PRICING INFLATION-INDEXED SWAPS AND SWAPTIONS USING AN HJM MODEL

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ABSTRACT

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Inflation-indexed instruments provide a real return and protect investors from the erosion of the purchasing power of money. Hence, inflation-indexed markets grow very fast day by day. In this thesis, we focus on pricing of the inflation-indexed swaps and swaptions which are the most liquid derivative products traded in the inflation-indexed markets. Firstly, we review the Hull-White extended Vasicek model in the HJM framework. Then, we use this model to price inflation-indexed swaps. Also, pricing of inflation-indexed swaptions is given using Black's market model.

Keywords: Inflation-indexed swaps, swaptions, HJM framework, Hull-White extended Vasicek model

ÖZ

ENFLASYONA ENDEKSLİ SWAP VE SWAP ÜZERİNE YAZILAN OPSİYONLARIN HJM MODELİ KULLANILARAK FİYATLANDIRILMASI

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Enflasyona endeksli enstrümanlar reel getiri kazanma imkanı sağlar ve yatırımcıları paranın alım gücünde meydana gelen aşınmadan korur. Bu nedenle, enflasyona endeksli piyasalar her geçen gün hızlı bir şekilde büyümektedir. Bu çalışmada, enflasyona endeksli piyasalarda en likit türev ürünleri olan enflasyona endeksli swap ve enflasyona endeksli swap üzerine yazılan opsiyonların fiyatlandırılması üzerinde çalışılmıştır. İlk olarak, Hull-White genişletilmiş Vasicek modeli HJM çerçevesinde incelenmiştir. Daha sonra, enflasyona endeksli swapları fiyatlamak için bu model kullanılmıştır. Ayrıca, enflasyona endeksli swap üzerine yazılan opsiyonları fiyatlama formülü Black piyasa modeli kullanılarak elde edilmiştir.

Anahtar Kelimeler: Enflasyona endeksli swap, Enflasyona endeksli swap üzerine yazılan opsiyon, HJM çerçevesi, Hull-White genişletilmiş Vasicek modeli

To my family

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CHAPTER 1

INTRODUCTION

People are saving their money for future consumption. The savers are concerned only with the real purchasing power of their savings. If individuals can be certain that deferred consumption is equivalent to more consumption or similarly corporations can be sure that the future real value of their capital is increased, then these would provide a powerful incentive to save. The uncertainty of future changes in the level of prices and its effect in the purchasing power of money is one of the forces that is boosting the inflation-indexed markets.

Inflation linked derivatives market has grown so rapidly that, by mid-2003, in some markets has become a significant proportion of that in the underlying bond markets. As is the case with all types of derivatives, inflation indexed derivatives are designed to help fill the gaps in the market for the underlying securities. The market participants still think the inflation linked securities market as having a large potential market growth worldwide.

The main objectives of this thesis are to search the finance literature for the studies of inflation linked securities, to review Heath-Jarrow-Morton framework, to price inflation-indexed swaps using Hull-White extended Vasicek model and to price inflation-indexed swaptions using the defined inflation-indexed swap market model.

The organization of this thesis is as follows: In the first chapter, we give the definition of inflation, used inflation indexes, and an overview for the inflation-indexed securities and derivatives which are the concern of this thesis. In Chapter 2, we present a review for the studies about pricing or the working principles of inflation-indexed securities. In Chapter 3, the fundamental definitions and theorems of mathematical finance, bonds, interest rates and change of numeriare techniques are given. Chapter 4, reviews the HJM framework. In Chapter 5, Hull-White extended Vasicek model is introduced in the HJM framework, then the dynamics in the real economy and also in the forward measure are given. In Chapter 6, the derivation for prices of inflation-indexed swaps using the model given in chapter 5 and prices of inflation-indexed swaptions using the Black's market model are presented. Chapter 7, concludes the thesis with a short summary.

1.1 Inflation

Inflation is defined as an increase in the level of prices in an economy, and therefore in effect with the real value of money. If prices are not fixed over time, the value of money will float, usually upwards, but rarely the prices decrease, then it is called deflation.

The reasons behind inflation are complex; so in the literature there are various theories using macroeconomic and microeconomic analysis. Some factors that cause inflation can be listed as follows: The level of monetary demand in the economy is one of the main reasons of inflation. When at the current price level, demand for the goods and services in the economy is greater than the economy's ability to produce them; inflation tends to rise. A rise in production costs or labor costs can also lead to inflation. If raw materials increase in price or workers demand wage increases, these lead to an increase in the cost of the product. Then, to provide stability in profits, the companies choose to pass these costs to their customers, finally the price of final product increases and inflation occurs. Another cause of inflation is international lending and national debts. As countries borrow money, prices rise to cope with their debts and interests.

Inflation can be measured in different ways. The commonly used measures for inflation are:

- Consumer Price Index (CPI): measures prices of a selection of goods and services purchased by a consumer. The basket includes hundreds of things; from basic items to new products. Prices are collected every month over the country and all these prices are combined to produce an index of prices. Then the inflation rate is the percentage rate of change of a price index over a period, generally twelve months.

- Producer Price Index (PPI): which measures changes in prices received by domestic producers for their output. This differs from the CPI in that profits and taxes may cause the amount received by the producer to differ from what the consumer paid. There is a delay between an increase in the PPI and any final increase in the CPI. Producer price index measures the pressure being put on producers by the costs of their raw materials. This could be passed on to consumers or it could be absorbed by profits, or could be compensated by increasing productivity.

- Core Price Index: Because food and oil prices can change quickly due to changes in supply and demand conditions in the food and oil markets, it can be difficult to detect the trend in price levels when those are included. Therefore a measure of core inflation is reported which removes the most volatile components (such as food and oil) from a price index like CPI. Because core inflation is less affected by short run supply and demand conditions in markets, central banks rely on it to better measure the inflationary impact of current monetary policy.

- Wholesale Price Index (WPI): is the price of a representative basket of wholesale goods. This index focuses on the price of goods traded between corporations rather than goods bought by consumers which is measured by the CPI. The purpose of the WPI is to monitor price movements that reflect supply and demand in industry, manufacturing and construction.

- GDP deflator: is a measure of the price of all the goods and services included in Gross Domestic Product (GDP).

Since almost all price indexed securities are linked to CPI, consumer prices are the only concern of this thesis.

1.2 Inflation-linked Securities

Indexation firstly used in the beginning of the 18th century. Deacon, Derry and Mirfenderesky present the history of indexation: In 1707, Bishop William Fleetwood produced a study into the erosion of the purchasing power of money, for a fellowship established in 1450 whose membership was restricted to those with an annual income of less than £5. He examined changes in the prices of corn, meat, drink, cloth between 1450 - 1700 and he found that there had been a huge increase in the prices. He concluded that the acceptance rule for the fellowship must be the real annual income of an individual would be less than £5, not his nominal annual income. Then in 1742, the State of Massachusetts issued bills of public credit which were linked to the cost of silver in the London Exchange. Since the price of silver appreciated more rapidly than the general price level, in 1747 the Parliament passed a law telling that a larger group of commodities should be used for indexation in future debt. After this law, the State of Massachusetts issued Depreciation Notes in 1780 for soldiers during the American Revolution and therefore a basket of goods was firstly defined.

Economists made so many researchs about indexation and they published books and articles with developing concept of indexation. Sir George Shuckburgh Evelyn (1798), Joseph Lowe (1822) and G. Povlett Scrope (1833) were first economists who discussed the construction of an index to represent the general level of prices. In 1875, W. Stanley Jevons suggested to use gold prices for indexation and told the benefits of indexation in his works. Alfred Marshall (1886), John Maynard Keynes (1924), Richard Musgrave, Milton Friedman and Robert Barro were also strong supporters of the concept to index wages and financial instruments.

Although so many investigations were made for indexation earlier, indexed debt was important in financial markets in the second half of the 20th century. Table 1.1 shows some countries issued indexed public sector bonds and which index they used for these bonds[1].

Inflation indexed securities are instruments that protect the investors from changes in the general level of prices in the economy. Since inflation indexed securities have an inflation adjustment, their yields are considered to be real yields. In these securities, the security holder earns this real return plus the inflation realized over the life of the security. But in nominal securities, the return that the investor gets by holding this security to the maturity is equal to its yield.

The yield on a nominal security has two components: real yield and inflation expectation over the life of the security. Since the investors have different expectations of future inflation, both nominal and real markets have some investors. For example; in Turkish bond market, the difference between the yields on the 4-year Treasury and 4-year Treasury inflation protected bonds is $4,62$ %. If an investor expects inflation to be higher than $4,62$ % over the life of the bonds which is 4 year here, then that investor would prefer owning the inflation protected bond. On the contrary, if the investor expects inflation to be lower than $4,62$ % over 4 years,

[1] Mark Deacon, Andrew Derry and Dariush Mirfendereski. Inflation Indexed Securities. Wiley-Finance, 2004 [9].

Table 1.1: Some countries in which indexed public sector bonds have been issued

Country	*Issue Date*	*Index used*
Australia	1983-2003	Consumer prices
	1991	Average weekly earnings
Austria	1953	Electricity prices
	2003-	Consumer prices
Brazil	1964-1990	Wholesale prices
	1991-	General prices
	2002-	Consumer prices
Canada	1991-	Consumer prices
France	1952, 1973	Gold price
	1956	Level of industrial production
	1956	Average value of French securities
	1957	Price of equities
	1998-	Domestic consumer prices
	2001-	European consumer prices
Germany	2002, 2003	European consumer prices
Italy	1983	Deflator of GDP at factor cost
	2003-	European consumer prices
South Africa	2000-	Consumer prices
Turkey	1994-1997	Wholesale prices
	1997-	Consumer prices
USA	1742, 1780	Commodity prices
	1997-	Consumer prices

then the investor would prefer owning the nominal bond.

Inflation-linked securities have some drawbacks for investors. In most countries, the CPI for a given month is announced in the middle of the following month. So, the cash flows of indexed securities can not be adjusted by inflation up to the moment which they are paid. There must be a lag between the actual movements in the price index and the inflation adjustments to the cash flows. When inflation is volatile, these lags can be problem for investors. Another factor is that the inflation-linked securities are taxed disadvantageously. Since cash flows are taxed on a nominal (not real) basis, the real returns of these instruments after tax are uncertain. Also, inflation indexed securities have a less liquid market than nominal securities. In spite of these drawbacks, investors prefer inflation indexed securities for their portfolios. Because, there is no asset class which is able to provide such a protection against the erosion of purchasing power and since these securities are generally purchased by buy-and-hold investors, the lower liquidity problem is unlikely to be significant.

Inflation linked securities have advantages for the issuers at the same time. The main advantage of issuance inflation indexed securities by governments is that it allows to reduce the cost of financing. Cost savings can be in several ways. Firstly, if investors are willing to pay premium for protection against inflation, then this premium will be reflected in a lower yield paid by the government on the instruments that provide such protection. Also, if inflation over the life of the security turns out to be lower than the market had expected at the time of issuance, then indexed debt again provides cheaper funding than conventional debt. Second advantage of indexing the government's debt is that it allows a more precise matching of the government's assets and liabilities.

1.3 Inflation-linked Derivatives

Derivatives in the inflation market provide many opportunities to investors. They meet the needs of investors that the bond market unable to provide. When investor demands and issuer needs can not be matched, inflation linked derivatives can be used to remove the mismatches. Derivatives help to match the maturity, frequency of cash flows, size, index and timing.

1.3.1 Inflation-indexed Swaps and Swaptions

A swap is a transaction in which two counterparties agree to pay each other a series of cash flows over a specified period of time. The four types of swaps are currency swaps, interest rate swaps, equity swaps and commodity swaps. In an inflation indexed swap, at least one of the cash flows is tied to inflation.

A receiver swap is a swap where the holder at each payment date receives a fixed amount and pays a floating amount which is the inflation rate in this thesis. A payer swap is a swap where the holder receives the inflation rate and pays a fixed rate.

A swaption is an option to enter into a swap at a pre specified date for a specified swap rate. The right to enter into a swap paying a fixed rate is called a payer swaption, and the right to enter into a swap receiving a fixed rate is called a receiver swaption. An inflation indexed swaption is a swaption where the underlying swap is an inflation indexed swap.

CHAPTER 2

REVIEW OF LITERATURE

There are so many researches, articles and books about inflation linked securities in the finance literature. Advantages and disadvantages of inflation indexed securities for issuers and investors, pricing of these instruments are investigated around the world, but in Turkey there is a little work about this subject.

One of the early studies about inflation derivatives was made by Hughston [21]. He introduced a methodology based on the foreign-currency analogy. Here nominal assets are thought of as domestic assets, real assets as foreign assets and the consumer price index is thought as a kind of exchange rate to determine the nominal payout of a real bond at maturity. In his article, he assumed a complete market with no arbitrage. For consumer price index, real and nominal bond prices, Heath-Jarrow-Morton model are used. Pricing formulas are given for inflation linked derivatives, but all the formulas are given in the closed form.

