PRICING AND HEDGING OF CONSTANT PROPORTION DEBT OBLIGATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF PHILOSOPHY OF DOCTORATE
IN
FINANCIAL MATHEMATICS

FEBRUARY 2011

Approval of the thesis:

PRICING AND HEDGING OF CONSTANT PROPORTION DEBT OBLIGATIONS

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ABSTRACT

PRICING AND HEDGING OF CONSTANT PROPORTION DEBT OBLIGATIONS

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February 2011, 84 pages

A Constant Proportion Debt Obligation is a credit derivative which has been introduced to generate a surplus return over a riskless market return. The surplus payments should be obtained by synthetically investing in a risky asset (such as a credit index) and using a linear leverage strategy which is capped for bounding the risk.

In this thesis, we investigate two approaches for investigation of constant proportion debt obligations. First, we search for an optimal leverage strategy which minimises the mean-square distance between the final payment and the final wealth of constant proportion debt obligation by the use of optimal control methods. We show that the optimal leverage function for constant proportion debt obligations in a mean-square sense coincides with the one used in practice for geometric type diffusion processes. However, the optimal strategy will lead to a shortfall for some cases.

The second approach of this thesis is to develop a pricing formula for constant proportion debt obligations. To do so, we consider both the early defaults and the default on the final payoff features of constant proportion debt obligations. We observe that a constant proportion debt obligation can be modelled as a barrier option with rebate. In this respect, given the knowledge on barrier options, the pricing equation is derived for a particular leverage strategy.

Keywords: Credit Risk, Credit Derivatives, Stochastic Control, Laplace Transforms, Jump Diffusion Processes

ÖZ

SABİT ORANLI BORÇ YÜKÜMLÜLÜKLERİ FİYATLAMASI VE RİSK MİNİMİZASYONU

İşcanoğlu Çekiç, Ayşegül
Doktora, Finansal Matematik Bölümü
Tez Yöneticisi : Doç. Dr. Ömür Uğur
Ortak Tez Yöneticisi : Prof. Dr. Ralf Korn

Şubat 2011, 84 sayfa

Sabit oranlı borç yükümlülükleri, piyasada gözlenen risksiz faiz oranı üzerinde bir getiri sağlama amacı ile piyasaya sürülmüş bir kredi türevidir. Bu türev ürünü yatırımcılarına dağıtmakla yükümlü olduğu ekstra faizi sentetik olarak yaratmayı hedeflemektedir. Bu hedef doğrultusunda lineer bir borçlanma stratejisi izleyerek riskini sınırlayan bir portföy oluşturur. Sabit oranlı borç yükümlülüklerinin karakterini temel olarak bu portföy stratejisi oluşturur.

Bu tezde biz, sabit oranlı borç yükümlülüklerini iki farklı açıdan inceledik. İlk olarak sabit oranlı borç yükümlülükleri için optimal bir borçlanma stratejisi bulmayı hedefledik. Bu hedefte optimal kontrol metodu kullanarak vade sonu ödemesi ile vade sonunda gözlenecek olan portföy değeri arasındaki farkı minimum yapacak bir borçlanma strateji bulduk öyleki bu strateji geometrik tipte difüzyon süreçleri için piyasada uygulanan strateji ile birebir örtüştü. Fakat, uygulamada optimal strateji kullanarak oluşturulmuş portföyler herzaman vade sonu ödemesini karşılayacak miktarı üretmeyi başaramadı.

İkinci yaklaşım olarak sabit oranlı borç yükümlülüklerini için bir fiyatlama denklemi geliştirmeyi hedefledik. Bu doğrultuda erken temerrüte düşme olasılığını ve vade sonu ödemesenini karşılayamama koşullarını da modele ekledik. Bazı işlemlerden sonra sabit oranlı borç yükümlülüklerinin aslında bir bariyer opsiyon olduğunu gösterdik ve belirli bir borçlanma stratejisi için fiyatlama denklemini türettik.

Anahtar Kelimeler: Kredi Riski, Kredi Türevleri, Stokastik Kontrol, Laplace Dönüşümü, Sıçramalı Difüzyon Süreçleri

To my Family...

ACKNOWLEDGMENTS

I would like to thank my advisor Assoc. Prof. Dr. Ömür Uğur for his great motivation, invaluable guidance and helpful suggestions. His careful proofreading and useful comments improved this thesis. For me he is the most friendly and generous advisor that anyone can have. Without his great understanding and support I have not been able to continue and complete successfully my PhD studies.

I would like to express my sincere gratitude to my co-advisor Prof. Dr. Ralf Korn for patiently guiding and motivating me during my PhD studies. My special thanks to him also for the effort and time he spent for my improvement in research. I will never forget his endless support and friendship. I also thank to him for his hospitality that made the time in Kaiserslautern pleasant for me. He was the one who introduced the literature on credit derivatives and gave the idea of this thesis.

I am especially grateful to Prof. Dr. Gerhard Wilhelm Weber for his endless support, motivation and proofreading during my studies.

I wish to thank Technische Universität Kaiserslautern, Germany, for their financial support and hospitality.

All financial and organisational support of Selçuk University and Middle East Technical University is gratefully acknowledged.

I also gratefully acknowledge the financial support of TÜBİTAK (Turkish Scientific and Technical Research Council).

I would like to thank all members of the Institute of Applied Mathematics at Middle East Technical University for the pleasant atmosphere, and everybody who helped me directly or indirectly to accomplish this thesis.

Deep is my gratitude to my family: to my admirable mother Süheyla Akdaş and to my dear brother Kerem İşcanoğlu for their love, generous support and great patience.

Finally, I am thankful by heart to my dear husband Hüseyin Çekiç for his ever undying love, enormous support and great patience. With his endless understanding and generous encouragement, I have been able to continue my career in a sweet and peaceful environment even in my difficult times.

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CHAPTER 1

INTRODUCTION

The main subject that finance has dealt with since the inroduction of the first financial instrument is the *risk*. Risk is inherent in all financial instruments due to its nature, hence, it cannot be avoided completely. It can however be hedged in some respects with the use of other instruments. In this perspective, everyday some new products, which may be superior to older ones, are introduced to prevent some sort of risks. The risk is separated in groups according to its source and products. Main risk groups are interest rate risk, market risk, operational risk, liquidity risk, currency risk, and credit risk.

Here we are interested mainly in credit risk due to the underlying credit products called credit derivatives. First, we give the definition of credit risk in Section 1.1 and we explain the credit derivatives briefly in Section 1.2. Then, we specialize our subject on Constant Proportion Debt Obligations in Section 1.2.1. Finally, in Section 1.3 we give an outline of the present thesis.

1.1 Credit Risk

Credit risk is the possibility that a counterparty will not meet his/her obligations written on a contract. If this is the case, we say that the counterparty defaults on the promised payments. As a financial terminology, the default mainly refers to firms or companies to default on all their obligations and also their future operations because of country laws and regulations. However, for an individual we can talk about a default event of a single obligation.

The default event is an important occasion for the financial markets because it loads extra cost to the creditors so to the market. In financial markets credit is always in a circulation. Therefore, creditors of defaulters have also some obligations to satisfy. If creditors have to met an obligation with the counterparties' payments and if the counterparty defauts then it may cause creditors to default or force them to lower their business circles. In other words, a counterparty's default may start a chain of defaults or disabilities. Therefore, managing credit risk is very important for creditors. According to the operation area of creditors the source of credit risk and precautions taken to deal with credit risk show differences: for example, the credit agencies such as banks are the institutions that are most likely to face defaults on their loans. However, other sources of default also exist for banks, namely, interbank transactions, bonds, equities, options, trade financing and so on. The rules suggested by the Basel Committee regulate the principles for managing credit risk for banks. Rules suggest that every bank should have a credit risk strategy that shows the bank's maximum risk expectations. Furthermore, all banks should identify the risks on their activities, portfolios and products [14]. The regulations also state that banks should collect sufficient information about counterparties which includes at least the necessity of credit, counterparties' repayment history on their other credits, risk profile of the counterparties' operational areas [14].

On the other hand, in the international area countries face credit risks because of the possibility of sovereign defaults. Sovereigns usually default on their domestic debts or international debts. For example, in 1998 Russia defaulted on not the whole but the crucial part of its domestic debts. Again, at the end of the same year Pakistan defaulted on both its international and domestic debts. Indeed, there are various such examples of sovereign defaults. However, the default of Argentine in 2001 should be considered carefully: this default event is studied in the economic literature as the largest default which has ever seen in the history of countries [44]. The default of Argentine and its restructuring process have charged about $74 billion to its international lenders and an indirect cost of at least $63 billion to the citizens of the world [52]. Since sovereign defaults cause worldwide consequences managing and assessing them become an important issue in international lending: The Moody's, S&P, and Fitch, are some of the well-known credit rating agencies who try to measure the risk on debts. They assign credit rating values to each security, investor and country to indicate the credit worthiness of the counterparty or country. A worldwide look to a country's performance is basically achieved by looking at its credit rating.

1.1.1 Measure and Management of Credit Risk

As mentioned above, managing and measuring credit risk are necessary for credit agencies and even for countries in order to continue their operations. However, credit risk has many components. A good management process indeed starts with an identification of the risk before processing it [50].

The main components of credit risk are arrival risk, exposure risk, recovery risk, and correlated default risk. In reality, the default events are rare and it is hard to predict the number of defaults in a given period of time. These types of uncertainties are called *arrival risk* of a credit risk and they are measured by calculating the probability of default [50]. Accordingly, it is also uncertain when a default event occurs. If one wants to predict the time of default, one has to collect all information about the creditor, and has to transfer it into a suitable probability density function for default time. This uncertainty on the time of default is defined as *exposure risk* [50]. Another component of credit risk is the unpredictability of the size of losses before a default event occurs. The uncertainty of the recovery rate among the total credits is referred to *recovery risk* and it is determined by the severity and the fraction of losses faced in the case of a default. The basic tool for calculating the recovery risk is to assign a probability density function to the recovery rate [25, 50].

The components of credit risk mentioned until now only concern with a single event of default. The main measure of the risk is the probability and it is calculated according to the historical data without considering the cause of default. However, the default may be a consequence of other defaults. Therefore, default correlations in the area of business should also be considered. If correlations exist then the probability calculation should be made by taking these correlations into account. This type of risk is regarded as *correlated defaults risk* [50].

Having identified the risk components, selection of a reliable counterparty or of products among the possible choices comes next for managing credit risk [25]. All the components of credit risk should then be calculated for each counterparty. Accordingly, with a counterparty satisfying all the requirements of the creditor, the contract should be signed.

As another management technique, credit agencies can also use some instruments according to the components of credit risk inherited in the contract. Such instruments are called *credit derivatives*. In Section 1.2, we examine such credit derivatives in detail.

1.2 Credit Derivatives

Credit derivatives are financial hedging instruments which are in fact a commercialized version of credit risk [33]. Therefore by using them someone can sell his/her credit risk or buy other parties' credit risk in return of a premium.

As a formal definition, a credit derivative is a financial derivative that allows one party (protection buyer) to get a payoff if default occurs on the predefined credit risk of a company. Here, the protection seller, on the other hand, collects a periodic premium from the protection buyer for a notional value written on the contract [50].

The first credit derivative, a so-called *credit default swap (CDS)*, was traded in the mid 1990s in London and NewYork [50] and after the first trades, it became the fastest developing derivative in the market. In the global size the credit derivative market in notional value ranges from level of $180 billions to $28 trillions in 10 years (from 1996 to 2006) [19]. Moreover, until the financial crisis in 2008, credit derivative markets have lived their golden age. Especially, from 2006 to 2008, the market grew three times in notional value [1]. This increase has caused credit agencies, especially, banks to loose important amounts of their reserves and has triggered the 2008 financial crisis [19]. After the crisis, the market's nominal value decreased to $41 trillions at the end of the same year and by the end of 2009, the total trade in global size observed as $32 trillions [2].

1.2.1 Types of Credit Derivatives

A guarantee is the simplest and earliest form of credit derivatives. However, with the developments in financial sectors it is seen that the use of a simple guarantee does not have enough potential in supplying user's needs. Credit derivatives are more complex tools for guarantees. In contracts users can specify protected asset or portfolio, credit risk, transaction details, and others. We refer to [33] and the references therein.

Below we introduce some types of credit derivatives.

Credit Default Swap

As mentioned above credit default swap is the first credit derivatives in the market. Credit default swap is a contract between two parties: the protection seller and the protection buyer. The protection buyer makes a periodic payment called CDS spread and the protection seller makes payment for the principal of underlying asset in case of default specified in the contract [6, 33].

Total Return Swap

Total return swap is a contract on the total return of a reference asset. In total return swap the protection buyer has the asset (e.g., bond financial asset, basket of assets, etc.) and transfers all the interest and capitals gained from the reference asset to the protection seller. On the other hand, the protection seller makes fixed or floating rate payments to the protection buyer and he/she also satisfies the negative price changes in the asset [6, 33].

Collateralised Debt Obligation

Collateralised debt obligations are firstly introduced at the end of 1980s. It is defined as a form of credit risk transfer tools. Credit debt obligations transfer risks on a pool of debt instruments to a fixed income securities called tranches. Coupon and principal payments depend on the performance and priority of tranches. Senior, mezzanine and equity tranches are names of these tranches and listed according to their priority on coupon and principal payments. i.e. Holders of senior tranche notes receive payments prior to the mezzanine tranche note holders. However, equity tranche note holders are paid if and only if there exists enough cash after all the payments are made. Therefore, equity tranche exposes the highest risk.

Constant Proportion Debt Obligations

Constant Proportion Debt Obligations are newly introduced structured credit products which have structural properties that are different from other credit derivatives. Unlike the traditional credit derivatives constant proportion debt obligations restructure the asset side of the issuer. They are a mixture of credit default swap, investment and leveraging. In Figure 1.1, the mechanisms of constant proportion debt obligations are shown: constant proportion debt obligations are issued by means of the Special Purpose Vehicle (SPV). At issuance, the proceeds are either deposited on a risk free account or invested in an arranging bank account with regular interest payments. Then, the arranging bank enters into a credit default swap on a risky reference portfolio. For this type of transactions, usually, the credit default swap index is taken as the reference portfolio. The notion of this swap is a multiple of the collateral taken from note holders. This is called a *leverage function* of the constant proportion debt obligations. The arranging bank also enters a total return swap with the SPV on the risky reference portfolio. The coupons gained from the total return swap and deposit account are paid to the note holder. Then, the deposit account is rearranged according to the mark-to-market gains and losses [26].

The structure of constant proportion debt obligations shows that there is only one asset that is obtained at issuance. Moreover, constant proportion debt obligations have an obligation to pay fixed coupons, risk free rate (or LIBOR) plus additional interest during the life of the note and the principle at maturity. The shortfall between the asset and obligations of the constant proportion debt obligations is covered from the leverage function [38]. If the present value of the collateral in the deposit account is enough in order to pay all future obligations then, the leverage will be reduced to zero. This situation is referred to as a "*cash-in*" event for constant proportion

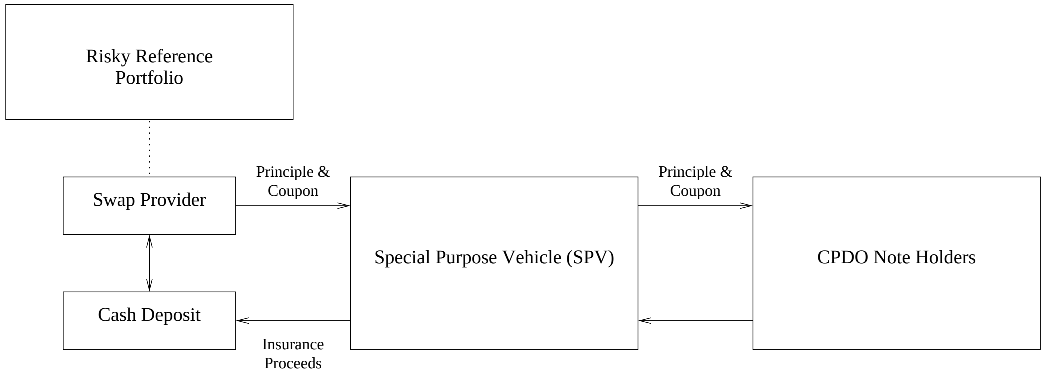

Figure 1.1: Structure at Closing of a Typical CPDO Transaction [26]

debt obligation. Similarly, constant proportion debt obligation is said to experience a "*cash-out*" if the shortfall exceeds a given threshold (level).

The literature on constant proportion debt obligations on one hand is concerned with the pricing of constant proportion debt obligations and on determining optimal leverage functions on the other. An example of pricing is found in [20]. In this paper, Dorn found a closed form valuation formula for constant proportion debt obligation notes by using structural approach. Moreover, he investigated the relations between constant proportion debt obligations and inverse Constant Proportion Portfolio Insurance. He also specified the inherit risks on constant proportion debt obligations.

In order to determine the optimal leverage functions for constant proportion debt obligations, Baydar, Di Graziano and Korn [8] used stochastic control methods: optimality is defined as maximizing the utility of the final payoff from the constant proportion debt obligation at maturity while guaranteeing the surplus payments. In their study leverage changes continuously and the risky reference portfolio is assumed to follow a Brownian motion with drift.

Later, Çekiç, Korn, and Uğur [22] performed the same problem for the geometric Brownian risky index and make a sensitivity analysis with respect to the parameters included in the system. Recently, Cont and Jessen [15] included the modelling of default risk, loss distribution and other risk factors to the problem of constant proportion debt obligations. They examine all the movements of the credit default swap markets deeply and they use them in the pricing of constant proportion debt obligations.

In 2010, Çekiç, Korn and Uğur [23] implemented a study that includes the minimization of the mean-square distance between the promissed final payoff and the final wealth of constant proportion debt obligation by using both the martingale method and the optimal control method for a geometric Brownian risky index. In the study, they show that the optimal leverage function for constant proportion debt obligations in a mean-square sense coincides with the leverage factor used in the industry.

1.3 Outline

In the thesis we only deal with constant proportion debt obligations. In calculations we use continuous time trading and we take the wealth definition stated in [8]. Accordingly we denote the wealth at time t by V_t and we denote the initial investment of note holders, which should be paid at maturity, by F_T. In reality, F_T is a constant principal. At issuance issuers pay some costs therefore we assume that initial wealth is $V_0 = v \leq F_T$. Then, wealth is defined as follows

$$dV_t = r_t V_t dt + \ell_t dS_t - F_T(r_t + \nu)dt, \quad V_0 = v. \tag{1.1}$$

In this definition, the first part of (1.1) shows the risk free deposit account of the constant proportion debt obligations, and r_t represents the risk free interest rate at time t. Moreover, in the second part unlike the definition stated in [15] all the markt-to-market gains and losses caused from index movements, losses observed because of the defaults of credit default swaps included in the index, and the changes observed in the index value caused from rearrangement of the index are all considered with a single definition of changes denoted by dS_t. Here, ℓ_t is the leverage function of constant proportion debt obligations, i.e., the position taken in the index. The third part of (1.1) is the reduction part of the wealth representing coupon payments of constant proportion debt obligations which are greater than the risk free rate for each unit of money invested by note holders and ν denotes this surplus return.

According to the characteristics of constant proportion debt obligations, the final payment and coupon payments are all dependent on the observed value of this wealth equation. Constant proportion debt obligations can default in two possible ways: Default on the Final Payment and Early Default. Therefore, at this step we classify constant proportion debt obligations as portfolio based notes with default possibility.

Default on Final Payment. If a constant proportion debt obligation lives until maturity time T and if we define $\varphi(V_T, T)$ as the final payment of constant proportion debt obligations, then

$$\varphi(V_T, T) = \begin{cases} F_T, & \text{if } V_T \geq F_T, \\ V_T, & \text{otherwise.} \end{cases} \tag{1.2}$$

Early Default. If the wealth of a constant proportion debt obligation falls under a predetermined barrier then the constant proportion debt obligation defaults on all its future obligations including coupon payments and final payment. We define this case in the following manner,

$$\text{If } V_t \leq \beta_t, \text{ then, early default occurs.} \tag{1.3}$$

where β_t is the predetermined barrier. In applications for easy calculations β is taken as a constant value, however for some cases it is also assigned as a small multiple of all future obligations. In other words, if we define F_t as present value of all future obligations including coupons and final payment, where

$$F_t = F_T e^{-\int_t^T r_s ds} + F_T (r + \nu) \int_t^T e^{-r_s(s-t)} ds, \quad F_0 = f, \tag{1.4}$$

then $\beta_t = aF_t$.

As it is seen, constant proportion debt obligations may default according to movements of the wealth and the main characteristic that affects the continuance of the note is the leverage factor. This factor keeps the balance between earnings and future obligations. The usual leverage factor used in industry is given as

$$\ell_t^U = \max\left\{ \frac{(F_t - V_t)}{PVS_t} \times m, \ 0 \right\}, \tag{1.5}$$

where PVS_t is the present value of the index and m is a constant predetermined multiple. With this definition of leverage, actually downside risk is preserved because if the gap between present value of future obligations, F_t and the net asset value, V_t increases then the issuer should enter more swap contracts to close this gap, which means taking more risky positions in the market. However, this situation may add an advantage to the issuers. By taking more risks, issuers may benefit from upward movements in the index value.

In Chapter 2, we examine constant proportion debt obligations and the effect of leverage in detail. Then, in Chapter 3 we obtain the closed form formulae for the fair price of constant proportion debt obligations under the Laplace domain.

CHAPTER 2

HEDGING OF CONSTANT PROPORTION DEBT OBLIGATIONS WITH DYNAMIC PROGRAMMING APPROACH

In the literature, the leverage function stated in (1.5) is used in the pricing of constant proportion debt obligations. The value of m in the formula should be predetermined according to the issuer's inside information and the market's behaviour at the time of issuance. However, the selection of this factor may not always be optimal. The market expectations of the issuer may fall and constant proportion debt obligation may face big losses. Therefore, the optimal selection of m is an important issue for issuers. A systematic way of selecting an optimal value for m is to solve an optimization problem by assuming a suitable model for the index.

This chapter introduces the optimization criterion and aims to hedge constant proportion debt obligations by selecting a leverage and a value for m optimally. Specifically, in Section 2.1 a review of the dynamic programming technique is given for general portfolio problems. In Section 2.2, the hedging problem for constant proportion debt obligations is stated in the general form and the Hamilton-Jacobi-Bellmann equation is derived. In addition, the solution of the problem is shown in Sections 2.2.1, 2.2.2 and 2.2.3 for cases when the risky index follows some special processes: a geometric Brownian motion with constant parameters, a Vasicek type stochastic differential equation, a geometric jump diffusion process with normal jump sizes, respectively. To illustrate the behaviour of some characteristic features corresponding to our optimization criterion of the constant proportion debt obligations the application studies are also given for each model.

