

COMPLETION, PRICING AND CALIBRATION
IN A LEVY MARKET MODEL

A THESIS SUBMITTED TO
THE INSTITUTE OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

BÜŞRA ZEYNEP YILMAZ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
THE DEPARTMENT OF FINANCIAL MATHEMATICS

SEPTEMBER 2010

Approval of the thesis:

**COMPLETION, PRICING and CALIBRATION
IN A LEVY MARKET MODEL**

submitted by **BÜŞRA ZEYNEP YILMAZ** in partial fulfillment of the requirements for the degree of **Master of Science in Department of Financial Mathematics, Middle East Technical University** by,

Prof. Dr. Ersan Akyıldız
Director, Graduate School of **Applied Mathematics**

Assoc. Prof. Dr. Işıl Erol
Head of Department, **Financial Mathematics**

Assoc. Prof. Dr. Azize Hayfavi
Supervisor, **Department of Financial Mathematics, METU**

Assoc. Prof. Dr. Işıl Erol
Co-Supervisor, **Department of Economics, METU**

Examining Committee Members:

Prof. Dr. Gerhard Wilhelm Weber
Department of Financial Mathematics, METU

Assoc. Prof. Dr. Azize Hayfavi
Department of Financial Mathematics, METU

Prof. Dr. Gül Ergün
Department of Statistics, Hacettepe University

Assoc. Prof. Dr. Işıl Erol
Department of Economics, METU

Dr. Yeliz Yolcu Okur
Department of Financial Mathematics, METU

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : BÜŞRA ZEYNEP YILMAZ

Signature :

ABSTRACT

COMPLETION, PRICING AND CALIBRATION IN A LEVY MARKET MODEL

Yılmaz, Büşra Zeynep

M.Sc., Department of Financial Mathematics

Supervisor : Assoc. Prof. Dr. Azize Hayfavi

Co-Supervisor: Assoc. Prof. Dr. Işıl Erol

September 2010, 106 pages

In this thesis, modelling with Lévy processes is considered in three parts. In the first part, the general geometric Lévy market model is examined in detail. As such markets are generally incomplete, it is shown that the market can be completed by enlarging with a series of new artificial assets called “power-jump assets” based on the power-jump processes of the underlying Lévy process. The second part of the thesis presents two different methods for pricing European options: the martingale pricing approach and the Fourier-based characteristic formula method which is performed via fast Fourier transform (FFT). Performance comparison of the pricing methods led to the fact that the fast Fourier transform produces very small pricing errors so the results of both methods are nearly identical. Throughout the pricing section jump sizes are assumed to have a particular distribution. The third part contributes to the empirical applications of Lévy processes. In this part, the stochastic volatility extension of the jump diffusion model is considered and calibration on Standard&Poors (S&P) 500 options data is executed for the jump-diffusion model, stochastic volatility jump-diffusion model of Bates and the Black-Scholes model. The model parameters are estimated by using an optimization algorithm. Next, the effect

of additional stochastic volatility extension on explaining the implied volatility smile phenomenon is investigated and it is found that both jumps and stochastic volatility are required. Moreover, the data fitting performances of three models are compared and it is shown that stochastic volatility jump-diffusion model gives relatively better results.

Keywords: Lévy Processes, Power-Jump Assets, Complete markets, Martingale representation property, Jump-Diffusion Processes, Stochastic Volatility Jump-Diffusion Processes, Fast Fourier Transform, Calibration

ÖZ

LEVY PİYASASI MODELİNDE TAMLAMA, FİYATLAMA VE KALİBRASYON

Yılmaz, Büşra Zeynep

Yüksek Lisans, Finansal Matematik Bölümü

Tez Yöneticisi : Doç. Dr. Azize Hayfavi

Ortak Tez Yöneticisi: Doç. Dr. Işıl Erol

Eylül 2010, 106 sayfa

Bu tezde, Lévy süreçleri ile modelleme üç kısımda ele alınmıştır. Birinci kısımda genel geometrik Lévy piyasa modeli detaylı olarak incelenmiştir. Bu tip modeller genellikle tam olmadıkları için, piyasanın, Lévy süreçlerine dayalı kuvvet-sıçrama süreçleri ile genişletilerek tam hale getirilebileceği gösterilmiştir. Tezin ikinci kısmı, Avrupa tipi opsiyonlar için iki farklı fiyatlama metodu sunmaktadır, bu metodlar martingale yaklaşımı ve hızlı Fourier dönüşümü kullanılarak hesaplanan Fourier metoduna dayalı karakteristik formül metodudur. Fiyatlama metodlarının performans karşılaştırması, hızlı Fourier dönüşümünün ürettiği hataların çok küçük olduğu bu yüzden iki metodun neredeyse özdeş sonuçlar verdiği sonucuna götürmüştür. Fiyatlama bölümü boyunca Lévy sürecinin sıçrama büyüklüklerinin belirli bir dağılıma sahip olduğu varsayılmıştır. Üçüncü kısım, Lévy süreçlerinin deneysel uygulamalarına katkıda bulunmaktadır. Bu bölümde sıçrama-difüzyon modellerinin stokastik volatilité genişlemesi ele alınmış ve Standard&Poors (S&P) 500 endeksi opsiyon verisi üzerinde sıçrama-difüzyon modeli, Bates stokastik volatilité sıçrama-difüzyon modeli ve Black-Scholes modelinin

kalibrasyonu yapılmıştır. Model parametreleri bir optimizasyon algoritması yardımıyla tahmin edilmiştir. Stokastik volatilitenin genişletmesinin etkileri araştırılmış ve zımnî volatilitenin grafiğini açıklayabilmek için sıçrama ve stokastik volatilitenin gerekli olduğu bulunmuştur. Ek olarak, üç modelin veri uyum performansları karşılaştırılmış ve stokastik volatilitenin sıçrama-difüzyon modelinin daha iyi sonuçlar verdiği gösterilmiştir.

Keywords: Lévy Süreçleri, Kuvver-Sıçrama Süreçleri, Tam Piyasalar, Martingale Temsili Özelliği, Sıçrama-Difüzyon Süreçleri, Stokastik Volatilitenin Sıçrama-Difüzyon Süreçleri, Hızlı Fourier Dönüşümü, Kalibrasyon

To My Family

ACKNOWLEDGEMENTS

I would like to thank all those people who have helped in the preparation of this thesis. I am grateful to Assoc. Prof. Dr. Azize Hayfavi for her guidance, wisdom and encouragement that provided me resolution and excitement throughout this study.

Special thanks to my co-advisor Assoc. Prof. Dr. Işıl Erol for her guidance and valuable advise.

Also I would like to thank Prof. Dr. Wilhelm Weber, Prof. Dr. Gül Ergün, Dr. Yeliz Yolcu Okur and Assist. Prof. Kasırğa Yıldırak for their kindness and support.

And, I would like to express my gratefulness to my family for their endless support, care, help and love.

My thanks for my friends for the will power and appreciations for the members of the Institute of Applied Mathematics for their kindness.

TABLE OF CONTENTS

PLAGIARISM	iv
ABSTRACT	v
ÖZ	vii
DEDICATION	viii
ACKNOWLEDGEMENTS	x
TABLE OF CONTENTS	xi
LIST OF TABLES	xiii
LIST OF FIGURES	xiv
1. INTRODUCTION	1
1.1. Motivation	1
1.2. Imperfections of the Black-Scholes Model	3
1.2.1. Log-normally Distributed Returns Assumption	4
1.2.2. Excess Kurtosis and Skewness	6
1.2.3. The Volatility Smile	7
2. PRELIMINARIES	9
2.1. Basic tools of Stochastic Calculus	9
2.2. Lévy Processes and Infinite Divisibility	12
2.2.1. Some Examples of Lévy Processes	13
2.2.2. Random Measures and Point Processes	18
2.3. Martingale Measures and Market Completeness	25
3. THE LÉVY MARKET MODEL	28
3.1. Introduction	28
3.2. The Stock Price Process	32
3.3. Equivalent Martingale Measures	33

4.	COMPLETION OF THE LÉVY MARKET MODEL	39
4.1.	Power-Jump Processes.....	39
4.2.	Volatility Trading with Power-Jump Assets	41
4.3.	Enlarging the Lévy Market Model.....	42
4.4.	Market Completeness.....	44
4.5.	Hedging Portfolios	49
5.	ARBITRAGE	57
5.1.	A Completion Example: Brownian Motion plus a finite number of Poisson Processes ...	59
6.	PRICING THE CONTINGENT CLAIM	61
6.1.	Martingale Pricing Approach.....	63
6.2.	Characteristic Formula via Fast Fourier Transform Method.....	69
6.2.1.	Time value of an option.....	74
6.2.2.	Option Pricing Using the FFT	76
6.3.	Comparison of Pricing Methods	80
7.	CALIBRATION.....	81
7.1.	Stochastic Volatility Extension	81
7.1.1.	Option Pricing	82
7.1.2.	Characteristic function of the log-price	82
7.2.	Calibration of the Models	85
7.2.1.	Data Set	85
7.2.2.	Parameter Estimation	86
7.2.3.	The Algorithm	86
7.2.4.	Results	88
8.	CONCLUSION	98
	APPENDIX.....	100
	REFERENCES	101

LIST OF TABLES

Table 1.1: Mean, standard deviation, skewness and kurtosis of S&P 500 index.....	4
Table 7.1: Separately calibrated jump-diffusion model parameters for different times to maturity.....	93
Table 7.2: Simultaneously calibrated jump-diffusion model parameters for different times to maturity.....	94
Table 7.3: Simultaneously calibrated Bates model parameters for different times to maturity ..	94
Table 7.4: Historical volatilities of S&P 500 index used as input in the Black-Scholes model.	95
Table 8.1: SPX option prices	100

LIST OF FIGURES

Figure 1.1: Time series of S&P 500 Index data.....	4
Figure 1.2: Q-Q Plot of S&P 500 log-returns data.	5
Figure 2.1: Sample paths of a Poisson process.....	16
Figure 2.2: Sample path of a compensated Poisson process.....	16
Figure 2.3: Sample path of a compound Poisson process with Gaussian distribution of jump sizes.	17
Figure 2.4: Sample path of a jump-diffusion process (Brownian motion + compound Poisson) 17	
Figure 6.1: Original Call Price.....	80
Figure 6.2: FFT - Call price.....	80
Figure 7.1: Two dimensional division example of DIRECT algorithm.....	88
Figure 7.2: Implied volatility surface generated by the Jump-Diffusion process. ($r = 0.05$, $\lambda = 0.1$, $\sigma = 0.15$, mean jump size $m = -0.05$, jump size standard deviation $v = 0.4$)	89
Figure 7.3: Implied volatility surface generated by Bates model.	89
Figure 7.4: Calibration of jump-diffusion model to market data separately for each maturity. Top: maturity 51 days. Bottom: maturity 203 days.	90
Figure 7.5: Calibration of jump-diffusion model to market data separately for each maturity. Top: maturity 662 days. Bottom: maturity 845 days.	91
Figure 7.6: Calibration of jump-diffusion model simultaneously to 8 maturities.....	91
Figure 7.7: Calibration of the Bates stochastic volatility jump-diffusion model simultaneously to 8 maturities. (initial volatility $\sqrt{V_0} = 0.0633$, rate of volatility mean reversion $\xi = 0.7446$, long-run volatility $\sqrt{\eta} = 0.01$, volatility of volatility $\theta = 0.8673$, correlation $\rho = -0.578$, jump intensity $\lambda = 0.0624$, mean jump size $\mu_j = 0.2147$, jump size standard deviation $v_j = 0.0815$)......	92
Figure 7.8: Historical volatility and price of S&P 500 index in last 4 months.	94
Figure 7.9: Calibration of Black-Scholes model simultaneously to 8 maturities. Top: historical volatility 11.46%. Bottom: historical volatility 18.47%.	95
Figure 7.10: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 51 days.	96

Figure 7.11: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 203 days.....	96
Figure 7.12: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 478 days.....	97
Figure 7.13: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 845 days.....	97

CHAPTER 1

INTRODUCTION

1.1. Motivation

During the past few decades financial analysts experienced significant success by using simple diffusion models to approximate the asset returns. The most popular example is the Black-Scholes (BS) model [1] which is the best known of all Lévy processes. However some well-known contradictions arised by Black-Scholes revealed that a simple geometric Brownian motion (GBM) process fails to represent some important features of the data. High frequency returns data display excess kurtosis (fat tailed distributions), skewness, and volatility clustering. Also as Cont&Tankov discuss in [13] various empirical results show that sudden downward jumps have been observed in stock price processes. To capture these essential characteristics and improve on the pricing and hedging performance of the Black-Scholes model, recently the majority of the research has been done on alternative pricing models such as exponential-Lévy models that incorporate jumps. Exponential-Lévy models consist of two categories. In the first category, called jump-diffusion models (Merton [4] or Kou [5]), the evolution of prices is given by a diffusion process, intervened by jumps at random times where the jumps represent rare events. Infinite activity models fall into the second category, in these models the paths have infinitely many jumps in any finite time interval. Brownian component is not needed in infinite activity models as the jump component covers small fluctuations. Also to be able to prevent the occurrence of the volatility smile phenomenon, which appears due to the constant volatility assumption of Black-Scholes, stochastic volatility models have been proposed (Hull and White [2] or Heston [3]). Lastly models that combine jumps and stochastic volatility draw attention because of their good performances on market data. (Bates [6, 7] or Duffie, Pan, and Singleton [8]). Empirical work on these models has generally supported the need for both stochastic

volatility and jumps at the same time. The additional stochastic differential equation in volatility gives a tractable system of differential equations for pricing options. This extension of the GBM, called geometric Lévy model or exponential Lévy model, is able to incorporate several stylized features of asset prices such as heavy tails, high-kurtosis, and asymmetry of log returns. For more extensive reviews of different types of geometric Lévy models in finance see Kyprianou [12] and Cont and Tankov [13].

An important feature of the standard Black-Scholes model is that market is *complete*, that is, any contingent claim can be replicated by a self-financing portfolio. In such a market there exists a unique equivalent martingale measure under which the unique price of a contingent claim can be calculated as an expectation. Incompleteness arises in a market when the sources of randomness are more than the number of assets available.

In this thesis we study under more general Lévy processes that incorporate jumps. In Lévy markets, as in the most realistic models, there are many equivalent martingale measures thus such markets are incomplete. There are a few different approaches towards the completion of a Lévy Market in the literature. Leon *et al.* [16] approximate the Lévy process by a sum of a Brownian motion and a countable number of compensated Poisson processes. They then introduce enough additional securities to complete the market. Via Malliavin calculus they calculate the hedging portfolio in the approximated market. On the other hand Corcuera *et al.* [14] enlarged the market by a series of very special assets (power-jump assets) related to the power-jump processes of the underlying Lévy process suitable compensated. To obtain a predictable representation-like property in the general Lévy case, Nualart & Schoutens [24] proved the existence of a Chaotic Representation Property, which states that every square integrable random variable adapted to the filtration generated by a Lévy process can be represented as its expectation plus an infinite sum of zero mean stochastic integrals with respect to the orthogonalised compensated power-jump processes of the underlying Lévy process. Hence, the market can be completed even in the case of a general Lévy process if trades in these processes are allowed.

In the next chapter, we present the basic tools of stochastic calculus and jump processes that will be used further.

In Chapter 3, we describe the set up of geometric Lévy model and discuss the so-called Power-Jump processes. By its definition the power-jump process of order two is just the quadratic variation process (see, *e.g.*, [60,61] and [62]), and is closely linked with the so-called realized variance. Contracts on realized variance have managed to get into OTC markets and are traded regularly now. Higher order power-jump processes have a similar relationship with

realized skewness and realized kurtosis processes. However contracts on these objects are uncommon. Other than these, Carr *et al.* [63] study contracts on the quadratic variation processes in a model driven by a so-called Sato process.

In Chapter 4, we enlarge the market by adding “Power-Jump Assets” as suggested in [14]. We showed that the enlarged market is complete by the Martingale Representation Property [24, 64]. The notion of completeness used is equivalent to the notion of approximate completeness of Björk *et al.* [26]. Moreover the explicit hedging portfolios for claims whose payoff function depends on the prices of the stock are given.

Chapter 5 contains equivalent martingale measure and absence of arbitrage conditions. Ultimately it is shown that the enlarged market is complete and arbitrage-free.

Then in Chapter 6, we determine a jump-size distribution for the jump component and obtain prices of European options by using two different methods, Martingale approach and characteristic formula via fast Fourier transform (FFT). We also compare the speeds and results of these methods.

Chapter 7 includes a stochastic volatility extension of jump-diffusion: Bates model. Then both JD and Bates models are calibrated to the S&P 500 option data. The optimization algorithm used for parameter estimation is presented and numerical results of the calibration procedure are given. The results support that both stochastic volatility and jumps are needed at the same time to describe market behaviour.

Lastly, we conclude the thesis with Chapter 8.

1.2. Imperfections of the Black-Scholes Model

Empirical evidence shows that the classical Black–Scholes model does not describe the statistical properties of financial time series very well. We will focus on two main problems. A more extended list of stylized features of financial data can be found in [13].

In Table 1.1, we summarize the empirical mean and the standard deviation for S&P 500 index daily returns over the period 1989–2010.

Table 1.1: Mean, standard deviation, skewness and kurtosis of S&P 500 index

Index	Mean	Std. Dev.	Skewness	Kurtosis
S&P 500 (1989–2010)	0,000271346	0,011730694	-0,037727667	9,12247009

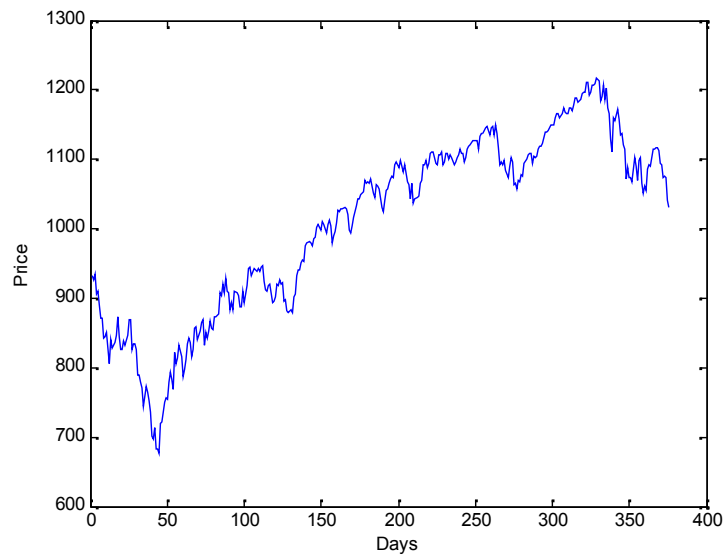


Figure 1.1: Time series of S&P 500 Index data

1.2.1. Log-normally Distributed Returns Assumption

In Black-Scholes model the stock price follows a geometric Brownian motion therefore the logarithmic return is normally distributed. Figure 1.2 reveals that the returns of the S&P 500 index are not log-normally distributed. Also comparing Figures 1.3 and 1.4 we can reach to the same conclusion.

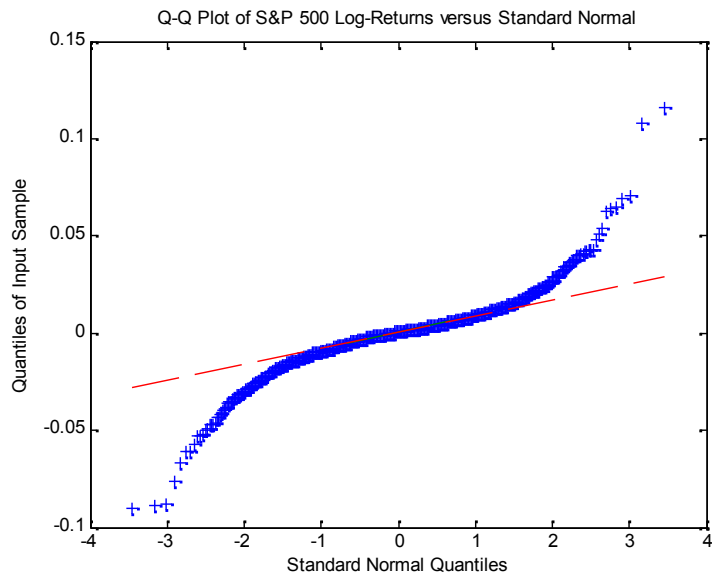


Figure 1.2: Q-Q Plot of S&P 500 log-returns data.

It is shown in Figure 1.3 that price changes in consecutive days are small in some periods and large in other periods. This inconsistency of returns leads to volatility clustering. This feature is not taken into account by geometric Brownian motion i.e. Black-Scholes model.

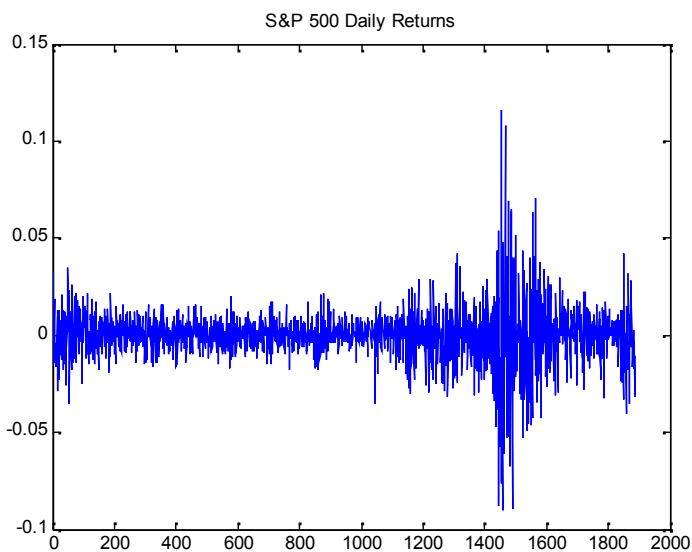


Figure 1.3: S&P 500 Index daily returns from Jan. 2, 2003 to Jun. 30, 2010.

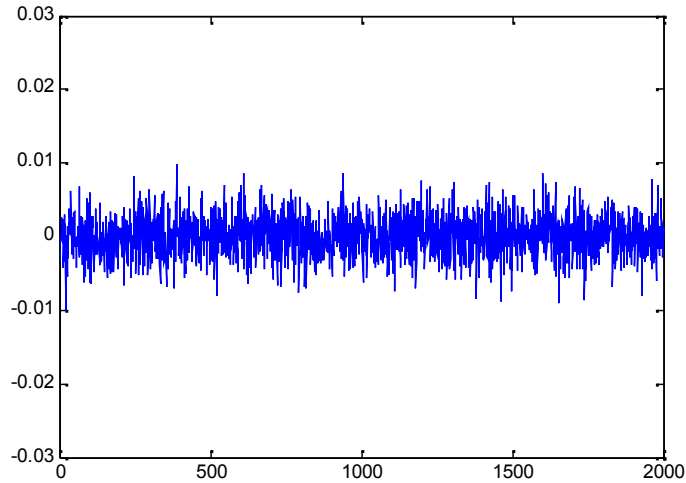


Figure 1.4: Returns generated from a geometric Brownian motion.

1.2.2. Excess Kurtosis and Skewness

In financial data, it is often observed that large movements in asset price occur more frequently than in a model with Normal distributed increments. This feature is named as excess kurtosis or fat tails and it is the main reason for researchers to consider asset price processes with jumps.

A way of measuring this fat tail behaviour is to look at the kurtosis, which is defined by

$$\frac{E\left((X - \mu)^4\right)}{\text{Var}(X)^2}.$$

For the Normal distribution, the kurtosis is 3. For our case from Table 1.1 we see that our data give a kurtosis bigger than 3, indicating that the tails of the Normal distribution go to zero much faster than the tails of empirical. Thus the empirical distribution is much more peaked than the Normal distribution. Also, for the Normal distribution the skewness is zero but we observe a significant negative skewness when we look at the daily log returns of S&P 500 index.

The fat tail of the data and bigger kurtosis can also be observed from Figure 1.5.

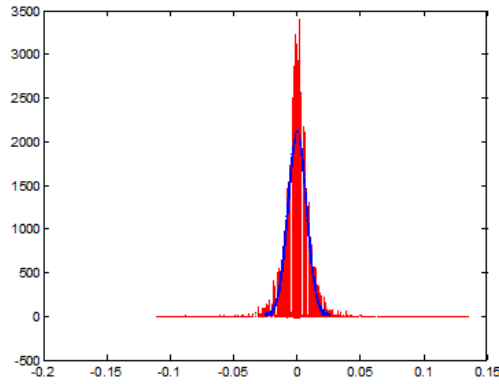


Figure 1.5: Empirical density function for the S&P 500 Returns vs. the Normal Distribution

1.2.3. The Volatility Smile

Another well-known contradiction arised by the Black-Scholes model is the so-called volatility smile. The implied volatility is the volatility that, when used in Black-Scholes model, yields the theoretical value of the option equal to the current market price. By its definition, we expect the implied volatilities to be the same over the periods because the model presumes constant volatility. Nevertheless, this is not the case when one observes real market prices. The implied volatilities of options on the same underlying with different strike prices tend to change.