Jarrow and Yıldırım [22] developed a three factor HJM model in order to price treasury inflation protected securities (TIPS) and options which are linked to the inflation index. They consider a cross-currency economy under no-arbitrage assumption. They assumed that the volatilities of all asset prices and the consumer price index are deterministic. When the bond prices are Gaussian, they obtained the pricing formulas and apply the HJM model to price a call option linked to the CPI-U[1] index.

Belgrade-Benhamou-Koehler [4] introduced a new market model to obtain a link between zero coupons and year-on-year swaps which is disregarded in the Jarrow-Yıldırım model. Their model is robust and simple which has only few parameters. The main hypothesis is that the market model for inflation considers forward inflation index return as a diffusion with

[1] Consumer price index for all urban consumers.

deterministic volatility structure. Their model is more suited in markets there is enough information from zero coupon and year-on-year swaps and also it is computationally intensive.

Mercurio [28] studied on pricing of zero coupon inflation indexed swaps, year-on-year inflation indexed swaps, inflation indexed caplets and floorlets. Firstly, the swaps are priced using the Jarrow-Yıldırım model with Hull-White parametrization and then he introduced two different market models for pricing swaps. The difference between two market models is that the dependence on the volatility of real rates which may be hard to estimate. Also, he finally tested the performance of the models using the Euro inflation indexed swaps market data.

The other study on pricing inflation indexed derivatives was made by Malvaez [26]. The aim of his work is to apply the Jarrow-Yıldırım model for pricing inflation indexed derivatives, especially swaps, European and Bermudan swaptions. He also tested the performance of the model calibrating to the Mexican market data, estimated parameters of the models and gave the analysis of results for European swaption.

Hinnerich [18] suggested an extended HJM model to price inflation linked swaps. The model includes jump components, also consumer price index and the forward rates are allowed to be driven by both Wiener process and a general marked point process. Volatilities of all asset prices and the consumer price index are deterministic with respect to the Wiener process and the point process. The important difference in the article is that derivatives are priced in HJM framework without assuming the foreign-currency analogy. Inflation indexed swap market model is introduced to price inflation indexed swaptions. Therefore, this work proved the validity of foreign-currency analogy.

Dodgson and Kainth [10] used a correlated Hull-White model to price inflation derivatives. They also priced complex derivatives using Monte Carlo sampling. Inflation swaps, options, cap and floors are included in this work with a two process short rate model.

Stewart [32] started to his work giving the working principles of inflation indexed markets. Then he reviewed the two currency Heath-Jarrow-Morton framework and he derived the prices of inflation indexed derivatives using the Hull-White extended Vasicek model. He finished his work calibrating to market data.

Deacon, Derry and Mirfendereski [9] wrote a book about inflation indexed securities. This book tells us these securities and the markets trading these instruments around the world in detail. The problems in indexation, tax regulations, advantages and disadvantages of issuing or investing in indexed securities, the principles of indexed bonds and derivatives market are included in the book.

Mauri and Mercurio [27] reviewed the approach given by Jarrow-Yıldırım to model inflation and nominal rates, to price inflation indexed swap and they introduced a constant volatility LIBOR market model to price inflation.

Costantini, D'Ippoliti and Papi [8] worked on valuation of inflation derivatives with payoff depending on European inflation, on European Central Bank official interest rate and on the short term interest rate at times between the present and the maturity date. They investigated the relationship between these three quantities in a stochastic time setting for this reason.

The last study about inflation derivatives has made by Leung and Wu [25]. In the article, they present extended Heath-Jarrow-Morton model in terms of continuous compounding nominal and inflation forward rates. They price several inflation derivatives except swaptions under this model. Then they introduced a lognormal model for displaced forward inflation rates for simple compounding and use this model to price inflation indexed swaption. They conclude with calibration results of the market model.

Garcia and Rixtel [11] present the inflation linked bonds from a central bank perspective. They gave the development of inflation-linked bond markets, the arguments for and against issuing inflation-linked bonds both from the perspective of the issuer and the investor and finally they told the uses of inflation-linked bonds to show investors' inflation expectations and the outlook for economic growth. European Central Bank's experiences are used in this work.

The only detailed work in Turkey is made by Tekmen [33]. In this work, history of inflation linked bonds in Turkey and other bond markets around the world are told in detail. He gave the advantages of issuing inflation indexed bond and a regression analysis is included in the study.

CHAPTER 3

PRELIMINARIES

In this chapter, some basic definitions and theorems that we need in later chapters are reviewed.

3.1 Fundamentals of Mathematical Finance

Definitions and theorems in this section are mainly taken from Lamberton and Lapeyre [24], Björk [6], Shreve [31], Brigo and Mercurio [7] and Protter [30].

Definition 3.1.1 *Consider the probability space* $(\Omega, \mathcal{A}, \mathbf{P})$*. A filtration* $(\mathcal{F}_t)_{t\geq 0}$ *is an increasing family of* σ *- algebras included in* $\mathcal{A}$*.*
The σ*- algebra* $\mathcal{F}_t$ *represents the information available at time t. A process* $(X_t)_{t\geq 0}$ *is adapted to* $(\mathcal{F}_t)_{t\geq 0}$*, if for any t,* X_t *is* $\mathcal{F}_t$ *- measurable.*

Definition 3.1.2 *A probability measure* **Q** *is called a* **martingale measure** *or* **risk neutral measure** *if the following condition holds:*

$$S_0 = \frac{1}{1+R} E^{\mathbf{Q}}[S_1].$$

Proposition 3.1.3 *The market model is arbitrage free if and only if there exists a martingale measure* **Q**.

Definition 3.1.4 *A* **Brownian motion** *is a real-valued, continuous stochastic process* $(X_t)_{t\geq 0}$ *with independent and stationary increments.*

- *Continuity:* $\mathbf{P}$ *- a.s. the map* $s \mapsto X_s(w)$ *is continuous.*

- *Independent increments: If* $s \leq t$ *then* $X_t - X_s$ *is independent of* $\mathcal{F}_s = \sigma\,(X_u, u \leq s)$*.*

- *Stationary increments: If* $s \leq t$ *then* $X_t - X_s$ *and* $X_{t-s} - X_0$ *have the same probability law.*

Definition 3.1.5 *A Brownian motion is standard if*

- $X_0 = 0$ $\mathbf{P}$ *- a.s.*

- $E(X_t) = 0$

- $Var(X_t) = E(X_t^2) = t$.

Definition 3.1.6 *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space, let T be a fixed positive number and let* $(\mathcal{F}_t)$*,* $0 \leq t \leq T$*, be a filtration of sub* σ *- algebras of* $\mathcal{F}$*. An adapted stochastic process* (M_t)*,* $0 \leq t \leq T$*, is :*

- *a* **martingale***, if* $E(M_t \mid \mathcal{F}_s) = M_s$ *for all* $0 \leq s \leq t \leq T$*,*
  It has no tendency to rise or fall.

- *a* **submartingale***, if* $E(M_t \mid \mathcal{F}_s) \geq M_s$ *for all* $0 \leq s \leq t \leq T$*,*
  It has no tendency to fall, it may have a tendency to rise.

- *a* **supermartingale***, if* $E(M_t \mid \mathcal{F}_s) \leq M_s$ *for all* $0 \leq s \leq t \leq T$*,*
  It has no tendency to rise, it may have a tendency to fall.

Theorem 3.1.7 *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space. Let* $\widetilde{\mathbf{P}}$ *be another probability measure on* $(\Omega, \mathcal{F})$ *that is equivalent to* $\mathbf{P}$ *and let* $\mathcal{Z}$ *be an almost surely positive random variable that relates* $\mathbf{P}$ *and* $\widetilde{\mathbf{P}}$*.*
Then $\mathcal{Z}$ *is called the* **Radon-Nikodym derivative** *of* $\widetilde{\mathbf{P}}$ *with respect to* $\mathbf{P}$*, and we write*

$$Z = \frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}}.$$

Theorem 3.1.8 (Girsanov Theorem) *Let* $(W_t)_{0\leq t\leq T}$ *be a Brownian motion on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and let* $(\mathcal{F}_t)_{0\leq t\leq T}$ *be a filtration for this Brownian motion.*

Let $(\theta_t)_{0\le t\le T}$ *be an adapted measurable process satisfying* $\int_0^t \theta_s^2\, ds < \infty$ *a.s. and such that the process* $(Z_t)_{0\le t\le T}$ *defined by*

$$Z_t \;=\; \exp\,(-\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du)$$

is a martingale. Then, under the probability measure $\widetilde{\mathbf{P}}$ *with density* $Z(T)$ *relative to* $\mathbf{P}$*, the process defined by*

$$\widetilde{W}(t) = W(t) + \int_0^t \theta_u\, du$$

is a standard Brownian motion.

Theorem 3.1.9 (Girsanov Theorem in Corollary Case) *Let* X_t *be a measurable process adapted to the natural filtration. Define*

$$Z_t \;=\; \varepsilon(X_t) = \exp\,(X_t - \frac{1}{2}\langle X_t, X_t\rangle)$$

where $\varepsilon(X)$ *is the Doléans-Dade exponential of* X *with respect to* W*.*
If Z_t *is a martingale, then a probability measure* $\widetilde{\mathbf{P}}$ *can be defined on* $(\Omega\,,\mathcal{F})$ *such that we have Radon-Nikodym derivative*

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} \mid F_t = Z_t = \varepsilon(X_t).$$

Then, if X *is a continuous process and* W *is Brownian motion under measure* $\mathbf{P}$*, then*

$$\widetilde{W}_t = W_t - [W, X]_t$$

is a Brownian motion under $\widetilde{\mathbf{P}}$*.*

Definition 3.1.10 *Let* $(\Omega\,,\,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\,\mathbf{P})$ *be a filtered probability space and* $(W_t)_{t\ge 0}$ *be an* $\mathcal{F}$*-Brownian motion.* $(X_t)_{0\le t\le T}$ *is an* $\mathbf{R}$*-valued Ito process if it can be written as* $\mathbf{P}$ *a.s.* $\forall$ $t \le T$

$$X_t \;=\; X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s, \tag{3.1}$$

where

- X_0 *is* $\mathcal{F}_0$*- measurable.*
- $(K_t)_{0\le t\le T}$ *and* $(H_t)_{0\le t\le T}$ *are* $\mathcal{F}_t$*-adapted processes.*

- $\int_0^T |K_s|\, ds < \infty$ **P** *a.s.*

- $\int_0^T |H_s|^2\, ds < \infty$ **P** *a.s.*

Theorem 3.1.11 (Ito-Doeblin Formula for an Ito-process) *Let $(X_t)_{0\leq t\leq T}$ be an Ito process*

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s$$

and f be a twice continuously differentiable function, then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X, X\rangle_s \tag{3.2}$$

where, by definition

$$\langle X, X\rangle_t = \int_0^t H_s^2\, ds.$$

Likewise, if $(t, x) \mapsto f(t, x)$ is a function which is twice differentiable with respect to x and once with respect to t, and if these partial derivatives are continuous with respect to (t,x), Ito formula becomes

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t f_s(s, X_s) ds + \int_0^t f_x(s, X_s)\, dX_s \\ &\quad + \frac{1}{2}\int_0^t f_{xx}(s, X_s) d\langle X, X\rangle_s. \end{aligned} \tag{3.3}$$

Proposition 3.1.12 (Integration by Parts Formula) *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two Ito processes such that*

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s$$

and

$$Y_t = Y_0 + \int_0^t K'_s\, ds + \int_0^t H'_s\, dW_s,$$

then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t \tag{3.4}$$

with $\langle X, Y\rangle_t = \int_0^t H_s H'_s\, ds$.

Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process, also known as the mean-reverting process, is a stochastic process r_t given by the following stochastic differential equation:

$$dr_t \;=\; \theta(\mu - r_t)dt + \sigma dW_t.$$

For the solution of this equation, we apply Ito's lemma to the function:

$$f(r_t, t) \;=\; r_t e^{\theta t},$$

$$\begin{aligned} df(r_t, t) &= \theta r_t e^{\theta t} dt + e^{\theta t} dr_t \\ &= e^{\theta t}\theta\mu dt + \sigma e^{\theta t} dW_t. \end{aligned}$$

Integrating from 0 to t, we get

$$r_t e^{\theta t} \;=\; r_0 + \int_0^t e^{\theta s}\theta\mu\, ds + \int_0^t \sigma e^{\theta s}\, dW_s$$

and, then,

$$r_t \;=\; r_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}\, dW_s.$$

Definition 3.1.13 *The Markov property means that the future behaviour of the process* $(X_t)_{t\geq 0}$ *after t depends only on the value* X_t *and that is not influenced by the history of the process before t.*

An $\mathcal{F}_t$*-adapted process* $(X_t)_{t\geq 0}$ *satisfies the Markov-property if, for any bounded Borel function f and for any s and t such that* $s \leq t$*, we have*

$$E(f(X_t)|F_s) = E(f(X_t)|X_s).$$

Theorem 3.1.14 *Let*

$$X_t \;=\; Z + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s$$

and r(s,x) be a non-negative measurable function.
For t>s, **P** *a.s.*

$$E(e^{-\int_s^t r(u,X_u)\, du} f(X_t)|F_s) \;=\; E(e^{-\int_s^t r(u,X_u^{s,x})\, du} f(X_t^{s,x}))|x = X_s.$$

Remark 3.1 *If b and σ are independent of x and f is a bounded measurable function, then*

$$E(f(X_{s+t}^{s,x})) \;=\; E(f(X_t^{0,x})).$$

In that case for t>s,

$$E(e^{-\int_s^t r(X_u)\,du} f(X_t)|F_s) \;=\; E(e^{-\int_0^{t-s} r(X_u^{0,x})\,du} f(X_{t-s}^{0,x}))|x = X_s.$$

3.2 Bonds and Interest Rates

In this section, we present some known definitions and results following Brigo and Mercurio [7] and Björk [6].

Definition 3.2.1 (Bank account/Money-market account) *Let r_t be a positive function of time. The value of a bank account at time $t \geq 0$ is defined by*

$$B(t) \;=\; \exp(\int_0^t r_s\, ds).$$

Then, the bank account satisfies the following differential equation:

$$dB(t) \;=\; r_t B(t)\, dt,$$

$$B(0) \;=\; 1.$$

Definition 3.2.2 (Discount factor) *The discount factor $D(t,T)$ between two instants t and T is the amount at time t, that is equivalent to one unit of currency payable at time T, and it is given by*

$$D(t,T) \;=\; \frac{B(t)}{B(T)} \;=\; \exp(-\int_t^T r_s\, ds).$$

Definition 3.2.3 (Zero-coupon bond) *A T-maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by P(t,T). Clearly, $P(T,T) = 1$ for all T.*

Definition 3.2.4 *The* **simple forward rate** *for $[S,T]$ contracted at t is defined as*

$$L(t;S,T) \;=\; -\frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}.$$

Definition 3.2.5 *The* **simple spot rate** *for* $[S, T]$ *is defined as*

$$L(S, T) = -\frac{P(S, T) - 1}{(T - S)P(S, T)}.$$

Definition 3.2.6 *The* **continuously compounded forward rate** *for* $[S, T]$ *contracted at* t *is defined as*

$$R(t; S, T) = -\frac{\log P(t, T) - \log P(t, S)}{T - S}.$$

Definition 3.2.7 *The* **continuously compounded spot rate** $R(S, T)$*, for the period* $[S, T]$ *is defined as*

$$R(S, T) = -\frac{\log P(S, T)}{T - S}.$$

Definition 3.2.8 *The* **instantaneous forward rate** *with maturity* T*, contracted at* t*, is defined by*

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.$$

Definition 3.2.9 *The* **instantaneous short rate** *at time* t*, is defined by*

$$r(t) = f(t, t).$$

Lemma 3.2.10 *For* $t \leq s \leq T$*,*

$$P(t, T) = \exp(-\int_t^T f(t, s)\, ds).$$

3.3 Change of Numeraire

This section deals with change of numeraire techniques in risk neutral and forward measure following Brigo and Mercurio [7], Björk [6] and Shreve [31].

Definition 3.3.1 *A numeriare is any strictly positive, non-dividend paying asset in which other assets are denominated.*

Theorem 3.3.2 (Stochastic representation of assets) *Let N be a strictly positive price process for a non-dividend paying asset, either primary or derivative, in the multidimensional market model. Then there exists a vector volatility process*

$$\nu(t) = (\nu_1(t), .., \nu_d(t))$$

such that

$$dN(t) = R(t)N(t)dt + N(t)\nu(t)d\widetilde{W}(t).$$

This equation is equivalent to each of the following equations:

$$\begin{aligned} d(D(t)N(t)) &= D(t)N(t)\nu(t)\, d\widetilde{W}(t), \\ D(t)N(t) &= N(0)\exp\Big(\int_0^t \nu(u)\, d\widetilde{W}(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\, du\Big), \\ N(t) &= N(0)\exp\Big(\int_0^t \nu(u)\, d\widetilde{W}(u) + \int_0^t (R(u) - \frac{1}{2}\|\nu(u)\|^2)\, du\Big). \end{aligned}$$

According to Girsanov Theorem, we can use the volatility vector of $N(t)$ to change the measure. Define

$$\widetilde{W}_j^{(N)}(t) = -\int_0^t \nu_j(u)\, du + \widetilde{W}_j(t), \;\; j = 1, ..., d$$

and a new probability measure

$$\widetilde{\mathbf{P}}^{(N)}(A) = \frac{1}{N(0)}\int_A D(T)N(T)d\widetilde{\mathbf{P}}, \;\; for all A \in \mathcal{F}.$$

Theorem 3.3.3 (Change of risk-neutral measure) *Let S(t) and N(t) be the prices of two assets denominated in a common currency and let $\sigma(t) = (\sigma_1(t), .., \sigma_d(t))$ and $\nu(t) = (\nu_1(t), .., \nu_d(t))$ denote their respective volatility vector processes:*

$$\begin{aligned} d(D(t)S(t)) &= D(t)S(t)\sigma(t)d\widetilde{W}(t), \\ d(D(t)N(t)) &= D(t)N(t)\nu(t)d\widetilde{W}(t). \end{aligned}$$

Take N(t) as the numeriare, so the price of S(t) becomes

$$S^{(N)}(t) = \frac{S(t)}{N(t)}.$$

Under the measure $\widetilde{\mathbf{P}}^{(N)}$, the process $S^{(N)}(t)$ is a martingale. Moreover,

$$dS^{(N)}(t) = S^{(N)}(t)[\sigma(t) - \nu(t)]d\widetilde{W}^{(N)}(t).$$

3.3.1 Zero-Coupon Bond as Numeriare in Forward Measure

A zero-coupon bond is an asset and therefore the discounted bond price $D(t)P(t,T)$ must be a martingale under the risk neutral measure $\widetilde{\mathbf{P}}$. According to stochastic representation of assets theorem[31], there is a volatility process $\sigma^*(t,T)$ for the bond such that

$$d(D(t)P(t,T)) \;=\; \sigma^*(t,T)D(t)P(t,T)d\widetilde{W}(t).$$

Definition 3.3.4 *Let T be a fixed maturity date. We define T-forward measure* $\widetilde{\mathbf{P}}^T$ *by for the bond such that*

$$\widetilde{\mathbf{P}}^T(A) \;=\; \frac{1}{P(0,T)}\int_A D(T)d\widetilde{\mathbf{P}},\; forallA \in \mathcal{F}.$$

The T-forward measure corresponds to taking as numeriare $N(t) = P(t,T)$. According to change of risk neutral measure theorem, the process

$$\widetilde{W}^T(t) \;=\; \int_0^t \sigma^*(u,T)\,du + \widetilde{W}(t)$$

is a Brownian motion under $\widetilde{\mathbf{P}}^T$.

Furthermore, under the T-forward measure, all assets denominated in units of the zero-coupon bond maturing at time T are martingale. In other words, T-forward prices are martingales under the T-forward measure $\widetilde{\mathbf{P}}^T$.

CHAPTER 4

HEATH-JARROW-MORTON FRAMEWORK

Since short rate models have some drawbacks, such as an exact calibration to the initial curve of discount factors and a clear understanding of the covariance structure of forward rates are both difficult to achieve, various authors tried to propose models alternative to short rate models.

In 1986, Ho and Lee modeled the evolution of the entire yield curve in a binomial-tree setting. Then in 1992, Heath, Jarrow and Morton developed a quite general framework for modeling of interest rate dynamics in continuous time. By choosing the instantaneous forward rates as fundamental quantities to represent the yield curve, they derived an arbitrage free framework where the forward rate dynamics are fully specified through their instantaneous volatility structures. In other words, they showed that there is a relationship between the drift and volatility parameters of the forward rate dynamics in an arbitrage free market.

In this chapter, we review the HJM forward-rate dynamics and give the arbitrage-free condition under an objective and a risk-neutral measure.

4.1 The HJM Forward-Rate Dynamics

Let $f(0, T)$, $0 \leq T \leq \overline{T}$, be the initial forward rate curve which is known at time 0. In the HJM model, the forward rate at later times t for investing at later times T is given by

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T)\, du + \int_0^t \sigma(u, T)\, dW(u). \tag{4.1}$$

When we differentiate with respect to the variable t, the variable T is being held constant, the above equation can be written as

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t) \quad , \; 0 \le t \le T. \tag{4.2}$$

Here $W(u)$ is a Brownian motion under the measure $\mathbf{P}$. The processes $\alpha(t,T)$ and $\sigma(t,T)$ may be random and for each fixed T, they are adapted processes in t-variable. We assume that the forward rate is driven by a single Brownian motion.

The differential of $-\int_t^T f(t,u)du$ is given by

$$d(-\int_t^T f(t,u)du) = f(t,t)dt - \int_t^T df(t,u)du. \tag{4.3}$$

From the definition of the instantaneous short rate, we know that

$$r(t) = f(t,t).$$

Using this and forward rate dynamics in differential form, we have

$$d(-\int_t^T f(t,u)du) = r(t)dt - \int_t^T [\alpha(t,u)\,dt + \sigma(t,u)\,dW(t)]du. \tag{4.4}$$

Define

$$\alpha^*(t,T) \quad = \quad \int_t^T \alpha(t,u)du, \tag{4.5}$$

and

$$\sigma^*(t,T) \quad = \quad \int_t^T \sigma(t,u)du. \tag{4.6}$$

By the Fubini theorem, we get

$$d(-\int_t^T f(t,u)\,du) \quad = \quad r(t)dt - \int_t^T \alpha(t,u)\,du\,dt - \int_t^T \sigma(t,u)\,du\,dW(t)$$

and using (4.5) and (4.6), we have

$$d(-\int_t^T f(t,u)\,du) \quad = \quad r(t)dt - \alpha^*(t,T)\,dt - \sigma^*(t,T)\,dW(t). \tag{4.7}$$

Let $g(x) = e^x$. We know

$$P(t,T) = \exp(-\int_t^T f(t,u)\,du) = g(-\int_t^T f(t,u)du).$$

By Ito-Doeblin formula, the dynamics of the bond prices is given by

$$\begin{aligned} dP(t,T) &= g'(-\int_t^T f(t,u)\,du)\,d(-\int_t^T f(t,u)du) \\ &\quad +\frac{1}{2}g''(-\int_t^T f(t,u)\,du)\,[\,d(-\int_t^T f(t,u)\,du)\,]^2 \\ &= P(t,T)\,[\,r(t)dt - \alpha^*(t,T)\,dt - \sigma^*(t,T)\,dW(t)\,] \\ &\quad +\frac{1}{2}P(t,T)\sigma^*(t,T)^2\,dt \\ &= P(t,T)\,[\,r(t) - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\,]\,dt \\ &\quad -\sigma^*(t,T)P(t,T)\,dW(t). \end{aligned} \tag{4.8}$$

4.2 HJM Drift Condition

We will show that there is no opportunity for arbitrage by trading the zero-coupon bonds with maturity T for all $T \in [0, \overline{T}]$ in the HJM model.

According to the first fundamental theorem of asset pricing, if a market model has a risk-neutral probability measure, then it does not admit arbitrage. So, we need a probability measure $\widetilde{\mathbf{P}}$ under which discounted bond prices

$$D(t)P(t,T) = \exp(-\int_0^t r(u)\,du)P(t,T),\ 0 \le t \le T$$

are martingales. Since

$$dD(t) = d(\exp(-\int_0^t r(u)\,du)) = -r(t)D(t)\,dt$$

and by using Ito's integration by parts formula, the differential form of the discounted bond price is given by

$$\begin{aligned} d(D(t)P(t,T)) &= dD(t)P(t,T) + D(t)dP(t,T) \\ &= -r(t)D(t)P(t,T)\,dt + D(t)dP(t,T) \\ &= D(t)P(t,T)\,[\,(-\alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\,)\,dt \\ &\quad -\sigma^*(t,T)\,dW(t)\,]. \end{aligned} \tag{4.9}$$

We want to write the term in square brackets as

$$-\sigma^*(t,T)\,[\Theta(t)\,dt + dW(t)\,]$$

and we can apply Girsanov's theorem to change to a probability measure $\widetilde{\mathbf{P}}$, where

$$\widetilde{W}(t) = \int_0^t \Theta(u)\,du + W(t)$$

is a Brownian motion.

