all $i \in S \setminus \{n\}$, and $a_{nS} \leq a_{nT}$.

Let $v \in G^N$ be a big boss game with n as a big boss. An imputation $b \in I(v)$ is bi-mas extendable if there exists a bi-mas $a = (a_{iS})_{i \in S, S \in P_n}$ such that $a_{iN} = b_i$ for each $i \in N$.

The next proposition and the definition of suitable marginal games for big boss games are obtained from Propositions 2 and 3 in Branzei, Dimitrov and Tijs (2006) with $\{n\}$ in the role of C.

Proposition 1.2.3 *Let* $< N, v > \in MV^{N,\{n\}}$*. Then the following assertions are equivalent:*

(i) $< N, v >$ *is a (total) big boss game with big boss* n*;*

(ii) $< N \setminus \{n\}, v^{\{n\}} >$ *is a concave game;*

(iii) $< N \setminus (\{n\} \cup T), \left(v^{\{n\}}\right)^T >$ *is a subadditive game for each* $T \subset N \setminus \{n\}$*;*

(iv) $< N \setminus (\{n\} \cup T), v^{\{n\} \cup T} >$ *is a subadditive game for each* $T \subset N \setminus \{n\}$*.*

Here, $MV^{N,\{n\}}$ is the set of all monotonic games on N satisfying the big boss property with respect to the big boss n. Given a game $< N, v > \in MV^{N,\{n\}}$ and a coalition $T \in 2^{N \setminus \{n\}}$, the *n-based T-marginal game* $(v^{\{n\}})^T : 2^{N \setminus T} \to \mathbb{R}$ is defined by

$$(v^{\{n\}})^T(S) = v(S \cup T \cup \{n\}) - v(T \cup \{n\})$$

for each $S \subset N \setminus T$.

We notice that since here the set of players is very crucial, we refer to the game $v \in G^N$ as $< N, v >$ and to its subgames as $< T, v >$ for each $T \subset N$. Moreover, we accordingly adjust the notation for the used notions that were defined previously.

Let $v \in G^N$. For each $i \in N$, the *marginal contribution of player i to the grand coalition N* is $M_i(N, v) := v(N) - v(N \setminus \{i\})$.

The core $C(N, v)$ of a traditional big boss game is always nonempty and equal to

$$\{x \in I(N, v) | 0 \leq x_i \leq M_i(N, v) \text{ for each } i \in N \setminus \{n\}\}.$$

For a (big boss) subgame $< T, v >$ (with n as a big boss) of $v \in G^N$ two particular elements of its core are the *big boss point* $B(T, v)$ defined by

$$B_j(T,v) := \begin{cases} 0, & \text{if } j \in T \setminus \{n\} \\ v(T), & \text{if } j = n, \end{cases}$$

and the *union point* $U(T, v)$ defined by

$$U_j(T,v) := \begin{cases} M_j(T,v), & \text{if } j \in T \setminus \{n\} \\ v(T) - \sum_{i \in T \setminus \{n\}} M_i(T,v), & \text{if } j = n. \end{cases}$$

A game $v \in G^N$ is called *quasi-balanced* if $m(N, v) \leq M(N, v)$ and $\sum_{i=1}^n m_i(N, v) \leq v(N) \leq \sum_{i=1}^n M_i(N, v)$, where for each $i \in N$ we put

$$m_i(N,v) := \max\{R(S,i) | i \in S, S \subset N\}$$

with

$$R(S,i) := v(S) - \sum_{j \in S \setminus \{i\}} M_j(N,v).$$

The *τ-value* or *compromise value* (Tijs (1981)) is defined on the class of quasi-balanced games. Specifically, for each quasi-balanced game $< N, v >$ its τ-value, $\tau(N, v)$, is a feasible compromise between the upper vector $M(N, v) := (M_i(N, v))_{i \in N}$ and the lower vector $m(N, v) := (m_i(N, v))_{i \in N}$ of a game satisfying $\sum_{i \in N} \tau_i(N, v) = v(N)$.
For a big boss game with n as a big boss the *τ-value* of v is given by

$$\tau(N,v) := (\frac{1}{2}M_1(N,v), \frac{1}{2}M_2(N,v), \ldots, v(N) - \sum_{i \in N \setminus \{n\}} \frac{1}{2}M_i(N,v)).$$

Now, let $\sigma = (\sigma(1), \sigma(2) \ldots, \sigma(k), \sigma(k+1), \ldots, \sigma(n))$ be an ordering of the players in $N = \{1, 2, \ldots, n\}$. The *lexicographic maximum* of the core $C(N, v)$ of a balanced game $< N, v >$ with respect to σ is denoted by $L^\sigma(N, v)$. Then, the *average lexicographic value* $AL(N, v)$ (Tijs (2005)) of $v \in G^N$ is the average of all lexicographically maximal vectors of the core of the game, i.e., $AL(N, v) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(N, v)$. For a big boss game with n as a big boss, $L^\sigma(N, v)$ is equal to

$$L^\sigma_{\sigma(i)}(N,v) := \begin{cases} M_{\sigma(i)}(N,v), & i < k \\ 0, & i > k \\ v(N) - \sum_{i=1}^{k-1} M(N,v), & i = k, \end{cases}$$

if $\sigma(k) = n$.
It is known that the AL-value coincides with the τ-value on the class of (total) big boss games (Tijs (2005)).

1.3 INTERVAL CALCULUS

In this thesis, tools of interval calculus play an important role. In this section we give some notions that we have used along the thesis.

An *interval* is a closed and bounded set of real numbers, $\left[\underline{I}, \overline{I}\right] = \left\{x \in \mathbb{R} | \underline{I} \leq x \leq \overline{I}\right\}$ for any $\underline{I}, \overline{I} \in \mathbb{R}$ with $\underline{I} \leq \overline{I}$.

Let $I(\mathbb{R})$ be the set of all closed and bounded intervals in $\mathbb{R}$ and $I(\mathbb{R}_+)$ be the set of all closed and bounded intervals in $\mathbb{R}_+$. We assume that 0 is an element of $\mathbb{R}_+$.

We define an *addition* between (ordered) pairs which are elements of $I(\mathbb{R})$, and a *multiplication* of an interval with a positive scalar. Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

(i) $+ : I(\mathbb{R}) \times I(\mathbb{R}) \to I(\mathbb{R})$ with $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$;

(ii) $\cdot : \mathbb{R}_+ \times I(\mathbb{R}) \to I(\mathbb{R})$ with $\alpha I = \left[\alpha \underline{I}, \alpha \overline{I}\right]$.

By (i) and (ii), $I(\mathbb{R})$ has a cone structure.

Let $I, J, K \in I(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}_+$. Then,

1. $I + J = J + I$;
2. $(I + J) + K = I + (J + K)$;
3. $I + [0, 0] = I$;
4. λI is an interval;
5. $(\lambda\mu)I = \lambda(\mu I)$;
6. $(\lambda + \mu)I = \lambda I + \mu I$;
7. $\lambda(I + J) = \lambda I + \lambda J$;
8. $1 \cdot I = I$.

Let I and $J \in I(\mathbb{R})$. Then, the subtraction operator (Moore (1979)) is defined by $I \ominus J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$.

Example 1.3.1 $[6,8]\ominus[2,5]=[1,6]$ *and* $[2,5]\ominus[6,8]=[-6,-1]$.

Along this thesis, in the context of vector notation we use a coordinate-wise subtraction operator (Alparslan Gök, Branzei and Tijs (2008b)). In the sequel, we define a *subtraction* between (ordered) pairs (I,J) which are elements of the set $D:=\{(I,J)\in I(\mathbb{R})\times I(\mathbb{R})\,|\,|I|\geq|J|\}$. Let $(I,J)\in D$. Then,

$-:D\to I(\mathbb{R})$ with $I-J=[\underline{I}-\underline{J},\overline{I}-\overline{J}]$.

Notice that if we make a comparison with Example 1.3.1, then in our case $[6,8]-[2,5]$ is not defined. But, $[2,5]-[6,8]$ is defined.

Note that $I-J\in I(\mathbb{R})$ and that $J+(I-J)=I$. Note also that $\underline{I}-\underline{J}\leq\overline{I}-\overline{J}$ and $|I+J|\leq|I|+|J|$.

We define a *multiplication* between (ordered) pairs which are elements of $I(\mathbb{R}_+)$. Let $I,J\in I(\mathbb{R}_+)$. Then,

$\cdot:I(\mathbb{R}_+)\times I(\mathbb{R}_+)\to I(\mathbb{R}_+)$ with $I\cdot J=[\underline{I}\,\underline{J},\overline{I}\,\overline{J}]$.

We define a *division* between (ordered) pairs which are elements of the set

$Q:=\left\{(I,J)\in I(\mathbb{R}_+)\times I(\mathbb{R}_+\setminus\{0\})|\underline{I}\overline{J}\leq\overline{I}\underline{J}\right\}$.

Let $(I,J)\in Q$. Then,

$\div:Q\to I(\mathbb{R}_+)$ with $\frac{I}{J}=[\frac{\underline{I}}{\underline{J}},\frac{\overline{I}}{\overline{J}}]$.

Note that $\frac{I}{J}$ is defined if there is an interval K such that $I=J\cdot K$. Notice that $\frac{[1,1]}{[2,3]}$ is undefined, but $\frac{[2,3]}{[1,1]}$ is defined.

Let I, J and $K\in I(\mathbb{R}_+)$. Then,

1. $I\cdot[1,1]=I$;
2. $I\cdot J=J\cdot I$;
3. $I\cdot[0,0]=[0,0]$;
4. $(I\cdot J)\cdot K=I\cdot(J\cdot K)$;
5. $(I+J)\cdot K=(I\cdot K)+(J\cdot K)$.

Let $I,J\in I(\mathbb{R})$. We say that I is *left to* J, denoted by $I\underline{\prec}J$, if for each $a\in I$ and for each $b\in J$, $a\leq b$, and we say that I is *weakly better* than J, which we denote by $I\succcurlyeq J$, if and only if $\underline{I}\geq\underline{J}$ and $\overline{I}\geq\overline{J}$. Note that in case $I\succcurlyeq J$, then for each $a\in J$ there exists $b\in I$ such that $a\leq b$, and for each $b\in I$ there exists $a\in J$ such that $a\leq b$. We say that I is *better* than J, which we denote by $I\succ J$, if and only if $I\succcurlyeq J$ and $I\neq J$. We also use the reverse notation

$I \preccurlyeq J$, if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$, and the notation $I \prec J$, if and only if $I \preccurlyeq J$ and $I \neq J$.
Note that $I \succcurlyeq J$ does not imply $|I| \geq |J|$; e.g., $[1, 1] \succcurlyeq [0, 1]$.
Note also that $I \succcurlyeq J$, $J \succcurlyeq K$ implies $I \succcurlyeq K$ (transitivity) and that $I \succcurlyeq J$, $J \succcurlyeq I$ implies $I = J$.
Let $I, J \in I(\mathbb{R})$. We say that I and J are *disjoint* if $I \cap J = \emptyset$. For example, the intervals $[1, 3]$ and $[4, 7]$ are disjoint.
Let $I, J \in I(\mathbb{R})$. We define the *minimum* of the two intervals, $I \wedge J$, by $I \wedge J = I$ if $I \preccurlyeq J$, and their *maximum*, $I \vee J$, by $I \vee J = J$ if $I \preccurlyeq J$.
Note that $I \wedge (J - K) = (I \wedge J) - (I \wedge K)$, $\underline{I} \wedge \underline{J} \leq \overline{I} \wedge \overline{J}$, $\underline{I} \wedge \underline{J} \leq \underline{I} \leq \overline{I}$ and $\underline{I} \wedge \underline{J} \leq \underline{J} \leq \overline{J}$.
In general, let $I_1, \ldots, I_k \in I(\mathbb{R})$. Suppose that $I_j \succcurlyeq I_r$ for each $r \in \{1, \ldots, k\}$. Then, we say that $I_j := \max\{I_1, \ldots, I_k\}$. If $I_s \preccurlyeq I_r$ for each $r \in \{1, \ldots, k\}$, then $I_s := \min\{I_1, \ldots, I_k\}$.
For example, let $I_1 = [0, 1]$, $I_2 = [-1, 2]$ and $I_3 = [3, 5]$. Then, $I_3 = \max\{I_1, I_2, I_3\}$, whereas $\max\{I_1, I_2\}$ does not exist. Similarly, $I_2 = \min\{I_2, I_3\}$, but $\min\{I_1, I_2, I_3\}$ does not exist.
In this thesis, n-tuples of intervals $I = (I_1, I_2, \ldots, I_n)$ where $I_i \in I(\mathbb{R})$ for each $i \in N = \{1, 2, \ldots, n\}$, will play a key role. For further use we denote by $I(\mathbb{R})^N$ the set of all n-dimensional vectors whose components are elements in $I(\mathbb{R})$. Let $I = (I_1, I_2, \ldots, I_n)$, $J = (J_1, J_2, \ldots, J_n) \in I(\mathbb{R})^N$ and $\alpha \in \mathbb{R}_+$. Then, $I(\mathbb{R})^N$ has a cone structure with respect to addition and multiplication with a positive scalar:

(i) $I + J = (I_1 + J_1, I_2 + J_2, \ldots, I_n + J_n)$;

(ii) $\alpha I = (\alpha I_1, \alpha I_2, \ldots, \alpha I_n)$.

In the following, some properties of intervals are introduced. The proofs of the relations below are straightforward.

- Let $I, J \in I(\mathbb{R})^N$. Then $I_i \succcurlyeq J_i$ implies $\sum_{i=1}^{n} I_i \succcurlyeq \sum_{i=1}^{n} J_i$.
- Let $A, B, C, D \in I(\mathbb{R})$, $|B| \leq |A|$ and $|D| \leq |C|$. Then, $A - B \preccurlyeq C - D$ if and only if $A + D \preccurlyeq C + B$.
- Let $A, B, C \in I(\mathbb{R})$ with $|B| \leq |A|$ and $|C| \leq |A|$. Then, $B \succcurlyeq C$ implies $A - B \preccurlyeq A - C$.
- Let $I, J \in I(\mathbb{R})$ and let $|I| \geq |J|$. Then, $\underline{I - J} = \underline{I} - \underline{J}$ and $\overline{I - J} = \overline{I} - \overline{J}$.
- Let $I, J, K \in I(\mathbb{R})$ such that $I = J + K$. Then, $|I| = |J| + |K|$.

Next we introduce the *square operator* (Alparslan Gök, Branzei and Tijs (2008b)), which assigns to each pair $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ with $a \leq b$ an element of $I(\mathbb{R})^N$. For some classical solutions for TU-games one can with the aid of this square operator define a corresponding square solution on suitable classes of interval games.

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ with $a \leq b$. Then, a and b determine a hypercube $\mathcal{H} = \{x \in \mathbb{R}^n | a_i \leq x_i \leq b_i \text{ for each } i \in \{1, \ldots, n\}\}$.

We denote by $a \Box b$ the vector $(I_1, \ldots, I_n) \in I(\mathbb{R})^N$ generated by the pair $(a, b) \in \mathbb{R}^n$, $a \leq b$ with $I_i = [a_i, b_i]$ for each $i \in \{1, \ldots, n\}$. Let $A, B \subset \mathbb{R}^n$. Then, we denote by $A \Box B$ the subset of $I(\mathbb{R})^N$ defined by $A \Box B := \{a \Box b | a \in A, b \in B, a \leq b\}$.

To sum up, in this chapter, first, we have tried to give an answer to the question: Why is the class of cooperative interval games important? In this thesis, classical cooperative game theory and the arithmetic of intervals play an important role since the model of cooperative interval games is an extension of cooperative games in coalitional form. So, secondly, basic definitions and useful results from the theory of classical cooperative games were given. Finally, preliminaries from basic interval calculus became established. In the following chapter, we will intensively study the model and the solution concepts for the class of cooperative interval games which is the skeleton of this pioneering work.

CHAPTER 2

COOPERATIVE INTERVAL GAMES

2.1 MODEL, EXAMPLES, BASIC DEFINITIONS

In this section, the model of the cooperative interval games and basic definitions will be given. A *cooperative n-person* interval game in coalitional form is an ordered pair $< N, w >$, where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the *characteristic function* which assigns to each coalition $S \in 2^N$ a closed interval $w(S) \in I(\mathbb{R})$, such that $w(\emptyset) = [0, 0]$. For each $S \in 2^N$, the *worth set* (or *worth interval*) $w(S)$ of the coalition S in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the lower bound and $\overline{w}(S)$ is the upper bound of $w(S)$. In other words, $\underline{w}(S)$ is the minimal reward which coalition S could receive on its own and $\overline{w}(S)$ is the maximal reward which coalition S could get. The family of all interval games with player set N is denoted by IG^N. Note that if all the worth intervals are *degenerate intervals*, i.e., $\underline{w}(S) = \overline{w}(S)$, then the interval game $< N, w >$ corresponds to the classical cooperative game $< N, v >$ where $v(S) = \underline{w}(S)$. This means that traditional cooperative games can in a natural way be embedded into the class of cooperative interval games. Given a game $w \in IG^N$ and a coalition $\{1, \ldots, k\} \subset N$, we will often write $w(i, \ldots, k)$ instead of $w(\{i, \ldots, k\})$.

Example 2.1.1 *(Interval glove game) Let $N = \{1, 2, 3\}$ consisting of two disjoint subsets L and R. The members of L possess each one left-hand glove, the members of R one right-hand glove. A single glove is worth nothing, a right-left pair of gloves is worth between 10 and 20 Euros. In case $L = \{1, 2\}$ this situation can be modeled as a three-person interval game with $w(1, 3) = w(2, 3) = w(1, 2, 3) = [10, 20]$ and $w(S) = [0, 0]$, otherwise.*

Example 2.1.2 *(Landlord peasants game) Let us consider a production economy with one landlord and many peasants. Let* $N = \{1, 2, \ldots, n\}$ *be the player set, where n is the landlord who cannot produce anything alone, and* $1, 2, \ldots, n-1$ *are landless peasants.*
Let $f : [0, n-1] \to I(\mathbb{R})$ *be the production function with interval data, where* $f(s)$ *is the interval reward* $[f_1(s), f_2(s)] \succcurlyeq [0, 0]$ *if s peasants are hired by the landlord, where* $f(0) = [0, 0]$*,* f_1 *and* $f_2 - f_1$ *are concave with* $f_2 - f_1 \geq 0$*. This situation corresponds to an interval game* $< N, w >$*, where* $N = \{1, 2, \ldots, n\}$ *and the characteristic function is given by*

$$w(S) := \begin{cases} [0, 0], & n \notin S \\ f(|S| - 1), & n \in S. \end{cases}$$

In this thesis, some classical TU-games associated with an interval game $w \in IG^N$ will play a key role, namely the *border* games $< N, \underline{w} >$, $< N, \overline{w} >$ and the *length* game $< N, |w| >$, where $|w|(S) := \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$.
Let $J \in I(\mathbb{R})$ with $J \succcurlyeq [0, 0]$ and let $T \in 2^N \setminus \{\emptyset\}$. The *unanimity interval game* based on J and T is defined by

$$u_{T,J}(S) := \begin{cases} J, & T \subset S \\ [0, 0], & \text{otherwise,} \end{cases}$$

for each $S \in 2^N$.
For a game $w \in IG^N$ and a coalition $S \in 2^N \setminus \{\emptyset\}$, the *subgame* with player set T is the game w_T defined by $w_T(S) := w(S)$ for all $S \in 2^T$. So, w_T is the restriction of w to the set 2^T. We refer to such subgames by $< T, w >$.
We say that a game $< N, w >$ is *supermodular* if

$$w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N, \quad (2.1.1)$$

and a game $< N, w >$ is called *submodular* if

$$w(S) + w(T) \succcurlyeq w(S \cup T) + w(S \cap T) \text{ for all } S, T \in 2^N. \quad (2.1.2)$$

A game $w \in IG^N$ is said to be *superadditive* if for all $S, T \subset N$ with $S \cap T = \emptyset$ the following two conditions hold:

$$w(S \cup T) \succcurlyeq w(S) + w(T); \quad (2.1.3)$$

$$|w|(S \cup T) \geq |w|(S) + |w|(T);$$

it is called *subadditive* if for all $S, T \subset N$ with $S \cap T = \emptyset$, the following two conditions are satisfied:

$$w(S \cup T) \preccurlyeq w(S) + w(T);$$

$$|w|(S \cup T) \leq |w|(S) + |w|(T).$$

Next we give the arithmetics of interval games.

For $w_1, w_2 \in IG^N$ we say that $w_1 \preccurlyeq w_2$ if $w_1(S) \preccurlyeq w_2(S)$, for each $S \in 2^N$.

For $w_1, w_2 \in IG^N$ and $\lambda \in \mathbb{R}_+$ we define $< N, w_1 + w_2 >$ and $< N, \lambda w >$ by $(w_1 + w_2)(S) = w_1(S) + w_2(S)$ and $(\lambda w)(S) = \lambda \cdot w(S)$ for each $S \in 2^N$. So, we conclude that IG^N endowed with $\preccurlyeq$ is a *partially ordered set* and has a *cone structure* with respect to addition and multiplication with non-negative scalars, as described above. For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) := w_1(S) - w_2(S)$.

2.2 SELECTION-BASED SOLUTION CONCEPTS

This section is based on the paper Alparslan Gök, Miquel and Tijs (2009). Here, the notion of a selection of an interval game is the building block of the theory.

Let $< N, w >$ be an interval game, then $v : 2^N \to \mathbb{R}$ is called a *selection* of w if $v(S) \in w(S)$ for each $S \in 2^N$. We denote the set of selections of w by $Sel(w)$.

Next we define solution concepts for interval games which are based on selections.

The imputation set of an interval game $< N, w >$ is defined by

$$I(w) := \cup \{I(v) | v \in Sel(w)\}.$$

The *core set* of an interval game $< N, w >$ is defined by

$$C(w) := \cup \{C(v) | v \in Sel(w)\}.$$

We see directly that $C(w) \neq \emptyset$ if and only if there exists a $v \in Sel(w)$ with $C(v) \neq \emptyset$.

If all the worth intervals of an interval game $w \in IG^N$ are degenerate intervals, then

$I(w) = I(\underline{w}) = I(\overline{w})$ and $C(w) = C(\underline{w}) = C(\overline{w})$.

Note that $v(S) \in w(S)$ is a real number, but $w(S) = [\underline{w}(S), \underline{w}(S)]$ is a degenerate interval which is a set consisting of one element.

An interval game $< N, w >$ is *strongly balanced* if for each balanced map λ it holds that $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\overline{w}(S) \leq \underline{w}(N)$. The family of all strongly balanced interval games with player set N is denoted by BIG^N.

Proposition 2.2.1 *Let $< N, w >$ be an interval game. Then, the following three statements are equivalent:*

(i) *For each $v \in Sel(w)$ the game $< N, v >$ is balanced.*

(ii) *For each $v \in Sel(w)$, $C(v) \neq \emptyset$.*

(iii) *The interval game $< N, w >$ is strongly balanced.*

Proof. $(i) \Leftrightarrow (ii)$ follows from Theorem 1.2.1.
$(i) \Leftrightarrow (iii)$ follows using the inequalities $\underline{w}(N) \leq v(N) \leq \overline{w}(N)$ and
$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\underline{w}(S) \leq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)v(S) \leq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\overline{w}(S)$ for each balanced map λ. ■

It follows from Proposition 2.2.1 that for a strongly balanced game $< N, w >$, $C(w) \neq \emptyset$ since for all $v \in Sel(w)$, $C(v) \neq \emptyset$.
We call an interval game $< N, w >$ *strongly unbalanced*, if there exists a balanced map λ such that $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\underline{w}(S) > \overline{w}(N)$. Then, $C(v) = \emptyset$ for all $v \in Sel(w)$, which implies that $C(w) = \emptyset$.
If all the worth intervals of an interval game $< N, w >$ are degenerate intervals, then strongly balancedness corresponds to balancedness and strongly unbalancedness corresponds to unbalancedness in a classical cooperative game $< N, v >$.
Note that strongly balancedness means that for all $v \in Sel(w)$, $< N, v >$ has a nonempty core, because for all λ

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)v(S) \leq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\overline{w}(S) \leq \underline{w}(N) \leq v(N).$$

Proposition 2.2.2 *Let $v^0(S) = \overline{w}(S)$ for all $S \in 2^N \setminus \{\emptyset\}$, $v^0(N) = \underline{w}(N)$. Then, all selections are balanced if and only if v^0 is balanced.*

Proof.

(i) Suppose that each $v \in Sel(w)$ is balanced. Then, trivially v^0 is balanced.

(ii) Suppose that v^0 is balanced. Take $v \in Sel(w)$ and a balanced map λ. We have to prove that $\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)v(S) \leq v(N)$. Indeed, it holds

$$\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)v(S) \leq \sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)v^0(S) \leq v^0(N) = \underline{w}(N) \leq v(N),$$

where the second inequality follows from the fact that v^0 is balanced. ■

We call $w \in IG^N$ *weakly balanced* if there is at least one selection $v \in Sel(w)$ which is balanced. Let $v^1(S) = \underline{w}(S)$ and $v^1(N) = \overline{w}(N)$. Then, it is obvious that $< N, w >$ is weakly balanced if and only if v^1 is balanced.

The rest of this section deals with two-person interval games. We start with balancedness and related topics.

Let $< N, w >$ be a two-person interval game. Then, we define:

(i) the *pre-imputation set*

$$I^*(w) := \left\{x \in \mathbb{R}^2 | x_1 + x_2 \in w(1,2)\right\},$$

(ii) the *imputation set*

$$I(w) := \left\{x \in \mathbb{R}^2 | x_1 \geq \underline{w}(1), x_2 \geq \underline{w}(2), x_1 + x_2 \in w(1,2)\right\},$$

(iii) the *mini-core set*

$$MC(w) := \left\{x \in \mathbb{R}^2 | x_1 \geq \overline{w}(1), x_2 \geq \overline{w}(2), x_1 + x_2 \in w(1,2)\right\},$$

(iv) the *core set*

$$C(w) := \left\{x \in \mathbb{R}^2 | x_1 \geq \underline{w}(1), x_2 \geq \underline{w}(2), x_1 + x_2 \in w(1,2)\right\}.$$

Notice that for two-person interval games the imputation set and the core set are equal. Moreover, if an interval game is strongly balanced, then its mini-core set is nonempty and it is a subset of the core set of the game.

The next example is intended to give insight into the core set and mini-core set of a two-person (strongly balanced) game $< N, w >$.

Example 2.2.1 *Let $N = \{1, 2\}$, $w \in IG^{\{1,2\}}$ such that*

$$w(\emptyset) = [0, 0],\ w(1) = [1, 3],\ w(2) = [2, 5],\ w(1, 2) = [10, 12].$$

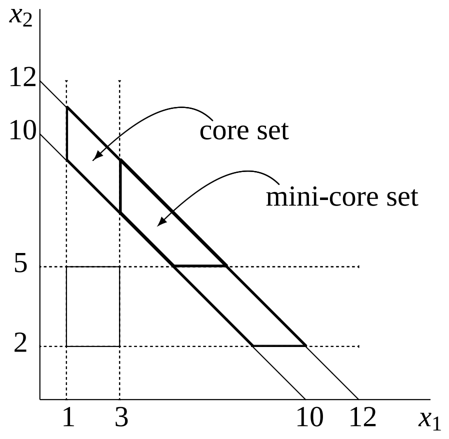

Figure 2.1: The mini-core set and the core set of a strongly balanced game.

In Figure 2.1, the mini-core set and the core set are depicted. This is a strongly balanced game since $\overline{w}(1) + \overline{w}(2) = 3 + 5 \leq \underline{w}(1, 2) = 10.$

Now, we describe the core set and the mini-core set of a two-person interval game in terms of its selections.

Let us introduce names of elements of $w(1)$, $w(2)$ and $w(1, 2)$ as follows:

$$s_1 \in w(1) = [\underline{w}(1), \overline{w}(1)], s_2 \in w(2) = [\underline{w}(2), \overline{w}(2)], t \in w(1, 2) = [\underline{w}(1, 2), \overline{w}(1, 2)]$$

and denote by $w^{s_1,s_2,t}$ the *selection* of w corresponding to s_1, s_2 and t. Then,

$$C(w) = \cup \left\{ C(w^{s_1,s_2,t}) | (s_1, s_2, t) \in w(1) \times w(2) \times w(1, 2) \right\}.$$

Furthermore,

$$MC(w) = \cup \left\{ C(w^{s_1,s_2,t}) | s_1 \in [\overline{w}(1), \overline{w}(1)], s_2 \in [\overline{w}(2), \overline{w}(2)], t \in w(1, 2) \right\}.$$

So,

$$MC(w) \subset \cup \left\{ C(w^{s_1,s_2,t}) | s_1 \in w(1), s_2 \in w(2), t \in w(1, 2) \right\},$$

i.e., $MC(w) \subset C(w)$.

The mini-core set $MC(w)$ is interesting because for each s_1, s_2 and t, all points in $MC(w)$ with $x_1 + x_2 = t$ are also in $C(w^{s_1,s_2,t})$. Note that all points in the mini-core set of w are individually rational points for each selection $w^{s_1,s_2,t}$, and each selection $w^{s_1,s_2,t}$ can be written as a linear combination of unanimity games in the following way:

$$w^{s_1,s_2,t} = s_1 u_{\{1\}} + s_2 u_{\{2\}} + (t - s_1 - s_2) u_{\{1,2\}}.$$

If $w \in IG^{\{1,2\}}$ is a superadditive game, then for each s_1, s_2 and t we have $s_1 + s_2 \leq t$. So, each selection $w^{s_1,s_2,t}$ of w is balanced. We conclude that if
$\overline{w}(1) + \overline{w}(2) \leq \underline{w}(1,2)$ is satisfied, then each selection $w^{s_1,s_2,t}$ of w is superadditive.
Hence, a two-person interval game $< N, w >$ is superadditive if and only if $< N, w >$ is strongly balanced. Here, optimism vectors will play a role.
Let $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, which we call the *optimism vector*, and $w \in IG^{\{1,2\}}$. We define:

$$s_1^{\alpha_1}(w) := \alpha_1 \overline{w}(1) + (1 - \alpha_1)\underline{w}(1),\ s_2^{\alpha_2}(w) := \alpha_2 \overline{w}(2) + (1 - \alpha_2)\underline{w}(2).$$

We are interested in maps $\kappa : [a,b] \to \mathbb{R}^2$, where $[a,b]$ is a closed interval in $\mathbb{R}$ with properties:

(i) for each $a \leq x_1 \leq x_2 \leq b,\ \kappa_1(x_1) \leq \kappa_1(x_2),\ \kappa_2(x_1) \leq \kappa_2(x_2)$;

(ii) for each $x \in [a,b]$, $\kappa_1(x) + \kappa_2(x) = x$.

In the following, we call such maps *monotonic curves*, and we denote by $\mathbb{K}(\mathbb{R}^2)$ the set of all monotonic curves in $\mathbb{R}^2$.
A map $F : IG^{\{1,2\}} \to \mathbb{K}(\mathbb{R}^2)$ assigning to each interval game w a unique curve
$F(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathbb{R}^2$ for $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ in $\mathbb{K}(\mathbb{R}^2)$ is called a *solution*.
We say that $F : IG^{\{1,2\}} \to \mathbb{K}(\mathbb{R}^2)$ has the property of

(i) *efficiency (EFF)*, if for all $w \in IG^{\{1,2\}}$, $t \in [\underline{w}(1,2), \overline{w}(1,2)]$:
$\sum_{i \in N} F(w)(t)_i = t$.