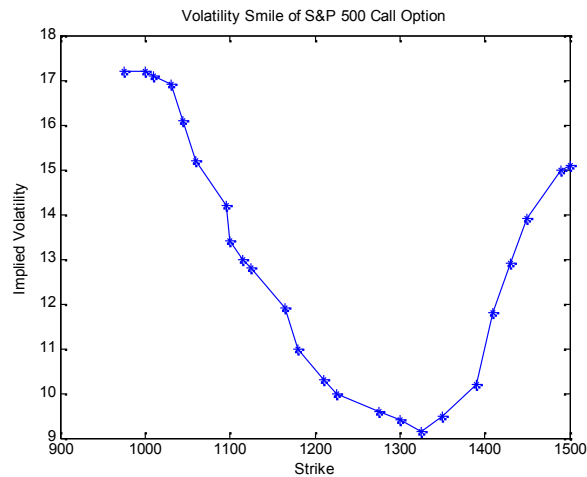


Figure 1.6: Volatility smile. Implied Volatilities for S&P 500 call options with maturity Jun. 17, 2005. Valuation date is Feb. 24, 2005. On that day the S&P 500 index is 1200.20.

It is usually revealing to plot implied volatility as a function of both strike price and time to maturity since implied volatilities also change with maturity. The implied volatility surface simultaneously shows the volatility smile and the term structure of volatility.

Obviously, volatility smile phenomenon is an inconsistency of the Black-Scholes model.

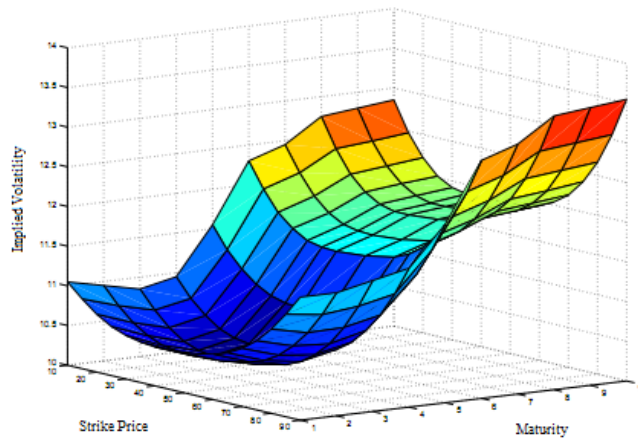


Figure 1.7: Volatility surface. Implied volatilities of vanilla options on the EUR/USD exchange rate on Nov. 5, 2001.

Recent studies in mathematical finance focused models that reflect statistical facts of the market data. Lévy processes appear as a natural candidate with all the required properties to describe the price process of a stock price. In addition to Lévy processes, stochastic volatility models and models that combine both stochastic volatility and jumps are focal points.

CHAPTER 2

PRELIMINARIES

In this section we give some definitions and fundamental results of stochastic calculus, Lévy processes and martingales. More details can be found in Sato [41], Applebaum [54] and Kyprianou [55]. Furthermore recent overviews of applications of Lévy processes are provided in Schoutens [15], Cont&Tankov [13].

We consider a filtered probability space $(\Omega, F, \mathfrak{F}, P)$ where the filtration $\mathfrak{F} = (F_t)_{t \geq 0}$ satisfies the following *usual hypotheses*,

- (i) F_0 contains all the P -null sets of \mathfrak{F} ,
- (ii) $F_t = \bigcap_{u>t} F_u, \forall t, 0 \leq t \leq \infty$; i.e. the filtration \mathfrak{F} is right continuous.

The probability space $(\Omega, F, \mathfrak{F}, P)$ is said to be P -complete if for each $E \subset H \in \mathfrak{F}$ such that $P(H) = 0$, we have that $E \in \mathfrak{F}$. By a filtration we mean a family of σ -algebras $\mathfrak{F} = (F_t)_{t \geq 0}$ that is increasing, i.e., $F_s \subset F_t$ for $s < t$.

Throughout this study we will work on a complete filtered probability space.

2.1. Basic tools of Stochastic Calculus

Definition 2.1.1 A stochastic process is family $X = (X_t)_{t \geq 0}$ of random variables on \mathbb{R}^d with parameter $t \in [0, \infty)$ defined on a probability space.

Definition 2.1.2 Two stochastic processes X and Y are *modifications* if $X_t = Y_t$ a.s., each t . Two processes X and Y are *indistinguishable* if a.s., for all t , $X_t = Y_t$.

Definition 2.1.3 A stochastic process $X = (X_t)_{t \geq 0}$ on $(\Omega, F, \mathfrak{F}, P)$ is said to be *adapted* to the filtration \mathfrak{F} if $X \in F_t$ that is, X is F_t measurable for each t .

If X is adapted then we have $E(X_s | F_s) = X_s$ a.s., i.e., F_s contains all the required information to predict the behaviour of X up to and including time s .

Definition 2.1.4 A stochastic process $X = (X_t)_{t \geq 0}$ is said to be *càdlàg* or *right-continuous with left limits (RCLL)* if it a.s. has sample paths which are right continuous, with left limits.

For each t the limits $X_{t-} = \lim_{s \rightarrow t, s < t} X_s$, $X_{t+} = \lim_{s \rightarrow t, s > t} X_s$ exist and $X_t = X_{t+}$. Every continuous function is càdlàg but càdlàg functions can have discontinuities. If t is a discontinuity point we define the expression $\Delta X_t = X_t - X_{t-}$ as the *jump* of X at t . The most remarkable property of a càdlàg process is that it can have at most a countable number of discontinuities. We can interpret this as such: it has finite number of large jumps on a time interval $[0, T]$. Càdlàg processes are thence natural models for the trajectories of processes with jumps.

Definition 2.1.5 If a stochastic process $X = (X_t)_{t \geq 0}$ satisfies $\lim_{s \rightarrow t} P(|X_t - X_s| > \varepsilon) = 0$ for every $t \geq 0$ and $\varepsilon > 0$; it is said to be *stochastically continuous* or *continuous in probability*.

Definition 2.1.6 Let I be some index set and $X = (X_t)_{t \geq 0}$ be a family of random variables. X is said to be *uniformly integrable* if $\lim_{n \rightarrow \infty} \sup_{i \in I} E(|X_i| 1_{\{|X_i| > n\}}) = 0$.

A process $X = (X_t)_{t \geq 0}$ is said to be *integrable* if $E(|X_t|) < \infty$ for all $t > 0$.

Theorem 2.1.1 (Lévy's Convergence Theorem) Let $(X_n)_{n \geq 1}$ be a uniformly integrable sequence of random variables that satisfies $X_n \xrightarrow{a.s.} X$. Then it follows that:

- (i) $E(|X|) < \infty$,
- (ii) $E(X_n) \rightarrow E(X)$,
- (iii) $E(|X - X_n|) \rightarrow 0$.

This theorem is a special case of *Lebesgue's Dominated Convergence Theorem* in measure theory.

Definition 2.1.7 (Martingale) A real-valued adapted process $X = (X_t)_{t \geq 0}$ is called an F_t -martingale if the following conditions hold:

- (i) X is adapted to the filtration $\mathfrak{F} = (F_t)_{t \in [0, T]}$.
- (ii) For all $t \in [0, T]$, $E(|X_t|) < \infty$.
- (iii) $\forall s \leq t$, $E(X_t | F_s) = X_s$.

A process satisfying the inequality $E(X_t | F_s) \leq X_s$ for every $s \leq t$ is called a *supermartingale*, and a process satisfying $E(X_t | F_s) \geq X_s$ is called a *submartingale*.

A commonly known example of a martingale is the Brownian motion which is defined below.

Definition 2.1.8 (Brownian Motion) A real-valued process $W = (W_t)_{t \geq 0}$ is said to be a *Brownian motion* if the following hold:

- (i) The paths of W are P -almost surely continuous.
- (ii) $P(W_0 = 0) = 1$.
- (iii) For $0 \leq s \leq t$, $W_t - W_s$ is equal in distribution to W_{t-s} .
- (iv) For $0 \leq s \leq t$, $W_t - W_s$ is independent of the $(W_u)_{u \leq s}$.
- (v) For each $t > 0$, W_t is equal in distribution to a normal variable with variance t .

2.2. Lévy Processes and Infinite Divisibility

In this subsection we first identify infinitely divisible distributions and point out their relationship with Lévy processes. Then we define Lévy processes and give some examples that are very commonly used in modeling underlying price processes.

Definition 2.2.1 (Infinite Divisibility) A real-valued random variable Y has an *infinitely divisible* distribution if for each $n = 1, 2, \dots$, there exist a sequence of i.i.d. random variables

Y_1, Y_2, \dots, Y_n such that $Y \stackrel{d}{=} Y_1 + Y_2 + \dots + Y_n$ where the sign $\stackrel{d}{=}$ represents equality in distribution. Alternatively, we can express this relation in terms of probability laws. That is to say, the law Ξ of a real-valued random variable is infinitely divisible if for each $n = 1, 2, \dots$ there exists another law Ξ_n of a real valued random variable such that $\Xi = \Xi_n^{*n}$ where Ξ_n^{*n} denotes the n -fold convolution of Ξ_n .

Another relation can be constructed in terms of characteristic functions. The law of a random variable Y is infinitely divisible, if for all $n \in \mathbb{N}$, there exists a random variable Y_n such that

$\varphi_Y(u) = [\varphi_{Y_n}(u)]^n$ where φ_Y and φ_{Y_n} are characteristic functions of Y and Y_n , respectively.

Definition 2.2.2 (Lévy Process). A càdlàg, adapted, real-valued stochastic process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ a.s. is said to be a *Lévy process* if the following conditions are satisfied:

- (i) X has independent increments, i.e. $X_t - X_s$ is independent of F_s for any $0 \leq s < t$.
- (ii) X has stationary increments, i.e. for any $0 \leq s, t \leq T$ the distribution of $X_{t+s} - X_s$ does not depend on t .
- (iii) X is stochastically continuous, i.e. for every $t \geq 0$ and $\varepsilon > 0$:

$$\lim_{s \rightarrow t} P(|X_t - X_s| > \varepsilon) = 0.$$

Proposition 2.2.1 Let $X = (X_t)_{t \geq 0}$ be a Lévy process. Then for every t , X_t has an infinitely divisible distribution. Conversely, if Ξ is an infinitely divisible distribution then there exists a Lévy process $X = (X_t)_{t \geq 0}$ such that the distribution of X_1 is given by Ξ .

Definiton 2.2.1 implies that one way to examine whether a given random variable has an infinitely divisible distribution or not is via its characteristic exponent. Suppose that Y has characteristic exponent,

$$\psi(u) = -\log E(e^{iuY})$$

for all $u \in \mathbb{R}$.

Then Y has an infinitely divisible distribution if for all $n \geq 1$ there exists a characteristic exponent of a probability distribution, say ψ_n , such that

$$\psi(u) = n\psi_n(u)$$

for all $u \in \mathbb{R}$.

Proposition 2.2.2 (Characteristic function of a Lévy process) Let $X = (X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . There exists a continuous function $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$ called the characteristic exponent of X , such that:

$$E(e^{iuX_t}) = e^{t\psi(u)}, \quad u \in \mathbb{R}^d.$$

The full extent to infinitely divisible distributions is described by an expression known as the Lévy Khintchine formula.

2.2.1. Some Examples of Lévy Processes

We proceed our introduction to Lévy processes with some prominent examples which are mostly used as base processes for jump models.

Definition 2.2.3 (Poisson Process) A càdlàg, adapted stochastic process $N = (N_t)_{t \geq 0}$ is said to be a *Poisson process* with intensity $\lambda > 0$ if the following hold:

- (i) The paths of N are P -almost surely right continuous with left limits.
- (ii) $P(N_0 = 0) = 1$.
- (iii) For $0 \leq s \leq t$, $N_t - N_s$ is equal in distribution to N_{t-s} .
- (iv) For $0 \leq s \leq t$, $N_t - N_s$ is independent of $(N_u)_{u \leq s}$.
- (v) For each $t > 0$, N_t is equal in distribution to a Poisson random variable with parameter λt .

We alternatively can define Poisson process as in the following definition.

Definition 2.2.4 (Poisson Process) Let $\tau = (\tau_i)_{i \geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^n \tau_i$.

The process $N = (N_t)_{t \geq 0}$ defined by $N_t = \sum_{n \geq 1} I_{\{t \geq T_n\}}$ is called a *Poisson process* with intensity λ .

The Poisson process is therefore defined as a counting process: it counts the number of random times (T_n) which occur between 0 and t , where $(T_n - T_{n-1})_{n \geq 1}$ is an i.i.d. sequence of exponential variables.

Characteristic function of Poisson process can be obtained as follows,

$$\begin{aligned}
 E(e^{iuN_t}) &= \sum_{k \geq 0} e^{iuk} P(N_t = k) = \sum_{k \geq 0} e^{iuk} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
 &= e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t e^{iu})^k}{k!} \\
 &= e^{-\lambda t} e^{\lambda t e^{iu}} \\
 &= e^{-\lambda t(1 - e^{iu})}.
 \end{aligned}$$

Definition 2.2.5 (Compensated Poisson Process) Let $N = (N_t)_{t \geq 0}$ be a Poisson process with parameter λ . We shall call the process $M = (M_t)_{t \geq 0}$ where

$$M_t := N_t - \lambda t$$

a compensated Poisson process.

Proposition 2.2.3 The compensated Poisson process is a martingale.

Note that the compensated Poisson process is no longer integer valued like the Poisson process, it is not a counting process. Its rescaled version $\frac{M_t}{\sqrt{\lambda}}$ has the same first two moments as a

standard Brownian motion: $E\left(\frac{M_t}{\sqrt{\lambda}}\right) = 0, Var\left(\frac{M_t}{\sqrt{\lambda}}\right) = t$.

Definition 2.2.6 (Compound Poisson Process) A compound Poisson process with intensity $\lambda > 0$ and jump size distribution F_Y is a stochastic process Q_t defined as

$$Q_t = \sum_{i=0}^{N_t} Y_i,$$

where jumps sizes Y_i are i.i.d. with distribution F_Y and $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity λ , independent from $(Y_i)_{i \geq 1}$.

Characteristic function of a compound poisson process has the following representation:

$$\begin{aligned} E\left(e^{iuQ_t}\right) &= E\left(E\left(e^{iuQ_t} \mid N_t = n\right)\right) \\ &= \sum_{n \geq 0} E\left(e^{iuQ_t}\right) P(N_t = n) \\ &= \sum_{n \geq 0} E\left(e^{iuQ_t}\right) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= \sum_{n \geq 0} \left(\int_{\mathfrak{R}} e^{iuy} F_Y(dy) \right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \end{aligned}$$

Using the fact that N has stationary independent increments and mutual independency of the random variables $(Y_i)_{i \geq 1}$, for $0 \leq s < t$ by writing

$$Q_t = Q_s + \sum_{i=N_s+1}^{N_t} Y_i$$

it is clear that Q_t is the sum of Q_s and an independent representative of Q_{t-s} . Right continuity with left limits of the Poisson process also guarantee the right continuity with left limits of Q . Thus compound Poisson processes are Lévy processes.

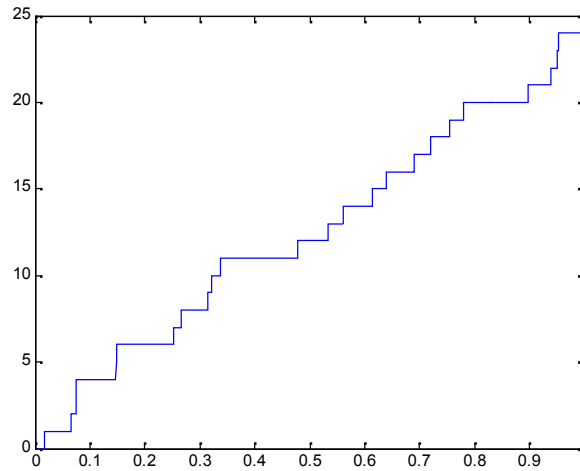


Figure 2.1: Sample paths of a Poisson process.

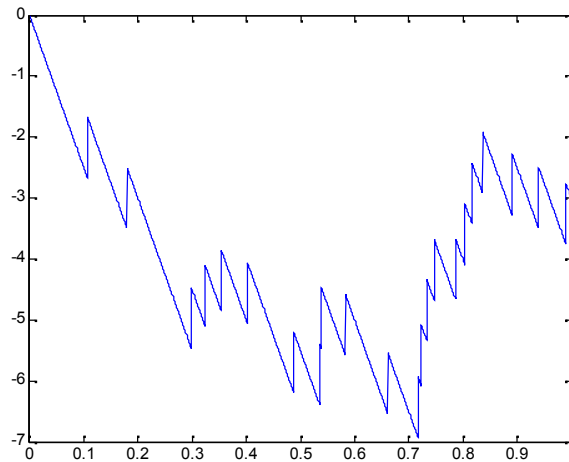


Figure 2.2: Sample path of a compensated Poisson process.

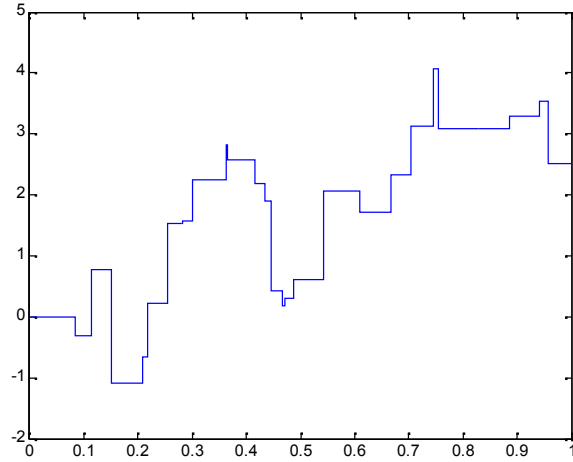


Figure 2.3: Sample path of a compound Poisson process with Gaussian distribution of jump sizes.

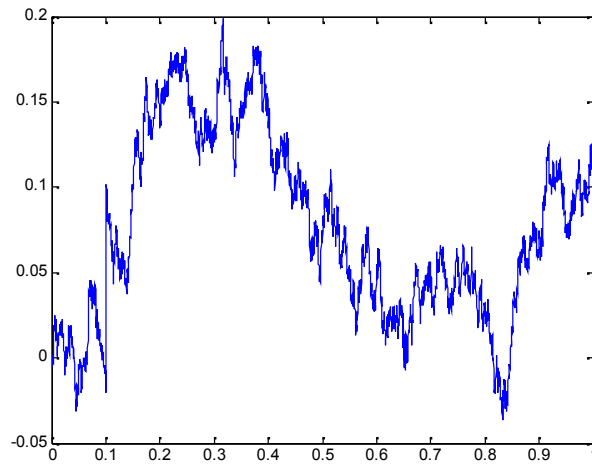


Figure 2.4: Sample path of a jump-diffusion process (Brownian motion + compound Poisson).

Definition 2.2.7 (Lévy measure) Let $X = (X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . The measure ν on \mathbb{R}^d defined by

$$\nu(A) = E\left(\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}\right), \quad A \in \mathcal{B}(\mathbb{R}^d)$$

is called the Lévy measure of X .

Namely, $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A .

2.2.2. Random Measures and Point Processes

A useful tool for analyzing the jumps of a Lévy process is the *random jump measure* of the process.

Consider a set $A \subset \mathbb{R} - \{0\}$ such that $0 \notin \bar{A}$ and a Lévy process $X = (X_t)_{t \geq 0}$, let $0 \leq t \leq T$.

The random measure of the jumps of the process X is defined by

$$J_X(\omega; t, A) = \#\{0 \leq s \leq t : \Delta X_s(\omega) \in A\} = \sum_{s \leq t} 1_A(\Delta X_s(\omega)).$$

The measure $J_X(\omega; t, A)$ counts the jumps of the process X of size in A up to time t .

Here the path is fixed and $J_X(\omega; t, \cdot)$ is a measure on $[0, T] \times \mathbb{R} - \{0\}$, on the other hand when the set A is fixed $J_X(t, A)$ becomes a random variable. Moreover, since $J_X(t, A) - J_X(s, A)$ is independent of F_s and it equals the number of jumps of $X_{s+u} - X_u$ for $0 \leq u \leq t - s$, $J_X(\cdot; t, A)$ has independent and stationary increments. Therefore $J_X(\cdot; t, A)$ is a Poisson process and J_X is said to be a *Poisson random measure*.

The following definition describes a Poisson random measure in detail.

Definition 2.2.8 (Poisson random measure) Let (Ω, F, P) be a probability space, $E \subset \mathbb{R}$ and μ a given (positive) Radon measure, μ on (E, ξ) . A Poisson random measure on E with intensity measure μ is an integer valued random measure:

$$J_X : \Omega \times \xi \rightarrow \mathbb{N}$$

such that

- (i) For $\omega \in \Omega$, $J_X(\omega; t, \cdot)$ is an integer valued Radon measure on E .

- (ii) For each measurable set $A \subset E$, $J_X(\cdot; t, A) = J_X(t, A)$ is a Poisson random variable with parameter $\mu(A)$.
- (iii) For disjoint measurable sets $A_1, A_2, \dots, A_n \in \mathcal{E}$, the variables $J_X(t, A_1), \dots, J_X(t, A_n)$ are independent.

We can define integrals with respect to $J_X(\omega; t, \cdot)$. Consider a finite, Boreal measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$ and a set A determined as above. Then the integral with respect to the Poisson random measure $J_X(\omega; t, \cdot)$ is defined as the following

$$\int_A f(x) J_X(\omega; t, dx) = \sum_{s \leq t, \Delta X_s \neq 0} f(\Delta X_s) 1_A(\Delta X_s(\omega)).$$

Integrating f with respect to J_X up to time t produces a stochastic process as follows

$$Y_t = \int_0^t \int_{\mathbb{R}^d - \{0\}} f(x) J_X(ds, dx).$$

Theorem 2.2.4 Consider a set $A \subset \mathbb{R} - \{0\}$ such that $0 \notin \bar{A}$ and a Lévy process $X = (X_t)_{t \geq 0}$ Lévy measure ν . Then the following is satisfied

$$E \left(\int_0^t \int_A f(x) J_X(ds, dx) \right) = t \int_A f(x) \nu(dx).$$

Theorem 2.2.1 (Lévy-Khintchine formula) A probability law P of a real-valued random variable is infinitely divisible with characteristic exponent ψ ,

$$E(e^{iuX_t}) = \int_{\mathbb{R}} e^{iux} \mu(dx) = e^{t\psi(u)} \quad \text{for all } u \in \mathbb{R},$$

if and only if there exists a triplet (α, c^2, ν) where $\alpha \in \mathbb{R}$, $c \geq 0$ and ν is a measure concentrated $\mathbb{R} - \{0\}$ on satisfying $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty$, such that

$$\psi(u) = i\alpha u - \frac{1}{2}u^2 c^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux 1_{\{|x|<1\}}) \nu(dx)$$

for every $u \in \mathbb{R}$, where μ is the distribution of X .

Theorem 2.2.2 (Lévy-Itô Decomposition) Consider a triplet (α, c^2, ν) where, $\alpha \in \mathbb{R}$, $c \geq 0$ and ν is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty$. Then, there exists a probability space (Ω, F, P) on which three independent Lévy processes exist $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ where $X^{(1)}$ is a Brownian motion with drift, $X^{(2)}$ is a compound Poisson process and $X^{(3)}$ is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Taking $X = X^{(1)} + X^{(2)} + X^{(3)}$, we have that there exists a probability space on which a Lévy process $X = (X_t)_{t \geq 0}$ with characteristic exponent given by Theorem 2.2.1.

Remark 2.2.1 The measure ν is called the Lévy measure. ν is a positive measure on P but not a probability measure since $\int \nu(dx) = \lambda \neq 1$.

Let us now discuss in further detail the relationship between infinitely divisible distributions and processes with stationary independent increments. From the definition of a Lévy process we see that for any $t > 0$, X_t is a random variable belonging to the class of infinitely divisible distributions. This follows from the fact that for any $n=1, 2, \dots$, consider a Lévy process

$$X = (X_t)_{t \geq 0} \text{ and a partition of } 0 < \frac{t}{n} < \frac{2t}{n} < \dots < \frac{(n-1)t}{n} < t.$$

$$X_t = (X_{t/n} - X_0) + (X_{2t/n} - X_{t/n}) + \dots + (X_{(n-1)t/n} - X_{(n-2)t/n}) + (X_t - X_{(n-1)t/n}).$$

By the stationary increments property of the Lévy processes, this equation results in:

$$X_t \stackrel{d}{=} X_{t/n} + X_{t/n} + \dots + X_{t/n}. \quad (2.1)$$

Suppose now that we define for all $u \in \mathbb{R}$, $t \geq 0$,

$$\psi_t(u) = \log E(e^{iuX_t})$$

Then using (2.1) we have for any two positive integers m, n that

$$m\psi_1(u) = \psi_m(u) = n\psi_{m/n}(u),$$

and hence for any rational $t > 0$,

$$\psi_t(u) = t\psi_1(u).$$

Theorem 2.2.3 For any Lévy process $X = (X_t)_{t \geq 0}$ we have,

$$E(e^{iuX_t}) = e^{t\psi(u)},$$

where $\psi(u)$ is the characteristic exponent of X_1 , a random variable with an infinitely divisible distribution.

Hence by Lévy Khintchine formula we can determine the characteristic function of a Lévy process if its triplet is known.

Definition 2.2.10 (Subordinator) A subordinator is an a.s. increasing Lévy process. For L to be a subordinator, the triplet must satisfy $\nu((-\infty, 0)) = 0$, $c = 0$.

Subordinators can be used as time changes for other Lévy process so they are very important ingredients for building Lévy-based models in finance.

Definition 2.2.11 Let $\bar{P} = \{a = t_1 < t_2 < \dots < t_{n+1} = b\}$ be a partition on the interval $[a, b] \subset \mathfrak{R}$. The variation of a function f over the partition \bar{P} is defined by

$$\text{var}_{\bar{p}}(f) = \sum_{i=1}^n |f(t_{i+1}) - f(t_i)|.$$

If the supremum over all partitions is finite, $\sup_{\bar{p}} \text{var}_{\bar{p}}(f) < \infty$, f is said to be of finite variation on $[a, b]$. If this is not the case, the function is said to be of infinite variation.