Then the discounted bond price formula becomes

$$d(D(t)P(t,T)) = -D(t)P(t,T)\sigma^*(t,T)\,d\widetilde{W}(t). \tag{4.10}$$

It is obvious that $D(t)P(t,T)$ is a martingale under the measure $\widetilde{\mathbf{P}}$.

Now, to find $\Theta(t)$ we have to solve the equation

$$(-\alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\,)\,dt - \sigma^*(t,T)\,dW(t) \quad = \quad -\sigma^*(t,T)\,(dW(t) + \Theta(t)\,dt)$$

and, equivalently,

$$(-\alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\,) \quad = \quad -\sigma^*(t,T)\,\Theta(t). \tag{4.11}$$

Here, the process $\Theta(t)$ is the market price of risk and there are infinitely many solutions of these market price of risk equations for each maturity $T \in [0,\overline{T}]$. But in our case, since we assume there is only one Brownian motion, there is only one process $\Theta(t)$.

When we differentiate the equations (4.5) and (4.6) with respect to T we have;

$$\frac{\partial}{\partial T}\alpha^*(t,T) = \alpha(t,T)$$

and

$$\frac{\partial}{\partial T}\sigma^*(t,T) = \sigma(t,T).$$

Then we differentiate (4.11) with respect to T and we get

$$-\alpha^*(t,T) + \sigma^*(t,T)\sigma(t,T) = -\sigma(t,T)\Theta(t)$$

and, equivalently,

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,[\,\sigma^*(t,T) + \Theta(t)\,]. \tag{4.12}$$

Then we can write the Heath-Jarrow-Morton no-arbitrage condition:

Theorem 4.2.1 (HJM Drift Condition) *A term structure model for zero coupon bond prices of all maturities in* $(0, \overline{T}]$ *and driven by a single Brownian motion does not admit arbitrage if there exists a process* $\Theta(t)$ *such that*

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,[\,\sigma^*(t,T) + \Theta(t)\,] \tag{4.13}$$

holds for all $0 \le t \le T \le \overline{T}$*. Here,* $\alpha(t)$ *and* $\sigma(t)$ *are the drift and the diffusion, respectively, of the forward rate,* $\sigma^*(t,T) = \int_t^T \sigma(t,u)du$ *and* $\Theta(t)$ *is the market price of risk.*

4.3 HJM Under Risk-Neutral Measure

Under the risk-neutral measure, the local rate of return has to be equal to the short rate r. So we take $\Theta(t) = 0$ in HJM drift condition equation (4.13) under the martingale measure.

Theorem 4.3.1 (HJM Drift Condition Under Risk-Neutral Measure) *Under the risk-neutral measure* $\widetilde{\mathbf{P}}$*, the processes* $\alpha(t,T)$ *and* $\sigma(t,T)$ *satisfy the following equation, for every t and every* $T \ge t$*:*

$$\alpha(t,T) \quad = \quad \sigma(t,T)\,\sigma^*(t,T). \tag{4.14}$$

Then the forward rate dynamics under the risk-neutral measure is given by

$$df(t,T) = \sigma(t,T)\,\sigma^*(t,T)\,dt + \sigma(t,T)\,d\widetilde{W}(t). \tag{4.15}$$

The zero-coupon bond price dynamics under the risk-neutral measure $\widetilde{\mathbf{P}}$ is as follows

$$dP(t,T) = r(t)P(t,T)\,dt - \sigma^*(t,T)P(t,T)\,d\widetilde{W}(t), \tag{4.16}$$

where $\widetilde{W}(t)$ is a Brownian motion under $\widetilde{\mathbf{P}}$, $r(t) = f(t,t)$ and $\sigma^*(t,T) = \int_t^T \sigma(t,u)du$.

And, finally, the discounted bond prices satisfy

$$d(D(t)P(t,T)) = -\sigma^*(t,T)P(t,T)\,d\widetilde{W}(t), \tag{4.17}$$

where $D(t) = \exp(-\int_0^t r(u)du)$ is the discount process.

CHAPTER 5

THE HULL-WHITE EXTENDED VASICEK MODEL IN THE HJM FRAMEWORK

The poor fitting of the initial term-structure of interest rates implied by the existing models was realized by some authors and they tried to propose exogenous term-structure models as opposed models that endogenously produce the current term-structure of rates.

Ho and Lee (1986) have been the first to propose a model like this. Their model was based on the assumption of a binomial tree governing the evolution of the entire term-structure of rates and its continuous-time limit. Because of the lack of mean reversion in the short rate dynamics, their model cannot be regarded as a proper extension of the previous models.

Then, in 1990, Hull and White introduced a time-varying parameter in the Vasicek model and the short rate process for the Hull-White extended Vasicek model is given by

$$dr(t) = (\theta(t) - a(t)r(t))\, dt + \sigma(t)\, dW(t),$$

where θ, a and σ are deterministic functions of time.

Such a model can be fitted to the term-structure of interest rates and the term-structure of spot or forward rate volatilities. But to do an exact calibration to the current yield curve, this model has some drawbacks:

- Since some market sectors are less liquid, the volatilities that are quoted in the market cannot be informative or reliable.
- Future volatility structures implied by this model are unlikely to be realistic in that they do not conform to typical market shapes.

Because of these drawbacks, in 1994 the Hull-White introduced the following extension of Vasicek model and the dynamics of the short rate is given by

$$dr(t) = (\theta(t) - ar(t))\, dt + \sigma\, dW(t),$$

where a and σ are positive constants, and θ is chosen so as to exactly fit the term-structure of interest rates being currently observed in the market.

Since the dynamics of this model allow for the derivation of explicit prices for derivative securities, we concentrate on this model in the following chapter.

This chapter reviews the model which is analyzed by Stewart [32]. The first section of this chapter describes how the Hull-White extended Vasicek model can be defined in the HJM framework. Forward rate and bond price dynamics by using this model are given in this section. The next section introduces the inflation index and the real economy into the model. Foreign-currency analogy introduced by Jarrow and Yıldırım [22] is used in this section. In the last section, the dynamics under the forward measures are given.

5.1 The Model in the HJM Framework

The stochastic differential equation for the short rate in the extended Vasicek model is given by

$$dr(t) = (\theta(t) - a(t)r(t))\, dt + \sigma(t)\, dW(t), \tag{5.1}$$

where $W(t)$ is a Brownian motion under the martingale measure. When writing the equation (5.1) in the form

$$dr(t) = a(t)\left(\frac{\theta(t)}{a(t)} - r(t)dt\right) + \sigma(t)dW(t), \tag{5.2}$$

we get the Ornstein-Uhlenbeck process. To solve this equation, let

$$X(t) = r(t)e^{\int_0^t a(s)\,ds}. \tag{5.3}$$

Applying Ito's lemma to (5.3),

$$dX(t) = e^{\int_0^t a(s)\,ds}\, dr(t) + r(t)e^{\int_0^t a(s)\,ds} a(t)\, dt.$$

Using (5.2) in the above equation, we get

$$\begin{aligned} dX(t) &= e^{\int_0^t a(s)\,ds}\left[a(t)\left(\frac{\theta(t)}{a(t)} - r(t)\right) dt + \sigma(t)\ dW(t)\right] dr(t) + r(t)e^{\int_0^t a(s)\,ds}a(t)\,dt \\ &= e^{\int_0^t a(s)\,ds}\theta(t)\,dt + e^{\int_0^t a(s)\,ds}\sigma(t)\ dW(t). \end{aligned} \tag{5.4}$$

Integrating (5.4) from 0 to t, we obtain

$$r(t)e^{\int_0^t a(s)\,ds} = r(0) + \int_0^t e^{\int_0^u a(s)\,ds}\theta(u)\,du + \int_0^t e^{\int_0^u a(s)\,ds}\sigma(u)\ dW(u).$$

Let $\alpha(t)=\int_0^t a(s)\ ds$. Then the solution of (5.1) is given by

$$\begin{aligned} r(t)e^{\alpha(t)} &= r(0) + \int_0^t e^{\alpha(u)}\theta(u)\,du + \int_0^t e^{\alpha(u)}\sigma(u)\ dW(u) \\ r(t) &= e^{-\alpha(t)}\left[r(0) + \int_0^t e^{\alpha(u)}\theta(u)\,du + \int_0^t e^{\alpha(u)}\sigma(u)\ dW(u)\right]. \end{aligned} \tag{5.5}$$

Hence,

$$\int_0^T r(s)\,ds = \int_0^T e^{-\alpha(s)}\left[r(0) + \int_0^s e^{\alpha(u)}\theta(u)\,du\right] ds + \int_0^T \left[\int_0^s e^{-\alpha(s)}e^{\alpha(u)}\sigma(u)\ dW(u)\right] ds$$

and by the Fubini theorem we have

$$\int_0^T r(s)\,ds = \int_0^T e^{-\alpha(s)}\left[r(0) + \int_0^s e^{\alpha(u)}\theta(u)\,du\right] ds + \int_0^T \left[\int_u^T e^{-\alpha(s)}\,ds\right] e^{\alpha(u)}\sigma(u)\ dW(u).$$

We know that the bond price formula is

$$P(t,T) = E\left[\exp\left(-\int_t^T r(s)\,ds\right)|F_t\right]$$

and the bank account is given by

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right).$$

Since $W(t)$ is a Brownian motion under the martingale measure, the process $\frac{P(t,T)}{B(t)}$ must be a martingale under this measure.

$$\frac{P(t,T)}{B(t)} = \exp\left(-\int_0^t r(s)\,ds\right) E\left[\exp\left(-\int_t^T r(s)\,ds\right)|F_t\right]. \tag{5.6}$$

By the Markov property, we can write

$$\frac{P(t,T)}{B(t)} = \exp\left(-\int_0^t r(s)\,ds\right) E\left[\exp\left(-\int_0^\theta r(s)\,ds\right)\right],$$

where $\theta = T - t$.

By Laplace transform, we can calculate the above expectation:

$$E\left(e^{-\int_0^\theta r(s)\,ds}\right) = \exp\left(-E\left(\int_0^\theta r(s)\,ds\right) + \frac{1}{2}var\left(\int_0^\theta r(s)\,ds\right)\right).$$

Firstly,

$$\begin{aligned} E\left(\int_0^\theta r(s)\,ds\right) &= E(\int_0^\theta e^{-\alpha(s)}\left[r(0) + \int_0^s e^{\alpha(u)}\theta(u)\,du\right]ds \\ &\quad + \int_0^\theta \left[\int_u^\theta e^{-\alpha(s)}\,ds\right] e^{\alpha(u)}\sigma(u)\ dW(u)). \\ &= \int_0^\theta e^{-\alpha(s)}\left[r(0) + \int_0^s e^{\alpha(u)}\theta(u)\,du\right]ds. \end{aligned}$$

and, then,

$$\begin{aligned} var\left(\int_0^\theta r(s)\,ds\right) &= E\left(\left(\int_0^\theta r(s)\,ds - E\left(\int_0^\theta r(s)\,ds\right)\right)^2\right) \\ &= E\left(\left(\int_0^\theta \left[\int_u^\theta e^{-\alpha(s)}\,ds\right] e^{\alpha(u)}\sigma(u)\ dW(u)\right)^2\right). \end{aligned}$$

Let $\phi_t{=}\int_0^t e^{-\alpha(u)}\ du$ and $g_t{=}e^{\alpha(t)}\sigma(t)$.