(ii) *α-symmetry (α-SYM)*, if for all $w \in IG^{\{1,2\}}$ with $s_1^{\alpha_1}(w) = s_2^{\alpha_2}(w)$ and for all $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we have $F(w)(t)_1 = F(w)(t)_2$;

(iii) *covariance with respect to translations (COV)*, if for all

$$w \in IG^{\{1,2\}}, t \in [\underline{w}(1,2), \overline{w}(1,2)] \text{ and } a = (a_1, a_2) \in \mathbb{R}^2$$

we have $F(w + \hat{a})(a_1 + a_2 + t) = F(w)(t) + a$.

Here, $\hat{a} \in IG^{\{1,2\}}$ is defined by

$$\hat{a}(\{1\}) := [a_1, a_1],\ \hat{a}(\{2\}) := [a_2, a_2],\ \hat{a}(\{1,2\}) := [a_1 + a_2, a_1 + a_2],$$

and $w + \hat{a} \in IG^{\{1,2\}}$ is defined by

$$(w + \hat{a})(S) := w(S) + \hat{a}(S) \text{ for } S \in \{\{1\}, \{2\}, \{1,2\}\}.$$

For each $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we define the map $\psi^{\alpha} : IG^{\{1,2\}} \to \mathbb{K}(\mathbb{R}^2)$ such that

$$\psi^{\alpha}(w)(t) := (s_1^{\alpha_1}(w) + \beta, s_2^{\alpha_2}(w) + \beta),$$

where $\beta = \beta(t, w) := \frac{1}{2}(t - s_1^{\alpha_1}(w) - s_2^{\alpha_2}(w))$.

The next example illustrates the solution ψ^{α} with $\alpha = (0,0)$ and its relations with the mini-core set.

Example 2.2.2 *(A bankruptcy situation with an uncertain estate) Consider a bankruptcy situation given by two claimants with demands $d_1 = 70$ and $d_2 = 90$ and (uncertain) estate $E = [100, 120]$.*

Then, the characteristic function of the interval game is as follows:

$$w(\emptyset) = [0,0], \ w(1) = [(\underline{E} - d_2)_+, (\overline{E} - d_2)_+] = [10, 30],$$

$$w(2) = [(\underline{E} - d_1)_+, (\overline{E} - d_1)_+] = [30, 50], \ w(1,2) = [100, 120],$$

where $x_+ = \max\{x, 0\}$.

This is a strongly balanced game, since $\overline{w}(1) + \overline{w}(2) = 30 + 50 \leq \underline{w}(1,2) = 100$,

$$\psi^{(0,0)}(w)(t) = (10 + \beta, 30 + \beta) \textit{ with } \beta = \frac{1}{2}(t - 40) \textit{ and } t \in [100, 120].$$

Figure 2.2 illustrates that for all $t \in [100, 120]$, $\psi^{(0,0)}(w)(t) \in MC(w^{(0,0,t)})$; in this figure L denotes the set $\{\psi^{(0,0)}(w)(t) | t \in [100, 120]\}$. In the following, we show the ψ^{α}-values for some realizations t of $w(N)$:

$$\begin{array}{l} t \\ \beta \\ \psi^{\alpha}(w)(t) \end{array} \left[\begin{array}{ccccc} 100 & 106 & 110 & 114 & 120 \\ 30 & 33 & 35 & 37 & 40 \\ (40,60) & (43,63) & (45,65) & (47,67) & (50,70) \end{array} \right].$$

Next, we give an axiomatic characterization of the ψ^{α}-value for $\alpha \in [0,1] \times [0,1]$.

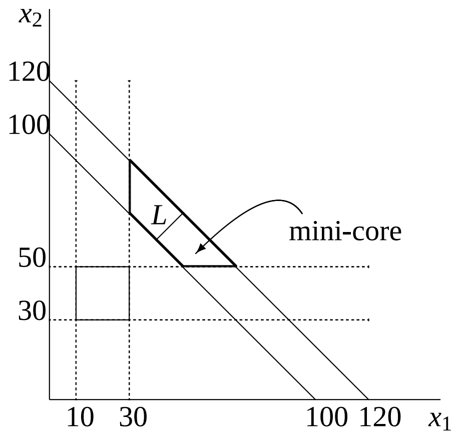

Figure 2.2: The mini-core set and the $\psi^{(0,0)}$-values of the game $< N, w >$.

Proposition 2.2.3 *The ψ^{α}-value satisfies the properties EFF, α-SYM and COV.*

Proof.

(i) For all $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, the solution ψ^{α} satisfies the efficiency (EFF) property since

$$\psi^{\alpha}(w)(t)_1 + \psi^{\alpha}(w)(t)_2 = s_1^{\alpha_1}(w) + s_2^{\alpha_2}(w) + 2\beta = t.$$

(ii) For all $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$, the solution ψ^{α} satisfies the α-symmetry (α-SYM) property since $s_1^{\alpha_1}(w) = s_2^{\alpha_2}(w)$ implies

$$\psi^{\alpha}(w)(t)_1 = s_1^{\alpha_1}(w) + \beta = s_2^{\alpha_2}(w) + \beta = \psi^{\alpha}(w)(t)_2.$$

(iii) Take $w \in IG^{\{1,2\}}$, $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ and $a \in \mathbb{R}^2$. The solution ψ^{α} satisfies the covariance with respect to translations (COV) property since

$$\psi^{\alpha}(w + \hat{a})(a_1 + a_2 + t) = (s_1^{\alpha_1}(w + \hat{a}) + \hat{\beta}, s_2^{\alpha_2}(w + \hat{a}) + \hat{\beta}).$$

Then,

$$\psi^{\alpha}(w + \hat{a})(a_1 + a_2 + t) = (s_1^{\alpha_1}(w) + \beta, s_2^{\alpha_2}(w) + \beta) + (a_1, a_2) = \psi^{\alpha}(w)(t) + a.$$

Note that

$$\beta = \hat{\beta} = \frac{1}{2}(\hat{t} - s_1^{\alpha_1}(w + \hat{a}) - s_2^{\alpha_2}(w + \hat{a})),$$

where $\hat{t} = a_1 + a_2 + t$. ■

Theorem 2.2.4 *The ψ^α-value is the unique solution satisfying the EFF, α-SYM and COV properties.*

Proof. Suppose the solution $F : IG^{\{1,2\}} \to \mathbb{K}(\mathbb{R}^2)$ satisfies the three properties above. We show that $F = \psi^\alpha$.

Take $w \in IG^{\{1,2\}}$ and let $a = (s_1^{\alpha_1}(w), s_2^{\alpha_2}(w))$. Then, $s^\alpha(w - \hat{a}) = (0, 0)$.

By α-SYM and EFF, for each $\tilde{t} = t - a_1 - a_2$ with $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we have $F(w - \hat{a})(\tilde{t}) = (\frac{1}{2}\tilde{t}, \frac{1}{2}\tilde{t}) = \psi^\alpha(w - \hat{a})(\tilde{t})$. Hence, $F(w - \hat{a}) = \psi^\alpha(w - \hat{a})$.

By COV of F and ψ^α we obtain

$$F(w)(t) = F(w - \hat{a})(\tilde{t}) + a = \psi^\alpha(w - \hat{a})(\tilde{t}) + a = \psi^\alpha(w)(t)$$

for each $w \in IG^{\{1,2\}}$ and $t \in [\underline{w}(1,2), \overline{w}(1,2)]$.

From Proposition 2.2.3 it follows that ψ^α satisfies EFF, α-SYM and COV.

So, ψ^α is the only solution with these three properties. ■

The *marginal curves* for a two-person game $< N, w >$ are defined by $m^{\sigma,\alpha}(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathbb{R}^2$, where

$$m^{(1,2),\alpha}(w)(t) := (s_1^{\alpha_1}(w), t - s_1^{\alpha_1}(w)),\ m^{(2,1),\alpha}(w)(t) := (t - s_2^{\alpha_2}(w), s_2^{\alpha_2}(w)).$$

The *Shapley-like solution* ψ^α is equal to

$$\psi^\alpha(w) = \frac{1}{2}(m^{(1,2),\alpha}(w) + m^{(2,1),\alpha}(w)).$$

Note that each point of the marginal curve $m^{(1,2),\alpha}(w) : [\underline{w}(1,2), \overline{w}(1,2)] \to \mathbb{R}^2$ corresponds to a marginal vector of a selection of w, since for all $\alpha \in [0,1] \times [0,1]$ and for all $t \in [\underline{w}(1,2), \overline{w}(1,2)]$ we have $m^{(1,2),\alpha}(w)(t) = m^{(1,2)}(v)$, where $v : 2^{\{1,2\}} \to \mathbb{R}$ is the characteristic function of the game with

$$v(\emptyset) := 0,\ v(1) := s_1^{\alpha_1}(w)(t),\ v(2) := s_2^{\alpha_2}(w)(t) \text{ and } v(1,2) := t.$$

Similarly, $m^{(2,1),\alpha}(w)(t) = m^{(2,1)}(v)$ for all $\alpha \in [0,1] \times [0,1]$ and for all $t \in [\underline{w}(1,2), \overline{w}(1,2)]$.

In case where $w(S)$ is a degenerate interval for each $S \in 2^N$, we have $m^{\sigma,\alpha}(w)(t) = m^\sigma(v)$ for all $\alpha \in [0,1] \times [0,1]$ and for all $t \in [\underline{w}(1,2), \underline{w}(1,2)]$ with

$$v(\emptyset) := 0, v(1) := \underline{w}(1), v(2) := \underline{w}(2) \text{ and } v(1,2) := \underline{w}(1,2).$$

Let us consider the *Shapley-like solutions* of the form
$\varphi^{\alpha} : IG^{\{1,2\}} \to \mathcal{K}(\mathbb{R}^2)$ defined by

$$\varphi^{\alpha}(w) := \frac{1}{2}(m^{(1,2),\alpha}(w) + m^{(2,1),\alpha}(w))$$

for each $w \in IG^{\{1,2\}}$ and for each $\alpha \in [0,1] \times [0,1]$.
Then, for each $t \in [\underline{w}(1,2), \underline{w}(1,2)]$ it holds that $\varphi^{\alpha}(w)(t) = \psi^{\alpha}(w)(t)$. So, φ^{α} coincides with ψ^{α}.

2.3 INTERVAL SOLUTION CONCEPTS

This section is based on Alparslan Gök, Branzei and Tijs (2008a,b).
Recall that a solution concept for classical n-person cooperative games associates with each such game a (possibly empty) set of n-dimensional real-valued vectors whose i-th component indicates the payoff for player i when the worth of the grand coalition is distributed among the n players. In case of n-person cooperative interval games the players have to cope with the division of the worth of the grand coalition when they only know its lower and upper bounds. As a consequence of the interval uncertainty regarding the realized value of the grand coalition, before cooperation starts, players' payoffs can be rather expressed as intervals of real numbers than as real numbers, i.e. each player might know at this stage only his/her minimum and maximum potential payoffs. Thus, an interval solution concept associates to each n-person cooperative interval game a (possibly empty) set of interval payoff vectors.

Let I_i be the interval payoff of player i, and let $I = (I_1, I_2, \ldots, I_n)$ be an interval payoff vector. Then, according to Moore (1979), we have $\sum_{i\in S} I_i = \left[\sum_{i\in S} \underline{I}_i, \sum_{i\in S} \overline{I}_i\right] \in I(\mathbb{R})$ for each $S \in 2^N \setminus \{\emptyset\}$. An interval solution concept $\mathcal{F}$ on IG^N is a map assigning to each interval game $w \in IG^N$ a set of n-dimensional vectors whose components belong to $I(\mathbb{R})$. Here, we define interval solution concepts for interval games $w \in IG^N$.
The *interval imputation set* $\mathcal{I}(w)$ of the interval game w, is defined by

$$\mathcal{I}(w) := \left\{(I_1, I_2, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i\in N} I_i = w(N), w(i) \preccurlyeq I_i, \text{ for all } i \in N\right\}.$$

We note that $\sum_{i\in N} I_i = w(N)$ is equivalent with $\sum_{i\in N} \overline{I}_i = \overline{w}(N)$ and $\sum_{i\in N} \underline{I}_i = \underline{w}(N)$, and $w(i) \preccurlyeq I_i$ is equivalent with $\underline{w}(i) \leq \underline{I}_i$ and $\overline{w}(i) \leq \overline{I}_i$, for each $i \in N$.

Furthermore, $\sum_{i\in N} I_i = w(N)$ implies that for all $i \in N$ and for all $t \in w(N)$, there exists $x_i \in I_i$, $i \in N$, such that $\sum_{i\in N} x_i = t$. Notice that the interval uncertainty of coalition values propagates into the interval uncertainty of individual payoffs and we obtain interval payoff vectors as building blocks of interval solutions. The interval imputation set consists of those interval payoff vectors which assure the distribution of the uncertain worth of the grand coalition such that each player can expect a weakly better interval payoff that what he/she can expect of his/her own.

Proposition 2.3.1 *Let $w \in IG^N$. The interval imputation set $\mathcal{I}(w)$ of w is nonempty if and only if $w(N) \succcurlyeq \sum_{i\in N} w(i)$.*

Proof. First, suppose that $\mathcal{I}(w) \neq \emptyset$. Take $I = (I_1, I_2, \ldots, I_n) \in \mathcal{I}(w)$. Then $I_i \succcurlyeq w(i)$ for each $i \in N$. So, $\sum_{i\in N} I_i \succcurlyeq \sum_{i\in N} w(i)$ by interval calculus. Now, we use $\sum_{i\in N} I_i = w(N)$. Next suppose that $w(N) \succcurlyeq \sum_{i\in N} w(i)$. Then, $I = (w(1), w(2), \ldots, w(n-1), I_n)$, where $I_n = [\underline{I}_n, \overline{I}_n] = [\underline{w}(n)+\delta, \overline{w}(n)+\epsilon]$ with $\epsilon = \overline{w}(N) - \sum_{i\in N} \overline{w}(i) \geq 0$ and $\delta = \underline{w}(N) - \sum_{i\in N} \underline{w}(i) \geq 0$, is an element of the interval imputation set. ■

The *interval core* $\mathcal{C}(w)$ of the interval game w, is defined by

$$\mathcal{C}(w) := \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i\in N} I_i = w(N), \sum_{i\in S} I_i \succcurlyeq w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

The interval core consists of those interval payoff vectors which assure the distribution of the uncertain worth of the grand coalition such that each coalition of players can expect a weakly better interval payoff than what that group can expect on its own, implying that no coalition has any incentives to split off. Here, $\sum_{i\in N} I_i = w(N)$ is the *efficiency condition* and $\sum_{i\in S} I_i \succcurlyeq w(S)$, $S \in 2^N \setminus \{\emptyset\}$, are the *stability conditions* of the interval payoff vectors. Clearly, $\mathcal{C}(w) \subset \mathcal{I}(w)$ for each $w \in IG^N$. Notice that for two-person cooperative interval games the interval imputation set coincides with the interval core.

Example 2.3.1 *Let $< N, w >$ be a three-person interval game with*

$$w(1,3) = w(2,3) = w(1,2,3) = J \succcurlyeq [0,0] \text{ and } w(S) = [0,0] \text{ otherwise.}$$

The interval core is $\mathcal{C}(w) = \{([0,0],[0,0],J)\}$.

Let $< N, d >$ be an interval cost game. Then, the *interval core* $\mathcal{C}(d)$ is defined by

$$\mathcal{C}(d) := \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i\in N} I_i = d(N), \sum_{i\in S} I_i \preccurlyeq d(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

The interval core $\mathcal{C}(d)$ consists of those interval payoff vectors which assure the distribution of the uncertain cost of the grand coalition, $d(N)$, such that each coalition of players S can expect a weakly better interval cost, $\sum_{i\in S} I_i$, than what that group can expect on its own, implying that no coalition has any incentives to split off. We refer to $\sum_{i\in N} I_i = d(N)$ as the *efficiency condition* and to $\sum_{i\in S} I_i \preccurlyeq d(S)$, $S \in 2^N \setminus \{\emptyset\}$, as the *stability conditions of the interval payoff vectors*.

Remark 2.3.1 *Elements of the interval core $\mathcal{C}(w)$, can be computed by solving a system of linear inequalities of the form: $\sum_{i\in N} \underline{I}_i = \underline{w}(N)$; $\sum_{i\in N} \overline{I}_i = \overline{w}(N)$ and $\sum_{i\in S} \underline{I}_i \geq \underline{w}(S)$; $\sum_{i\in S} \overline{I}_i \geq \overline{w}(S)$, for each $S \in 2^N \setminus \{\emptyset\}$. We notice that the time complexity of the algorithm for computing the interval core $\mathcal{C}(w)$ for $w \in IG^N$ is the same as the time complexity [1] of the algorithm for computing the core $C(v)$ for $v \in G^N$.*

Remark 2.3.2 *We notice that the elements of the sets $C(w)$ and $\mathcal{C}(w)$ are of different types, implying that we cannot compare the sets with respect to the inclusion relation. Specifically, the elements of $C(w)$ are vectors $x \in \mathbb{R}^N$, whereas the elements of $\mathcal{C}(w)$ are vectors $I \in I(\mathbb{R})^N$. But, if all the worth intervals of the interval game $< N, w >$ are degenerate intervals, then the interval core $\mathcal{C}(w)$ corresponds in a natural way to the core $C(w)$, since $([a_1, a_1], \ldots, [a_n, a_n])$ is in the interval core $\mathcal{C}(w)$ if and only if $(a_1, \ldots, a_n)$ is in the core $C(w)$ for each $a_i \in \mathbb{R}$ and $i \in N$. Furthermore, we could have situations in which $\mathcal{C}(w) = \emptyset$ and $C(w) \neq \emptyset$, as Example 2.3.2 illustrates.*

Remark 2.3.3 *Note also that if the worth of the grand coalition is given by a degenerate interval then the elements of the interval core are tuples of degenerate intervals. Under this assumption, the necessary and sufficient condition for the nonemptiness of the interval core is the balancedness of the upper game.*

The interval core is defined as the set of efficient n-person interval payoff vectors that satisfy coalitional rationality (or split-off stability) in the interval setting. An algorithm for computing elements of the interval core of a cooperative interval game based on Remark 2.3.1 is provided in Drechsel and Kimms (2008). There is a fundamental difference between the interval core $\mathcal{C}(w)$ and the core $C(w)$ as we emphasized in Remark 2.3.2. Now, we notice that the interval

[1] For details on complexity theory we refer to Garey and Johnson (1979).

core of n-person cooperative interval games can generate via selections $(x_1, x_2, \ldots, x_n) \in (I_1, I_2, \ldots, I_n) \in \mathcal{C}(w)$ a set which has the same type of elements as $C(w)$. The two sets do not coincide for arbitrary cooperative interval games, but they coincide in case where all the coalitional worth values are degenerate intervals.

Example 2.3.2 *Let $< N, w >$ be a two-person interval game with $w(1,2) = [6,8]$, $w(1) = [2,4]$, $w(2) = [5,6]$ and $w(\emptyset) = [0,0]$. For this game $\mathcal{C}(w) = \emptyset$. But, $C(w) \neq \emptyset$ since $C(v) \neq \emptyset$ for some selections $v \in Sel(w)$.*

Proposition 2.3.2 *Let $w \in IG^N$. If the interval core $\mathcal{C}(w)$ is nonempty, then the core $C(w)$ is nonempty.*

Proof. Take $(I_1, I_2, \ldots, I_n) \in \mathcal{C}(w)$. Then, $\sum_{i \in N} I_i = w(N)$, meaning that $\sum_{i \in N} \underline{I}_i = \underline{w}(N)$ and $\sum_{i \in N} \overline{I}_i = \overline{w}(N)$, and $\sum_{i \in S} I_i \succcurlyeq w(S)$, implying that $\sum_{i \in S} \underline{I}_i \geq \underline{w}(S)$ and $\sum_{i \in S} \overline{I}_i \geq \overline{w}(S)$. Let $< N, v >$ be the selection of w with $v(S) = \underline{w}(S)$, $v(N) = \underline{w}(N)$ and let $x_i = \underline{I}_i$, $i \in S$. Then, $\sum_{i \in S} x_i \geq \underline{w}(S)$ and $\sum_{i \in N} x_i = \underline{w}(N)$ which shows that $C(\underline{w}) \neq \emptyset$ and $C(\overline{w}) \neq \emptyset$ implying that $C(w)$ is nonempty. ■

Some basic properties of the interval core are straightforward extensions of the corresponding properties of the core of traditional cooperative games (Gillies (1959)) as Proposition 2.3.3 and Proposition 2.3.4 illustrate. In Proposition 2.3.4, we extend to interval games the property of relative invariance with respect to strategic equivalence for the core. For this extension, we need the notion of additive interval games. A game $< N, a >$ is called an *additive interval game* if for each $S \in 2^N$ it holds $a(S) = \sum_{i \in S} a(\{i\})$. For such a game, $\mathcal{C}(a) = \{(a(\{1\}), a(\{2\}), \ldots, a(\{n\}))\}$.

Proposition 2.3.3 *Let $w \in IG^N$. Then, the interval core $\mathcal{C}(w)$ of w is a convex set.*

Proposition 2.3.4 *The interval core $\mathcal{C} : IG^N \twoheadrightarrow I(\mathbb{R})^N$ is relative invariant with respect to strategic equivalence, i.e., for each $w, a \in IG^N$ with a being an additive interval game, and for each $k > 0$ we have $\mathcal{C}(kw + a) = k\mathcal{C}(w) + \mathcal{C}(a)$.*

Proposition 2.3.5 *Let $w \in IG^N$. Then the interval core correspondence $\mathcal{C} : IG^N \twoheadrightarrow I(\mathbb{R})^N$ is a superadditive map.*

Proof. We have to prove that $C(w_1) + C(w_2) \subset C(w_1 + w_2)$ for each $w_1, w_2 \in IG^N$. First, we note that the inclusion holds if $C(w_1) = \emptyset$ or $C(w_2) = \emptyset$. Otherwise, consider that w_1, $w_2 \in IG^N$ and take $(I_1, I_2, \ldots, I_n) \in C(w_1)$ and $(J_1, J_2, \ldots, J_n) \in C(w_2)$. Then,

$$\sum_{k\in N} I_k + \sum_{k\in N} J_k = w_1(N) + w_2(N) \Rightarrow \sum_{k\in N} (I_k + J_k) = (w_1 + w_2)(N),$$

and, for each $S \in 2^N \setminus \{\emptyset\}$, $\sum_{k\in S} I_k \succcurlyeq w_1(S)$ and $\sum_{k\in S} J_k \succcurlyeq w_2(S)$, implying that $\sum_{k\in S} \overline{I}_k \geq \overline{w}_1(S)$ and $\sum_{k\in S} \overline{J}_k \geq \overline{w}_2(S)$. Then, for each $S \in 2^N \setminus \{\emptyset\}$,

$$\sum_{k\in S} \overline{I}_k + \sum_{k\in S} \overline{J}_k \geq \overline{w}_1(S) + \overline{w}_2(S) \Rightarrow \sum_{k\in S} (\overline{I}_k + \overline{J}_k) \geq (\overline{w}_1 + \overline{w}_2)(S).$$

Similarly, $\sum_{k\in S} (\underline{I}_k + \underline{J}_k) \geq (\underline{w}_1 + \underline{w}_2)(S)$. Hence, the interval core correspondence is a superadditive map. ■

We call a game $w \in IG^N$ an *exact interval game* if for each $S \in 2^N$ it holds:

(i) there exists an $I = (I_1, \ldots, I_n) \in C(w)$ such that $\sum_{i\in S} I_i = w(S)$;

(ii) there exists an $x \in C(|w|)$ such that $\sum_{i\in S} x_i = |w|(S)$.

Note that (ii) expresses the exactness of the length game $< N, |w| >$.

Other interesting interval type solution concepts for interval games like the interval dominance core and stable sets based on a dominance relation are introduced in the following.
Let $w \in IG^N$, $I = (I_1, \ldots, I_n), J = (J_1, \ldots, J_n) \in \mathcal{I}(w)$ and $S \in 2^N \setminus \{\emptyset\}$. We say that I *dominates J via coalition S*, and denote it by $I \operatorname{dom}_S J$, if:

(i) $I_i \succ J_i$ for all $i \in S$,

(ii) $\sum_{i\in S} I_i \preccurlyeq w(S)$.

For $S \in 2^N \setminus \{\emptyset\}$ we denote by $D(S)$ the set of those elements of $\mathcal{I}(w)$ which are dominated via S. For $I, J \in \mathcal{I}(w)$, we say that I *dominates* J and denote it by $I \operatorname{dom} J$ if there is an $S \in 2^N \setminus \{\emptyset\}$ such that $I \operatorname{dom}_S J$.
Furthermore, I is called *undominated* if there does not exist J and a coalition S such that $J \operatorname{dom}_S I$.
The *interval dominance core* $\mathcal{DC}(w)$ of an interval game $w \in IG^N$ consists of all undominated elements in $\mathcal{I}(w)$, i.e., it is the complement of $\cup \left\{D(S) | S \in 2^N \setminus \{\emptyset\}\right\}$ in $\mathcal{I}(w)$.
For $w \in IG^N$ a subset A of $\mathcal{I}(w)$ is an *interval stable set* if the following conditions hold:

(i) (*Internal stability*) There do not exist $I, J \in A$ such that I dom J or J dom I.

(ii) (*External stability*) For each $I \notin A$ there exists a $J \in A$ such that J dom I.

Next, we study relations between the interval core, interval dominance core and stable sets for interval games.

Theorem 2.3.6 *Let $w \in IG^N$ and let A be a stable set of w. Then, $\mathcal{C}(w) \subset \mathcal{DC}(w) \subset A$.*

Proof. In order to show that $\mathcal{C}(w) \subset \mathcal{DC}(w)$, let us assume that there is an $I \in \mathcal{C}(w)$ such that $I \notin \mathcal{DC}(w)$. Then, there are a $J \in \mathcal{I}(w)$ and a coalition $S \in 2^N \setminus \{\emptyset\}$ such that J dom$_S$ I. Thus, $I(S) \prec J(S) = \sum_{i \in S} J_i \preccurlyeq w(S)$ and $J_i \succ I_i$ for all $i \in S$ implying that $I \notin \mathcal{C}(w)$. From this contradiction it follows that $\mathcal{C}(w) \subset \mathcal{DC}(w)$.
To prove next that $\mathcal{DC}(w) \subset A$, it is sufficient to show $\mathcal{I}(w) \setminus A \subset \mathcal{I}(w) \setminus \mathcal{DC}(w)$. Take $I \in \mathcal{I}(w) \setminus A$. By the external stability of A there is a $J \in A$ with J dom I. The elements in $\mathcal{DC}(w)$ are not dominated. So, $I \notin \mathcal{DC}(w)$, i.e., $I \in \mathcal{I}(w) \setminus \mathcal{DC}(w)$. ■

The inclusions stated in the previous theorem may be strict. The following example, inspired by Tijs (2003), illustrates that the inclusion of $\mathcal{C}(w)$ in $\mathcal{DC}(w)$ might be strict.

Example 2.3.3 *Let $< N, w >$ be the three-person interval game with $w(1,2) = [2,2]$, $w(N) = [1,1]$ and $w(S) = [0,0]$ if $S \neq \{\{1,2\}, N\}$. Then, $\mathcal{C}(w) = \emptyset$ because the game is not $\mathcal{I}$-balanced (note that $w(1,2) + w(3) \succ w(N)$). Further, $D(S) = \emptyset$ if $S \neq \{1,2\}$ and $D(\{1,2\}) = \{I \in \mathcal{I}(w) | I_3 \succ [0,0]\}$. The elements I in $\mathcal{I}(w)$ which are undominated satisfy $I_3 = [0,0]$. Since the interval dominance core is the set of undominated elements in $\mathcal{I}(w)$, the interval dominance core of this game is nonempty.*

2.4 HANDLING INTERVAL SOLUTIONS

This section is based on Branzei, Tijs and Alparslan Gök (2008b). Its goal is to provide a basic guide for handling interval solution concepts. We want to make clear that interval allocations and protocols to handle them are interrelated and their choice has to be made under interval uncertainty of coalition values. First, the commonly chosen interval solution concept and protocol provide support for the players' decision making regarding the most suitable coalition to form under interval uncertainty of coalition values. Second, after the group of cooperating players is fixed, the same interval solution concept gives *a priori* interval-type information regarding the potential reward/cost shares for cooperating individuals. *A posteriori*, when uncertainty on the outcome(s) of cooperation is removed, the ranges of potential individual shares are processed according to the chosen protocol to determine uncertainty-free individual shares. We notice that usually only uncertainty about the outcome of the grand coalition is removed. For this scenario we propose several procedures for solving the difficult task of distributing the effective total profit/cost among the cooperating players consistently with all their previous decisions. We cope with two basic ways for evaluating the actual outcome of the grand coalition:

(i) in one step, when the joint enterprise is finished;

(ii) in several steps, at a priori fixed moments of time when the progress of the joint enterprise is evaluated.

The suitability of a particular procedure relies on the nature of the situation modeled as a cooperative interval game and also on players' joint decision about when they should receive their uncertainty-free shares.

The players who like to cooperate in a situation with interval data can at a first stage consider the corresponding cooperative interval game and a cooperative interval solution. The interval allocation obtained by the commonly agreed upon interval solution concept has at this stage a two-fold use:

(a) to assist people or businesses in taking optimal decisions regarding cooperation under interval uncertainty;

(b) to prescribe before cooperation starts minimal and maximal values for individual shares for cooperating players such that the interval reward/cost of the grand coalition is cleared.

We notice that once it was agreed upon which coalition to form, the jointly chosen interval solution concept provides the ranges of potential individual shares. This means that before cooperation starts the agents are uncertain about their rights or liabilities, respectively. Clearly, the outcome of cooperation will be known with certainty at some future moment(s). However, the agents have to sign at this stage a contingent contract, rather than waiting until the uncertainty on their joint rewards/costs is removed. This contract should also specify the protocol to handle the potential interval shares when the uncertainty on the outcome of the grand coalition is removed. We notice that such a protocol should be jointly chosen before cooperation starts but used when the outcome of the grand coalition is known with certainty. Depending on the dimension of the joint enterprise, the evaluation of the achieved profit or cost, respectively, by the grand coalition and its distribution among players can be completely done either in one step, after the joint enterprise is finished, or in several steps corresponding to commonly agreed upon mile stones during the carrying out process.

We focus first on the case when the evaluation of the achieved reward of the whole cooperating group takes place only once, after the joint enterprise is finished. We denote by R the achieved joint financial outcome, and look at it as the realization of the value $w(N)$ of the grand coalition in the cooperative interval game $< N, w >$. Notice that in case the cooperating group of agents is a proper subset of the initial set of people or businesses considering cooperation under interval uncertainty, the interval game $< N, w >$ is nothing else than a proper subgame of the initial interval game arising from the analyzed situation with interval data.

The problem which agents face when the uncertainty on $w(N)$ is removed is how to allocate this total payoff R. At this stage, uncertainty on individual shares should be removed as well, i.e., uncertainty-free individual shares should be determined based on the protocol and in accordance with the individual interval shares specified in the binding contract.