Proposition 2.2.4 Let $X = (X_t)_{t \geq 0}$ be a Lévy process with triplet (α, c^2, ν) .

- (i) If $c^2 = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$ then almost all paths of X have finite variation.
- (ii) If $c^2 \neq 0$ or $\int_{|x| \leq 1} |x| \nu(dx) = \infty$ then almost all paths of X have infinite variation.

We say that a Lévy process $X = (X_t)_{t \geq 0}$ is of finite variation if the sample paths are of finite variation with probability 1. Again if this is not the case, we say that the process is of infinite variation.

Representative examples of finite and infinite variation processes are Poisson process and Brownian motion, respectively. Because the Brownian motion is of infinite variation, a Lévy process with a Brownian component is also of infinite variation.

By Proposition 2.2.5 a pure jump Lévy process, i.e. one with no Brownian component,

with $\int_{|x| < 1} |x| \nu(dx) = \infty$ is of infinite variation. In that case special attention has to be paid to the

small jumps since the sum of all jumps smaller than some $\varepsilon > 0$ does not converge. So to make it converge we should compensate the sum of the jumps by a compensator term which is in fact their mean $iux1_{\{|x| < 1\}}$.

Proposition 2.2.5 Let $X = (X_t)_{t \geq 0}$ be a Lévy process with triplet (α, c^2, ν) .

- (i) If $\nu(\mathbb{R}) < \infty$ then almost all paths of X have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.
- (ii) If $\nu(\mathbb{R}) = \infty$ then almost all paths of X have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.

Three most important examples of such pure jump infinite activity Lévy processes which are widely used in financial modeling are the normal inverse Gaussian (NIG) model of Barndorff-Nielsen [31], the symmetric variance gamma (VG) model studied by Madan and Seneta [56] and the CGMY model developed by Carr, Geman, Madan, and Yor (CGMY) [57], which further generalizes the VG model.

They find that the empirical performance of these models is generally not improved by adding a diffusion component for returns. These results arise the question that do we need diffusion components when we model asset returns.

An important goal of stochastic calculus is to give the meaning of the stochastic integral $\int_0^t f(s) dX_s$ for a proper class of adapted processes X . The processes defined below are ideally suited for the role of integrators.

Definition 2.2.12 (Semimartingale) An adapted stochastic process $X = (X_t)_{t \geq 0}$ is said to be a *semimartingale* if for each $t \geq 0$ it can be written as follows,

$$X_t = X_0 + M_t + C_t,$$

where $X_0 \in F_0$, $M = (M_t)_{t \geq 0}$ is a local martingale and $C = (C_t)_{t \geq 0}$ is an adapted process of finite variation.

Proposition 2.2.6 Every Lévy process is a semimartingale.

Proposition 2.2.7 Every finite variation process is a semimartingale.

Proposition 2.2.8 Every square integrable martingale is a semimartingale.

The proofs of the Propositions 2.2.6, 2.2.7 and 2.2.8 can be found in [13], Chapter 8.

A Lévy process can be split into a sum of a square integrable martingale and a finite variation process which is Lévy-Itô decomposition (Theorem 2.2.2). Thus, every Lévy process is a semimartingale.

The proof of the next Proposition can be found in [13], Chapter 8.

Proposition 2.2.9 (Itô formula for semimartingales) Let $X = (X_t)_{t \geq 0}$ be a semimartingale.

For any $C^{1,2}$ function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial s}(s, X_{s-}) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-}) d[X, X]_s^c \\ &\quad + \sum_{\substack{0 \leq s \leq t \\ \Delta X_s \neq 0}} \left(f(s, X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-}) \right). \end{aligned}$$

Definition 2.2.13 Consider the Lévy process $X = (X_t)_{t \geq 0}$ with triplet (α, c^2, ν) . The quadratic variation process of X is given by

$$[X, X]_t = ct + \int_0^t \int_{\mathbb{R}} x^2 J_X(ds, dx).$$

Definition 2.2.14 A semimartingale is called *quadratic pure jump* if $[X, X]^c = 0$.

Definitions and theorems below can be found in [21], Chapter 5.

Definition 2.2.15 (g-moment) Let $g(x)$ be a nonnegative measurable function on \mathbb{R} . We call $\int g(x)\mu(dx)$ the *g-moment* of a measure μ on \mathbb{R} . We call $E(g(x))$ the *g-moment* of a random variable X on \mathbb{R} .

Definition 2.2.16 (Submultiplicativity) A function $g(x)$ on \mathbb{R} is called *submultiplicative* if it is non negative and there is a constant $a > 0$ such that

$$g(x+y) \leq ag(x)g(y) \text{ for } x, y \in \mathbb{R}.$$

Remark 2.2.2 A function bounded on every compact set is called *locally bounded*.

Theorem 2.3.1 (g-Moment). Let g be a submultiplicative, locally bounded, measurable function on \mathbb{R} . Then, finiteness of g-moment is not a time dependent property in the class of Lévy processes. Let $X = (X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R} with Lévy measure ν . Then, X_t has finite g-moment for every $t > 0$ if and only if $\int_{\{|x|<1\}} g(x)\nu(dx) < \infty$.

2.3. Martingale Measures and Market Completeness

Definition 2.3.1 Consider a probability space (Ω, F, P) . A probability measure Q defined on (Ω, F) is said to be an *equivalent martingale* measure if

- (i) Q is equivalent to P , i.e. they have the same null sets: for any $A \in \Omega$

$$P(A) = 1 \Leftrightarrow Q(A) = 1.$$
- (ii) The discounted stock-price process $\tilde{S} = (\tilde{S}_t)_{t \geq 0} = (e^{-rt}S_t)_{t \geq 0}$ is a martingale under Q .

The existence of an equivalent martingale measure Q allows one to obtain the price of options on the risky asset by calculating the expected values of the discounted payoffs with respect to

Q . When we work under Q , we are in a risk-neutral world, since under Q the expected return of the stock equals the risk-free return of the bank account:

$$e^{-rt} E^Q(S_t | F_0) = S_0.$$

Equivalent martingale measure is of high importance since its existence is related to the absence of arbitrage, while its uniqueness is equivalent to market completeness.

Besides absence of arbitrage, another important problem is hedging. A contingent claim can be perfectly hedged if there exists a self-financing strategy ϕ that can replicate the claim in the sense that there is a dynamic portfolio, investing in the bank account and the stock, such that at every time point the value of the portfolio matches the value of the claim. Moreover, the strategy must also be admissible, that is the portfolio's value must be bounded from below by a constant.

$$H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_t^0 dS_t^0.$$

Therefore in a complete market a contingent claim can be valued only one way, the value of any contingent claim is given by the initial capital V_0 needed to set up a perfect hedge for H . So all equivalent martingale measures give the same pricing rules. Therefore market completeness seems to imply the uniqueness of pricing rules/equivalent martingale measures.

Proposition 2.3.1 A market defined by the assets $(S_t^0, S_t^1, \dots, S_t^d)$, $t \in [0, T]$ described as stochastic processes on $(\Omega, F, \mathfrak{F}, P)$ is complete if and only if there is a unique martingale measure Q equivalent to P .

In Black-Scholes model the question of completeness is identified with the uniqueness of the martingale measure, which is in turn linked with the mathematical predictable representation property (PRP) of a martingale.

Definition 2.3.2 (Predictable Representation Property) Let $M = (M_t)_{t \geq 0}$ be a martingale. M is said to have the *predictable representation property* if, for any square-integrable random variable $H \in F_T$, we have

$$H = E(H) + \int_0^T a_s dM_s,$$

for some predictable process $a = \{a_s, 0 \leq s \leq T\}$.

The predictable process $a = \{a_s, 0 \leq s \leq T\}$ represents the weights of the assets in the portfolio, hence gives us the necessary self-financing admissible strategy.

An important point is that the PRP is an exceptional property, which only a few martingales possess. Examples include Brownian motion, the compensated Poisson process, and the Azéma martingale (see [58]). So uniqueness of an equivalent martingale measure implies the PRP which in turn implies market completeness. Since in the markets driven by Lévy processes, contingent claims do not generally possess the PRP the market is incomplete.

To obtain a similar property in the general Lévy case, Nualart & Schoutens [24] proved the existence of a Chaotic Representation Property, which states that every square integrable random variable adapted to the filtration generated by a Lévy process can be represented as its expectation plus an infinite sum of zero mean stochastic integrals with respect to the orthogonalised compensated power-jump processes of the underlying Lévy process. Hence, the market can be completed even in the case of a general Lévy process if trades in these processes are allowed.

When we show the completeness of the market in Section 4.2, we will make use of Martingale Representation Property.

CHAPTER 3

THE LÉVY MARKET MODEL

3.1. Introduction

We consider a continuous time Lévy market model with finite horizon T . The uncertainty in the market is modelled by the complete probability space (Ω, F, P) and the market consists of a bond (risk-free asset) and a stock (risky asset). We will model the stock price process $S = \{S_t, t \geq 0\}$ with the following Lévy-driven stochastic differential equation (SDE)

$$\frac{dS_t}{S_{t-}} = bdt + dX_t, \quad S_0 > 0, \quad (3.1)$$

whereas the bond price process is given by $B = \{B_t, t \geq 0\}$.

The solution to equation (3.1) with initial condition $S_0 = 1$ is called the stochastic exponential of the process $X_t + bt$ and is given by

$$\varepsilon(X_t + bt) = e^{X_t + bt - \frac{1}{2}[X^c, X^c]_t} \prod_{s \leq t, \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s}.$$

Here if the Lévy process X_t is a Brownian motion, the model becomes the classical Black-Scholes model [1], which is *complete*. In complete markets all the contingent claims in the market can be replicated by a self-financing portfolio, moreover there exist a unique equivalent

martingale measure under which the expectation of the discounted payoff at maturity is equal to the unique price of a contingent claim.

In our study we assume that X_t is a Lévy process which incorporates jumps and we consider the càdlàg version of X_t . Here the process $J_t = \Delta X_t$ represents the jump size process of X_t and may or may not have a specified distribution.

Like most of the Lévy models, the above described model is *incomplete*. So as suggested in [14], we will complete the market by enlarging it with a series of very special assets related to the power-jump processes of the underlying Lévy process, called power-jump assets.

The completion procedure will be done in the following order: First we assume that the unique equivalent martingale measure is given and then we enlarge the market in such a way that it will remain-arbitrage free.

Since X_t is a Lévy process by the Lévy-Kintchine formula its characteristic exponent has the following form:

$$\psi(u) = i\alpha u - \frac{1}{2}u^2c^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|<1\}}) \nu(dx),$$

where (α, c^2, ν) is the characteristic triplet of X_t with $\alpha \in \mathbb{R}$, $c \geq 0$. ν is the Lévy measure of X on $\mathbb{R} - \{0\}$ with $\int_{-\infty}^{+\infty} (1 \wedge x^2) \nu(dx) < \infty$. Again from the Lévy-Kintchine formula it can be shown that X has the following decomposition,

$$X_t = cW_t + L_t, \tag{3.2}$$

where $W = (W_t)_{t \geq 0}$ is a Brownian motion, $L = (L_t)_{t \geq 0}$ is a pure jump process and the processes W and L are independent. By the Lévy-Itô Decomposition we can express L as a sum of three independent Lévy processes. Taking a constant drift $L_t^1 = \alpha t$, a Brownian motion, a compound Poisson process $L_t^2 = \int_{(0,t] \times \{|x| \geq 1\}} xN(ds, dx)$ and a square integrable martingale with

an a.s. countable number of jumps on each finite interval of magnitude less than one,

$$L_t^3 = \int_{(0,t] \times \{|x| < 1\}} x(N(ds, dx) - t\nu(dx)) \text{ we have } L = L^1 + L^2 + L^3.$$

Thus we can write:

$$L_t = \int_{(0,t] \times \{|x| < 1\}} x(N(ds, dx) - t\nu(dx)) + \int_{\{|x| \geq 1\}} xN(ds, dx) + \alpha t \quad (3.3)$$

where $N(ds, dx)$ is a Poisson random measure on $(0, +\infty) \times \mathbb{R} - \{0\}$ with intensity $dt \times \nu$.

Here dt denotes the Lebesgue measure and ν is the Lèvy measure of X .

In this model it is supposed that the Lèvy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$,

$$\int_{(-\varepsilon, \varepsilon)^c} \exp(\lambda|x|)\nu(dx) < \infty. \quad (3.4)$$

Here $g(x) = \exp(\lambda|x|)$ is submultiplicative, therefore we deduce that L_t has finite g -moment for every $t > 0$ or that exponential moments $E(\exp(\lambda|x|))$ exist, (see Theorem 25.3 of [21]).

Since we can obtain all moments $E(|x|^i)$ by simply differetiating exponential moments, it is clear that all moments of L_t exist.

This implies that

$$\int_{-\infty}^{+\infty} |x|^i \nu(dx) < \infty, \quad i \geq 2,$$

and that the characteristic function $E(\exp(iuL_t))$ is analytic in a neighborhood of 0. Consider selection of $u = ih$ thus there exist $0 < h_1, h_2 \leq \infty$ such that $E(\exp(-hX_1)) < \infty$ for all $h \in (-h_1, h_2)$. By stationary increments property of X_t it can be shown that $E(\exp(-hX_t)) < \infty$ for all t . Hence all moments of X_t also exist.

By the equation (3.3) we have,

$$\begin{aligned} E(L_t) &= E\left(\int_{(0,t] \times \{|x|<1\}} x(N(ds, dx) - tv(dx))\right) + E\left(\int_{\{|x|\geq 1\}} xN(ds, dx) + \alpha t\right) \\ &= E\left(\int_{\{|x|\geq 1\}} xN(ds, dx)\right) + \alpha t. \end{aligned}$$

Since the first expectation equals to zero due to the fact that compensated compound Poisson process is a martingale.

From this last equation we have

$$\alpha = E(L_1) - \int_{\{|x|\geq 1\}} xv(dx).$$

Since every Lévy process is a semimartingale, by semimartingale decomposition theorem L_t can under these assumptions be written as a sum of a martingale and a predictable of finite variation as follows:

$$L_t = M_t + at, \tag{3.5}$$

where $M = (M_t)_{t \geq 0}$ is a martingale and $E(L_1) = a$.

Consider the measure $\tilde{N}(dt, dx) := N(dt, dx) - dtv(dx)$. Since the second term is the intensity measure of the Poisson random measure, $\tilde{N}(dt, dx)$ is a compensated Poisson random measure and we can write the martingale part of L_t in terms of $M(dt, dx)$ as

$$M_t = \int_{(0,t] \times \mathbb{R}} x\tilde{N}(ds, dx).$$

Ultimately, our Lévy process X_t takes the form

$$X_t = cW_t + \int_{(0,t] \times \{|x| < 1\}} x \tilde{N}(ds, dx) + at,$$

and the SDE of S_t becomes

$$\frac{dS_t}{S_{t-}} = (a+b)dt + cdW_t + dM_t. \quad (3.6)$$

Since we have described the properties of the Lévy processes we will work, we can now obtain the stock price process.

3.2. The Stock Price Process

In this subsection we make use of Itô's formula (see [22]) for càdlàg semimartingales to obtain the solution of equation (3.6)

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S_{s-}) dS_s^c + \frac{1}{2} \int_0^t f''(S_{s-}) d[S, S]_s^c \\ &\quad + \sum_{0 < s \leq t} \left\{ f(S_s) - f(S_{s-}) - f'(S_{s-}) \Delta S_s \right\}. \end{aligned}$$

Applying Itô's formula to equation (3.6) for $f(S_t) = \log(S_t)$ we get:

$$\begin{aligned} \log(S_t) &= \log(S_0) + \int_0^t \frac{1}{S_{s-}} dS_s^c + \frac{1}{2} \int_0^t \left(-\frac{1}{S_{s-}^2} \right) d[S, S]_s^c \\ &\quad + \sum_{0 < s \leq t} \left\{ \log(S_s) - \log(S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right\} \\ &= \log(S_0) + \int_0^t \frac{1}{S_{s-}} S_{s-} ((a+b)dt + cdW_t) - \frac{1}{2} \int_0^t \frac{1}{S_{s-}^2} S_{s-}^2 c^2 ds \end{aligned}$$

$$+ \sum_{0 < s \leq t} \left\{ \log \left(\frac{S_s}{S_{s-}} \right) - \frac{1}{S_{s-}} \Delta S_s \right\}.$$

Note that $d[S, S]_s^c = S_{s-}^2 c^2 ds$ and $\Delta S_s = S_{s-} \Delta M_s$ with $\Delta S_s = S_s - S_{s-}$. These equations

imply that $S_s = S_{s-}(1 + \Delta M_s)$ and $\log \left(\frac{S_s}{S_{s-}} \right) = \log(1 + \Delta M_s)$.

By using these relations in the above equation we can easily deduce the explicit solution of (3.6) as

$$S_t = S_0 \exp \left(cW_t + M_t + \left(a + b - \frac{c^2}{2} \right) t \right) \prod_{0 < s \leq t} (1 + \Delta M_s) \exp(-\Delta M_s). \quad (3.7)$$

Since stock price process has to be non-negative for all t , in order to guarantee that $1 + \Delta M_t \geq 0$ for all $t \geq 0$ a.s., we should put the constraint that $\Delta M_t \geq -1$. Thus we need our Lévy measure to have a domain that is a subset of $[-1, +\infty)$.

The market consists of one risky and one riskless assets, a stock and a bank account or a riskfree bond. Throughout the study we will assume that the riskless interest rate r is constant. So, the value of the bank account or risk-free bond at time t is given by, $B_t = \exp(rt)$.

3.3. Equivalent Martingale Measures

In this section we investigate structure-preserving P -equivalent measures which enable us to stay within an analytically tractable family of models. Under the structure-preserving P -equivalent martingale measures, X remains a Lévy process and the discounted stock price process $\tilde{S} = \{ \tilde{S}_t = \exp(-rt)S_t, t \geq 0 \}$ is a martingale.

As we are working on a market with finite horizon T , locally equivalence will be regarded the same as equivalence.

The following theorem (see Proposition 9.8 in [13] or Theorems 33.1 and 33.2 in [21]) gives the relations between the triplets of X according to the equivalent probability measures and an expression for the densities on the path space.

Theorem 3.3.1 *Let X be a Lévy process with Lévy triplet (α, c^2, ν) under the probability measure P .*

Then the following two conditions are equivalent.

(a) *There exists a probability measure Q , locally equivalent to P for any $t \geq 0$, such that X is a Q -Lévy process with triplet $(\tilde{\alpha}, \tilde{c}^2, \tilde{\nu})$.*

(b) *All of the following conditions hold.*

(i) $\tilde{\nu}(dx) = H(x)\nu(dx)$ where $H: \mathbb{R} \rightarrow (0, \infty)$ is a Borel function satisfying

$$\int_{-\infty}^{+\infty} \left(1 - \sqrt{H(x)}\right)^2 \nu(dx) < \infty.$$

(ii) $\tilde{\alpha} = \alpha + \int_{-\infty}^{+\infty} x 1_{\{|x| \leq 1\}} (H(x) - 1) \nu(dx) + Gc$ for some $G \in \mathbb{R}$.

(iii) $\tilde{c} = c$.

When P and Q are equivalent, the Radon-Nikodym derivative is given by

$$\frac{dQ|F_t}{dP|F_t} = e^{U_t},$$

where

$$U_t = GL_t^c - \frac{G^2 c^2 t}{2} - G\alpha t + \lim_{\varepsilon \rightarrow 0} \left(\int_{(0,t] \times \{|x| > \varepsilon\}} \ln H(x) N(ds, dx) - t \int_{|x| > \varepsilon} (H(x) - 1) \nu(dx) \right).$$

Here the process L_t^c represents the continuous part of L_t and U_t is a Lévy process with characteristic triplet (α_U, c_U^2, ν_U) given by:

$$\begin{aligned} c_U^2 &= c^2 G^2, \\ \nu_U &= \nu \gamma^{-1}, \\ \alpha_U &= -\frac{1}{2} G^2 - \int_{-\infty}^{+\infty} \left(e^y - 1 - y 1_{\{|y| \leq 1\}} \right) (\nu \gamma^{-1})(dy), \end{aligned}$$

where $\gamma := \ln H(x)$.

The above theorem shows that we are able to change the Lévy measure, while retaining the equivalence of measures, but, if a diffusion component is not present, we cannot freely change the drift.

Consider the compensated Poisson random measure $\tilde{M}(dt, dx) = M(dt, dx) - dt\tilde{\nu}(dx)$, where $M(dt, dx)$ is a Poisson random measure on $(0, +\infty) \times \mathbb{R}$ under Q with intensity $dt \times \tilde{\nu}(dx)$.

Suppose the equivalent conditions in the above theorem hold, and X be a Lévy process with triplet $(\tilde{\alpha}, \tilde{c}^2, \tilde{\nu})$. Then we the following results:

By Lévy-Itô decomposition we can write X as

$$X_t = c\tilde{W}_t + \tilde{L}_t, \quad t \geq 0$$

where \tilde{L}_t , the pure jump process with respect to measure Q , is defined by:

$$\tilde{L}_t = \int_{(0,t] \times \{|x| \geq 1\}} x M(ds, dx) + \int_{(0,t] \times \{|x| < 1\}} x (M(ds, dx) - t\tilde{\nu}(dx)) + \tilde{\alpha}t$$

with $\tilde{\alpha} = E_Q(\tilde{X}_1) - \int_{\{|x| \geq 1\}} x\tilde{\nu}(dx)$.

By equations (3.2) and (3.3) we have that,

$$X_t - \tilde{L}_t = cW_t + L_t - \tilde{L}_t = c(W_t - Gt)$$

thus Brownian motion with respect to Q is defined by $\tilde{W}_t = W_t - Gt$. Moreover if the condition $\int_{(-\varepsilon, \varepsilon)^c} \exp(\tilde{\lambda}|x|)\tilde{\nu}(dx) < \infty$ is satisfied, then we have:

$$\tilde{L}_t = \int_{(0, t] \times \{|x| \geq 1\}} xM(ds, dx) + \int_{(0, t] \times \mathbb{R}} x(M(ds, dx) - t\tilde{\nu}(dx)) + \tilde{\alpha}t$$

with

$$\tilde{\alpha} = E_Q(\tilde{L}_1) - \int_{\{|x| \geq 1\}} x\tilde{\nu}(dx). \quad (3.7)$$

Hence the Doob-Meyer decomposition of \tilde{L}_t is given by:

$$\tilde{L}_t = \tilde{M}_t + \tilde{\alpha}t, \quad (3.8)$$

where

$$\tilde{M}_t := \int_{(0, t] \times \mathbb{R}} x\tilde{M}(ds, dx), \quad (3.9)$$

is a Q -martingale, moreover by (3.7) we have

$$\tilde{\alpha} = \tilde{\alpha} + \int_{\{|x| \geq 1\}} x\tilde{\nu}(dx). \quad (3.10)$$

If the equivalent conditions of the previous theorem hold, Brownian motion under Q , \tilde{W} can be defined as:

$$\tilde{W}_t = W_t - Gt. \quad (3.11)$$

Equations (3.8) and (3.10) results in $\tilde{L}_t = \tilde{M}_t + \left(\tilde{\alpha} + \int_{\{|x| \geq 1\}} x \tilde{\nu}(dx) \right) t$. Furthermore if condition

(ii) is satisfied, L is a quadratic pure jump process with Doob-Meyer decomposition

$$L_t = \tilde{M}_t + \left(a + \int_{-\infty}^{+\infty} x(H(x)-1)\nu(dx) \right) t, \quad (3.12)$$

where \tilde{M} is a Q -martingale and the Lèvy measure is given by $\tilde{\nu}(dx) = H(x)\nu(dx)$.

By equations (3.12) and (3.5) we have the relation $\tilde{M}_t = M_t - \int_{-\infty}^{+\infty} x(H(x)-1)\nu(dx)$.

Now, our aim is to find an equivalent martingale measure Q under which the discounted price process \tilde{S} is a martingale. By the above theorem, under such a Q , X has the Doob-Meyer decomposition given by (3.12).

Noting that $\Delta M_t = \Delta \tilde{M}_t$, and using equations (3.6), (3.11) and (3.12) the discounted asset prices under Q can be written as follows:

$$\begin{aligned} \tilde{S}_t = S_0 \exp & \left(c\tilde{W}_t + \tilde{M}_t + \left(a + b - r + cG - \frac{c^2}{2} \right) t \right) \\ & \times \exp \left(t \int_{-\infty}^{+\infty} x(H(x)-1)\nu(dx) \right) \prod_{0 < s \leq t} (1 + \Delta \tilde{M}_s) \exp(-\Delta \tilde{M}_s). \end{aligned}$$

So a necessary and sufficient condition for \tilde{S} to be a Q -martingale is the existence of G and

$H(x)$ with the condition $\int_{-\infty}^{+\infty} \left(1 - \sqrt{H(x)} \right)^2 \nu(dx) < \infty$ such that

$$cG + a + b - r + \int_{-\infty}^{+\infty} x(H(x)-1)\nu(dx) = 0. \quad (3.13)$$

By using equations (3.2), (3.11), (3.12) and (3.13) we deduce that, $X_t = \sigma \tilde{W}_t + \tilde{M}_t + (r-b)t$. Since process \tilde{W}_t and \tilde{M}_t are Q -martingales, the process $\tilde{X}_t := X_t - (r-b)t$ is also a Q -martingale and hence $E_Q(\tilde{X}_t) = 0$.