Then, using these and the calculated expectation, we can write (5.6) as

$$\frac{P(t,T)}{B(t)} = P(0,T)\exp\left(-\int_0^t (\phi_T - \phi_u)g_u\ dW(u) - \frac{1}{2}\int_0^t (\phi_T - \phi_u)^2 g_u^2\ du\right). \tag{5.7}$$

From the definition of the instantaneous forward rate, we get

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\log P(t,T) \\ &= -\frac{\partial}{\partial T}\log B(t)\frac{P(t,T)}{B(t)} \\ &= -\frac{\partial}{\partial T}\log B(t) - \frac{\partial}{\partial T}\log\frac{P(t,T)}{B(t)} \\ &= -\frac{\partial}{\partial T}\log\frac{P(t,T)}{B(t)}. \end{aligned}$$

Taking the logarithms and differentiating with respect to T, we get

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\log P(0,T) + \frac{\partial}{\partial T}\left(\int_0^t (\phi_T-\phi_u)g_u\, dW(u) + \frac{1}{2}\int_0^t (\phi_T-\phi_u)g_u^2\, du\right) \\ &= f(0,T) + \frac{\partial \phi_T}{\partial T}\int_0^t g_u\, dW(u) + \frac{\partial \phi_T}{\partial T}\int_0^t (\phi_T-\phi_u)g_u^2\, du. \qquad (5.8) \end{aligned}$$

Under the martingale measure, dynamics of the forward rate, it holds

$$df(t,T) = g_t\frac{\partial \phi_T}{\partial T}\, dW(t) + \frac{\partial \phi_T}{\partial T}\int_0^t (\phi_t-\phi_T)\, dt.$$

Then the extended Vasicek model is an HJM model with the volatility of the instantaneous forward rate given by

$$\begin{aligned} \sigma(t,T) = g_t\frac{\partial \phi_T}{\partial T} &= e^{\alpha(t)}\sigma(t)\frac{\partial}{\partial T}\left(\int_0^T e^{-\alpha(u)}\, du\right) \\ &= \sigma(t)e^{\alpha(t)-\alpha(T)}. \end{aligned}$$

Zero coupon bond price formula

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\, ds\right).$$

When we use the equation (5.8), the dynamics of the zero coupon bond in terms of the extended Vasicek parameters under the martingale measure is represented by

$$P(t,T) = \exp\left[-\int_t^T \left(f(0,s) + \frac{\partial \phi_s}{\partial s}\int_0^t g_u\, dW(u) + \frac{\partial \phi_s}{\partial s}\int_0^t (\phi_s-\phi_u)g_u^2\, du\right) ds\right].$$

By the Fubini theorem, we obtain

$$\begin{aligned} P(t,T) &= \frac{\exp(-\int_0^T f(0,s)\, ds)}{\exp(-\int_0^t f(0,s)\, ds)}\exp\left[-\int_0^t\left(\int_t^T \frac{\partial \phi_s}{\partial s}\, ds\right)g_u\, dW(u)\right] \\ &\quad \exp\left[-\int_0^t\left(\int_t^T \frac{\partial \phi_s}{\partial s}(\phi_s-\phi_u)\, ds\right)g_u^2\, du\right], \end{aligned}$$

and taking the integrals we get

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)}\exp\left(-\int_0^t (\phi_T-\phi_t)g_u\, dW(u)\right)\exp\left(-\int_0^t \frac{1}{2}(\phi_T-\phi_t)(\phi_T+\phi_t)g_u^2\, du\right) \\ &\quad \exp\left(\int_0^t (\phi_T-\phi_t)\phi_u g_u^2\, du\right) \\ &= \frac{P(0,T)}{P(0,t)}\exp[-(\phi_T-\phi_t)(\int_0^t g_u\, dW(u) \\ &\quad +\frac{1}{2}\int_0^t (\phi_T+\phi_t-2\phi_u)g_u^2\, du)]. \qquad (5.9) \end{aligned}$$

Now, we derive the dynamics of the zero coupon bond in terms of extended Vasicek parameters when discounted by the zero coupon bond $P(t, T)$ under the T-forward measure. Under the T-forward measure $\mathbf{P^T}$, the expression $\frac{P(t,S)}{P(t,T)}$ is a martingale.
By equation (5.9), the bond prices are given by

$$\begin{aligned} P(t, S) &= \frac{P(0, S)}{P(0, t)} \exp[-(\phi_S - \phi_t)(\int_0^t g_u \, dW(u) \\ &+ \frac{1}{2} \int_0^t (\phi_S + \phi_t - 2\phi_u) g_u^2 \, du)]. \end{aligned}$$

and

$$\begin{aligned} P(t, T) &= \frac{P(0, T)}{P(0, t)} \exp[-(\phi_T - \phi_t)(\int_0^t g_u \, dW(u) \\ &+ \frac{1}{2} \int_0^t (\phi_T + \phi_t - 2\phi_u) g_u^2 \, du)]. \end{aligned}$$

Then, the expression $\frac{P(t,S)}{P(t,T)}$ is:

$$\begin{aligned} \frac{P(t, S)}{P(t, T)} &= \frac{P(0, S)}{P(0, T)} \exp(-(\phi_S - \phi_T) \int_0^t g_u \, dW(u) \\ &+ \int_0^t \frac{(\phi_T^2 - \phi_S^2)}{2} g_u^2 \, du + \int_0^t (\phi_S - \phi_T) \phi_u g_u^2 \, du) \\ &= \frac{P(0, S)}{P(0, T)} \exp(-(\phi_S - \phi_T) \int_0^t g_u \, dW(u) \\ &- \frac{1}{2} (\phi_S - \phi_T) \int_0^t (\phi_T + \phi_S - 2\phi_u) g_u^2 \, du)). \end{aligned}$$

When we change the measure from risk neutral to T-forward measure, by Girsanov's Theorem we can use volatility of the bond price dynamics (5.9):

$$\begin{aligned} \frac{P(t, S)}{P(t, T)} &= \frac{P(0, S)}{P(0, T)} \exp(-(\phi_S - \phi_T) \int_0^t g_u \, dW^T(u) \\ &- \frac{1}{2} (\phi_S - \phi_T)^2 \int_0^t g_u^2 \, du), \end{aligned} \tag{5.10}$$

where $W^T(u)$ is a Brownian motion under $\mathbf{P^T}$.
As we introduced before, ε is the Doléans Dade exponential and $\varepsilon(X_t)$ is defined by

$$\varepsilon(X_t) = \exp\left(X_t - \frac{1}{2} \langle X_t, X_t \rangle\right). \tag{5.11}$$

Then the equation (5.10) can be rewritten by using Doléans Dade exponential:

$$\frac{P(t, S)}{P(t, T)} = \frac{P(0, S)}{P(0, T)} \varepsilon\left(-(\phi_S - \phi_T) \int_0^t g_u \, dW^T(u)\right). \tag{5.12}$$

5.2 Foreign-Currency Analogy

In this section, the real economy and the inflation index are introduced into the model that we have presented in the previous chapter. Under the no-arbitrage assumption, a cross-currency economy as also known foreign-currency analogy is considered.

In the foreign currency analogy, the nominal interest rates correspond to the interest rates in the domestic economy, the real interest rates correspond to the interest rates in the foreign economy and the inflation index corresponds to the exchange rate between two economies. So, every asset in the real economy can be converted into an asset in the nominal economy using the inflation index.

We defined the notation used in the model as in Jarrow and Yıldırım model [22]:

- r is used for real, n is used for nominal and I is used for the inflation index.
- $(\Omega, \mathcal{F}, \mathbf{P})$ is the probability space where Ω is a state space, $\mathcal{F}$ is a σ-algebra and $\mathbf{P}$ is the probability measure on $(\Omega, \mathcal{F})$.
- $\{\mathcal{F}_t, t \in [0,T]\}$ is the filtration generated by three Brownian motions $(W^n(t), W^r(t), W^I(t) : t \in [0, T])$.
- $W^n(t)$, $W^r(t)$ and $W^I(t)$ are standard Brownian motions initialized at zero and the correlations between them are given by:

$$\begin{aligned} dW^n(t)dW^r(t) &= \rho^{nr}dt, \\ dW^n(t)dW^I(t) &= \rho^{nI}dt, \\ dW^r(t)dW^I(t) &= \rho^{rI}dt. \end{aligned}$$

- $I(t)$ is the inflation index at time t, which is considered as CPI in this thesis.
- $P^n(t,T)$ is the time t price of a nominal zero-coupon bond that pays out one unit of nominal currency at maturity T.
- $P^r(t,T)$ is the time t price of a real zero-coupon bond that pays out one unit of real currency at maturity T.

- $f^k(t,T)$ is the time t forward rate for date T where $k \in \{r,n\}$:

$$\begin{aligned} df^k(t,T) &= \alpha^k(t,T)dt + \sigma^k(t,T)dW^k(t), \\ P^k(t,T) &= \exp\left(-\int_t^T f^k(t,u)du\right). \end{aligned}$$

- $r^k(t)$ is the time t instantaneous short rate where $k \in \{r,n\}$:

$$r^k(t) = f^k(t,t).$$

- $B^k(t)$ is the time t money market account value for $k \in \{r,n\}$:

$$B^k(t) = \exp\left(\int_0^t r^k(u)du\right).$$

Every nominal tradable asset when discounted by the nominal money market account must be a martingale under the measure $\mathbf{P^n}$. So, for the process $I(t)B^r(t)$, there must exist a process $\sigma^I(t)$ such that

$$\begin{aligned} \frac{I(t)B^r(t)}{B^n(t)} &= \frac{I(0)B^r(0)}{B^n(0)}\varepsilon\left(\int_0^t \sigma^I(s)\, dW^I(s)\right) \\ &= I(0)\varepsilon\left(\int_0^t \sigma^I(s)\, dW^I(s)\right), \end{aligned} \tag{5.13}$$

where $W^I(s)$ is a Brownian motion under the nominal risk neutral measure $\mathbf{P^n}$, and we know $B^r(0) = B^n(0) = 1$.

When we normalize (5.13) by $I(0)$, we have the Radon-Nikodym density of the $\mathbf{P^r}$ measure with respect to the $\mathbf{P^n}$, measure and it is given by

$$\begin{aligned} Z(t) &= \frac{I(t)B^r(t)}{I(0)B^n(t)} \\ &= \varepsilon\left(\int_0^t \sigma^I(s)dW^I(s)\right). \end{aligned} \tag{5.14}$$

Also; from the definition of the martingale measure $\mathbf{P^n}$, $I(t)P^r(t,T)$ is a martingale under $\mathbf{P^n}$ measure when discounted by the nominal money market account $B^n(t)$. We know from the previous section that:

$$\frac{P(t,T)}{B(t)} = P(0,T)\exp\left(-\int_0^t (\phi_T - \phi_u)g_u\, dW(u) - \frac{1}{2}\int_0^t (\phi_T - \phi_u)g_u^2\, du\right).$$

Using this and the Hull-White parametrization which is introduced in the previous section, the dynamics is given by

$$\begin{aligned} \frac{I(t)P^r(t,T)}{B^n(t)} &= I(0)P^r(0,T) \\ &\quad \varepsilon\left(-\int_0^t (\phi_T^r - \phi_u^r) g_u^r \, dW^r(u) + \int_0^t \sigma^I(u) dW^I(u)\right). \end{aligned} \tag{5.15}$$

We know that Heath, Jarrow and Morton approach models the instantaneous forward rate as:

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW(u).$$

Let

$$\begin{aligned} A(t,T) &= \int_t^T \alpha(t,s)\,ds, \\ \Sigma(t,T) &= \int_t^T \sigma(t,s)\,ds. \end{aligned}$$

Then the zero-coupon bond price $P(t,T)$ can be expressed in terms of the forward rate:

$$\begin{aligned} P(t,T) &= \exp\left(-\int_t^T f(t,u)\,du\right) \\ &= \exp\left(-\int_t^T \left(f(0,s) + \int_0^t \sigma(u,s)\,dW(u) + \int_0^t \alpha(u,s)\,du\right) ds\right) \\ &= \frac{P(0,T)}{P(0,t)} \exp\left(-\int_0^t \left(\int_t^T \sigma(u,s)\,ds\right) dW(u) - \int_0^t \left(\int_t^T \alpha(u,s)\,ds\right) du\right) \\ &= \frac{P(0,T)}{P(0,t)} \exp\left(\int_0^t (\Sigma(u,T) - \Sigma(u,t))\,dW(u) - \int_0^t (A(u,T) - A(u,t))\,du\right). \end{aligned}$$

And the money market account $B(t)$ can be written in terms of the forward rate as:

$$\begin{aligned} B(t) &= \exp\left(\int_0^t r(s)\,ds\right) = \exp\left(\int_0^t f(s,s)\,ds\right) \\ &= \exp\left(\int_0^t \left(f(0,s) + \int_0^s \sigma(u,s)\,dW(u) + \int_0^s \alpha(u,s)\,du\right) ds\right) \\ &= \frac{1}{P(0,t)} \exp\left(\int_0^t \left(\int_u^t \sigma(u,s)\,ds\right) dW(u) + \int_0^t \left(\int_u^t \alpha(u,s)\,ds\right) du\right) \\ &= \frac{1}{P(0,t)} \exp\left(-\int_0^t \Sigma(u,t)\,dW(u) + \int_0^t A(u,t)\,du\right). \end{aligned}$$

Then, the bond price discounted by the money market account is given by

$$\frac{P(t,T)}{B(t)} = P(0,T)\exp\left(\int_0^t \Sigma(u,T)\,dW(u) - \int_0^t A(u,T)\,du\right).$$

Hence, we can also write (5.15) as:

$$\begin{aligned}\frac{I(t)P^r(t,T)}{B^n(t)} &= I(0)P^r(0,T)\\ &\quad \varepsilon\left(\int_0^t \Sigma^r(s,T)\,dW^r(s) + \int_0^t \sigma^I(s)dW^I(s)\right). \qquad (5.16)\end{aligned}$$

5.3 Forward Measure

In this section, the dynamics in the forward measure and mainly the Radon-Nikodym density of the real forward measure $\mathbf{P^{r,T}}$ with respect to the nominal forward measure $\mathbf{P^{n,T}}$ are given.