To be more concrete, let $< N, w >$ be the interval game, and let ψ be the solution concept on which the decision to start cooperation of all players in N was based. Here, we suppose that the interval game is of reward type. The final uncertainty-free individual shares will depend on $J_i = \psi_i(w) \in I(\mathbb{R})$ for all $i \in N$ and on the reward $R \in \mathbb{R}$ achieved by N. The players have to cope with the question: How to divide R according to the given $J = (J_1, \ldots, J_n)$? This issue is solved using the protocol chosen before cooperation starts. In the sequel, we offer some possible candidates for such protocols.

First, since R appears as a realization of $w(N)$, one can naturally expect that

$$\underline{w}(N) \leq R \leq \overline{w}(N). \quad (2.4.1)$$

One idea is to determine $\lambda \in [0, 1]$ such that

$$R = \lambda \underline{w}(N) + (1 - \lambda)\overline{w}(N), \quad (2.4.2)$$

and give to each $i \in N$ the payoff $x_i = \lambda \underline{J}_i + (1-\lambda)\overline{J}_i$.

Note that $\underline{J}_i \leq x_i \leq \overline{J}_i$ and

$$\sum_{i \in N} x_i = \lambda \sum_{i \in N} \underline{J}_i + (1-\lambda) \sum_{i \in N} \overline{J}_i = \lambda \underline{w}(N) + (1-\lambda)\overline{w}(N) = R.$$

So, x is a contract-consistent and an efficient payoff vector corresponding to R.

Now, note that we can also write $x = \underline{J} + (1-\lambda)(\overline{J} - \underline{J})$. So, the payoff for player $i \in N$ can be given in the following manner: first (eventually even before cooperation starts), each player $i \in N$ is given the amount $\underline{J}_i$; later on (when R is known), the amount $R - \sum_{i \in N} \underline{J}_i$ is distributed over the players proportionally with their residual contractual rights, $\overline{J}_i - \underline{J}_i$, $i \in N$. This is equivalent with using the bankruptcy rule $PROP$ for a standard bankruptcy problem (E, d), where the estate E equals $R - \sum_{i \in N} \underline{J}_i$ and the claims d_i with $d_1 \leq \ldots \leq d_n$ are equal to $\overline{J}_i - \underline{J}_i$ for each $i \in N$. Note that (2.4.1) implies that $E < \sum_{i \in N} d_i$; so, in this case, we deal with a standard bankruptcy problem. Recall that the rule $PROP$ is defined by $PROP_i(E, d) := \frac{d_i}{\sum_{j \in N} d_j} E$ for each bankruptcy problem (E, d) and all $i \in N$.

Furthermore, we can extend the previous bankruptcy approach by considering also other well-known bankruptcy rules such as the *constrained equal awards (CEA) rule* and the *constrained equal losses (CEL) rule*. Recall that the bankruptcy rule CEA is defined by $CEA_i(E, d) := \min\{d_i, \alpha\}$, where $\alpha \in [0, d_n]$ is determined by $\sum_{i \in N} CEA_i(E, d) = E$ for each bankruptcy problem (E, d) and all $i \in N$, while the bankruptcy rule CEL is defined by $CEL_i(E, d) := \max\{d_i - \beta, 0\}$, where $\beta \in [0, d_n]$ is determined by $\sum_{i \in N} CEL_i(E, d) = E$, for each bankruptcy problem (E, d) and all $i \in N$. For details about bankruptcy problems and rules we refer the reader to Aumann and Maschler (1985), Curiel, Maschler and Tijs (1987), Kaminsky (2000), O'Neill (1982) and Thomson (2003).

Denote $F := \{CEA, CEL, PROP\}$ and let $f \in F$. Then, we can divide the amount R achieved by N by handing out the amount $\underline{J}_i + f_i(E, d)$ to each player $i \in N$, where $E = R - \sum_{i \in N} \underline{J}_i$ and $d_i = \overline{J}_i - \underline{J}_i$ for each $i \in N$.

Next, we illustrate such one-step procedures.

Example 2.4.1 *Let* $< N, w >$ *be the three-person interval game with* $w(S) = [0, 0]$ *if* $3 \notin S$, $w(\emptyset) = w(3) = [0, 0]$, $w(1, 3) = [20, 30]$ *and* $w(N) = w(2, 3) = [50, 90]$. *We consider that the interval Shapley value was chosen as an interval solution concept and the decision of full cooperation was taken. Then,* $\Phi(w) = ([3\frac{1}{3}, 5], [18\frac{1}{3}, 35], [28\frac{1}{3}, 50])$. *Further, we assume that the realization of* $w(N)$ *is* $R = 60$. *First, note that condition (2.4.1) is satisfied. From (2.4.2)* *we obtain* $\lambda = \frac{3}{4}$ *implying that the payoff vector is* $x = (3\frac{3}{4}, 22\frac{1}{2}, 33\frac{3}{4})$.

Now, we determine the individual uncertainty-free shares by using PROP, CEA and CEL to distribute the amount $R - (\underline{J}_1 + \underline{J}_2 + \underline{J}_3) = 10$ *among the three agents. Note that we deal here with a classical bankruptcy problem* (E, d) *with* $E = 10$, $d = (1\frac{1}{3}, 16\frac{2}{3}, 21\frac{2}{3})$. *We obtain*

$PROP(E, d)$	$CEA(E, d)$	$CEL(E, d)$
$(\frac{5}{12}, 4\frac{1}{6}, 5\frac{5}{12})$	$(1\frac{2}{3}, 4\frac{1}{6}, 4\frac{1}{6})$	$(0, 2\frac{1}{2}, 7\frac{1}{2})$.

Then, we can divide the amount $R = 60$ *achieved by* N *by handing out the payoffs* $(3\frac{1}{3}, 18\frac{1}{3}, 28\frac{1}{3}) + f(10, (1\frac{1}{3}, 16\frac{2}{3}, 21\frac{2}{3}))$, $f \in F$, *shown in the next table:*

f	$PROP(E, d)$	$CEA(E, d)$	$CEL(E, d)$
x	$(3\frac{3}{4}, 22\frac{1}{2}, 33\frac{3}{4})$	$(5, 22\frac{1}{2}, 32\frac{1}{2})$	$(3\frac{1}{3}, 20\frac{5}{6}, 35\frac{5}{6})$.

A comparison of the payoff vectors obtained using PROP, CEA and CEL can be useful in practice to support the choice of the preferred bankruptcy rule to be implemented.

Next, we focus on the case when the evaluation of the achieved reward of the whole group takes place along the carrying out process of the joint enterprise. Let $T_1, \ldots, T_K$ be the time moments when the financial progress is evaluated and, thus, the current valuation of the joint outcome is known with certainty. We denote by R_k the realization of $w(N)$ at moment T_k, where $k \in \{1, 2, \ldots, K\}$, and focus on the situation when $R_1 < R_2 < \ldots < R_K$. We notice that $\underline{w}(N)$ can be viewed as the realization R_0 of the grand coalition N at the initial moment T_0, i.e., before starting cooperation. Clearly, the uncertainty about the outcome of cooperation is reduced at each time moment T_k, $k \in \{1, 2, \ldots, K-1\}$, being completely removed at moment T_k.

The problem here is to determine individual portions $p_i^{(k)}$, $i \in N$, at each moment T_k, $k \in \{1, 2, \ldots, K\}$, based on the history of the allocation process and on the financial fluctuations of the joint outcome.

Thus, the participants face the problem of distributing among them at each time moment T_k the amount $R_k - R_{k-1}$, where $R_0 = \underline{w}(N)$, by taking into account their adjusted individual entitlements at step k.

We assume that participants receive individual portions $p_i^{(0)} = \underline{J}_i$, $i \in N$, at moment T_0. Then, the adjusted individual entitlements at moment T_1 are $d_i^{(1)} = \overline{J}_i - p_i^{(0)}$ for $i \in N$. Now, we describe our procedure more formally.

Step 0. The portion $p_i^{(0)} = \underline{J}_i$ is handed out to agent i, $i \in N$, obtaining the individual portions $p_i^{(1)}$, $i \in N$.

Step 1. The amount $R_1 - R_0$ is distributed over agents in N by taking into account their adjusted rights $d_i^{(1)}$, $i \in N$.

Step k. The amount $R_k - R_{k-1}$ is distributed over agents in N according to adjusted rights $d_i^{(k)} = d_i^{(k-1)} - p_i^{(k-1)}$, $i \in N$, obtaining the individual portions $p_i^{(k)}$, $i \in N$.

Each player $i \in N$ receives in total the amount $x_i = \underline{J}_i + \sum_{k=1}^{K} p_i^{(k)}$. We note that each bankruptcy rule f in F can be used in our multi-step procedure if in each step the division problem at stake is bankruptcy-like, i.e., for all $k \in \{1, 2, \ldots, K\}$ we have $R_k - R_{k-1} > 0$, $d_i^{(k)} > 0$ for all $i \in N$ and $R_k - R_{k-1} < \sum_{i \in N} d_i^{(k)}$.
Next, we illustrate our multi-step procedure using the *PROP* rule (which is one of the most often used rule in real life).

Example 2.4.2 *Consider the interval game and the Shapley value as in Example 2.4.1. But, suppose there are 3 steps for evaluating the actual outcome of the grand coalition and consider a scenario with $R_1 = 60; R_2 = 65$ and $R_3 = 80$ ($R_0 = 50$). The reader can easily check that for this scenario, in each step we deal with a classical bankruptcy problem, for which we use the classical bankruptcy rule PROP.*

Step 0. *The portion $p^{(0)} = (3\frac{1}{3}, 18\frac{1}{3}, 28\frac{1}{3})$ is handed out.*

Step 1. *The amount $R_1 - R_0 = 10$ is distributed over agents in N by taking into account their adjusted rights $d^{(1)} = (1\frac{1}{3}, 16\frac{2}{3}, 21\frac{2}{3})$. Then, $p^{(1)} = (\frac{5}{12}, 4\frac{1}{6}, 5\frac{5}{12})$.*

Step 2. *The amount $R_2 - R_1 = 5$ is distributed over agents in N according to the adjusted rights $d^{(2)} = (1\frac{1}{4}, 12\frac{1}{2}, 16\frac{1}{4})$. Then, $p^{(2)} = (\frac{5}{24}, 2\frac{1}{12}, 2\frac{17}{24})$.*

Step 3. *The amount $R_3 - R_2 = 15$ is distributed over agents in N according to the adjusted rights $d^{(3)} = (1\frac{1}{24}, 10\frac{5}{12}, 13\frac{13}{24})$. Then, $p^{(3)} = (\frac{5}{8}, 6\frac{1}{4}, 8\frac{1}{8})$.*

Finally, $x = (4\frac{7}{12}, 30\frac{5}{6}, 44\frac{7}{12})$.

Note that the assumption $R_1 < R_2 < \ldots < R_K$ corresponds to the case where the joint outcomes have an increasing trend. However, it may happen to have ups and downs for the joint outcomes along the sequence $T_1, T_2 \ldots, T_K$, implying that at a certain moment T_k, $k = 1, 2, \ldots, K$, the amount to be distributed among participants might be negative. Moreover, at any T_k some individual entitlements could be negative, implying a redistribution of agents' holdings. The *rights-egalitarian rule* (Herrero, Maschler and Villar (1999)) could be a good candidate for solving the sequence of division problems obtained during our multi-stage procedure in the general case. This rule is defined by $f_i^{RE}(E, d) := d_i + \frac{1}{n}(E - \sum_{i \in N} d_i)$, for each division problem (E, d) and all $i \in N$. The rights-egalitarian rule divides equally among the agents the difference between the total entitlement and the available amount, being suitable for all circumstances of division problems. In particular, the amount to be divided can be either positive or negative, the rights may have negative components, and the amount to be divided may exceed or fall short of the aggregate rights. For these reasons the rights-egalitarian rule is always applicable in the multi-stage procedure. A negative amount $R_k - R_{k-1}$ to be distributed in some step k of our procedure means that a deficit has to be shared. A negative right $d_i^{(k)}$ for some player i in some step k of our procedure corresponds to a debt. If in some step k the amount $R_k - R_{k-1}$ to be distributed is greater (respectively smaller) than the aggregate right $\sum_{i \in N} d_i^{(k)}$, we cope with a problem of distributing a surplus (respectively sharing a deficit). We leave as an exercise for the reader to apply the rights-egalitarian rule in the 3-step procedure of Example 2.4.2 as an alternative to *PROP*. Next, we illustrate some shortcomings of the rights-egalitarian rule by considering the scenario $R_1 = 85; R_2 = 55$ and $R_3 = 60$ (i.e., ups and downs) for the interval game in Example 2.4.2 with the interval Shapley value as the chosen solution concept.

Step 0. The portion $p^{(0)} = (3\frac{1}{3}, 18\frac{1}{3}, 28\frac{1}{3})$ is handed out.

Step 1. The amount $R_1 - R_0 = 35$ is distributed over agents in N by taking into account their adjusted rights $d^{(1)} = (1\frac{2}{3}, 16\frac{2}{3}, 21\frac{2}{3})$. Then, $p^{(1)} = (0, 15, 20)$.

Step 2. The amount $R_2 - R_1 = -30$ is distributed over agents in N according to adjusted rights $d^{(2)} = (1\frac{2}{3}, 1\frac{2}{3}, 1\frac{2}{3})$. Then, $p^{(2)} = (-10, -10, -10)$.

Step 3. The amount $R_3 - R_2 = 5$ is distributed over agents in N according to adjusted rights $d^{(3)} = (11\frac{2}{3}, 11\frac{2}{3}, 11\frac{2}{3})$. Then, $p^{(3)} = (1\frac{2}{3}, 1\frac{2}{3}, 1\frac{2}{3})$.

Finally, $x = (-5, 25, 40)$.

Note that the final payoff for player 1 does not belong to $[3\frac{1}{3}, 5]$; moreover, it is negative. The fact that individual shares obtained via the rights-egalitarian rule may lie outside of the a priori intervals generated by the chosen solution ψ is a shortcoming of this rule. There is a need to design division rules which are more suitable for our multi-step procedure than the rights-egaliterian rule for the general case.

We conclude this section with some hints about how to handle abnormal cases regarding the realization of the total outcome and briefly discuss rare cases where realizations of more coalition values are known a posteriori.

Occasionally, R might not belong to the interval $w(N)$. Even in these abnormal cases the distribution of R among the players can be done consistently with the vector J of intervals obtained by the jointly chosen interval solution concept and specified in the binding contract. Our idea to handle interval solutions here is as follows:

- Suppose that the joint enterprise proved to be very profitable generating $R > \overline{w}(N)$. In this case, all agents benefit from the unexpected profit. Our proposal is each player $i \in N$ to receive the maximum expected from cooperation in N, $\overline{J}_i$, and something more which is calculated as equal share of the unexpected profit. In the formula, each player $i \in N$ will receive the amount $\overline{J}_i + \frac{1}{n}(R - \overline{w}(N))$. This is in the spirit of the rights-egalitarian allocation rule from the classical division problems literature (Herrero, Mascher and Villar (1999)) with R in the role of E and $d_i = \overline{J}_i$ for each $i \in N$, in case $E > \sum_{i \in N} d_i$.

- Suppose now that the joint enterprise was bankrupt and the amount left, R, is less than $\underline{w}(N)$. In this case, we have a division problem under interval uncertainty of claims for which rules in Branzei et al. (2004) can be helpful. We notice that the rights-egalitarian rule can also be applied to this case because agents hold collective responsibility for the losses; in formula, each player $i \in N$ will receive the amount $\underline{J}_i + \frac{1}{n}(R - \underline{w}(N))$.

An alternative approach for designing one-step and multi-step protocols is to use taxation rules instead of bankruptcy rules by handing out first $\overline{J}_i$ and, then take away with the aid of a taxation rule the deficit $T = \sum_{i \in N} \overline{J}_i - R$ based on $d_i = \overline{J}_i - \underline{J}_i$ for each $i \in N$.

Now, we briefly discuss about how to handle interval solutions in rare cases where besides the realization of $w(N)$ also realizations of $w(S)$ for some $S \subset N$ are known. Suppose first that the uncertainty on all outcomes is removed, implying that a selection of the initial interval game is available. Then, we can use for this selection a suitable classical solution to determine a posteriori uncertainty-free individual shares. Now, suppose that only the uncertainty of the outcomes of a few coalitions (including the total outcome) was removed. In such situations, we deal with a classical cooperative game with restricted cooperation and we can determine a posteriori uncertainty-free individual shares by using for this game a suitable classical solution concept.

Finally, in case the situation with interval data at stake is modeled as a cost interval game, any interval solution concept defined on suitable subclasses of such games is a candidate, and similar procedures with those described in this section are applicable.

Briefly summarizing, in Chapter 2, the model of cooperative interval games has been introduced. Selection-type and interval-type solution concepts for cooperative interval games were intensively studied. We have also focussed on the essential issue of handling interval solutions. In the next chapter, we introduce the notion of $\mathcal{I}$-balancedness and give some results.

CHAPTER 3

$\mathcal{I}$-BALANCED INTERVAL GAMES

An interval game $w \in IG^N$ is called *$\mathcal{I}$-balanced*, if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)w(S) \preccurlyeq w(N)$. The class of $\mathcal{I}$-balanced interval games is denoted by $\mathcal{I}BIG^N$ and a game $w \in IG^N$ for which all subgames are $\mathcal{I}$-balanced is called a *totally $\mathcal{I}$-balanced* game. The class of totally $\mathcal{I}$-balanced games is denoted by $T\mathcal{I}BIG^N$. In the following proposition, a relation between balancedness in terms of selections and $\mathcal{I}$-balancedness is given.

Proposition 3.1 *Let $< N, w >$ be a strongly balanced interval game; then $< N, w >$ is $\mathcal{I}$-balanced.*

Proof. Take a balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$. Then

$$\overline{w}(N) \geq \underline{w}(N) \geq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\overline{w}(S) \geq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\underline{w}(S).$$

So, $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)w(S) \preccurlyeq w(N)$. Hence, $< N, w >$ is $\mathcal{I}$-balanced. ■

Note that the converse of the Proposition 3.1 is not true since there exists $v \in Sel(w)$ with $C(v) \neq \emptyset$, implying that the core $C(w)$ is nonempty, but the interval core may be empty as we learn from Example 2.3.2.

In the next theorem, we extend to interval games the well-known result of classical cooperative game theory that a game $v \in G^N$ is balanced if and ony if $C(v)$ is nonempty (see Theorem 1.32 in Branzei, Dimitrov and Tijs (2005)) by using the duality theorem from linear programming theory.

Theorem 3.1 *Let $w \in IG^N$. Then the following two assertions are equivalent:*

(i) $C(w) \neq \emptyset$;

(ii) *The game w is $\mathcal{I}$-balanced.*

Proof. First, using Remark 2.3.1, we note that $C(w) \neq \emptyset$ if and only if the following two equalities hold simultaneously:

$$\underline{w}(N) = \min\left\{\sum_{i\in N}\underline{I}_i \mid \sum_{i\in S}\underline{I}_i \geq \underline{w}(S), \text{for each } S \in 2^N \setminus \{\emptyset\}\right\}, \quad (3.1)$$

$$\overline{w}(N) = \min\left\{\sum_{i\in N}\overline{I}_i \mid \sum_{i\in S}\overline{I}_i \geq \overline{w}(S), \text{for each } S\right\}. \quad (3.2)$$

We consider the matrix A whose columns are the characteristic vectors e^S, $S \in 2^N \setminus \{\emptyset\}$, and apply the duality theorem from linear programming theory (Dantzig (1963), Gale, Kuhn and Tucker (1951)). Then, (3.1) holds true if and only if

$$\underline{w}(N) = \max\left\{\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)\underline{w}(S) \mid \sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)e^S = e^N, \lambda \geq 0\right\}, \quad (3.3)$$

and (3.2) is satisfied if and only if

$$\overline{w}(N) = \max\left\{\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)\overline{w}(S) \mid \sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)e^S = e^N, \lambda \geq 0\right\}. \quad (3.4)$$

Now, note that (3.3) holds if and only if

$$\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)\underline{w}(S) \leq \underline{w}(N), \text{ for each } \lambda \geq 0 \text{ such that } \sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)e^S = e^N, \quad (3.5)$$

whereas (3.4) is guaranteed if and only if

$$\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)\overline{w}(S) \leq \overline{w}(N), \text{ for each } \lambda \geq 0 \text{ such that } \sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)e^S = e^N. \quad (3.6)$$

Finally, we note that (3.5) and (3.6) together express the $\mathcal{I}$-balancedness of w. ■

Let us note that the interval game in Example 2.3.2 is not $\mathcal{I}$-balanced since $w(1) + w(2) \not\succcurlyeq w(1,2)$. According to Theorem 3.1 we conclude that $C(w) = \emptyset$.

Remark 3.1 *If $C(w)$ is not empty, then $C(\underline{w})$ and $C(\overline{w})$ are both nonempty.*

We note that if all the worth intervals of the interval game $< N, w >$ are degenerate intervals, then strongly balancedness and $\mathcal{I}$-balancedness of the game also correspond to the classical balancedness.

The next proposition gives a description of the interval core of a unanimity interval game and shows that on the class of unanimity games the interval core and the interval dominance core coincide. We define $\mathcal{K}$ as follows:

$$\mathcal{K} := \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = J, \underline{I}_i \geq 0, \forall i \in N, I_i = [0,0] \text{ for } i \in N \setminus T \right\}.$$

Proposition 3.2 *Let $< N, u_{T,J} >$ be the unanimity interval game based on the coalition T and the payoff interval $J \succcurlyeq [0,0]$. Then, $\mathcal{DC}(u_{T,J}) = C(u_{T,J}) = \mathcal{K}$.*

Proof. First, we prove that the interval core of $u_{T,J}$ can be described as the set $\mathcal{K}$. In order to show that $C(u_{T,J}) \subset \mathcal{K}$, let $(I_1, \ldots, I_n) \in C(u_{T,J})$. Clearly, for each $i \in N$ we have $I_i \succcurlyeq u_{T,J}(\{i\})$ and $u_{T,J}(\{i\}) \succcurlyeq [0,0]$. So, $\underline{I}_i \geq 0$ for all $i \in N$. Furthermore, $\sum_{i \in N} I_i = u_{T,J}(N) = J$. Since also $\sum_{i \in T} I_i \succcurlyeq J$, we conclude that $I_i = 0$ for $i \in N \setminus T$. So, $(I_1, \ldots, I_n) \in \mathcal{K}$. In order to show that $\mathcal{K} \subset C(u_{T,J})$, let $(I_1, \ldots, I_n) \in \mathcal{K}$. So, $\underline{I}_i \geq 0$ for all $i \in N$, $I_i = [0,0]$ if $i \in N \setminus T$, $\sum_{i \in N} I_i = J$. Then $(I_1, \ldots, I_n) \in C(u_{T,J})$, because it also holds:

(i) $\sum_{i \in S} I_i \succcurlyeq [0,0] = u_{T,J}(S)$ if $T \setminus S \neq \emptyset$,

(ii) $\sum_{i \in S} I_i = \sum_{i \in N} I_i = u_{T,J}(N) = J = u_{T,J}(S)$ if $T \subset S$.

Next, we prove that $C(u_{T,J}) = \mathcal{DC}(u_{T,J})$. Note first that $C(u_{T,J}) \subset \mathcal{DC}(u_{T,J})$ by Theorem 2.3.6. We only have to prove that $\mathcal{DC}(u_{T,J}) \subset C(u_{T,J})$ or we need to show that for each $I \notin C(u_{T,J})$ we have $I \notin \mathcal{DC}(u_{T,J})$. Take $I \notin C(u_{T,J})$. Then, there is a $k \in N \setminus T$ with $I_k \neq [0,0]$. Then, $I' \text{ dom}_T\, I$, where $I'_i = [0,0]$ for $i \in N \setminus T$ and $I'_i = I_i + \frac{1}{|T|} I_k$ for $i \in T$. So, $I \notin \mathcal{DC}(u_{T,J})$. ■

Notice that Proposition 3.2 shows that unanimity interval games are $\mathcal{I}$-balanced games.

Remark 3.2 *From Proposition 3.2 we obtain that the core $C(u_{T,[1,1]})$ of the unanimity interval game $u_{T,J}$ with $J = [1,1]$ is*

$$C(u_{T,[1,1]}) = \left\{ I \in I(\mathbb{R})^N | \sum_{i \in N} I_i = [1,1], \underline{I}_i \geq 0, \forall i \in N, I_i = [0,0] \textit{ for } i \in N \setminus T \right\}.$$

We notice that the interval core of the unanimity interval game based on the degenerate interval $J = [1,1]$ corresponds to the core of the unanimity game in the traditional case because all I_i are degenerate for $I \in \mathcal{C}(u_{T,[1,1]})$.

The next example illustrates the fact that the interval core might coincide with the interval dominance core also for games which are not unanimity interval games.

Example 3.1 *Consider the game w in Example 2.3.1. We will show that $\mathcal{DC}(w) = \mathcal{C}(w)$. Take $I = (I_1, I_2, I_3) \in \mathcal{I}(w)$. Note that if $I_1 \neq [0,0]$ then $([0,0], I_2 + \frac{1}{2}I_1, I_3 + \frac{1}{2}I_1)\ dom_{\{2,3\}}\ (I_1, I_2, I_3)$. So, $I \notin \mathcal{DC}(w)$. Similarly, if $I_2 \neq [0,0]$, then $I \notin \mathcal{DC}(w)$. Hence, $\mathcal{DC}(w) \subset \{([0,0],[0,0],J)\} = \mathcal{C}(w)$ by Example 2.3.1. On the other hand we know, in view of Theorem 2.3.6, that $\mathcal{C}(w) \subset \mathcal{DC}(w)$. So, we conclude that $\mathcal{DC}(w) = \mathcal{C}(w)$.*

In the next proposition, we connect the $\mathcal{I}$-balancedness of $< N, w >$ with the balancedness of its border games.

Proposition 3.3 *If $< N, w >$ is $\mathcal{I}$-balanced, then the border games $< N, \underline{w} >$ and $< N, \overline{w} >$ are balanced.*

Proof. Let $< N, w >$ be $\mathcal{I}$-balanced. Then, for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) w(S) \preccurlyeq w(N)$ implying that $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \underline{w}(S) \leq \underline{w}(N)$ and $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \overline{w}(S) \leq \overline{w}(N)$, which express the balancedness of the border games of w. ■

We define the *square interval core* $\mathcal{C}^{\square} : IG^N \twoheadrightarrow I(\mathbb{R})^N$ by $\mathcal{C}^{\square}(w) := C(\underline{w}) \square C(\overline{w})$ for each $w \in IG^N$. We notice that a necessary condition for the non-emptiness of the square interval core is the balancedness of the border games.

Proposition 3.4 *Let $w \in \mathcal{I}BIG^N$. Then, $\mathcal{C}(w) = \mathcal{C}^{\square}(w)$.*

Proof. $(I_1, \ldots, I_n) \in \mathcal{C}(w)$, if and only if $(\underline{I}_1, \ldots, \underline{I}_n) \in C(\underline{w})$ and $(\overline{I}_1, \ldots, \overline{I}_n) \in C(\overline{w})$, if and only if $(I_1, \ldots, I_n) = (\underline{I}_1, \ldots, \underline{I}_n) \square (\overline{I}_1, \ldots, \overline{I}_n) \in \mathcal{C}^{\square}(w)$. ■

We define the *square Weber set* $\mathcal{W}^{\square} : IG^N \twoheadrightarrow I(\mathbb{R})^N$ by $\mathcal{W}^{\square}(w) := W(\underline{w}) \square W(\overline{w})$ for each $w \in IG^N$.

Theorem 3.2 *Let $w \in IG^N$. Then, $\mathcal{C}(w) \subset \mathcal{W}^{\square}(w)$.*

Proof. If $\mathcal{C}(w) = \emptyset$ the inclusion holds true. Suppose $\mathcal{C}(w) \neq \emptyset$ and let $(I_1, \ldots, I_n) \in \mathcal{C}(w)$. Then, by Proposition 3.4, $(\underline{I}_1, \ldots, \underline{I}_n) \in C(\underline{w})$ and $(\overline{I}_1, \ldots, \overline{I}_n) \in C(\overline{w})$, and, since $C(v) \subset W(v)$ for each $v \in G^N$, we obtain $(\underline{I}_1, \ldots, \underline{I}_n) \in W(\underline{w})$ and $(\overline{I}_1, \ldots, \overline{I}_n) \in W(\overline{w})$. Hence, we get $(I_1, \ldots, I_n) \in \mathcal{W}^{\square}(w)$. ■

In the sequel, we introduce the notion of (interval) *population monotonic allocation scheme (pmas)* for totally $\mathcal{I}$-balanced interval games, which is a direct extension of pmas for classical cooperative games (Sprumont (1990)).

We say that for a game $w \in T\mathcal{I}BIG^N$ a scheme $A = (A_{iS})_{i\in S, S\in 2^N\setminus\{\emptyset\}}$ with $A_{iS} \in I(\mathbb{R})^N$ is a pmas of w if

(i) $\sum_{i\in S} A_{iS} = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $A_{iS} \preccurlyeq A_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

As a conclusion in this chapter, we have put the emphasis on $\mathcal{I}$-balanced interval games. We proved that the interval core of a cooperative interval game is nonempty if and only if the game is $\mathcal{I}$-balanced. The notion of population monotonic allocation scheme (pmas) in the interval setting became introduced. The next chapter deals with size monotonic interval games.

CHAPTER 4

SIZE MONOTONIC INTERVAL GAMES

We call a game $< N, w >$ *size monotonic* if $< N, |w| >$ is monotonic, i.e., $|w|(S) \le |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set N.

We notice that size monotonic games may have an empty interval core. In this chapter, we introduce marginal operators on the class of size monotonic interval games, define the Shapley value and the Weber set on this class of games.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$ of N. Let $w \in SMIG^N$. We introduce the notions of *interval marginal operator* corresponding to σ, denoted by m^σ, and of *interval marginal vector* of w with respect to σ, denoted by $m^\sigma(w)$. The marginal vector $m^\sigma(w)$ corresponds to a situation, where the players enter a room one by one in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$, and each player is given the marginal contribution he/she creates by entering. If we denote the set of predecessors of i in σ by $P_\sigma(i) := \left\{r \in N|\sigma^{-1}(r) < \sigma^{-1}(i)\right\}$, where $\sigma^{-1}(i)$ denotes the entrance number of player i, then $m^\sigma_{\sigma(k)}(w) = w(P_\sigma(\sigma(k)) \cup \{\sigma(k)\}) - w(P_\sigma(\sigma(k)))$ or in short $m^\sigma_i(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$. We notice that $m^\sigma(w)$ is an efficient interval payoff vector for each $\sigma \in \Pi(N)$. For size monotonic games $< N, w >$, $w(T) - w(S)$ is defined for all $S, T \in 2^N$ with $S \subset T$ since $|w(T)| = |w|(T) \ge |w|(S) = |w(S)|$. Now, we notice that for each $w \in SMIG^N$ the interval marginal vectors $m^\sigma(w)$ are defined for each $\sigma \in \Pi(N)$, because the monotonicity of $|w|$ implies $\overline{w}(S \cup \{i\}) - \underline{w}(S \cup \{i\}) \ge \overline{w}(S) - \underline{w}(S)$, which can be rewritten as $\overline{w}(S \cup \{i\}) - \overline{w}(S) \ge \underline{w}(S \cup \{i\}) - \underline{w}(S)$. So, $w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \notin S$. The following example illustrates that for interval games which are not size monotonic it might happen that some interval marginal vectors do not exist.