The risk neutral dynamics of \tilde{S} can be written in terms of \tilde{X}_t as follows:

$$\frac{d\tilde{S}_t}{\tilde{S}_{t-}} = c\tilde{W}_t + d\tilde{M}_t = d\tilde{X}_t$$

with solution

$$\tilde{S}_t = S_0 \exp\left(c\tilde{W}_t + \tilde{M}_t - \frac{c^2}{2}t\right) \times \prod_{0 < s \leq t} (1 + \Delta\tilde{M}_s) \exp(-\Delta\tilde{M}_s).$$

Note that S has the dynamics

$$\frac{dS_t}{S_{t-}} = rdt + c\tilde{W}_t + d\tilde{M}_t = rdt + d\tilde{X}_t, \text{ under } Q.$$

with solution $S_t = S_0 \exp\left(c\tilde{W}_t + \tilde{M}_t + \left(r - \frac{c^2}{2}\right)t\right) \times \prod_{0 < s \leq t} (1 + \Delta\tilde{M}_s) \exp(-\Delta\tilde{M}_s)$. (3.14)

Remark 3.3.1 If there exists a (non-structure preserving) equivalent martingale measure Q_1 under which Z is not a Lèvy process, there always exists a (structure preserving) equivalent martingale measure Q_2 under which Z is a Lèvy process (see, e.g., [48]).

CHAPTER 4

COMPLETION OF THE LÉVY MARKET MODEL

4.1. Power-Jump Processes

In this part we introduce *power-jump processes* which are transformations of the Lévy process $X = (X_t)_{t \geq 0}$ and will be used to construct a series of special artificial assets the next chapters.

Consider the process

$$X_t^{(i)} = \sum_{0 < s \leq t} (\Delta X_s)^i, \quad i \geq 2, \quad (4.1)$$

where $\Delta X_s = X_s - X_{s-}$ is the amplitude of jump at time s . For convenience we set $X_t^{(1)} = X_t$.

Notice that the equation $\Delta X_s = X_s - X_{s-}$ is not satisfied all the time, it is only satisfied if X_s has bounded variation i.e. in the case when c is equal to zero.

Consider the processes $L_t^{(i)}$ defined as follows:

$$L_t^{(i)} = \sum_{0 < s \leq t} (\Delta L_s)^i, \quad L_t^{(1)} = L_t,$$

since $\Delta X_t = \Delta L_t$ is satisfied by equation (3.2), we have

$$L_t^{(i)} = X_t^{(i)}, \quad i \geq 2.$$

It is clear that $[L, L]_t = L_t^{(2)}$ as $[L, L]_t = \sum_{0 \leq s \leq t} (L_s - L_{s-})^2 = \sum_{0 \leq s \leq t} (\Delta L_s)^2$.

The processes $L^{(i)} = \left(L_t^{(i)}\right)_{t \geq 0}$ are Lévy processes for $i \geq 2$ and are called the *ith-power-jump processes*. They jump at the same points as the original Lévy process X , but the jump sizes are the i -th power of the jump size of the original Lévy process.

Corollary 4.1.1 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and vanish in a neighborhood of 0.*

Then

$$E\left(\sum_{0 < s \leq t} f(\Delta X_s)\right) = t \int_{-\infty}^{+\infty} f(x) \nu(dx).$$

By the corollary and the definition of $L_t^{(i)}$, we have $E(L_t) = E(L_t^i) = ta = tm_1 < \infty$, where $m_1 < \infty$ is a constant and also

$$E_Q\left(X_t^{(i)}\right) = a^{(i)}t, \quad i \geq 2. \quad (4.2)$$

Let $Y_t^{(i)}$ denote the *compensated power-jump processes* (or *Teugels martingales*) for $i \geq 1$ defined by:

$$Y_t^{(i)} = L_t^{(i)} - E\left(L_t^{(i)}\right) = L_t^{(i)} - m_i t. \quad (4.3)$$

As $Y_t^{(i)}$ are martingale for all $i \geq 1$, they have zero mean.

According to the procedure described to orthonormalize the sequence of martingales $\left(Y^{(i)}\right)_{i \geq 1}$ in [24], by taking a suitable linear combination of $Y^{(i)}$'s, we can obtain a set of pairwise strongly orthonormal martingales $\left(T^{(i)}\right)_{i \geq 1}$. Thus, for $i \neq j$, the process $T^{(i)}T^{(j)}$ is a martingale (see Leon [16]).

$\left(T^{(i)}\right)_{i \geq 1}$ represents a basis for L^2 space and each T^i is a linear combination of the Y^j , $j = 1, 2, \dots, i$:

$$T^{(i)} = c_{i,i} Y^{(i)} + c_{i,i-1} Y^{(i-1)} + \dots + c_{i,1} Y^{(1)}, \quad i \geq 1.$$

It is proved in [24] that, this orthogonality ensures the existence of an orthogonal family of polynomials $\{p_n, n \geq 0\}$ with respect to the measure $\mu(dx) = x^2 \nu(dx) + c^2 \delta_0(dx)$ and $c_{i,j}$ corresponds to the coefficients of the orthonormalization of the polynomials $\{p_n, n \geq 0\}$.

The polynomials, defined by $p_n(x) := \sum_{j=1}^n c_{n,j} x^{j-1}$ are orthogonal with respect to the measure μ :

$$\int_{\mathbb{R}} p_n(x) p_m(x) \mu(dx) = 0, \quad n \neq m.$$

The resulting processes $T^{(i)} = (T_t^{(i)})_{i \geq 0}$ are called *the orthonormalized i th-power-jump processes*.

4.2. Volatility Trading with Power-Jump Assets

Volatility trading is a trading strategy that is designed to speculate on changes in the volatility of the market rather than the direction of the market. Besides a few custom volatility indices, the volatility can be traded by combining stable positions in options with dynamic trading in the underlying. Neuberger [58] showed that by delta-hedging a contract, the hedging error accumulates to the difference between the realized variance and the fixed variance used in the delta-hedge. Therefore a hedged position makes money if the realized volatility exceeds implied volatility with an amplitude depending on option's curvature.

Similar strategy is valid for the trade in the power-jump assets as it is too a trade based on volatility of the stock. The 2th-power-jump asset, which is the quadratic variation of the Lévy process L , in a sense quantifies the volatility of the stock because it accounts for the square of the jumps. If one believes that in the future the market will be more volatile than the current market's prediction, trading the 2th-power-jump asset can be logical. Moreover if one wants protection against periods of high (or low) volatility, buying (or selling) 2th-power-jump assets can cover the possible losses due to such unfavorable periods.

The same strategy works for the higher order variation assets. The 3th-power-jump asset measures a kind of asymmetry therefore it can be traded based on expectations of realized skewness. Similarly the 4th-power-jump process is measuring extremal movements so a trade in this asset can be of use if one likes to bet on the realized kurtosis of the stock. These positions can be hold when one believes that the market is not adequately taking asymmetry and possible extremal moves into account.

Furthermore, an insurance against a crash can be easily built by trading in the 4th- or higher-power-jump assets.

4.3. Enlarging the Lévy Market Model

In this subsection we consider a fixed finite planning horizon T . Suppose we have an equivalent martingale measure Q under which X maintains being a Lévy process with characteristic triplet $(\tilde{\alpha}, \tilde{c}^2, \tilde{\nu})$ and the discounted stock price process $\tilde{S} = \{\tilde{S}_t = \exp(-rt)S_t, 0 \leq t \leq T\}$ is a martingale. In the previous section we have showed that the process $\tilde{X}_t := X_t - (r-b)t$ is a Q -martingale. Moreover, since Q is a structure-preserving equivalent martingale measure it is also a Lévy process.

It is clear that jump sizes of \tilde{X} and X are equal that is, $\Delta\tilde{X}_t = \Delta X_t$. This result leads us to the fact that the power-jump processes transformed by them are also equal $\tilde{X}_t^{(i)} = X_t^{(i)}, i \geq 2$.

Now let us consider the i th-power-jump processes $Y^{(i)} = (Y_t^{(i)})_{0 \leq t \leq T}$ based on $\tilde{X} = (\tilde{X}_t)_{0 \leq t \leq T}$ and their orthonormalized version $T^{(i)} = (T_t^{(i)})_{0 \leq t \leq T}$ for $i \geq 2$.

By using the i th-power-jump processes, Nualart and Schoutens (see [14]) built up a new series of assets called *i th-power-jump assets* and introduced to the market. The price process $H^{(i)} = \left(H_t^{(i)} \right)_{0 \leq t \leq T}$ of the assets is defined as follows:

$$H_t^{(i)} = \exp(rt) Y_t^{(i)}, \quad i \geq 2,$$

and the orthonormalized version is denoted by $\bar{H}^{(i)} = \left(\bar{H}_t^{(i)} \right)_{0 \leq t \leq T}$ with

$$\bar{H}_t^{(i)} = \exp(rt) T_t^{(i)}, \quad i \geq 2.$$

The compensators are $m_i t = t E_Q \left(\tilde{L}_1^{(i)} \right)$. Note that for $i \geq 2$, $m_i = \int_{-\infty}^{+\infty} x^i \tilde{\nu}(dx)$ and we will require $\tilde{\nu}$ to fulfill (3.4).

It is clear that the discounted price processes $\tilde{H}_t^{(i)} = \exp(-rt) H_t^{(i)}$ which are in fact the power-jump processes, are martingales under Q .

We will enlarge the Lévy market with i th-power-jump assets with price processes,

$$H_t^{(i)} = \exp(rt) \left(L_t^{(i)} - a^{(i)} t \right), \quad i \geq 2.$$

The constants $a^{(i)}$, $i \geq 2$ are chosen in such a way that the market remains arbitrage-free. This is a delicate selection as $a^{(i)}$ may lead or prevent arbitrage opportunities.

For example, if $a^{(i)}$ and risk-free rate r are chosen as zero, this may cause arbitrage opportunities since the processes $H_t^{(i)}$ for even i are strictly increasing and starting at zero.

So the choice of the constants $a^{(i)}$ will be discussed later in Chapter 5.

4.4. Market Completeness

Market completeness is often identified with the *predictable representation property* (PRP) which states that any square integrable Q -martingale M in can be represented as follows

$$M = E(M) + \int_0^T \phi_s d\tilde{S}_t.$$

It was shown in [25] that PRP holds when \tilde{S} is a Brownian motion or a Brownian stochastic integral, but it fails to hold for most discontinuous models used in finance like non-Gaussian Lévy processes.

Even though the predictable representation property holds, it does not automatically lead to market completeness by itself. For a predictable process ϕ_t to be interpreted as a trading strategy, we must be able to approximate its value process using an admissible portfolio.

The next theorem presents the chaotic representation property (CRP) in terms of orthogonalised compensated power-jump processes which is derived by Nualart & Schoutens [45]. The CRP is important since it implies the predictable representation property (PRP), which provides the hedging portfolio for a contingent claim. Based on the PRP of Lévy processes, Corcuera et al. (2005) completed the market by introducing power-jump assets.

Theorem 4.4.1 (Chaotic Representation Property) *Every square integrable random variable H has a representation of the form*

$$H = E(H) + \sum_{j=1}^{\infty} \sum_{i_1, \dots, i_j \geq 1} \int_0^{t_1} \int_0^{t_1} \dots \int_0^{t_{j-1}} f_{(i_1, \dots, i_j)}(t_1, \dots, t_j) dT_{t_j}^{i_j} \dots dT_{t_2}^{i_2} dT_{t_1}^{i_1}$$

where $f_{(i_1, \dots, i_j)}$'s are \mathbb{R}_+^j -valued square integrable functions and the processes

$T^{(i)} = \left(T_t^{(i)} \right)_{0 \leq t \leq T}$ the orthonormalized power-jump processes.

The *Martingale Representation Property (MRP)*, which is obtained by Nualart and Schoutens in [24], is an immediate result of the CRP.

Theorem 4.4.2 (Martingale Representation Property) *Every square integrable Q -martingale*

$M = (M_t)_{0 \leq t \leq T}$ *has the following representation*

$$M_t = M_0 + \int_0^t h_s d\tilde{X}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} dT_s^{(i)}$$

where h_s and $h_s^{(i)}$ are predictable processes, satisfying the conditions

$$E \left(\int_0^t |h_s|^2 ds \right) < \infty \quad \text{and} \quad E \left(\int_0^t \sum_{i=2}^{\infty} |h_s^{(i)}|^2 ds \right) < \infty.$$

Martingale representation property allows any square integrable Q -martingale to be represented as an orthogonal sum of stochastic integrals with respect to the orthonormalized power-jump processes $T^{(i)} = (T_t^{(i)})_{0 \leq t \leq T}$.

Next theorem states that the market enlarged with power-jump assets is complete in the sense that every square-integrable contingent claim X is reachable that is, it can be replicated by a sequence of self-financing portfolios whose values converge in space of square integrable Q -martingales to X .

Self-financing portfolios will consist of finite number of bonds, stocks and i th power-jump assets. This notion of completeness is equivalent to the *approximately complete* which is studied by Björk given in [26].

Theorem 4.4.3 *The Lévy market model enlarged with the i th-power-jump assets is complete, in the sense that any square integrable contingent claim X can be replicated.*

Proof: Consider a square-integrable contingent claim X under Q with maturity T .

Define the process M_t as

$$M_t = E_Q(\exp(-rT)X | F_t),$$

where r is the constant risk-free rate.

M_t is a square-integrable martingale since the following conditions hold:

- (i) $M_t \in F_t$ (adaptedness)
- (ii) $E_Q(M_t) < \infty$ since $E_Q(M_t) = E_Q(E_Q(\exp(-rT)X | F_t))$

$$= E_Q(\exp(-rT)X)$$

$$= \exp(-rT)E_Q(X) < \infty$$
- (iii) $E_Q(M_t | F_s) = E_Q(E_Q(\exp(-rT)X | F_t) | F_s), s \leq t$

$$= E_Q(\exp(-rT)X | F_s)$$

$$= M_s$$

By MRP we write M_t in the following form:

$$M_t = M_0 + \int_0^t h_s d\tilde{X}_s + \sum_{i=2}^{\infty} \int_0^t h_s^{(i)} dT_s^{(i)}.$$

If we define

$$M_t^N := M_0 + \int_0^t h_s d\tilde{X}_s + \sum_{i=2}^N \int_0^t h_s^{(i)} dT_s^{(i)}$$

we have that

$$\lim_{N \rightarrow \infty} M_t^N = M_t$$

in space of square integrable Q -martingales.

Define the sequence of portfolios in terms of the orthonormalized i th-power-jump assets as

$$\phi^N := \left\{ \phi_t^N = \left(\alpha_t^N, \beta_t, \beta_t^{(2)}, \beta_t^{(3)}, \dots, \beta_t^{(N)} \right), t \geq 0 \right\}, N \geq 2,$$

where the weights of the assets in the portfolio are as follows

$$\alpha_t^N = M_{t-}^N - \beta_t S_{t-} e^{-rt} - e^{-rt} \sum_{i=2}^N \beta_t^{(i)} \bar{H}_{t-}^{(i)},$$

$$\beta_t = e^{rt} h_t S_{t-}^{-1},$$

$$\beta_t^{(i)} = h_t^{(i)}, \quad i = 2, 3, \dots, N.$$

Here α_t^N corresponds to the number of bonds at time t ; β_t is the number of stocks at that time and $\beta_t^{(i)}$ is the number of assets $\bar{H}^{(i)}$, $i = 2, 3, \dots, N$ one needs to hold at time t .

The claim is that $\{\phi^N, N \geq 2\}$ is the sequence of self-financing portfolios which replicates the contingent claim X .

Let V_t^N denote the value of ϕ^N at time t , then the value at time t is given by

$$V_t^N = \alpha_t^N e^{rt} + \beta_t S_t + \sum_{i=2}^N \beta_t^{(i)} \bar{H}_t^{(i)} = e^{rt} M_t^N.$$

This last equation shows that the value of the portfolio at time t is equal to

$$e^{rt} M_t^N = E_Q \left(\exp(-r(T-t)) X \mid F_t \right).$$

which is in fact value of the contingent claim at time t . So the sequence of portfolios $\{\phi^N, N \geq 2\}$ replicates the claim.

Now let's show that the portfolio is also self-financing.

Denote by

$$G_u^N = r \int_0^u \alpha_t^N e^{rt} dt + \int_0^u \beta_t dS_t + \sum_{i=2}^N \int_0^u \beta_t^{(i)} d\bar{H}_t^{(i)}$$

the gain process, that is the gains or losses obtained up to time u by following ϕ^N .

We should show that $G_u^N + M_0 = M_u^N e^{ru}$ that is, the change in value of the portfolio only occurs as the price of the participating securities changes.

$$G_u^N = r \int_0^u M_{t-}^N e^{rt} dt - \int_0^u h_t e^{rt} dt - r \sum_{i=2}^N \int_0^u h_t^{(i)} \bar{H}_{t-}^{(i)} dt + \int_0^u h_t e^{rt} S_{t-}^{-1} dS_t + \sum_{i=2}^N \int_0^u h_t^{(i)} d\bar{H}_t^{(i)}. \quad (4.4)$$

By using integration by parts formula we can write

$$r \int_0^u M_t^N e^{rt} dt = e^{ru} M_u^N - M_0 - \int_0^u h_t e^{rt} d\tilde{X}_t - \sum_{i=2}^N \int_0^u h_t^{(i)} e^{rt} dT_t^{(i)}, \quad (4.5)$$

substituting (4.5) into (4.4) yields

$$\begin{aligned} G_u^N &= e^{ru} M_u^N - M_0 - r \int_0^u h_t e^{rt} dt - r \sum_{i=2}^N \int_0^u h_t^{(i)} \bar{H}_{t-}^{(i)} dt - \int_0^u h_t e^{rt} d\tilde{X}_t \\ &\quad + \int_0^u h_t e^{rt} S_{t-}^{-1} dS_t + \sum_{i=2}^N \int_0^u h_t^{(i)} d\bar{H}_t^{(i)} - \sum_{i=2}^N \int_0^u h_t^{(i)} e^{rt} dT_t^{(i)} \\ &= e^{ru} M_u^N - M_0 - r \int_0^u h_t e^{rt} dt - \int_0^u h_t e^{rt} d\tilde{X}_t + \int_0^u h_t e^{rt} S_{t-}^{-1} dS_t \\ &= e^{ru} M_u^N - M_0. \end{aligned}$$

Thus every square-integrable contingent claim it can be replicated by a sequence of self-financing portfolios.

4.5. Hedging Portfolios

In this section we investigate the sequence of self-financing portfolios that replicates the contingent claim X which has a payoff only a function of the value at maturity of the stock price, i.e. $X = f(S_T)$.

The price function $F(t, S_t)$ of the contingent claim X at time t is given by

$$F(t, S_t) = E_Q \left(\exp(-r(T-t)) f(S_T) \middle| \mathcal{F}_t \right).$$

Now we will derive the Partial Differential Integral Equation (PDIE) that $F(t, S_t)$ will satisfy in the Lèvy market setting.

It is clear that the discounted option price process $e^{-rt}F(t, S_t)$ is square-integrable Q -martingale. Hence the process $e^{-rt}F(t, S_t)$ is a semimartingale and has the following decomposition $e^{-rt}F(t, S_t) = K_t + A_t$ where K_t is a local martingale and A_t is a finite variation predictable process.

Applying Ito formula for semimartingales to $e^{-rt}F(t, S_t)$ (see [13] Proposition 8.19) yields

$$\begin{aligned} F(t, S_t) - F(0, S_0) &= \int_0^t \frac{\partial F(s, S_s)}{\partial s} ds + \int_0^t \frac{\partial F(s, S_{s-})}{\partial S_s} dS_s + \frac{1}{2} \int_0^t \frac{\partial^2 F(s, S_{s-})}{\partial S_s^2} d[S, S]_s^c \\ &\quad + \sum_{0 < s \leq t} \left[F(s, S_s) - F(s, S_{s-}) - \Delta S_s \frac{\partial F(s, S_{s-})}{\partial S_s} \right]. \end{aligned}$$

If we differentiate both sides the following equation is obtained in terms of differential operators

$$\begin{aligned} dF(t, S_t) &= D_1 F(t, S_t) dt + D_2 F(t, S_{t-}) dS_t + \frac{1}{2} D_2^2 F(t, S_{t-}) d[S, S]_t^c \\ &\quad + \int_{\mathbb{R}} \left(F(t, S_{t-}(1+y)) - F(t, S_{t-}) - S_{t-} y \frac{\partial F(s, S_{t-})}{\partial S_t} \right) M(dt, dy), \quad (4.6) \end{aligned}$$

where D_1 denotes the differential operator with respect to the time variable, and D_2 denotes the differential operator with respect to the space variable i.e. the stock price.

Now we will apply Itô's formula to the product $e^{-rt}F(t, S_t)$:

$$d(e^{-rt}F(t, S_t)) = -re^{-rt}F(t, S_t)dt + e^{-rt}dF(t, S_t). \quad (4.7)$$

Substituting (4.6) into (4.7) we obtain

$$\begin{aligned} d(e^{-rt}F(t, S_t)) &= -re^{-rt}F(t, S_t)dt + e^{-rt} \left\{ D_1F(t, S_t)dt + D_2F(t, S_{t-})dS_t \right. \\ &\quad \left. + \frac{1}{2}D_2^2F(t, S_{t-})d[S, S]_t^c \right. \\ &\quad \left. + \int_{\mathbb{R}} \left(F(t, S_{t-}(1+y)) - F(t, S_{t-}) - S_{t-}y \frac{\partial F(s, S_{t-})}{\partial S_t} \right) M(dt, dy) \right\}. \end{aligned}$$

The right hand side can be written as the sum of a local martingale and a finite variation predictable process as

$$\begin{aligned} d(e^{-rt}F(t, S_t)) &= -re^{-rt}F(t, S_t)dt + e^{-rt} \left\{ D_1F(t, S_t)dt + cS_{t-}D_2F(t, S_{t-})d\tilde{W}_t \right. \\ &\quad \left. + rS_{t-}D_2F(t, S_{t-})dt + \frac{1}{2}S_{t-}^2D_2^2F(t, S_{t-})c^2dt + S_{t-}D_2F(t, S_{t-})d\tilde{M}_t \right. \\ &\quad \left. + \int_{\mathbb{R}} \left(F(t, S_{t-}(1+y)) - F(t, S_{t-}) - S_{t-}y \frac{\partial F(s, S_{t-})}{\partial S_t} \right) \tilde{\nu}(dy) \right\}, \end{aligned}$$

where $\tilde{\nu}(dy)$ is the compensator of the Poisson random measure $M(dt, dy)$. Since $e^{-rt}F(t, S_t)$ is a Q -martingale, the finite variation part in the equation must be equal to zero.

Then the price function (at time t) $F(t, x)$ satisfies the following PDIE

$$D_1F(t, x) + rxD_2F(t, x) + \frac{1}{2}c^2x^2D_2^2F(t, x) + DF(t, x) = rF(t, x) \quad (4.8)$$

where $DF(t, x) = \int_{\mathbb{R}} \left(F(t, x(1+y)) - F(t, x) - xy \frac{\partial F(t, x)}{\partial x} \right) \tilde{\nu}(dy)$ with

$$F(T, S_T) = f(S_T).$$

Following lemma will enable us to calculate the sequence of self-financing portfolios that replicates X .

Lemma 4.5.1 Consider a real function $h(s, x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ which is infinitely differentiable in the y variable and satisfies $h(s, x, 0) = 0$, and $\frac{\partial h}{\partial y}(s, x, 0) = 0$.

Set

$$a_i(s, x) = \frac{1}{i!} \frac{\partial^i h}{\partial y^i}(s, x, 0),$$

and assume that

$$\sup_{x < K, s \leq s_0} \sum_{i=2}^{\infty} |a_i(s, x)| R^i < \infty, \quad (4.9)$$

for all $K, R > 0, s_0 > 0$. Then we have

$$\sum_{t < s \leq T} h(s, S_{s-}, \Delta L_s) = \sum_{i=2}^{\infty} \int_t^T \frac{1}{i!} \frac{\partial^i}{\partial y^i} h(s, S_{s-}, 0) dY_s^{(i)} + \int_t^{T+\infty} \int_{-\infty}^{\infty} \frac{1}{i!} \frac{\partial^i}{\partial y^i} h(s, S_{s-}, y) \tilde{\nu}(dy) ds.$$

Proof: Consider the Taylor expansion of $h(s, x, y)$ in the neighborhood of 0

$$h(s, x, y) = \sum_{i=2}^{\infty} a_i(s, x) y^i.$$

Then we have

$$\begin{aligned}\sum_{t < s \leq T} h(s, S_{s-}, \Delta L_s) &= \sum_{t < s \leq T} \sum_{i=2}^{\infty} a_i(s, S_{s-}) (\Delta L_s)^i \\ &= \sum_{i=2}^{\infty} \int_t^T a_i(s, S_{s-}) dL_s^{(i)}.\end{aligned}$$

By equation (4.3) this equals to the following summation,

$$\sum_{i=2}^{\infty} \int_t^T a_i(s, S_{s-}) dY_s^{(i)} + \sum_{i=2}^{\infty} \int_t^T a_i(s, S_{s-}) m_i ds,$$

where the sums converge for every $\omega \in \Omega$ because of (4.9).

Since m_i is defined by $m_i = \int_{-\infty}^{+\infty} y^i \tilde{\nu}(dy)$, we have

$$\begin{aligned}\sum_{t < s \leq T} h(s, S_{s-}, \Delta L_s) &= \sum_{i=2}^{\infty} \int_t^T a_i(s, S_{s-}) dY_s^{(i)} + \int_t^{T+\infty} \int_{-\infty}^{+\infty} a_i(s, S_{s-}) y^i \tilde{\nu}(dy) ds \\ &= \sum_{i=2}^{\infty} \int_t^T a_i(s, S_{s-}) dY_s^{(i)} + \int_t^{T+\infty} \int_{-\infty}^{+\infty} h(s, S_{s-}, y) \tilde{\nu}(dy) ds.\end{aligned}$$

Next theorem specifies the weights of the assets in the self-financing portfolio and can be found in [14].