2.1 Dynamic Programming Technique

Dynamic Programming is one of the popular techniques used for solving continuous time portfolio optimization problems. The Dynamic Programming technique includes a powerfull recursive algorithm that selects the optimal policy for the current state on an optimized path of future policies. For detailied information, one refers to [9, 29, 41, 43]. In this section we present the theoretical aspect of stochastic optimal control problems used in financial applications. Following two subsections, we review the idea of stochastic control applied to diffusion processes and jump diffusion processes, respectively. We mainly refer to [27, 41, 43, 49, 54].

2.1.1 Stochastic Control of Diffusion Processes

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and let $\mathbf{B} \in \mathbb{R}^m$ be a m-dimensional Brownian motion on this space. Define $Y_t = Y_t^{(c)} \in \mathcal{Y} \subset \mathbb{R}^k$ ("the state process") as a stochastic diffusion process of the form

$$dY_t^{(c)} = b\left(Y_t^{(c)}, c_t\right) dt + \sigma\left(Y_t^{(c)}, c_t\right) d\mathbf{B}_t, \quad Y_0^{(c)} = y \in \mathbb{R}^k, \tag{2.1}$$

where $c \in \mathcal{C}$ is a control process, $b : \mathbb{R}^k \times \mathcal{C} \to \mathbb{R}^k$ is a controlled drift function, and $\sigma : \mathbb{R}^k \times \mathcal{C} \to \mathbb{R}^{k\times m}$ is a controlled diffusion function. Then, $Y_t = Y_t^{(c)}$ is called controlled diffusion process.

Here,

i. $\mathcal{C}$ is the space of control processes which is closed, and compact subset of $\mathbb{R}^k$.

ii. An admissible control c_t, $t \in [0, T]$, is a progressively measurable, $\mathcal{C}-$valued process.

iii. b and σ are continuous functions of class $\mathbb{C}^1$, the class of the first order continuously differentiable functions, for all $c \in \mathcal{C}$. Moreover, there exists a constant $M > 0$ such that

$$\begin{aligned} |b_t| + |b_y| \leq M \quad |\sigma_t| + |\sigma_y| &\leq M, \\ |b(y,c)| \leq M(1 + |y| + |c|), \quad |\sigma(y,c)| &\leq M(1 + |y| + |c|) \end{aligned} \tag{2.2}$$

then (2.1) possesses a unique solution Y_t.

We define cost/performance function $J = J^{(c)}(y)$ of the form

$$J^{(c)}(y) = \mathbb{E}^y \left[\int_0^T f_1 \left(Y_t^{(c)}, c_t, t \right) dt + f_2 \left(Y_T^{(c)} \right) \right], \tag{2.3}$$

where f_1 and f_2 are given continuous functions. Then, optimal control problem is written as follows

$$\Phi(y,0) = \sup_{c \in \mathcal{C}} \quad J^{(c)}(y) = J^{(c^*)}(y),$$

subject to

$$dY_t^{(c)} = b\left(Y_t^{(c)}, c_t\right) dt + \sigma\left(Y_t^{(c)}, c_t\right) d\mathbf{B}_t, \quad Y_0^{(c)} = y \in \mathbb{R}^k, \tag{2.4}$$

where $\Phi(y,t)$ denotes the value function and c^* is the optimal control (for simplicity we may assume that it exists).

If $\Phi(y,t) \in \mathbb{C}^{1,2}$, then the Hamilton-Jacobi-Bellman equation obeyed by the optimal value function $\Phi(y,t)$ is

$$\sup_{c \in \mathcal{C}} \mathcal{D}^c \Phi(y,t) + f_1(y,c,t) = 0, \quad \Phi(y,T) = f_2(y), \tag{2.5}$$

where $\mathcal{D}^c$ is the Dynkin operator and has the form

$$\mathcal{D}^c \Phi(y,t) = \frac{1}{2} tr\left[\sigma(y,c)\, \sigma^T(y,c)\, \Phi_{yy}(y,t)\right] + \Phi_t(y,t) + \Phi_y(y,t) b(y,c). \tag{2.6}$$

The solution of (2.5) gives the solution of the problem (2.4) for some specific diffusion processes. However, in general a verification theorem is needed. Verification theorem is a theorem which provides conditions such that the solution of the Hamilton-Jacobi-Bellman equation with optimal control c^* is the solution of the optimal control problem given in (2.4) [49].

Theorem 2.1 (Verification Theorem [49]) *Let* $\|\sigma(y,c)\|^2 \leq M_\sigma \left(1 + \|y\|^2 + \|c\|^2\right)$ *and* $\|f_1(y,c,t)\|^2 \leq M_{f_1} \left(1 + \|y\|^2 + \|c\|^2\right)$ *hold for some nonnegative* M_σ *and* M_{f_1}*, where for any* $x = (x_1, x_2, \cdots, x_n)^T$*,* $\|x\| := \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$*. Then,*

i. Assume that $\psi \in \mathbb{C}^{1,2}$ *with condition* $\|\psi(y,t)\| \leq M_\psi \left(1 + \|y\|^2\right)$ *is continuous function and satisfies the Hamilton-Jacobi-Bellmann equation with boundary condition given in* (2.5)*. Then for all* $t \in [0,T]$ *and* $y \in \mathbb{R}^k$ *we write*

$$\psi(y,t) \geq \Phi(y,t). \tag{2.7}$$

ii. *Suppose a maximizer,* $\hat{c}(y,t)$*, of the functional* $c \mapsto \mathcal{D}^c\psi(y,t) + f_1(y,c,t)$ *exits satisfying* $c^* = (c_t^*)_{t\in[0,T]}$*, and suppose that* $c_t^* = \hat{c}(Y_t^*,t)$ *is admissible, then* $\psi(y,t) = \Phi(y,t)$ *for all* $t \in [0,T]$*,* $y \in \mathbb{R}^k$ *and* c^* *is an optimal strategy.*

2.1.2 Stochastic Control of Jump Diffusion Processes

In a fixed domain $\mathcal{S} \in \mathbb{R}^k$ define $Y_t = Y_t^{(c)}$ as a stochastic jump diffusion process of the form

$$\begin{aligned} dY_t^{(c)} &= b\left(Y_t^{(c)}, c_t\right)dt + \sigma\left(Y_t^{(c)}, c_t\right)d\mathbf{B}_t + \int_{\mathbb{R}^k} Q\left(Y_{t^-}^{(c)}, c_{t^-}, z\right)\bar{N}(dt,dz), \\ Y_0^{(c)} &= y \in \mathbb{R}^k, \end{aligned} \tag{2.8}$$

where $c \in \mathcal{C}$ is the set of controls, $b : \mathbb{R}^k \times \mathcal{C} \to \mathbb{R}^k$ is a controlled drift function, $\sigma : \mathbb{R}^k \times \mathcal{C} \to \mathbb{R}^{k\times m}$ is a controlled diffusion function, $Q : \mathbb{R}^k \times \mathcal{C} \times \mathbb{R}^k \to \mathbb{R}^{k+n}$, and $N(t,z)$ is a Possion random measure. Then, $Y_t = Y_t^{(c)}$ is called a controlled jump diffusion process.

For the jump diffusion case we also define our cost/performance function as given in (2.3) and we assume the following conditions:

i. $\mathcal{C}$ is the space of control processes which is closed, and compact subset of $\mathbb{R}^k$,

ii. An admissible control c_t, $t \in [0,T]$, is a progressively measurable, $\mathcal{C}-$valued process,

iii. b and σ are continuous functions of class $\mathbb{C}^1$ for all $c \in \mathcal{C}$. Moreover, there exists a constant $M > 0$, such that

$$\begin{aligned} |b_t| + |b_y| \le M, \quad |\sigma_t| + |\sigma_y| &\le M, \\ |b(y,c)| \le M(1+|y|+|c|), \quad |\sigma(y,c)| &\le M(1+|y|+|c|), \end{aligned} \tag{2.9}$$

iv. Let us say that the Lévy measure is a positive finite measure on $\mathbb{R}^k$, with a singularity in 0 satisfying

$$\int_{|z|\ge 0} m(dz) < +\infty. \tag{2.10}$$

Then, there exists $\rho : \mathbb{R}^k \to \mathbb{R}_+$ with $\int_{\mathbb{R}^k} \rho^2(z)m(dz) < +\infty$, such that

$$\begin{aligned} |Q(x,c,z) - Q(y,c,z)| &\le \rho(z)\,|x-y|, \\ |Q(y,c,z)| &\le \rho(z)\,|y|, \end{aligned} \tag{2.11}$$

for all $x,\, y \in \mathbb{R}^k$,

such that (2.8) possesses a unique solution Y_t.

Then, the optimal control problem is given as follows:

$$
\begin{aligned}
&\Phi(y,0) = \sup_{c\in\mathcal{C}} \quad J^{(c)}(y) = J^{(c^*)}(y), \\
&\quad \text{subject to} \\
&\qquad\qquad dY_t^{(c)} = b\left(Y_t^{(c)}, c_t\right) dt + \sigma\left(Y_t^{(c)}, c_t\right) d\mathbf{B}_t \\
&\qquad\qquad + \int_{\mathbb{R}^k} Q\left(Y_{t^-}^{(c)}, c_{t^-}, z\right) \bar{N}(dt,dz), \quad Y_0^{(c)} = y \in \mathbb{R}^k,
\end{aligned}
\tag{2.12}
$$

where $\Phi(y,t)$ denotes the value function and c^* is the optimal control.

If $\Phi(y,t) \in \mathbb{C}^{1,2}$, then the Hamilton-Jacobi-Bellmann equation obeyed by the optimal value function $\Phi(y,t)$ is written as follows:

$$
\sup_{c\in\mathcal{C}} \mathcal{D}^c\Phi(y,t) + f_1(y,c,t) = 0, \quad \Phi(y,T) = f_2(y), \tag{2.13}
$$

where $\mathcal{D}^c$ is the Dynkin operator and has the form

$$
\begin{aligned}
\mathcal{D}^c\Phi(y,t) \quad &= \quad \tfrac{1}{2} tr\left[\sigma(y,c)\,\sigma^T(y,c)\,\Phi_{yy}(y,t)\right] + \Phi_t(y,t) + \Phi_y(y,t) b(y,c) \\
&+ \quad \textstyle\sum_{j=1}^n \int_{\mathbb{R}} \left\{\Phi\left(y + Q^j(y, c(y), z_j), t\right) - \Phi(y,t)\right. \\
&- \quad \left.\nabla\Phi(y,t)\cdot Q^j(y, c(y), z_j)\right\} m_j(dz_j).
\end{aligned}
\tag{2.14}
$$

Now, as an example we give the verification theorem for optimal portfolio consumption problem in the following subsection.

An Example: Verification Theorem For Optimal Portfolio Consumption Problems.

Let us define the dynamics of risky and riskless stock at time t as follows:

$$
dS_t = S_{t^-}\left(b(t)dt + \sigma(t)dW_t + Q(t)dN_t\right), \tag{2.15}
$$

and

$$
dB_t = B_t r(t)dt, \tag{2.16}
$$

respectively. Then, the corresponding wealth equation which is a special case of (2.8) is given in the form of

$$
dY_t = Y_t[(l - \pi_t)r(t) + \pi_t b(t)]dt - c_t dt + Y_t\pi_t\sigma(t)dW(t) + Y_t\pi_t Q(t)dN_t, \tag{2.17}
$$

where the consumption rate is denoted by c_t, and the proportion of the money invested in the risky stock at time t is given by π_t. Here, X_t is the state variable, and c_t, π_t are the control variables.

For optimal portfolio consumption problems the cost/performance function is similar to function given in (2.3) where f_1 and f_2 are utility functions and satisfy the conditions as follows:

1. $f_1,\ f_2 \in \mathbb{C}^{0,1}$,
2. f_1, and f_2 are strictly monotone and concave,
3. $\lim_{c\to\infty} \frac{\partial f_1(y,c,t)}{\partial c} = 0$.

Then, the verification theorem is given as follows:

Theorem 2.2 (Verification Theorem [54]) *Suppose there exists a function ψ such that its partial derivatives $\frac{\partial\psi(y,t)}{\partial t}$, $\frac{\partial\psi(y,t)}{\partial y}$, and $\frac{\partial^2\psi(y,t)}{\partial y^2}$ exist and are continuous. Moreover, there exist some constants d and g satisfying $|\psi(y,t)| \le d\,(1+|y|^g)$ and let us assume that $\psi(y,t)$ obeys the Hamilton-Jacobi-Bellmann equation given as follows*

$$\sup_{\pi,c\in\mathcal{C}} \mathcal{D}^{\pi}\Phi(y,t) + f_1\,(c,t) = 0, \quad \Phi(y,T) = f_2(y), \tag{2.18}$$

where $\mathcal{D}^c$ is the Dynkin operator and has the form

$$\begin{aligned} \mathcal{D}^c\Phi(y,t) &= \tfrac{1}{2}\Phi_{yy}(y,t)y^2\pi^2\sigma^2 + \Phi_t(y,t) + \Phi_y(y,t)y\,(\pi\,(b-r)+r) \\ &- \Phi_y(y,t)c + \lambda(t)\,[\phi(y+y\pi Q,t) - \phi(y,t)]\,. \end{aligned} \tag{2.19}$$

If $(c^*,\pi^*) \in \operatorname{Arg\,max} \mathcal{D}^{\pi}\psi(y,t) + f_1\,(c,t)$, then $\psi(y,t) = \Phi(y,t)$ and (c^*,π^*) is the optimal strategy for (2.12).

2.2 Optimizing the Leverage Function of a Constant Proportion Debt Obligation

As the promissed payments of a constant proportion debt obligation are deterministic and as their present value at time 0 exceeds the initial capital of F_T, it is clear that in an arbitrage free market it is not possible that the issuer can hedge the risk of the future payments completely. Therefore, we have set up a problem where this shortfall risk should be minimized. For this problem we select cost/performance function stated in Section 2.1.1, Section 2.1.2 as in the classic mean-square distance problem. The idea of the problem is to minimize the squared distance between the promissed final payment and the final wealth of constant proportion debt obligations by rearranging leverage. We call it $\mathcal{L}^2-$ utility criterion. Then, we define $f_1 = 0$, and $f_2 = (F_T - v)^2$. For an initial investment $V_0 = v$ the optimal leverage $\ell^* \in L(v)$ solves the problem as follows:

$$\begin{aligned} \underset{\ell \in L(v)}{\text{maximise}} \quad & \bar{J}^{(\ell)}(v) = -\tfrac{1}{2}\mathbb{E}\left[F_T - V_T\right]^2, \\ \text{subject to} \quad & (V_t)_{t\in[0,T]} \geq 0, \\ & dV_t = rV_t dt + \ell_t dS_t - F_T(r+\nu)dt, \quad V_0 = v, \\ & dF_t = rF_t dt - F_T(r+\nu)dt, \quad F_0 = f. \end{aligned} \tag{2.20}$$

In this problem, we do not consider early default and r is assumed to be constant. Then, we write the value function $\bar{\phi}(v, s, t)$ as

$$\bar{\phi}(v, s, t) = \sup_{\ell \in L(v)} \bar{J}^{(\ell)}(v) = \bar{J}^{(\ell^*)}(v), \tag{2.21}$$

subject to the stochastic differential equation

$$dV_t = rV_t dt + \ell_t dS_t - F_T(r+\nu)dt, \quad V_0 = v.$$

In (2.21), s is the initial value of the index and ℓ^* is the optimal control. The corresponding terminal condition is then,

$$\bar{\phi}(v, s, T) = -\frac{1}{2}\left(F_T - v\right)^2. \tag{2.22}$$

To simplify calculations, we continue our calculations with the discounted wealth, $\widetilde{V}$ hereafter. We rearrange the optimization problem and the value function according to the discounted wealth satisfying the stochastic differential equation as follows

$$d\widetilde{V}_t = \widetilde{\ell}_t dS_t - F_T e^{-rt}(r+\nu)dt, \quad \widetilde{V}_0 = v, \tag{2.23}$$

where $\widetilde{\ell} = e^{-rt}\ell$. Then, the optimization problem is

$$\underset{\widetilde{\ell} \in L(\widetilde{v})}{\text{maximise}} \; J^{(\widetilde{\ell})}(\widetilde{v}) = -\frac{1}{2}\mathbb{E}\left[F_T - e^{rT}\widetilde{V}_T\right]^2, \tag{2.24}$$

with the value function

$$\phi(\widetilde{v}, s, t) = \sup_{\widetilde{\ell} \in L(\widetilde{v})} J^{(\widetilde{\ell})}(\widetilde{v}) = J^{(\widetilde{\ell^*})}(\widetilde{v}), \tag{2.25}$$

and the terminal condition

$$\phi(\widetilde{v}, s, T) = -\frac{1}{2}\left(F_T - e^{rT}\widetilde{v}\right)^2, \tag{2.26}$$

where $\bar{\phi}(v, s, t) \equiv \phi(\widetilde{v}, s, t)$.

In accordance with the dynamic programming technique we derive the Hamilton-Jacobi-Bellman equation of the form

$$\begin{aligned} \sup_{\widetilde{\ell}\in L(\widetilde{v})} \mathcal{D}^{\widetilde{\ell}}\phi(\widetilde{v}, s, t) &= 0, \\ \phi(\widetilde{v}, s, T) &= -\tfrac{1}{2}\left(F_T - e^{rT}\widetilde{v}\right)^2, \end{aligned} \tag{2.27}$$

where the form of the Dynkin operator, $\mathcal{D}^{\widetilde{\ell}}$ is given by the chosen dynamics of the index, S_t. Standard verification theorems ensure that a smooth solution to the Hamilton-Jacobi-Bellman equation yield the value function and the optimal leverage.

Now we give the solution of the problem by solving the Hamilton-Jacobi-Bellman equation under the different assumptions on the dynamics of the index in order to make a comparison.

2.2.1 Solution when the Risky Index Follows a Geometric Brownian Motion

We start with the simplest selection of index dynamics. Let us assume that the index, S_t follows the Markov stochastic differential equation

$$dS_t = S_t\left(\mu dt + \sigma dW_t^s\right), \tag{2.28}$$

where W^s is the one-dimensional standard Brownian motion, μ is the constant drift parameter and σ is the constant volatility. Under this geometric Brownian motion assumption the index is more suitable to describe stock indexes rather than credit indexes and actually we can only capture the normal changes in the index. Large changes caused from defaults in the index and rearrangement movements are not well represented. However, because of its simplicity it provides exact solutions for most of the problems. Moreover, since it makes the market complete, the verification of the solution can easily be given.

Inserting (2.28) into (2.23) we obtain

$$d\widetilde{V}_t = \left[\widetilde{\ell}_t\mu S_t - F_T e^{-rt}(r+\nu)\right]dt + \widetilde{\ell}_t\sigma S_t dW_t^s, \quad \widetilde{V}_0 = v. \tag{2.29}$$

Then, the corresponding Dynkin operator, $\mathcal{D}^{\widetilde{\ell}}$ (for our problem it is equal to the Hamilton-Jacobi-Bellman equation) is of the form

$$\begin{aligned}\sup_{\widetilde{\ell}\in L(\widetilde{v})}\mathcal{D}^{\widetilde{\ell}}\phi(\widetilde{v},s,t) &= \tfrac{\partial\phi}{\partial t}+\sup_{\widetilde{\ell}\in L(\widetilde{v}))}\Big\{\phi_{\widetilde{v}}\left(\widetilde{\ell}\mu s - F_T e^{-rt}(r+\nu)\right) \\ &\quad +\tfrac{1}{2}\phi_{\widetilde{v}\widetilde{v}}\widetilde{\ell}^2\sigma^2 s^2+\phi_s\mu s+\tfrac{1}{2}\phi_{ss}\sigma^2 s^2+\phi_{s\widetilde{v}}\widetilde{\ell}\sigma^2 s^2\Big\}, \\ &= 0.\end{aligned} \tag{2.30}$$

Standard verification theorems [30] ensure that a smooth solution to (2.27) indeed coincides with the value function. Note that $\widetilde{v}=\widetilde{v}_t$ and $s=S_t$ are the state variables, and $\widetilde{\ell}=\widetilde{\ell}_t$ is considered to be the control variable at time t. In (2.27), $\phi_{\widetilde{v}}$, $\phi_{\widetilde{v}\widetilde{v}}$, ϕ_s, ϕ_{ss} and $\phi_{\widetilde{v}s}$ denote the partial derivatives of the relevant order with respect to the discounted wealth $\widetilde{v}$ and the risky index s.

Then, the first order optimality condition for (2.30) implies that we have

$$\widetilde{\ell}_t=-\frac{\mu}{\sigma^2 s}\frac{\phi_{\widetilde{v}}}{\phi_{\widetilde{v}\widetilde{v}}}-\frac{\phi_{\widetilde{v}s}}{\phi_{\widetilde{v}\widetilde{v}}}. \tag{2.31}$$

Hence, inserting this optimality condition into the Hamilton-Jacobi-Bellman equation, the following partial differential equation is obtained:

$$\begin{aligned}0 &= \tfrac{\partial\phi}{\partial t}-\tfrac{\mu^2}{2\sigma^2}\tfrac{\phi_{\widetilde{v}}^2}{\phi_{\widetilde{v}\widetilde{v}}}-\mu s\tfrac{\phi_{\widetilde{v}}\phi_{\widetilde{v}s}}{\phi_{\widetilde{v}\widetilde{v}}} \\ &\quad -F_T e^{-rt}(r+\nu)\phi_{\widetilde{v}}-\tfrac{1}{2}s^2\sigma^2\tfrac{\phi_{\widetilde{v}s}^2}{\phi_{\widetilde{v}\widetilde{v}}}+\mu s\phi_s+\tfrac{1}{2}\sigma^2 s^2\phi_{ss}.\end{aligned} \tag{2.32}$$

Proposition 2.3 *The closed-form solution of* (2.32) *with terminal condition* (2.26) *is given by*

$$\begin{aligned}\bar{\phi}(V_t,S_t,t) &\equiv \phi(\widetilde{V}_t,S_t,t) \\ &= -\tfrac{1}{2}\left[F_t-V_t\right]^2 e^{(2r+\frac{\mu^2}{\sigma^2})(T-t)},\end{aligned} \tag{2.33}$$

where $F_t=F_T\left(e^{-r(T-t)}+\frac{r+\nu}{r}\left(1-e^{-r(T-t)}\right)\right)$ *under the assumption of constant interest rate,* $r_t=r$.