Example 4.1 *Let $< N, w >$ be the interval game with $N = \{1, 2\}$, $w(1) = [1, 3]$, $w(2) = [0, 0]$ and $w(1, 2) = [2, 3\frac{1}{2}]$. This game is not size monotonic. Note that $m^{(12)}(w)$ is not defined because $w(1, 2) - w(1)$ is undefined since $|w(1, 2)| < |w(1)|$.*

Now, we straightforwardly extend for size monotonic interval games two important solution concepts in cooperative game theory which are based on marginal worth vectors: the Shapley value (Shapley (1953)) and the Weber set (Weber (1988)).

The *interval Weber set* $\mathcal{W}$ on the class of size monotonic interval games is defined by $\mathcal{W}(w) := conv\{m^\sigma(w)|\sigma \in \Pi(N)\}$ for each $w \in SMIG^N$. We notice that for traditional TU-games we have $W(v) \neq \emptyset$ for all $v \in G^N$, while for arbitrary interval games it might not exist (in case none of the interval marginal vectors $m^\sigma(w)$ is defined). Clearly, $\mathcal{W}(w) \neq \emptyset$ for all $w \in SMIG^N$. The *interval Shapley value* $\Phi : SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N. \quad (4.1)$$

We can write (4.1) as follows

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))). \quad (4.2)$$

The terms after the summation sign in (4.2) are of the form $w(S \cup \{i\}) - w(S)$, where S is a subset of N not containing i.

Note that there are exactly $|S|!(n - 1 - |S|)!$ orderings for which one has $P^\sigma(\{i\}) = S$. The first factor, $|S|!$, corresponds to the number of orderings of S and the second factor, $(n - 1 - |S|)!$, is just the number of orderings of $N \setminus (S \cup \{i\})$. Using this, we can rewrite (4.2) as

$$\Phi_i(w) = \sum_{S: i \notin S} \frac{|S|!(n - 1 - |S|)!}{n!} (w(S \cup \{i\}) - w(S))). \quad (4.3)$$

Note that

$$\sum_{S: i \notin S} \frac{|S|!(n - 1 - |S|)!}{n!} = 1. \quad (4.4)$$

Proposition 4.1 *The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R})^N$ is additive.*

Proof. First, we show that for each $\sigma \in \Pi(N)$ the interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ is additive, i.e., for all $w_1, w_2 \in SMIG^N$, $m^\sigma(w_1 + w_2) = m^\sigma(w_1) + m^\sigma(w_2)$.

Let $\sigma \in \Pi(N)$ and $k \in N$. Then,

$$\begin{aligned}
m^{\sigma}_{\sigma(k)}(w_1 + w_2) &= (w_1 + w_2)(\sigma(1), \ldots, \sigma(k)) \\
&- (w_1 + w_2)(\sigma(1), \ldots, \sigma(k-1)) \\
&= w_1(\sigma(1), \ldots, \sigma(k)) - w_1(\sigma(1), \ldots, \sigma(k-1)) \\
&+ w_2(\sigma(1), \ldots, \sigma(k)) - w_2(\sigma(1), \ldots, \sigma(k-1)) \\
&= m^{\sigma}_{\sigma(k)}(w_1) + m^{\sigma}_{\sigma(k)}(w_2).
\end{aligned}$$

Now, using the additivity property of interval marginal operators we obtain that $\Phi : SMIG^N \rightarrow I(\mathbb{R})^N$ is an *additive* map, i.e.,

$$\begin{aligned}
\Phi(w_1 + w_2) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(w_1 + w_2) \\
&= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(w_1) + \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(w_2) \\
&= \Phi(w_1) + \Phi(w_2),
\end{aligned}$$

for all $w_1, w_2 \in SMIG^N$. ∎

Let $w \in SMIG^N$ and $i, j \in N$. Then, i and j are called *symmetric players*, if $w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S)$, for each S with $i, j \notin S$. We leave the proof of the following proposition to the reader.

Proposition 4.2 *Let $i, j \in N$ be symmetric players in $w \in SMIG^N$. Then, $\Phi_i(w) = \Phi_j(w)$.*

Let $w \in SMIG^N$ and $i \in N$. Then, i is called a *dummy player* if $w(S \cup \{i\}) = w(S) + w(\{i\})$, for each $S \in 2^{N \setminus \{i\}}$.

Proposition 4.3 *The interval Shapley value $\Phi : SMIG^N \rightarrow I(\mathbb{R})^N$ has the dummy player property, i.e. $\Phi_i(w) = w(\{i\})$ for all $w \in SMIG^N$ and for all dummy players i in w.*

Proof. This follows from (4.3) by taking (4.4) into account. ∎

Proposition 4.4 *The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R})^N$ is efficient, i.e., $\sum_{i\in N} \Phi_i(w) = w(N)$.*

Proof. First, we show that for each $\sigma \in \Pi(N)$ the interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ is efficient, i.e. $\sum_{i\in N} m_i^\sigma(w) = w(N)$.
Let $w \in SMIG^N$ and $\sigma \in \Pi(N)$. Then,

$$\begin{aligned}\sum_{i\in N} m_i^\sigma(w) &= \sum_{k=1}^{N} m_{\sigma(k)}^\sigma(w)\\ &= w(\sigma(1)) + \sum_{k=2}^{n} w(\sigma(1),\dots,\sigma(k)) - w(\sigma(1),\dots,\sigma(k-1))\\ &= w(\sigma(1)) + w(\sigma(1),\dots,\sigma(n)) - w(\sigma(1)) = w(N).\end{aligned}$$

Now, using the efficiency of interval marginal operators, we obtain that $\Phi : SMIG^N \to I(\mathbb{R})^N$ is an efficient map, i.e.,

$$\sum_{i\in N} \Phi_i(w) = \frac{1}{n!}\sum_{i\in N}\sum_{\sigma\in\Pi(N)} m_i^\sigma(w) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)}\sum_{i\in N} m_i^\sigma(w) = \frac{1}{n!} n! w(N) = w(N),$$

for each $w \in SMIG^N$. ■

Proposition 4.5 *Let $w \in SMIG^N$ and let $\sigma \in \Pi(N)$. Then, $m_i^\sigma(w) = \left[m_i^\sigma(\underline{w}), m_i^\sigma(\overline{w})\right]$ for all $i \in N$.*

Proof. By definition,

$$m^\sigma(\underline{w}) = (\underline{w}(\sigma(1)), \underline{w}(\sigma(1),\sigma(2)) - \underline{w}(\sigma(1)), \dots, \underline{w}(\sigma(1),\dots,\sigma(n)) - \underline{w}(\sigma(1),\dots,\sigma(n-1))),$$

and

$$m^\sigma(\overline{w}) = (\overline{w}(\sigma(1)), \overline{w}(\sigma(1),\sigma(2)) - \overline{w}(\sigma(1)), \dots, \overline{w}(\sigma(1),\dots,\sigma(n)) - \overline{w}(\sigma(1),\dots,\sigma(n-1))).$$

Now, we prove that $m^\sigma(\overline{w}) - m^\sigma(\underline{w}) \geq 0$. Since $|w| = \overline{w} - \underline{w}$ is a classical convex game, we have for each $k \in N$

$$\begin{aligned}m_{\sigma(k)}^\sigma(\overline{w}) - m_{\sigma(k)}^\sigma(\underline{w}) &= (\overline{w} - \underline{w})(\sigma(1),\dots,\sigma(k)) - (\overline{w} - \underline{w})(\sigma(1),\dots,\sigma(k-1))\\ &= |w|(\sigma(1),\dots,\sigma(k)) - |w|(\sigma(1),\dots,\sigma(k-1)) \geq 0,\end{aligned}$$

where the inequality follows from the monotonicity of $|w|$. So, $m_i^\sigma(\underline{w}) \leq m_i^\sigma(\overline{w})$ for all $i \in N$, and

$$\left(\left[m_i^\sigma(\underline{w}), m_i^\sigma(\overline{w})\right]\right)_{i\in N} = (w(\sigma(1)),\dots,w(\sigma(1),\dots,\sigma(n)) - w(\sigma(1),\dots,\sigma(n-1))) = m^\sigma(w).$$

■

Proposition 4.6 *Let $w \in SMIG^N$ and let $\sigma \in \Pi(N)$. Then, $\Phi_i(w) = \left[\phi_i(\underline{w}), \phi_i(\overline{w})\right]$ for all $i \in N$.*

Proof. From (4.1) and Proposition 4.5 we have

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^{\sigma}(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \left[m_i^{\sigma}(\underline{w}), m_i^{\sigma}(\overline{w})\right] =$$

$$\left[\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^{\sigma}(\underline{w}), \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^{\sigma}(\overline{w})\right] = \left[\phi_i(\underline{w}), \phi_i(\overline{w})\right].$$

■

This chapter has been devoted to size monotonic games. The Weber set and the Shapley value have been defined on this class, and their relations were studied. The next chapter will conside intensively an interesting class of cooperative interval games, called convex interval games.

CHAPTER 5

CONVEX INTERVAL GAMES

This chapter is based on Alparslan Gök, Branzei and Tijs (2008b) and Branzei, Tijs and Alparslan Gök (2008a).

5.1 DEFINITION AND RELATIONS WITH OTHER CLASSES OF GAMES

We introduce the notion of convex interval game and denote by CIG^N the class of convex interval games with player set N. We call a game $w \in IG^N$ *convex* if $< N, w >$ is supermodular and $|w|(S) + |w|(T) \leq |w|(S \cup T) + |w|(S \cap T)$ for all $S, T \in 2^N$.
An interval game $< N, w >$ is called *concave*, if $< N, w >$ is submodular and $|w|(S)+|w|(T) \geq |w|(S \cup T) + |w|(S \cap T)$ for all $S, T \in 2^N$.
Next we give as a motivating example a situation with an economic flavour leading to a convex interval game.

Example 5.1.1 *Let $N = \{1, 2, \dots, n\}$ and let $f : [0, n] \to I(\mathbb{R})$ be such that $f(x) = [f_1(x), f_2(x)]$ for each $x \in [0, n]$ and $f(0) = [0, 0]$. Suppose that $f_1 : [0, n] \to \mathbb{R}$, $f_2 : [0, n] \to \mathbb{R}$ and $(f_2 - f_1) : [0, n] \to \mathbb{R}$ are convex monotonic increasing functions. Then, we can construct a corresponding interval game $w : 2^N \to I(\mathbb{R})$ such that $w(S) = f(|S|) = [f_1(|S|), f_2(|S|)]$ for each $S \in 2^N$. It is easy to show that w is a convex interval game with the symmetry property $w(S) = w(T)$ for each $S, T \in 2^N$ with $|S| = |T|$.*
We can regard $< N, w >$ as a production game if we interpret $f(s)$ for $s \in N$ as the interval reward which s players in N can produce by working together.

Convex games are useful for modeling economic and OR situations like public good situations (Moulin (1988)) and sequencing situations (Curiel, Pederzoli and Tijs (1989)). In case where the parameters determining such situations are not numbers but intervals, under certain conditions also convex interval games may appear. Also, special bankruptcy situations (O'Neill (1982), Aumann and Maschler (1985) and Curiel, Maschler and Tijs (1987)) when the estate of the bank and the claims are intervals give rise in a natural way to convex interval games. Some economic and OR situations with interval data lead to concave interval games instead of convex interval games.

Note that the nonempty set CIG^N is a subcone of IG^N. The next proposition shows that traditional convex games can in a natural way be embedded into the class of convex interval games. The proof of the next proposition is straightforward.

Proposition 5.1.1 *If $v \in G^N$ is convex, then the corresponding game $w \in IG^N$ which is defined by $w(S) := [v(S), v(S)]$ for each $S \in 2^N$ is also convex.*

Let us note the fact that $< N, |w| >$ is supermodular implies that $< N, |w| >$ is monotonic, because for each $S, T \in 2^N$ with $S \subset T$ we have

$$|w|(T) + |w|(\emptyset) \geq |w|(S) + |w|(T \setminus S),$$

and from this inequality follows $|w|(S) \leq |w|(T)$ since $|w|(T \setminus S) \geq 0$. So, $CIG^N \subset SMIG^N$. Then we obtain from Proposition 4.5 that $m_i^{\sigma}(w) = \left[m_i^{\sigma}(\underline{w}), m_i^{\sigma}(\overline{w})\right]$ for each $w \in CIG^N$, $\sigma \in \Pi(N)$ and for all $i \in N$. From Proposition 4.6 we obtain that for each $w \in CIG^N$ we have $\Phi_i(w) = \left[\phi_i(\underline{w}), \phi_i(\overline{w})\right]$ for all $i \in N$.

5.2 CHARACTERIZATIONS OF CONVEX INTERVAL GAMES

Proposition 5.2.1 gives some characterizations of supermodular and convex games $w \in IG^N$ based on their related length game $|w| \in G^N$ and border games $\underline{w}, \overline{w} \in G^N$.

Proposition 5.2.1 *Let $w \in IG^N$ and its related games $|w|, \underline{w}, \overline{w} \in G^N$. Then the following assertions hold:*

(i) *A game $< N, w >$ is supermodular if and only if its border games $< N, \underline{w} >$ and $< N, \overline{w} >$ are convex.*

(ii) *A game $< N, w >$ is convex if and only if its length game $< N, |w| >$ and its border games $< N, \underline{w} >, < N, \overline{w} >$ are convex.*

(iii) *A game $< N, w >$ is convex if and only if its border game $< N, \underline{w} >$ and the game $< N, \overline{w} - \underline{w} >$ are convex.*

Proof.

(i) This assertion follows from formula (2.1.1).

(ii) By definition $< N, w >$ is convex if and only if $< N, w >$ and $< N, |w| >$ are both supermodular. By (i), $< N, w >$ is supermodular if and only if its border games are convex. Now, since supermodularity of $< N, |w| >$ is equivalent with its convexity, we conclude that $< N, w >$ is convex if and only if $< N, \underline{w} >, < N, \overline{w} >$ and $< N, |w| >$ are convex.

(iii) This assertion follows easily from (ii) by noting that $< N, |w| >, < N, \underline{w} >$ and $< N, \overline{w} >$ are convex if and only if $< N, \overline{w} - \underline{w} >$ and $< N, \underline{w} >$ are convex because $\overline{w} = \underline{w} + |w|$.

■

Remark 5.2.1 *First, we note that (2.1.3) is equivalent to the superadditivity of the lower game and the upper game. Additionally, notice that, by Proposition 5.2.1, if $w \in CIG^N$, then $< N, w >$ is superadditive; further, $< N, |w| >, < N, \overline{w} >$ and $< N, \underline{w} >$ are superadditive.*

Proposition 5.2.2 gives some characterizations of submodular and concave games $w \in IG^N$ based on their related length game $|w| \in G^N$ and border games $\underline{w}, \overline{w} \in G^N$.

Proposition 5.2.2 *Let $< N, w >$ be an interval game. Then the following assertions hold:*

(i) *A game $< N, w >$ is submodular if and only if $< N, \underline{w} >$ and $< N, \overline{w} >$ are concave (or submodular).*

(ii) *A game $< N, w >$ is concave if and only if $< N, |w| >$ and $< N, \underline{w} >, < N, \overline{w} >$ are concave (or submodular).*

(iii) *A game* $< N, w >$ *is concave if and only if* $< N, \underline{w} >$ *and* $< N, |w| >$ *are concave (or submodular).*

Proof.

(i) This assertion follows from formula (2.1.2).

(ii) By definition $< N, w >$ is concave if and only if $< N, w >$ and $< N, |w| >$ are both submodular. By (i), $< N, w >$ is submodular if and only if its border games are concave (or submodular). Now, since submodularity of $< N, |w| >$ is the same with its concavity, we conclude that $< N, w >$ is concave if and only if $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| >$ are concave (or submodular).

(iii) This assertion follows easily from (ii) by noting that $< N, |w| >$, $< N, \underline{w} >$ and $< N, \overline{w} >$ are concave (or submodular) if and only if $< N, |w| >$ and $< N, \underline{w} >$ are concave (or submodular) because $\overline{w} = \underline{w} + |w|$.

■

The next example shows that a supermodular interval game is not necessarily convex.

Example 5.2.1 *Let* $< N, w >$ *be the two-person interval game with* $w(\emptyset) = [0, 0]$*,* $w(1) = w(2) = [0, 1]$ *and* $w(1, 2) = [3, 4]$*. Here,* $< N, w >$ *is supermodular and the border games are convex, but* $|w|(1) + |w|(2) = 2 > 1 = |w|(1, 2) + |w|(\emptyset)$*. Hence,* $< N, w >$ *is not convex.*

The next example shows that an interval game whose length game is supermodular is not necessarily convex.

Example 5.2.2 *Let* $< N, w >$ *be the three-person interval game with* $w(i) = [1, 1]$ *for each* $i \in N$*,* $w(N) = w(1, 3) = w(1, 2) = w(2, 3) = [2, 2]$ *and* $w(\emptyset) = [0, 0]$*. Here,* $< N, w >$ *is not convex, but* $< N, |w| >$ *is supermodular, since* $|w|(S) = 0$*, for each* $S \in 2^N$*.*

Interesting examples of convex interval games are unanimity interval games. Clearly, $< N, \left|u_{T,J}\right| >$ is supermodular.

The supermodularity of $< N, u_{T,J} >$ can be checked by considering the following case study:

	$u_{T,J}(A \cup B)$	$u_{T,J}(A \cap B)$	$u_{T,J}(A)$	$u_{T,J}(B)$
$T \subset A, T \subset B$	J	J	J	J
$T \subset A, T \not\subset B$	J	$[0,0]$	J	$[0,0]$
$T \not\subset A, T \subset B$	J	$[0,0]$	$[0,0]$	J
$T \not\subset A, T \not\subset B$	J or $[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$.

Theorem 5.2.3 *Let $w \in IG^N$ be such that $|w| \in G^N$ is supermodular. Then, the following three assertions are equivalent:*

(i) *$w \in IG^N$ is convex.*

(ii) *For all $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$ we have*

$$w(S_1 \cup U) - w(S_1) \preccurlyeq w(S_2 \cup U) - w(S_2). \quad (5.2.1)$$

(iii) *For all $S_1, S_2 \in 2^N$ and $i \in N$ such that $S_1 \subset S_2 \subset N \setminus \{i\}$ we have*

$$w(S_1 \cup \{i\}) - w(S_1) \preccurlyeq w(S_2 \cup \{i\}) - w(S_2).$$

Proof. We show (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i). Suppose that (i) holds. To prove (ii) take $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$. From (2.1.1) with $S_1 \cup U$ in the role of S and S_2 in the role of T we obtain (5.2.1) by noting that $S \cup T = S_2 \cup U$, $S \cap T = S_1$. Hence, (i) implies (ii).

That (ii) implies (iii) is straightforward (take $U = \{i\}$).

Now, suppose that (iii) holds. To prove (i) take $S, T \in 2^N$. Clearly, (2.1.1) holds if $S \subset T$. Suppose that $T \setminus S$ consists of the elements $i_1, \ldots, i_k$ and let $D = S \cap T$. Then, from (*iii*) follows that

$$\begin{aligned}
w(S) - w(S \cap T) &= w(D \cup \{i_1\}) - w(D) \\
&+ \sum_{s=2}^{k} w(D \cup \{i_1, \ldots, i_s\}) - w(D \cup \{i_1, \ldots, i_{s-1}\}) \\
&\preccurlyeq w(T \cup \{i_1\}) - w(T) \\
&+ \sum_{s=2}^{k} w(T \cup \{i_1, \ldots, i_s\}) - w(T \cup \{i_1, \ldots, i_{s-1}\}) \\
&= w(S \cup T) - w(T),
\end{aligned}$$

for each $S \in 2^N$. ■

We notice that the characterizations of convex interval games in Theorem 1.2.2 are inspired by Shapley (1971). The next proposition provides additional characterizations of concave interval games.

Proposition 5.2.4 *Let $w \in IG^N$ be such that $|w| \in G^N$ is submodular. Then, the following three assertions are equivalent:*

(i) *$w \in IG^N$ is concave.*

(ii) *For all $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$ we have*

$$w(S_1 \cup U) - w(S_1) \succcurlyeq w(S_2 \cup U) - w(S_2).$$

(iii) *For all $S_1, S_2 \in 2^N$ and $i \in N$ such that $S_1 \subset S_2 \subset N \setminus \{i\}$ we have*

$$w(S_1 \cup \{i\}) - w(S_1) \succcurlyeq w(S_2 \cup \{i\}) - w(S_2).$$

Proof. To prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i) we simply replace the inequality sign $\preccurlyeq$ in the proof of Theorem 5.2.3 by the inequality sign $\succcurlyeq$. ■

A characterization of convex interval games with the aid of interval marginal vectors is given in the next theorem.

Theorem 5.2.5 *Let $w \in IG^N$. Then, the following assertions are equivalent:*

(i) *w is convex.*

(ii) *$|w|$ is supermodular and $m^\sigma(w) \in \mathcal{C}(w)$ for all $\sigma \in \Pi(N)$.*

Proof. (i) $\Rightarrow$ (ii) Let $w \in CIG^N$, let $\sigma \in \Pi(N)$ and take $m^\sigma(w)$. Clearly, $\sum_{k\in N} m_k^\sigma(w) = w(N)$. To prove that $m^\sigma(w) \in \mathcal{C}(w)$ we have to show that for $S \in 2^N$, $\sum_{k\in S} m_k^\sigma(w) \succcurlyeq w(S)$.

Let $S = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\}$ with $i_1 < i_2 < \ldots < i_k$. Then,

$$\begin{aligned} w(S) &= w(\sigma(i_1)) - w(\emptyset) \\ &+ \sum_{r=2}^{k} (w(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)) - w(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{r-1}))) \\ &\preccurlyeq w(\sigma(1), \ldots, \sigma(i_1)) - w(\sigma(1), \ldots, \sigma(i_1 - 1)) \\ &+ \sum_{r=2}^{k} (w(\sigma(1), \sigma(2), \ldots, \sigma(i_r)) - w(\sigma(1), \sigma(2), \ldots, \sigma(i_r - 1))) \\ &= \sum_{r=1}^{k} m^{\sigma}_{\sigma(i_r)}(w) = \sum_{k \in S} m^{\sigma}_{k}(w), \end{aligned}$$

where the inequality follows from Theorem 5.2.3 (iii) applied to $i = \sigma(i_r)$ and

$$S_1 := \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{r-1})\} \subset S_2 := \{\sigma(1), \sigma(2), \ldots, \sigma(i_r - 1)\}$$

for $r \in \{1, 2, \ldots, k\}$. Further, by convexity of w, $|w|$ is supermodular.

(ii) $\Rightarrow$ (i) From $m^{\sigma}(w) \in \mathcal{C}(w)$ for all $\sigma \in \Pi(N)$ it follows that $m^{\sigma}(\underline{w}) \in C(\underline{w})$ and $m^{\sigma}(\overline{w}) \in C(\overline{w})$ for all $\sigma \in \Pi(N)$. Now, by Theorem 1.2.2 we obtain that $< N, \underline{w} >$ and $< N, \overline{w} >$ are convex games. Since $< N, |w| >$ is convex by hypothesis, we learn from Proposition 5.2.1 (ii) that $< N, w >$ is convex. ■

However, the well-known result in Theorem 1.2.2 (v) can not be extended to convex interval games as we prove in the following proposition.

Proposition 5.2.6 *Let $w \in CIG^N$. Then, $\mathcal{W}(w) \subset \mathcal{C}(w)$.*

Proof. By Theorem 5.2.5 we have $m^{\sigma}(w) \in \mathcal{C}(w)$ for each $\sigma \in \Pi(N)$. Now, we use the convexity of $\mathcal{C}(w)$. ■

The following example shows that the inclusion in Proposition 5.2.6 might be strict, i.e., with $\neq$, different from Theorem 1.2.2 (v).

Example 5.2.3 *Let $N = \{1, 2\}$ and let $w : 2^N \to I(\mathbb{R})$ be defined by $w(1) = w(2) = [0, 1]$ and $w(1, 2) = [2, 4]$. This game is convex. Further, $m^{(1,2)}(w) = ([0, 1], [2, 3])$ and $m^{(2,1)}(w) = ([2, 3], [0, 1])$, belong to the interval core $\mathcal{C}(w)$ and $\mathcal{W}(w) = conv\left\{m^{(1,2)}(w), m^{(2,1)}(w)\right\}$. Notice that $([\frac{1}{2}, 1\frac{3}{4}], [1\frac{1}{2}, 2\frac{1}{4}]) \in \mathcal{C}(w)$ and there is no $\alpha \in [0, 1]$ such that*

$$\alpha m^{(1,2)}(w) + (1 - \alpha) m^{(2,1)}(w) = ([\frac{1}{2}, 1\frac{3}{4}], [1\frac{1}{2}, 2\frac{1}{4}]).$$

Hence, $\mathcal{W}(w) \subset \mathcal{C}(w)$ and $\mathcal{W}(w) \neq \mathcal{C}(w)$.

Since $\Phi(w) \in \mathcal{W}(w)$ for each $w \in SMIG^N$, by Proposition 5.2.6 we have $\Phi(w) \in C(w)$ for each $w \in CIG^N$.

From Theorem 3.2 and Proposition 5.2.6 we obtain that $\mathcal{W}(w) \subset \mathcal{W}^{\square}(w)$ for each $w \in CIG^N$. This inclusion might be strict as Example 5.2.3 illustrates.

Given a game $< N, w >$ and a coalition $T \subset N$, the T*-marginal interval game* $w^T : 2^{N\setminus T} \to I(\mathbb{R})$ is defined by $w^T(S) := w(S \cup T) - w(T)$ for each $S \subset N \setminus T$.

Proposition 5.2.7 *Let $< N, w >$ be a convex game and $T \subset N$. Then, $< N \setminus T, w^T >$ (see Section 2.1) is a convex game.*

Proof. Let $w \in CIG^N$. Then, $< N, w >$ and $< N, |w| >$ are supermodular. From this we obtain the supermodularity of $< N \setminus T, w^T >$ as follows. Take $S_1, S_2 \subset N \setminus T$. Then,

$$\begin{aligned} w^T(S_1 \cup S_2) + w^T(S_1 \cap S_2) &= w(S_1 \cup S_2 \cup T) - w(T) + w((S_1 \cap S_2) \cup T) - w(T) \\ &= w((S_1 \cup T) \cup (S_2 \cup T)) - w(T) \\ &+ w((S_1 \cup T) \cap (S_2 \cup T)) - w(T) \\ &\succcurlyeq w(S_1 \cup T) - w(T) + w(S_2 \cup T) - w(T) \\ &= w^T(S_1) + w^T(S_2). \end{aligned}$$

Similarly, the supermodularity (convexity) of $< N \setminus T, \left|w^T\right| >$ follows from the supermodularity (convexity) of $< N, |w| >$. Hence, $w^T \in CIG^{N\setminus T}$. ■

Theorem 5.2.8 *Let $w \in IG^N$. Then, the following assertions are equivalent:*

(i) $w \in CIG^N$.

(ii) *$< N \setminus T, w^T >$ is superadditive for each $T \subset N$.*

Proof. First, we notice that by Proposition 5.2.1 $w \in CIG^N$ if and only if $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| >$ are convex games. Now, using the characterization of classical convex games based on the superadditivity of marginal games (Branzei, Dimitrov and Tijs (2004), Martinez-Legaz (1997, 2006)), we obtain that $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| >$ are convex if and only if for each $T \subset N$, $< N \setminus T, \overline{w}^T >$, $< N \setminus T, \underline{w}^T >$ and $< N \setminus T, \left|w^T\right| >$ are superadditive games. Further, by Proposition 5.2.1 and Remark 5.2.1 this is equivalent to the superadditivity of $< N \setminus T, w^T >$ for each $T \subset N$. ■

Proposition 5.2.9 *Each convex interval game $w \in IG^N$ is an exact interval game.*

Proof. First, the convexity of $w \in IG^N$ implies by Theorem 5.2.5 that $|w|$ is supermodular (and consequently monotonic) and $m^\sigma(w) \in \mathcal{C}(w)$ for each $\sigma \in \Pi(N)$. So, let $S = \{s_1, \dots, s_k\}$ and $\sigma \in \Pi(N)$ be such that $\sigma(r) = s_r$ for $r = 1, \dots, k$. Then, $\sum_{i \in S} m_i^\sigma(w) = w(S)$. Further, the convexity of $w \in IG^N$ implies that $< N, |w| >$ is convex and, consequently, it is an *exact* game, i.e., for each $S \in 2^N$ there exists $x \in C(|w|)$ such that $\sum_{i \in S} x_i = |w|(S)$. ■

Remark 5.2.2 *For a given $S \in 2^N$ and $I = (I_1, \dots, I_n) \in \mathcal{C}(w)$, $\sum_{i \in S} I_i = w(S)$ also delivers $(\underline{I}_1, \dots, \underline{I}_n) \in C(\underline{w})$, $(\overline{I}_1, \dots, \overline{I}_n) \in C(\overline{w})$ and $(\overline{I}_1 - \underline{I}_1, \dots, \overline{I}_n - \underline{I}_n) \in C(|w|)$, with $\sum_{i \in S} \underline{I}_i = \underline{w}(S)$, $\sum_{i \in S} \overline{I}_i = \overline{w}(S)$ and $\sum_{i \in S} (\overline{I}_i - \underline{I}_i) = |w|(S)$. This can be used for extending the characterization of Biswas et al. (1999) to interval games.*

Theorem 5.2.10 *Let $w \in IG^N$. Then, the following assertions are equivalent:*

(i) $w \in CIG^N$.

(ii) $< T, w_T >$ *is exact for each* $T \subset N$.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 5.2.9 because each subgame of a convex interval game is convex and, hence, exact.
(ii) $\Rightarrow$ (i) From the exactness of each interval subgame $< T, w_T >$ we obtain that $< N, \overline{w}_T >$, $< N, \underline{w}_T >$ and $< N, |w|_T >$ are exact games for each $T \subset N$. Now, we use the result of Biswas et al. (1999) and obtain that the games $< N, \overline{w} >$, $< N, \underline{w} >$ and $< N, |w| >$ are all convex. By Proposition 5.2.1 we obtain that $w \in CIG^N$. ■

Now, we notice that $CIG^N \subset \mathcal{I}BIG^N$ and obtain that $\mathcal{C}(w) = C^\square(w)$ for each $w \in CIG^N$.
The two theorems in the next section are very interesting because they extend for interval games, with the square interval Weber set in the role of the Weber set, the well-known results of classical cooperative game theory that $C(v) \subset W(v)$ for each $v \in G^N$ (Weber (1988)) and $C(v) = W(v)$ if and only if v is convex (Ichiishi (1981)). We cope with similar issues in the interval setting. Note that $C^\square(w) = \mathcal{W}^\square(w)$ if $w \in CIG^N$.

5.3 PROPERTIES OF INTERVAL SOLUTION CONCEPTS

Theorem 5.3.1 *Let $w \in \mathcal{I}BIG^N$. Then, the following assertions are equivalent:*

(i) *w is convex.*

(ii) *$|w|$ is supermodular and $\mathcal{C}(w) = \mathcal{W}^{\square}(w)$.*

Proof. By Proposition 5.2.1, w is convex if and only if $|w|$, $\underline{w}$ and $\overline{w}$ are convex. Clearly, the convexity of $|w|$ is equivalent with its supermodularity. Further, $\underline{w}$ and $\overline{w}$ are convex if and only if $W(\underline{w}) = C(\underline{w})$ and $W(\overline{w}) = C(\overline{w})$. These equalities are equivalent with $\mathcal{W}^{\square}(w) = \mathcal{C}^{\square}(w)$. Finally, since w is $\mathcal{I}$-balanced by hypothesis, we have by Proposition 3.4 that $\mathcal{C}(w) = \mathcal{W}^{\square}(w)$.