$\frac{P^r(t,S)}{P^r(t,T)}$ is a martingale in the real T-forward measure $\mathbf{P^{r,T}}$. We know from the first section that:

$$\begin{aligned}\frac{P(t,S)}{P(t,T)} &= \frac{P(0,S)}{P(0,T)}\exp(-(\phi_S - \phi_T)\int_0^t g_u\,dW^T(u)\\ &\quad -\frac{1}{2}(\phi_S - \phi_T)^2\int_0^t g_u^2\,du).\end{aligned}$$

Using this and the Hull-White parametrization, the dynamics of this ratio is given by

$$\begin{aligned}\frac{P^r(t,S)}{P^r(t,T)} &= \frac{P^r(0,S)}{P^r(0,T)}\\ &\quad \varepsilon\left(-(\phi_S^r - \phi_T^r)\int_0^t g_u^r\,dW^{r,T}(u)\right). \qquad (5.17)\end{aligned}$$

where $W^{r,T}(u)$ is a Brownian motion under the real T-forward measure $\mathbf{P^{r,T}}$.
Let $t = T$ in (5.17). The real zero-coupon bond $P_r(t,T)$ pays 1 unit of real currency at maturity T. Hence, the price of the real zero-coupon bond at time T which pays 1 unit of real currency at time S can be written as

$$\begin{aligned}P^r(T,S) &= \frac{P^r(0,S)}{P^r(0,T)}\\ &\quad \varepsilon\left(-(\phi_S^r - \phi_T^r)\int_0^T g_u^r\,dW^{r,T}(u)\right). \qquad (5.18)\end{aligned}$$

By Girsanov's Theorem, it is known that the volatility term is not affected by the change of measure, only the drift term changes. Then (5.17) and (5.18) can equally be written as shown in the previous section:

$$\frac{P^r(t,S)}{P^r(t,T)} = \frac{P^r(0,S)}{P^r(0,T)} \varepsilon\left(\int_0^t (\Sigma^r(s,S) - \Sigma^r(s,T))\, dW^{r,T}(s)\right). \tag{5.19}$$

And when $t = T$ it is given by

$$P^r(T,S) = \frac{P^r(0,S)}{P^r(0,T)} \varepsilon\left(\int_0^T (\Sigma^r(s,S) - \Sigma^r(s,T))\, dW^{r,T}(s)\right). \tag{5.20}$$

Also, $I(t)P^r(t,S)$ is a martingale under the nominal T-forward measure $\mathbf{P^{n,T}}$ when discounted by the nominal zero-coupon bond price $P^n(t,T)$. So; the dynamics is given by

$$\frac{I(t)P^r(t,S)}{P^n(t,T)} = \frac{I(0)P^r(0,S)}{P^n(0,T)} \varepsilon\left(\int_0^t \sigma^I(s)\, dW^{I,T}(s) + \int_0^t \Sigma^r(s,S)\, dW^{r,T}(s) - \int_0^t \Sigma^n(s,T)\, dW^{n,T}(s)\right), \tag{5.21}$$

where $W^{I,T}(s)$, $W^{r,T}(s)$ and $W^{n,T}(s)$ are correlated $\mathbf{P^{n,T}}$ Brownian motions.

Hence, the Radon-Nikodym density of the real T-forward measure $\mathbf{P^{r,T}}$ with respect to the nominal T-forward measure $\mathbf{P^{n,T}}$ is given by

$$Z(t,T) = \varepsilon\left(\int_0^t \sigma^I(s)\, dW^{I,T}(s) + \int_0^t \Sigma^r(s,T)\, dW^{r,T}(s) - \int_0^t \Sigma^n(s,T)\, dW^{n,T}(s)\right). \tag{5.22}$$

CHAPTER 6

PRICING INFLATION-INDEXED SWAPS AND SWAPTIONS

In the inflation indexed derivatives market, swaps and swaptions are the most liquid instruments. In this chapter, derivation for the prices of zero-coupon and year-on-year inflation-indexed swaps is given using the Hull-White extended Vasicek model. A market model is introduced which was defined by Hinnerich [18]. Then this market model is used for pricing inflation-indexed swaptions.

6.1 Inflation-Indexed Swaps

In this section, firstly we will price the zero-coupon inflation-indexed swaps which provides a simple, model independent price formula. Then, a formula for the price of the year-on-year swaps is given explicitly and the convexity adjustment that is needed to price the swaps is introduced.

6.1.1 Zero-Coupon Inflation-Indexed Swaps

In a zero-coupon inflation-indexed swap, at maturity of the swap which is time T, Party A pays Party B the inflation rate over a fixed period of time, and Party B pays Party A a fixed rate k. The inflation rate is calculated as the percentage return of the CPI index over the time.

As its name indicates, in a zero-coupon inflation-indexed swap there is only one time interval with payments at time T and there are no intermediary payments. In other words, that is, cash flows are exchanged only once.

Assume the maturity of the contract is equal to m years. $I(0)$ is the CPI index when the swap is initially traded and $I(T)$ is the CPI index at maturity.

Let us call as the floating leg, the leg paying inflation and the fixed leg, the leg paying the fixed rate k. At maturity, the fixed leg of the swap contract pays

$$(1+k)^m - 1$$

and the floating leg pays

$$\frac{I(T)}{I(0)} - 1.$$

Let $ZCIIS(t, T, I(0))$ denote the price of the inflation leg in a zero-coupon inflation-indexed swap at time t. The payoff is given by

$$ZCIIS(t, T, I(0)) = P^n(t, T)E^{n,T}\left(\frac{I(T)}{I(0)} - 1|F_t\right),$$

where $E^{n,T}$ denotes the expectation with respect to the nominal T-forward measure $\mathbf{P^{n,T}}$. Since $I(0)$ is $\mathcal{F}_t$-measurable,

$$ZCIIS(t, T, I(0)) = P^n(t, T)\frac{1}{I(0)}E^{n,T}(I(T)|F_t) - P^n(t, T). \tag{6.1}$$

By the foreign-currency analogy, the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the CPI index at bond maturity which is time T. Mathematically, for each $t < T$,

$$\begin{aligned} I(t)P^r(t, T) &= I(t)E^r\left(\exp\left(-\int_t^T r^r(s)\,ds\right)|F_t\right) \\ &= E^n\left(\exp\left(-\int_t^T r^n(s)\,ds\right)I(T)|F_t\right) \\ &= P^n(t, T)E^n(I(T)|F_t). \end{aligned} \tag{6.2}$$

Then, from (6.2)

$$E^n(I(T)|F_t) = \frac{I(t)P^r(t, T)}{P^n(t, T)}. \tag{6.3}$$

Therefore; when we use (6.3) in (6.1) we have:

$$\begin{aligned} ZCIIS(t, T, I(0)) &= P^n(t, T)\frac{1}{I(0)}\frac{I(t)P^r(t, T)}{P^n(t, T)} - P^n(t, T) \\ &= \frac{I(t)P^r(t, T)}{I(0)} - P^n(t, T). \end{aligned} \tag{6.4}$$

At time $t = 0$, when the swap is initially traded, $I(t) = I(0)$. Then, the value of the floating leg is given by

$$ZCIIS(0, T, I(0)) = P^r(0, T) - P^n(0, T). \tag{6.5}$$

The value of the fixed leg at $t = 0$ is

$$P^n(0, T)[(1 + k)^m - 1]. \tag{6.6}$$

Let $Z_0(t, T)$ denote the price of a receiver zero-coupon inflation-indexed swap at time t. Since in a receiver swap, the holder receives the fixed rate and pays the inflation rate; $Z_0(t, T)$ can be written as:

$$\begin{aligned} Z_0(t, T) &= P^n(0, T)[(1 + k)^m - 1] - [P^r(0, T) - P^n(0, T)] \\ &= P^n(0, T)(1 + k)^m - P^r(0, T). \end{aligned} \tag{6.7}$$

The fixed rate k is chosen so as to make the value of the fixed leg equal to that of the floating leg when the swap is initially traded. In other words, it is chosen so as to make the swap has zero value at time zero. And inflation-indexed swaps are quoted in terms of the fixed rate. Hence; the fixed rate k is given by

$$\begin{aligned} Z_0(t, T) &= P^n(0, T)(1 + k)^m - P^r(0, T) = 0, \\ k &= \left(\frac{P^r(0, T)}{P^n(0, T)}\right)^{1/m} - 1. \end{aligned} \tag{6.8}$$

Regardless of whether the martingale property or the replicating argument is used to price the zero-coupon inflation-indexed swaps, no assumptions on the dynamics of the assets are needed. Hence, this result shows that the price of the zero-coupon inflation-indexed swap is model independent.

6.1.2 Year-on-Year Inflation-Indexed Swaps

A year-on-year inflation-indexed swap starts at time T_1 and has payment dates $T_1, T_2, .., T_N$. Assume that the payment frequency is annual.

A year-on-year inflation-indexed swap also has a fixed and a floating leg. At the end of each period, Party A pays Party B the inflation rate which is calculated as the percentage return of the CPI index over the previous period. As in the zero-coupon inflation-indexed swap, Party B pays the fixed rate k at each payment date.

The floating payment that starts at time T_{i-1} and paid at time T_i is

$$\frac{I(T_i)}{I(T_{i-1})} - 1.$$

The fixed leg pays the fixed rate k at the end of each period since the swap has annual payment dates and so $m = 1$ in $(1 + k)^m - 1$.

Let $YYIIS(t, T_1, T_2)$ denote the price at time t of the floating payment reset at time T_1 and paid at time T_2.

Each floating payment on a year-on-year swap can be considered as the floating payment on a forward starting zero-coupon swap. Then $YYIIS(t, T_1, T_2)$ can be written as

$$YYIIS(t, T_1, T_2) = P^n(t, T_1)E^{n,T_1}[ZCIIS(T_1, T_2, I(T_1))], \quad (6.9)$$

where E^{n,T_1} denotes the expectation with respect to the nominal T_1-forward measure $\mathbf{P^{n,T_1}}$.

Since $I(T_1)$ is known at time T_1, the price of the zero-coupon inflation-indexed swap (6.5) can be used here. Then,

$$YYIIS(t, T_1, T_2) = P^n(t, T_1)E^{n,T_1}[(P^r(T_1, T_2) - P^n(T_1, T_2))|F_t]. \quad (6.10)$$

Since $P^n(T_1, T_2)$ is a martingale in the nominal $\mathbf{P^{n,T_1}}$ measure, it holds

$$P^n(t, T_1)E^{n,T_1}(P^n(T_1, T_2)) = P^n(t, T_1)P^n(T_1, T_2) = P^n(t, T_2). \quad (6.11)$$

Using (6.11) in the floating payment of the year-on-year inflation-indexed swap at time t (6.10), we get

$$\begin{aligned} YYIIS(t, T_1, T_2) &= P^n(t, T_1)E^{n,T_1}[(P^r(T_1, T_2)|F_t] - P^n(t, T_1)E^{n,T_1}(P^n(T_1, T_2)) \\ &= P^n(t, T_1)E^{n,T_1}[(P^r(T_1, T_2)|F_t] - P^n(t, T_2). \quad (6.12) \end{aligned}$$

The first term in (6.12) is the nominal discounted price of the payoff which is equal to the price of the real zero-coupon bond $P^r(T_1, T_2)$. If we assumed that the real rates were deterministic, then this price would be equal to the present value of the forward price of the real bond in nominal terms:

$$
\begin{aligned}
P^n(t,T_1)E^{n,T_1}[(P^r(T_1,T_2)|F_t] &= P^n(t,T_1)P^r(T_1,T_2) \\
&= P^n(t,T_1)\frac{P^r(t,T_2)}{P^r(t,T_1)}.
\end{aligned}
$$

But, the real rates in this model are stochastic. It is known that $P^r(T_1,T_2)$ is a martingale under the real T_1-forward measure $\mathbf{P^{r,T_1}}$. So, to find the expectation of $P^r(T_1,T_2)$ in the $\mathbf{P^{n,T_1}}$ measure, we will use the Radon-Nikodym density for the $\mathbf{P^{r,T_1}}$ measure with respect to the $\mathbf{P^{n,T_1}}$ measure which was introduced in the previous chapter. When we change the measure using this, the forward price of the real bond must be corrected by a factor which is known as convexity adjustment.

We know from the previous chapter that $\frac{P^r(t,T_2)}{P^r(t,T_1)}$ is a martingale under the real T_1-forward measure $\mathbf{P^{r,T_1}}$:

$$
\begin{aligned}
\frac{P^r(t,T_2)}{P^r(t,T_1)} &= \frac{P^r(0,T_2)}{P^r(0,T_1)} \\
&\quad \varepsilon\left(\int_0^t (\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\, dW^{r,T_1}(s)\right)
\end{aligned} \tag{6.13}
$$

and when $t = T_1$ the price of the real zero-coupon bond at time T_1 which pays 1 unit of real currency at time T_2 can be written as

$$
\begin{aligned}
P^r(T_1,T_2) &= \frac{P^r(0,T_2)}{P^r(0,T_1)} \\
&\quad \varepsilon\left(\int_0^{T_1} (\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\, dW^{r,T_1}(s)\right),
\end{aligned} \tag{6.14}
$$

where $W^{r,T}(s)$ is a Brownian motion under the real T_1-forward measure $\mathbf{P^{r,T_1}}$.