Proof. We assume the form of $\bar{\phi}$ to be

$$\bar{\phi}(V_t,S_t,t)=-\frac{1}{2}\left[F_t-V_t\right]^2 e^{2r(T-t)}A(t,s),$$

where $A(t,s)$ is sufficiently smooth function. By taking partial derivatives, of ϕ above, with respect to t, v, s, vv, ss, and vs and inserting them into (2.32), we find the partial differential equation

$$0 = \frac{\partial A}{\partial t} - \frac{1}{2}\sigma^2 s^2 A_{ss} + \sigma^2 s^2 \frac{A_s^2}{A} + \mu s A_s + \frac{\mu^2}{\sigma^2} A, \tag{2.34}$$

which the function $A(t,s)$ should satisfy together with the terminal condition $A(T,s) = 1$. Let $A = e^{\frac{\mu^2}{\sigma^2}\tau} \frac{1}{g(\tau,z)}$ where $\tau = T - t$, and $z = \ln(s) + (\mu + \frac{1}{2}\sigma^2)\tau$. Then, we obtain the partial derivatives of A with respect to g as follows:

$$\begin{aligned} \frac{\partial A}{\partial t} &= \left[-\frac{\mu^2}{\sigma^2}\frac{1}{g(\tau,z)} + \left(\frac{\partial g(\tau,z)}{\partial \tau} + \frac{\partial g(\tau,z)}{\partial z}(\mu + \tfrac{1}{2}\sigma^2) \right) \frac{1}{g^2(\tau,z)} \right] e^{\frac{\mu^2}{\sigma^2}\tau}, \\ A_s &= -\frac{\partial g(\tau,z)}{\partial z}\frac{1}{s}\frac{1}{g^2(\tau,z)} e^{\frac{\mu^2}{\sigma^2}\tau}, \\ A_{ss} &= \frac{1}{s^2}[2\left(\frac{\partial g(\tau,z)}{\partial z}\right)^2 \frac{1}{g(\tau,z)} - \frac{\partial^2 g(\tau,z)}{\partial z^2} + \frac{\partial g(\tau,z)}{\partial z}]\frac{1}{g^2(\tau,z)} e^{\frac{\mu^2}{\sigma^2}\tau}. \end{aligned} \tag{2.35}$$

Putting (2.35) into (2.34) we observe a partial differential equation for $g(\tau,z)$ of the following form

$$\begin{aligned} \frac{\partial g(\tau,z)}{\partial \tau} &= -\frac{1}{2}\sigma^2 \frac{\partial^2 g(\tau,z)}{\partial z^2}, \\ g(0,z) &= 1. \end{aligned} \tag{2.36}$$

The problem given in (2.36) is heat equation and since the final condition is independent of the $s-$variable the solution is simply found as equal to $g(\tau,z) = 1$. Then, $A(t,s)$ is of the form

$$A(t,s) = e^{\frac{\mu^2}{\sigma^2}(T-t)}, \tag{2.37}$$

which completes the proof. ■

Proposition 2.3 shows us that the solution does not depend on s-variable. Therefore, we can conclude that for geometric type index processes the same solution can be reached by simply assuming $\ell_t = \ell_t/S_t$ and removing partial derivatives with respect to s-variable from the Hamilton-Jacobi-Bellman equation.

Moreover, using Equation (2.31), Proposition 2.3 has an immediate corollary.

Corollary 2.1 *The corresponding optimal leverage is*

$$\ell_t^* = \frac{\mu}{S_t \sigma^2}\left[F_t - V_t^*\right]. \tag{2.38}$$

Corollary 2.1 states that the optimal leverage preserves the classical form of leverage factor given in (1.5). This is actually an interesting result. With this form of the optimal leverage we may say that the constant leverage factor used by investors is a natural selection when the index follows a geometric Brownian motion and when we choose an $\mathcal{L}^2-$ utility criterion. We state this result in the following corollary.

Corollary 2.2 *The optimal leverage strategy* ℓ_t^* *is of the form given in* (1.5) *with*

$$m^* = \frac{\mu}{\sigma^2}. \tag{2.39}$$

Remark 2.1 *Comparing the often proposed form of the leverage strategy given in* (1.5) *with our optimal strategy as described in* (2.38)*, we realize that the standard form uses the present value of the asset price as denominator. This, however, is only a formal difference as in an arbitrage free market, the price of an asset and its present value have to coincide. Further, as we will demonstrate below, the wealth process corresponding to our leverage strategy never exceeds the present value of the future payments. Therefore, our strategy is always non-negative and thus coincides with its positive part.*

To continue with our considerations, we first derive the stochastic differential equation for the wealth process and then calculate its first and second moment, together with the expected shortfall.

Proposition 2.4 *Let* $m^* = \frac{\mu}{\sigma^2}$ *be the multiplier in the leverage strategy* ℓ_t^**. Let further* F_t *denote the present value at time* t *of the future obligations of constant proportion debt obligation. Then we have:*

$$\begin{aligned} V_t^* &= F_t + (v_0 - F_0)\, e^{\left(r - m^*\mu - \frac{1}{2}(m^*)^2\sigma^2\right)t - m^*\sigma W_t^s}, && (2.40)\\ \mathbb{E}\left(V_t^*\right) &= F_t + (v_0 - F_0)\, e^{(r-m^*\mu)t}, && (2.41)\\ Var\left(V_t^*\right) &= (v_0 - F_0)^2\, e^{2(r-m^*\mu)t}\left(e^{(m^*)^2\sigma^2 t} - 1\right), && (2.42)\\ \mathbb{E}\left(V_T^* - F_T \middle| \mathcal{F}_t\right) &= (V_t - F_t)\, e^{(r-m^*\mu)(T-t)}. && (2.43) \end{aligned}$$

Proof. Differential equations for V_t and F_t stated in (2.20) directly lead to

$$d\left(V_t^* - F_t\right) = \left(V_t^* - F_t\right)\left((r - m^*\mu)\, dt - m^*\sigma dW_t^s\right), \tag{2.44}$$

which implies

$$V_t^* - F_t = (v_0 - F_0)\, e^{\left(r - m^*\mu - \frac{1}{2}(m^*)^2\sigma^2\right)t - m^*\sigma W_t^s}. \tag{2.45}$$

From this and the well-known properties of geometric Brownian motion all the assertions of the proposition follow. ■

Remark 2.2 *1. Note first that for a similar leverage strategy, but with an arbitrary constant multiplier m, we obtain exactly the same results as in Proposition 2.4. In particular, we obtain the limiting behaviour for the first two moments of the corresponding wealth process as*

$$\lim_{m\to\infty} \mathbb{E}(V_t) = F_t, \tag{2.46}$$

$$\lim_{m\to\infty} Var(V_t) = \infty. \tag{2.47}$$

Thus, Equation (2.46) *yields that if m is big enough then the leverage strategy is in the mean able to generate the necessary income needed for the obligations. However, the variance will then tends to ∞ according to* (2.47) *which, on one hand, demonstrates the enormous risk of such a big leverage, and, on the other hand, illustrates how our $\mathcal{L}^2$-criterion balances between risk (expressed in terms of variance) and return (expressed in the mean shortfall).*

2. Another effect of our results derived so far is that for a given upper bound for the expected shortfall of

$$\gamma := (v_0 - F_0)\, e^{(r-m^*\mu)T}, \tag{2.48}$$

the leverage strategy ℓ_t^* is also mean-variance efficient.

An Application

In the sequel, we illustrate the behaviour of some characteristic features corresponding to our hedging activities of constant proportion debt obligation via some numerical examples. They will include dependence of the optimal strategy to the volatility of the risky index, dependence of the leverage to the volatility, and the dependence of the expected wealth to the leverage leverage multiplier m. In the analysis, we assume the following set of parameters:

$$\mu = 0.06,\ r = 0.05,\ T = 10,\ \nu = 0.025,\ F_T = 1, \tag{2.49}$$

and let σ takes on the different values, such as $\sigma = 0.025, 0.05$, and 0.25. We first look at the evolution of the expected wealth $\mathbb{E}(V_t^*)$ of the optimal hedging strategy over time as given by Proposition 2.4 for the three different values of σ.

As illustrated in Figure 2.1, using the fact that for a (very) small volatility a high leverage of $m^* = 96$ can be used to make gains with a high probability, the expected wealth increases very fast and ends up with approximately 1. Thus, there is nearly no shortfall in the end (note that we have shown that the wealth process never exceeds 1 at maturity!). A similar behaviour can be seen for the choice of $\sigma = 0.05$. On the other hand, for the high volatility $\sigma = 0.25$, it seems to be clear from the beginning that the final wealth does not get very close to 1. Even more, the expected wealth never gets above 1. This is in particular due to the fact that the optimal multiplier m^* in this case takes the value of $m^* = 0.96$ which simply is not enough to generate the required surplus of $\nu = 0.025$ for our set of parameters. A higher m^* would be considered as too risky under our $\mathcal{L}^2$-criterion.

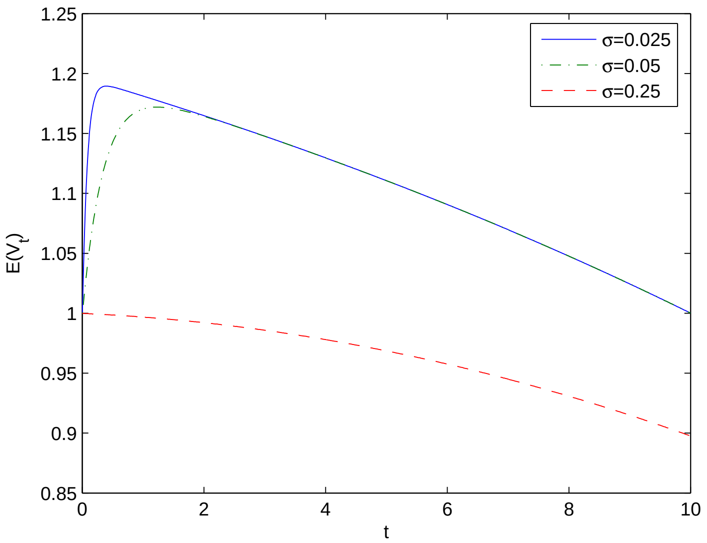

Figure 2.1: Optimal Wealth over Time for different σ

Moreover, in Figure 2.2 this effect is emphasized by the expected amount of money optimally invested in the risky asset over time for the different volatilities. For small volatilities the investment of a lot of money in the risky asset at the beginning of the investment period pays out quickly. The money needed to satisfy all future obligations can be collected quite fast and the risky positions are (nearly) closed after short time passes. However, for $\sigma = 0.25$, the risky position has to be held until maturity as – due to the small number m^* – not enough surplus can be generated.

Finally, Figures 2.3 and 2.4 demonstrate the evolution of the wealth and the amount of money in the stock (the leverage) corresponding to constant proportion debt obligation

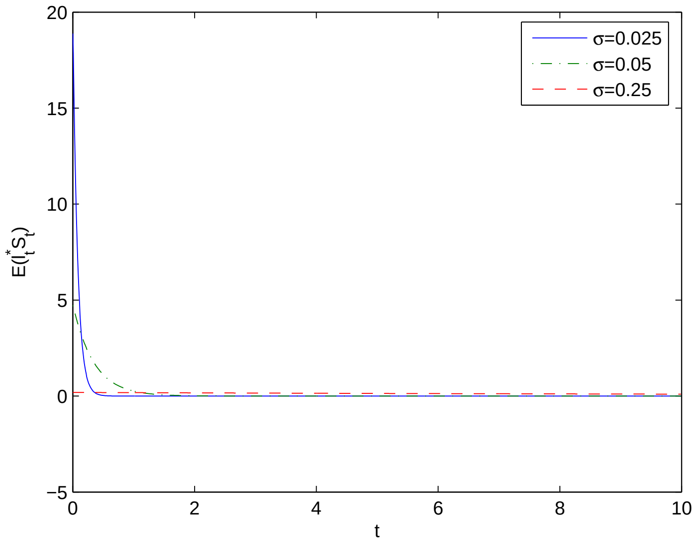

Figure 2.2: Optimal Amount of Money in the Risky Asset over Time for different σ

hedging strategy for different values of m and the optimal $m^* = \mu/\sigma^2$ for $\sigma = 0.025$. Again, we note the importance of selecting an appropriate value m to satisfy the obligations. Note that we computed the expectations without taking the positive part in the definition of the leverage strategy. For the non-optimal m values the wealth can exceed the capital needed to satisfy all future obligations. Note further that only a sufficiently big multiplier m can ensure that the (mean) final payment is close to 1.

Compared to Figure 2.3, the situation changes completely when the volatility is increased to $\sigma = 0.2$ as is done in Figure 2.5. In this case, the optimal multiplier $m^* = 1.5$ is not big enough to generate the necessary surplus to ensure that the final payment of 1 can be delivered, at least in the mean. To ensure the latter, one has to use a multiplier of at least $m = 10$ which bears a far too high risk in terms of the $\mathcal{L}^2$-criterion.

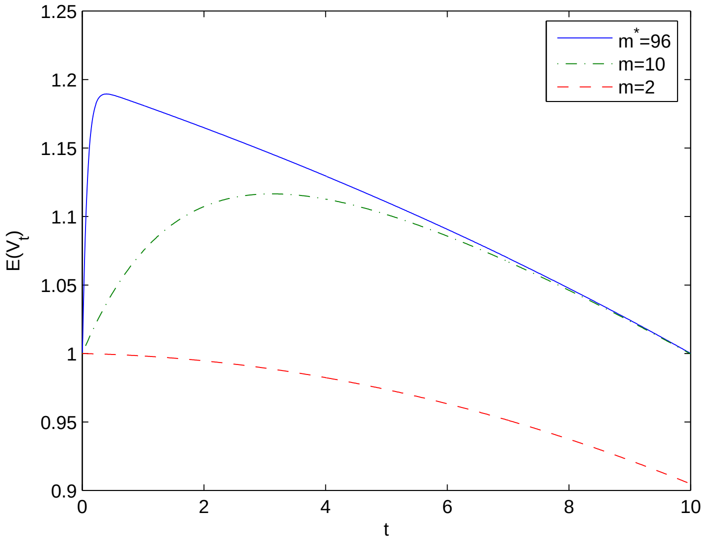

Figure 2.3: Optimal Wealth over Time for different m

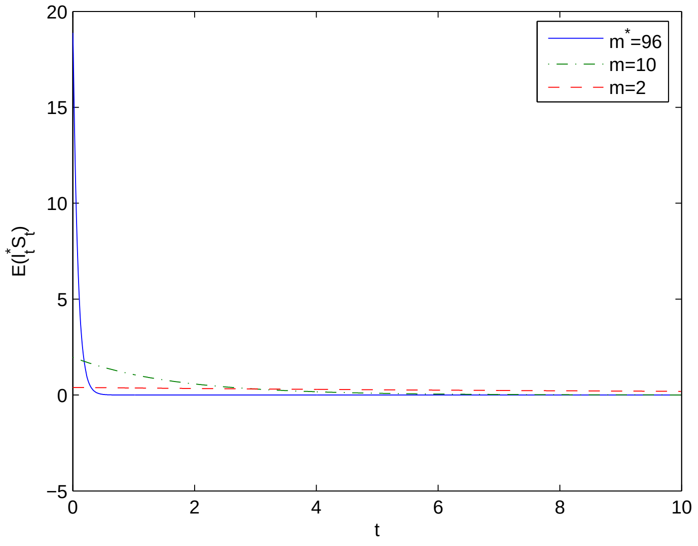

Figure 2.4: Optimal Amount of Money in the Risky Asset over Time for different m

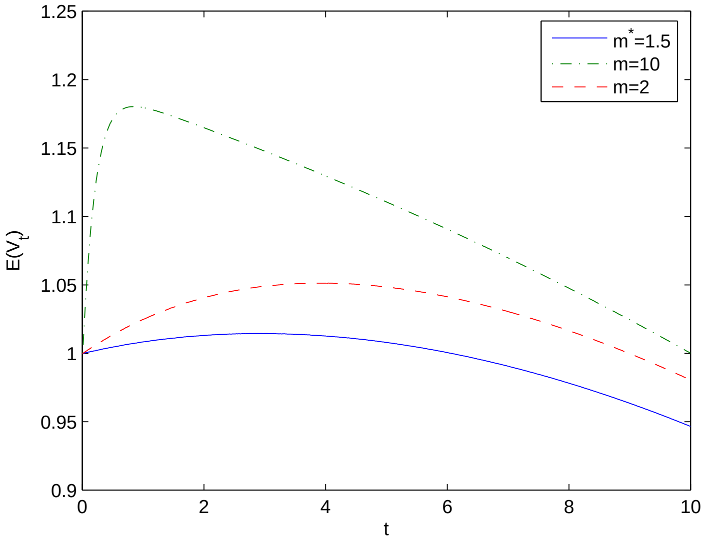

Figure 2.5: Optimal Wealth over Time for different m ($\sigma = 0.2$)

2.2.2 Solution when the Risky Index Follows a Vasicek Type Stochastic Differential Equation

The study mentioned above assumes that the risky credit swap index follows a geometric Brownian motion. However, the index is a portfolio of swap rates which are actually interest rates and so it may be more suitable to use one of the popular interest rate models to explain the dynamics of the credit swap index. For these reasons, in this section, we find a solution to the minimum mean-square distance problem in which the index follows

$$dS_t = [\mu - \kappa S_t]\, dt + \sigma dW_t^s, \tag{2.50}$$

with W^s a one-dimensional standard Brownian motion, μ and κ are non-negative constants and σ is the constant volatility parameter of the index.

With this index dynamics, the discounted wealth equation of the constant proportion debt obligation is given by

$$d\widetilde{V}_t = \left[\widetilde{\ell}_t \left(\mu - \kappa S_t\right) - F_T e^{-rt}(r+\nu)\right] dt + \widetilde{\ell}_t \sigma dW_t^s, \quad \widetilde{V}_0 = v. \tag{2.51}$$

By the dynamic programming principle of optimality we derive the Hamilton-Jacobi-Bellman (HJB) equation for the problem (2.24) as

$$\begin{aligned} \sup_{\widetilde{\ell}\in L(\widetilde{v})} \mathcal{D}^{\widetilde{\ell}}\phi(\widetilde{v}, s, t) &= \tfrac{\partial \phi}{\partial t} + \sup_{\widetilde{\ell}\in L(\widetilde{v})} \Big\{ \phi_{\widetilde{v}} \left[\widetilde{\ell}\left(\mu - \kappa s\right) - F_T e^{-rt}(r+\nu)\right] \\ &\quad + \tfrac{1}{2}\phi_{\widetilde{v}\widetilde{v}}\widetilde{\ell}^2\sigma^2 + \phi_s\left(\mu - \kappa s\right) + \tfrac{1}{2}\phi_{ss}\sigma^2 + \phi_{s\widetilde{v}}\widetilde{\ell}\sigma^2 \Big\}, \\ &= 0, \end{aligned} \tag{2.52}$$

where $\widetilde{V}_t = \widetilde{v}$, $S_t = s$ are the state variables and $\widetilde{\ell}_t = \widetilde{\ell}$ is the control variable at time t.

When formally solving the optimization problem in (2.52), the first order optimality conditions imply that a candidate for the optimal leverage is given by

$$\widetilde{\ell}_t = -\frac{\mu - \kappa s}{\sigma^2}\frac{\phi_{\widetilde{v}}}{\phi_{\widetilde{v}\widetilde{v}}} - \frac{\phi_{\widetilde{v}s}}{\phi_{\widetilde{v}\widetilde{v}}}. \tag{2.53}$$

Inserting (2.53) into the Hamilton-Jacobi-Bellman equation (2.52), and dropping the supremum operator, we obtain that the value function should satisfy the partial differential equation given as follows:

$$\begin{aligned} 0 &= \tfrac{\partial \phi}{\partial t} - \tfrac{(\mu-\kappa s)^2}{2\sigma^2}\tfrac{\phi_{\widetilde{v}}^2}{\phi_{\widetilde{v}\widetilde{v}}} - \left(\mu - \kappa s\right)\tfrac{\phi_{\widetilde{v}}\phi_{\widetilde{v}s}}{\phi_{\widetilde{v}\widetilde{v}}} \\ &\quad - F_T e^{-rt}(r+\nu)\phi_{\widetilde{v}} - \tfrac{1}{2}\sigma^2\tfrac{\phi_{\widetilde{v}s}^2}{\phi_{\widetilde{v}\widetilde{v}}} + \left(\mu - \kappa s\right)\phi_s + \tfrac{1}{2}\sigma^2\phi_{ss}. \end{aligned} \tag{2.54}$$

Proposition 2.5 *The closed-form solution of problem* (2.54) *with terminal condition* (2.26) *is as follows:*

$$\bar{\phi}(V_t, S_t, t) \equiv \phi(\widetilde{V}_t, S_t, t) = -\frac{1}{2}\left[F_t - V_t\right]^2 e^{2r(T-t)} A(t, S_t), \tag{2.55}$$

where

$$\begin{aligned} A(t, S_t) = &\sqrt{\cos\left(\kappa(T-t)\right) - \sin\left(\kappa(T-t)\right)} \\ &\times \exp\left\{-\tfrac{1}{2}(\gamma(t)-1)\tfrac{(\mu-\kappa S_t)^2}{\kappa\sigma^2} + \tfrac{1}{2}\kappa(T-t)\right\} \end{aligned} \tag{2.56}$$

and

$$\gamma(t) = \frac{1}{\cos^2\left(\kappa(T-t)\right) - \cos\left(\kappa(T-t)\right)\sin\left(\kappa(T-t)\right)} + \tan(\kappa(T-t)), \tag{2.57}$$

provided that $0 < \kappa(T-t) < \frac{\pi}{4}$.

Proof. The form of ϕ in (2.55) includes both an explicit part and an unknown function $A(t, S_t)$. By taking the partial derivatives of ϕ in (2.55) with respect to t, w, s, ww, ss, and ws and inserting them into (2.54) we reach the partial differential equation for function $A(t, S_t)$,

$$0 = \frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 A_{ss} - \sigma^2\frac{A_s^2}{A} - (\mu - \kappa s)\, A_s - \frac{(\mu-\kappa s)^2}{\sigma^2}A, \tag{2.58}$$

satisfying the terminal condition $A(T, s) = 1$.