■

Now, we define the *square interval dominance core* $\mathcal{DC}^{\square} : IG^N \twoheadrightarrow I(\mathbb{R})^N$ by

$$\mathcal{DC}^{\square}(w) := DC(\underline{w})\square DC(\overline{w})$$

for each $w \in IG^N$ and notice that for convex interval games we have

$$\mathcal{DC}^{\square}(w) = DC(\underline{w})\square DC(\overline{w}) = C(\underline{w})\square C(\overline{w}) = \mathcal{C}^{\square}(w) = \mathcal{C}(w),$$

where the second equality follows from the well-known result in the theory of TU-games that for convex games the core and the dominance core coincide, and the last equality follows from Proposition 3.4. From $\mathcal{DC}^{\square}(w) = \mathcal{C}(w)$ for each $w \in CIG^N$ and $\mathcal{C}(w) \subset \mathcal{DC}(w)$ for each $w \in IG^N$ we obtain $\mathcal{DC}(w) \supset \mathcal{DC}^{\square}(w)$ for each $w \in CIG^N$. We notice that this inclusion might be strict (see Example 2.3.3).
Finally, we will show that the interval core is additive on the class of convex interval games with the aid of Theorem 5.3.1, which is inspired by Dragan, Potters and Tijs (1989).

Proposition 5.3.2 *The interval core $\mathcal{C} : CIG^N \twoheadrightarrow I(\mathbb{R})^N$ is an additive map.*

Proof. The interval core is a superadditive solution concept for all interval games (Proposition 2.3.5). Therefore, we need to show the subadditivity of the interval core. We have to prove that $\mathcal{C}(w_1 + w_2) \subset \mathcal{C}(w_1) + \mathcal{C}(w_2)$. Note that $m^{\sigma}(w_1 + w_2) = m^{\sigma}(w_1) + m^{\sigma}(w_2)$ for each

$w_1, w_2 \in CIG^N$. By definition of the square interval Weber set we have $\mathcal{W}^{\square}(w_1 + w_2) = W(\underline{w}_1 + \underline{w}_2)\square W(\overline{w}_1 + \overline{w}_2)$. By Theorem 5.3.1, we therefore learn:

$$C(w_1 + w_2) = \mathcal{W}^{\square}(w_1 + w_2) \subset \mathcal{W}^{\square}(w_1) + \mathcal{W}^{\square}(w_2) = C(w_1) + C(w_2).$$

■

5.4 POPULATION INTERVAL MONOTONIC ALLOCATION SCHEMES

In this section, we focus on pmas on the class of convex interval games. Notice that the total $\mathcal{I}$-balancedness of an interval game is a necessary condition for the existence of a pmas for that game. A sufficient condition is the convexity of the interval game. Recall that all subgames of a convex interval game are also convex, and that for a game $w \in CIG^N$ an imputation $I = (I_1, \ldots, I_n) \in \mathcal{I}(w)$ is called *pmas extendable* if there exists a pmas $A = (A_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ such that $A_{iN} = I_i$ for each $i \in N$ (see Chapter 3). In the sequel, we show that the interval Shapley value has the population monotonicity property and, consequently, it generates a pmas.

Proposition 5.4.1 *The interval Shapley value has the population monotonicity property on the class of convex interval games.*

Proof. Let $w \in CIG^N$. We have to prove that for all $S, T \in 2^N$ such that $S \subset T$ and for each $i \in N$ the relation $\Phi_i(S, w_S) \preccurlyeq \Phi_i(T, w_T)$ holds, where (S, w_S) and (T, w_T) are the corresponding subgames. We know that $\Phi_i(w) = [\phi_i(\underline{w}), \phi_i(\overline{w})]$ for each $w \in CIG^N$ and for all $i \in N$. Further, the fact that the classical Shapley value ϕ has the population monotonicity property on CG^N implies that for each $S, T \in 2^N$ such that $S \subset T$ and for each $i \in N$, $\phi_i(S, \underline{w}_S) \leq \phi_i(T, \underline{w}_T)$ and $\phi_i(S, \overline{w}_S) \leq \phi_i(T, \overline{w}_T)$, from which follows

$$[\phi_i(S, \underline{w}_S), \phi_i(S, \overline{w}_S)] = \Phi_i(S, w_S) \preccurlyeq \Phi_i(T, w_T) = [\phi_i(T, \underline{w}_T), \phi_i(T, \overline{w}_T)].$$

■

Now, since $\Phi(w)$ of $w \in CIG^N$ is an element of the interval Weber set and is pmas extendable, the question arises whether each element of the interval Weber set is pmas extendable. The next theorem clarifies this issue.

Theorem 5.4.2 *Let $w \in CIG^N$. Then, each element I of $\mathcal{W}(w)$ is extendable to a pmas of w.*

Proof. Let $w \in CIG^N$. First, we show that for each $\sigma \in \Pi(N)$, $m^\sigma(w)$ is extendable to a pmas. We know that the interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ is efficient for each $\sigma \in \Pi(N)$. Then, for each $S \in 2^N$, $\sum_{i \in S} m_i^\sigma(w) = \sum_{k \in S} m_{\sigma(k)}^\sigma(w) = w(S)$ holds, where (S, w_S) is the corresponding (convex) subgame. Further, by convexity, $m_i^\sigma(w_S) \preccurlyeq m_i^\sigma(w_T)$ for each $i \in S \subset T \subset N$, where (S, w_S) and (T, w_T) are the corresponding subgames.
Second, each $I \in \mathcal{W}(w)$ is a convex combination of $m^\sigma(w)$, $\sigma \in \Pi(N)$, i.e.,
$I = \sum_{\sigma \in \Pi(N)} \alpha_\sigma m^\sigma(w)$ with $\alpha_\sigma \in [0, 1]$ and $\sum_{\sigma \in \Pi(N)} \alpha_\sigma = 1$.
Now, since all $m^\sigma(w)$ are pmas extendable, we obtain that I is pmas extendable as well. ■

From Theorem 5.4.2 we obtain that the *total interval Shapley value*, i.e., the interval Shapley value applied to the game itself and all its subgames, generates a pmas for each convex interval game. We illustrate this in the following example, where the calculations are based on Proposition 4.6.

Example 5.4.1 *Let $w \in CIG^N$ with $w(\emptyset) = [0, 0]$, $w(1) = w(2) = w(3) = [0, 0]$, $w(1, 2) = w(1, 3) = w(2, 3) = [2, 4]$ and $w(1, 2, 3) = [9, 15]$. It is easy to check that for this game the interval Shapley value generates the pmas depicted as*

$$\begin{array}{l} \\ \\ N \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1\} \\ \{2\} \\ \{3\} \end{array} \left[\begin{array}{ccc} & & \\ 1 & 2 & 3 \\ [3,5] & [3,5] & [3,5] \\ [1,2] & [1,2] & * \\ [1,2] & * & [1,2] \\ * & [1,2] & [1,2] \\ [0,0] & * & * \\ * & [0,0] & * \\ * & * & [0,0] \end{array}\right].$$

We refer the reader to Yanovskaya, Branzei and Tijs (2008) for an alternative proof of the population monotonicity of the interval Shapley value on the class of convex interval games and for other monotonicity properties of value-type interval solutions on this class of games.

To summarize in this chapter, convex (concave) interval games have been introduced and characterizations were given. The relations of the Weber set with the interval core for convex interval games were established. It was proved that each element of the Weber set of a convex interval game is extendable to such a pmas. In the next chapter, we introduce another interesting class of cooperative interval games, called big boss interval games.

CHAPTER 6

BIG BOSS INTERVAL GAMES

This chapter is based on Alparslan Gök, Branzei and Tijs (2008c) and Branzei, Tijs and Alparslan Gök (2008a). We notice that because here sets of players have an important role, we refer to the game $w \in IG^N$ as $< N, w >$ and to its subgames as $< T, w >$ for each $T \subset N$. Moreover, we adjust accordingly the notation for the used notions that were defined previously.

6.1 DEFINITION AND RELATIONS WITH OTHER CLASSES OF GAMES

Let $w \in IG^N$ and let $< N, |w| >$ be the corresponding length game. Then, we call a game $< N, w >$ a *big boss interval game* if its border game $< N, \underline{w} >$ and the game $< N, |w| >$ are classical (total) big boss games. We denote by $BBIG^N$ the set of all big boss interval games with player set N (without loss of generality we denote the big boss by n). Note that $BBIG^N$ is a subcone of IG^N.

The interval game in the next example is not a big boss interval game since the related length game is not a big boss game.

Example 6.1.1 *Let* $< N, w >$ *be a three-person interval game with* $w(1) = w(2) = w(3) = w(1,2) = [0,0], w(2,3) = [5,6], w(1,3) = [6,6]$ *and* $w(N) = [9,11]$. *Here,* $< N, \underline{w} >$ *is a big boss game, but the length game* $< N, |w| >$ *is not because it does not satisfy the condition* (iii) *for classical big boss games (see Chapter 1.2, take* $S = \{1\}$*).*

Next we notice that the Example 2.1.2 leads to a big boss interval game.

6.2 CHARACTERIZATIONS OF BIG BISS INTERVAL GAMES

In the following propositions and theorems, characterizations for big boss interval games are given.

Proposition 6.2.1 *Let $w \in IG^N$ and its related games $|w|, \underline{w}, \overline{w} \in G^N$. Then, $w \in BBIG^N$ if and only if its length game $< N, |w| >$ and its border games $< N, \underline{w} >$, $< N, \overline{w} >$ are big boss games.*

Proof. The proof is straightforward. Note that $\overline{w} = \underline{w} + |w|$ is a big boss game because classical big boss games form a cone. ■

Theorem 6.2.2 *Let $w \in SMIG^N$. Then, the following two assertions are equivalent:*

(i) $w \in BBIG^N$.

(ii) $< N, w >$ *satisfies:*

(a) *Big boss property:*
$w(S) = [0, 0]$ *for each* $S \in 2^N$ *with* $n \notin S$;

(b) *Monotonicity property:*
$w(S) \preccurlyeq w(T)$ *for each* $S, T \in 2^N$ *with* $n \in S \subset T$;

(c) *Union property:*

$$w(T) - w(S) \succcurlyeq \sum_{i \in T \setminus S} (w(T) - w(T \setminus \{i\})) \text{ for all } S, T \text{ with } n \in S \subset T.$$

Proof. By Proposition 6.2.1, $w \in BBIG^N$ if and only if $< N, \underline{w} >$, $< N, \overline{w} - \underline{w} >$ and $< N, \overline{w} >$ are classical big boss games. Now, using Definition 1.2.1, $w \in BBIG^N$ if and only if $< N, w >$ satisfies (a), (b) and (c). ■

In the following, we use the marginal contributions of a player $i \in N$ to coalitions T, with $T \subset N$, in the game $< T, w >$ given by $M_i(T, w) := w(T) - w(T \setminus \{i\})$.
Further, we give a concavity property for big boss interval games with n as a big boss:

(d) *n-concavity property:* $w(S \cup \{i\}) - w(S) \succcurlyeq w(T \cup \{i\}) - w(T)$, for all $S, T \in 2^N$ with $n \in S \subset T \subset N \setminus \{i\}$.

The following theorem which is inspired by Branzei, Tijs and Timmer (2001b) shows that (c) and (d) are equivalent if (a) and (b) hold.

Theorem 6.2.3 *Let $w \in IG^N$ and let* (a) *and* (b) *hold. Then,* (c) *implies* (d)*, and conversely.*

Proof.

(i) Suppose that (d) holds. Let S, T be such that $n \in S \subset T$. Suppose $T \setminus S = \{i_1, \ldots, i_h\}$. Then,

$$\begin{aligned} w(T) - w(S) &= w(S \cup \{i_1\}) - w(S) \\ &+ \sum_{r=2}^{h} (w(S \cup \{i_1, \ldots, i_r\}) - w(S \cup \{i_1, \ldots, i_{r-1}\})) \\ &= \sum_{r=1}^{h} M_{i_r}(S \cup \{i_1, \ldots, i_r\}, w) \\ &\succcurlyeq \sum_{r=1}^{h} M_{i_r}(T, w) = \sum_{i \in T \setminus S} M_i(T, w), \end{aligned}$$

where "the inequality" follows from (d). So, (d) implies (c).

(ii) Suppose that (c) holds. Then,

$$w(U \cup \{j\}) - w(U \setminus \{i\}) \succcurlyeq M_j(U \cup \{j\}, w) + M_i(U \cup \{j\}, w). \quad (6.2.1)$$

But,

$$w(U \cup \{j\}) - w(U \setminus \{i\}) = w(U \cup \{j\}) - w(U) + w(U) - w(U \setminus \{i\})$$

$$= M_j(U \cup \{j\}, w) + M_i(U, w). \quad (6.2.2)$$

From (6.2.1) and (6.2.2) we obtain

$$M_i(U, w) \succcurlyeq M_i(U \cup \{j\}, w) \quad (6.2.3)$$

for all $U \subset N$ and $i, j \in N \setminus \{n\}$ such that $\{i, n\} \subset U \subset N \setminus \{j\}$. Now, take $S, T \subset N$ with $\{i, n\} \subset S \subset T$ and suppose that $T \setminus S = \{i_1, \ldots, i_h\}$. If we apply (6.2.3) h times, then we have $M_i(S, w) \succcurlyeq M_i(S \cup \{i_1\}, w) \succcurlyeq M_i(S \cup \{i_1, i_2\}, w) \succcurlyeq \ldots \succcurlyeq M_i(T, w)$.

So, (c) implies (d). ■

In the sequel, we use the two characterizations of convex interval games provided by Theorems 5.2.8 and 5.2.10 to derive new characterizations of big boss interval games based on the notions of subadditivity and exactness.

Remark 6.2.1 *In view of Theorem 5.2.8, we obtain that a game $w \in IG^N$ is concave if and only if for each $T \in 2^N$ the marginal interval game $< N \setminus T, w^T >$ is subadditive.*

Remark 6.2.2 *In view of Theorem 5.2.10, a game $w \in IG^N$ is concave if and only if $< T, w_T >$ is exact for each $T \subset N$.*

We denote by $MIG^{N,\{n\}}$ the set of all size monotonic interval games on N that satisfy the big boss property with respect to n (the big boss player).

Proposition 6.2.4 *Let $w \in MIG^{N,\{n\}}$. Then, $w \in BBIG^N$ if and only if the marginal interval game $< N \setminus \{n\}, w^{\{n\}} >$ is a concave interval game.*

Proof. Let $w \in BBIG^N$. By Proposition 6.2.1 this is equivalent to $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| >$ being (total) big boss games with n as a big boss, which implies that $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| > \in MV^{N,\{n\}}$. Now, by Proposition 1.2.3 we obtain that $< N, \underline{w} >$, $< N, \overline{w} >$ and $< N, |w| >$ are (total) big boss games, if and only if $< N \setminus \{n\}, \underline{w}^{\{n\}} >$, $< N \setminus \{n\}, \overline{w}^{\{n\}} >$ and $< N \setminus \{n\}, |w|^{\{n\}} >$ are concave, which is equivalent with the marginal game $< N \setminus \{n\}, w^{\{n\}} >$ being a concave interval game. ■

Proposition 6.2.5 *Let $w \in MIG^{N,\{n\}}$. Then, the following assertions are equivalent:*

(i) $w \in BBIG^N$.

(ii) *Each marginal interval game of $< N \setminus \{n\}, w^{\{n\}} >$ is subadditive.*

(iii) *Each (interval) subgame of $< N \setminus \{n\}, w^{\{n\}} >$ is exact.*

Proof. (i) ⇔ (ii) follows from Proposition 6.2.4 and Remark 6.2.1.
(i) ⇔ (iii) follows from Proposition 6.2.4 and Remark 6.2.2. ■

Now, in the context of cooperative interval games, we extend the definition of the n-based T-marginal game $(v^{\{n\}})^T$, where $T \in 2^{N\setminus\{n\}}$. Let $< N, w > \in MIG^{N,\{n\}}$ and $T \in 2^{N\setminus\{n\}}$. The *$n$-based T-marginal interval game* $(w^{\{n\}})^T : 2^{N\setminus T} \to I(\mathbb{R})$ is defined by

$$(w^{\{n\}})^T(S) := w(S \cup T \cup \{n\}) - w(T \cup \{n\})$$

for each $S \subset N \setminus T$.

Based on the characterization of big boss interval games using its border and length games, we can easily extend Proposition 1.2.3 from classical cooperative games to cooperative interval games.

Proposition 6.2.6 *Let $< N, w > \in MIG^{N,\{n\}}$. Then, the following assertions are equivalent:*

(i) *$< N, w >$ is a big boss interval game with big boss n.*

(ii) *$< N \setminus \{n\}, w^{\{n\}} >$ is a concave game.*

(iii) *$< N \setminus (\{n\} \cup T), \left(w^{\{n\}}\right)^T >$ is a subadditive game for each $T \subset N \setminus \{n\}$.*

(iv) *$< N \setminus (\{n\} \cup T), w^{\{n\}\cup T} >$ is a subadditive game for each $T \subset N \setminus \{n\}$.*

Proof. (i) $\Leftrightarrow$ (ii) follows from Proposition 6.2.4;
(ii) $\Leftrightarrow$ (iii) holds by Remark 6.2.2;
(iii) $\Leftrightarrow$ (iv) follows from the definition of the n-based T-marginal interval game. ■

6.3 THE CORE OF BIG BOSS INTERVAL GAMES

We define the set $\Gamma(T, w)$ for each subgame $< T, w >$ of $< N, w >$ by

$$\Gamma(T, w) := \{(I_1, \ldots, I_n) \in \mathcal{I}(T, w) \mid [0, 0] \preccurlyeq I_i \preccurlyeq M_i(T, w) \text{ for each } i \in T \setminus \{n\}\}.$$

The next proposition gives a characterization of the interval core of a big boss interval game by using marginal contributions of the players.

Proposition 6.3.1 *Let $w \in BBIG^N$. Then,*

$$C(T, w) = \Gamma(T, w). \quad (6.3.1)$$

Proof. It is sufficient to show $\mathcal{C}(T, w) = \Gamma(T, w)$ for $T = N$.

(i) Suppose that $I = (I_1, \ldots, I_n) \in \mathcal{C}(N, w)$.

Then, $w(N) = \sum_{i \in N} I_i$ and $\sum_{j \in N \setminus \{i\}} I_j \succcurlyeq w(N \setminus \{i\})$, for all $i \in N \setminus \{n\}$. Further,

$$I_i = \sum_{j \in N} I_j - \sum_{j \in N \setminus \{i\}} I_j = w(N) - \sum_{j \in N \setminus \{i\}} I_j \preccurlyeq w(N) - w(N \setminus \{i\}) = M_i(N, w),$$

where the second equality follows from efficiency and "the inequality" follows from stability (see Section 2.3). Clearly, $I_i \succcurlyeq [0, 0] = w(i)$ for $i \in N \setminus \{n\}$. So, $I \in \Gamma(N, w)$. Therefore, $\mathcal{C}(N, w) \subset \Gamma(N, w)$ holds.

(ii) Suppose that $I = (I_1, \ldots, I_n) \in \Gamma(N, w)$. Then, for a coalition S which does not contain n, one finds that $\sum_{i \in S} I_i \succcurlyeq [0, 0] = w(S)$. To prove that $\sum_{i \in S} I_i \succcurlyeq w(S)$ for S such that $n \in S$, we first show that $w(N) - w(S) \succcurlyeq \sum_{i \in N \setminus S} M_i(N, w)$. Let $N \setminus S = \{i_1, \ldots, i_k\}$. Then, in a similar way as in the proof of Theorem 6.2.3 (i) with N in the role of T, we have

$$\begin{aligned} w(N) - w(S) &= w(S \cup \{i_1\}) - w(S) \\ &+ \sum_{s=2}^{k} (w(S \cup \{i_1, \ldots, i_s\}) - w(S \cup \{i_1, \ldots, i_{s-1}\})) \\ &= \sum_{s=1}^{k} M_{i_s}(S \cup \{i_1, \ldots, i_s\}, w) \\ &\succcurlyeq \sum_{s=1}^{k} M_{i_s}(N, w) = \sum_{i \in N \setminus S} M_i(N, w), \end{aligned}$$

where "the inequality" follows from the n-concavity property. Then, using the definition of $\Gamma(N, w)$ we have

$$w(S) \preccurlyeq w(N) - \sum_{i \in N \setminus S} M_i(N, w) \preccurlyeq w(N) - \sum_{i \in N \setminus S} I_i = \sum_{i \in S} I_i.$$

So, $I \in \mathcal{C}(N, w)$. Therefore, $\Gamma(N, w) \subset \mathcal{C}(N, w)$ holds. ■

Next, we define for a (big boss) subgame $< T, w >$ (with n as a big boss) of $w \in BBIG^N$ two particular elements of its interval core, which we call the big boss interval point and the union interval point. These points will play an important role regarding the description of the interval core. The *big boss interval point* $\mathcal{B}(T, w)$ is defined by

$$\mathcal{B}_j(T, w) := \begin{cases} [0, 0], & \text{if } j \in T \setminus \{n\} \\ w(T), & \text{if } j = n, \end{cases}$$

and the *union interval point* $\mathcal{U}(T, w)$ is defined by

$$\mathcal{U}_j(T, w) := \begin{cases} M_j(T, w), & \text{if } j \in T \setminus \{n\} \\ w(T) - \sum_{i \in T \setminus \{n\}} M_i(T, w), & \text{if } j = n. \end{cases}$$

Theorem 6.3.2 *Let $w \in IG^N$ be such that if property* (a) *in Theorem 6.2.2 holds. Then, $w \in BBIG^N$ if and only if for each $T \subset N$ with $n \in T$ the big boss interval point $\mathcal{B}(T, w)$ and the union interval point $\mathcal{U}(T, w)$ belong to the interval core of $< T, w >$.*

Proof. If $w \in BBIG^N$, then by Proposition 6.3.1, it is clear that $\mathcal{B}(T, w)$ and $\mathcal{U}(T, w) \in C(T, w)$ for each $T \subset N$ with $n \in T$.

Conversely, assume that for each $T \subset N$ with $n \in T$ the points $\mathcal{B}(T, w)$ and $\mathcal{U}(T, w)$ belong to the interval core. Since by hypothesis $< N, w >$ satisfies (a), we only need to show that (b) and (c) hold.

First, take $n \in T$. Since $\mathcal{B}(T, w) \in C(T, w)$, we have

$$w(S) \preccurlyeq \sum_{i \in S} \mathcal{B}_i(T, w) = \mathcal{B}_n(T, w) + \sum_{i \in S \setminus \{n\}} \mathcal{B}_i(T, w) = w(T) + [0, 0] = w(T).$$

So, (b) is satisfied.

Second, take S such that $n \in S \subset T$. Since $\mathcal{U}(T, w) \in C(T, w)$ we have

$$\begin{aligned} w(S) \preccurlyeq \sum_{i \in S} \mathcal{U}_i(T, w) = \mathcal{U}_n(T, w) + \sum_{i \in S \setminus \{n\}} \mathcal{U}_i(T, w) &= \\ (w(T) - \sum_{i \in T \setminus \{n\}} M_i(T, w)) + \sum_{i \in S \setminus \{n\}} M_i(T, w) = w(T) - \sum_{i \in T \setminus S} M_i(T, w).& \end{aligned}$$

So, (c) is satisfied. ■

From the above theorem we learn that big boss interval games are totally $\mathcal{I}$-balanced games. Note that $\mathcal{B} : BBIG^N \to I(\mathbb{R})^N$ and $\mathcal{U} : BBIG^N \to I(\mathbb{R})^N$ are additive maps.

6.4 BI-MONOTONIC INTERVAL ALLOCATION SCHEMES OF BIG BOSS INTERVAL GAMES

In this section, we introduce bi-monotonic allocation schemes (bi-mas) for big boss interval games. We denote by P_n the set $\{S \subset N | n \in S\}$ of all coalitions containing the big boss.

Take a game $w \in BBIG^N$ with n as a big boss. We call a scheme $B := (B_{iS})_{i \in S, S \in P_n}$ an *(interval) allocation scheme* for w if $(B_{iS})_{i\in S}$ is an interval core element of the subgame $< S, w >$ for each coalition $S \in P_n$. Such an allocation scheme $B = (B_{iS})_{i\in S, S\in P_n}$ is called a *bi-monotonic (interval) allocation scheme (bi-mas)* for w if for all $S, T \in P_n$ with $S \subset T$ we have $B_{iS} \succcurlyeq B_{iT}$ for all $i \in S \setminus \{n\}$, and $B_{nS} \preccurlyeq B_{nT}$. In a bi-mas the big boss is weakly better off in larger coalitions, while the other players are weakly worse off.

We say that for a game $w \in BBIG^N$ with n as a big boss, an imputation $I = (I_1, \ldots, I_n) \in \mathcal{I}(w)$ is *bi-mas extendable* if there exists a bi-mas $B = (B_{iS})_{i\in S, S\in P_n}$ such that $B_{iN} = I_i$ for each $i \in N$. The next theorem is inspired by Voorneveld, Tijs and Grahn (2003).

Theorem 6.4.1 *Let $w \in BBIG^N$ with n as a big boss and let $I \in C(N, w)$. Then, I is bi-mas extendable.*

Proof. Since $I \in C(N, w)$, by (6.3.1), we can find for each $i \in N \setminus \{n\}$ an $\alpha_i \in [0, 1]$, such that $I_i = \alpha_i M_i(N, w)$, and then $I_n = w(N) - \sum_{i\in N\setminus\{n\}} \alpha_i M_i(N, w)$. We will show that $(B_{iS})_{i\in S, S\in P_n}$, defined by $B_{iS} = \alpha_i M_i(S, w)$ for each S and i such that $i \in S \setminus \{n\}$, and $B_{nS} = w(S) - \sum_{i\in S\setminus\{n\}} \alpha_i M_i(S, w)$ is a bi-mas.

Take $S, T \in P_n$ with $S \subset T$ and $i \in S \setminus \{n\}$. We have to prove that $B_{iS} \succcurlyeq B_{iT}$ for $i \in N \setminus \{n\}$ and $B_{nS} \preccurlyeq B_{nT}$. First, $B_{iS} := \alpha_i M_i(S, w) \succcurlyeq \alpha_i M_i(T, w) = B_{iT}$, where "the inequality" follows from (d). Second,

$$
\begin{aligned}
B_{nT} &= w(T) - \sum_{i\in T\setminus\{n\}} \alpha_i M_i(T, w) \\
&\succcurlyeq (w(S) + \sum_{i\in T\setminus S} M_i(T, w)) - \sum_{i\in T\setminus\{n\}} \alpha_i M_i(T, w) \\
&= (w(S) - \sum_{i\in S\setminus\{n\}} \alpha_i M_i(T, w)) + \sum_{i\in T\setminus S} (1 - \alpha_i) M_i(T, w) \\
&\succcurlyeq (w(S) - \sum_{i\in S\setminus\{n\}} \alpha_i M_i(S, w)) + \sum_{i\in T\setminus S} (1 - \alpha_i) M_i(T, w) \\
&= B_{nS} + \sum_{i\in T\setminus S} (1 - \alpha_i) M_i(T, w) \succcurlyeq B_{nS},
\end{aligned}
$$

where the first follows from (c), the second follows from (d), and the third follows from $\alpha_i \leq 1$ and the nonincreasing of the interval marginal contribution vectors. So, $B_{nT} \succcurlyeq B_{nS}$. ■

Example 6.4.1 *Consider the interval game in Example 2.1.2. We illustrate Theorem 6.4.1 by using the special interval core elements $\mathcal{B}(T, w)$ and $\mathcal{U}(T, w)$. For each $i \neq n$ and for each $S \subset T$, $\mathcal{B}_i(S, w) = \mathcal{B}_i(T, w) = [0, 0]$; for $i = n$ and for each $S \subset T$, $\mathcal{B}_n(S, w) = f(|S| - 1) \preccurlyeq$*

$f(|T|-1) = \mathcal{B}_n(T,w)$. *For each* $i \neq n$ *and for each* $S \subset T$,

$$\mathcal{U}_i(S,w) = M_i(S,w) = w(S) - w(S \setminus \{i\}) \succcurlyeq w(T) - w(T \setminus \{i\}) = M_i(T,w) = \mathcal{U}_i(T,w);$$

for $i = n$ *and for each* $S \subset T$,

$$\mathcal{U}_n(S,w) = w(S) - \sum_{i \in S \setminus \{n\}} M_i(S,w) \preccurlyeq w(T) - \sum_{i \in T \setminus \{n\}} M_i(T,w) = \mathcal{U}_n(T,w).$$

6.5 THE $\mathcal{T}$-VALUE AND THE INTERVAL AL-VALUE OF BIG BOSS INTERVAL GAMES

Now, we introduce on the class of big boss interval games an interval compromise-like solution concept, called the $\mathcal{T}$-value, and the interval AL-value inspired by Tijs (2005), and show that the $\mathcal{T}$-value equals the interval AL-value.
Let $w \in BBIG^N$. The $\mathcal{T}$*-value* of w is defined by

$$\mathcal{T}(N,w) := \frac{1}{2}(\mathcal{U}(N,w) + \mathcal{B}(N,w)).$$

Note that $\mathcal{T} : BBIG^N \to I(\mathbb{R})^N$ has some trade-off flavour, because $\mathcal{T}(N,w)$ is the average of the union point $\mathcal{U}(N,w)$ and the big boss interval point $\mathcal{B}(N,w)$ for each $w \in BBIG^N$.
Next, we consider a holding situation with interval data and construct a holding interval game which turns out to be a big boss interval game. Player 3 is the owner of a holding house which has capacity for one container. Players 1 and 2 have each one container which they want to store. If player 1 is allowed to store his/her container, then the benefit belongs to $[10,30]$ and if player 2 is allowed to store his/her container then the benefit belongs to $[50,70]$. The situation described above corresponds to an interval game which is studied in the following example.

Example 6.5.1 *(A big boss interval game) The interval game* $< N, w >$ *with* $N = \{1,2,3\}$ *and* $w(S) = [0,0]$ *if* $3 \notin S$, $w(\emptyset) = w(3) = [0,0]$, $w(1,3) = [10,30]$, *and* $w(N) = w(2,3) = [50,70]$ *is a big boss interval game with player 3 as big boss, because the properties* (a), (b) *and* (c) *in Theorem 6.2.2 are satisfied. The* $\mathcal{T}$*-value, in case of full cooperation, generates the interval allocation* $\mathcal{T}(N,w) = ([0,0],[20,20],[30,50])$, *which indicates sharp shares for players 1 and 2 equal to* 0 *and* 20, *respectively. The payoff for player 3 depends, in this case, only* *on the realization R of $w(N)$. Assuming that $R = 60$ player 3 will receive a payoff equal to 40. However, in general, the actual players' shares when R is known depend not only R, but also on the vector allocation agreed upon before starting cooperation. For details regarding the use of interval solutions for determining the distribution of achieved common gains see Section 2.4. Finally, the total $\mathcal{T}$-value generates a bi-mas represented by the following matrix:*

$$\begin{array}{l} \\ N \\ \{1,3\} \\ \{2,3\} \\ \{3\} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ [0,0] & [20,20] & [30,50] \\ [5,15] & * & [5,15] \\ * & [25,35] & [25,35] \\ * & * & [0,0] \end{bmatrix}.$$

Such a bi-mas extension of the interval core element $\mathcal{T}(N,w)$ might be helpful in the decision making process regarding which coalitions should form and how to distribute the collective gains among the participants.