Theorem 4.5.1 *The sequence of self-financing portfolios replicating a contingent claim X , with a payoff only depending on the stock value at maturity and a price function $F(t, x) \in C^{1,\infty}$ which satisfies*

$$\sup_{x < K, t \leq t_0} \sum_{n=2}^{\infty} |D_2^n F(t, x)| R^n < \infty, \quad (4.10)$$

for all $K, R > 0, t_0 > 0$, is given at time t by:

$$\text{number of bonds } \alpha_t^N = B_t^{-1} (F(t, S_{t-}) - S_{t-} D_2 F(t, S_{t-})) - B_t^{-1} \sum_{i=2}^N \frac{S_{t-}^i D_2^i F(t, S_{t-})}{i! B_t} H_{t-}^{(i)},$$

$$\text{number of stocks } \beta_t = D_2 F(t, S_{t-}),$$

number of i th-power-jump assets $\beta_t^{(i)} = \frac{S_t^i D_2^i F(t, S_{t-})}{i! B_t}$, $i = 2, 3, \dots, N$.

Proof: Consider a portfolio which has the value $P(t, S_t)$ at time t . Our market consists of a risk-free bond, a stock and i th-power-jump assets. Assume that portfolio $P(t, S_t)$ contains α_t^N numbers of risk-free bond, β_t numbers of stock and $\beta_t^{(i)}$ numbers of i th-power-jump assets. Then its value at time t is given by,

$$P(t, S_t) = \alpha_t^N B_t + \beta_t S_t + \sum_{i=1}^{\infty} \beta_t^{(i)} H_s^{(i)}.$$

In order $P(t, S_t)$ to replicate the contingent claim X , its value should be equal to the price of X at each time t . Thus the equation,

$$P(t, S_t) = F(t, S_t), \text{ for } t \geq 0,$$

or equivalently $F(t, S_t) = \alpha_t^N B_t + \beta_t S_t + \sum_{i=1}^{\infty} \beta_t^{(i)} H_s^{(i)}$ must be satisfied.

To determine the weights of the portfolio, first, apply Itô's formula to $F(T, S_T)$.

Then we have the following

$$\begin{aligned} F(T, S_T) - F(t, S_t) &= \int_t^T D_1 F(s, S_{s-}) ds + \frac{c^2}{2} \int_t^T S_{s-}^2 D_2^2 F(s, S_{s-}) ds \\ &\quad + \int_t^T D_2 F(s, S_{s-}) dS_s + \sum_{t < s \leq T} (F(s, S_s) - F(s, S_{s-}) - D_2 F(s, S_{s-})) \Delta S_s. \end{aligned}$$

Noting that $\Delta S_s = S_{s-} \Delta L_s$ this becomes

$$\begin{aligned} F(T, S_T) - F(t, S_t) &= \int_t^T D_1 F(s, S_{s-}) ds + \frac{c^2}{2} \int_t^T S_{s-}^2 D_2^2 F(s, S_{s-}) ds + \int_t^T D_2 F(s, S_{s-}) dS_s \\ &\quad + \sum_{t < s \leq T} (F(s, S_{s-} (1 + \Delta L_s)) - F(s, S_{s-}) - D_2 F(s, S_{s-})) \Delta L_s S_{s-}. \end{aligned}$$

Now, define the function $h(t, x, y)$ as follows

$$h(t, x, y) := F(t, x(1+y)) - F(t, x) - xyD_2F(t, x).$$

Then the following results can be deduced easily,

$$\begin{aligned} h(t, x, 0) &= 0, \\ \frac{\partial}{\partial y} h(t, x, 0) &= 0, \\ \frac{\partial^n}{\partial y^n} h(t, x, 0) &= x^n D_2^n F(t, x), n \geq 2. \end{aligned}$$

As condition (4.10) is satisfied, $h(t, x, y)$ satisfies the conditions in the previous Lemma.

Thus applying Lemma 2 gives,

$$\begin{aligned} F(T, S_T) - F(t, S_t) &= \int_t^T D_1 F(s, S_{s-}) ds + \frac{c^2}{2} \int_t^T S_{s-}^2 D_2^2 F(s, S_{s-}) ds \\ &\quad + \int_t^T D_2 F(s, S_{s-}) dS_s + \sum_{i=2}^{\infty} \int_t^T \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i!} dY_s^{(i)} \\ &\quad + \int_t^{T+\infty} \int_{-\infty}^{\infty} (F(t, x(1+y)) - F(t, x) - xyD_2F(t, x)) \tilde{\nu}(dy) ds \\ &= \int_t^T \frac{D_1 F(s, S_{s-}) + S_{s-}^2 \frac{c^2}{2} D_2^2 F(s, S_{s-}) + DF(t, x)}{rB_s} dB_s \\ &\quad + \int_t^T D_2 F(s, S_{s-}) dS_s + \sum_{i=2}^{\infty} \int_t^T \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i! B_s} dH_s^{(i)} \\ &\quad - \sum_{i=2}^{\infty} \int_t^T \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i! B_s} Y_{s-}^{(i)} dB_s. \end{aligned}$$

Finally by using the PDIE equation (4.8) for the price, we obtain

$$F(T, S_T) - F(t, S_t) = \int_t^T \frac{F(s, S_{s-}) - S_{s-} D_2^2 F(s, S_{s-}) - \sum_{i=2}^{\infty} \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i! B_s} H_{s-}^{(i)}}{B_s} dB_s \\ + \int_t^T D_2 F(s, S_{s-}) dS_s + \sum_{i=2}^{\infty} \int_t^T \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i! B_s} dH_{s-}^{(i)}.$$

Hence it can be seen from equation (4.7) that the weights of the portfolio are found as given in the theorem.

In the light of this theorem we can now determine the sequence of self-financing portfolios.

Contrary to our model, Black-Scholes model is complete and the risk-neutral dynamics of the stock price is given by,

$$\frac{dS_t}{S_t} = \left(r - \frac{1}{2} \sigma^2 \right) dt + dW_t, \quad S_0 > 0,$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion. Hedging portfolio consists of $\frac{F(s, S_s) - S_s D_2 F(s, S_s)}{B_s}$ number of bonds and $D_2 F(s, S_s)$ number of stocks.

In the Poissonian case however the risk-neutral dynamics of the stock price is the following

$$\frac{dS_t}{S_t} = (r - \lambda) dt + dN_t, \quad S_0 > 0,$$

where $N = (N_t)_{t \geq 0}$ is Poisson process with intensity parameter $\lambda > 0$.

Consider a contingent claim with a payoff $F(T, S_T)$ that only depends on the value of the stock price at maturity T . In this case since the payoff also depends on the Poisson process at time T , we can represent the payoff as $G(T, N_T) = F(T, S_T)$.

Since the stock price follows the process $S_t = S_0 \exp((r - \lambda)t) 2^{N_t}$, payoff function becomes

$$G(T, N_T + 1) = F(T, 2S_T).$$

Considering these relations we have,

$$\begin{aligned} \sum_{i=2}^{\infty} \frac{S_{s-}^i D_2^i F(s, S_{s-})}{i!} &= F(s, 2S_{s-}) - F(s, S_{s-}) - S_{s-} D_2 F(s, S_{s-}) \\ &= G(s, N_{s-} + 1) - G(s, N_{s-}) - S_{s-} D_2 F(s, S_{s-}). \end{aligned}$$

Thus the equality (4.7) becomes

$$\begin{aligned} F(T, S_T) - F(t, S_t) &= \int_t^T \frac{F(s, S_{s-}) - G(s, N_{s-} + 1) + G(s, N_{s-})}{B_s} dB_s \\ &\quad + \int_t^{T-1} (G(s, N_{s-} + 1) - G(s, N_{s-})) S_{s-}^{-1} dS_s. \end{aligned}$$

Hence, it is clear that market is already complete and that an enlargement is not necessary.

Moreover the hedging portfolio is given by $\frac{2G(s, S_{s-}) - G(s, N_{s-} + 1)}{B_s}$ number of bonds and

$(G(s, N_{s-} + 1) - G(s, N_{s-})) S_{s-}^{-1}$ number of stocks.

CHAPTER 5

ARBITRAGE

In this section we proceed assuming that the market is enlarged by power-jump assets with the constants $a^{(i)}$, $i \geq 2$. So the trade in the bond, the stock and the power-jump assets are allowed in the market where the power-jump assets have the price process

$$H_t^{(i)} = \exp(rt)(L_t^{(i)} - a^{(i)}t), \quad i \geq 2.$$

Now our aim is to examine whether after this enlargement the market remains arbitrage-free or not. It is shown in [46] that the existence of an equivalent martingale measure in continuous time is a sufficient but not a necessary condition to ensure no-arbitrage condition.

So to ensure that the enlarged market is free of arbitrage, we should show the existence of an equivalent martingale measure Q making \tilde{S} and the discounted $H^{(i)}$'s martingales.

The martingale condition for \tilde{S} has already been derived and given by (3.13).

On the other hand discounted $H^{(i)}$'s are martingale only if $L_t^{(i)} - a^{(i)}t$ are martingales for $i \geq 2$. Again by Theorem 10 we know that the Lévy measure under Q is given by $\tilde{\nu}(dx) = H(x)\nu(dx)$, hence we can write

$$\int_{-\infty}^{+\infty} x^i H(x)\nu(dx) = a^{(i)}, \quad i \geq 2. \quad (5.1)$$

The existence question of G and $H(x)$ is associated with the moment problem which is the problem of finding necessary and sufficient conditions for the existence of equivalent martingale measure with μ_n as n-th moments, given a series of number $\{\mu_n\}$.

A result is that if the moment problem has a solution with bounded support, then the solution will be unique (see [47]).

The following proposition ensures the uniqueness of such measure.

Proposition 5.1.1 Suppose that the measure $\nu(dx)$ has compact support. Then, if there exists a martingale measure in the market enlarged with the power-jump assets, the martingale measure is unique, structure preserving and the market is complete.

Proof: If there is a martingale measure in the enlarged market, using the same arguments as in [48], there exists an $H(x)$ verifying (3.13) and (5.1) with $H(x) > 0$. The measure $\mu(dx) = x^2 H(x) \nu(dx)$ is finite and has a bounded support. Then $H(x)$ is determined by condition (5.1). Also, as the support is bounded, $H(x) \nu(dx)$ verifies (3.4) and hence the model enlarged with the power-jump assets is complete. Lastly, as the contingent claim $B_T 1_A$ with $A \in F$ can be replicated, the uniqueness of its initial arbitrage price $E(1_A)$ implies the uniqueness of the martingale measure.

Generally, it is known that uniqueness of the martingale measure implies completeness.

Proposition 5.1.2 If the probability measure under which the discounted stock price and the power-jump assets are martingales is unique, the market is complete.

Proof: Assume that Q is a martingale measure in the market. If the market is not complete, there exists a contingent claim $X \geq 0, X \in L^2(Q)$, not zero for every t , which is orthogonal to any replicable contingent claim. If we define, $Q^*(d\omega) := (1 + X)Q(d\omega)$ is a martingale measure different from Q .

In fact, for any $s \leq t$, and $A \in F_s$ we have

$$E_{Q^*} \left(1_A \left(Y_t^{(i)} - Y_s^{(i)} \right) \right) = E_Q \left(1_A \left(Y_t^{(i)} - Y_s^{(i)} \right) \right) + E_Q \left(X 1_A \left(Y_t^{(i)} - Y_s^{(i)} \right) \right) = 0$$

and $\{Y_t^{(i)}, t \geq 0\}$ are Q^* -martingales for all $i \geq 2$.

Hence, it is clear that \tilde{S} is also a Q^* -martingale.

5.1. A Completion Example: Brownian Motion plus a finite number of Poisson Processes

In this example we will complete a market where the Lévy process L_t is a compound Poisson process with finite number of jumps.

Thus the process X_t is sum of a Brownian motion and the compound Poisson process given by

$$X_t = cW_t + \sum_{j=1}^n c_j N_{j,t}$$

where $c \neq 0$, $W = \{W_t, t \geq 0\}$ a standard Brownian motion and $N_j = \{N_{j,t}, t \geq 0\}$ are independent Poisson processes with intensities $a_j \geq 0$. The constants c_j are assumed to be non-zero and different from each other for $j = 1, \dots, n$. Then we have $E(L_1) = \sum_{j=1}^n c_j a_j = a$.

In this market the price processes of power-jump assets are given by

$$H_t^{(i)} = \exp(rt) \left(\sum_{j=1}^n c_j N_{j,t} - a^{(i)} t \right) \text{ for } i = 2, 3, \dots$$

It is shown in [16] that for $i > n + 1$, $H_t^{(i)}$ can be written as a linear combination of the $H_t^{(i)}$, $i = 2, 3, \dots, n + 1$. So clearly in this case we can enlarge the market with only n assets which are driven by $H_t^{(i)}$ $i = 2, 3, \dots, n + 1$.

Similarly for an equivalent measure to exist in the market the conditions (3.13) and (5.1) must be satisfied.

The support set of $H_t^{(i)}$ is now the set $\{c_2, \dots, c_{n+1}\}$ and the equations (3.13), (5.1) reduce to

$$\sum_{j=1}^n c_j H(c_j) a_j = r - cG - b,$$

$$\sum_{j=1}^n c_j^i H(c_j) a_j = a^{(i)}, \quad i = 2, \dots, n+1.$$

The existence of an equivalent martingale measure is ensured if the following system of equations for $H(c_j)$, $j = 1, \dots, n$ has a positive solution.

$$\begin{bmatrix} c_1^2 a_1 & c_2^2 a_2 & \dots & c_n^2 a_n \\ c_1^3 a_1 & c_2^3 a_2 & \dots & c_n^3 a_n \\ \dots & \dots & \dots & \dots \\ c_1^{n+1} a_1 & c_2^{n+1} a_2 & \dots & c_n^{n+1} a_n \end{bmatrix} \times \begin{bmatrix} H(c_1) \\ H(c_2) \\ \dots \\ H(c_n) \end{bmatrix} = \begin{bmatrix} a^{(2)} \\ a^{(3)} \\ \dots \\ a^{(n+1)} \end{bmatrix}.$$

By Proposition 8, the existence of a positive solution $H(c_j)$, $j = 1, \dots, n$ can be represented by to condition

$$C^{-1} \cdot a' > 0,$$

where C^{-1} is the inverse of the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_n \\ \dots & \dots & \dots & \dots \\ c_1^{n-1} & c_2^{n-1} & \dots & c_n^{n-1} \end{bmatrix},$$

and a' is the transpose of $[a^{(2)} \dots a^{(n+1)}]$. Since all c_i 's are different from each other $\det C \neq 0$.

For calculation of the inverse of Vandermonde matrices see [49] and for further details and applications of Vandermonde matrices see [50].

CHAPTER 6

PRICING THE CONTIGENT CLAIM

In this section we discuss two predominant methods for pricing European options on assets driven by Lèvy processes and obtain the explicit prices under these methods. The methods are Martingale pricing approach, fast Fourier transform based characteristic formula method.

Throughout the pricing section, we assume that the jump size of the compound Poisson process has a particular distribution which will be clarified below. Ultimately we will obtain the value at time t of a European option with strike price K and payoff function $f(S_T)$ only depending on the stock price at maturity.

By Proposition 3.5 in [13] we know that the compensated compound Poisson process \tilde{M}_t , defined by equation (3.9), can be represented in the following form

$$\tilde{M}_t = \int_{(0,t] \times \mathbb{R}} x \tilde{M}(ds, dx) = \sum_{0 < s \leq t} \Delta \tilde{M}_t - \tilde{\lambda} E^Q(\Delta \tilde{M}_t) t. \quad (6.1)$$

Here, we have only rewritten the process \tilde{M}_t as the sum of its jumps. Since a compound Poisson process has almost surely a finite number of jumps in interval $(0, t]$, the summation is finite, so there are no convergence problems.

Now let the sequence of independent random variables $U = (U_i)_{i \geq 1}$ denote the jump size process of \tilde{M}_t i.e., $U_i = \Delta \tilde{M}_t$ for $i \geq 1$.

Then equation (6.1) takes the form

$$\tilde{M}_t = \int_{(0,t] \times \mathbb{R}} x \tilde{M}(ds, dx) = \sum_{i=1}^{N(t)} U_i - \tilde{\lambda} E^Q(U_i) t.$$

Under these circumstances risk-neutral dynamics of the stock price process can be rewritten as:

$$\begin{aligned} S_t &= S_0 \exp\left(c\tilde{W}_t + \tilde{M}_t + \left(r - \frac{c^2}{2}\right)t\right) \times \prod_{0 < s \leq t} (1 + \Delta\tilde{M}_s) \exp(-\Delta\tilde{M}_s) \\ &= S_0 \exp\left(c\tilde{W}_t + \left(\sum_{i=1}^{N(t)} U_i - \tilde{\lambda} E^Q(U_i) t\right) + \left(r - \frac{c^2}{2}\right)t\right) \times \prod_{i=1}^{N(t)} (1 + U_i) e^{-U_i} \\ &= S_0 \exp\left(c\tilde{W}_t + \left(\sum_{i=1}^{N(t)} U_i - \tilde{\lambda} E^Q(U_i) t\right) + \left(r - \frac{c^2}{2}\right)t + \ln\left(\prod_{i=1}^{N(t)} (1 + U_i) e^{-U_i}\right)\right) \\ S_t &= S_0 \exp\left(c\tilde{W}_t - \tilde{\lambda} E^Q(U_i) t + \left(r - \frac{c^2}{2}\right)t + \sum_{i=1}^{N(t)} \ln(1 + U_i)\right). \end{aligned}$$

Moreover let us now assume that the intensity process of the jumps modelled by the variable $(1+U)$ follows a log-normal distribution with mean m and volatility v^2 , i.e.,

$\ln(1+U) \sim N(m, v^2)$. The expectation of U is thus given by $E^Q(U) = e^{m + \frac{v^2}{2}} - 1$.

Finally the stock price process under Q becomes

$$S_t = S_0 \exp\left(c\tilde{W}_t + \left(r - \tilde{\lambda} \left(e^{m + \frac{v^2}{2}} - 1\right) - \frac{c^2}{2}\right)t + \sum_{i=1}^{N(t)} \ln(1 + U_i)\right). \quad (6.2)$$

Under these assumptions S_t has now jump-diffusion dynamics.

6.1. Martingale Pricing Approach

The value at time t of a contingent claim X with a payoff function $f(S_T)$ is defined as

$$F(t, S_t) = \exp(-r(T-t)) E^Q(f(S_T) | F_t) \quad (6.3)$$

Using equation (3.14) the stock price process at time T takes the following form under Q ,

$$S_T = S_t \exp\left(c(\tilde{W}_T - \tilde{W}_t) + (\tilde{M}_T - \tilde{M}_t) + \left(r - \frac{c^2}{2}\right)(T-t)\right) \prod_{t < s \leq T} (1 + \Delta \tilde{M}_s) \exp(-\Delta \tilde{M}_s).$$

Substituting this into equation (6.3) we have

$$F(t, S_t) = \exp(-r(T-t)) E^Q \left(f \left(S_t e^{c(\tilde{W}_T - \tilde{W}_t) + (\tilde{M}_T - \tilde{M}_t) + \left(r - \frac{c^2}{2}\right)(T-t)} \prod_{t < s \leq T} (1 + \Delta \tilde{M}_s) e^{-\Delta \tilde{M}_s} \right) \middle| F_t \right).$$

By the stationary increments properties of $(\tilde{W}_t)_{t \geq 0}$ and $(\tilde{M}_t)_{t \geq 0}$, the equations $\tilde{W}_{T-t} = \tilde{W}_T - \tilde{W}_t$ and $\tilde{M}_{T-t} = \tilde{M}_T - \tilde{M}_t$ are satisfied. Also noting that the σ -algebra F_t is the knowledge of $S_t = x$ we obtain,

$$F(t, x) = \exp(-r(T-t)) E^Q \left(f \left(x e^{c\tilde{W}_{T-t} + \tilde{M}_{T-t} + \left(r - \frac{c^2}{2}\right)(T-t)} \prod_{t < s \leq T} (1 + \Delta \tilde{M}_s) e^{-\Delta \tilde{M}_s} \right) \right).$$

Now consider the price of the option under the Black-Scholes model (with volatility c):

$$F_{BS}(t, x) = \exp(-r(T-t)) E^Q \left(f \left(x \exp \left(c\tilde{W}_{T-t} + \left(r - \frac{c^2}{2}\right)(T-t) \right) \right) \right).$$

Then we can write the value of the option in terms of $F_{BS}(t, x)$ as

$$F(t, x) = E_Q \left(F_{BS} \left(t, x e^{\tilde{M}_{T-t}} \prod_{0 < s \leq T-t} (1 + \Delta \tilde{M}_s) e^{-\Delta \tilde{M}_s} \right) \right).$$

The derivative with respect to x , which is also needed in the formula for the number of n -th power-jump assets in the replicating portfolio, is given by

$$D_2^n F(t, x) = E_Q \left(e^{n\tilde{M}_{T-t}} \prod_{0 < s \leq T-t} (1 + \Delta \tilde{M}_s)^n e^{-n\Delta \tilde{M}_s} \times D_2^n F_{BS} \left(t, x e^{\tilde{M}_{T-t}} \prod_{0 < s \leq T-t} (1 + \Delta \tilde{M}_s) e^{-\Delta \tilde{M}_s} \right) \right).$$

In the Black-Scholes case the derivative $D_2^n F_{BS}$ are very simple. For instance for a European call the first two derivatives are given in terms of the cumulative probability distribution function $N(x)$ and the density function $n(x)$ of a Standard Normal random variable by

$$D_2^1 F_{BS}(t, x) = N(d_1) = N \left(\frac{\log \left(\frac{x}{T-t} \right) + \left(r + \frac{c^2}{2} \right) (T-t)}{c\sqrt{T-t}} \right) \quad (6.4)$$

and

$$D_2^2 F_{BS}(t, x) = N(d_2) = \frac{n(d_1)}{xc\sqrt{T-t}}, \quad (6.5)$$

where $d_1 = \frac{\log \left(\frac{x}{T-t} \right) + \left(r + \frac{c^2}{2} \right) (T-t)}{c\sqrt{T-t}}$ and $d_2 = d_1 - c\sqrt{\tau}$.

$N(d_1)$ and $N(d_2)$ are known as the *delta* and the *gamma* of the option, respectively.

In equation (6.2) it is given that under the certain assumptions the stock price process at time t has the following form under Q ,

$$S_t = S_0 \exp \left(c\tilde{W}_t + \left(r - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) - \frac{c^2}{2} \right) t + \sum_{i=1}^{N(t)} \ln(1 + U_i) \right).$$

Then by (6.3) we can write the option price as

$$F(t, S_t) = \exp(-r(T-t)) E^Q \left(f \left(S_t \exp \left(c\tilde{W}_{T-t} + \left(r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right) (T-t) + \sum_{i=1}^{N(T-t)} \ln(1+U_i) \right) \right) \middle| F_t \right).$$

Since F_t can be viewed as $\{S_t = x\}$,

$$\begin{aligned} F(t, x) &= \exp(-r(T-t)) E^Q \left(f \left(x \exp \left(c\tilde{W}_{T-t} + \left(r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right) (T-t) + \sum_{i=1}^{N(T-t)} \ln(1+U_i) \right) \right) \right) \\ &= \exp(-r(T-t)) \sum_{n=0}^{+\infty} E^Q \left(f \left(x \exp \left(c\tilde{W}_{T-t} + \left(r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right) (T-t) + \sum_{i=1}^{N(T-t)} \ln(1+U_i) \right) \right) \middle| N(T-t) = n \right) \\ &\quad \times Q(N(T-t) = n). \end{aligned}$$

Setting $\tau = T - t$ we have,

$$F(t, x) = \exp(-r\tau) \sum_{n=0}^{+\infty} E^Q \left(f \left(x \exp \left(c\tilde{W}_\tau + \left(r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right) \tau + \sum_{i=1}^n \ln(1+U_i) \right) \right) \right) \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!}.$$

The process inside of the exponential function is normally distributed so we have,

$$\left\{ r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right\} \tau + c\tilde{W}_\tau + \sum_{i=1}^n \ln(1+U_i) \sim N \left(\left\{ r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right\} \tau + nm, c^2\tau + n\nu^2 \right),$$

We can rewrite the process such that its distribution will remain the same,

$$\left\{ r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right\} \tau + nm + \sqrt{\frac{c^2\tau + n\nu^2}{\tau}} \tilde{W}_\tau \sim N \left(\left\{ r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+\nu^2}{2}} - 1 \right) \right\} \tau + nm, c^2\tau + n\nu^2 \right).$$

Now we are able to express the equation (6.3) in the following form

$$F(t, x) = \exp(-r\tau) \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} E^Q \left(f \left(x \exp \left(\left(r - \frac{c^2}{2} - \tilde{\lambda} \left(e^{\frac{m+v^2}{2}} - 1 \right) \right) \tau + nm + \sqrt{\frac{c^2\tau + nv^2}{\tau}} \tilde{W}_\tau \right) \right) \right).$$

By adding and subtracting $\frac{nv^2}{2\tau}$ into the exponential function we get

$$F(t, x) = \exp(-r\tau) \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} E^Q \left(f \left(x \exp \left(\left(r - \frac{c^2}{2} + \frac{nv^2}{2\tau} - \frac{nv^2}{2\tau} - \tilde{\lambda} \left(e^{\frac{m+v^2}{2}} - 1 \right) \right) \tau + nm + \sqrt{\frac{c^2\tau + nv^2}{\tau}} \tilde{W}_\tau \right) \right) \right)$$

$$F(t, x) = \exp(-r\tau) \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} E^Q \left(f \left(x \exp \left(\left(r - \frac{1}{2} \left(c^2 + \frac{nv^2}{2\tau} \right) + \frac{nv^2}{2\tau} - \tilde{\lambda} \left(e^{\frac{m+v^2}{2}} - 1 \right) \right) \tau + nm + \sqrt{\frac{c^2\tau + nv^2}{\tau}} \tilde{W}_\tau \right) \right) \right).$$

By setting $c_n^2 = c^2 + \frac{nv^2}{\tau}$ above equation becomes

$$F(t, x) = \exp(-r\tau) \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} E^Q \left(f \left(x \exp \left(\left(r - \frac{1}{2} c_n^2 + \frac{nv^2}{2\tau} - \tilde{\lambda} \left(e^{\frac{m+v^2}{2}} - 1 \right) \right) \tau + nm + c_n \tilde{W}_\tau \right) \right) \right)$$

$$F(t, x) = \exp(-r\tau) \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} E^Q \left(f \left(x \exp \left(\left(nm + \frac{nv^2}{2\tau} - \tilde{\lambda} \left(e^{\frac{m+v^2}{2}} - 1 \right) \right) \tau \right) \times \exp \left(\left(r - \frac{1}{2} c_n^2 \right) \tau + c_n \tilde{W}_\tau \right) \right) \right).$$

We know that in the Black-Scholes model the price formula is given by,

$$F_{BS}(t, S_t; c) = \exp(-r\tau) E^Q \left(f \left(x \exp \left(cW_\tau + \left(r - \frac{c^2}{2} \right) \tau \right) \right) \right).$$

Hence our price formula $F(t, x)$ can be represented as a weighted average of Black-Scholes price in terms of number of jumps n ,

$$F(t, S_t) = \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} F_{BS}(\tau, S_n; c_n) \quad (6.6)$$

where $S_n = S_t \exp \left(nm + \frac{nv^2}{2\tau} - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) \tau \right)$ and $c_n = \sqrt{\frac{c^2\tau + nv^2}{\tau}}$.