To reduce this partial differential equation we define

$$\begin{aligned} \tau = &\ T - t, \\ z = &\ -\tfrac{\sqrt{2}}{\sigma\kappa}\left(\mu - \kappa s\right), \end{aligned} \tag{2.59}$$

and we assume that $A(t, S_t)$ is of the form

$$A(t, s) = -\frac{1}{g(\tau, z)}\exp\left\{-\frac{s}{\sigma^2}\left(\mu - \frac{\kappa s}{2}\right)\right\}. \tag{2.60}$$

By taking the partial derivatives of (2.60) we obtain the corresponding derivatives for function $g(\tau, z)$ as follows:

$$\begin{aligned} \frac{\partial A}{\partial t} &= -\frac{\partial g(\tau, z)}{\partial \tau}\frac{1}{g^2(\tau, z)}\exp\left\{-\frac{s}{\sigma^2}\left(\mu - \frac{\kappa s}{2}\right)\right\}, \\ A_s &= \left(\frac{\partial g(\tau, z)}{\partial z}\frac{\sqrt{2}}{\sigma} + g(\tau, z)\frac{\mu - \kappa s}{\sigma^2}\right)\frac{1}{g^2(\tau, z)}\exp\left\{-\frac{s}{\sigma^2}\left(\mu - \frac{\kappa s}{2}\right)\right\}, \\ A_{ss} &= \frac{1}{\sigma^2}\Bigg(-4\left(\frac{\partial g(\tau, z)}{\partial z}\right)^2\frac{1}{g(\tau, z)} - \frac{\partial g(\tau, z)}{\partial z}\frac{2\sqrt{2}}{\sigma}(\mu - \kappa s) \\ &\quad -g(\tau, z)\left(\frac{(\mu-\kappa s)^2}{\sigma^2} + \kappa\right) + 2\frac{\partial^2 g(\tau, z)}{\partial z^2}\Bigg)\frac{1}{g^2(\tau, z)}\exp\left\{-\frac{s}{\sigma^2}\left(\mu - \frac{\kappa s}{2}\right)\right\}. \end{aligned} \tag{2.61}$$

Then putting the derivatives given in (2.61) and the definitions of variables in (2.59) into (2.58) we find

$$\frac{\partial g(\tau,z)}{\partial \tau} = \frac{\partial^2 g(\tau,z)}{\partial z^2} + \frac{\kappa}{2}\left(1/2\kappa z^2 - 1\right) g(\tau,z) \tag{2.62}$$

with the initial condition

$$g(0,z) = -\exp\left\{\frac{1}{4}\left(\kappa z^2 - \frac{2\mu^2}{\sigma^2\kappa}\right)\right\}. \tag{2.63}$$

Moreover, referring to [45], we introduce

$$\begin{aligned} \widetilde{\tau} &= \tfrac{1}{\kappa}\tan(\kappa\tau), \\ \widetilde{z} &= \tfrac{z}{\cos(\kappa\tau)}, \end{aligned} \tag{2.64}$$

and assign

$$g(\tau,z) = h(\widetilde{\tau},\widetilde{z})\frac{1}{\sqrt{|\cos(\kappa\tau)|}}\exp\left\{\frac{\kappa}{2}\left(z^2/2\tan(\kappa\tau) - \tau\right)\right\}, \tag{2.65}$$

for the form of $g(\tau,z)$ where

$$h(0,\widetilde{z}) \equiv g(0,z). \tag{2.66}$$

Henceforth, by plugging the derivatives of (2.65) into (2.62) and replacing τ and z with $\widetilde{\tau}$ and $\widetilde{z}$, respectively; we finally obtain the standard heat problem that has to be satisfied by $h(\widetilde{\tau},\widetilde{z})$:

$$\begin{aligned} \frac{\partial h(\widetilde{\tau},\widetilde{z})}{\partial\widetilde{\tau}} &= \frac{\partial^2 h(\widetilde{\tau},\widetilde{z})}{\partial\widetilde{z}^2}, \\ h(0,\widetilde{z}) &= -\exp\left\{1/4\left(\kappa\widetilde{z}^2 - \frac{2\mu^2}{\sigma^2\kappa}\right)\right\}. \end{aligned} \tag{2.67}$$

Then, we obtain the solution of (2.67) as

$$h(\widetilde{\tau},\widetilde{z}) = -\frac{1}{\sqrt{1-\kappa\widetilde{\tau}}}\exp\left\{-1/2\left(\frac{\mu^2}{\sigma^2\kappa} - \frac{\kappa}{1-\kappa\widetilde{\tau}}\frac{\widetilde{z}^2}{2}\right)\right\}, \tag{2.68}$$

where $0 < \tan(\kappa\tau) < 1$ is assigned to provide a finite solution and refers to the condition given in proposition. Finally, putting (2.68) into (2.65) and inserting the form of $g(\tau,z)$ into $A(t,s)$ complete the proof. ■

Corollary 2.3 *The corresponding optimal leverage is*

$$\ell_t^* = \gamma(t)\frac{(\mu - \kappa S_t)}{\sigma^2}\left[F_t - V_t\right]. \tag{2.69}$$

Proof. We obtain the form of ℓ^* from $\widetilde{\ell}^*$ given in (2.53) as follows:

$$\ell_t^* = -\frac{\mu - \kappa s}{\sigma^2}\frac{\bar{\phi}_v}{\bar{\phi}_{vv}} - \frac{\bar{\phi}_{vs}}{\bar{\phi}_{vv}}. \tag{2.70}$$

Hence, inserting $\bar{\phi}_v$, $\bar{\phi}_{vs}$ and $\bar{\phi}_{vv}$ into (2.70) completes the proof. ■

Remark 2.3 *Here are some remarks:*

1. *Similarly, as in Section 2.2.1, comparing the positive part of the often proposed form of the leverage strategy given in* (1.5) *and the optimal strategy for the geometric Brownian case given in* (2.38) *with the optimal strategy obtained for the Vasicek case stated in* (2.69)*, we observe that the constant multiplier assumption fails to explain all the movements of the leverage for the Vasicek case. In other words, the optimal multiplier for the Vasicek case indeed is a function of the observed index value and time. Then, we define the optimal multiplier as follows:*

$$m^*(t, S_t) = \gamma(t)\frac{S_t(\mu - \kappa S_t)}{\sigma^2}. \tag{2.71}$$

2. *For the Vasicek case we cannot mention the non-negativity of the strategy because it includes the stochastic variable S_t.*

An Application

In this section, to figure out the behaviour of constant proportion debt obligations when the index satisfies the Vasicek type stochastic differential equation, we carry out a similar numerical analysis as in Section 2.2.1. In the analysis below, we use the set of parameters assigned to be

$$\mu = 0.08,\ \kappa = 0.02,\ r = 0.05,\ T = 10,\ \nu = 0.025,\ F_T = 1, \tag{2.72}$$

which are convenient with the parameters specified for geometric Brownian case and we assume the same values for $\sigma = 0.025, 0.05$, and 0.25. Within this application we use the optimal m^* as in (2.71) without any positivity constraints.

Figure 2.6 demonstrates the optimal expected wealth over time for different levels of σ. With this analysis we try to estimate the effect of volatility of the index to the optimal wealth. In the figure we see that for small and medium volatilities like $\sigma = 0.025, 0.05$ hedging works perfect and at maturity there is no shortfall. In fact, by using the optimal m^* at maturity constant proportion debt obligation generates the obligated amount F_T. However, for high volatility, $\sigma = 0.25$ the shortfall exists at maturity but it is not so big.

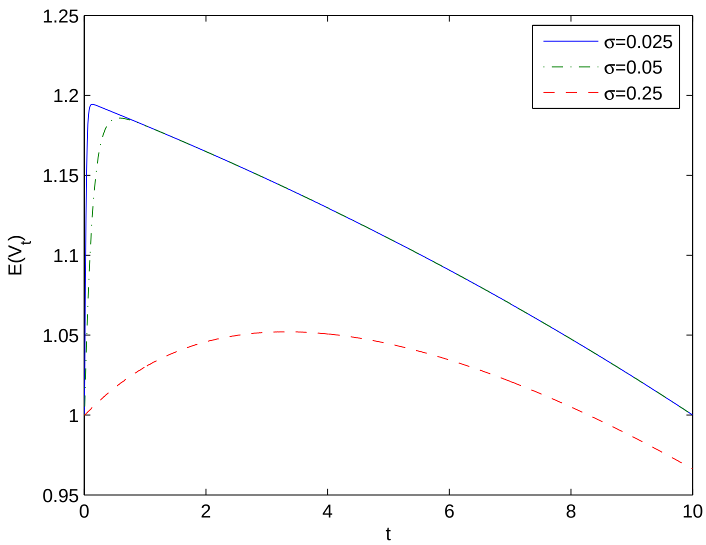

Figure 2.6: Optimal Wealth over Time for different σ

Moreover, in Figure 2.7 we see the effect of volatility to the optimal expected amount of money which is invested in the index. For $\sigma = 0.025, 0.05$ large amounts of money invested in the index at the beginning of the investment period. However, with this strategy the amount of money needed for paying obligation is earned quickly and position is closed without putting any positivity constraint on optimal leverage. In addition, for $\sigma = 0.25$ amount of money invested to the index is nearly 0 due to the high risk exposed.

In Figures 2.8 and 2.9, we look at the effect of optimal and arbitrary selections of m^* on the expected optimal wealth and the optimal amount of money in the index over time. We take $\sigma = 0.025$. As seen in Figures 2.8 and 2.9, optimal selection of m^* has a purpose of paying obligations only. It preserves the issuer from unnecessary risks taken to obtain large amounts of money.

On the other hand, in Figure 2.10, for $\sigma = 0.2$ the optimal multiplier m^* is not enough

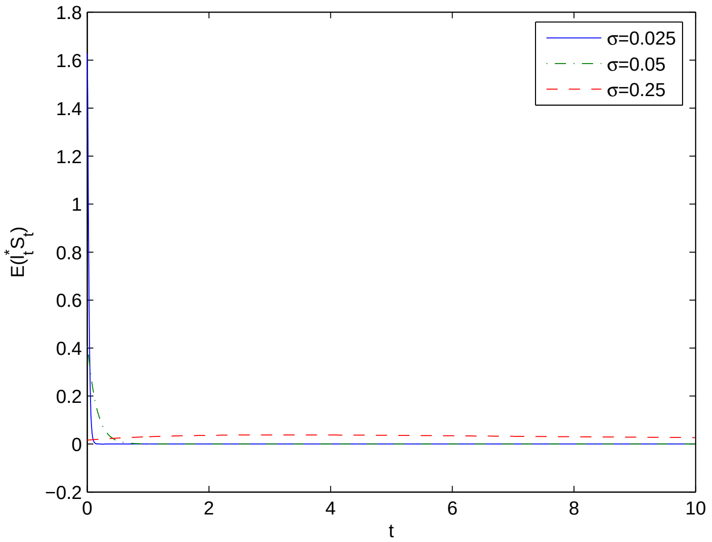

Figure 2.7: Optimal Amount of Money in the Risky Asset over Time for different σ

for satifying necessary surplus. However, the necessary surplus can be generated by selecting a multiplier at least $m = 10$, which seems too risky for our $\mathcal{L}^2$-criterion.

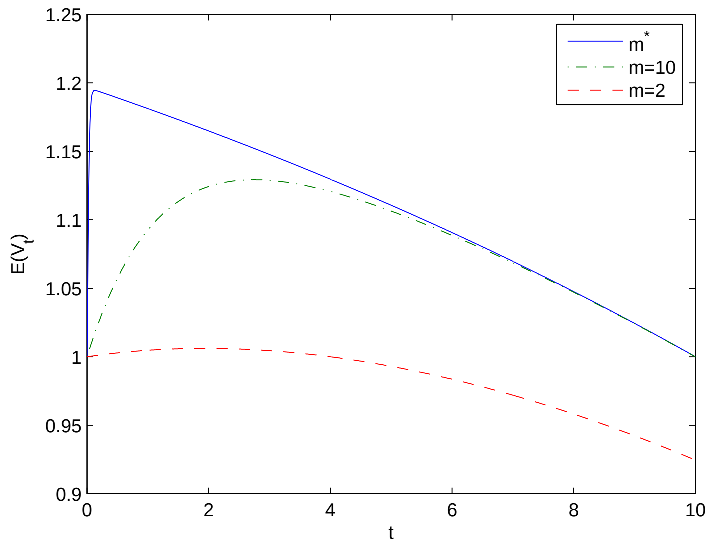

Figure 2.8: Optimal Wealth over Time for different m

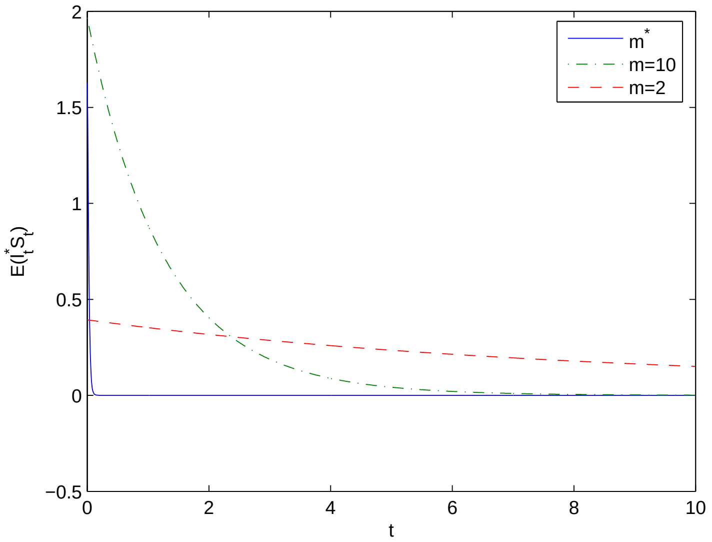

Figure 2.9: Optimal Amount of Money in the Risky Asset over Time for different m

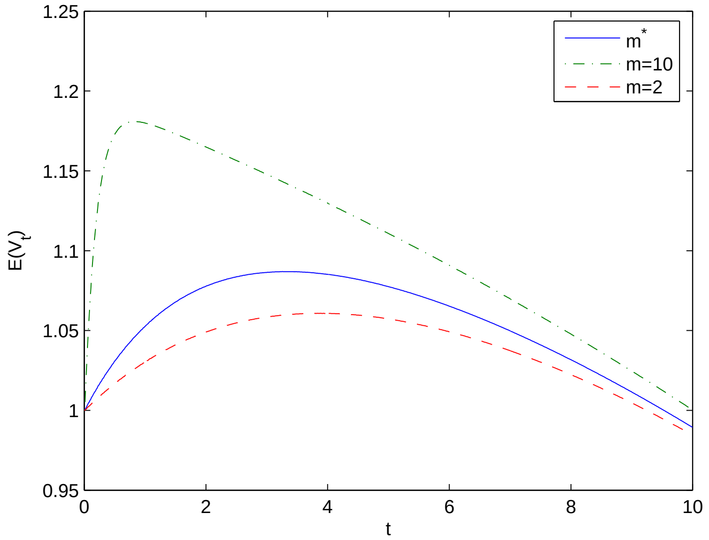

Figure 2.10: Optimal Wealth over Time for different m ($\sigma = 0.2$)

2.2.3 Solution when the Risky Index Follows a Geometric Jump Diffusion Process

In this part of the study we assume that the index follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t^s + Q(J) S_{t-} dN_t, \quad S_0 = s, \tag{2.73}$$

where μ and σ are drift and volatility parameters, W^s is a one-dimensional Brownian motion, N is a one-dimensional Poisson process with constant intensity λ and J is a random jump size with probability density function $\delta(J)$.

In the previous sections the processes assumed for the index changes had problems in explaining all the movements of the index. They could only explain normal index movements. The selection of the jump process provides to capture the jumps caused from defaults of credit default swaps included in the index and the adaptation of the index value in the rearrangement periods.

Inserting (2.73) into (1.1), we obtain

$$dV_t = [\ell_t \mu S_t - F_T(r+\nu)]\, dt + \ell_t \sigma S_t dW_t^s + +\ell Q(J) S_{t-} dN_t, \quad V_0 = v. \tag{2.74}$$

Here, the index follows a geometric type stochastic differential equation. Therefore, according to results stated in Section 2.2.1 for the geometric type processes we redefine the leverage factor at time t as $\ell_t = \ell_t / S_t$. Then, by applying the Ito formula for jump processes to $\widetilde{V}_t = e^{-rt} V_t$ and using the new definition of ℓ in its place we obtain the stochastic differential equation for $\widetilde{V}_t$ as

$$d\widetilde{V}_t = \left[\widetilde{\ell}_t \mu - F_T e^{-rt}(r+\nu)\right] dt + \widetilde{\ell}_t \sigma dW_t^s + \widetilde{\ell} Q(J) dN_t, \quad \widetilde{V}_0 = v. \tag{2.75}$$

The Hamilton-Jacobi-Bellman equation is as follows:

$$\begin{aligned} \sup_{\widetilde{\ell} \in L(\widetilde{v})} \mathcal{D}^{\widetilde{\ell}} \phi(\widetilde{v}, t) &= \tfrac{\partial \phi}{\partial t} + \sup_{\widetilde{\ell} \in L(\widetilde{v})} \Big\{ \phi_{\widetilde{v}} \left[\widetilde{\ell} \mu - F_T e^{-rt}(r+\nu)\right] \\ &\qquad + \tfrac{1}{2} \phi_{\widetilde{v}\widetilde{v}} \widetilde{\ell}^2 \sigma^2 + \lambda \mathbb{E}\left[\phi\left(\widetilde{v} + \widetilde{\ell} Q(J), t\right) - \phi(\widetilde{v}, t)\right], \\ &= 0, \end{aligned} \tag{2.76}$$

where $\widetilde{V}_t = \widetilde{v}$, $S_t = s$ are the state variables and $\widetilde{\ell}_t = \widetilde{\ell}$ is the control variable at time t.

Proposition 2.6 *The closed-form solution of problem* (2.54) *with terminal condition* (2.26) *is*

$$\bar{\phi}(V_t, S_t, t) \equiv \phi(\widetilde{V}_t, S_t, t) = -\frac{1}{2}\left[F_t - V_t\right]^2 e^{2r(T-t)} e^{-\frac{(\mu+\lambda\mathbb{E}(J))^2}{\sigma^2+\lambda\mathbb{E}(J^2)}(T-t)}. \tag{2.77}$$

Proof. As in previous solutions we assume the form of $\bar{\phi}$ as follows:

$$\bar{\phi}(V_t, S_t, t) = -\frac{1}{2}B^2(t, V_t)A(t), \tag{2.78}$$

where $A(t)$ is an unknown smooth function and

$$B(t, V_t) = \left[F_t - V_t\right] e^{r(T-t)}. \tag{2.79}$$

By taking the partial derivatives, of $\bar{\phi}$, with respect to the t, v, vv and inserting them into (2.76) we find that it holds:

$$\begin{aligned} 0 \quad &= -\tfrac{1}{2}B^2(t,v)A'(t) - \tfrac{1}{2}A(t)e^{-2rT}\widetilde{\ell}^2\left(\sigma^2 + \lambda\mathbb{E}(J^2)\right) \\ &+B(t,v)A(t)e^{rT}\widetilde{\ell}\left(\mu + \lambda\mathbb{E}(J)\right). \end{aligned} \tag{2.80}$$

Applying the first order optimality condition to (2.80) we obtain

$$\widetilde{\ell} = B(t,v)e^{-rT}\frac{\mu + \lambda\mathbb{E}(J)}{\sigma^2 + \lambda\mathbb{E}(J^2)}. \tag{2.81}$$

Putting (2.81) into (2.80) the first order ordinary differential equation for $A(t)$ is found to be

$$A'(t) = \frac{(\mu + \lambda\mathbb{E}(J))^2}{\sigma^2 + \lambda\mathbb{E}(J^2)}A(t), \tag{2.82}$$

with the terminal condition $A(T) = 1$. The solution of (2.82) is simply given as follows

$$A(t) = e^{-\frac{(\mu+\lambda\mathbb{E}(J))^2}{\sigma^2+\lambda\mathbb{E}(J^2)}(T-t)}. \tag{2.83}$$

Hence, inserting (2.83) into (2.78) completes the proof. ■

Proposition 2.6 and the use of (2.81) yield the following corollary.

Corollary 2.4 *The corresponding optimal leverage is*

$$\ell_t^* = \frac{1}{S_t}\frac{\mu + \lambda\mathbb{E}(J)}{\sigma^2 + \lambda\mathbb{E}(J^2)}\left[F_t - V_t\right]. \tag{2.84}$$

As stated in Section 2.2.1 the geometric type diffusion processes satisfies the proposed form of leverage strategy given in (1.5). For this problem, with Corollary 2.4 the optimal multiplier is written in the following corollary.

Corollary 2.5 *The optimal leverage strategy* ℓ_t^* *is of the form given in* (2.84) *with*

$$m^* = \frac{\mu + \lambda \mathbb{E}(J)}{\sigma^2 + \lambda \mathbb{E}(J^2)}. \tag{2.85}$$

Here, we can not say anything about the non-negativity of the strategy because of the jump movements.