Given a game $w \in BBIG^N$ and an ordering

$$\sigma = (\sigma(1), \sigma(2) \dots, \sigma(k), \sigma(k+1), \dots, \sigma(n)),$$

with $\sigma(k) = n$ of the players in $N = \{1, 2, \dots, n\}$, the *lexicographic maximum* of the interval core $C(N,w)$ of $< N, w >$ with respect to σ, which we denote by $L^{\sigma}(N,w)$, is defined as follows:

$$L^{\sigma}_{\sigma(i)}(N,w) := \begin{cases} M_{\sigma(i)}(N,w), & \text{if } \ i < k \\ [0,0]\,, & \text{if } \ i > k \\ w(N) - \sum_{j=1}^{k-1} M_j(N,w), & \text{if } \ i = k. \end{cases} \quad (6.5.1)$$

We notice that L^{σ} is additive on $BBIG^N$.

The *interval average lexicographic value* of $w \in BBIG^N$ is defined by

$$AL(N,w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^{\sigma}(N,w),$$

where $\Pi(N)$ is the set of permutations $\sigma : N \to N$.

Applying (6.5.1) we obtain

$$AL(N,w) = (\frac{1}{2}M_1(N,w), \dots, \frac{1}{2}M_{n-1}(N,w), w(N) - \frac{1}{2}\sum_{i=1}^{n-1} M_i(N,w)).$$

So, we have $AL(N,w) = \mathcal{T}(N,w)$. Summarizing, we give the following theorem:

Theorem 6.5.1 *Let $w \in BBIG^N$ with n as a big boss. Then, $\mathcal{T}(N, w) = AL(N, w) \in C(N, w)$ and the (total) AL-value generates a bi-mas for $w \in BBIG^N$.*

To sum up this chapter, big boss interval games have been introduced and various characterizations were given. We related big boss interval games with concave interval games and obtained characterizations of big boss interval games in terms of subadditivity and exactness. The structure of the core of a big boss interval game was explicitly described and we showed that it plays an important role relative to interval-type bi-monotonic allocation schemes for such games. Specifically, each element of the interval core of a big boss interval game is extendable to a bi-monotonic allocation scheme. Furthermore, on the class of big boss interval games two interval solution concepts of value type were defined which can be seen as extensions to the interval setting of the compromise value and the AL-value for classical games. It turns out that these interval solutions coincide and generate bi-monotonic allocation schemes for each big boss interval game. A small but interesting class of cooperative interval games is that of interval peer group games (Branzei, Mallozzi and Tijs (2008)) which is a subclass of CIG^N and has nonempty intersection with $BBIG^N$. In the next chapter, we continue with applications of cooperative interval games.

CHAPTER 7

ECONOMIC AND OR SITUATIONS AND RELATED COOPERATIVE INTERVAL GAMES

7.1 AIRPORT INTERVAL GAMES AND THEIR SHAPLEY VALUE

This section is based on Alparslan Gök, Branzei and Tijs (2008d). The major topic is to present and identify the *interval Baker-Thompson rule*.

In literature much attention is paid to airport situations and related games. We refer here to Littlechild and Owen (1973), Littlechild and Thompson (1977) and Driessen (1988). In the sequel, we summarize the classical airport situation, its classical airport cost game and the Baker-Thompson rule. Consider the aircraft fee problem of an airport with one runway. Suppose that the planes which are to land are classified into m types. For each $1 \leq j \leq m$, denote the set of landings of planes of type j by N_j and its cardinality by n_j. Then $N = \cup_{j=1}^{m} N_j$ represents the set of all landings. Let c_j represent the cost of a runway adequate for planes of type j. We assume that the types are ordered such that $0 = c_0 < c_1 < \ldots < c_m$. We consider the runway divided into m consecutive pieces P_j, $1 \leq j \leq m$, where P_1 is adequate for landings of planes of type 1; P_1 and P_2 together for landings of planes of type 2, and so on. The cost of piece P_j, $1 \leq j \leq m$, is the marginal cost $c_j - c_{j-1}$. The economists Baker (1965) and Thompson (1971) proposed an appealing rule now called the *Baker-Thompson rule*, given by $\beta_i := \sum_{k=1}^{j}[\sum_{r=k}^{m} n_r]^{-1}(c_k - c_{k-1})$ whenever $i \in N_j$. That is, every landing of planes of type j contributes to the cost of the pieces P_k, $1 \leq k \leq j$, equally allocated among its users $\cup_{r=k}^{m} N_r$. We denote the marginal costs $c_k - c_{k-1}$ by t_k, $1 \leq k \leq m$. The classical airport TU game $< N, c >$ is given by $c(S) := \max\{c_k | 1 \leq k \leq m, S \cap N_k \neq \emptyset\}$ for all $S \subset N$. It is well-known that airport games are concave and the Shapley value (Shapley (1953)) of a concave game belongs to the core of the game. Littlechild and Owen (1973) showed that for this game the Shapley value agrees with the Baker-Thompson rule.

In this section, we consider airport situations where cost of pieces of the runway are intervals. Then, we associate as in the classical case to such a situation an interval cost game and extend to airport interval games the results presented above.

Let $I \in I(\mathbb{R}_+)$, $T \in 2^N \setminus \emptyset$, and let $u_T^* : 2^N \to \mathbb{R}$ be the classical dual unanimity game based on T. Here, the interval game $< N, Iu_T^* >$ defined by $(Iu_T^*)(S) := u_T^*(S)I$ for each $S \in 2^N$ will play an important role. We notice that the $\Phi(Iu_T^*)$ for the interval game $< N, Iu_T^* >$ is related with the Shapley value $\phi(u_T^*)$ of the classical game $< N, u_T^* >$ as follows:

$$\Phi_i(Iu_T^*) = \phi_i(u_T^*)I = \begin{cases} I/\,|T|\,, & i \in T \\ [0,0]\,, & i \in N \setminus T. \end{cases} \quad (7.1.1)$$

Consider the aircraft fee problem of an airport with one runway. Suppose that the planes which are to land are classified into m types. For each $1 \le j \le m$, denote the set of landings of planes of type j by N_j and its cardinality by n_j. Then $N = \cup_{j=1}^m N_j$ represents the set of all landings. Consider that the runway is divided into m consecutive pieces P_j, $1 \le j \le m$, where P_1 is sufficient for landings of planes of type 1, P_1 and P_2 together for landings of planes of type 2, and so on. Let the interval T_j with non-negative finite bounds represent the interval cost of piece P_j, $1 \le j \le m$.

Next we propose an interval cost allocation rule γ, which we call the *interval Baker-Thompson rule*. For a given airport interval situation $(N, (T_k)_{k=1,\dots,m})$ the Baker-Thompson allocation for each player $i \in N_j$ is given by:

$$\gamma_i := \sum_{k=1}^{j} (\sum_{r=k}^{m} n_r)^{-1} T_k. \quad (7.1.2)$$

Note that for the piece P_k of the runway the users are $\cup_{r=k}^m N_r$, i.e., there are $\sum_{r=k}^m n_r$ users. So, $(\sum_{r=k}^m n_r)^{-1} T_k$ is the equal cost share of each user of the piece P_k. This means that a player $i \in N_j$ contributes to the cost of the pieces $P_1, \dots, P_j$. The characteristic cost function d (see Section 1.2) of the airport interval game $< N, d >$ is given by $d(\emptyset) := [0,0]$ and $d(S) := \sum_{k=1}^j T_k$ for all coalitions $S \subset N$ satisfying $S \cap N_j \neq \emptyset$ and $S \cap N_k = \emptyset$ for all $j+1 \le k \le m$ (since such coalition S needs the pieces P_k, $1 \le k \le j$ of the runway). Now, we give the description of the airport interval game as follows:

$$d = \sum_{k=1}^{m} T_k u^*_{\cup_{r=k}^m N_r}. \quad (7.1.3)$$

In the following proposition, we show that the interval Baker-Thompson allocation for the airport situation with interval data coincides with the interval Shapley value of the corresponding airport interval game.

Proposition 7.1.1 *Let $< N, d >$ be an airport interval game. Then, the interval allocation γ of (7.1.2) agrees with the interval Shapley value $\Phi(d)$.*

Proof. For $i \in N_j$ we have

$$\begin{aligned} \Phi_i(d) &= \Phi_i(\sum_{k=1}^{m} T_k u^*_{\cup_{r=k}^m N_r}) = \sum_{k=1}^{m} \Phi_i(T_k u^*_{\cup_{r=k}^m N_r}) \\ &= \sum_{k=1}^{j} (\sum_{r=k}^{m} n_r)^{-1} T_k = \gamma_i, \end{aligned}$$

where the equalities follow from (7.1.3), the additivity of the interval Shapley value Φ, (7.1.1) and (7.1.2) respectively. ■

Note that if we consider the special case $N_1 = \{1\}, N_2 = \{2\}, \ldots, N_n = \{n\}$. Then, $\gamma = (\frac{T_1}{n}, \frac{T_1}{n} + \frac{T_2}{n-1}, \ldots, \frac{T_1}{n} + \frac{T_2}{n-1} + \ldots + \frac{T_n}{1})$. Here, each piece of the runway is completely paid by the users and all users of the same piece contribute equally.

Example 7.1.1 *Let $< N, d >$ be a three-person airport interval game corresponding to the airport interval situation depicted in Figure 7.1. The interval costs of the pieces are given by $T_1 = [30, 45]$, $T_2 = [20, 40]$ and $T_3 = [100, 120]$. Then, $d(\emptyset) = [0, 0]$, $d(1) = [30, 45]$, $d(2) = d(1, 2) = [50, 85]$ and $d(3) = d(1, 3) = d(2, 3) = d(N) = [150, 205]$.*

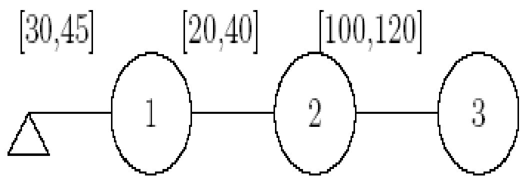

Figure 7.1: An airport situation with interval data

The following table shows the interval marginal vectors of the game, where the rows correspond to orderings of the players and the columns correspond to the players

$$\begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{bmatrix} [30,45] & [20,40] & [100,120] \\ [30,45] & [0,0] & [120,160] \\ [0,0] & [50,85] & [100,120] \\ [0,0] & [50,85] & [100,120] \\ [0,0] & [0,0] & [150,205] \\ [0,0] & [0,0] & [150,205] \end{bmatrix}.$$

Note that $d = [30,45]u^*_{\{1,2,3\}} + [20,40]u^*_{\{2,3\}} + [100,120]u^*_{\{3\}}$ *and*

$$\Phi(d) = ([10,15]\,, [20,35]\,, [120,155]).$$

Notice also that

$$\begin{aligned}\Phi(d) &= \Phi(\sum_{k=1}^{3} T_k u^*_{\cup_{r=k}^{3} N_r}) = \Phi(T_1 u^*_{\{1,2,3\}}) + \Phi(T_2 u^*_{\{2,3\}}) + \Phi(T_3 u^*_{\{3\}}) \\ &= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})[30,45] + (0, \frac{1}{2}, \frac{1}{2})[20,40] + (0,0,1)[100,120] = \gamma.\end{aligned}$$

In the following proposition, we show that airport interval games are concave.

Proposition 7.1.2 *Let* $< N, d >$ *be an airport interval game. Then,* $< N, d >$ *is concave.*

Proof. It is well-known that non-negative multiples of classical dual unanimity games are concave (or submodular). By (7.1.3) we have, $\underline{d} = \sum_{k=1}^{m} \underline{T}_k u^*_{k,m}$ and $|d| = \sum_{k=1}^{m} |T_k|\, u^*_{k,m}$ are concave, because $\underline{T}_k \geq 0$ and $|T_k| \geq 0$ for each k. By Proposition 5.2.2, $< N, d >$ is concave.

■

Note that the interval game $< N, d >$ in Example 7.1.1 is concave by Proposition 7.1.2.

Proposition 7.1.3 *Let* $(N, (T_k)_{k=1,\ldots,m})$ *be an airport situation with interval data and* $< N, d >$ *be the related airport interval game. Then, the interval Baker-Thompson rule applied to this airport situation provides an allocation which belongs to* $C(d)$.

Proof. First, by Proposition 7.1.2 the airport game $< N, d >$ is concave. We prove that $m^\sigma(d) \in C(d)$ for all $\sigma \in \Pi(N)$. Let $\sigma \in \Pi(N)$ and take $m^\sigma(d)$. Clearly, we have $\sum_{k \in N} m^\sigma_k(d) =$

$d(N)$. To prove that $m^\sigma(d) \in C(d)$ we have to show that for $S \in 2^N$, $\sum_{k\in S} m_k^\sigma(d) \preccurlyeq d(S)$. Let $S := \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\}$ with $i_1 < i_2 < \ldots < i_k$. Then,

$$\begin{aligned}
d(S) &= d(\sigma(i_1)) - d(\emptyset) \\
&+ \sum_{r=2}^{k} (d(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)) - d(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{r-1}))) \\
&\succcurlyeq d(\sigma(1), \ldots, \sigma(i_1)) - d(\sigma(1), \ldots, \sigma(i_1 - 1)) \\
&+ \sum_{r=2}^{k} (d(\sigma(1), \sigma(2), \ldots, \sigma(i_r)) - d(\sigma(1), \sigma(2), \ldots, \sigma(i_r - 1))) \\
&= \sum_{r=1}^{k} m_{\sigma(i_r)}^\sigma(d) = \sum_{k\in S} m_k^\sigma(d),
\end{aligned}$$

where the inequality follows from Proposition 5.2.4 (iii) applied to $i = \sigma(i_r)$ and

$$S_1 = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{r-1})\} \subset S_2 = \{\sigma(1), \sigma(2), \ldots, \sigma(i_r - 1)\}$$

for $r \in \{1, 2, \ldots, k\}$.

Further, since the interval Shapley value of d is the average of all marginal interval vectors of d and by convexity of $C(d)$ we obtain $\Phi(d) \in C(d)$. Now, we apply Proposition 7.1.1. ■

An alternative proof of Proposition 7.1.3: [1] By Proposition 7.1.1 the Baker-Thompson allocation is efficient. We need only to check the stability conditions for the interval Baker-Thompson allocation. Consider the airport interval game $< N, d >$ and any coalition $S \subset N, S \neq \emptyset$. Say $d(S) = \sum_{r=1}^{j} T_r$, that is, $S \cap N_j \neq \emptyset$ and $S \cap N_p = \emptyset$ for all $j < p \leq m$. Then, we obtain, for $i \in N_k$, $\gamma_i = \sum_{r=1}^{k} \frac{T_r}{n_r+\ldots+n_m}$. Thus,

$$\sum_{i\in S} \gamma_i = \sum_{k=1}^{j} \left(|S \cap N_k| \sum_{r=1}^{k} \frac{T_r}{n_r + \ldots + n_m} \right) = \sum_{r=1}^{j} \left(\frac{T_r}{n_r + \ldots + n_m} \sum_{k=r}^{j} |S \cap N_k| \right).$$

Note that $\sum_{k=r}^{j} |S \cap N_k| \leq n_r + \ldots + n_j \leq n_r + \ldots + n_m$. From this, we conclude that $\sum_{i\in S} \gamma_i \preccurlyeq \sum_{r=1}^{j} T_r = d(S)$ by taking care of the ordering of intervals through their lower and upper borders. ■

We notice that the interval Baker-Thompson rule is useful at an ex-ante stage to inform users about what they can expect to pay, between two bounds, for the construction of the runway. At an ex-post stage when all costs are known with certainty, the classical Baker-Thompson rule can be applied to pick up effective costs $x_i \in \gamma_i$ for each $i \in N$ such that $\sum_{i\in N} x_i$ equals the realization $\tilde{d} \in [\underline{d}(N), \overline{d}(N)]$.

[1] This direct proof was provided by one of the anonymous referees of Alparslan Gök, Branzei and Tijs (2008d).

7.2 BANKRUPTCY PROBLEMS WITH INTERVAL UNCERTAINTY

This section is based on Branzei and Alparslan Gök (2008). It focuses on bankruptcy situations with interval data and related cooperative interval games and considers bankruptcy situations where the estate and (some of the) claims vary within closed and bounded intervals, which we call bankruptcy interval situations.

Classical bankruptcy problems and bankruptcy games have been intensively studied. We refer here to O'Neill (1982), Aumann and Maschler (1985), Herrero, Maschler and Villar (1999) and Young (1987). In a classical bankruptcy situation, a certain amount of money (estate) has to be divided among some people (claimants) who have individual claims on the estate, and the total claim is weakly larger than the estate.

A bankruptcy situation with set of claimants N is a pair (E, d), where $E \geq 0$ is the estate to be divided and $d \in \mathbb{R}^N_+$ is the vector of claims such that $\sum_{i \in N} d_i > E$. We assume without loss of generality that $d_1 \leq d_2 \leq \ldots \leq d_n$ and denote by BR^N the set of bankruptcy situations with player set N. The total claim is denoted by $D = \sum_{i \in N} d_i$. A bankruptcy rule is a function $f : BR^N \to \mathbb{R}^N$ which assigns to each bankruptcy situation $(E, d) \in BR^N$ a payoff vector $f(E, d) \in \mathbb{R}^N$ such that $0 \leq f(E, d) \leq d$ (*reasonability*) and $\sum_{i \in N} f_i(E, d) = E$ (*efficiency*). Here, we are interested in bankruptcy rules that are coordinate-wise (weakly) increasing in E. The proportional rule (*PROP*) (see Chapter 2.4) is one of the most often used in real-life. Another interesting bankruptcy rule is the rights-egalitarian rule as a division rule for all circumstances of division problems.

To each bankruptcy situation $(E, d) \in BR^N$ one can associate a pessimistic bankruptcy game $v_{E,d}$ defined by $v_{E,d}(S) = (E - \sum_{i \in N \setminus S} d_i)_+$ for each $S \in 2^N$, where $x_+ = \max\{0, x\}$. The game $v_{E,d}$ is convex and the bankruptcy rules *PROP* and f^{RE} provide allocations in the core of the game.

Cooperative interval games arising from bankruptcy situations where the claims can vary within closed intervals are introduced and analyzed in Branzei, Dimitrov and Tijs (2003). A bankruptcy situation where the claims are certain but the available estate can vary within a closed interval is used in Example 2.2.2 to illustrate cores for two-person interval games.

It is important to consider interval claims because in various disputes including inheritance (O'Neill (1982)) claimants face uncertainty regarding their effective rights and, as a result, individual claims can be expressed in the form of closed intervals without any probability distributions attached to them. In such situations, our model based on interval claims fits better than the more standard claims approach with reality and, additionally, offers flexibility in conflict resolution under interval uncertainty of the estate at stake. Economic applications of our approach include funds' allocation of a firm among its divisions (Pulido, Sánchez-Soriano and Llorca (2002), Pulido et al. (2008)), priority problems (Moulin (2000)), distribution of penalty costs in delayed projects (Branzei et al. (2002)) and disputes related to cooperation in joint projects where agents have restricted willingness to pay (Tijs and Branzei (2004)).

A *bankruptcy interval situation* with a fixed set of claimants $N = \{1, 2, \ldots, n\}$ is a pair $(E, d) \in I(\mathbb{R}) \times I(\mathbb{R})^N$, where $E = [\underline{E}, \overline{E}] \succcurlyeq [0, 0]$ is the estate to be divided and d is the vector of interval claims with i-th coordinate $d_i = [\underline{d}_i, \overline{d}_i]$, $i \in N$, such that $[0, 0] \preccurlyeq d_1 \preccurlyeq d_2 \preccurlyeq \ldots \preccurlyeq d_n$ and $\overline{E} < \sum_{i=1}^{n} \underline{d}_i$. We note that all selections $(\tilde{E}, \tilde{d})$, where $\underline{E} < \tilde{E} < \overline{E}$ and $\underline{d}_i < \tilde{d}_i < \overline{d}_i$, for all $i \in N$, are traditional bankruptcy situations. We denote by $\underline{d}(N)$ the total lower claim and by $\overline{d}(N)$ the total upper claim. We also use the notations $\underline{d}(S) := \sum_{i \in S} \underline{d}_i$ and $\overline{d}(S) := \sum_{i \in S} \overline{d}_i$ for $S \subset N$. By BRI^N we denote the family of bankruptcy interval situations with set of claimants N.

A *bankruptcy interval rule* for bankruptcy interval situations is a function $\mathcal{F} : BRI^N \to I(\mathbb{R})^N$ assigning to each bankruptcy interval situation $(E, d) \in BRI^N$ a vector $\mathcal{F}(E, d) = (\mathcal{F}_1(E, d), \ldots, \mathcal{F}_n(E, d)) \in I(\mathbb{R})^N$, such that

(i) $[0, 0] \preccurlyeq \mathcal{F}_i(E, d) \preccurlyeq d_i$ for each $i \in N$ (*reasonability*);

(ii) $\sum_{i=1}^{n} \mathcal{F}_i(E, d) = E$ (*efficiency*).

Now, we look at the bankruptcy rules *PROP* and f^{RE} and extend them to the interval setting. By BRI_1^N we denote the family of all bankruptcy situations $(E, d) \in BRI^N$ which satisfy the condition

$$\underline{E}/\underline{d}(N) \leq \overline{E}/\overline{d}(N), \quad (7.2.1)$$

and by BRI_2^N the family of all bankruptcy situations $(E, d) \in BRI^N$ which satisfy the condition

$$|E| \geq |d(N)|. \quad (7.2.2)$$

Condition (7.2.1) can be read as follows: The available amount per-unit of lower-estate is weakly smaller than the available amount per-unit of upper estate. Condition (7.2.2) can be read so: The spread of uncertainty regarding the estate is weakly larger than the total spread of uncertainty regarding the claims. Note that the conditions (7.2.1) and (7.2.2) are satisfied for any bankruptcy interval situations where all the claim intervals are *degenerate*, i.e., $\underline{d}_i = \overline{d}_i$

for all $i \in N$. Bankruptcy interval situations where the estate is a nondegenerate interval, i.e. $\underline{E} < \overline{E}$, and all claims are uncertainty-free are studied in Branzei and Dall'Aglio (2008).
The inclusion $BRI_1^N \subset BRI^N$ might be strict as the following example illustrates.

Example 7.2.1 *Let (E, d) be a three-person bankruptcy situation. We suppose that the claims of the players are closed intervals with $d_1 = [10, 20]$, $d_2 = [30, 50]$ and $d_3 = [30, 70]$, respectively, and the estate is $E = [60, 100]$. Then, we obtain $\underline{E}/\underline{d}(N) = 6/7 > 5/7 = \overline{E}/\overline{d}(N)$.*

The inclusion $BRI_2^N \subset BRI^N$ might also be strict as we can see from Example 7.2.1, where $|E| = 40 < 70 = |d(N)|$. In the following, we extend the proportional rule and the rights-egalitarian rule to the interval setting.
First, note that

$$PROP_i(\underline{E}, \underline{d}) = (\underline{d}_i/\underline{d}(N))\underline{E} \leq (\underline{d}_i/\overline{d}(N))\overline{E} \leq (\overline{d}_i/\overline{d}(N))\overline{E} = PROP_i(\overline{E}, \overline{d})$$

for each $i \in N$, where the first inequality follows from condition (7.2.1) and the second inequality follows from $[\underline{d}_i, \overline{d}_i] \in I(\mathbb{R})$.
We define the *proportional interval rule* $\mathcal{PROP} : BRI_1^N \to I(\mathbb{R})^N$ by

$$\mathcal{PROP}_i(E, d) := [PROP_i(\underline{E}, \underline{d}), PROP_i(\overline{E}, \overline{d})],$$

for each $(E, d) \in BRI_1^N$ and all $i \in N$. Second, note that

$$f_i^{RE}(\underline{E}, \underline{d}) = \underline{d}_i + \frac{1}{n}(\underline{E} - \underline{d}(N)) \leq \underline{d}_i + \frac{1}{n}(\overline{E} - \overline{d}(N)) \leq \overline{d}_i + \frac{1}{n}(\overline{E} - \overline{d}(N)) = f_i^{RE}(\overline{E}, \overline{d})$$

for each $i \in N$, where the first inequality follows from condition (7.2.2) and the second inequality follows from $[\underline{d}_i, \overline{d}_i] \in I(\mathbb{R})$.
We define the *rights-egalitarian interval rule* $\mathcal{F}^{RE} : BRI_2^N \to I(\mathbb{R})^N$ by

$$\mathcal{F}_i^{RE}(E, d) := [f_i^{RE}(\underline{E}, \underline{d}), f_i^{RE}(\overline{E}, \overline{d})],$$

for each $(E, d) \in BRI_2^N$ and all $i \in N$. The next proposition shows that $\mathcal{PROP}$ and $\mathcal{F}^{RE}$ are bankruptcy interval rules.

Proposition 7.2.1 *Let $\mathcal{B} = \left\{\mathcal{PROP}, \mathcal{F}^{RE}\right\}$. Then, each interval rule $\mathcal{F} \in \mathcal{B}$ is efficient and reasonable.*

Proof. The efficiency of $\mathcal{F}$ follows from the efficiency of corresponding classical bankruptcy rule $f \in \left\{PROP, f^{RE}\right\}$, i.e., $\sum_{i\in N} f_i(\underline{E}, \underline{d}) = \underline{E}$ and $\sum_{i\in N} f_i(\overline{E}, \overline{d}) = \overline{E}$. Further, the reasonability of $\mathcal{F}$ results from

$$0 \leq f_i(\underline{E}, \underline{d}) \leq \underline{d}_i \text{ and } 0 \leq f_i(\overline{E}, \overline{d}) \leq \overline{d}_i \text{ for each } i \in N.$$

■

Subsequently, we define a subclass of BRI^N, denoted by $SBRI^N$, consisting of all bankruptcy interval situations such that

$$\text{for each } S \in 2^N \text{ with } \underline{d}(N \setminus S) \leq \underline{E} \text{ it holds } |d(N \setminus S)| \leq |E|. \quad (7.2.3)$$

We call a bankruptcy interval situation in $SBRI^N$ a *strong bankruptcy interval situation*. With each $(E, d) \in SBRI^N$ we associate a cooperative interval game $< N, w_{E,d} >$ defined by $w_{E,d}(S) := [v_{\underline{E},\underline{d}}(S), v_{\overline{E},\overline{d}}(S)]$ for each $S \subset N$.
Note that (7.2.3) implies $v_{\underline{E},\underline{d}}(S) \leq v_{\overline{E},\overline{d}}(S)$ for each $S \in 2^N$. We denote by $SBRIG^N$ the family of all bankruptcy interval games $w_{E,d}$ with $(E, d) \in SBRI^N$. We notice that $w_{E,d} \in SBRIG^N$ is supermodular because $v_{\underline{E},\underline{d}}$ and $v_{\overline{E},\overline{d}} \in G^N$ are convex (see Proposition 5.2.1). The following example illustrates that $w_{E,d} \in SBRIG^N$ is supermodular but not necessarily convex.

Example 7.2.2 *Let (E, d) be a two-person bankruptcy situation. We suppose that the claims of the players are closed intervals $d_1 = [70, 70]$ and $d_2 = [80, 80]$, respectively and the estate is $E = [100, 140]$. Then, for each $i = 1, 2$ the corresponding game $< N, w_{E,d} >$ is given by $w_{E,d}(\emptyset) = [0, 0]$, $w_{E,d}(1) = [20, 60]$, $w_{E,d}(2) = [30, 70]$ and $w_{E,d}(1, 2) = [100, 140]$. This game is supermodular, but is not convex because $\left|w_{E,d}\right| \in G^N$ is not convex.*

In the following, we consider the restriction of the interval rule $\mathcal{PROP}$ to $SBRI_1^N = BRI_1^N \cap SBRI^N$, and the restriction of the interval rule $\mathcal{F}^{RE}$ to $SBRI_2^N = BRI_2^N \cap SBRI^N$. In the next proposition, we consider $(E, d) \in SBRI_1^N$ if $\mathcal{F}$ is $\mathcal{PROP}$, and $(E, d) \in SBRI_2^N$ if $\mathcal{F}$ is $\mathcal{F}^{RE}$.

Proposition 7.2.2 *Let $\mathcal{F} \in \mathcal{B}$. Then, $\mathcal{F}(E, d) \in C(w_{E,d})$ for each $w_{E,d} \in SBRIG^N$.*

Proof. First, we have

$$\sum_{i=1}^{n} \mathcal{F}_i(E, d) = E = E - \sum_{i\in\emptyset} d_i = w_{E,d}(N),$$

where the first equality follows from efficiency of the bankruptcy interval rules.
Second, take $S \subset N$. Then,

$$\sum_{i\in S}\mathcal{F}_i(E,d) = w_{E,d}(N) - \sum_{i\in N\setminus S}\mathcal{F}_i(E,d) \succcurlyeq E - \sum_{i\in N\setminus S} d_i,$$

where the equality follows from efficiency and the inequality follows from reasonability of the bankruptcy interval rules. Also, $\sum_{i\in S}\mathcal{F}_i(E,d) \succcurlyeq [0,0]$ by reasonability. So, $\sum_{i\in S}\mathcal{F}_i(E,d) \succcurlyeq w_{E,d}(S)$. Hence, $\mathcal{F}(E,d) \in C(w_{E,d})$. ■

The use of the allocations generated by the rules $\mathcal{PROP}$ and $\mathcal{F}^{RE}$ in practical bankruptcy-like situations with interval uncertainty is two-fold. Firstly, these interval allocations are used to inform claimants about what they can expect, between two boundaries, from the division problem at stake. Secondly, when the realization of the estate occurs, they are used to obtain standard allocations. We refer the reader to Section 2.4 for ways to transform vectors of intervals into vectors of real numbers. Further, in Branzei, Dall'Aglio and Tijs (2008) interval bankruptcy rules which are interesting from the game-theoretic point of view are introduced and studied.

7.3 SEQUENCING INTERVAL SITUATIONS AND RELATED GAMES

This section is based on Alparslan Gök et al. (2008). We consider one-machine sequencing situations with interval data. We present different possible scenarios and extend classical results on well-known rules and on sequencing games to the interval setting.

Sequencing situations arise in several instances of real-life. Here, we refer to the classical scheduling of a sequence of jobs and the waiting line in front of a counter. The use of an optimal ordering may reduce the cost connected with the time spent in the system and is particularly interesting in sequencing situations where several agents are involved. In such situations, the optimal order is good for the agents as a whole (because it increases the efficiency of the system), but since agents are basically interesting in their individual benefit, an agreement is equally important. The agreement includes how to compensate those agents that are required to spend more time in the system and how to share the joint cost savings. In the classical approach to the problem, the processing time of each job and the cost per unit of time associated with it are supposed to be known with certainty. It should be clear that the optimality of an ordering may be affected when the actual processing times and/or the

unitary costs are different from the forecasted ones. Here, we simply require an estimation of intervals of values for the processing times and/or unitary costs, avoiding the difficulties of associating a reasonable probability distribution. In this setting, the optimal order may be difficult to reach, but the agents may accept to switch their position in the queue in change of an adequate compensation. Depending on the agents' attitude towards risk, various possibilities could be considered to settle the agreement, both for improving the ordering (with more switches) and for sharing the joint cost savings.
First we recall that a one-machine sequencing situation arises when a set of ordered jobs has to be processed sequentially on a machine. The basic issue is to determine the optimal order of the jobs to be processed taking into account the individual processing times and the costs per unit of time. Formally, a sequencing situation is a 4-tuple (N, σ_0, α, p) where:

- $N = \{1, 2, \ldots, n\}$ is the set of jobs;
- $\sigma_0 : N \to \{1, 2, \ldots, n\}$ is a permutation that defines the initial order of the jobs;
- $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}^n_+$ is a non-negative real vector, where α_i is the cost per unit of time of job i;
- $p = (p_i)_{i \in N} \in \mathbb{R}^n_+$ is a positive real vector, where p_i is the processing time of job i.