Alternatively, we can write

$$F(t, S_t) = \exp(-r\tau) \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} E^Q \left(f \left(x \exp \left(\left(r - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) + \frac{n(m+v^2/2)}{\tau} - \frac{1}{2}c_n^2 \right) \tau + nm + c_n \tilde{W}_\tau \right) \right) \right),$$

which yields

$$F(t, S_t) = \sum_{n=0}^{+\infty} \frac{(\bar{\lambda}\tau)^n e^{-\bar{\lambda}\tau}}{n!} F_{BS}(\tau, S_t; c_n, r_n),$$

where $r_n = r - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) + \frac{n(m+v^2/2)}{\tau}$ and $\bar{\lambda} = \tilde{\lambda} e^{m+\frac{v^2}{2}}$.

This formula is the same as the one derived from solving the PDE by forming a risk-free portfolio. This relation shows that PDE approach and Martingale approach give the same result.

By equation (6.6)

$$D_2^1 F(\tau, x) = \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n e^{-\tilde{\lambda}\tau}}{n!} \left[\exp\left(nm + \frac{nv^2}{2\tau} - \tilde{\lambda}\left(e^{\frac{m+v^2}{2}} - 1 \right)\tau \right) \times D_2^1 F_{BS}(\tau, S_n; c_n) \right],$$

where $S_n = S_t \exp\left(nm + \frac{nv^2}{2\tau} - \tilde{\lambda}\left(e^{\frac{m+v^2}{2}} - 1 \right)\tau \right)$ and $c_n = \sqrt{\frac{c^2\tau + nv^2}{\tau}}$.

By (6.4) and (6.5) we know that

$$D_2^1 F_{BS}(t, x) = N(d_1) = N\left(\frac{\log\left(\frac{x}{\tau}\right) + \left(r + \frac{c^2}{2}\right)\tau}{c\sqrt{\tau}} \right) \quad (6.7)$$

and

$$D_2^2 F_{BS}(t, x) = N(d_2) = \frac{n(d_1)}{xc\sqrt{\tau}}. \quad (6.8)$$

Then the delta and the gamma of the option under Black-Scholes model are as follows

$$N(d_1) = N\left(\frac{\log\left(\frac{S_n}{\tau}\right) + \left(r + \frac{c_n^2}{2}\right)\tau}{c_n\sqrt{\tau}} \right) \text{ and } N(d_2) = \frac{n(d_1)}{S_n c_n \sqrt{\tau}}. \quad (6.9)$$

Hence by Proposition 7.10 in [28] the price of a European call option with strike price K is

$$F^C(\tau, S_t) = S_t N(d_1) - Ke^{-r\tau} N(d_2),$$

where $N(d_1)$ and $N(d_2)$ are given as in (6.9).

Similarly by using Put-Call Parity (see [28], Proposition 9.2) European put option price can be obtained as

$$F^P(\tau, S_t) = Ke^{-r\tau}N(-d_2) - S_tN(-d_1).$$

6.2. Characteristic Formula via Fast Fourier Transform Method

In this section we will obtain the price of a European call option by using fast Fourier transform (FFT) method. This method has significant advantages compared to the classical martingale approach. First of all, when the risk neutral density is unknown, as is very often the case, we can find the price by using the Fourier transform of S_t which is known from the Lévy Khintchine representation. Besides, the algorithms used for the inversion of the Fourier transform are fast and optimized which enables us to price options with different strikes in a single calculation.

We again consider the European call option with payoff function maturity T and strike price K , written on S_T . Assume S_t has the form specified in the previous section, that is, has log-normally distributed jumps. Then as shown above we know that

$$S_t = S_0 \exp \left(c\tilde{W}_t + \left(r - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) - \frac{c^2}{2} \right) t + \sum_{i=1}^{N(t)} \ln(1+U_i) \right).$$

The characteristic function of the log-return process $s_T = \ln(S_T)$ can be found explicitly as follows:

$$\begin{aligned} \phi_T(u) &= E(\exp(ius_T)) = E \left(\exp \left(iu \left(c\tilde{W}_t + \left(r - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) - \frac{c^2}{2} \right) t + \sum_{i=1}^{N(t)} \ln(1+U_i) \right) \right) \right) \\ &= \exp \left(-\frac{1}{2}u^2c^2 + iu \left(r - \frac{c^2}{2} \right) t - \tilde{\lambda} \left(e^{m+\frac{v^2}{2}} - 1 \right) t + \tilde{\lambda} t \left(e^{ium - \frac{1}{2}u^2v^2} \right) \right). \quad (6.10) \end{aligned}$$

$$\phi_T(u) = \exp\left(iu\left(r - \frac{c^2}{2}\right)t - \frac{1}{2}u^2c^2t - iu\tilde{\lambda}t\left(e^{m+\frac{v^2}{2}} - e^{i\mu m - \frac{1}{2}u^2v^2}\right)\right)$$

Let k denote the log of the strike price K , and let $C_T(k)$ be the desired value of a T maturity call option with strike $\exp(k)$. Let the risk-neutral density of the log price be $q_T(s)$.

The characteristic function of $S_T = \ln(S_T)$ is defined by:

$$\phi_T(u) = \int_{-\infty}^{+\infty} e^{ius} q_T(s) ds.$$

The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(s)$ by the following steps:

$$C_T(k) = E^Q\left(e^{-rT} (S_T - K)_+ | F_0\right). \quad (6.11)$$

Clearly $(S_T - K)_+ = S_T - K$ under the condition that $S_T > K$ or, equivalently, $\ln(S_T) > \ln(K)$.

Thus equation (6.11) can be written as

$$C_T(k) = \int_k^{\infty} e^{-rT} (e^s - e^k) q_T(s) ds. \quad (6.12)$$

Here note that $C_T(k)$, tends to S_0 does not decay as $k \rightarrow -\infty$. Since it does not decay at the negative log strike axis $C_T(k)$ is not integrable. So the Fourier transform of $C_T(k)$, which is defined as

$$\psi_T^C(u) = \int_{-\infty}^{+\infty} e^{iuk} C_T(k) dk,$$

does not exist. To overcome this problem Carr and Madan used a dampening coefficient $\alpha > 0$ and defined a new square integrable modified call price $\bar{C}_T(k)$ as

$$\bar{C}_T(k) \equiv \exp(\alpha k) C_T(k) \text{ for } \alpha > 0. \quad (6.13)$$

To show that $\bar{C}_T(k)$ does have a Fourier transform well-defined by

$$\psi_T(u) = \int_{-\infty}^{+\infty} e^{iuk} \bar{C}_T(k) dk,$$

provided that $\alpha > 0$ is chosen appropriately, we consider the next theorem.

Theorem: Consider the payoff function $f(x, k) = (e^{b_0 x} - e^{b_1 k})_+$ where $b_0, b_1 \in \mathbb{R}^n$ are arbitrary constants. Assume that f satisfies $b_1 \in A$. Then there exists $\alpha > 0$ with $\alpha b_0 + b_1 \in A$. For any such α , the Fourier transform $\psi_T(u)$ of $\bar{C}_T(k)$ exists and is given by

$$\psi_T(u) = \frac{e^{-rT} \phi_T(u - i(1 + \alpha))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}.$$

Proof: Proof is presented in the Appendix section. For details see [42].

Consider now the Fourier transform of $\bar{C}_T(k)$ defined by:

$$\psi_T(u) = \int_{-\infty}^{+\infty} e^{iuk} \bar{C}_T(k) dk.$$

By the inversion formula it is known that the following satisfied

$$\bar{C}_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iuk} \psi_T(u) du,$$

and using equality (6.13) we have

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{+\infty} e^{-iuk} \psi_T(u) du.$$

Since the integrand is even in its real part $C_T(k)$ can be written as

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{-iuk} \psi_T(u) du. \quad (\text{DFT}) \quad (6.14)$$

Here the integration (6.14) is a direct Fourier transform which is adapted to an application of the FFT.

Now we derive an analytical expression of $\psi_T(u)$ in terms of $\phi_T(u)$ and then obtain call prices numerically using the inverse transform.

The expression for $\psi_T(u)$ is established as follows:

$$\psi_T(u) = \int_{-\infty}^{+\infty} e^{iuk} \bar{C}_T(k) dk = \int_{-\infty}^{+\infty} e^{iuk} \left\{ \int_k^{+\infty} e^{\alpha k} e^{-rT} (e^s - e^k) q_T(s) ds \right\} dk.$$

By Fubini's Theorem (see [22], Theorem 64) above equation takes the form

$$\psi_T(u) = \int_k^{+\infty} e^{-rT} (e^{\alpha k+s} - e^{k(1+\alpha)}) q_T(s) \left\{ \int_{-\infty}^{+\infty} e^{iuk} dk \right\} ds.$$

The boundaries $k < s < +\infty$ and $-\infty < k < +\infty$ of the integrals yield $-\infty < k < s < +\infty$.

$$\begin{aligned} \psi_T(u) &= \int_{-\infty}^{+\infty} e^{-rT} q_T(s) \left\{ \int_{-\infty}^s e^{iuk} (e^{\alpha k+s} - e^{k(1+\alpha)}) dk \right\} ds \\ &= \int_{-\infty}^{+\infty} e^{-rT} q_T(s) \left\{ \int_{-\infty}^s (e^{(iu+\alpha)k+s} - e^{k(1+\alpha+iu)}) dk \right\} ds \\ &= \int_{-\infty}^{+\infty} e^{-rT} q_T(s) \left\{ \frac{1}{iu+\alpha} e^{s(1+iu+\alpha)} - \frac{1}{1+\alpha+iu} e^{s(1+\alpha+iu)} \right\} ds \\ &= \int_{-\infty}^{+\infty} e^{-rT} q_T(s) \frac{e^{s(1+\alpha+iu)}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} ds \\ &= \frac{e^{-rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \int_{-\infty}^{+\infty} e^{i(u-i(1+\alpha))s} q_T(s) ds \\ &= \frac{e^{-rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \phi_T(u - i(1+\alpha)). \end{aligned}$$

where $\phi_T(u - i(1+\alpha))$ is the characteristic function of $s_T = \ln(S_T)$ for $u' = u - i(1+\alpha)$.

Hence we obtained the following analytical relation between $\psi_T(u)$ and $\phi_T(u)$,

$$\psi_T(u) = \frac{e^{-rT} \phi_T(u - i(1 + \alpha))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}. \quad (6.15)$$

To obtain the option price we will substitute (6.15) into (6.14) and perform the required integration.

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{-iuk} \frac{e^{-rT} \phi_T(u - i(1 + \alpha))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u} du.$$

Here notice that the denominator vanishes when $u = 0$ and causes a singularity in the integrand when $\alpha = 0$. So the factor $\exp(\alpha k)$ necessary since the FFT evaluates the integrand at $u = 0$. Now consider the issue of the appropriate choice of the coefficient α . A range of positive values for α provide the integrability of $\bar{C}_T(k)$ over the negative log strike axis, but for the positive log strike axis the same condition is not satisfied. For $\bar{C}_T(k)$ to be integrable in the entire log strike axis, a sufficient condition is provided by $\psi(0)$ being finite. By equation (6.15) the condition $\psi(0) < \infty$ yields that $\phi_T(-i(1 + \alpha)) < \infty$.

From the definition of the characteristic function, this inequality is equivalent to:

$$E\left(e^{i(-i(\alpha+1))s_T}\right) = E\left(e^{(\alpha+1)\ln(S_T)}\right) = E\left(S_T^{\alpha+1}\right) < \infty. \quad (6.16)$$

Using (6.15) and (6.16) an upper bound on α can be determined and Carr-Madan find that one fourth of this upper bound is a good choice.

The next issue to be considered is the infinite upper limit of integration in (6.14). The absolute value of ϕ_t is bounded by $E(S_T^{\alpha+1})$ which is independent of u , so by equation (6.15)

$$|\psi_T(u)|^2 \leq \frac{E(S_T^{\alpha+1})}{(\alpha^2 + \alpha - u^2)^2 + (2\alpha + 1)u^2} \leq \frac{A}{u^4}$$

for some constant A .

The above inequality can be written as

$$|\psi(u)| < \frac{\sqrt{A}}{u^2}.$$

Thus the integral of the upper tail by can be bounded as;

$$\int_a^\infty |\psi(u)| du < \frac{\sqrt{A}}{a}.$$

With this bound establishing a truncation procedure. Since the integral of the tail in computing the transform of (6.14) is bounded by $\frac{\sqrt{A}}{a}$ the truncation error is bounded by:

$$\frac{\exp(-\alpha k)}{\pi} \frac{\sqrt{A}}{a}.$$

By choosing $a < \frac{\exp(-\alpha k)}{\pi} \frac{\sqrt{A}}{\varepsilon}$ the truncation error can be made smaller than ε .

6.2.1. Time value of an option

The purpose of this section is to derive an alternative approach, introduced by Carr and Madan [61], which works for only out-of-the money options since for the out-of-the money options the intrinsic value is zero. The formula derived in the previous section depends on the intrinsic value of the option so a pricing formula for out-of-the money options is needed but since the authors do not see a big difference between the two methods, we will use the formula of the previous section.

Again set $s_T = \ln(S_T)$, $k = \ln(K)$, where K is the strike price of the option and S_0 the initial spot price.

$z_T(k)$ is assumed to be the price of a T maturity put when $k < \ln(S_0)$ and a T maturity call price when $k > \ln(S_0)$. The price function $z_T(k)$ is peaked as $k = \ln(S_0)$ and declines in both directions as k goes to $-\infty$ and to $+\infty$.

Next, we derive an analytic expression between the Fourier transform of $z_T(k)$ and the characteristic function of the log of the terminal stock price. The Fourier transform of $z_T(k)$ is given by

$$\zeta_T(u) = \int_{-\infty}^{+\infty} e^{iuk} z_T(k) dk$$

and by inversion formula

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iuk} \zeta_T(u) du.$$

Assuming that $S_0 = 1$, we can define $z_T(k)$ as

$$z_T(k) = e^{-rT} \int_{-\infty}^{+\infty} \left((e^k - e^s) 1_{\{s < k, k > 0\}} + (e^s - e^k) 1_{\{s > k, k > 0\}} \right) q_T(s) ds.$$

Then we have

$$\begin{aligned} \zeta_T(u) &= \int_{-\infty}^{+\infty} e^{iuk} e^{-rT} \int_{-\infty}^{+\infty} \left((e^k - e^s) 1_{\{s < k, k > 0\}} + (e^s - e^k) 1_{\{s > k, k > 0\}} \right) q_T(s) ds dk \\ &= \int_{-\infty}^{+\infty} e^{iuk} e^{-rT} dk \int_{-\infty}^k (e^k - e^s) q_T(s) ds + \int_{-\infty}^0 e^{iuk} e^{-rT} dk \int_k^{+\infty} (e^s - e^k) 1_{\{s > k, k > 0\}} q_T(s) ds \\ &= \int_{-\infty}^0 e^{-rT} q_T(s) ds \int_s^{+\infty} (e^{(1+iu)k} - e^s e^{iuk}) dk + \int_0^{+\infty} e^{-rT} q_T(s) ds \int_k^{+\infty} (e^s e^{iuk} - e^{(1+iu)k}) 1_{\{s > k, k > 0\}} dk \end{aligned}$$

Simplifying the above integral and writing the outer integration in terms of characteristic functions, we obtain

$$\zeta_T(u) = e^{-rT} \left[\frac{1}{1+iu} - \frac{e^{rT}}{iu} - \frac{\phi_T(u-i)}{u^2 - iu} \right]. \quad (6.17)$$

Since the function $z_T(k)$ approximates the shape of a Dirac delta function near $k = 0$ when maturity is small and thus the transform is wide and oscillatory.

So it is useful in this case to consider the transform of $\sinh(\alpha k)z_T(k)$ as this function vanishes at $k = 0$.

Define

$$\begin{aligned}\gamma_T(u) &:= \int_{-\infty}^{+\infty} e^{iuk} \sinh(\alpha k) z_T(k) dk \\ &= \int_{-\infty}^{+\infty} e^{iuk} \frac{e^{\alpha k} - e^{-\alpha k}}{2} z_T(k) dk \\ &= \frac{\zeta_T(u - i\alpha) - \zeta_T(u + i\alpha)}{2}.\end{aligned}$$

The time value is then given by:

$$z_T(k) = \frac{1}{\sinh(\alpha k)} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iuk} \gamma_T(u) du,$$

where $\gamma_T(u) = \frac{\zeta_T(u - i\alpha) - \zeta_T(u + i\alpha)}{2}$ and $\zeta_T(u) = e^{-rT} \left[\frac{1}{1+iu} - \frac{e^{rT}}{iu} - \frac{\phi_T(u-i)}{u^2 - iu} \right]$.

6.2.2. Option Pricing Using the FFT

The FFT is an efficient algorithm for computing the sum:

$$\omega(k) = \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j) \text{ for } k = 1, \dots, N. \quad (6.18)$$

where N is typically a power of 2. The algorithm reduces the number of multiplications in the required N summations from an order of N^2 to that of $N \ln_2(N)$, a very considerable

reduction. We present in this section the details for writing the integration (6.14) as an application of the summation (6.18). We will make use of the following theorem to approximate the summation.

Theorem (Trapezoidal Rule): Consider $y = f(x)$ over $[x_0, x_1]$ where $x_1 = x_0 + h$. The

$$\text{trapezoidal rule is } \int_{x_0}^{x_1} f(x) dx \approx \frac{h}{2} (f(x_0) + f(x_1)).$$

Dividing the interval into N segments of width η and setting $u_j = \eta(j-1)$ yields

$$u_1 = 0, u_2 = \eta, u_3 = 2\eta, \dots, u_N = (N-1)\eta.$$

Using the Trapezoidal rule for the integral on the right hand side of (6.14), we have

$$\int_0^{\infty} e^{-iuk} \psi(u) du = \eta \left(\frac{e^0 \psi(0)}{2} + \sum_{j=1}^{N-1} e^{-iuk} \psi(u_j) \right).$$

Thus an approximation for $C_T(k)$ is obtained as follows

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N-1} e^{-iu_j k} \psi_T(u_j) \eta. \quad (6.19)$$

The effective upper limit for the integration is now: $a = N\eta$. (6.20)

We are mainly interested in at-the-money call values $C_T(k)$ which correspond to k near 0. The FFT returns N values of k and we employ a regular spacing of size λ so that, our values for k are:

$$k_\nu = -b + \lambda(\nu - 1) \text{ for } \nu = 1, \dots, N. \quad (6.21)$$

This gives us log strike levels ranging from $-b$ to b where

$$b = \frac{N\lambda}{2}. \quad (6.22)$$

Substituting (6.21) into (6.19) yields

$$C_T(k_\nu) \approx \frac{e^{-\alpha k_\nu}}{\pi} \sum_{j=1}^{N-1} e^{-iu_j(-b+\lambda(\nu-1))} e^{ibu_j} \psi_T(u_j) \eta \quad \text{for } u = 1, \dots, N.$$

Noting that $u_j = \eta(j-1)$ we write

$$C_T(k_\nu) \approx \frac{e^{-\alpha k_\nu}}{\pi} \sum_{j=1}^{N-1} e^{-i\lambda\eta(j-1)(\nu-1)} e^{ibu_j} \psi_T(u_j) \eta. \quad (6.23)$$

To apply the fast Fourier transform, we note from (6.18) that

$$\lambda\eta = \frac{2\pi}{N}. \quad (6.24)$$

Hence if we choose η small in order to obtain a fine grid for the integration, then we observe call prices at strike spacings that are relatively large, with few strikes lying in the desired region near the stock price. We would therefore like to obtain an accurate integration with larger values of η and for this purpose, we incorporate Simpson's rule weightings into our summation. With Simpson's rule weightings and the restriction (6.24), we may write our call price as,

$$C(k_\nu) = \frac{e^{-\alpha k_\nu}}{\pi} \sum_{j=1}^{N-1} e^{-i\frac{2\pi}{N}(j-1)(\nu-1)} e^{ibu_j} \psi(u_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1}), \quad (6.25)$$

where δ_n is the Kronecker delta function that is unity for $n=0$ and zero otherwise. The summation in (6.25) is an exact application of the FFT. One needs to make the appropriate choices for η and α .

Now we will discuss these issues within the frame of the Lévy market model which was described in the previous section. By (6.10) we know that

$$\phi_T(u-i(1+\alpha)) = \exp\left(i(u-i(1+\alpha))\left(r-\frac{c^2}{2}\right)t - \frac{1}{2}(u-i(1+\alpha))^2 c^2 t - \tilde{\lambda}t \left(e^{\frac{m+\nu^2}{2}} - e^{i(u-i(1+\alpha))m-\frac{1}{2}u^2\nu^2}\right)\right) \quad (6.26)$$

By substituting (6.26) into (6.15) we get the fourier transform of $\psi_T(u)$ in terms of characteristic function of $s_T = \ln(S_T)$ as

$$\psi_T(u) = \frac{\exp\left(i(u-i(1+\alpha))\left(r-\frac{c^2}{2}\right)t - \frac{1}{2}(u-i(1+\alpha))^2 c^2 t - \tilde{\lambda}t \left(e^{\frac{m+\nu^2}{2}} - e^{i(u-i(1+\alpha))m-\frac{1}{2}u^2\nu^2}\right) - rT\right)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \quad (6.27)$$

Finally substituting (6.27) into (6.25) yields the call price.

$$C(k_\nu) = \frac{e^{-\alpha k_\nu}}{\pi} \sum_{j=1}^{N-1} \left\{ \frac{e^{\left(i(u_j-i(1+\alpha))\left(r-\frac{c^2}{2}\right)t - \frac{1}{2}(u_j-i(1+\alpha))^2 c^2 t - \tilde{\lambda}t \left(e^{\frac{m+\nu^2}{2}} - e^{i(u_j-i(1+\alpha))m-\frac{1}{2}u_j^2\nu^2}\right)\right)}}{\alpha^2 + \alpha - u_j^2 + i(2\alpha+1)u_j} \times \frac{e^{-rT+ibu_j-i\frac{2\pi}{N}(j-1)(\nu-1)}}{\alpha^2 + \alpha - u_j^2 + i(2\alpha+1)u_j} \frac{\eta}{3} \left(3 + (-1)^j - \delta_{j-1}\right) \right\}. \quad (6.28)$$

6.3. Comparison of Pricing Methods

In this section we compare the results of the pricing methods. We perform the FFT-price formula given by equation (6.28) and the original price formula

$$F(t, S_t) = \sum_{n=0}^{+\infty} \frac{(\bar{\lambda}\tau)^n e^{-\bar{\lambda}\tau}}{n!} F_{BS}(\tau, S_t; c_n, r_n)$$

for $\alpha = 1$. For the comparison we used common parameters and variables fixed $S_0 = 50$, $c = 0.02$, $r = 0.05$, $T = 20/252$, $\lambda = 1$, $m = -0.1$ and $\nu = 0.1$. As shown in Figures 6.1 and 6.2 the outcomes of both methods are one and the same and this is the expected result since in fact equation (6.28) is just a series representation of the original price formula.