Also we know that the Radon-Nikodym density of the real T_1-forward measure $\mathbf{P^{r,T_1}}$ with respect to the nominal T_1-forward measure $\mathbf{P^{n,T_1}}$:

$$
Z(t,T_1) = \varepsilon\left(\int_0^t \sigma^I(s)\, dW^{I,T_1}(s) + \int_0^t \Sigma^r(s,T_1)\, dW^{r,T_1}(s) - \int_0^t \Sigma^n(s,T_1)\, dW^{n,T_1}(s)\right), \tag{6.15}
$$

where $W^{r,T_1}(s)$, $W^{n,T_1}(s)$ and $W^{I,T_1}(s)$ are correlated $\mathbf{P^{n,T_1}}$ Brownian motions.

By Girsanov's Theorem in corollary case, the dynamics of $P^r(T_1,T_2)$ under the nominal T_1-forward measure can be written by using (6.14) and (6.15), we obtain

$$\begin{aligned}
P^r(T_1,T_2) &= \frac{P^r(0,T_2)}{P^r(0,T_1)}\,\varepsilon\left(\int_0^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\,dW^{r,T_1}(s)\right)\\
&= \frac{P^r(0,T_2)}{P^r(0,T_1)}\,\varepsilon(\int_0^{T_1}[(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\\
&\quad (\,d\widetilde{W}^{r,T_1}(s)-\langle dW^{r,T_1}(s),\\
&\quad \sigma^I(s)\,dW^{I,T_1}(s)+\Sigma^r(s,T_1)\,dW^{r,T_1}(s)-\Sigma^n(s,T_1)\,dW^{n,T_1}(s)\rangle_s\,)\,]\,).
\end{aligned}$$

Using the correlation coefficients between the Brownian motions, we have

$$\begin{aligned}
P^r(T_1,T_2) &= \frac{P^r(0,T_2)}{P^r(0,T_1)}\,\varepsilon(\int_0^{T_1}[(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\,d\widetilde{W}^{r,T_1}(s)\\
&\quad (\Sigma^r(s,T_2)-\Sigma^r(s,T_1))(\sigma^I(s)\rho^{rI}(s)+\Sigma^r(s,T_1)-\Sigma^n(s,T_1)\rho^{nr}(s)\,)ds\,]\,).
\end{aligned} \tag{6.16}$$

If X_t has only drift term, since $\langle X_t,X_t\rangle=0$ then

$$\varepsilon(X_t)=\exp\left(X_t-\frac{1}{2}\langle X_t,X_t\rangle\right)=\exp(X_t) \tag{6.17}$$

and

$$\varepsilon(X_t+Y_t)=\varepsilon(X_t)\,\varepsilon(Y_t). \tag{6.18}$$

Using (6.18), (6.16) can be written as

$$\begin{aligned}
P^r(T_1,T_2) &= \frac{P^r(0,T_2)}{P^r(0,T_1)}\,\varepsilon(\int_0^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\,d\widetilde{W}^{r,T_1}(s)\,)\\
&\quad \varepsilon(\int_0^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))(\Sigma^n(s,T_1)\rho^{nr}(s)-\sigma^I(s)\rho^{rI}(s)-\Sigma^r(s,T_1))\,)ds\,).
\end{aligned}$$

Hence when (6.17) is used, the dynamics of $P^r(T_1,T_2)$ under the nominal T_1-forward measure is given by

$$
\begin{aligned}
P^r(T_1,T_2) &= \frac{P^r(0,T_2)}{P^r(0,T_1)}\,\varepsilon\Big(\int_0^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\,d\widetilde{W}^{r,T_1}(s)\,\Big)\\
&\quad \exp\Big(\int_0^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))(\Sigma^n(s,T_1)\rho^{nr}(s)-\sigma^I(s)\rho^{rI}(s)-\Sigma^r(s,T_1)))ds\Big).
\end{aligned}
\tag{6.19}
$$

Since we need the price at time t, and using the definition of the Doléans Dade exponential the equation (6.19) can be written at time t as

$$
\begin{aligned}
P^r(T_1,T_2) &= \frac{P^r(t,T_2)}{P^r(t,T_1)}\,\exp\Big(\int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))\,d\widetilde{W}^{r,T_1}(s)\,\Big)\\
&\quad \exp\Big(-\frac{1}{2}\int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))^2\,ds\,\Big)\\
&\quad \exp\Big(\int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))(\Sigma^n(s,T_1)\rho^{nr}(s)-\sigma^I(s)\rho^{rI}(s)-\Sigma^r(s,T_1)))ds\Big).
\end{aligned}
\tag{6.20}
$$

Now, we want to find $E^{n,T_1}(P^r(T_1,T_2)|F_t)$. When $Y\sim N(\mu,\sigma^2)$, then the moment generating function is given by

$$
E(e^{tY}) = e^{\mu t+\frac{1}{2}t^2\sigma^2}. \tag{6.21}
$$

Then, since

$$
\int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))d\widetilde{W}^{r,T_1}(s)\sim N\left(0,\int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))^2ds\right),
$$

using (6.21),

$$
E^{n,T_1}\left(\exp\int_t^{T_1}\Big((\Sigma^r(s,T_2)-\Sigma^r(s,T_1))d\widetilde{W}^{r,T_1}(s)\Big)\right)=\exp\left(\frac{1}{2}\int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))^2ds\right).
$$

Hence, the expectation $E^{n,T_1}(P^r(T_1,T_2)|F_t)$ can be written as

$$
E^{n,T_1}(P^r(T_1,T_2)|F_t) = \frac{P^r(t,T_2)}{P^r(t,T_1)}e^{C(t,T_1,T_2)}, \tag{6.22}
$$

where

$$
C(t,T_1,T_2) = \int_t^{T_1}(\Sigma^r(s,T_2)-\Sigma^r(s,T_1))(\Sigma^n(s,T_1)\rho^{nr}(s)-\sigma^I(s)\rho^{rI}(s)-\Sigma^r(s,T_1))\,ds. \tag{6.23}
$$

So, the expectation of a real zero-coupon bond price under a nominal forward measure is equal to the current forward price of the real bond multiplied by a correction factor.

Then the floating payment of the year-on-year inflation-indexed swap at time t is

$$\begin{aligned} YYIIS(t, T_1, T_2) &= P^n(t, T_1) E^{n,T_1}[(P^r(T_1, T_2)|F_t] - P^n(t, T_2) \\ &= P^n(t, T_1)\left(\frac{P^r(t, T_2)}{P^r(t, T_1)} e^{C(t,T_1,T_2)} - \frac{P^n(t, T_2)}{P^n(t, T_1)}\right) \end{aligned} \tag{6.24}$$

Here, the correction factor $C(t, T_1, T_2)$ is the convexity adjustment which depends on the volatilities of the nominal rate, the real rate and the CPI, on the correlation between nominal and real rates and on the correlation between the real rate and the CPI.

Since we use the Hull-White extended Vasicek model, an analytical formula for the convexity adjustment and so for the year-on-year inflation-indexed swap can be derived:

In the Hull-White extended Vasicek model, since a and σ are constant parameters, we have the short rate process

$$dr(t) = (\theta(t) - ar(t))\, dt + \sigma\, dW(t),$$

and

$$\alpha(t) = \int_0^t a\, ds = at.$$

Then, the volatility of the forward rate is given by

$$\begin{aligned} \sigma(t, T) &= \sigma e^{\alpha(t)-\alpha(T)} \\ &= \sigma e^{-a(T-t)}, \end{aligned}$$

and the volatility of the zero-coupon bond can be written as

$$\begin{aligned} \Sigma(t, T) &= -\int_t^T \sigma(t, u)\, du \\ &= -\int_t^T \sigma e^{-a(u-t)}\, du \\ &= \frac{\sigma}{a}(e^{-a(T-t)} - 1). \end{aligned}$$

Let us review the zero-coupon bond price formula in the Hull-White model which we will use to find the formula of the convexity adjustment. For $i \in \{r, n\}$,

$$
\begin{aligned}
P_i(t,T) &= A_i(t,T)e^{-B_i(t,T)r(t)}, \\
B_i(t,T) &= \frac{1}{a_i}(1 - e^{-a_i(T-t)}), \\
A_i(t,T) &= \frac{P_i^m(0,T)}{P_i^m(0,t)} \exp\left(B_i(t,T) f_i^m(0,t) - \frac{\sigma_i^2}{4a_i}(1 - e^{-2a_i t}) B_i(t,T)^2 \right),
\end{aligned}
$$

where $f_i^m(0,t)$ and $P_i^m(0,T)$ are the market instantaneous forward rates and the market value of T-bonds, respectively, at time 0 for all maturities T.

Then the convexity adjustment $C(t, T_1, T_2)$ is:

$$
\begin{aligned}
C(t,T_1,T_2) &= \int_t^{T_1} (\Sigma^r(s,T_2) - \Sigma^r(s,T_1))(\Sigma^n(s,T_1)\rho_{nr}(s) - \sigma_I(s)\rho_{rI}(s) - \Sigma^r(s,T_1))\, ds \\
&= \int_t^{T_1} \frac{\sigma_r}{a_r}\left(e^{-a_r(T_2-s)} - e^{-a_r(T_1-s)}\right) \\
&\quad \left(\frac{\sigma_n \rho_{nr}}{a_n}(e^{-a_n(T_1-s)} - 1) - \sigma_I \rho_{Ir} - \frac{\sigma_r}{a_r}(e^{-a_r(T_1-s)} - 1) \right) ds \\
&= \frac{\sigma_r \sigma_n \rho_{nr}}{a_r a_n} \int_t^{T_1} (\; e^{-a_r(T_2-s) - a_n(T_1-s)} \\
&\quad - e^{-a_r(T_2-s)} - e^{-a_r(T_1-s) - a_n(T_1-s)} + e^{-a_r(T_1-s)} \;)\, ds \\
&\quad - \frac{\sigma_r \sigma_I \rho_{Ir}}{a_r} \int_t^{T_1} \left(e^{-a_r(T_2-s)} - e^{-a_r(T_1-s)}\right) ds \\
&\quad - \frac{\sigma_r^2}{a_r^2} \int_t^{T_1} \left(e^{-a_r(T_1+T_2-2s)} - e^{-2a_r(T_1-s)} - e^{-a_r(T_2-s)} + e^{-a_r(T_1-s)}\right) ds
\end{aligned}
$$

Taking the integrals, we have

$$
\begin{aligned}
C(t,T_1,T_2) &= \frac{\sigma_r\sigma_n\rho_{nr}}{a_r a_n}[\ \frac{1}{a_r+a_n}e^{-a_r(T_2-s)-a_n(T_1-s)} - \frac{1}{a_r}e^{-a_r(T_2-s)} \\
& -\frac{1}{a_r+a_n}e^{-a_r(T_1-s)-a_n(T_1-s)} + \frac{1}{a_r}e^{-a_r(T_1-s)}\ ]_t^{T_1} \\
& -\frac{\sigma_r\sigma_I\rho_{Ir}}{a_r}\left[\frac{1}{a_r}e^{-a_r(T_2-s)} - \frac{1}{a_r}e^{-a_r(T_1-s)}\right]_t^{T_1} \\
& -\frac{\sigma_r^2}{a_r^2}[\ \frac{1}{2a_r}e^{-a_r(T_1+T_2-2s)} - \frac{1}{2a_r}e^{-2a_r(T_1-s)} \\
& -\frac{1}{a_r}e^{-a_r(T_2-s)} + \frac{1}{a_r}e^{-a_r(T_1-s)}\ ]_t^{T_1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma_r\sigma_n\rho_{nr}}{a_r a_n}[\ \frac{1}{a_r+a_n}e^{-a_r(T_2-T_1)} - \frac{1}{a_r}e^{-a_r(T_2-T_1)} - \frac{1}{a_r+a_n} + \frac{1}{a_r} \\
& -\frac{1}{a_r+a_n}e^{-a_r(T_2-t)-a_n(T_1-t)} + \frac{1}{a_r}e^{-a_r(T_2-t)} \\
& +\frac{1}{a_r+a_n}e^{-a_r(T_1-t)-a_n(T_1-t)} - \frac{1}{a_r}e^{-a_r(T_1-t)}\ ] \\
& -\frac{\sigma_r\sigma_I\rho_{Ir}}{a_r^2}\left[e^{-a_r(T_2-T_1)} - 1 - e^{-a_r(T_2-t)} + e^{-a_r(T_1-t)}\right] \\
& -\frac{\sigma_r^2}{a_r^3}[\ \frac{1}{2}e^{-a_r(T_2-T_1)} - \frac{1}{2} - e^{-a_r(T_2-T_1)} + 1 \\
& -\frac{1}{2}e^{-a_r(T_1+T_2-2t)} + \frac{1}{2}e^{-2a_r(T_1-t)} + e^{-a_r(T_2-t)} - e^{-a_r(T_1-t)}\ ].
\end{aligned}
$$

Using

$$
B_i(t,T) = \frac{1}{a_i}(1-e^{-a_i(T-t)})\,, \quad i=\{n,r\},
$$

in the Hull-White model, the convexity adjustment is given by

$$
\begin{aligned}
C(t, T_1, T_2) \;=\; & \sigma_r B_r(T_1, T_2) B_r(t, T_1) \frac{\rho_{nr}\sigma_n}{a_n + a_r} (1 + a_r B_n(t, T_1)) \\
& -\sigma_r B_r(T_1, T_2) \frac{\rho_{nr}\sigma_n}{a_n + a_r} B_n(t, T_1) \\
& +\sigma_r \sigma_I \rho_{rI} B_r(T_1, T_2) B_r(t, T_1) \\
& -\frac{1}{2}\sigma_r^2 (B_r(t, T_1))^2 B_r(T_1, T_2). \qquad (6.25)
\end{aligned}
$$

6.2 Inflation-Indexed Swaptions

In this section, we will price firstly the year-on-year inflation-indexed swaptions and then the zero-coupon inflation-indexed swaptions using a defined swap market model. This section reviews the model which is analyzed by Hinnerich [18].