Given a sequencing situation and an ordering σ of the jobs, we can associate to it the cost C_σ defined by the sum of the costs of the jobs, where the cost of job $i \in N$ is given by the product of its unitary cost α_i and the time that it spends in the system, i.e., its processing time p_i plus the waiting time for completing all the jobs preceding i in the queue. In formula, $C_\sigma := \sum_{i \in N} \alpha_i \left(\sum_{j \in P(\sigma, i)} p_j + p_i \right)$, where $P(\sigma, i)$ is the set of jobs preceding i, according to the order σ.

The optimal order of the jobs σ^* produces the minimum cost

$$C_{\sigma^*} := \sum_{i \in N} \alpha_i \left(\sum_{j \in P(\sigma^*, i)} p_j + p_i \right)$$

or the maximum cost saving $C_{\sigma_0} - C_{\sigma^*}$. Smith (1956) proved that an optimal order can be obtained reordering the jobs according to decreasing urgency indices, where the urgency index of job $i \in N$ is defined as $u_i = \frac{\alpha_i}{p_i}$ (of course, if this condition holds for the initial order, no reordering of jobs is necessary).

If the jobs belong to the same agent, he will agree to reorder them optimally, according to Smith's result. The situation is completely different when each job belongs to a different agent. In this case, a reordering requires that at least the agents that change their position agree on the new order. So, we can say that a switch among two jobs is always possible if they are consecutive in the current order or if all the agents that own one of the jobs in between the two that are switched agree.

The following question arises: Is it possible to share this cost savings $C_{\sigma_0} - C_{\sigma^*}$ among the agents in such a way that the new order results to be stable? In other words, we want to find fair shares of the overall cost savings to be given to the different agents, in such a way that all of them agree on the optimal order and have no incentive to recede from the agreement. This question finds its natural habitat in cooperative game theory.

In 1989, Curiel, Pederzoli and Tijs introduced the class of sequencing games. An updated survey on these games can be found in Curiel, Hamers and Klijn (2002). See also the survey on Operations Research Games (Borm, Hamers and Hendricks (2001)). A sequencing game is a pair $< N, v >$ where N is the set of players, that coincides with the set of jobs, and the characteristic function v assigns to coalition S the maximal cost savings that the members of S can obtain by reordering only their jobs. We say that a set of jobs T is *connected according to an order* σ if for all $i, j \in T$ and $k \in N, \sigma(i) < \sigma(k) < \sigma(j)$ implies $k \in T$.

Switching two connected jobs i, j, the cost associated to the ordering varies by the value $\alpha_j p_i - \alpha_i p_j$. The variation is positive if and only if the urgency indices verify $u_i < u_j$. Clearly, if $\alpha_j p_i - \alpha_i p_j$ is negative it is not beneficial for i and j to switch their positions. We denote the gain of the switch as

$$g_{ij} := (\alpha_j p_i - \alpha_i p_j)_+ = max\{0, \alpha_j p_i - \alpha_i p_j\}$$

and, consequently, the gain of a connected coalition T according to an order σ is defined by $v(T) := \sum_{j \in T} \sum_{i \in P(\sigma,j) \cap T} g_{ij}$.

If S is not a connected coalition, the order σ induces a partition into connected components, denoted by S/σ. In view of this, the characteristic function v of the sequencing game can be defined as $v(S) := \sum_{T \in S/\sigma} v(T)$ for each $S \subset N$ or, equivalently, as $v = \sum_{i,j \in N: i<j} g_{ij} u_{[i,j]}$, where

$u_{[i,j]}$ is the unanimity game defined as:

$$u_{[i,j]}(S) := \begin{cases} 1, & \text{if } \{i, i+1, ..., j-1, j\} \subset S \\ 0, & \text{otherwise.} \end{cases}$$

Curiel, Pederzoli and Tijs (1989) show that sequencing games are convex games and, consequently, their core is nonempty. Moreover, it is possible to determine a core allocation without computing the characteristic function of the game. They propose to share equally between the players i, j the gain g_{ij} produced by the switch, and they call this rule the *Equal Gain Splitting (EGS) rule*. It can be computed by $EGS_i := \frac{1}{2}\sum_{k\in P(\sigma,i)} g_{ki} + \frac{1}{2}\sum_{j:i\in P(\sigma,j)} g_{ij}$ for each $i \in N$. There exist two other simple allocation rules, denoted by $\mathcal{P}$ and $\mathcal{S}$, respectively. According to the first rule, the gain of each switch is assigned to the predecessor in the initial order, while the second rule assigns the gain to the successor. We can write $\mathcal{P}_i := \sum_{j:i\in P(\sigma,j)} g_{ij}$ and $\mathcal{S}_i := \sum_{j\in P(\sigma,i)} g_{ji}$ for each $i \in N$, and it is easy to see that $EGS = \frac{1}{2}(\mathcal{P} + \mathcal{S})$, understood in the vectorial sense based on these members.

In a similar way, we can define the *EGS^{ε} solution* for each $\varepsilon \in [0, 1]$ as $EGS^{\varepsilon} := \varepsilon\mathcal{P}+(1-\varepsilon)\mathcal{S}$. Clearly, for $\varepsilon = 0$ we get $\mathcal{S}$, for $\varepsilon = \frac{1}{2}$ we get EGS, and for $\varepsilon = 1$ we get $\mathcal{P}$.

In this section, we drop the hypothesis of complete knowledge of the parameters of a sequencing situation, in order to better fit the real-world situations. In particular, we suppose that the processing time and/or the cost per unit of time of each job are represented by intervals. In fact, each agent may have some difficulties in evaluating the actual duration of his/her job and the unitary cost. On the other hand, it is often possible to assign minimal and maximal values for both elements. We consider three scenarioes: In the first one, the processing time of each job is a positive real number but its unitary cost is an interval of positive real values. In the second one, the unitary costs are positive real numbers and the processing times are intervals of positive real values. In the last one, both elements are intervals of positive real values.

1. *The first scenario:*

   A one-machine sequencing situation with interval-uncertain costs per unit of time can be described as a 4-tuple (N, σ_0, α, p), where N, σ_0 and p are the same as in the classical case and $\alpha = ([\underline{\alpha}_i, \overline{\alpha}_i])_{i\in N} \in I(\mathbb{R}_+)^N$ is a vector of intervals. Here, $\underline{\alpha}_i$ is the minimal unitary cost and $\overline{\alpha}_i$ is the maximal unitary cost of job i.

In this situation, the arithmetics of intervals allows us to compute the *urgency index* of the jobs, $u_i := \frac{\alpha_i}{p_i} = \left[\frac{\underline{\alpha}_i}{p_i}, \frac{\overline{\alpha}_i}{p_i}\right]$, $i \in N$.

To use *Smith's result* for finding the optimal order we need not only to compare u_i and u_j to check if $u_i \preccurlyeq u_j$ for any two possible candidates i and j to a neighbor switch, but also that these intervals are disjoint, i.e., $\overline{u}_i < \underline{u}_j$. This setting corresponds to the maximal risk aversion of the agents that agree on a switch of their job only if it is surely profitable.

Example 7.3.1 *Consider the sequencing interval situation with* $N = \{1, 2\}$, $\sigma_0 = \{1, 2\}$, $p = (2, 3)$ *and* $\alpha = ([2, 4], [12, 21])$. *The urgency indices are* $u_1 = [1, 2]$ *and* $u_2 = [4, 7]$*; so the two jobs may be switched.*

Now, the question is how to share among the switching agents i and j the gain arising from their switch. We consider two possible approaches.

First, the agents i and j may agree on the dictatorial solution for agent i, i.e., the compensation corresponds to the upper bound $\overline{\alpha}_i p_j$; this means that agent i asks to be fully compensated referring to his maximal unitary cost, plus the possibility of an extra gain if the actual cost per unit of time is lower.

Second, the agents i and j could determine the individual compensation when the jobs are performed and realizations of the unitary costs are available. This leads to a classical sequencing situation and the agents may agree on one of the existing allocation rules, e.g., the EGS-rule.

Example 7.3.2 *Referring to the situation in Example 7.3.1, the dictatorial approach assigns to agent 1 a compensation* $\overline{\alpha}_2 p_1 = 21 \times 2 = 42$ *and* 0 *to agent 2. The realization approach may be performed only when the two jobs are processed. Suppose that the realization of the unitary cost is 4 for agent 1 and 16 for agent 2. The* EGS*-rule for the resulting classical sequencing situation assigns to both agents a compensation of* 10.

2. *The second scenario:*

   We describe a one-machine sequencing situation with interval-uncertain processing time as a 4-tuple (N, σ_0, α, p), where N, σ_0 and α are as in the classical case and $p = ([\underline{p}_i, \overline{p}_i])_{i \in N} \in I(\mathbb{R}_+)^N$ is the vector of intervals where $\underline{p}_i$ is the minimal processing time and $\overline{p}_i$ is the maximal processing time of job i.

In this situation, the arithmetics of intervals does not allow us to compute the urgency index of a job, as we cannot divide a real number by an interval, so we introduce the notion of *relaxation index of job i*, defined by $r_i := \alpha_i^{-1} p_i = \left[\frac{\underline{p}_i}{\alpha_i}, \frac{\overline{p}_i}{\alpha_i}\right]$ for all $i \in N$.

We notice that the relaxation index is the inverse of the urgency index in the classical case, so we may reformulate for this scenario the rule of Smith saying that to obtain an optimal order, the jobs have to be ordered according to increasing relaxation indices. Two jobs $i, j \in N$ may be switched only if $r_i \succcurlyeq r_j$ and the intervals are disjoint, i.e., $\underline{r}_i > \overline{r}_j$.

We can consider the same sharing approaches of the first scenario, with suitable modification.

3. *The third scenario:*

   Here, a one-machine sequencing interval situation is described as a 4-tuple (N, σ_0, α, p), where N and σ_0 are as usual, whereas $\alpha = ([\underline{\alpha}_i, \overline{\alpha}_i])_{i\in N} \in I(\mathbb{R}_+)^N$ and $p = ([\underline{p}_i, \overline{p}_i])_{i\in N} \in I(\mathbb{R}_+)^N$ are the vectors of intervals with $\underline{\alpha}_i$, $\overline{\alpha}_i$ representing the minimal and maximal unitary cost of job i, respectively. Here, $\underline{p}_i$, $\overline{p}_i$ representing the minimal and maximal processing time of job i, respectively.

   To handle such sequencing situations we propose to use either the approach based on urgency indices or the approach based on relaxation indices. This requires to be able to compute either $u_i = \left[\frac{\underline{\alpha}_i}{\underline{p}_i}, \frac{\overline{\alpha}_i}{\overline{p}_i}\right]$ for all $i \in N$ or $r_i = \left[\frac{\underline{p}_i}{\underline{\alpha}_i}, \frac{\overline{p}_i}{\overline{\alpha}_i}\right]$ for all $i \in N$, i.e., for each such an index the lower bound has to be less than or equal to the upper bound. Example 7.3.5 shows that this could be impossible. When all indices of a certain type can be calculated, they are useful to find an optimal order only in case they can be ordered properly and are also disjoint. Example 7.3.3 illustrates a successful use of the urgency indices, while Example 7.3.4 shows that although the relaxation indices can be computed and compared, they are not useful to find an optimal order because they are not disjoint.

Example 7.3.3 *Consider the two-agent situation with* $p_1 = [1, 4], p_2 = [6, 8], \alpha_1 = [5, 25], \alpha_2 = [10, 30]$. *We can compute* $u_1 = \left[5, \frac{25}{4}\right], u_2 = \left[\frac{5}{3}, \frac{15}{4}\right]$ *and use them to reorder the jobs as the intervals are disjoint.*

Example 7.3.4 *Consider the two-agent situation with* $p_1 = [1, 3], p_2 = [4, 6], \alpha_1 = [5, 6], \alpha_2 = [11, 12]$. *Here, we can compute* $r_1 = \left[\frac{1}{5}, \frac{1}{2}\right], r_2 = \left[\frac{4}{11}, \frac{1}{2}\right]$, *but we cannot reorder the jobs as the intervals are not disjoint.*

Example 7.3.5 *Consider the two-agent situation with* $p_1 = [1,3], p_2 = [5,8], \alpha_1 = [5,6], \alpha_2 = [10,30]$. *Now,* r_1 *is defined but* r_2 *is undefined; on the other hand,* u_1 *is undefined and* u_2 *is defined, so no comparison is possible and, consequently, the reordering cannot take place.*

If two jobs may be switched, we can use the sharing approaches introduced above. In particular, we may have no total order, as some pairs of jobs cannot be compared, but we may reach just a partial optimal order and share the associated gains.

Remark 7.3.1 *Allowing degenerate intervals* $[a,a] \in I(\mathbb{R}_+)$ *leads to the possibility of a unique game-theoretic treatment of all three scenarios of sequencing situations with interval data, based on the third scenario. In fact, in the first scenario we may consider the vector of real numbers* $p = (p_i)_{i\in N}$ *as a vector of degenerate intervals* $p = ([p_i, p_i])_{i\in N}$. *Analogously, in the second scenario we may consider the vector of real numbers* $\alpha = (\alpha_i)_{i\in N}$ *as a vector of degenerate intervals* $\alpha = ([\alpha_i, \alpha_i])_{i\in N}$.

Next we introduce the class of cooperative sequencing interval games. In view of Remark 7.3.1, we refer to the general situation presented in the third scenario.
Let $i, j \in N$. We define the *interval gain of the switch of jobs* i and j by

$$G_{ij} := \begin{cases} \alpha_j p_i - \alpha_i p_j & \text{if jobs } i \text{ and } j \text{ switch} \\ [0,0] & \text{otherwise.} \end{cases}$$

The sequencing interval game associated to a one-machine sequencing situation (N, σ_0, α, p) is defined by

$$w := \sum_{i,j\in N: i<j} G_{ij} u_{[i,j]},$$

provided that $G_{ij} \in I(\mathbb{R})$ for all switching jobs $i, j \in N$.

Remark 7.3.2 *The condition* $G_{ij} \in I(\mathbb{R})$ *is equivalent to* $\underline{G}_{ij} \leq \overline{G}_{ij}$. *Note that for the first two scenarioes this condition may be written as* $\frac{|\alpha_i|}{p_i} \leq \frac{|\alpha_j|}{p_j}$ *and* $\frac{|p_i|}{\alpha_i} \geq \frac{|p_j|}{\alpha_j}$, *respectively, and such conditions may not be satisfied. Consider the sequencing interval situation with* $N = \{1,2\}$, $\sigma_0 = \{1,2\}$, $p = ([2,2],[3,3])$ *and* $\alpha = ([2,4],[12,13])$. *The urgency indices are* $u_1 = [1,2]$ *and* $u_2 = \left[4, \frac{13}{3}\right]$, *so the switch is profitable, since* u_2 *is larger than* $u_1 = [1,2]$. *Moreover, the intervals are disjoint but* $\frac{|\alpha_1|}{p_1} = 1 \geq \frac{|\alpha_2|}{p_2} = \frac{1}{3}$, *implying that* $G_{12} = [18,14]$, *i.e., it is not an interval.*

In the following, we show that each sequencing interval game is convex.

Proposition 7.3.1 *Let $< N, w >$ be a sequencing interval game. Then, $< N, w >$ is convex.*

Proof. By definition $G_{ij} \succcurlyeq [0,0]$. So, $\underline{G}_{ij} \geq 0$ and $\left|G_{ij}\right| \geq 0$ for all (i, j). It is well known that classical unanimity games are convex. Then, $\underline{w} = \sum_{i,j \in N: i<j} \underline{G}_{ij} u_{[i,j]}$ and $|w| = \sum_{i,j \in N: i<j} \left|G_{ij}\right| u_{[i,j]}$ are convex games, in the classical sense. So, $w = \sum_{i,j \in N: i<j} G_{ij} u_{[i,j]}$ is convex (see Proposition 5.2.1 (iii)). ■

The *interval equal gain splitting rule* is defined by

$$IEGS_i := \frac{1}{2} \sum_{j \in N: i<j} G_{ij} + \frac{1}{2} \sum_{j \in N: i>j} G_{ij}$$

for each $i \in N$.

Proposition 7.3.2 *Let $< N, w >$ be a sequencing interval game. Then,*

i) $IEGS(w) = \frac{1}{2}(m^{(1,2\ldots,n)}(w) + m^{(n,n-1,\ldots,1)}(w))$,

ii) $IEGS(w) \in C(w)$.

Proof.

i) If $\sigma = (1, 2, \ldots, n)$, then

$$m^{(1,2\ldots,n)}(w) = ([0,0], G_{12}, G_{13} + G_{23}, G_{14} + G_{24} + G_{34}, \ldots, G_{1n} + \ldots + G_{n-1,n}).$$

If $\sigma = (n, n-1, \ldots, 1)$, then

$$m^{(n,n-1,\ldots,1)}(w) = (G_{12} + \ldots + G_{1,n}, \ldots, G_{n-1,n}, [0,0]).$$

ii) In Proposition 5.2.5, it is proved that the interval marginal vectors are interval core elements for convex interval games. The proof follows immediately as the sequencing interval games are convex by Proposition 7.3.1 and the interval core is a convex set (see Proposition 2.3.3).

■

Example 7.3.6 *Referring to the situation in Example 7.3.1, the interval gain is $G_{12} = [18, 30]$, the sequencing interval game $< N, w >$ is $w(1) = w(2) = [0,0]$, $w(1,2) = [18,30]$ and $IEGS(w) = ([9,15],[9,15])$.*

As we have already seen, for some sequencing interval situations we may have difficulties in ordering the jobs using only the urgency indices or the relaxation indices. In such situations, we can (partially) reorder the jobs using a mixed approach: We can consider actually adjacent pairs of jobs i and j for which both u_i and u_j or both r_i and r_j are defined, and decide if they may be switched, i.e., if all the required conditions are satisfied. Consider the sequencing interval situation with $N = \{1, 2, 3, 4\}$, $\sigma_0 = \{1, 2, 3, 4\}$, $p = ([1, 6], [8, 15], [2, 3], [2, 7])$ and $\alpha = ([1, 3], [2, 3], [6, 12], [6, 8])$. We may compute $u_1 = [1, 2]$, $u_2 = [4, 5]$, $r_3 = [3, 4]$ and $r_4 = \left[\frac{1}{3}, \frac{1}{2}\right]$, while the other indices are undefined. We can observe that jobs 1 and 2 and jobs 3 and 4 may be switched, but we can say nothing about jobs 1 and 4, that become adjacent after the first two switches, as we have no common index. But we can go further in our analysis. In fact, it is easy to realize that the urgency of job 1 is a number in the interval $[1, 2]$ while the relaxation of job 4 is a number in the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. So, in any realization, the urgency of job 4 is a number in the interval $[2, 3]$ and, apparently, the switch is surely profitable.

In this chapter, we studied some economic and OR situations, and extended classical results on well-known rules to the interval setting. We have shown that some of these situations are modeled as cooperative interval games. Motivating examples for the model of cooperative interval games and discussions about potential applications can also be found in the papers listed in references and in further publications. In the next chapter, we will present some algorithmic results for cooperative interval games.

CHAPTER 8

ALGORITHMIC ASPECTS

In this chapter, we give some numerical results that we have obtained by using Matlab. All the m-files used in this chapter are given in the Appendix.

We start with some examples related with the interval Shapley value introduced in Chapter 4 with two, three and four players. We use the m-files `shapley2` for two-person case, `shapley3` for three-person case, and `shapley4` for four-person case. For these examples, we work on the class of size monotonic interval games, where the interval Shapley value is defined.

Example 8.1 *Let $< N, w >$ be a two-person cooperative interval game with*

$$w(1) = [5, 9], w(2) = [7, 13], w(N) = [20, 32].$$

Then, the algorithm gives the numerical result $([9.0000, 14.0000], [11.0000, 18.0000])$ *and draws the interval Shapley value of the game depicted in Figure 8.1.*

Example 8.2 *Let $< N, w >$ be a two-person cooperative interval game with*

$$w(1) = [0, 1], w(2) = [0, 2], w(N) = [4, 8].$$

Then, the algorithm gives the numerical result $([2.0000, 3.5000], [2.0000, 4.5000])$ *and draws the interval Shapley value of the game depicted in Figure 8.2.*

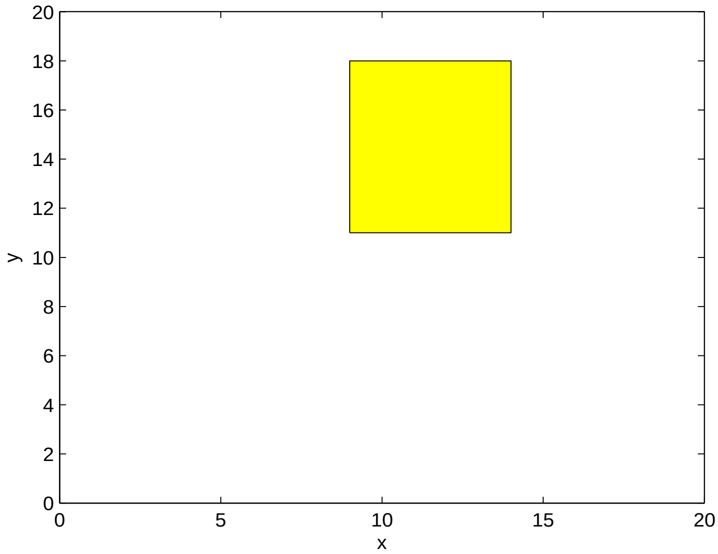

Figure 8.1: The interval Shapley value of the two-person cooperative interval game in Example 8.1.

Example 8.3 *Let $< N, w >$ be a three-person cooperative interval game with*

$$w(1) = [0, 0], w(2) = [0, 0], w(3) = [0, 0], w(1, 2) = [0, 0], w(1, 3) = [60, 75],$$

$w(2, 3) = [40, 55]$ *and* $w(N) = [100, 120]$. *Then, the algorithm gives the numerical result*

$$([30.0000, 34.1667], [20.0000, 24.1667], [50.0000, 61.6667])$$

and draws the interval Shapley value of the game depicted in Figure 8.3.

Example 8.4 *Let $< N, w >$ be a three-person cooperative interval game with*

$$w(1) = [0, 0], w(2) = [0, 0], w(3) = [0, 0], w(1, 2) = [0, 0], w(1, 3) = [1, 2],$$

$w(2, 3) = [1, 2]$ *and* $w(N) = [1, 2]$. *Then, the algorithm gives the numerical result*

$$([0.1667, 0.3333], [0.1667, 0.3333], [0.6667, 1.3333])$$

and draws the interval Shapley value of the game depicted in Figure 8.4.

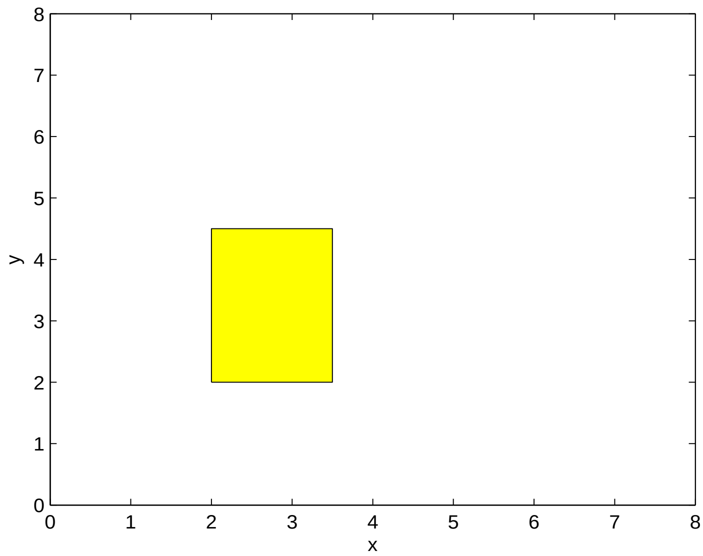

Figure 8.2: The interval Shapley value of the two-person cooperative interval game in Example 8.2.

Example 8.5 *Let $< N, w >$ be a three-person cooperative interval game with*

$$w(1) = [0, 0], w(2) = [0, 0], w(3) = [0, 0], w(1, 2) = [0, 0], w(1, 3) = [10, 30],$$

$w(2, 3) = [50, 70]$ *and* $w(N) = [50, 70]$. *Then, the interval Shapley value of the game is*

$$([1.6667, 5.0000], [21.6667, 25.0000], [26.6667, 40.0000]).$$

See Figure 8.5.

Example 8.6 *Let $< N, w >$ be a four-person cooperative interval game with $w(i) = [0, 0]$ for each $i = 1, 2, \ldots, 4$,*

$$w(1, 2) = [0, 0], w(1, 3) = [1, 2], w(1, 4) = [1, 2], w(2, 3) = [1, 2], w(2, 4) = [1, 2],$$

$$w(3, 4) = [1, 2], w(1, 2, 3) = [1, 2], w(1, 2, 4) = [1, 2], w(1, 3, 4) = [1, 2],$$

$w(2, 3, 4) = [1, 2]$ *and* $w(1, 2, 3, 4) = [2, 3]$. *Then, the algorithm gives the numerical result*

$$([0.4167, 0.5833], [0.4167, 0.5833], [0.5833, 0.9167], [0.5833, 0.9167])$$

which is the interval Shapley value of the interval game.

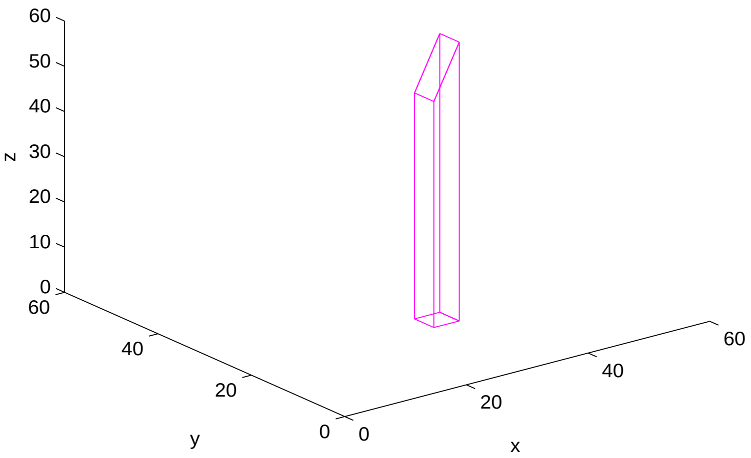

Figure 8.3: The interval Shapley value of the three-person cooperative interval game in Example 8.3.

Example 8.7 *Let* $< N, w >$ *be a four-person cooperative interval game with* $w(i) = [0, 0]$ *for each* $i = 1, 2, \ldots, 4$,

$$w(1, 2) = [0, 0], w(1, 3) = [6.5, 8], w(1, 4) = [6.5, 8], w(2, 3) = [6.5, 8], w(2, 4) = [6.5, 8],$$

$$w(3, 4) = [0, 0], w(1, 2, 3) = [10, 20], w(1, 2, 4) = [10, 20], w(1, 3, 4) = [10, 20],$$

$w(2, 3, 4) = [10, 20]$ *and* $w(1, 2, 3, 4) = [13, 34]$.
Then, the algorithm gives the numerical result

$$([3.2500, 8.5000], [3.2500, 8.5000], [3.2500, 8.5000], [3.2500, 8.5000])$$

which is the interval Shapley value of the interval game.

Our examples continue with the calculation of interval core elements for two-person and three-person cooperative interval games. The notion of the interval core introduced in Section 2.3 is one of the more interesting solution concepts on the class of cooperative interval games. Here, the nearest interval core element is obtained according to the initial guess that we have chosen. We use the m-files `icore2` for the two-person case and `icore3` for the three-person case.

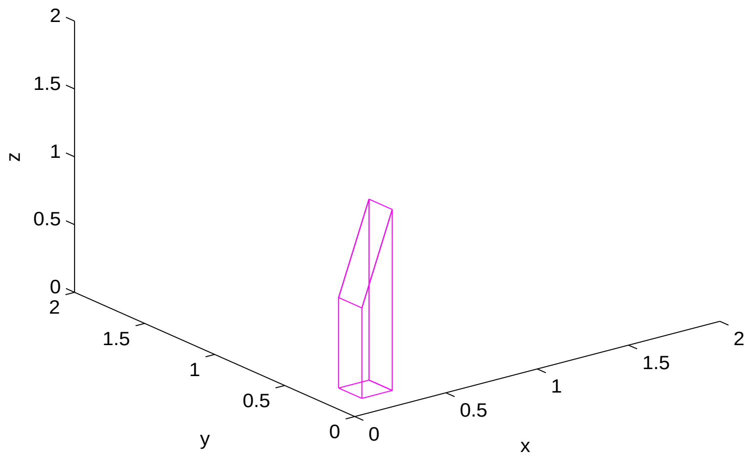

Figure 8.4: The interval Shapley value of the three-person cooperative interval game in Example 8.4.

Example 8.8 *Let* $< N, w >$ *be a two-person cooperative interval game with*

$$w(1) = [0, 0], w(2) = [0, 0], w(1, 2) = [18, 30].$$

Then, the algorithm gives the numerical result

$$([9.0000, 15.0000], [9.0000, 15.0000])$$

obtained by choosing the initial guess $[8.5; 8.5]$ *for the lower game and* $[14.5; 14.5]$ *for the upper game.*

Example 8.9 *Let* $< N, w >$ *be a three-person cooperative interval game with*

$$w(1) = [0, 0], w(2) = [0, 0], w(3) = [0, 0], w(1, 2) = [0, 0], w(1, 3) = [10, 30],$$

$w(2, 3) = [50, 70]$ *and* $w(N) = [50, 70]$. *Then, the algorithm gives the numerical result*

$$([0.0000, 0.0000], [25.0000, 35.0000], [25.0000, 35.0000])$$

obtained by choosing the initial guess $[0; 10; 10]$ *for the lower game and* $[0; 10; 10]$ *for the upper game. Another result obtaining by choosing the initial guess as* $[0; 15; 25]$ *for the lower game and* $[0; 15; 45]$ *for the upper game is*

$$([0.0000, 0.0000], [20.0000, 20.0000], [30.0000, 50.0000]).$$

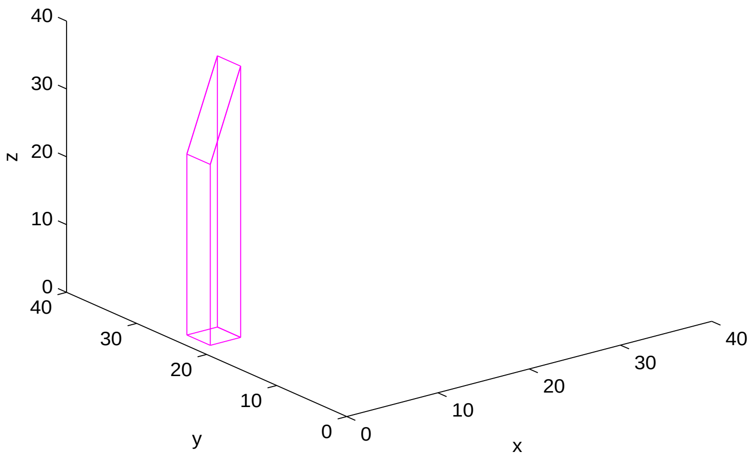

Figure 8.5: The interval Shapley value of the three-person cooperative interval game in Example 8.5.