Proposition 2.7 *Let* $m^* = \frac{\mu+\lambda\mathbb{E}(J)}{\sigma^2+\lambda\mathbb{E}(J^2)}$ *be the multiplier in the leverage strategy* ℓ_t^*. *Let further* F_t *denotes the present value at time* t *of the future obligations of constant proportion debt obligation. Then, we have*

$$\begin{aligned} V_t^* &= F_t + (v_0 - F_0)\, e^{\left(r - m^*\mu - \frac{1}{2}(m^*)^2\sigma^2\right)t - m^*\sigma W_t^s} && (2.86) \\ &\quad \times \prod_{i=1}^{N_t} \left(1 - mQ_i(J)\right), \\ \mathbb{E}\left(V_t^*\right) &= F_t + (v_0 - F_0)\, e^{(r - m^*\mu - m^*\lambda\mathbb{E}(Q_1(J)))t}, && (2.87) \\ Var\left(V_t^*\right) &= (v_0 - F_0)^2\, e^{2(r - m^*\mu - m^*\lambda\mathbb{E}(Q_1(J)))t} && (2.88) \\ &\quad \times \left(e^{(m^*)^2\left(\sigma^2 + \lambda\mathbb{E}\left(Q_1^2(J)\right)\right)t} - 1\right), \\ \mathbb{E}\left(V_T^* - F_T \middle| \mathcal{F}_t\right) &= (V_t - F_t)\, e^{(r - m^*\mu - m^*\lambda\mathbb{E}(Q_1(J)))(T-t)}. && (2.89) \end{aligned}$$

Proof. Differential equations for V_t and F_t stated in (2.20) directly lead to

$$d\left(V_t^* - F_t\right) = \left(V_t^* - F_t\right) \cdot \left((r - m^*\mu)\, dt - m^*\sigma dW_t^s - m^*Q(J)dN_t\right). \tag{2.90}$$

This implies that

$$\begin{aligned} V_t^* - F_t &= (v_0 - F_0)\, e^{\left(r - m^*\mu - \frac{1}{2}(m^*)^2\sigma^2\right)t - m^*\sigma W_t^s} \\ &\quad \times \prod_{i=1}^{N_t} \left(1 - mQ_i(J)\right). \end{aligned} \tag{2.91}$$

Consequently, for all the assertions of the proposition we refer to [28]. ■

An Application

To illustrate the effects that the jump risk adds to constant proportion debt obligations we make a similar analysis as in Sections 2.2.1 and 2.2.2. From the estimation theory of geometric Brownian motion and geometric jump diffusion we know that since the expectation and the variance of the data set are preserved the jump part affects the estimators of μ and σ. Therefore, to see the effect of jump part over the geometric Brownian motion and to make a visual comparision of the model we firstly assume the additional parameters included in the jump diffusion process given in (2.73) as follows:

$$\lambda = 0.17, \qquad \delta(Q) = N(\alpha, \beta^2) = N(-0.006, 0.0025). \tag{2.92}$$

Here, α is the mean and β is the standard deviation of the normal distribution. Then, to preserve the expectation and the variance of the S_t we obtain the drift and the volatility parameters for the jump diffusion process as stated subsequently:

$$\mu = 0.06102, \ \sigma = 0.0139, \ 0.0455, \ \text{and } 0.2491. \tag{2.93}$$

Other parameters are as in the geometric Brownian case.

Figure 2.11 presents the optimal wealth over time for three levels of σ. Accordingly, Figures 2.11 and 2.12 depict the same levels of volatility risks. The amount of money invested in the index shown in Figure 2.12 is also the same with that of geometric Brownian case. This shows us that the optimal leverage approach preserves the wealth of constant proportion debt obligation for a given index data by adapting m to the risks included in the underlying index.

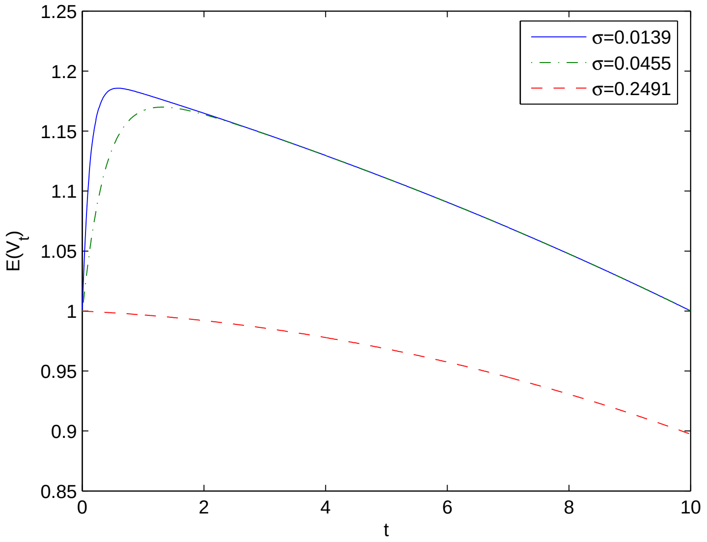

Figure 2.11: Optimal Wealth over Time for different σ

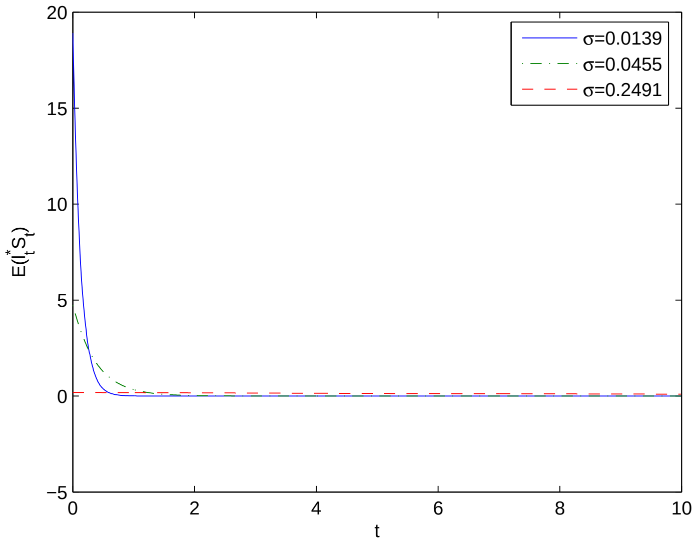

Figure 2.12: Optimal Amount of Money in the Risky Asset over Time for different σ

Then, for Figures 2.13 and 2.14 we assume $\mu = 0.06$ and $\sigma = 0.025$, for which the optimal value of m^* is calculated as 56. This is relatively smaller than that of the geometric Brownian case.

According to Figures 2.13 and 2.14 we can conclude that under our $\mathcal{L}^2$-criterion addition of new risks make the model more conservative with the selection of optimal m. With this strategy there is no shortfall for constant proportion debt obligations at maturity. However, for any arbitrary small selection of m we may still fall in trouble in fulfilling the obligations.

Figure 2.15 demonstrates a similar analysis for higher value of σ: $\sigma = 0.15$. At this volatility level, with the risk caused by the jumps, our model would not show high risk; however, it limits the optimal value to $m^* = 2.57$, which is not enough to pay all the obligations. Therefore, constant proportion debt obligation has a shortfall at maturity.

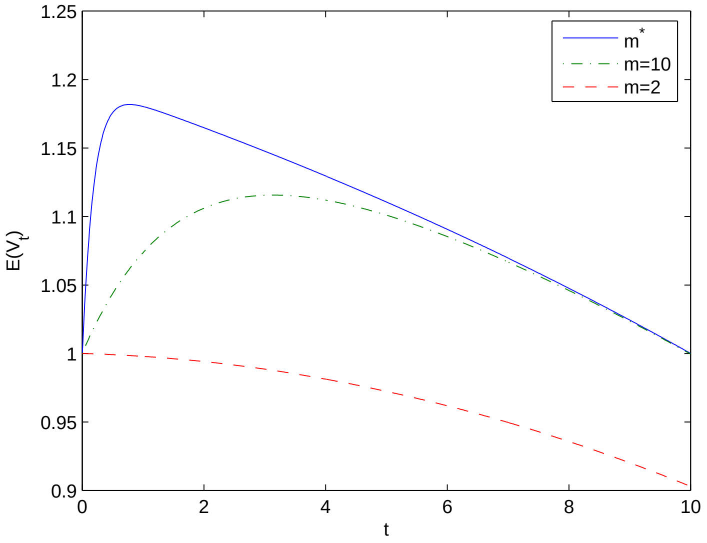

Figure 2.13: Optimal Wealth over Time for different m

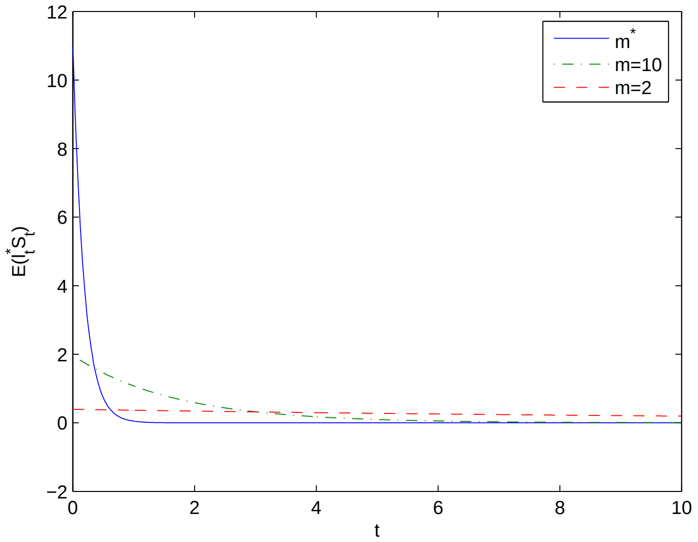

Figure 2.14: Optimal Amount of Money in the Risky Asset over Time for different m

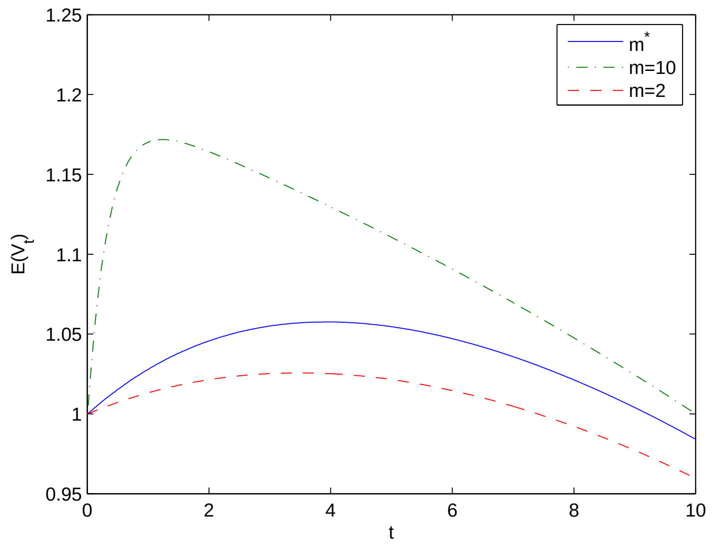

Figure 2.15: Optimal Wealth over Time for different m ($\sigma = 0.15$)

CHAPTER 3

PRICING OF CONSTANT PROPORTION DEBT OBLIGATIONS USING LAPLACE TRANSFORM TECHNIQUES

In Chapter 2, we considered the selection of the optimal leverage factor for constant proportion debt obligations. However, the fair price of the constant proportion debt obligations is not mentioned. Also, the early default possibility has not been included in the model. In this chapter, by considering early default with a particular leverage strategy given in (1.5) we classify constant proportion debt obligations as a double barrier option with rebate. For this purpose, we define two stopping times, one for the *early default* τ_{def} and one for the closing the *strategy* τ_{str} for constant proportion debt obligations, as

$$\tau_{def} = \inf \{t \in (0, T) \mid V_t < \beta_t\}, \tag{3.1}$$

$$\tau_{str} = \inf \{t \in (0, T) \mid V_t \geq F_t\}, \tag{3.2}$$

where β_t is either a predetermined constant or a variable default barrier. In other words:

(i) If $t \geq \tau_{def}$, then constant proportion debt obligation defaults in both coupon payments and final payment.

(ii) If $t \geq \tau_{str}$, then the issuers has enough money to pay all the obligations. Therefore, $\ell_s = 0$ for all $s \geq t$ and the equation of wealth process turns to be

$$dV_s = rV_s dt - F_T(r + \nu)ds, \; s \geq t. \tag{3.3}$$

In order to obtain the pricing equation for constant proportion debt obligation we consider two final conditions. The first condition is the final payment, written as $F_T - \max\{F_T - V_T, 0\}$. Note that with this condition we take into account that the final payment is guaranteed only when $V_T \geq F_T$. The second condition is the coupons distributed by constant proportion debt obligations if early default does not occur until maturity. This costs a total amount of $F_T \frac{r+\nu}{r}\left(1 - e^{-r(T-t)}\right)$ units of money at time t. Accordingly, we write the pricing equation for constant proportion debt obligation as follows: whenever $\tau_{def} \geq T$, we have

$$\begin{aligned} \varphi(v,t) &= F_T \tfrac{r+\nu}{r}\left(1 - e^{-r(T-t)}\right) + F_T e^{-r(T-t)} \\ &\quad -\mathbb{E}^Q\left(\max\{F_T - V_T, 0\}\, e^{-r(T-t)} | \mathcal{F}_t\right), \end{aligned} \tag{3.4}$$

together with the final (terminal) and the boundary conditions, respectively, defined to be

$$\begin{aligned} \varphi(v,T) &= F_T - \max\{F_T - v, 0\}, \\ \varphi(v,\tau_{def}) &= K, \end{aligned} \tag{3.5}$$

where K is either a constant or a variable rebate.

Remark 3.1 *Since the consequence of hitting τ_{str} coincides with the pricing equation given in* (3.4) *we do not need to define any boundary condition for it.*

Thus, to solve the problem we should investigate the first passage time distribution of hitting the default barrier,

$$\mathbb{P}\left(\tau_{def} \leq t\right) = \mathbb{P}\left\{\inf_{0\leq s\leq t} V_s \leq \beta_t\right\}, \tag{3.6}$$

and the joint distribution of the first passage time and the terminal value

$$\mathbb{P}\left\{\tau_{def} \leq t,\, V_t \geq a\right\}, \tag{3.7}$$

where a is a realization of the random variable V_t.

In cases without jumps, processes are commonly selected as geometric Brownian motions. Hence, the distributions given in (3.6) and (3.7) are easily obtained by using both the Girsanov Theorem and the reflection principle or, by applying Laplace transform technique. However, when processes include jumps, the first passage time distributions are not easily determined. Because a jump process will either hit at the

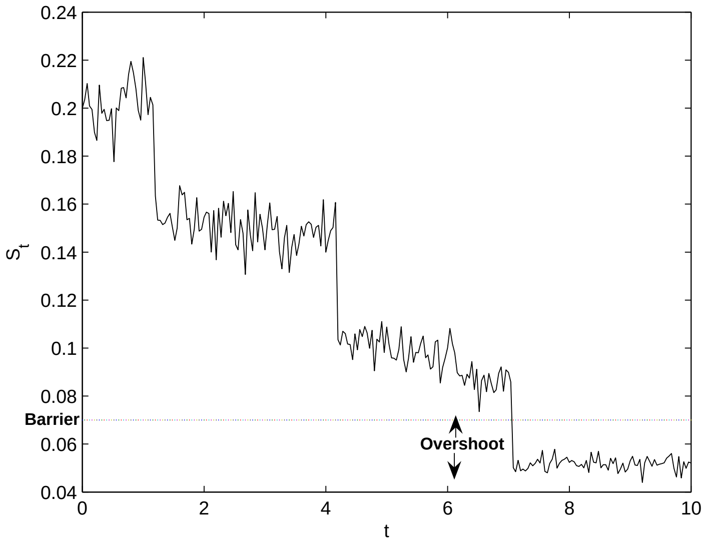

Figure 3.1: Overshoot Problem

boundary or will cross over the default barrier. The latter is generally refered to as "overshoot" the barrier [36], which is illustrated in Figure 3.1.

Overshoot is not a trivial problem to tackle with in order to derive the first passage time distribution. From the stochastic renewal theory it is known that the exact distribution of the overshoot is obtained only when jump sizes follow an exponential type distribution [35].

Remark 3.2 *The renewal theory is a class of the probability theory which mainly deals with the counting processes. In renewal theory it is assumed that the inter-arrival times are independently identically distributed with an arbitrary distribution.*

In this chapter we mainly focus on solutions of the pricing equation stated in (3.4): We consider the case for ordinary geometric Brownian motion and the jump diffusion processes with double exponential jump sizes.

Literature on the use of double exponential jump distribution in option pricing is given in Section 3.1, in which the contribution of double exponential jump distribution to the models is also explained and compared to other models in literature. Further, in Section 3.2 we briefly summarize the Rational Expectation settings that the double exponential jump distribution satisfies. The construction of the model for the problem is outlined in Section 3.3. Furthermore, solutions to the pricing problems in the cases of geometric Brownian and geometric jump diffusion model with double exponential jump sizes are given in Sections 3.4 and 3.5, respectively.

3.1 Literature and Contributions to Double Exponential Distribution on Option Pricing

With the seminal work of Merton on option pricing [40] jump diffusion processes entered the literature on the theory of derivative pricing. Moreover, because of the limitations of geometric Brownian motion and, hence, the Black and Scholes option pricing model [11], jump diffusion processes found great acceptance.

There are enormous amount of studies on the generalization of jump diffusion processes. According to the literature these generalizations may either be made by simply specifying drift and diffusion parameters of the jump diffusion processes or be made by assuming a different distribution of jump sizes. For example, Andersen and Andreasen [7] is one of the researches that investigated the jump processes. In their study, they formulated a forward Partial Integro-Differential Equation (PIDE) for European call option prices and performed a numeric study. Duffie, Pan, and Singleton defined processes in which all parts (drift, volatility, and jump) show affine dependence on the state variable [21]; such a model is refered to as a *affine jump diffusion model*. Furthermore, they derived a closed form formula for a general transform of affine jump diffusion processes, and they provided examples to validate their methodology for applications in option pricing.

Double exponential jump diffusion processes were used in option pricing by Kou [35] in 2002 for the first time. Therein an analytical pricing formulae for European options on assets as well as futures contracts were given. Main studies on option pricing with double exponential then followed: Sepp [51] applied the double exponential jump diffusions to the path dependent options like double barrier options and double touch options. The closed form pricing formulae for such derivatives in Laplace domain were also presented. Kou and Wang [37] in 2004 gave an approximation formula for pricing American options in the case of double exponential jump processes, and they derived analytical pricing formulae for lookback, barrier and perpetual American options. Meanwhile, Cont and Tankov calibrated option prices for various exponential Lévy models including double exponential jump processes by using a non-parametric method [16]. In 2009, Cai, Chen and Wan [13] worked on a more complex version of the double exponential distribution. In their paper, *hyper* exponential distribution, which is indeed a mixture of more than two exponential distributions, was used to provide an analytical pricing equation for the double barrier options.

In short, double exponential distribution is a special form of both affine jump diffusions and Lévy processes. Due to its desirable properties listed below, it has gained popularity in recent years [18, 34, 37]:

Leptokurtic property with two sided jumps. Distribution of double exponential jump diffusion process shows heavier tails and a higher peak than normal distribution. It is also more skewed. With these properties double exponential jump processes are more effective on capturing the movements on stock and index returns in the market.

Volatility smile. Emprical studies indicate that implied volatility curves of the options show convex nature, which is generally called a volatility *smile*. Double exponential jump size processes also enjoy from this feature.

Memoryless property. This is an important feature of the exponential distribution. In fact, beside its memoryless property, double exponential jump diffusion processes provide analytical solutions not only for simple European options but also for more complex, exotic and path dependent options.

3.1.1 Contributions of Double Exponential Jumps to the Models

Referring once again to [34–37], in this section, evaluation of the models with double exponential jump diffusion processes are discussed. Mainly, the contributions of double exponential jump diffusion processes to the underlying models are summed in the following groups.

Self Consistency. In financial modelling, an arbitrage-free model that satisfies the equilibrium settings is called *self-consistent.* Consequently, such a self-consistent equilibrium modelling allows closed form pricing formulae of derivatives easily. Moreover, an economy in presence of arbitrage causes infinite demand and supply for securities which is inconsistent with both preferences of individuals and equilibrium setting [47].

Double exponential jump diffusion process satisfies the rational expectation equilibrium setting (which will be explained in Section 3.2), and it does not allow arbitrage opportunities. Therefore, it is self-consistent [35].

Empirical Efficiency. In modelling another important thing is that the constructed models should represent and reflect the real world conditions to a certain extent. For example, most of the option pricing models assume a constant volatility scheme for options, however, the real world applications and observations prove that options show volatility smiles. Therefore, models with constant volatility are not considered empirically efficient in option pricing. Similarly, normality is the fundamental assumption for the models used in pricing. However, beside the considerable advantages, building models on normality assumption does not capture all movements of the real world. Having the leptokurtic property with two-sided jumps and satisfying volatility smile feature, double exponential jump diffusion is empirically efficient in modelling of options.

Simplicity. Simplicity of the calculations is as important as the efficiency. Consider a model that explains all emprical indicators of the real world but it is too complex to attain a closed form solution. The use of such a model in practice is not considered appropriate. However, double exponential jump diffusion model ensures closed form solutions not only for plain European options but also for more complex, exotic and path dependent ones.

Interpretability. This property with empirical efficiency provides models to interpret economic movements. Empirical indicators show that the market components like stocks and indexes are affected from over- and under-reactions subject to the economic news [35]. Most models do not cover such reactions of the market. According to empirical researches such reactions are observed with a large temporary motion. Double exponential jump diffusion model with leptokurtic property has a good advantage of interpreting such an economy.

3.1.2 Superiority of Double Exponential Jump Diffusion Model

In this section a comparison of the double exponential jump diffusion model with other well known models is made. Table 3.1 demonstrates and compares the most widely used models of option pricing in terms of their fundamental properties.

Table 3.1: Model Comparison

	Analytical Solutions		Leptokurtosis	Volatility Smile
	European Options	Path-Dependent Options		
GBM-BS	X	X	-	-
CEV	X	X	-	-
N-JD	X	-	X	X
DE-JD	X	X	X	X

We briefly describe the deficiencies and advantageous of the models introduced in Table 3.1 in the sequel.

Geometric Brownian Motion and Black-Scholes Model (GBM-BS). Geometric Brownian motion and the famous model of Black and Scholes [11] have found great acceptance in literature: Geometric Brownian motion allows analytical solutions for both basic European and complex options. Furthermore, the model is too simple to apply.

Black-Scholes model depends basically on normality assumption which does not hold for options in general. Options show leptokurtosis which generates biases in pricing and generate an implied volatility curve. However, in the Black-Scholes setting, the implied volatility is assumed to be constant, and hence the associated normal distribution does not show a leptokurtic behaviour.

The Constant Elasticity of Variance Model (CEV). The CEV model was first introduced by Cox [17] in 1975. This model is a generalized version of the model that uses a geometric Brownian motion. In this setting volatility changes with stock prices. Under the assumptions of the model, stock price process satisfies a stochastic differential equation of the form

$$dS_t = \mu S_t dt + \sigma S_t^{\beta} dW_t^S, \tag{3.8}$$

where $\beta > 0$ is a constant parameter.