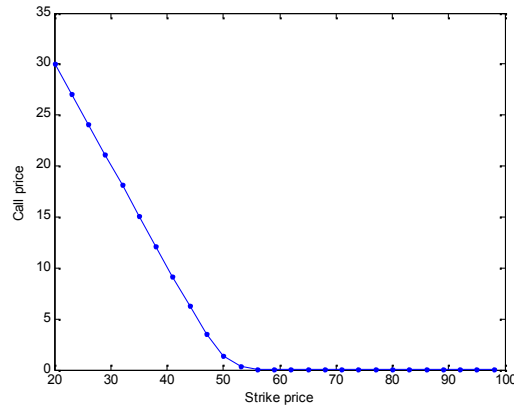


Figure 6.1: Original Call Price

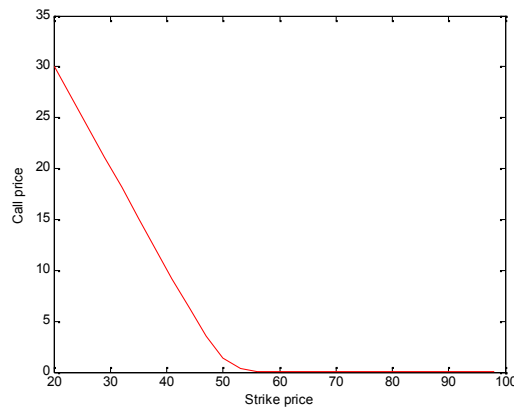


Figure 6.2: FFT - Call price

CHAPTER 7

CALIBRATION

7.1. Stochastic Volatility Extension

In the previous sections we studied under a geometric Lévy market model and in particular jump-diffusion model. We completed the Lévy market and valued European options under the complete jump-diffusion market model. Even though the class of Lévy processes is considerably rich, it is sometimes unsatisfactory in multiperiod financial modeling. First of all because of the stationarity of increments, the stock price returns for a fixed time horizon always have the same law. That is why we can not integrate any kind of new market information into the return distribution. On the other hand since the law of a Lévy process X_t for any given time horizon t is completely determined by the law of X_1 , moments and cumulants depend on time in a well-defined manner which does not always coincide with the empirically observed time dependence of these quantities [53].

For these reasons, several models which combines jumps and stochastic volatility attracted interest in the literature. The Bates model is one of the most popular examples of the class. Therefore in this chapter in addition to the Jump-Diffusion and Black-Scholes we will calibrate Stochastic Volatility Jump-Diffusion model of Bates [6] to compare the empirical findings of each models.

In Bates model an independent jump component is added to the Heston stochastic volatility model:

$$\frac{dS_t}{S_{t-}} = \mu dt + \sqrt{V_t} dW_t^1 + dQ_t,$$
$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t}dW_t^2,$$

where (W_t^1) and (W_t^2) are Brownian motions with non-zero correlation and Q_t is a compound Poisson process with intensity λ and log-normal distribution of jump sizes. Namely if U is its jump size then $\ln(1+U) \sim N\left(\ln(1+\bar{U}) - \frac{1}{2}\delta^2, \delta^2\right)$. By the no-arbitrage condition the drift term becomes $r - \lambda\bar{U}$ under the risk-neutral probability.

Applying Itô's lemma to Equation (9.1) we obtain the log-price equation

$$d \log(S_t) = \left(r - \lambda\bar{U} - \frac{1}{2}V_t \right) dt + \sqrt{V_t} dW_t^1 + d\tilde{Q}_t,$$

where $Q = (\tilde{Q}_t)_{t \geq 0}$ is a compound Poisson process with intensity $\tilde{\lambda}$ and Gaussian distributed jump sizes. This model can also be considered as a stochastic volatility extension of the Merton's jump-diffusion model. The only difference is jumps of the log-price process do not have to be Gaussian so they can be replaced by any other distribution.

7.1.1. Option Pricing

In this stochastic volatility model the characteristic function of the log-price is known in closed form which is derived below. Therefore, European options can be priced using the fast Fourier transform the process is described in Section 6.2.

7.1.2. Characteristic function of the log-price

The log-price $s_t := \log(S_t)$ can be written as a sum of a continuous part and a jump part. If we represent the continuous part of s_t as s_t^c , defining the function $f(x, \nu, t) = E\left(e^{ius_t^c} \mid s_t^c = x, V_t = \nu\right)$ and applying Itô's formula to $f(s_t^c, V_t, t) = E\left(e^{ius_t^c}\right)$ yields the characteristic function of the continuous part as

$$df(s_t^c, V_t, t) = \left(\frac{1}{2} V_t \frac{\partial^2 f}{\partial x^2} + \gamma \theta \frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2} \theta^2 V_t \frac{\partial^2 f}{\partial v^2} + \left(r - \lambda \bar{U} - \frac{V_t}{2} \right) \frac{\partial f}{\partial x} + \xi (\eta - V_t) \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} \right) dt + \sqrt{v} \frac{\partial f}{\partial x} dW^1 + \theta \sqrt{v} \frac{\partial f}{\partial v} dW^2,$$

where γ is the correlation between W^1 and W^2 .

As $f(s_t^c, V_t, t)$ is a martingale the drift term must be equal to zero. Then we get

$$\frac{1}{2} v \frac{\partial^2 f}{\partial x^2} + \gamma \theta \frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2} \theta^2 V_t \frac{\partial^2 f}{\partial v^2} + \left(r - \lambda \bar{U} - \frac{V_t}{2} \right) \frac{\partial f}{\partial x} + \xi (\eta - V_t) \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} = 0. \quad (7.1)$$

Moreover by definition we have the following terminal condition

$$f(x, u, T) = E\left(e^{ius_T^c} \mid s_T^c = x, V_T = u\right) = E\left(e^{iux}\right).$$

With this condition we are able to solve (7.1) and find the characteristic function of the log-price s_t .

Assume f is of the form

$$f(x, u, t) = \exp\{A(T-t) + vB(T-t) + iux\}, \quad (7.2)$$

where A and B are functions of time t only. Substituting this into equation (7.1) we obtain ordinary differential equations for A and B :

$$A'(s) = \xi \eta B(s) + iu(r - \lambda \bar{U})$$

$$B'(s) = \frac{1}{2} \theta^2 B^2(s) + (i\gamma \theta u - \xi) B(s) - \frac{u^2 + iu}{2}$$

with initial conditions $A(0) = B(0) = 0$. It is given in [29] that these equations have the solutions

$$A(s) = ius(r - \lambda\bar{U}) + \frac{\xi\eta s(\xi - i\gamma\theta u)}{\theta^2} - \frac{2\xi\eta}{\theta^2} \ln\left(\cosh\frac{\varsigma s}{2} + \frac{\xi - i\gamma\theta u}{\varsigma} \sinh\frac{\varsigma s}{2}\right)$$

$$B(s) = -\frac{u^2 + iu}{\varsigma \coth\frac{\varsigma s}{2} + \xi - i\gamma\theta u},$$

where $\varsigma = \sqrt{\theta^2(u^2 + iu) + (\xi - i\gamma\theta u)^2}$.

Now we have the explicit formula for the characteristic function of s_t^c . To incorporate the jump part, all we have to do is to multiply (7.1) with the characteristic function of the jumps since jumps are homogeneous and independent from the continuous part.

Let $\phi_t^J(u)$ denote the characteristic function of the jump part i.e. the compound Poisson process. Then we can write

$$\phi_t^J(u) = \exp\left(\lambda t(\varphi_U(u) - 1)\right),$$

where $\varphi_U(u) = \exp\left(-\delta^2 u^2 / 2 + i\left(\ln(1 + \bar{U}) - \frac{1}{2}\delta^2\right)u\right)$ is the characteristic function of the jump sizes which are distributed as $\ln(1 + U) \sim N\left(\ln(1 + \bar{U}) - \frac{1}{2}\delta^2, \delta^2\right)$.

Hence the characteristic function of the log-price is the following can be found as:

$$\phi_t(u) = \phi_t^J(u) \frac{\exp\left(\frac{\xi\eta t(\xi - i\gamma\theta u)}{\theta^2} + iut(r - \lambda\bar{U}) + iux_0\right)}{\left(\cosh\frac{\varsigma t}{2} + \frac{\xi - iu\gamma\theta}{\gamma} \cosh\frac{\varsigma t}{2}\right)^{\frac{2\xi\eta}{\theta^2}}}$$

$$\times \exp\left(-\frac{(u^2 + iu)\nu_0}{\gamma \coth\frac{\gamma t}{2} + \xi - iu\gamma\theta}\right).$$

The above characteristic function is exactly what is needed in the FFT option pricing formula (6.28). Therefore under this model a call option price can be easily obtained by substituting (7.2) into (6.28).

7.2. Calibration of the Models

In this section we illustrate the theory of the previous parts and calibrate the jump-diffusion (JD) model to an option set on Standard & Poors 500. We also calibrate the stochastic volatility jump-diffusion (SVJD) model of Bates [6] on the same data set to investigate the additional stochastic volatility effect and Black-Scholes model to compare the data fitting performances.

In the calibration of Lévy-based models, qualitative features of the model must be considered such as infinite/finite activity and time homogeneity. Moreover, as discussed in [38] testing the concerned data for the presence of jumps will provide accuracy in model choice. Several authors have investigated the presence of a jump component looking at the S&P 500 option data. They showed that next to a continuous diffusion component, the jump component exists but may not be present every single day in the sample (see [40]). Based on this study, it can be concluded that it is not possible to determine whether the jump component is of finite or infinite activity, as the small jumps are completely reflected by the diffusion part. It is shown in [41] that Lévy processes produce a reasonably better implied volatility smile for a single maturity, but when it comes to calibrating several maturities at the same time, the calibration by Lévy processes becomes much less precise. This difficulty of calibrating an exponential Lévy model to options of several different maturities arises due to independence and stationarity properties of the increments.

7.2.1. Data Set

The concerned models are calibrated to S&P 500 index option prices observed in the market on 20 August 2010 at 2.42 pm and at that time, the S&P 500 Index quote was 1075.63. The data consist of call options on with 21 different strike prices, ranging from 925 to 1500 and with the following maturities:

$T_1 = 21$ days, $T_2 = 51$ days, $T_3 = 82$ days, $T_4 = 203$ days, $T_5 = 295$ days, $T_6 = 478$ days, $T_7 = 662$ days and $T_8 = 845$ days.

Selected options have at least 20-days maturity since prices of the options near to the maturity are very close to the intrinsic value. Also options whose implied volatilities are unrealistic or cannot be calculated are eliminated such as Deep-ITM options (close to intrinsic value) and Deep-OTM (close to zero). Lastly, options with a very large bid-ask spread are ignored since it often implies inaccurate option price. The data is presented in the Appendix.

7.2.2. Parameter Estimation

The parameter estimation procedure is carried out by minimizing numerically the squared norm of the difference between market and model prices for both JD and SVJD models.

The vector of unknown parameters θ are thus determined by minimizing the following expression

$$\sum_{i=1}^N \left(C_{market}^i - C_{model}^{i,\theta} \right)^2.$$

The numerical method used in this process is the *DIRECT optimization algorithm*. In this next section we outline and give motivation of the algorithm.

7.2.3. The Algorithm

DIRECT algorithm [44] is a deterministic sampling method for finding the global minimum of a multivariate function subject to simple bounds. It was created in order to solve difficult global optimization problems with bound constraints and a real-valued objective function.

The algorithm attempts to solve the following problem:

Problem: Let $a, b \in \mathbb{R}^N$, $\Omega = \{x \in \mathbb{R}^N : a_i \leq x_i \leq b_i\}$ and $f : \Omega \rightarrow \mathbb{R}$ be Lipschitz continuous with constant α . Find $x_{opt} \in \Omega$ such that

$$f_{opt} = f(x_{opt}) \leq f^* + \varepsilon,$$

where ε is a given small positive constant.

Since DIRECT is a sampling algorithm, it doesn't require any knowledge of the objective function gradient. In place of this, it samples points in the domain, and uses the information it has obtained to decide where to search next. A global search algorithm like DIRECT can be very useful when the objective function is a simulation. The algorithm will globally converge to the minimal value of the objective function [3].

We consider a bound-constrained optimization problem,

$$\min_{x \in \Omega} f(x), \quad f : \mathbb{R}^N \rightarrow \mathbb{R}$$

where

$$\Omega = \{x \in \mathbb{R}^N : l \leq x \leq u\}$$

and f is Lipschitz continuous on Ω .

DIRECT begins by scaling the domain Ω , to the unit hypercube. Thus, we will assume that

$$\Omega = \{x \in \mathbb{R}^N : 0 \leq x_i \leq 1\}.$$

This transformation doesn't affect results, it just abbreviates the analysis. DIRECT's sample points are centers of hyperrectangles and it initiates its search by sampling the objective function at the center of Ω . The entire domain is treated as the first hyperrectangle, which DIRECT identifies as potentially optimal and divides.

In the division phase, it determines hyperrectangles that has the most potential to contain unsampled points and in the sampling stage it samples f at the centers of the newly-created hyperrectangles.

Figure 7.1 illustrates the division process.

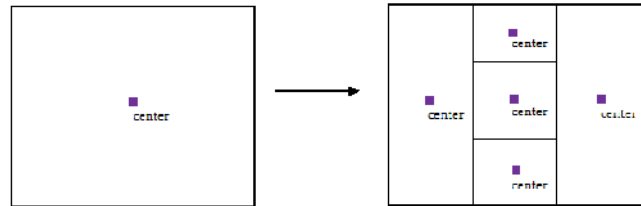


Figure 7.1: Two dimensional division example of DIRECT algorithm

7.2.4. Results

The main goal of models with stochastic volatility is to resolve the implied volatility smile phenomenon. These models provide explanations for the implied volatility smile phenomenon since the implied volatility is both different from the historical volatility and changes as a function of strike and maturity. But the performance of stochastic volatility models at short maturities is not very different from that of the Black-Scholes model, the effect of stochastic volatility becoming visible only at longer time scales: short-term skews cannot match empirically observed ones.

In Figures 7.2 and 7.3 we plot the implied volatility surfaces of jump-diffusion and stochastic volatility jump-diffusion models for fixed parameters.

Comparing the Figures, it can be seen that the stochastic volatility jump diffusion model yields a flatter smile however a smile still exists.

It can be seen from Figure 7.2 that the implied volatility surface generated by JD model shows a peak at the money for short maturities which is observed in real markets rarely.

On the other hand the implied volatility surface of Bates model, given by Figure 7.3, is more similar to surfaces observed on the market because the implied volatilities decrease for increasing time to maturity. Moreover, in this surface the smile flattens with increasing time to

maturity and this feature can be regarded as a stylized characteristic of implied volatility surfaces.

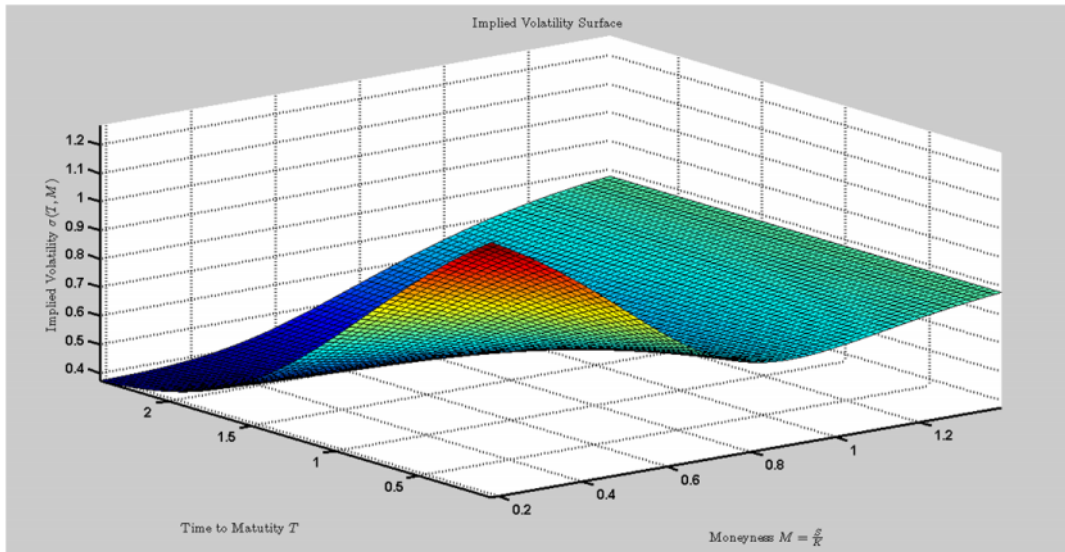


Figure 7.2: Implied volatility surface generated by the Jump-Diffusion process. ($r = 0.05$, $\lambda = 0.1$, $\sigma = 0.15$, mean jump size $m = -0.05$, jump size standard deviation $\nu = 0.4$)

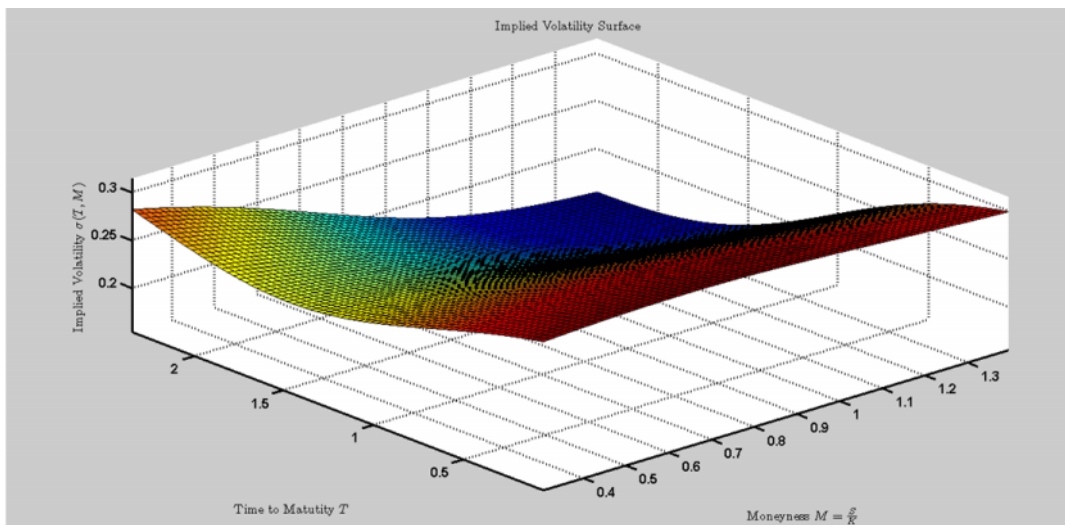


Figure 7.3: Implied volatility surface generated by Bates model.

(initial volatility $\sqrt{V_0} = 0.124$, rate of volatility mean reversion $\xi = 3.72$, long-run volatility $\sqrt{\eta} = 0.118$, volatility of volatility $\theta = 0.501$, correlation $\rho = -0.488$, jump intensity $\lambda = 0.038$, mean jump size $\mu_j = -0.05$, jump size standard deviation, $\nu_j = 0.4$)

As a result, the smile of JD model is more definite and its shape does not resemble that of the market. Therefore, explaining the implied volatility smile phenomenon requires a model that allows both jumps and stochastic volatility.

The calibration procedure of jump-diffusion model is carried out both simultaneously for all maturities and separately for each maturity in the S&P 500 option data sample. As seen in Figures 7.4 and 7.5, the calibration for each individual maturity gives quite good results, despite the fact that the options with different maturities correspond to the same underlying and the same trading day, the parameter values for each maturity are different, as seen from Table 7.1.

Figure 7.6 presents the result of simultaneous calibration of the model to 8 different maturities ranging from 1 month to 3 years. As can be seen from the Figure, the calibration error is much higher than in Figures 7.4 and 7.5.

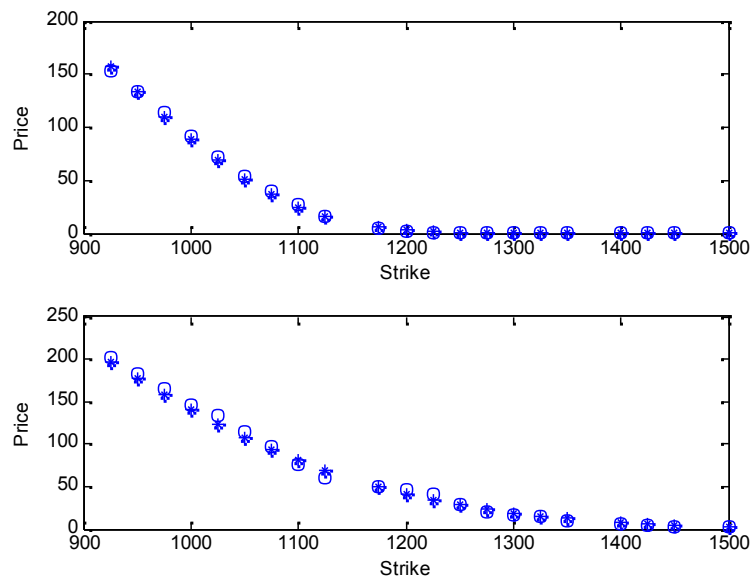


Figure 7.4: Calibration of jump-diffusion model to market data separately for each maturity. Top: maturity 51 days. Bottom: maturity 203 days. (circles are market prices, stars are model prices)

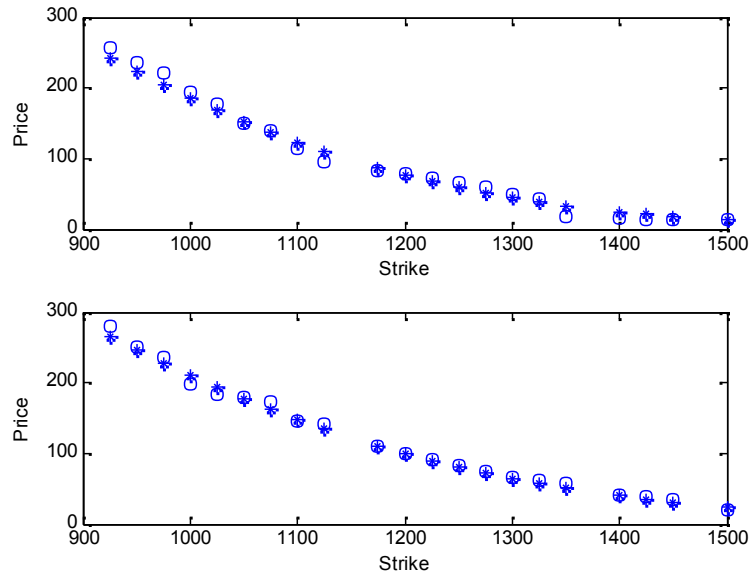


Figure 7.5: Calibration of jump-diffusion model to market data separately for each maturity. Top: maturity 662 days. Bottom: maturity 845 days. (circles are market prices, stars are model prices)

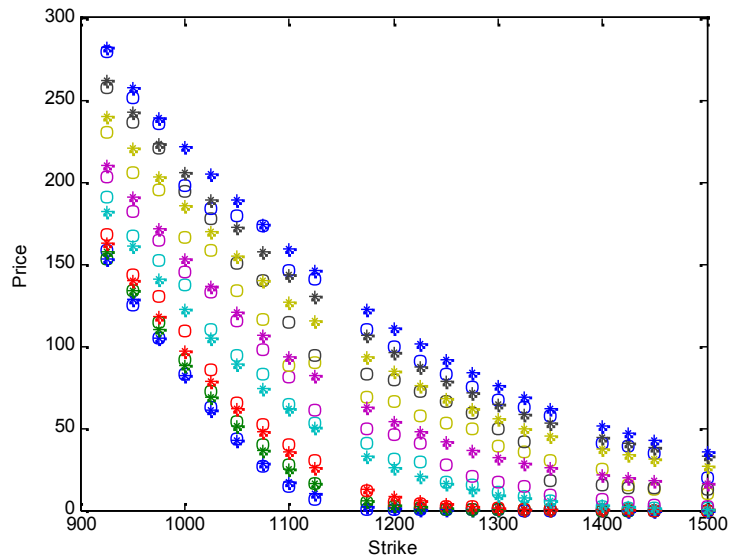


Figure 7.6: Calibration of jump-diffusion model simultaneously to 8 maturities. (circles are market prices, stars are model prices)

This problem arises due to the fact that log-price process, which is a Lévy process, has independent and stationary increments. So the law of the entire process is completely determined by its law at any given time t . If we have calibrated the model parameters for a single maturity T , this fixes completely the risk-neutral stock price distribution for all other maturities.

To adequately calibrate a jump-diffusion model to options of different maturities at the same time, the model must have a sufficient number of degrees of freedom to generate different term structures. The Bates model on the other hand has a significant number of degrees of freedom to generate different term structures, so calibrating simultaneously for different maturities is possible. The volatility smile for short maturities is accounted by the jump component while the smile for longer maturities and the term structure of implied volatility is taken into consideration using the stochastic volatility process.

Figure 7.7 shows the calibration of the Bates model to the same data set. As we see, the calibration quality is better and almost as good as when each maturity was calibrated separately for JD.

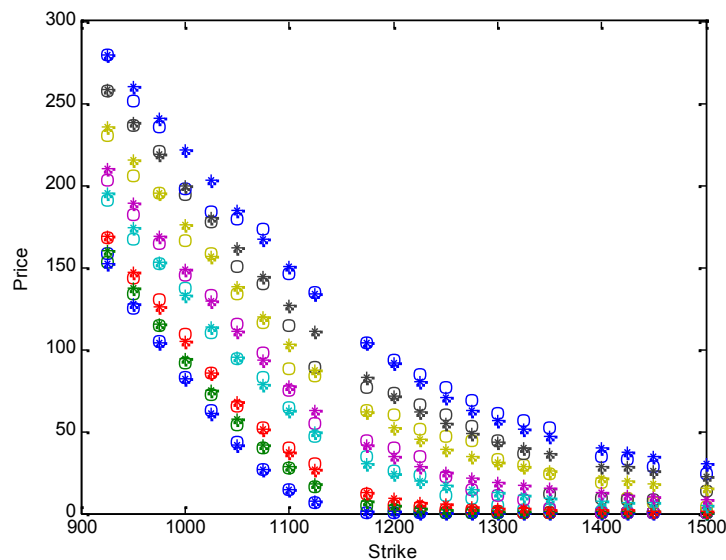


Figure 7.7: Calibration of the Bates stochastic volatility jump-diffusion model simultaneously to 8 maturities. (initial volatility $\sqrt{V_0} = 0.0633$, rate of volatility mean reversion $\xi = 0.7446$, long-run volatility $\sqrt{\eta} = 0.01$, volatility of volatility $\theta = 0.8673$, correlation $\rho = -0.578$, jump

intensity $\lambda = 0.0624$, mean jump size $\mu_J = 0.2147$, jump size standard deviation $v_J = 0.0815$.) (circles are market prices, stars are model prices)

Comparing the Figures 7.7 and 7.6 we can conclude that simultaneous calibration of the Bates model gives superior result than the simultaneous calibration of jump diffusion model.