By choosing different initial guesses we can obtain different interval core elements.

Example 8.10 *Let* $< N, w >$ *be a three-person cooperative interval game with*

$$w(1) = [0,0], w(2) = [0,0], w(3) = [0,0], w(1,2) = [0,0], w(1,3) = [1,2],$$

$w(2,3) = [1,2]$ *and* $w(N) = [1,2]$. *Then, the algorithm gives the numerical result*

$$([0.0000, 0.0000], [0.0000, 0.0000], [1.0000, 2.0000])$$

obtained by choosing the initial guess $[0;1;1]$ *for the lower game and* $[1;0;2]$ *for the upper game. Note that this game has only one element in the interval core.*

In this chapter, we intended to give a flavour of some numerical results related with the interval Shapley value and the interval core. In the next chapter, we shall conclude our studies by mentioning some open problems and future work for the class of cooperative interval games.

CHAPTER 9

CONCLUSION

In this thesis, we have developed the theory of cooperative interval games, which is a new area in cooperative game theory. We also aim to intensively present the state-of-the-art in this booming field of research and its applications. The reader is referred to Branzei, Tijs and Alparslan Gök (2008c) for a brief survey on cooperative interval games and interval solution concepts. This is a pioneering work on a promising topic, and there are still many interesting questions to be solved by further research such as the following ones.

A difficult topic might be to analyze under which conditions the set of payoff vectors generated by the interval core of a cooperative interval game coincides with the core of the game in terms of selections (of the interval game). The interval core is an appealing solution concept both from the theoretical point of view and from the respect of computational complexity. However, the use of elements of the interval core in practical situations requires to transform such an interval payoff vector into a traditional payoff vector. This can be done as in Chapter 2 where the reader can find a basic guide for handling interval solution concepts. A straightforward interpretation of interval core elements is questionable as discussed in Drechsel and Kimms (2008). The fact that there are interesting classes of cooperative interval games with nonempty interval cores like convex interval games and big boss interval games increases the interest in this solution concept.

A dominance relation in the interval setting is used to define the interval dominance core and interval stable sets for cooperative interval games. Relations between the interval core, the interval dominance core and interval stable sets of a cooperative interval game are studied. It is interesting to find sufficient conditions for the equality of the interval core and the interval dominance core, and to investigate whether for each interval game its interval dominance core is a convex set. Moreover, studying stable sets of a cooperative interval game in terms of selections of the game seems to be a valuable topic for the extension of the theory of cooperative interval games.

Further interesting questions are to study whether one can extend to interval games the well-known result of the traditional cooperative game theory that the core of a convex game is the unique stable set (Shapley (1971)) and to find an axiomatization of the interval Shapley value on the class of convex interval games. Other topics could be related to introducing new models in cooperative game theory by generalizing cooperative interval games. For example, the concepts and results on (convex) cooperative interval games could be extended to cooperative games in which the coalition values $w(S)$ are ordered intervals of the form $[u, v]$ of an (infinite dimensional) ordered vector space. Such generalization could give more applications to the interval game theory. Also to establish relations between convex interval games and convex games in other existing models of cooperative games would be interesting.

We notice that other OR situations and combinatorial optimization problems with interval data among which are flow situations, linear production situations and holding situations also could give rise to interesting interval games. The existing literature on related classical games can be an inspiration source for further research (Borm, Hamers and Hendrickx (2001), Curiel (1997), Kalai and Zemel (1982), Owen (1975), Tijs, Meca and López (2005). Weber, Alparslan Gök and Söyler (2007), Weber et al. (2008) and Weber, Alparslan Gök and Dikmen (2008) considered gene-network problems and environmental problems such as carbon dioxide emission reduction and fish quota with interval uncertainty. Weber, Kropat and Alparslan Gök (2008) show how advanced methods of continuous optimization contribute to modeling, learning and problem solution in areas of environmental protection, medicine and development under various kinds of uncertainty. It is a topic for further research to associate cooperative interval games with such situations.

For sequencing interval situations and related games, the approach using both urgency indices and relaxation indices when dealing with sequencing interval situations is a topic for further research. Other approaches for sharing the gain generated by a switch may be investigated. For example, it is possible to assign to each job its minimal compensation obtained supposing that its unitary cost and the processing time of the jobs involved in the switches coincide with the lower bound. After a realization, the difference between the actual cost savings and the sum of the shares already distributed over the switched jobs, can be allocated according to a fair division procedure or a bankruptcy rule.

In this thesis, we define two bankruptcy interval rules by extending the proportional rule and the rights-egalitarian rule to bankruptcy interval situations (see Chapter 7). An interesting topic is to extend to the interval setting the axiomatic characterizations of $PROP$ and f^{RE} and to compare them in the spirit of Herrero, Maschler and Villar (1999). Note that to compare $\mathcal{PROP}$ with $\mathcal{F}^{RE}$ we need to consider the restricted class $BRI_1^N \cap BRI_2^N$. The use of the allocations generated by the rules $\mathcal{PROP}$ and $\mathcal{F}^{RE}$ in practical bankruptcy-like situations with interval uncertainty is two-fold. Firstly, these interval allocations are used to inform claimants about what they can expect, between two boundaries, from the division problem at stake. Secondly, when the realization of the estate occurs, they are used to obtain standard allocations (see Chapter 2 for ways to transform vectors of intervals in vectors of real numbers).

Finally, Moretti et al. (2008) cope with uncertainty in cost allocation problems arising from connection configurations. Basically, they deal with minimum cost spanning tree situations where the costs are intervals and the agents may act optimistically or pessimistically. In the sequel, they briefly introduce a more complex problem as a possible topic for further research: how to deal with minimum interval cost spanning tree situations where not all the agents follow the same (pessimistic or optimistic) approach to make the decision on which spanning tree must be realized.

Consequently, cooperative interval games that we have developed in this thesis are a very valuable tool for modeling various economic and OR situations. The reader can find in Branzei, Tijs and Alparslan Gök (2008b) several protocols specifying how a certain interval solution, chosen for a specific situation to support decision making regarding cooperation might be used when uncertainty on payoffs is removed. The recent developments in the field of cooperative interval games offer new opportunities for game practice.

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APPENDIX

First, we present the Matlab m-files inspired from Mirás Calvo and Sánchez Rodriguez (2006) which are used to obtain Interval Shapley value.

shapley2.m

Interval Shapley value of a two-person cooperative interval game

```
function [sU,sL,mU,mL]=shapley2(vL1,vL2,vL12,vU1,vU2,vU12)

%shapley2 calculates and draws the interval Shapley value of a
%two-person cooperative interval game.
%The algorithm only works on the class of size monotonic interval
%games.
%The inputs vL1, vL2, vL12 are the lower bounds; vU1, vU2, vU12
%are the upper bounds of the characteristic function of the
%two-person cooperative interval game.
%From the outputs sL represents the lower bound, sU represents the
%upper bound of the Interval Shapley value and mL is the interval
%marginal vector of the lower game, mU is the interval marginal
%vector of the upper game.
%Here, X is the vector which is used to obtain the figure.

mL(1,:)=[vL1 vL12-vL1];
mL(2,:)=[vL12-vL2 vL2];
mU(1,:)=[vU1 vU12-vU1];
mU(2,:)=[vU12-vU2 vU2];
```

```
tL1=0;
tL2=0;
tU1=0;
tU2=0;

for i=1:2
    tL1=mL(i,1)+tL1;
    tL2=mL(i,2)+tL2;
    sL=[tL1;tL2]*(1/2)
    tU1=mU(i,1)+tU1;
    tU2=mU(i,2)+tU2;
    sU=[tU1;tU2]*(1/2)
end

Sh=[sL sU];

fprintf('IntervalShapley=([%6.4f,%6.4f],[%6.4f,%6.4f])\n',sL(1,1),
sU(1,1),sL(2,1),sU(2,1))

X=[sL(1) sL(2); sU(1) sL(2); sU(1) sU(2); sL(1) sU(2);
sL(1) sL(2)]

fill(X(:,1),X(:,2),'y')

axis([0 20 0 20])
xlabel('x')
ylabel('y')
```

shapley3.m

Interval Shapley value of a three-person cooperative interval game

```
function [sU,sL,mU,mL]=shapley3(vL1,vL2,vL3,vL12,vL13,vL23,vL123,
vU1,vU2,vU3,vU12,vU13,vU23,vU123)

%shapley3 calculates and draws the interval Shapley value of a th-
%ree-person cooperative interval game. The algorithm only works on
%the class of size monotonic interval games.
%The inputs vL1,vL2,vL3,vL12,vL13,vL23,vL123 are the lower bounds;
%vU1,vU2,vU3,vU12,vU13,vU23,vU123 are the upper bounds of the
%characteristic function of the three-person cooperative interval
%game.
%From the outputs sL represents the lower bound, sU represents the
%upper bound of the Interval Shapley value and mL is the interval
%marginal vector of the lower game, mU is the interval marginal
%vector of the upper game.
%Here, X is the vector which is used to obtain the figure.

mL(1,:)=[vL1 vL12-vL1 vL123-vL12];
mL(2,:)=[vL1  vL123-vL13 vL13-vL1];
mL(3,:)=[vL12-vL2 vL2 vL123-vL12];
mL(4,:)=[vL123-vL23  vL2 vL23-vL2];
mL(5,:)=[vL13-vL3  vL123-vL13 vL3];
mL(6,:)=[vL123-vL23 vL23-vL3 vL3];

mU(1,:)=[vU1 vU12-vU1 vU123-vU12];
mU(2,:)=[vU1  vU123-vU13 vU13-vU1];
mU(3,:)=[vU12-vU2 vU2 vU123-vU12];
mU(4,:)=[vU123-vU23  vU2 vU23-vU2];
mU(5,:)=[vU13-vU3  vU123-vU13 vU3];
mU(6,:)=[vU123-vU23 vU23-vU3 vU3];
```

```
tU=[0;0;0];
tL=[0;0;0];

for i=1:6
    tL(1)=mL(i,1)+tL(1);
    tL(2)=mL(i,2)+tL(2);
    tL(3)=mL(i,3)+tL(3)
    sL=[tL(1);tL(2);tL(3)]*(1/6)
    tU(1)=mU(i,1)+tU(1);
    tU(2)=mU(i,2)+tU(2);
    tU(3)=mU(i,3)+tU(3)
    sU=[tU(1);tU(2);tU(3)]*(1/6)
end

Sh=[sL sU];

fprintf('IntervalShapley=([%6.4f,%6.4f],[%6.4f,%6.4f],[%6.4f,%6.4f])
\n', sL(1,1),sU(1,1), sL(2,1),sU(2,1),sL(3,1),sU(3,1))

X=[sL(1) sL(2) 0; sU(1) sL(2) 0; sU(1) sU(2) 0; sL(1) sU(2) 0;
sL(1) sL(2) 0; sL(1) sL(2) sL(3); sU(1) sL(2) sU(3); sU(1)
sU(2) sU(3); sL(1) sU(2) sL(3); sL(1) sL(2) sL(3); sU(1) sL(2)
sU(3); sU(1) sL(2) 0; sU(1) sU(2) 0; sU(1) sU(2) sU(3); sL(1)
sU(2) sL(3); sL(1) sU(2) 0]

axis([0 40 0 40 0 40])
hold on
plot3(X(:,1),X(:,2),X(:,3),'m')
xlabel('x')
ylabel('y')
zlabel('z')
```

shapley4.m

Interval Shapley value of a four-person cooperative interval game

```
function [sU,sL,mU,mL]=shapley4(vL1,vL2,vL3,vL4,vL12,vL13,vL14,
vL23,vL24,vL34,vL123,vL124,vL134,vL234,vL1234,vU1,vU2,vU3,vU4,
vU12,vU13,vU14,vU23,vU24,vU34,vU123,vU124,vU134,vU234,vU1234)

%shapley4 calculates the interval Shapley value of a four-person
%cooperative interval game. The algorithm only works on the
%class of size monotonic interval games.
%The inputs vL1, vL2,vL3,vL4,vL12,vL13,vL14,vL23,vL24,vL34,vL123,
%vL124,vL134,%vL234,vL1234 are the lower bounds; vU1,vU2,vU3,vU4,
%vU12,vU13,vU14,vU23,vU24,vU34,vU123,vU124,vU134,vU234,vU1234
%are the upper bounds of the characteristic function of the
%four-person cooperative interval game.
%From the outputs sL represents the lower bound, sU represents
%the upper bound of the Interval Shapley value and mL is the
%interval marginal vector of the lower game, mU is the interval
%marginal vector of the upper game.

mL(1,:)=[vL1 vL12-vL1 vL123-vL12 vL1234-vL123];
mL(2,:)=[vL1 vL12-vL1 vL1234-vL124 vL124-vL12];
mL(3,:)=[vL1 vL123-vL13 vL13-vL1 vL1234-vL123];
mL(4,:)=[vL1 vL1234-vL134 vL13-vL1 vL134-vL13];
mL(5,:)=[vL1 vL124-vL14  vL1234-vL124 vL14-vL1];
mL(6,:)=[vL1 vL1234-vL134 vL134-vL14 vL14-vL1];

mL(7,:)=[vL12-vL2 vL2 vL123-vL12 vL1234-vL123];
mL(8,:)=[vL12-vL2 vL2 vL1234-vL124 vL124-vL12];
mL(9,:)=[vL123-vL23 vL2 vL23-vL2 vL1234-vL123];
mL(10,:)=[vL1234-vL234 vL2 vL23-vL2 vL234-vL23];
mL(11,:)=[vL124-vL24 vL2 vL1234-vL124 vL24-vL2];
```

```
mL(12,:)=[vL1234-vL234 vL2 vL234-vL24 vL24-vL2];

mL(13,:)=[vL13-vL3 vL123-vL13 vL3 vL1234-vL123];
mL(14,:)=[vL13-vL3 vL1234-vL134 vL3 vL134-vL13];
mL(15,:)=[vL123-vL23 vL23-vL3 vL3 vL1234-vL123];
mL(16,:)=[vL1234-vL234 vL23-vL3 vL3 vL234-vL23];
mL(17,:)=[vL134-vL34 vL1234-vL134 vL3 vL34-vL3];
mL(18,:)=[vL1234-vL234 vL234-vL34 vL3 vL34-vL3];

mL(19,:)=[vL14-vL4 vL124-vL14 vL1234-vL124 vL4];
mL(20,:)=[vL14-vL4 vL1234-vL134 vL134-vL14 vL4];
mL(21,:)=[vL124-vL24 vL24-vL4 vL1234-vL124 vL4];
mL(22,:)=[vL1234-vL234 vL24-vL4 vL234-vL24 vL4];
mL(23,:)=[vL134-vL34 vL1234-vL134 vL34-vL4 vL4];
mL(24,:)=[vL1234-vL234 vL234-vL34 vL34-vL4 vL4];

mU(1,:)=[vU1 vU12-vU1 vU123-vU12 vU1234-vU123];
mU(2,:)=[vU1 vU12-vU1 vU1234-vU124 vU124-vU12];
mU(3,:)=[vU1 vU123-vU13 vU13-vU1 vU1234-vU123];
mU(4,:)=[vU1 vU1234-vU134 vU13-vU1 vU134-vU13];
mU(5,:)=[vU1 vU124-vU14  vU1234-vU124 vU14-vU1];
mU(6,:)=[vU1 vU1234-vU134 vU134-vU14 vU14-vU1];

mU(7,:)=[vU12-vU2 vU2 vU123-vU12 vU1234-vU123];
mU(8,:)=[vU12-vU2 vU2 vU1234-vU124 vU124-vU12];
mU(9,:)=[vU123-vU23 vU2 vU23-vU2 vU1234-vU123];
mU(10,:)=[vU1234-vU234 vU2 vU23-vU2 vU234-vU23];
mU(11,:)=[vU124-vU24 vU2 vU1234-vU124 vU24-vU2];
mU(12,:)=[vU1234-vU234 vU2 vU234-vU24 vU24-vU2];

mU(13,:)=[vU13-vU3 vU123-vU13 vU3 vU1234-vU123];
mU(14,:)=[vU13-vU3 vU1234-vU134 vU3 vU134-vU13];
mU(15,:)=[vU123-vU23 vU23-vU3 vU3 vU1234-vU123];
```

```
mU(16,:)=[vU1234-vU234 vU23-vU3 vU3 vU234-vU23];
mU(17,:)=[vU134-vU34 vU1234-vU134 vU3 vU34-vU3];
mU(18,:)=[vU1234-vU234 vU234-vU34 vU3 vU34-vU3];

mU(19,:)=[vU14-vU4 vU124-vU14 vU1234-vU124 vU4];
mU(20,:)=[vU14-vU4 vU1234-vU134 vU134-vU14 vU4];
mU(21,:)=[vU124-vU24 vU24-vU4 vU1234-vU124 vU4];
mU(22,:)=[vU1234-vU234 vU24-vU4 vU234-vU24 vU4];
mU(23,:)=[vU134-vU34 vU1234-vU134 vU34-vU4 vU4];
mU(24,:)=[vU1234-vU234 vU234-vU34 vU34-vU4 vU4];

tU=[0;0;0;0];
tL=[0;0;0;0];

for i=1:24
    tL(1)=mL(i,1)+tL(1);
    tL(2)=mL(i,2)+tL(2);
    tL(3)=mL(i,3)+tL(3)
    tL(4)=mL(i,4)+tL(4)
    sL=[tL(1);tL(2);tL(3);tL(4)]*(1/24)
    tU(1)=mU(i,1)+tU(1);
    tU(2)=mU(i,2)+tU(2);
    tU(3)=mU(i,3)+tU(3);
    tU(4)=mU(i,4)+tU(4);
    sU=[tU(1);tU(2);tU(3);tU(4)]*(1/24)
end

Sh=[sL sU];

fprintf('IntervalShapley=([%6.4f,%6.4f],[%6.4f,%6.4f],[%6.4f,%6.4f],
[%6.4f,%6.4f])\n', sL(1,1),sU(1,1),sL(2,1),sU(2,1),sL(3,1),
sU(3,1),sL(4,1),sU(4,1))
```

Second, we present the Matlab m-files which find the nearest interval core element according to the initial guess chosen.

icore2.m

The interval core element of a two-person cooperative interval game

```
function[xL,xU]=icore2(vL1,vL2,vL12,xL0,vU1,vU2,vU12,xU0)

%icore2 finds an interval core element of a two-person coopera-
%tive interval game which is the nearest to the initial guess
%chosen.
%The inputs vL1, vL2, vL12 are the lower bounds; vU1, vU2, vU12
%are the upper bounds of the characteristic function of the
%two-person cooperative interval game.
%xL0 is the initial guess for the lower game and xU0 is the
%initial guess for the upper game.
%The output xL is the lower bound and the output xU is the
%upper bound of the  interval core element.
%For the lower game [xL,fval]=fsolve(@coremyfunL,xL0) returns
%the value of the objective function coremyfunL at the solution
%xL, i.e., the algorithm starts at xL0 and tries to find a zero
%of fL.
%For the upper game the procedure is similar as above.
%For details on fsolve we refer to Matlab Optimization Toolbox.

global vL12
global vU12

[xL,fval]=fsolve(@coremyfunL,xL0)

if xL(1)>=vL1 &   xL(2)>=vL2
    solution=xL
else
```

```
    fprintf('Change your initial guess')
end

[xU,fval]=fsolve(@coremyfunU,xU0)

if xU(1)>=vU1 &   xL(2)>=vU2
    solution=xU
else
    fprintf('Change your initial guess')
end

fprintf('Intervalcore element=([%6.4f,%6.4f],[%6.4f,%6.4f])\n',
xL(1,1),xU(1,1),xL(2,1),xU(2,1))

function fL =coremyfunL(x)
global vL12

fL =x(1)+x(2)-vL12

function fU=coremyfunU(x)
global vU12

fU=x(1)+x(2)-vU12
```

icore3.m

The interval core element of a three-person cooperative interval game

```
function[xL,xU]= icore3(vL1,vL2,vL3,vL12,vL13,vL23,vL123,xL0,vU1,
vU2,vU3,vU12,vU13,vU23,vU123,xU0)

%icore3 finds an interval core element of a three-person
%cooperative interval game which is the nearest to the
```

```
%initial guess chosen.
%The inputs vL1,vL2,vL3,vL12,vL13,vL23,vL123 are the lower bounds;
%vU1,vU2,vU3,vU12,vU13,vU23,vU123 are the upper bounds of the
%characteristic function of the three-person cooperative interval
%game.
%xL0 is the initial guess for the lower game and xU0 is the
%initial guess for the upper game.
%The output xL is the lower bound and the output xU is the upper
%bound of the  interval core element.
%For the lower game [xL,fvalL]=fmincon(@coremyfunL,xL0,A,bL,[],[],
%lbL) starts at xL0, attempts to find a minimum xL to the function
%described in coremyfunL subject to the linear inequalities AxL<=b
%and defines a set of upper bounds on the design variables in xL,
%so that the solution always satisfies lbL<=xL.
%For the upper game the procedure is similar as above.
%For details on fmincon we refer to Matlab Optimization Toolbox.

global vL123
global vU123

 A=[-1 -1 0;-1 0 -1;0 -1 -1]
 bL=[-vL12;-vL13;-vL23]
 lbL=[vL1;vL2;vL3]
 bU=[-vU12;-vU13;-vU23]
 lbU=[vU1;vU2;vU3]

[xL,fvalL]=fmincon(@coremyfunL,xL0,A,bL,[],[],lbL)
[xU,fvalU]=fmincon(@coremyfunU,xU0,A,bU,[],[],lbU)

fprintf('Intervalcore element=([%6.4f,%6.4f],[%6.4f,%6.4f],
[%6.4f,%6.4f])\n',xL(1,1),xU(1,1),xL(2,1),xU(2,1),xL(3,1),xU(3,1))
```

```
function fL=coremyfunL(x)
global vL123

fL =x(1)+x(2)+x(3)-vL123

function fU=coremyfunU(x)
global vU123

fU=x(1)+x(2)+x(3)-vU123
```

VITA

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EDUCATION

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Advisor: Prof. Dr. Gerhard Wilhelm Weber
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B.S. Department of Mathematics, June 2002
Ankara University-Ankara

High School Milli Piyango Anadolu High School
June 1998, Ankara

WORK EXPERIENCE

2004-Present Research Assistant
Department of Scientific Computing, Institute of Applied Mathematics,
Middle East Technical University-Ankara

2002-2004 Research Assistant

Department of Mathematics,

Middle East Technical University-Ankara

FOREIGN LANGUAGES

Turkish (native), English (High level), German (Middle level), Italian (Elementary level)

COMPUTER ABILITIES

Microsoft Office, Latex, Matlab

HOBBIES

tracking, swimming, cycling, cinema, theatre, strolling, travelling

PUBLICATIONS

(i) Alparslan Gök S.Z., Branzei R., Fragnelli V. and Tijs S., "Sequencing interval situations and related games", *preprint no.* ***113***, *Institute of Applied Mathematics, METU and Tilburg University, Center for Economic Research, The Netherlands, CentER DP 63* (2008).

(ii) Alparslan Gök S.Z., Branzei R. and Tijs S., "Cores and stable sets for interval-valued games", *preprint no.* ***90***, *Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 17* (2008a).

(iii) Alparslan Gök S.Z., Branzei R. and Tijs S., "Convex interval games", *preprint no.* ***100***, *Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 37* (2008b).

(iv) Alparslan Gök S.Z., Branzei R. and Tijs S., "Big boss interval games", *preprint no.* ***103***, *Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 47* (2008c).

(v) Alparslan Gök S.Z., Branzei R. and Tijs S., "Cooperative interval games arising from airport situations with interval data", *preprint no.* ***107***, *Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research*, *The Netherlands, CentER DP 57* (2008d) (The paper "Airport interval games and their Shapley value" which is based on this paper will appear in Operations Research and Decisions).

(vi) Alparslan Gök S.Z., Miquel S. and Tijs S., "Cooperation under interval uncertainty", *Mathematical Methods of Operations Research*, Vol. **69**, no. **1** (2009) 99-109.

(vii) Branzei R. and Alparslan Gök S.Z., "Bankruptcy problems with interval uncertainty", *Economics Bulletin*, Vol. **3**, no. **56** (2008) pp. 1-10.

(viii) Branzei R., Tijs S. and Alparslan Gök S.Z., "Some characterizations of convex interval games", AUCO Czech Economic Review, Vol. **2**, no. **3** (2008a) 219-226.

(ix) Branzei R., Tijs S. and Alparslan Gök S.Z., "How to handle interval solutions for cooperative interval games", *preprint no. **110**, Institute of Applied Mathematics, METU* (2008b).

(x) Branzei, R., Tijs, S., Alparslan Gök, S.Z., "Cooperative interval games: a survey", *preprint no. **104**, Institute of Applied Mathematics, METU* (2008c).

(xi) Moretti S., Alparslan Gök S.Z., Branzei R. and Tijs S., "Connection situations under uncertainty", *preprint no. **112**, Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 64* (2008).

(xii) Weber G.W., Alparslan Gök S.Z. and Dikmen N., "Environmental and life sciences: Gene-environment networks-optimization, games and control-a survey on recent achievements", *in the special issue of Journal of Organizational Transformation and Social Change, guest editor: D. DeTombe*, Volume **5**, no. **3** (2008) pp. 197-233.

(xiii) Weber G.W., Alparslan Gök S.Z. and Söyler B., "A new mathematical approach in environmental and life sciences: gene-environment networks and their dynamics", *preprint no. **69**, Institute of Applied Mathematics, METU* (2007) (to appear in Environmental Modeling and Assessment, DOI number: `http://dx.doi.org/doi:10.1007/s10666-007-9137-z`).

(xiv) Weber G.W., Kropat E. and Alparslan Gök S. Z., "Semi-infinite and conic optimization in modern human, life and financial sciences under uncertainty", *in the ISI Proceedings of 20th Mini - EURO conference, Continuous Optimization and Knowledge-Based Technologies, Neringa, Lithuania* (May 20-23, 2008) 180-185.

(xv) Weber G.W., Taylan P., Alparslan Gök S.Z., Özöğür Akyüz S. and Akteke Öztürk B., "Optimization of gene-environment networks in the presence of errors and uncertainty with Chebychev approximation", *TOP, the Operational Research journal of SEIO (Spanish Statistics and Operations Research Society*, Vol. **16**, no. **2** (2008) pp. 284-318.

PRESENTATIONS IN INTERNATIONAL SCIENTIFIC MEETINGS

(i) "Cooperative Interval Games", *ICSS, International Conference on Social Sciences 1*, İzmir, Turkey, August 21-22, 2008.

(ii) "Economic Situations and Cooperative Interval Games", *SING 4, Spain Italy Netherlands Meeting on Game Theory*, Wroclaw, Poland, June 26-28, 2008.

(iii) “Cooperative Interval Games-Part 2”, *Seminar on Game Theory*, Department of Mathematics (DIMA), Genoa, Italy, May 5, 2008.

(iv) “Cooperation under Interval Uncertainty”, *EURO XXII, European Conference on Operational Research*, Prague, Czech Republic, July 8-11, 2007.

(v) Various talks in Seminar of Applied Mathematics in Life and Human Sciences and Economy group,
`http://www.iam.metu.edu.tr/research/groups/compbio/seminars.html`).

(vi) Various joint talks with Prof. Dr. Gerhard Wilhelm Weber in Ballarat, Australia (2006), Alberta, Canada (2007), Kiev, Ukraine (2006, 2007, 2008), Rio de Janeiro, Brazil (2008), Braga, Portugal (2008) and Haifa, Israel (2008).

(vi) Various joint talks with Assoc. Prof. Dr. Rodica Branzei and Prof. Dr. Stef Tijs in Genoa, Italy (2008), Twente, the Netherlands (2008), Ischia, Italy (2008) and Lyon, France (2008).

RESEARCH VISITS

June 22 - July 19, 2005, Universität der Bundeswehr München, Germany, supervised by Prof. Dr. Stefan Pickl.

January 29 - March 14, 2007, Tilburg University, Netherlands, supervised by Prof. Dr. Stef Tijs.

January 14 - July 14, 2008, Genoa University, Italy, supervised by Prof. Dr. Stef Tijs.

PARTICIPATION IN INTERNATIONAL SCIENTIFIC MEETINGS

ICSS, International Conference on Social Sciences 1, İzmir, Turkey, August 21-22, 2008.

SING 4, Spain Italy Netherlands Meeting on Game Theory, Wroclaw, Poland, June 26-28, 2008.

Seminars on Game Theory, Department of Mathematics (DIMA), Genoa, Italy, June 16, May 5, April 15, March 27, February 21 and January 28, 2008.

First Workshop on Game Theory, University of Milan Bicocca, Milan, Italy, March 7-8, 2008.

EURO XXII, European Conference on Operational Research, Prague, Czech Republic, July 8-11, 2007.

Workshop on Sustainable Living at Turkish Rural Countryside, METU, Ankara, Turkey, June 8, 2007.

Workshop on Networks in Computational Biology, METU, Ankara, Turkey, September 10-12, 2006.

Workshop on Complex Societal Problems, Sustainable Living and Development, METU, Ankara, Turkey, April, 15-21, 2006.

EURO, Thirtieth Anniversary, Munich, Germany, July 1, 2005.

Workshop on Advances in Continuous Optimization, Istanbul, Turkey, July 4-5, 2003.

MEMBERSHIPS

EUROPT - The Continuous Optimization Working Group of EURO,
`http://www.iam.metu.edu.tr/EUROPT/`.

EURO - Association of European Operational Research Societies

SIAM - Society of Industrial and Applied Mathematics

Applied Mathematics in Life and Human Sciences and Economy, Institute of Applied Mathematics, METU,
`http://www.iam.metu.edu.tr/research/groups/compbio/index.html`

ORGANIZATION OF SCIENTIFIC EVENTS

Member of organizing commitee (stream organizer) of *EURO XXIII conference*, Bonn, Germany, July 5-8, 2009.

Member of organizing commitee of *Applied Mathematics in Life and Human Sciences and Economy group*, Institute of Applied Mathematics, METU, Ankara.

Member of organizing commitee of *Workshop on Sustainable Living at Turkish Rural Countryside*, METU, Ankara, June 8, 2007.

Member of organizing commitee of *Workshop on Complex Societal Problems, Sustainable Living and Development*, METU, Ankara, April 15-21, 2006.

Member of organizing commitee of *Workshop on The Mathematical Tool Box for the Financial Engineer*, METU, Ankara, March 21-22, 2005.

REFEREE ACTIVITIES

Journal of Intelligent and Fuzzy Systems

Journal of Mathematical Analysis and Applications

Special Issue on OR for Better Management of Sustainable Development of European Journal of Operational Research (EJOR)

Proceedings of the International Conference on Collaborative Decision Making, Toulouse, France

IEEE Transactions on Information Technology in Biomedicine