For the CEV model analytical solutions for European and path dependent options are available like the ones in the Black-Scholes setting. However, tails under the CEV model show differences according to the parameter, β. If $\beta < 1$, the left tail of the return distribution is heavier and the right tail is thinner than those in the normal distribution. If $\beta > 1$, tails present totally opposite behavior with thinner left tail and heavier right tail [46]. Moreover, the CEV model assumes an implied volatility, as a monotone function of the strike price; but this cannot capture the volatility smile [35].

Normal Jump Diffusion Model (N-JD). Normal jump diffusion model satisfies leptokuticity and has the implied volatility curve feature for options. However, closed form solutions cannot be found for path dependent options by using this model.

Due to the features described for each model above, the double exponential jump diffusion model is superior to other most widely used models in the literature of the path-dependent options. This is the reason why we select the geometric jump diffusion process with double exponential jump sizes.

3.2 Rational Expectations Equilibrium Setting

A rational expectation theory is introduced by Muth in 1960s [42]. Then, the idea is mathematically supported and developed by Lucas in 1978 [39]. The Lucas's theory depends on the idea that if expectations of investors in an economy is rational then the goverment effect cannot change the performance of the economy on average. In other words, an economy can only have one equilibrium and so the future state of the economy is equivalent to the investors' expectations about the equilibrium [3–5, 48].

Let $U(x, t)$ be the utility function of an investor which represents all the investors in the economy. Then, in a rational expectations economy the investor has an objective to maximize his/her utility gained from consumptions [35]. In other words, the utility maximization problem of the investor is defined as follows

$$\underset{c}{\text{maximise}}\ \mathbb{E}\left(\int_0^\infty U(c(t),t)dt\right), \tag{3.9}$$

where $c(t)$ is the consumption process.

In such an economy, it is also assumed that there is an endowment process $\delta(t)$ and the investors have an opportunity to invest in a security $p(t)$ which does not distribute dividends. Then, if $\delta(t)$ is Markovian, the equilibrium price of the security $p(t)$ in the rational expectations economy is given by the Euler equation,

$$p(t) = \frac{\mathbb{E}\left(U_c(\delta(T),T)p(T) \mid \mathcal{F}_t\right)}{U_c(\delta(t),t)}, \quad \forall T \in [t, T_0], \tag{3.10}$$

where U_c is the partial derivative of utility function U with respect to c and T_0 is a finite liquidation date [35].

In a rational expectations economy with an equivalent martingale measure Q, the stock price is transformed to a form in which the drift parameter is equal to r, and other parameters are suitably adapted to the new measure and the pricing equation is calculated by using the transformed stock price [31]. In order to illustrate this, suppose that the stock price follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t + \left(e^J - 1\right) S_{t^-} dN_t, \quad S_0 = s, \tag{3.11}$$

where W_t is a one-dimensional standard Brownian motion, N_t is a Poisson process with rate λ and J is a random jump size with density

$$f_J(j) = q_1 \frac{1}{\eta_1} e^{-\frac{j}{\eta_1}} \mathbb{I}_{j\geq 0} + q_2 \frac{1}{\eta_2} e^{\frac{j}{\eta_2}} \mathbb{I}_{j<0}, \quad 1 > \eta_1 > 0, \quad \eta_2 > 0, \tag{3.12}$$

where q_1, $q_2 \geq 0$, $q_1 + q_2 = 1$. Moreover, W_t, N_t, J are assumed to be independent processes.

Let us define Hyperbolic Absolute Risk Aversion (HARA) type utility functions as follows:

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^\gamma, \tag{3.13}$$

satisfying $\frac{ax}{1-\gamma} + b > 0$ where a, b and γ are constant parameters. Then, referring to [37] we can write (3.11) under the rational expectations assumptions with HARA type utility functions as

$$d\bar{S}_t = (r - \lambda^*\zeta^*)\bar{S}_t dt + \sigma \bar{S}_t dW_t^* + \left(e^{J^*} - 1\right) dN_t^*, \quad \bar{S}_0 = \bar{s}, \tag{3.14}$$

where W_t^* is is a one-dimensional standard Brownian motion, N_t^* is a Poisson process with intensity λ^* and W_t^*, N_t^*, J^* are independent processes under Q. Moreover, J^* follows a new double exponential distribution

$$f_{J^*}(j) = q_1^* \frac{1}{\eta_1^*} e^{-\frac{j}{\eta_1^*}} \mathbb{I}_{j \geq 0} + q_2^* \frac{1}{\eta_2^*} e^{\frac{j}{\eta_2^*}} \mathbb{I}_{j<0}, \qquad 1 > \eta_1^* > 0, \quad \eta_2^* > 0, \tag{3.15}$$

where q_1^*, $q_2^* \geq 0$, $q_1^* + q_2^* = 1$, and $\zeta^* := \mathbb{E}^*\left[e^{J^*}\right] - 1 = \frac{q_1^*}{1-\eta_1^*} + \frac{q_2^*}{1+\eta_2^*} - 1$. This fact will be used in Section 3.5 for the pricing problem in the double exponential jump setting.

3.3 Modelling the Problem

In continuous trading, constant proportion debt obligations under the assumption of a geometric Brownian asset with usual leverage factor given in (1.5) may not be able to generate additional interest without getting close to default. This is due to the fact that the constant proportion debt obligation has the wealth equation in the form

$$V_t = F_t + (v - F_0) e^{\left(r - m\mu - \frac{1}{2}m^2\sigma^2\right)t - m\sigma W_t^s}. \tag{3.16}$$

Since we know that $v - F_0 < 0$, future obligations are guaranteed asymptotically if and only if m grows unboundedly; and such a big selection of m may add other risks to the investors.

As a consequence of these drawbacks we change the strategy of constant proportion debt obligations in this chapter. Based on the approach stated in [32] to eliminate $F_T(r + \nu)dt$ term in (1.1), the strategy is splited into two parts: (i) the risk free investment for satisfying coupon payments, and (ii) the investment of the remaining assets in a portfolio. With this splitted strategy the initial wealth v may be written in the form

$$v = \widetilde{v} + \bar{v}, \tag{3.17}$$

where $\widetilde{v} = G\frac{(r+\nu)}{r}(1 - e^{-rT})$ is the necessary amount of money to guarantee all coupon payments, and $\bar{v} = v - \widetilde{v}$ is the remaining part for generating a portfolio strategy. Here, we assume that $F_T = G$. In order to satisfy $\bar{v} > 0$, we also assume that $v = G$ and $\nu < r\frac{e^{-rT}}{1-e^{-rT}}$.

Accordingly, the portfolio strategy satisfies the stochastic differential equation

$$d\bar{V}_t = r_t\bar{V}_t dt + \ell_t dS_t, \quad \bar{V}_0 = \bar{v}, \tag{3.18}$$

and the current value of the risk free investment strategy allocated for paying the coupons is found to be

$$\widetilde{V}_t = G\frac{(r+\nu)}{r}(1 - e^{-r(T-t)}), \quad \widetilde{V}_0 = \widetilde{v}, \tag{3.19}$$

with the condition that $V_t = \widetilde{V}_t + \bar{V}_t$.

Here, all the coupons are satisfied by $\widetilde{V}_t$, whereas, $\bar{V}_t$ is only responsible in fulfilling final payment G. Therefore, as in the usual strategy, it is not necessary to subtract the present value of the future obligations from the wealth. In other words, we assume that the amount of money invested in the risky asset is only a multiple of the current wealth over the current index value, namely, $\ell_t = m\frac{\bar{V}_t}{S_t}$.

Our only problem is then rearranging the definitions of default condition and the pricing equation of constant proportion debt obligation according to the changes made in the strategy. In settings of a constant proportion debt obligation, usually a small proportion α of the present value of the future obligations is taken as a default barrier, $B = \alpha F_t$. Thus, it is reasonable to take default barrier as $\alpha G e^{-r(T-t)}$. Consequently, the default time for constant proportion debt obligation is simply written as

$$\tau_{def} = \inf\left\{t \in (0,T) \mid \bar{V}_t < \alpha G e^{-r(T-t)}\right\}. \tag{3.20}$$

Since $\widetilde{V}_T = 0$, and hence $V_T = \bar{V}_T$, the pricing equation given in (3.4) is still valid and it can be re-expressed as

$$\varphi(V,t) = \widetilde{V}_t + Ge^{-r(T-t)} - \mathbb{E}^Q\left(\max\left\{G - \bar{V}_T, 0\right\} e^{-r(T-t)} \mid \mathcal{F}_t\right) \tag{3.21}$$

for $\tau_{def} \geq T$, where the terminal condition is given by

$$\varphi(V,T) = G - \max\{G - V, 0\}. \tag{3.22}$$

However, the boundary condition changes to

$$\varphi(V, \tau_{def}) = \widetilde{V}_t + R, \tag{3.23}$$

where R is a constant or a variable rebate.

It turns out that the only unknown term in pricing equation (3.21) is the conditional expectation

$$\phi(\bar{V},t) := \mathbb{E}^Q\left(\max\left\{G-\bar{V}_T,0\right\}e^{-r(T-t)} \mid \mathcal{F}_t\right).$$

This is in fact a barrier option written on a portfolio regarded as a trading asset. If $\bar{V}_T \leq G$ then option pays $G-\bar{V}_T$ amount of money, otherwise it pays nothing. Moreover, if $\bar{V}_t \leq \alpha G e^{-r(T-t)}$ option defaults and pays $Ge^{-r(T-t)}-R$ amounts of rebate. Therefore, a barrier option problem hidden in the pricing equation of constant proportion debt obligations is defined as follows:

$$\begin{aligned} \phi(\bar{V},t) &= \mathbb{E}^Q\left(\max\left\{G-\bar{V}_T,0\right\}e^{-r(T-t)} \mid \mathcal{F}_t\right), \\ d\bar{V}_t &= r\bar{V}_t dt + \ell_t dS_t, \quad \bar{V}_0 = \bar{v}, \end{aligned} \tag{3.24}$$

subject to the terminal and the boundary conditions

$$\begin{aligned} \phi(\bar{V},T) &= \max\left\{G-\bar{V},0\right\}, \\ \phi(\bar{V},\tau_{def}) &= Ge^{-r(T-\tau_{def})}-R, \end{aligned} \tag{3.25}$$

respectively. Solving this problem yields the pricing equation for constant proportion debt obligation when $\phi(\bar{V},t)$ is plugged into (3.21).

The following Sections 3.4 and 3.5 deal with the solution of the problem given in (3.24) under the assumptions of geometric Brownian and double exponential jump diffusion processes, respectively. In calculations we refer to the work of Sepp [51] and we use the following properties of Laplace transformation:

$$\begin{aligned} L(x,p) &= \mathcal{L}\left(f(x,t)\right) = \int_0^\infty f(x,t)e^{-pt}dt, \\ \mathcal{L}\left(\frac{\partial f(x,t)}{\partial t}\right) &= pL(x,p) - f(x,0), \\ \mathcal{L}\left(\frac{\partial^n f(x,t)}{\partial x^n}\right) &= \frac{\partial^n L(x,p)}{\partial x^n}, \end{aligned} \tag{3.26}$$

where $\mathcal{L}$ denotes the Laplace transformation.

3.4 Solution of the Hidden Barrier Pricing Problem with Geometric Brownian Motion

Similarly as in Chapter 2, we begin with the simplest selection of index dynamics: let the index S_t follow a Markov stochastic differential equation as in (2.28). Then the corresponding portfolio equation can be written as

$$d\bar{V}_t = (r + m\mu)\bar{V}_t dt + m\sigma\bar{V}_t dW_t, \quad \bar{V}_0 = \bar{v}. \tag{3.27}$$

Furthermore, under the equivalent martingale measure Q, dynamics of $\bar{V}_t$ becomes

$$d\bar{V}_t = r\bar{V}_t dt + m\sigma\bar{V}_t d\widetilde{W}_t, \quad \bar{V}_0 = \bar{v}, \tag{3.28}$$

where $d\widetilde{W}_t = dW_t + \mu/\sigma dt$ is the Brownian motion under Q. Consequently, the discounted value of the portfolio is a martingale under Q.

The value function $\phi(\bar{V}, t)$ of such a European barrier option therefore satisfies the partial differential equation

$$0 = \frac{\partial\phi}{\partial t} + \frac{1}{2}m^2\sigma^2\bar{V}^2\phi_{\bar{V}\bar{V}} + r\bar{V}\phi_{\bar{V}} - r\phi, \quad \bar{V}_d < \bar{V} < \infty, \tag{3.29}$$

with the corresponding terminal and the boundary conditions, respectively,

$$\begin{aligned} \phi(\bar{V}, T) &= \max\left\{G - \bar{V}, 0\right\}, \\ \phi(\bar{V}_d, t) &= Ge^{-r(T-t)} - R. \end{aligned} \tag{3.30}$$

Here the left boundary $\bar{V}_d$ is defined to be $\bar{V}_d = \alpha G e^{-r(T-t)}$ and $\phi_{\bar{V}}$ and $\phi_{\bar{V}\bar{V}}$ stand for the partial derivatives of $\phi(\bar{V}, t)$ with respect to $\bar{V}$.

To make the partial differential equation (3.29) a constant coefficient one we define the change of variables,

$$\tau = T - t, \qquad x = \ln\left(\frac{\bar{V}}{G}\right). \tag{3.31}$$

Hence, $\bar{\phi}(x, \tau) := \phi(\bar{V}(x), t(\tau))$ satisfies the partial differential equation

$$0 = -\frac{\partial\bar{\phi}}{\partial\tau} + \frac{1}{2}m^2\sigma^2\bar{\phi}_{xx} + \left(r - \frac{1}{2}m^2\sigma^2\right)\bar{\phi}_x - r\bar{\phi}, \quad x_d < x < \infty, \tag{3.32}$$

where $x_d = \ln\left(\frac{\bar{V}_d}{G}\right)$, and

$$\bar{\phi}\left(x, 0\right) = \max\left\{G\left(1 - e^x\right), 0\right\}, \quad \bar{\phi}(x_d, \tau) = Ge^{-r\tau} - R. \tag{3.33}$$

Since $\bar{V}_t$ follows a geometric Brownian motion the default event occurs exactly when $\bar{V}_t = V_d$. Therefore, it is reasonable to take default rebate as $R = \alpha G e^{-r\tau}$.

Now, referring to [51] we solve (3.32) together with the initial and boundary conditions (3.33) by applying Laplace transformation. In the Laplace domain, $L(x, p) = \mathcal{L}\left(\bar{\phi}(x, \tau)\right)$ satisfies the ordinary differential equation given by

$$\frac{1}{2}m^2\sigma^2 L_{xx} + \left(r - \frac{1}{2}m^2\sigma^2\right)L_x - (r + p)\,L = -\max\left\{G\left(1 - e^x\right), 0\right\}, \tag{3.34}$$

where L_x, L_{xx} represent, respectively, the first and the second partial derivatives of $L(x,p)$ with respect to the variable x. Moreover, the boundary condition is written as follows

$$L(x_d,p) = \mathcal{L}\left((1-\alpha)\,Ge^{-r(T-t)}\right) = \frac{(1-\alpha)\,G}{r+p}. \tag{3.35}$$

In order to find the solution of (3.34) we assume that it can be expressed as

$$L(x,p) = L^{wb}(x,p) + L^{b}(x,p), \tag{3.36}$$

where $L^{wb}(x,p)$ is a bounded solution discarding the boundary condition and $L^{b}(x,p)$ is a solution that takes the boundary condition into account. Thus, we seek the solution of (3.34) in two steps.

As a first step, we present the solution of the ordinary differential equation (3.34) for $L^{wb}(x,p)$, which discards the boundary condition in the following proposition.

Proposition 3.1 *Solution of the ordinary differential equation* (3.34) *is given by*

$$L^{wb}(x,p) = \begin{cases} C_1e^{\xi_1 x} - \frac{G}{p}e^{x} + \frac{G}{r+p}, & \text{if } x<0,\\ C_2e^{\xi_2 x}, & \text{if } x \geq 0, \end{cases} \tag{3.37}$$

where

$$\xi_{1,2} = \frac{-\left(r-\frac{1}{2}m^2\sigma^2\right) \pm \sqrt{\left(r-\frac{1}{2}m^2\sigma^2\right)^2 + 2m^2\sigma^2(r+p)}}{m^2\sigma^2}, \quad \xi_2 < 0 < \xi_1, \tag{3.38}$$

and

$$C_{1,2} = \frac{G}{\xi_1 - \xi_2}\left(\frac{1-\xi_{2,1}}{p} + \frac{\xi_{2,1}}{r+p}\right). \tag{3.39}$$

Proof. From the general theory of differential equations, a general solution to (3.34) is written as

$$L^{wb}(x,p) = L^{c}(x,p) + L^{p}(x,p), \tag{3.40}$$

where $L^{c}(x,p)$ and $L^{p}(x,p)$ are complementary and particular solutions, respectively.

On one hand, we write the complementary solution $L^{c}(x,p)$ as

$$L^{c}(x,p) = C_1e^{\xi_1 x} + C_2e^{\xi_2 x}, \tag{3.41}$$

where C_1 and C_2 are two constants and ξ_1 and ξ_2 are two distinct roots of the characteristic equation

$$\frac{1}{2}m^2\sigma^2\xi^2 + \left(r - \frac{1}{2}m^2\sigma^2\right)\xi - (r+p) = 0, \tag{3.42}$$

for the corresponding homogeneous equation for (3.34). The characteristic roots ξ_1 and ξ_2 of (3.42) are found to be the ones in (3.38).

On the other hand, the particular solution $L^p(x,p)$ assumes the form

$$L^p(x,p) = D_1 e^x + D_2, \tag{3.43}$$

where D_1 and D_2 are constants to be determined. Indeed, upon plugging the partial derivatives of (3.43) into (3.34) we obtain the system

$$\begin{aligned} -pD_1e^x - (r+p)\,D_2 &= Ge^x - G \quad \text{for} \quad x < 0, \\ -pD_1e^x - (r+p)\,D_2 &= 0 \qquad\qquad \text{for} \quad x \geq 0, \end{aligned} \tag{3.44}$$

whose solution is

$$\begin{aligned} D_1 = -\tfrac{G}{p}, \quad D_2 = \tfrac{G}{r+p}, \quad &\text{for} \quad x < 0, \\ D_1 = D_2 = 0, \quad &\text{for} \quad x \geq 0. \end{aligned} \tag{3.45}$$

Furthermore, to ensure the boundedness of (3.34), $C_2 = 0$ if $x < 0$, and similarly $C_1 = 0$ if $x \geq 0$, due to the fact that $\xi_2 < 0 < \xi_1$. Therefore, continuous differentiability assumptions of the solution, C_1 and C_2 should satisfy the systems of algebraic equations

$$\begin{aligned} C_1 - \tfrac{G}{p} + \tfrac{G}{r+p} &= C_2, \\ C_1\xi_1 - \tfrac{G}{p} &= C_2\xi_2. \end{aligned} \tag{3.46}$$

Consequently, this system has the solution as presented in (3.39). Thus, the proof is completed. ■

Towards the solution that also considers the boundary condition, we assume that the form of $L^b(x,p)$ in (3.36) is of the form

$$L^b(x,p) = C_3 e^{\xi_2 x}. \tag{3.47}$$

Now the problem is to determine the constant C_3: we know that when the value of the portfolio hits the boundary $x = x_d$, solution $L^b(x,p)$ should satisfy the equality

$$L^b(x_d,p) = L(x_d,p) - L^{wb}(x_d,p). \tag{3.48}$$

Hence, using (3.48), (3.35) and (3.37), and considering $x_d < 0$, we find

$$C_3 = e^{-\xi_2 x_d} \left(-\alpha \frac{G}{r+p} - C_1 e^{\xi_1 x_d} + \frac{G}{p} e^{x_d} \right). \tag{3.49}$$

Consequently, this, together with Proposition 3.1, proves the following result on the solution of (3.34).

Proposition 3.2 *Solution of the ordinary differential equation* (3.34) *that also satisfies the boundary condition at* $x = x_d$ *is*

$$L(x,p) = \begin{cases} C_1 e^{\xi_1 x} + C_3 e^{\xi_2 x} - \frac{G}{p} e^x + \frac{G}{r+p}, & \text{if } x < 0, \\ (C_2 + C_3)\, e^{\xi_2 x}, & \text{if } x \geq 0, \end{cases} \tag{3.50}$$

where $\xi_{1,2}$ *are given in* (3.38), $C_{1,2}$ *are given in* (3.39), *and* C_3 *is given in* (3.49).

An Application

We illustrate the risk neutral price of constant proportion debt obligation via some numerical examples. In the application we use Stehfest algorithm for inverse Laplace transformation stated in [53]. We examine the dependence of the pricing equation on the volatility of the risky index and on the leverage multiplier m. In the analysis, we assume the following set of parameters

$$\mu = 0.06,\ r = 0.05,\ T = 10,\ \nu = 0.025,\ F_T = 1. \tag{3.51}$$

We first look at the change observed in the price of constant proportion debt obligation over time under four values of σ: the volatility parameter is taken to be $\sigma = 0.025, 0.05,$ and 0.25 in the case when the leverage multiplier is $m = 2$. Furthermore, the latter is replaced with $\sigma = 0.1$ in the case when $m = 8$.

In Figure 3.2, for small volatilities with a low leverage of $m = 2$, the price is enough to pay all coupons until maturity and the final payment. This is due to the fact that the price of the barrier option hidden in the constant proportion debt obligation goes to 0 when maturity approaches. On the other hand, the hidden barrier option for high volatility, such as $\sigma = 0.25$, is in the money, Therefore, it seems to be clear from the beginning that the final price of constant proportion debt obligation will not get close to 1, and the shortfall is seen at maturity.

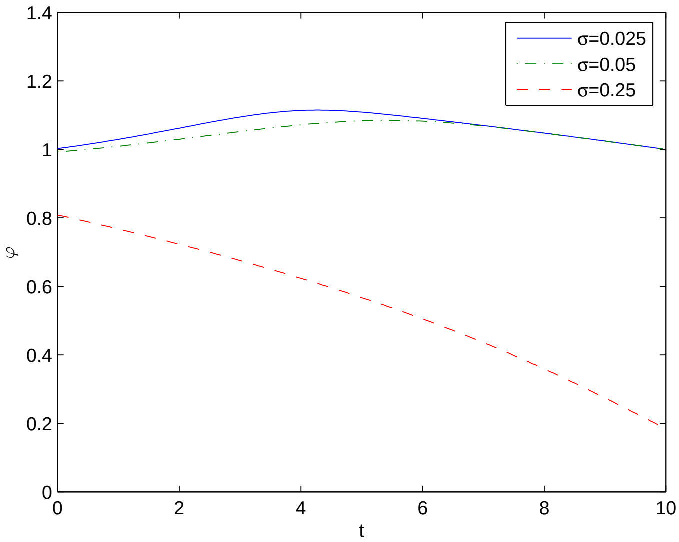

Figure 3.2: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different σ $(m = 2)$

Figure 3.3 is a replication of Figure 3.2 for $m = 8$. Accordingly, for small volatilities the price shows an increase until some point in time and then it closes with a price of 1 as in Figure 3.2. For higher volatilities, however, the price shows a decrease even from the beginning to the end and it closes with a high shorfall at maturity.