Estimated parameter values for separate and simultaneous calibration of JD and are given in Tables 7.1 and 7.2, respectively.

Table 7.1: Separately calibrated jump-diffusion model parameters for different times to maturity.

Maturity	σ	λ	μ_J	v_J
21 days	0.2550	0.0261	0.0750	0.1289
51 days	0.2006	0.1550	-0.2167	0.2600
82 days	0.2006	0.1550	-0.0500	0.2600
203 days	0.1824	0.2517	-0.1056	0.2067
295 days	0.1824	0.1550	0.0056	0.2067
478 days	0.1824	0.1872	-0.1611	0.3133
662 days	0.1461	0.1550	-0.2667	0.1000
845 days	0.1461	0.2194	0.1167	0.4733

As we can see from the estimated parameter values, the qualitative behaviour for short and long maturities is different. For longer maturities the jump intensity tends to increase while the mean jump size decreases for extensive holding period.

Table 7.2: Simultaneously calibrated jump-diffusion model parameters for different times to maturity.

	σ	λ	μ_J	ν_J
Estimates	0.2369	0.2517	-0.1015	0.42

Table 7.3 presents the parameter estimates for Bates model. The estimation procedure of the stochastic volatility parameters converged to the same estimates for quite different starting values, so these are relatively better estimations.

Table 7.3: Simultaneously calibrated Bates model parameters for different times to maturity.

	$\sqrt{V_0}$	$\sqrt{\eta}$	ρ	ξ	θ	λ	μ_J	ν_J
Estimates	0.0633	0.0100	-0.578	0.7446	0.8673	0.0624	0.2187	0.0815

We also test the fitting performance of the Black-Scholes (BS) model and compare the results with JD and SVJD models. The historical volatilities of the underlying index for the last 15 and 35 days are used as inputs for the model which are given Table 7.4. Figure 7.8 shows the graph of historical volatilities and prices of S&P 500 index in last four months.

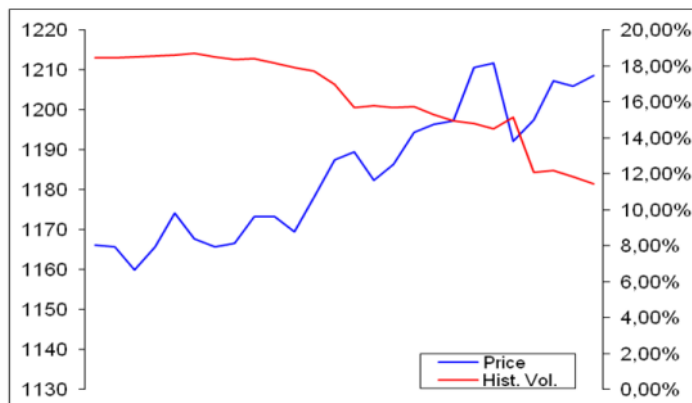


Figure 7.8: Historical volatility and price of S&P 500 index in last 4 months.

Table 7.4: Historical volatilities of S&P 500 index used as an input in the Black-Scholes model.

SPX	15 days	25 days	35 days
Hist.Vol.	11.467%	17.89%	18.47%

Figure 7.9 presents the calibration of BS model for two different historical volatilities (15 days and 35 days) used as an input.

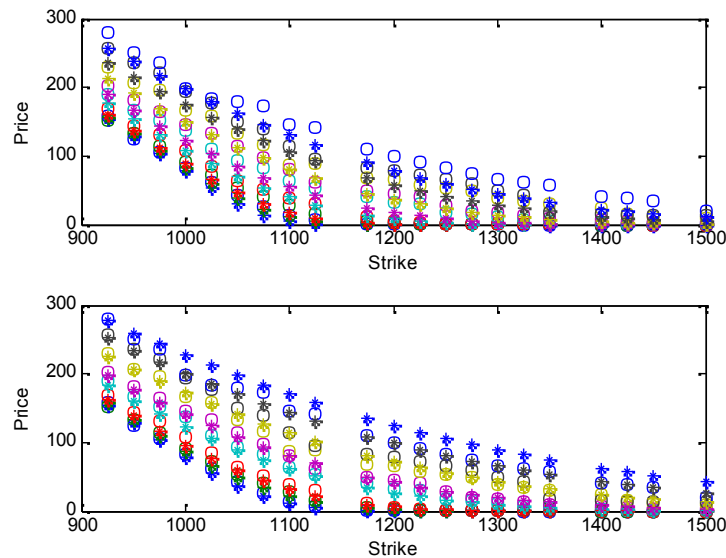


Figure 7.9: Calibration of Black-Scholes model simultaneously to 8 maturities. Top: historical volatility 11.46%. Bottom: historical volatility 18.47%. (circles are market prices, stars are model prices)

From the Figure 7.9 we can easily infer that the Black-Scholes (BS) model fails to fit market option prices, especially considering the performances of the JD and SVJD models. Calibrating the model with a σ parameter estimated from the market option prices is an alternative but we still get a really bad fitting.

Next, the call option prices generated by SVJD, JD and BS models are compared for the maturity times 51 days, 203 days, 478 days and 845 days, respectively. As it was expected, Figures 7.10-7.13 reveal that call prices of SVJD and JD models are higher than those of Black-Scholes model with respect to the strike price. The reason for this is the stochastic volatility and jump increase the risk premium. It is also found that the differences are bigger for longer the

maturity time. Moreover the SVJD and JD models have higher option prices, especially for longer maturity and near at-the-money strike price.

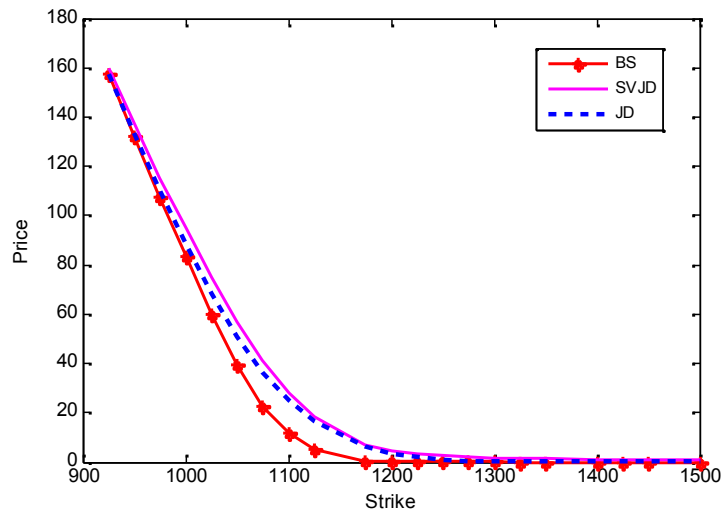


Figure 7.10: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 51 days.

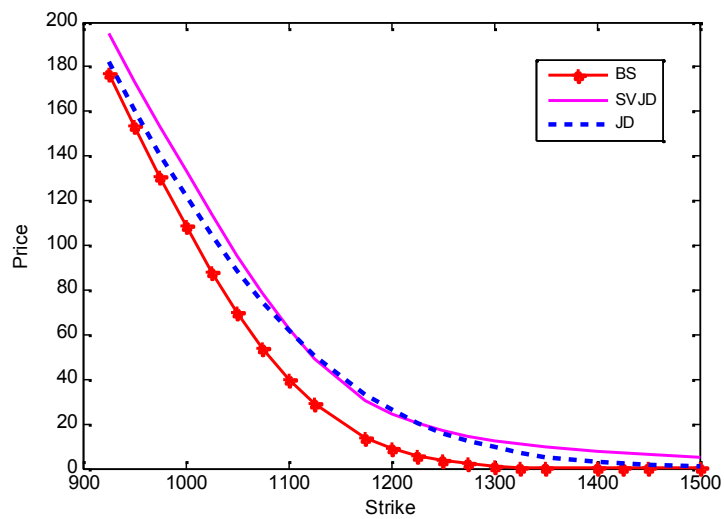


Figure 7.11: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 203 days.

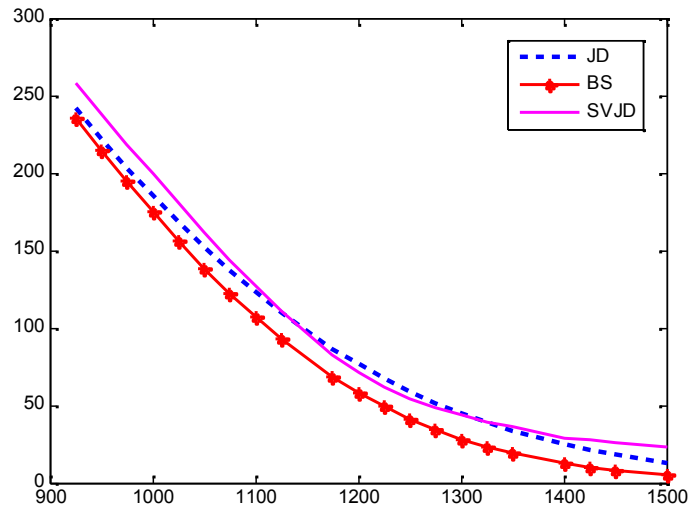


Figure 7.12: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 478 days.

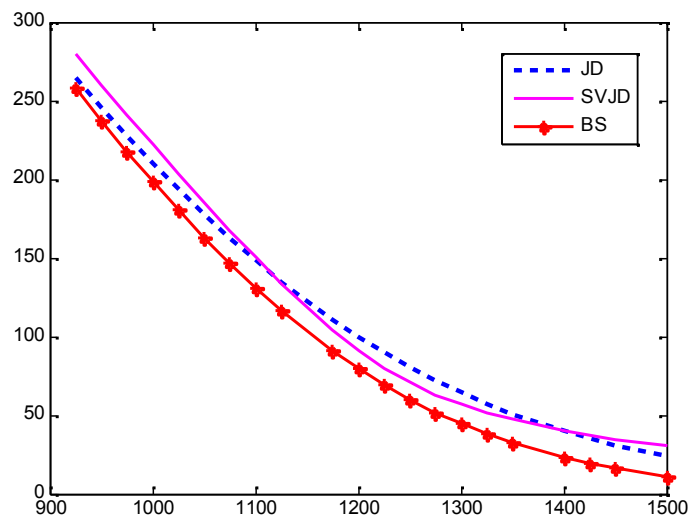


Figure 7.13: Call option prices for the SVJD, JD and BS models calculated with corresponding estimated parameters with maturity 845 days.

Moreover, Figures 7.11-7.13 display that prices generated by JD model are higher than prices of SVJD model for at the money options.

CHAPTER 8

CONCLUSION

In this thesis, a geometric Lévy market model is considered in three parts. In the first part, the market setup is examined. Since generally these models are incomplete, *i.e.*, all contingent claims cannot be replicated by a self-financing strategy, the market is enlarged by a series of artificial assets called “Power-Jump assets” which are related to the power-jump processes of the underlying Lévy process. These assets are linked to options on the stock and contracts on realized variance that are traded in OTC markets and thus can be traded for volatility expectation purposes.

It is shown by using the martingale representation property that the enlarged market is complete. Then the equivalent martingale measure conditions are given and market is shown to be arbitrage-free. Next the explicit hedging portfolios for contingent claims whose payoff function depends on the prices of the stock are derived.

In the second part, we obtained prices for European options by using two different methods under a specified jump-size distribution for the jump component. The methods were Martingale approach and characteristic formula via fast Fourier transform (FFT). Moreover, we made comparisons of the performances and speeds of these methods and found that the fast Fourier transform produces very small pricing errors so the results of both methods are nearly identical.

In the third part, we considered the stochastic volatility extension of the jump diffusion model and performed calibration of the jump-diffusion model, stochastic volatility jump-diffusion model of Bates and the Black-Scholes model on Standard&Poors (S&P) 500 options data. The optimization algorithm used is discussed and the parameter estimation values are presented. We found that the behaviour for short and long maturities is different. For longer maturities the jump intensity tends to increase meanwhile the mean jump size decreases for extensive holding period.

We also examined the effect of additional stochastic volatility assumption and reveal that explaining the implied volatility smile phenomenon requires a model that allows both jumps and stochastic volatility.

The calibration of jump-diffusion model is executed both simultaneously for all maturities and separately for each maturity in the S&P 500 option data sample. We found that the calibration for each individual maturity gives better results than separate calibration does.

The reason for this problem is the fact that log-price process, which is a Lévy process, has independent and stationary increments. So the law of the entire process is determined by its law at any given time t . That is why, when we calibrate the model parameters for a single maturity T , this fixes completely the risk-neutral stock price distribution for all other maturities.

Simultaneous calibration of SVJD model on the other hand gives quite good results since the model a significant number of degrees of freedom to generate different term structures. Also, we showed that the data fitting performance of SVJD model is better than that of the other models.

We compared the call option prices generated by SVJD, JD and BS models for the maturities 51 days, 203 days, 478 days, 845 days, respectively and showed that call prices of SVJD and JD models are higher than those of Black-Scholes model with respect to the strike price. This is observed because the stochastic volatility and jump increase the risk premium. It is also found that the differences are bigger for longer the maturity time.

To conclude, the calibration part revealed that both stochastic volatility and jump component are needed factors in a model in order to describe market behaviour adequately.

In this thesis, numerical methods for solving the PDIE's and pricing in time changed-Lévy models are not touched on. Thence accordingly, these subjects could be two possible extensions based on the results of this thesis.

APPENDIX

The following table contains the call option prices on the S&P 500 on 20 August 2010 at 2.42 pm. On that day, the S&P 500 Index closed at 1075.63. We set $q = 0$ and the risk free rate estimated with the future quotes was 5.0%.

Table 8.1: SPX option prices

STRIKE	TIME TO MATURITY							
	Sep.10	Oct.10	Nov.10	Mar.10	Jun.11	Dec.11	Jun.12	Dec.12
925	158.45	152.65	168.00	190.90	202.70	230.30	257.65	279.05
950	125.40	134.15	143.65	166.70	182.00	205.75	236.20	251.20
975	105.20	114.50	130.50	152.10	164.55	195.60	220.60	235.50
1000	82.55	91.50	108.90	137.20	145.60	165.90	194.24	198.15
1025	62.50	72.25	85.60	110.00	133.00	158.00	177.50	183.35
1050	43.65	54.20	65.00	94.00	115.00	134.00	150.00	179.50
1075	27.00	40.00	52.00	82.50	97.60	116.34	140.00	172.95
1100	14.50	27.50	39.50	64.90	81.23	88.00	114.10	145.80
1125	6.50	16.00	30.50	52.80	61.00	89.70	94.60	141.00
1175	0.80	5.00	12.00	40.80	50.00	68.64	83.00	110.00
1200	0.35	2.15	4.50	31.50	46.00	66.20	79.20	99.50
1225	0.15	1.15	4.00	29.50	41.00	57.10	72.00	90.50
1250	0.10	0.55	1.95	17.00	28.00	53.00	66.00	83.00
1275	0.05	0.45	1.80	15.00	20.65	50.00	59.30	74.70
1300	0.05	0.25	1.20	11.00	17.00	39.00	50.00	67.00
1325		0.24	0.60	8.00	14.20	35.10	42.00	62.45
1350		0.20	0.55	6.00	8.85	30.40	18.00	57.80
1400			0.30	2.32	6.30	25.00	15.60	40.60
1425			0.15	1.50	5.00	15.00	14.00	38.90
1450				1.20	3.40	12.54	13.90	34.55
1500				1.00	2.10	10.00	13.00	20.10

REFERENCES

- [1] Black, F., Scholes, M., The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, 637–654, (1973)
- [2] Hull J, and A. White. (1988), The Pricing of Options on Assets with Stochastic Volatility, *Journal of Finance*, Jun., 281-300.
- [3] Heston, S. (1993), A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6, 327-343.
- [4] Merton, R.C. (1976), Option Pricing When Underlying Asset Returns are Discontinuous, *Journal of Financial Economics*, 3, 125-144.
- [5] Barndorff-Nielsen, O.E. and Shephard, N. (2001b), Non-Gaussian OU based models and some of their uses in Financial economics. *Journal of the Royal Statistical Society* 63 (2), 167-241.
- [6] Bates, D. (1996), Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutschmark Options, *Review of Financial Studies*, 9, 69-108.
- [7] Bates, D. (2000), Post-'87 Crash fears in S&P 500 Futures Options, *Journal of Econometrics*, 94, 181-238.
- [8] Duffie, D., J. Pan, and K. Singleton (2000), Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica*, 68, 1343-1376.
- [9] Karatzas, I., and Shreve, S.E., *Methods of Mathematical Finance*, Springer-Verlag, New York, (1998).
- [10] Press, S.J., A compound event model for security prices. *The Journal of Business*, 40:317-335, 1967.

- [11] Mandelbrot, B., The variation of certain speculative prices. *The Journal of Business*, 36:394-419, 1963.
- [12] Kyprianou, A., Schoutens, W., and Wilmott, P., Exotic option pricing and advanced Lévy models. Wiley, 2005.
- [13] Cont, R. and Tankov, P., Financial modelling with Jump Processes. Chapman & Hall, 2004.
- [14] Corcuera, J.M., Nualart, D., Schoutens, W. (2005), Completion of a Lévy market by power-jump assets, *Finance and Stochastics*, 9(1), 109-127.
- [15] Schoutens, W., Lévy Processes in Finance, Pricing Financial Derivatives, Wiley, (2003).
- [16] Léon, J. A., Vives, J., Utzet, F. and Solé, J. L. (2002), On Lévy processes, Malliavin calculus and market models with jumps. *Finance and Stochastics* 6, 197–225.
- [17] Hull, John C. and White, A. (1987), “The Pricing of Options on Assets with Stochastic Volatilities”, *Journal of Finance* 42:281-300.
- [18] Wiggins, J. (1987), "Option values under stochastic volatility: Theory and empirical estimates", *Journal of Financial Economics* 19: 351 - 372.
- [19] Melino, A. and S. M. Turnbull (1990), "Pricing Foreign Currency Options with Stochastic Volatility", *Journal of Econometrics* 45: 239 - 265.
- [20] Papapantoleon, A. (2008)., An introduction to Lévy processes with applications in finance. Lecture notes, TU Vienna.
- [21] Sato, K. (1999)., Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press.
- [22] Protter, Ph., Stochastic integration and differential equations. Berlin Heidelberg New York: Springer 1990

- [23] Demeterfi, K., Derman, E., Kamal, M., Zou, J., A guide to variance swaps, *Journal of Derivatives*, 6(4), 9–32, (1999).
- [24] Nualart, D., Schoutens W., Chaotic and predictable representations for Lévy processes. *Stochast. Proc. Appl.* 90, 109–122 (2000)
- [25] Revuz, D. and Yor, M., *Continuous Martingales and Brownian Motion*, Springer: Berlin, 1999.
- [26] Björk, T., Di Masi, G., Kabanov, Y., Runggaldier, W., Towards a general theory of bond markets. *Finance Stochastics*.1, 141–174 (1997)
- [27] Shreve S.E., *Stochastic calculus for finance II. Continuous-time models* (Springer, 2004)
- [28] Björk, T. (1998), *Arbitrage Theory in Continuous Time*. Oxford: Oxford University Press.
- [29] Bates, D., Jumps and stochastic volatility: the exchange rate processes implicit in Deutchemark options. *Rev. Fin. Studies*, 9 A996), pp. 69-107.
- [30] Barndorff-Nielsen, O.E. and Shephard, N. (2001a), Modelling by Lévy processes for financial econometrics. In *Lévy processes*, pp. 238-318. Boston, MA: Birkhauser Boston.
- [31] Barndorff-Nielsen, O. E. (1998), Processes of normal inverse Gaussian type. *Finance and Stochastics* 2, 41–68.
- [32] Barndorff-Nielsen, O. E. and K. Prause (2001), Apparent Scaling. *Finance and Stochastics*. 5(1), 103-113.
- [33] Rydberg, T.H. (1999), Generalized hyperbolic diffusion processes with applications towards finance. *Mathematical Finance* 9(2), 183-201.

- [34] Barndorff-Nielsen, O. E., Nicolato, E. and Shephard, N. (2002), Some recent developments in stochastic volatility modelling. *Quantitative Finance* 2, 11–23.
- [35] Carr, P., H. Geman, D. Madan, and M. Yor (2001), The Fine Structure of Asset Returns: An Empirical Investigation, forthcoming in the Journal of Business.
- [36] Corcuera, J. M., Guillaume, F., Leoni, P. and Schoutens, W. (2009), Implied Lévy volatility, *Quantitative Finance*, 9: 4, 383 - 393
- [37] Carr, P., Geman, H., Madan, D. and Yor, M., Stochastic volatility for Lévy processes. *Mathematical Finance*, 13:345–382, 2003.
- [38] Barndorff-Nielsen, O.E. and Shephard, N., Econometrics of testing for jumps in financial economics using bipower variation, *Journal of Financial Econometrics*, 4 (2006), 1–30.
- [39] Jacod, J. and AIT-Sahalia, Y., Volatility estimators for discretely sampled Lévy processes, *Annals of Statistics*
- [40] Carr, P. and Wu, L., What type of process underlies options? A simple robust test, *J. Finance*, 58 (2003).
- [41] Cont, R. and Tankov, P., Non-parametric calibration of jump-diffusion option pricing models, *Journal of Computational Finance*, 7, (2004), 1–49.
- [42] Lee, R.W., Option pricing by transform methods: Extensions, unifications and error control, *Journal of Computational Finance* 7(3), (2004), 51-86.
- [43] Finkel, D. E. and Kelley, C. T., Convergence Analysis of the DIRECT algorithm, North Carolina State University, Center for Research in Scientific Computation, Raleigh, version of July 14, (2004).
- [44] Jones, D. R., Perttunen, C. C. and Stuckman, B. E., *Lipschitzian optimization without the Lipschitz constant*, *J. Optim. Theory Appl.*, 79, (1993), 157–181.

- [45] Shubert. B., A sequential method seeking the global maximum of a function. *SIAM J. Numer. Anal.*, 9, (1972), 379-388.
- [46] Delbaen, F., Schachermayer, W., A general version of the fundamental theorem of asset pricing. *Math. Ann.* 300, (1994), 463–520.
- [47] Shohat, J.A, Tamarkin, J.D., The problem of moments. New York: Mathematical Surveys No. 1, A.M.S. (1950).
- [48] Eberlein, E., Jacod, J., On the range of option prices. *Finance Stochast.* 1, (1997), 131–140.
- [49] Graybill, A., *Matrices with applications to statistics* (2nd ed.) Belmont: Wadsworth (1983).
- [50] Norberg, R., On the Vandermonde matrix and its role in mathematical finance. Laboratory of Actuarial Mathematics (working paper), University of Copenhagen (1999)
- [51] Carr, P. and Madan, D., Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2, (1999), 61–73.
- [52] Tankov, P. and Voltchkova, E., *Jump-diffusion models: a practitioner's guide*, Working Paper, (2007).
- [53] Bates, D. S., Testing option pricing models, in *Statistical Methods in Finance*, vol. 14 of *Handbook of Statistics*, North-Holland, Amsterdam, (1996), 567–611.
- [54] Applebaum. D., *Levy Processes and Stochastic Calculus*. Cambridge University Press, (2004).
- [55] Kyprianou. A. E., *Introductory Lectures on Fluctuations of Levy Processes*.
- [56] Madan D. B. and Seneta, E., The Variance Gamma (VG) Model for Share Market Rreturns," *Journal of Business*, (1990), 63, 511-524.

- [57] Carr, P., H. Geman, D. Madan, and M. Yor (2001), "The Fine Structure of Asset Returns: An Empirical Investigation," forthcoming in the *Journal of Business*.
- [58] Dritschel, M., Protter, P., Complete markets with discontinuous security prices, *Finance and Stochastics*, 3, 203–214, (1999).
- [59] Neuberger, A., Volatility Trading, London Business School working paper, (1990).
- [60] Barndorff-Nielsen, O.E., Shephard, N., Estimating quadratic variation using realized variance. *J. Appl. Econ.* 17, 457–477 (2002).
- [61] Barndorff-Nielsen, O.E., Shephard, N., Realised power variation and stochastic volatility models. *Bernoulli* 9, 243–265 (2003).
- [62] Barndorff-Nielsen, O.E., Shephard, N., Financial volatility: Stochastic volatility and Lévy based models. Cambridge: Cambridge University Press (2004).
- [63] Carr, P., Geman, H., Madan, D.H., Yor, M., Pricing options on realized variance. *Prepublications du Laboratoire de Probabilités et Modèles Aléatoires* 768, Universités de Paris 6 & Paris 7, Paris (2002).
- [64] Nualart, D., Schoutens W., Backwards stochastic differential equations and Feynman-Kac Formula for Lévy processes, with applications in finance, *Bernoulli*, 7(5), 761–776, (2001).

WEB REFERENCES

<http://www.reuters.com>

<http://finance.yahoo.com>

<http://www.standardandpoors.com>