6.2.1 Year-on-Year Inflation-Indexed Swaptions

Let $YYIISO(t, K)$ denote the time-t price of an option to enter into a payer year-on-year inflation-indexed swap at time T_m and with the fixed swap rate K. Then,

$$
YYIISO(t, K) \;=\; \max[YYIIS(t, K), 0], \qquad (6.26)
$$

where $YYIIS(t, K)$ is the time-t price of the payer year-on-year inflation-indexed swap with a fixed rate K.

Let $\Pi(t, X)$ denote the price of X at time t. Since in a payer year-on-year inflation indexed swap, in every period $[T_i, T_{i+1}]$ for $i = m, ..., M - 1$, the holder pays the fixed amount of K and receives the floating amount of $X_{i+1} - 1$ where

$$
X_{i+1} = \frac{I(T_{i+1})}{I(T_i)},
$$

then

$$\begin{aligned} YYIIS(t,K) &= \sum_{i=m}^{M-1}\Pi(t,X_{i+1}-1) - \sum_{i=m}^{M-1}\Pi(t,K) \\ &= \sum_{i=m}^{M-1}\Pi(t,X_{i+1}) - (1+K)\sum_{i=m}^{M-1}P(t,T_{i+1}). \end{aligned} \tag{6.27}$$

Define the forward swap rate of a year-on-year inflation-indexed swap to be the value of K for which the price of the swap is zero and denote this by $R_m^M(t)$. Then,

$$\begin{aligned} YYIIS(t,R_m^M(t)) &= 0, \\ R_m^M(t) &= \frac{\sum_{i=m}^{M-1}\Pi(t,X_{i+1}) - \sum_{i=m}^{M-1}P(t,T_{i+1})}{\sum_{i=m}^{M-1}P(t,T_{i+1})}. \end{aligned} \tag{6.28}$$

For each pair m,k such that $m<k$, define $S_m^k(t)$ by

$$S_m^k(t) = \sum_{i=m}^{k-1}P(t,T_{i+1}). \tag{6.29}$$

Using $S_m^k(t)$, the forward swap rate can be written as

$$R_m^M(t) = \frac{\sum_{i=m}^{M-1}\Pi(t,X_{i+1}) - S_m^M(t)}{S_m^M(t)}. \tag{6.30}$$

Then using the forward swap rate in (6.30), the price of the payer year-on-year inflation-indexed swap is given by

$$\begin{aligned} YYIIS(t,K) &= \sum_{i=m}^{M-1}\Pi(t,X_{i+1}) - (1+K)\sum_{i=m}^{M-1}P(t,T_{i+1}) \\ &= \sum_{i=m}^{M-1}\Pi(t,X_{i+1}) - (1+K)S_m^M(t) \\ &= \frac{\sum_{i=m}^{M-1}\Pi(t,X_{i+1}) - S_m^M(t)}{S_m^M(t)}S_m^M(t) - KS_m^M(t) \\ &= (R_m^M(t)-K)S_m^M(t). \end{aligned} \tag{6.31}$$

Hence, the price of the payer year-on-year inflation-indexed swaption is given by

$$YYIISO(t,K) = S_m^M(t)\max[(R_m^M(t)-K),0]. \tag{6.32}$$

Black's model will be used to price year-on-year inflation-indexed swaption. In this model, the key assumption is that the swap forward rates are lognormal distributed. Now, firstly an inflation-indexed swap market model is defined and it will be used to price the swaption:

Definition 6.2.1 *Given a set of increasing resettlement times $T_0, T_{1,..}, T_M$, let B be the set consisting of pairs (m, k) of positive integers such that $0 \leq m < k \leq M$. For any given pair (m, k) in B assume that the forward swap rate $R_m^k(t)$ has dynamics given by*

$$dR_m^k(t) = R_m^k(t)\sigma_m^k(t)dW_m^k(t) \tag{6.33}$$

where $W_m^k(t)$ is a multidimensional Brownian motion under $\mathbf{P}_m^M$-measure and $\sigma_m^k(t)$ is a vector of non-stochastic functions.

Proposition 6.2.2 *The price of a payer year-on-year inflation-indexed swaption at time t $YYIISO(t, K)$, where $t \leq T$, is given by*

$$YYIISO(t, K) = S_m^M(t)[R_m^M(t)N(d_1) - KN(d_2)], \tag{6.34}$$

where

$$d_1 = \frac{1}{\Sigma}\left(\ln\left(\frac{R_m^M}{K}\right) + \frac{1}{2}\Sigma^2\right),$$

$$d_2 = d_1 - \Sigma,$$

with

$$\Sigma^2 = \int_t^{T_m} ||\sigma_m^M(s)||^2\, ds.$$

Proof. By assumption, the forward swap rate $R_m^M(t)$ follows a geometric Brownian motion with the dynamics given by

$$dR_m^M(t) = R_m^M(t)\sigma_m^M(t)dW_m^M(t).$$

Then for the solution of $R_m^M(t)$, let

$$f(R_m^M(t)) = \log R_m^M(t).$$

Applying Ito's lemma;

$$\begin{aligned} d\log R_m^M(t) &= \frac{1}{R_m^M(t)} dR_m^M(t) - \frac{1}{2(R_m^M(t))^2} d\langle R_m^M(t), R_m^M(t)\rangle \\ &= \sigma_m^M(t) dW_m^M(t) - \frac{1}{2} \|\sigma_m^M(t)\|^2 \, dt. \end{aligned}$$

Hence,

$$R_m^M(T_M) = R_m^M(t) \exp\left(-\frac{1}{2}\int_t^{T_M} \|\sigma_m^M(s)\|^2 \, ds + \int_t^{T_M} \sigma_m^M(s) \, dW_m^M(s)\right). \tag{6.35}$$

We know that

$$YYIISO(T_M, K) = S_m^M(T_M) \max[R_m^M(T_M) - K, 0].$$

Then the value of the swaption at time t is

$$YYIISO(t, K) = S_m^M(t) E_t^{P_m^M}[\max(R_m^M(T_M) - K, 0)].$$

Using the equation (6.35),

$$\begin{aligned} YYIISO(t, K) &= S_m^M(t) E_t^{P_m^M}[(R_m^M(t) \\ & \quad \exp(-\frac{1}{2}\int_t^{T_M} \|\sigma_m^M(s)\|^2 \, ds + \int_t^{T_M} \sigma_m^M(s) \, dW_m^M(s)) - K)_+]. \end{aligned}$$

Then, this is equal in distribution to the formula

$$YYIISO(t, K) = S_m^M(t) E_t^{P_m^M}[(R_m^M(t) e^{-\frac{1}{2}\Sigma^2 + \Sigma x} - K)_+],$$

where $x \in N(0, 1)$.

To find the region where the expectation is defined:

$$R_m^M(t) e^{-\frac{1}{2}\Sigma^2 + \Sigma x} - K > 0.$$

Taking the logarithms of both sides, we have

$$\ln R_m^M(t) - \frac{\Sigma^2}{2} + \Sigma x > \ln K.$$

Hence

$$x > \frac{\ln\left(\frac{K}{R_m^M(t)}\right) + \frac{\Sigma^2}{2}}{\Sigma} = -d_2.$$

Then, the price of the swaption can be written as

$$YYIISO(t,K) \;=\; S_m^M(t) E_t^{P_m^M}[(R_m^M(t) e^{-\frac{\Sigma^2}{2}+\Sigma x} - K) 1_{(x>-d_2)}].$$

Writing the expectation in integral form, we have

$$\begin{aligned} YYIISO(t,K) &= S_m^M(t) \int_{-d_2}^{\infty} R_m^M(t) e^{-\frac{\Sigma^2}{2}+\Sigma x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &\quad - S_m^M(t) K \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= S_m^M(t) \left(R_m^M(t) \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\Sigma^2}{2}+\Sigma x - \frac{x^2}{2}}\, dx - K \int_{-d_2}^{\infty} f_X(x)\, dx \right), \end{aligned}$$

where $f_X(x)$ denotes the density function of a standard normal random variable.

Let $x' = x - \Sigma$, then $dx' = dx$. Using this, we get

$$YYIISO(t,K) \;=\; S_m^M(t) \left(R_m^M(t) \int_{-d_2-\Sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\Sigma^2}{2}+\Sigma(x'+\Sigma) - \frac{(x'+\Sigma)^2}{2}}\, dx' - KN(d_2) \right),$$

where N denotes the normal distribution.

After small algebra, we have

$$\begin{aligned} YYIISO(t,K) &= S_m^M(t) \left(R_m^M(t) \int_{-(d_2+\Sigma)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}}\, dx' - KN(d_2) \right) \\ &= S_m^M(t)(R_m^M(t) N(d_2+\Sigma) - KN(d_2)). \end{aligned}$$

Hence, when we define $d_1 = d_2 + \Sigma$, the price of the year-on-year inflation-indexed swaption is

$$YYIISO(t,K) \;=\; S_m^M(t)(R_m^M(t) N(d_1) - KN(d_2)). \tag{6.36}$$

6.2.2 Zero-Coupon Inflation-Indexed Swaptions

Let $ZCIISO(t, T_{m+1}, K)$ denote the time-t price of an option to enter into a payer zero-coupon inflation-indexed swap at time T_m and with the fixed swap rate K. A zero-coupon inflation indexed swaption is a special case of a year-on year inflation-indexed swaption. Then using the defined swap market model, the price of the zero-coupon inflation indexed swaption can be written as

$$ZCIISO(t, T_{m+1}, K) = YYIISO_m^{m+1}(t, K).$$

Hence,

$$ZCIISO(t, T_{m+1}, K) = S_m^{m+1}(t)(R_m^{m+1}(t)N(d_1) - KN(d_2)), \tag{6.37}$$

where

$$d_1 = \frac{1}{\Sigma}\left(\ln\left(\frac{R_m^{m+1}}{K}\right) + \frac{1}{2}\Sigma^2\right),$$

$$d_2 = d_1 - \Sigma,$$

with

$$\Sigma^2 = \int_t^{T_m} \|\sigma_m^{m+1}(s)\|^2 \, ds.$$

CHAPTER 7

CONCLUSION

Inflation linked securities are of interest to investors, especially in countries which have a high inflation like Turkey. These markets have grown so fast in the last years, and they still have a great growth potential. In this work, we study on pricing swaps and swaptions which are the most liquid products among the inflation linked derivative instruments in the market.

Firstly, we gave the history of indexation, the definitions of indexes for inflation which are used in the market and the characteristics of inflation-linked securities. We examined the studies about indexed instruments. Then, HJM framework and the drift condition for the market to satisfy no arbitrage are given. A Gaussian short rate model, Hull-White extended Vasicek model, is introduced in the HJM framework and the real prices and dynamics in the forward measure are given using the foreign-currency analogy approach. In this model, the domestic and foreign economies defined as HJM models and the spot FX rate between the two currencies which is inflation is modeled as a lognormal process. Using this model, explicit formulas for the prices of zero-coupon and year-on-year inflation-indexed swaps are derived. We review that the zero-coupon inflation-indexed swaps have model independent prices. But the price of year-on-year inflation-indexed swaps is model dependent and we need a correction factor to calculate the forward price of a real bond. Then finally, zero-coupon and year-on-year inflation-indexed swaptions are priced with Black's market model.

Our future work will be on pricing of the inflation-indexed caps and floors which are the other traded inflation-indexed derivatives in the market. Also an application to the swap market data is an another idea for future work.

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