Next, we examine the change observed in the price of constant proportion debt obligation over time under three values of the leverage multiplier $m = 2$, 10, 25. Figure 3.4 and Figure 3.5 demonstrate the price for small volatility $\sigma = 0.025$ and a relatively high volatility $\sigma = 0.05$, respectively.

Figure 3.4 shows that small values of m yield smoother price curves than those obtained for big values. Furthermore, for big values of m, the price of hidden barrier option is higher than those for small leverage multipliers. Since there is an inverse relation between the price of constant proportion debt obligations and the hidden barrier option, lower prices for constant proportion debt obligation at the beginning. However, for any values of m, the price of constant proportion debt obligation reaches to 1 at the end.

In Figure 3.5, for small and moderate values of the leverage multiplier $m = 2$ and

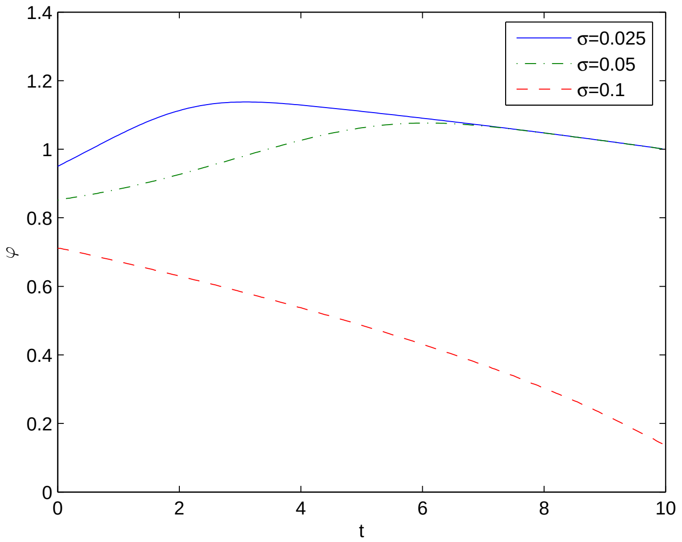

Figure 3.3: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different σ $(m = 8)$

$m = 10$, the price satisfies the final payment of 1. Unfortunately, however, for higher leverage multiplier $m = 25$, the price shows a linear decay within the first aproximately 8 years. Afterwards, althought some improvements occur in the prices, but far away from being close to 1.

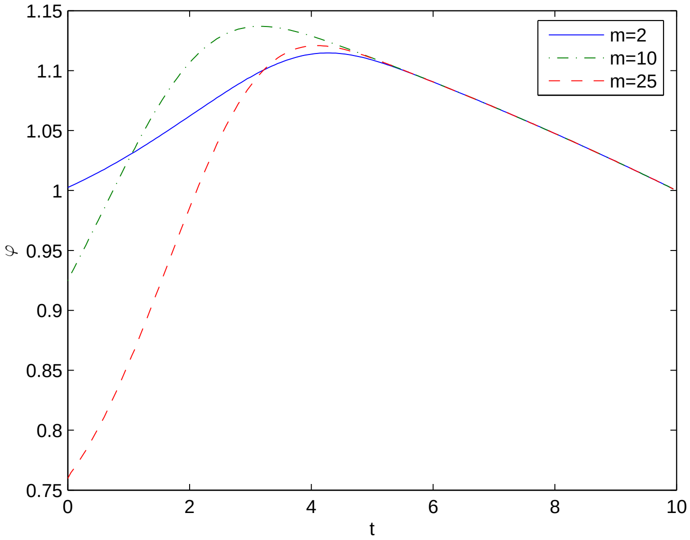

Figure 3.4: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different m ($\sigma = 0.025$)

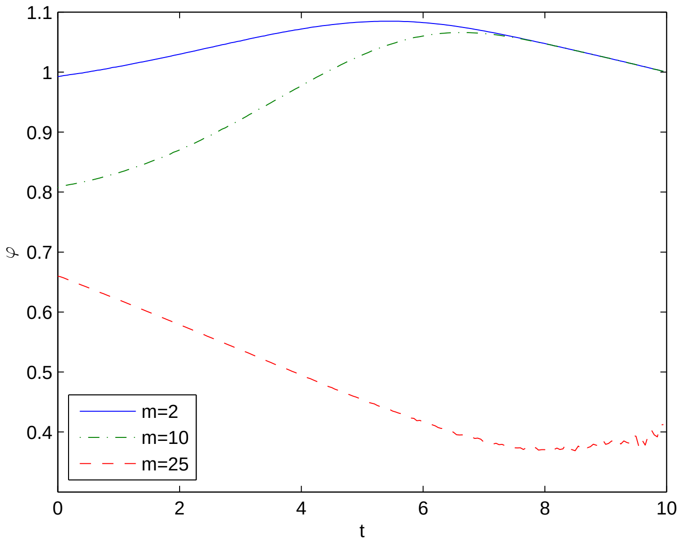

Figure 3.5: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different m ($\sigma = 0.05$)

3.5 Solution of the Hidden Barrier Pricing Problem Under Double Exponential Jumps

As a second model we assume that the index dynamic is of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t + 1/m\left(e^J - 1\right) S_{t^-} dN_t, \quad S_0 = s, \tag{3.52}$$

where W_t is one dimensional Brownian motion, N_t is a one-dimensional Poisson process with constant intensity λ and J is a random jump size. Moreover, W_t, N_t, J are independent processes.

Furthermore, we assume that J follows a double exponential distribution

$$f_J(j) = q_1 \frac{1}{\eta_1} e^{-\frac{j}{\eta_1}} \mathbb{I}_{j\geq 0} + q_2 \frac{1}{\eta_2} e^{\frac{j}{\eta_2}} \mathbb{I}_{j<0}, \quad 1 > \eta_1 > 0, \quad \eta_2 > 0, \tag{3.53}$$

where q_1, $q_2 \geq 0$, $q_1 + q_2 = 1$. Then, the corresponding portfolio process is given by

$$d\bar{V}_t = (r + m\mu)\bar{V}_t dt + m\sigma\bar{V}_t dW_t + \left(e^J - 1\right) dN_t, \quad \bar{V}_0 = \bar{v}. \tag{3.54}$$

Referring back to Section 3.2, under the rational expectations economy and equivalent martingale measure Q, we obtain the dynamics for $\bar{V}_t$ as follows

$$d\bar{V}_t = (r - \lambda^*\zeta^*)\bar{V}_t dt + m\sigma\bar{V}_t dW_t^* + \left(e^{J^*} - 1\right) dN_t^*, \quad \bar{V}_0 = \bar{v}, \tag{3.55}$$

where W_t^* is a standard Brownian motion, N_t^* is a Poisson process with intensity λ^* and W_t^*, N_t^*, J^* are independent under Q. Moreover, J^* follows a modified, but a double exponential distribution

$$f_{J^*}(j) = q_1^* \frac{1}{\eta_1^*} e^{-\frac{j}{\eta_1^*}} \mathbb{I}_{j\geq 0} + q_2^* \frac{1}{\eta_2^*} e^{\frac{j}{\eta_2^*}} \mathbb{I}_{j<0}, \qquad 1 > \eta_1^* > 0, \quad \eta_2^* > 0, \tag{3.56}$$

where q_1^*, $q_2^* \geq 0$, $q_1^* + q_2^* = 1$, and $\zeta^* := \mathbb{E}^*\left[e^{J^*}\right] - 1 = \frac{q_1^*}{1-\eta_1^*} + \frac{q_2^*}{1+\eta_2^*} - 1$.

The corresponding Generalized Black Scholes integro-partial differential equation stated in [12] is therefore

$$\begin{aligned} 0 \ = \ & \frac{\partial\phi}{\partial t} + \frac{1}{2}m^2\sigma^2\bar{V}^2\phi_{\bar{V}\bar{V}} + (r - \lambda^*\zeta^*)\,\bar{V}\phi_{\bar{V}} - r\phi \\ & + \lambda^* \int_{-\infty}^{\infty} \left[\phi\left(\bar{V}e^j\right) - \phi\left(\bar{V}\right)\right] f_j(j)dj, \quad \bar{V}_d < \bar{V} < \infty. \end{aligned} \tag{3.57}$$

Associated terminal and boundary conditions are

$$\begin{aligned} \phi(\bar{V}, T) &= \max\left\{G - \bar{V}, 0\right\}, \\ \phi(\bar{V}_d, t) &= Ge^{-r(T-t)} - R, \end{aligned} \tag{3.58}$$

respectively. Having applied the transformation in (3.31), we find an integro-partial differential equation for $\bar{\phi}(x, \tau)$ of the form

$$\begin{aligned} 0 &= -\frac{\partial \bar{\phi}}{\partial \tau} + \frac{1}{2} m^2 \sigma^2 \bar{\phi}_{xx} + \left(r - \lambda^* \zeta^* - \frac{1}{2} m^2 \sigma^2 \right) \bar{\phi}_x - r \bar{\phi}, \\ & + \lambda^* \int_{-\infty}^{\infty} \left[\bar{\phi}\left(x+j\right) - \bar{\phi}\left(x\right) \right] f_{J^*}(j) dj, \quad x_d < x < \infty, \end{aligned} \tag{3.59}$$

where $x_d = \ln\left(\frac{\bar{V}_d}{G}\right)$, and

$$\bar{\phi}\left(x, 0\right) = \max\left\{ G\left(1 - e^x\right), 0 \right\}, \quad \bar{\phi}(x_d, \tau) = G e^{-r\tau} - R.$$

Recall that $\bar{\phi}(x, \tau) := \phi(\bar{V}(x), \tau(t))$.

In case of default constant proportion debt obligation distributes the current value of the wealth process. For jump processes because of overshoot phenomena the rebate becomes too close to 0. Therefore, we may regard R as negligible and set $R = 0$. We apply the Laplace transformation to (3.59) and, consequently, transform it into an ordinary differential equation:

$$\begin{aligned} -\max\left\{ G\left(1 - e^x\right), 0 \right\} &= \tfrac{1}{2} m^2 \sigma^2 L_{xx} + \left(r - \lambda^* \zeta^* - \tfrac{1}{2} m^2 \sigma^2 \right) L_x \\ & - \left(r + p + \lambda^* \right) L + \lambda^* \int_{-\infty}^{\infty} L(x+j, p) f_{J^*}(j) dj. \end{aligned} \tag{3.60}$$

The boundary condition then becomes $L(x_d, p) = \mathcal{L}\left(e^{-r\tau}\right) = \frac{G}{r+p}$.

In the solution procedure we apply the same methodology as in Section 3.4. The solution $L^{wb}(x, p)$ is given in the following proposition.

Proposition 3.3 *Solution of the ordinary differential equation* (3.60) *is given by*

$$L^{wb}(x, p) = \begin{cases} C_1 e^{\xi_1 x} + C_2 e^{\xi_2 x} - \frac{G}{p} e^x + \frac{G}{r+p}, & \text{if } x < 0, \\ C_3 e^{\xi_3 x} + C_4 e^{\xi_4 x}, & \text{if } x \geq 0, \end{cases} \tag{3.61}$$

where ξ_1, ξ_2, ξ_3, and ξ_4 are four distinct roots of the characteristic equation

$$\begin{aligned} & \tfrac{1}{2} m^2 \sigma^2 \xi^2 + \left(r - \lambda^* \zeta^* - \tfrac{1}{2} m^2 \sigma^2 \right) \xi - \left(r + p + \lambda^* \right) \\ & \qquad + \lambda^* \left[\tfrac{q_1^*}{1 - \eta_1^* \xi} + \tfrac{q_2^*}{1 + \eta_2^* \xi} \right] = 0, \end{aligned} \tag{3.62}$$

and they are ordered as

$$-\infty < \xi_4 < -\frac{1}{\eta_2} < \xi_3 < 0 < \xi_2 < \frac{1}{\eta_1} < \xi_1 < \infty.$$

Moreover, C_1, C_2, C_3, and C_4 are obtained from the solution of the following system of equations

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ \xi_1 & \xi_2 & -\xi_3 & -\xi_4 \\ \frac{1}{1-\eta_1^*\xi_1} & \frac{1}{1-\eta_1^*\xi_2} & -\frac{1}{1-\eta_1^*\xi_3} & -\frac{1}{1-\eta_1^*\xi_4} \\ \frac{1}{1+\eta_2^*\xi_1} & \frac{1}{1+\eta_2^*\xi_2} & -\frac{1}{1+\eta_2^*\xi_3} & -\frac{1}{1+\eta_2^*\xi_4} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \frac{G}{p} - \frac{G}{r+p} \\ \frac{G}{p} \\ \frac{G}{p(1-\eta_1^*)} - \frac{G}{r+p} \\ \frac{G}{p(1+\eta_2^*)} - \frac{G}{r+p} \end{bmatrix}. \tag{3.63}$$

Proof. We write a general solution of (3.60) in the form

$$L^{wb}(x,p) = L^c(x,p) + L^p(x,p), \tag{3.64}$$

where $L^c(x,p)$ and $L^p(x,p)$ denote, respectively, a complementary and a particular solution. Note also that a complementary solution can be expressed as

$$L^c(x,p) = \begin{cases} C_1 e^{\xi_1 x} + C_2 e^{\xi_2 x}, & \text{if } x < 0, \\ C_3 e^{\xi_3 x} + C_4 e^{\xi_4 x}, & \text{if } x \geq 0, \end{cases} \tag{3.65}$$

due to the characteristic equation (3.62). Moreover, the consequent relation

$$-\infty < \xi_4 < -\frac{1}{\eta_2} < \xi_3 < 0 < \xi_2 < \frac{1}{\eta_1} < \xi_1 < \infty$$

ensures the boundedness of $L^c(x,p)$. Here, C_1, C_2, C_3, and C_4 are four constants that are to be determined later by imposing the continuity assumption on L^{wb}.

Having guessed the particular solution $L^p(x,p)$ of the form

$$L^p(x,p) = D_1 e^x + D_2, \tag{3.66}$$

where D_1 and D_2 are other two constants; similar to geometric Brownian case, we first calculate D_1 and D_2 by inserting derivatives of (3.66) into (3.60):

$$D_1 = -\tfrac{G}{p}, \quad D_2 = \tfrac{G}{r+p}, \quad \text{for} \quad x < 0,$$
$$D_1 = D_2 = 0, \quad \text{for} \quad x \geq 0.$$

Note that these are the same as the ones in (3.45), and they ensures the boundedness of L^{wb}.

Now, we use the continuity of the function $L^{wb}(x,p)$ and its derivative at $x = 0$ in determination of the contants $C_1, \ldots, C_4$. These conditions give

$$L^{wb(-)}(0,p) = L^{wb(+)}(0,p), \tag{3.67}$$

$$\frac{\partial L^{wb(-)}(x,p)}{\partial x}\Big|_{x=0^-} = \frac{\partial L^{wb(+)}(x,p)}{\partial x}\Big|_{x=0^+}, \tag{3.68}$$

from which we obtain

$$\begin{aligned} C_1 + C_2 - \tfrac{G}{p} + \tfrac{G}{r+p} &= C_3 + C_4, \\ C_1\xi_1 + C_2\xi_2 - \tfrac{G}{p} &= C_3\xi_3 + C_4\xi_4. \end{aligned} \tag{3.69}$$

Here, we should remark that $L^{wb^{(-)}}(x,p)$ and $L^{wb^{(+)}}(x,p)$ denote the function $L^{wb}(x,p)$ for $x < 0$ and $x \geq 0$, respectively. Also, equations in (3.69) correspond, respectively, to the first and second rows of the system of equations given in (3.63).

However, we have two equations with four unknown terms. Thus, we need to check whether or not the general form given in (3.61) satisfies the ordinary differential equation (3.60): since it includes integral term, we calculate the integral terms for both $x < 0$ and $x \geq 0$ at first. Below are the collected results.

(i) For $x < 0$,

$$\begin{aligned} &\int_{-\infty}^{\infty} L^{wb}(x+j,p) f_{J^*}(j)dj \\ &= \int_{-\infty}^{0} L^{wb}(x+j,p)\frac{q_2^*}{\eta_2^*}e^{\frac{j}{\eta_2^*}}dj + \int_{0}^{\infty} L^{wb}(x+j,p)\frac{q_1^*}{\eta_1^*}e^{-\frac{j}{\eta_1^*}}dj. \end{aligned} \tag{3.70}$$

The function $L^{wb}(x+j,p)$ gives two values according to value of $x+j$. If $j < -x$, then $L^{wb}(x+j,p) = L^{wb^{(-)}}(x+j,p) = C_1e^{\xi_1(x+j)} + C_2e^{\xi_2(x+j)} - \frac{G}{p}e^{x+j} + \frac{G}{r+p}$, otherwise; $L^{wb}(x+j,p) = L^{wb^{(+)}}(x+j,p) = C_3e^{\xi_3(x+j)} + C_4e^{\xi_4(x+j)}$.

Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} L^{wb}(x+j,p) f_{J^*}(j)dj &= \int_{-\infty}^{0} L^{wb^{(-)}}(x+j,p)\frac{q_2^*}{\eta_2^*}e^{\frac{j}{\eta_2^*}}dj \\ &\quad + \int_{0}^{-x} L^{wb^{(-)}}(x+j,p)\frac{q_1^*}{\eta_1^*}e^{-\frac{j}{\eta_1^*}}dj \\ &\quad + \int_{-x}^{\infty} L^{wb^{(+)}}(x+j,p)\frac{q_1^*}{\eta_1^*}e^{-\frac{j}{\eta_1^*}}dj \\ &= A_1 + A_2 + A_3, \end{aligned} \tag{3.71}$$

where A_1, A_2, and A_3 are found to be

$$A_1 = \sum_{i=1}^{2} C_i\frac{q_2^*}{1+\eta_2^*\xi_i}e^{\xi_i x} - \frac{G}{p}\frac{q_2^*}{1+\eta_2^*}e^{x} + \frac{G}{r+p}q_2^*, \tag{3.72}$$

$$\begin{aligned} A_2 &= \sum_{i=1}^{2} C_i\frac{q_1^*}{1-\eta_1^*\xi_i}e^{\xi_i x} - \frac{G}{p}\frac{q_1^*}{1-\eta_1^*}e^{x} + \frac{G}{r+p}q_1^* \\ &\quad - e^{\frac{x}{\eta_1^*}}\left[\sum_{i=1}^{2} C_i\frac{q_1^*}{1-\eta_1^*\xi_i} - \frac{G}{p}\frac{q_1^*}{1-\eta_1^*} + \frac{G}{r+p}q_1^*\right], \end{aligned} \tag{3.73}$$

$$A_3 = e^{\frac{x}{\eta_1^*}}\sum_{i=3}^{4} C_i\frac{q_1^*}{1-\eta_1^*\xi_i}. \tag{3.74}$$

(ii) Similary for $x \geq 0$, we write $\int_{-\infty}^{\infty} L^{wb}(x+j,p) f_{J^*}(j) dj$ as

$$\int_{-\infty}^{\infty} L^{wb}(x+j,p) f_{J^*}(j) dj = B_1 + B_2 + B_3, \tag{3.75}$$

where

$$\begin{aligned} B_1 &= \int_{-\infty}^{-x} L^{wb(-)}(x+j,p) \frac{q_2^*}{\eta_2^*} e^{\frac{j}{\eta_2^*}} dj \qquad (3.76) \\ &= e^{-\frac{x}{\eta_2^*}} \left[\sum_{i=1}^{2} C_i \frac{q_2^*}{1+\eta_2^* \xi_i} - \frac{G}{p} \frac{q_2^*}{1+\eta_2^*} + \frac{G}{r+p} q_2^* \right], \\ B_2 &= \int_{-x}^{0} L^{wb(+)}(x+j,p) \frac{q_2^*}{\eta_2^*} e^{\frac{j}{\eta_2^*}} dj \qquad (3.77) \\ &= \sum_{i=3}^{4} C_i \frac{q_2^*}{1+\eta_2^* \xi_i} e^{\xi_i x} - e^{-\frac{x}{\eta_2^*}} \sum_{i=3}^{4} C_i \frac{q_2^*}{1+\eta_2^* \xi_i}, \\ B_3 &= \int_{0}^{\infty} L^{wb(+)}(x+j,p) \frac{q_1^*}{\eta_1^*} e^{-\frac{j}{\eta_1^*}} dj \qquad (3.78) \\ &= \sum_{i=3}^{4} C_i \frac{q_1^*}{1-\eta_1^* \xi_i} e^{\xi_i x}. \end{aligned}$$

Since the terms $e^{\frac{x}{\eta_1^*}}$ and $e^{-\frac{x}{\eta_2^*}}$ in calculation of the integral $\int_{-\infty}^{\infty} L^{wb}(x+j,p) f_{J^*}(j) dj$ are not relevant with the solution, we assume

$$e^{\frac{x}{\eta_1^*}} \left[\sum_{i=1}^{2} C_i \frac{q_1^*}{1-\eta_1^* \xi_i} - \sum_{i=3}^{4} C_i \frac{q_1^*}{1-\eta_1^* \xi_i} - \frac{G}{p} \frac{q_1^*}{1-\eta_1^*} + \frac{G}{r+p} q_1^* \right] = 0, \tag{3.79}$$

and

$$e^{-\frac{x}{\eta_2^*}} \left[\sum_{i=1}^{2} C_i \frac{q_2^*}{1+\eta_2^* \xi_i} - \sum_{i=3}^{4} C_i \frac{q_2^*}{1+\eta_2^* \xi_i} - \frac{G}{p} \frac{q_2^*}{1+\eta_2^*} + \frac{G}{r+p} q_2^* \right] = 0, \tag{3.80}$$

due to the fact that $e^{\frac{x}{\eta_1^*}} \neq 0$ and $e^{-\frac{x}{\eta_2^*}} \neq 0$. These equations, indeed, correspond to the third and fourth row of the system of equations given in (3.63), respectively.

Consequently, the solution presented in (3.61) satisfies the ordinary differential equation (3.60), and the proof is completed. ■

On the other hand, we solve (3.60) by considering boundary conditions. Since we only have a single, down barrier, we assume that the solution $L^b(x,p)$ is of the form

$$L^b(x,p) = C_5 e^{\xi_3 x} + C_6 e^{\xi_4 x}, \tag{3.81}$$

where C_5 and C_6 are constants to be determined. We know that the solution considering the boundary condition should satisfy following system of equations, respectively, when process crosses or jumps over the barrier:

$$L^b(x,p) = \begin{cases} L(x_d,p) - L^{wb}(x,p), & \text{if } x \le x_d, \\ C_5 e^{\xi_3 x} + C_6 e^{\xi_4 x}, & \text{if } x_d < x \le \infty, \end{cases} \tag{3.82}$$

where $L(x_d,p) = \frac{G}{r+p}$. Similarly as in the geometric Brownian case, by imposing the first condition we obtain

$$C_1 e^{\xi_1 x_d} + C_2 e^{\xi_2 x_d} + C_5 e^{\xi_3 x_d} + C_6 e^{\xi_4 x_d} = \frac{G}{p} e^{x_d}, \tag{3.83}$$

however, this is not sufficient to find the two unknown constants, C_5 and C_6.

Meanwhile, the function $L^b(x,p)$ should satisfy the ordinary differential equation (3.60). Thus, we calculate the integral term

$$\begin{aligned} &\int_{-\infty}^{\infty} L^b(x+j,p) f_{J^*}(j) dj \\ &\quad = \int_{-\infty}^{0} L^b(x+j,p) \frac{q_2^*}{\eta_2^*} e^{\frac{j}{\eta_2^*}} dj + \int_{0}^{\infty} L^b(x+j,p) \frac{q_1^*}{\eta_1^*} e^{-\frac{j}{\eta_1^*}} dj. \end{aligned} \tag{3.84}$$

Since $L^b(x,p)$ changes its form accordingly with x_d, we rearrange (3.84) as

$$\int_{-\infty}^{\infty} L^b(x+j,p) f_{J^*}(j) dj = F_1 + F_2 + F_3. \tag{3.85}$$

Here, the calculations of F_1, F_2, and F_3 are not too difficult:

$$\begin{aligned} F_1 &= \int_{-\infty}^{x_d - x} \left[\frac{G}{r+p} - L^{wb(-)}(x+j,p) \right] \frac{q_2^*}{\eta_2^*} e^{\frac{j}{\eta_2^*}} dj, \qquad (3.86) \\ &= e^{-\frac{x}{\eta_2^*}} \left[-\sum_{i=1}^{2} C_i \frac{q_2^*}{1+\eta_2^* \xi_i} e^{\left(\xi_i + \frac{1}{\eta_2^*}\right) x_d} + \frac{G}{p} \frac{q_2^*}{1+\eta_2^*} e^{\left(1 + \frac{1}{\eta_2^*}\right) x_d} \right], \\ F_2 &= \int_{x_d - x}^{0} \left[C_5 e^{\xi_3 (x+j)} + C_6 e^{\xi_4 (x+j)} \right] \frac{q_2^*}{\eta_2^*} e^{\frac{j}{\eta_2^*}} dj, \qquad (3.87) \\ &= \sum_{i=5}^{6} C_i \frac{q_2^*}{1+\eta_2^* \xi_{i-2}} e^{\xi_{i-2} x} - e^{-\frac{x}{\eta_2^*}} \sum_{i=5}^{6} C_i \frac{q_2^*}{1+\eta_2^* \xi_{i-2}} e^{\left(\xi_{i-2} + \frac{1}{\eta_2^*}\right) x_d}, \end{aligned}$$

and

$$\begin{aligned} F_3 &= \int_{0}^{\infty} \left[C_5 e^{\xi_3 (x+j)} + C_6 e^{\xi_4 (x+j)} \right] \frac{q_1^*}{\eta_1^*} e^{-\frac{j}{\eta_1^*}} dj, \\ &= \sum_{i=5}^{6} C_i \frac{q_1^*}{1-\eta_1^* \xi_{i-2}} e^{\xi_{i-2} x}. \end{aligned} \tag{3.88}$$

Because $L^b(x,p)$ must satisfy (3.60), the term $e^{-\frac{x}{\eta_2^*}}$ should be omitted. Hence, this follows that

$$e^{-\frac{x}{\eta_2^*}}\Bigg[-\sum_{i=1}^{2}C_i\frac{q_2^*}{1+\eta_2^*\xi_i}e^{\left(\xi_i+\frac{1}{\eta_2^*}\right)x_d}-\sum_{i=5}^{6}C_i\frac{q_2^*}{1+\eta_2^*\xi_{i-2}}e^{\left(\xi_{i-2}+\frac{1}{\eta_2^*}\right)x_d}+\frac{G}{p}\frac{q_2^*}{1+\eta_2^*}e^{\left(1+\frac{1}{\eta_2^*}\right)x_d}\Bigg]=0, \tag{3.89}$$

and since $e^{-\frac{x}{\eta_2^*}}\neq 0$, we obtain

$$-\sum_{i=1}^{2}C_i\frac{1}{1+\eta_2^*\xi_i}e^{\left(\xi_i+\frac{1}{\eta_2^*}\right)x_d}-\sum_{i=5}^{6}C_i\frac{1}{1+\eta_2^*\xi_{i-2}}e^{\left(\xi_{i-2}+\frac{1}{\eta_2^*}\right)x_d}+\frac{G}{p}\frac{1}{1+\eta_2^*}e^{\left(1+\frac{1}{\eta_2^*}\right)x_d}=0. \tag{3.90}$$

The latter is another equation that C_5 and C_6 should satisfy.

As a summary of (3.83) and (3.90), we state the following proposition.

Proposition 3.4 *Solution of the ordinary differential equation* (3.60) *that satisfies the boundary condition* $L(x_d,p)=\frac{G}{r+p}$ *is given by*

$$L(x,p)=\begin{cases} C_1e^{\xi_1 x}+C_2e^{\xi_2 x}+C_5e^{\xi_3 x}+C_6e^{\xi_4 x}-\frac{G}{p}e^{x}+\frac{G}{r+p}, & \textit{if } x<0,\\ (C_3+C_5)\,e^{\xi_3 x}+(C_4+C_6)\,e^{\xi_4 x}, & \textit{if } x\geq 0,\end{cases} \tag{3.91}$$

where $\xi_1,\ldots,\xi_4$ *are the solutions of the characteristic equation* (3.62), $C_1,\ldots,C_4$ *are the solutions of the system* (3.63), *and* C_5, C_6 *are calculated from the system*

$$\begin{bmatrix} e^{\xi_3 x_d} & e^{\xi_4 x_d}\\ \frac{1}{1+\eta_2^*\xi_3}e^{\left(\xi_3+\frac{1}{\eta_2^*}\right)x_d} & \frac{1}{1+\eta_2^*\xi_4}e^{\left(\xi_4+\frac{1}{\eta_2^*}\right)x_d}\end{bmatrix}\begin{bmatrix} C_5\\ C_6\end{bmatrix} = \begin{bmatrix} \frac{G}{p}e^{x_d}-\sum_{i=1}^{2}C_ie^{\xi_i x_d}\\ \frac{G}{p}\frac{1}{1+\eta_2}e^{\left(1+\frac{1}{\eta_2^*}\right)x_d}-\sum_{i=1}^{2}C_i\frac{1}{1+\eta_2^*\xi_i}e^{\left(\xi_i+\frac{1}{\eta_2^*}\right)x_d}\end{bmatrix}. \tag{3.92}$$

An Application

In this section we apply a similar numerical study with Section 3.4. We assume that the double exponential jump size distribution satisfies the following parameters:

$$\lambda^*=0.17,\quad q_{1,2}^*=0.5,\quad \eta_{1,2}^*=0.2. \tag{3.93}$$

Other parameters are taken as in Section 3.4. Figures 3.6 and 3.7 demonstrate the changes in the price of the constant proportion debt obligation over time for three levels of σ and for two values of leverage multiplier $m = 2$ and 8, respectively.

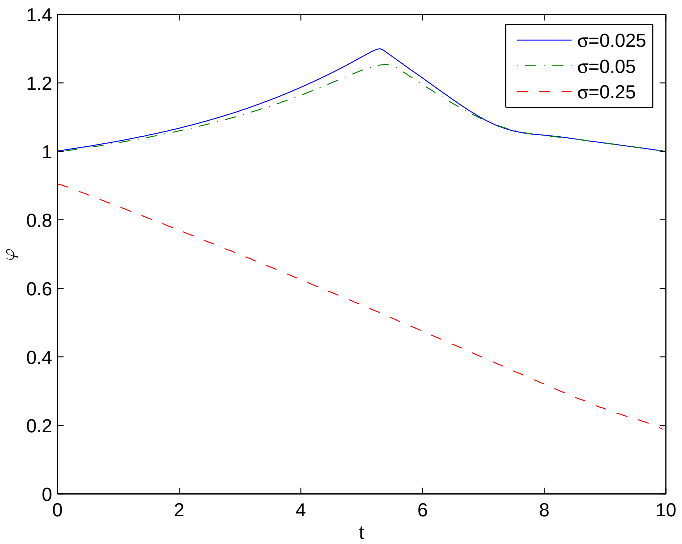

Figure 3.6: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different σ ($m = 2$)

Figure 3.6 shows that the price curves have peaks for small volatilities but they get close to 1 at maturity. For relatively higher volatility $\sigma = 0.25$, the constant proportion debt obligation is getting close to default until maturity.

In Figure 3.7, the price curves are more smoother for all levels of σ. Similarly, the price satisfies the final payment of 1 for small volatilities, however, for higher volatility the constant proportion debt obligation defaults at maturity.

In Figures 3.8 and 3.9, changes in the price of the constant proportion debt obligation over time for different levels of the leverage multiplier $m = 2$, 10 and 25. For small values of volatility, such as $\sigma = 0.025$, Figure 3.8 shows that for all levels of m the price curves firstly make a peak at first, then get close to 1 at maturity. However, for $\sigma = 0.05$ the final price of 1 is only satisfied for $m = 2$, and 10. See Figure 3.9. For $m = 25$ the prices show an increase when time to maturity decreases, but it is not enough in order to satisfy 1.

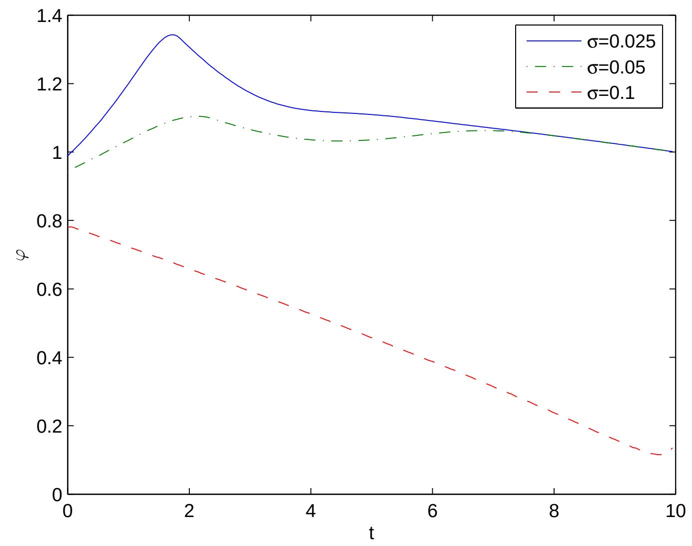

Figure 3.7: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different σ ($m = 8$)

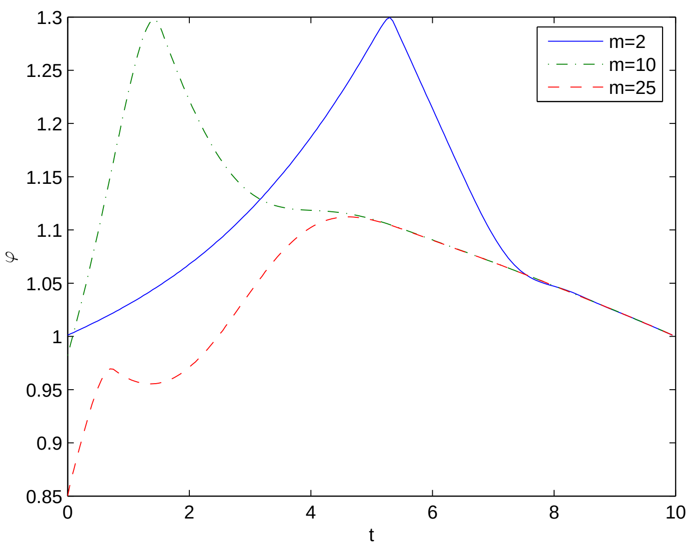

Figure 3.8: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different m ($\sigma = 0.025$)

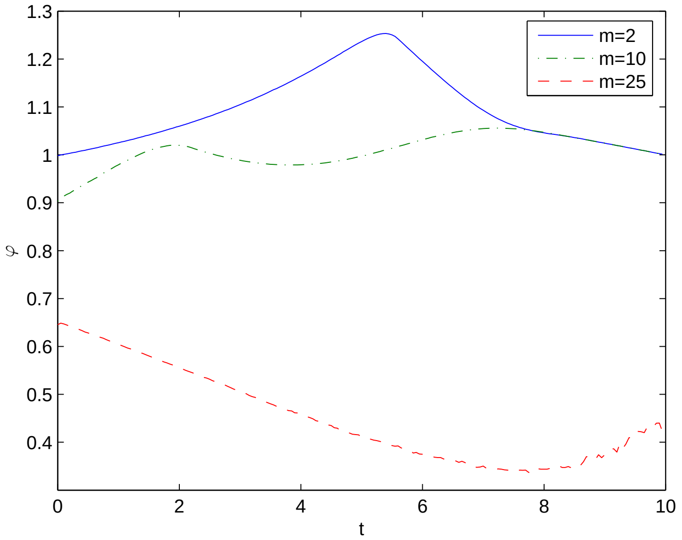

Figure 3.9: Risk Neutral Price of Constant Proportion Debt Obligation over Time for different m ($\sigma = 0.05$)

CHAPTER 4

CONCLUSION

Investors and issuers have to be aware of the fact that in order to obtain greater returns greater risks need to be taken. These risks sometimes are severe, however in the light of the recent credit crisis, they have often been hidden behind complex payoff structures. Some of them even seem to promise arbitrage opportunities and only very deep study the payoff structures reveal the underlying risks. One such example is constant proportion debt obligation.

In the thesis, constant proportion debt obligation is defined as a type of credit derivative which promises its owner a continuous coupon stream at a rate of $r + \nu$ until maturity T, and a final payment of 1 for an initial investment of 1 unit of money. Here, r is the riskless rate, $\nu > 0$ is the constant surplus rate. However, the final payment and the continuous coupon streams are at risk. The final payment is only guaranteed when the hedging portfolio followed by the issuer has a terminal value of at least 1. Otherwise, he simply receives the wealth of the hedging portfolio at time T. Moreover, the coupon payments are guaranteed if wealth of the hedging portfolio is over a predetermined level in whole life of constant proportion debt obligation. This hedging portfolio usually consists of continuously entering a swap contract position on a risky index on credit default swaps to get a premium which is then invested in a risk-free account and paid back at maturity. The position to deal with the shortfall between earnings and future obligations is called the leverage factor of constant proportion debt obligation and it forms the main characteristic of the hedging portfolio.

We firstly highlight the importance of the selection of the leverage factor optimally. For this purpose, we investigate the minimization of the mean-square distance between the promissed final payoff and the final wealth of constant proportion debt obligation by using the optimal control method. In this part of the study, we assume that there is no early default possibility, that is, only the final payment is at risk. For this study, the main problem is to decide a suitable model for the credit index dynamics to develop our hedging strategy. According to previous studies and various reasons, we perform our study for three cases: in which the risky index follows a geometric Brownian motion, a Vasicek type stochastic differential equation and a geometric jump diffusion process.

A geometric Brownian motion is selected because it is too simple to apply and it provides closed form solutions for most of the problems. Moreover, under the geometric Brownian motion market completeness is satisfied, and the verification theorems can easily be given. However, it is not a suitable model for credit indexes. Credit indexes usually do not show a strictly stationary log-normal behaviour. This may be classified as the main drawback of the solution. On the other hand, it is possible to make the hedging of the constant proportion debt obligations by using stock indexes rather than credit indexes and in literature geometric Brownian motion is most widely used model for stock indexes. Therefore, the use of geometric Brownian motion seems reliable for modelling.

As a second model we have chosen the Vasicek one because credit indexes like the credit swap indexes include the credit default swap spreads observed in the market, further, the literature shows that the use of mean reverting processes is more suitable for spread dynamics. The main factor in the selection of Vasicek model from various mean reverting processes is its simplicity in achieving closed form solutions like geometric Brownian motion for some models. However, Vasicek model also has a drawback: it does not explain the large movements of the spread caused by defaults observed in credit default swaps included in index.

The risk of unexpected large movements is included to the solution in the last part of the study by simply assuming a geometric jump diffusion process for the model of the index. Usually jump diffusion processes have drawbacks in generating closed form solutions for optimization problems, hence, numerical approaches are implemented. However, for our hedging problem geometric jump diffusion process allows us to derive an exact solution. This is the reason why we select the geometric jump diffusion process.

Our main findings in this part of study may be classified into four groups. Firstly, we show that the theoretically optimal leverage strategy for the geometric type diffusion processes coincides with the one used in practical applications (mostly named as "usual leverage"), only the leverage multiplier might differ. This conclusion can be generalized to Vasicek case. However, for Vasicek case we talk about a variable leverage multiplier which depends on both time and current index value. This contradicts to the constant multiplier assumption of the usual leverage factor.

Secondly, the final wealth of the optimal hedging strategy is highly sensitive to the volatility of the index and this causes a sure shortfall for high volatilies even in the Black-Scholes setting. In fact, one can reduce or even close the shortfall by simply selecting higher leverage multipliers without considering our hedging strategy. However, one should take into account that for higher volatilities large leverage multipliers are considered as too risky in terms of the $\mathcal{L}^2$-criterion. Therefore, the final decison is left to the issuers.

Thirdly, as an interesting finding our hedging strategy avoids the unnecessary risks taken to generate large amounts of money. In other words, in applications the strategy automatically near or equals to 0 even for Vasicek and jump cases without putting any positivity constraints on the optimal leverage when the necessary money is generated to satisfy all the future obligations.

Finally, special to geometric Brownian case we also show that the final wealth of the hedging strategy always leads to a shortfall. Because the limiting behaviour of the first and the second order moment of the wealth process shows that the hedging strategy is in the mean able to generate the necessary income needed for obligations if and only if leverage multiplier is big enough. However, such a big selection of leverage multiplier causes the enormous risk.

In the thesis, we further emphasize the importance of the fair price of constant proportion debt obligations which has not been considered in the literature as far as we know until now. As it is mentioned, constant proportion debt obligations have two important features: possibility of early default and possibility of closing the risky investment before maturity. By considering these features and using the usual leverage factor for modelling we classify constant proportion debt obligations as a double barrier option written on a trading portfolio.

At this stage of the thesis we perform a study for cases in which risky index follows a geometric Brownian motion and a geometric jump diffusion process with double exponential jump sizes. The reasons we use these dynamics for the index are the same with the ones explained above. However, as a special feature of this part there are some special motivations to use double exponential distribution for jump sizes. Main motivation is that double exponential jump size distribution allows to obtain closed form solutions for not only European options but also more complex ones such as path-dependent and barrier options.

In the modelling part of this stage, our first finding is that there is no need to impose a barrier for closing the strategy because the pricing equation of constant proportion debt obligations covers this barrier. This may be the reason of the third finding of the hedging part of the thesis. With this finding constant proportion debt obligation turns to be a single barrier option with rebate. Moreover, similar to the first part of the thesis, we also show that geometric Brownian motion with this strategy always generates a final shortfall. Therefore, we use a constant proportion of wealth strategy insted of the usual strategy and we develop our model with this strategy.

We realize that the pricing equation is a combination of a pricing problem (conditional expectation) and a static part that depends only on time. Accordingly, in the light of the study of Korn and Krekel [32] in 2001 we divide our portfolio into two parts: one for satisfying coupon payments and one for fulfilling the final payment. By adapting the model and the barrier conditions to these changes, the hidden pricing problem turns into a simple European barrier put option with rebate. In the solution part the completeness of the market is automatically satisfied and a unique risk neutral measure is found for geometric Brownian case. On the other hand, for geometric jump diffusion process with double exponential jump size, we use the equivalent martingale measure under the rational expectations equilibrium setting stated in [35]. The closed form formulae are found for both cases using the Laplace transformation as described in the approach of Sepp [51] in 2004.

For this stage of the thesis our main findings can be listed as follows:

I. It is shown that constant proportion debt obligation can be modelled as a barrier option with rebate. This provides us an opportunity to find a fair price for constant proportion debt obligation.

II. The inverse Laplace transform of the solution is easy to implement. This feature provides us an opportunity to perfom application studies.

III. The sensitivity of the constant proportion debt obligations is also validated with this model and pricing equation. For higher volatilities the early default is not observed in applications however decrease in price causes the issuers not to satisfy the final payment.

As a conclusion, constant proportion debt obligations seem to be not too risky in the case of low volatility, but carry extremely high risk of a final shortfall for medium to high sized volatility. The settings in the thesis allow us to obtain explicit forms of hedging strategies and the corresponding wealth processes. Moreover, the closed form pricing equations are also generated.

For the first part of the thesis, one might argue that $\mathcal{L}^2$-criterion we used is not appropriate and other criteria such as the one used in [8] might be better suited. However, for the $\mathcal{L}^2$-criterion we obtain an optimal leverage strategy that has exactly the same structure as of strategies used in practise. There are various generalizations where one can solve a comparable problem. They are left for future research.

For the second part of the thesis, one might also argue that the use of usual leverage (leverage used in practice) is more appropriate for pricing constant proportion debt obligations. However, by dividing the strategy we protect us from big losses that may be caused by the strategy used. In other words, within this framework we already put a floor constraint to the wealth invested in the risky index as in the definition of the usual leverage. On the other hand, various assumptions can be made on the strategies and various results can be obtained in the pricing of constant proportion debt obligations. As a suggestion under the assumption of geometric Brownian motion, the fair price of the constant proportion debt obligation can easily be found by using Black-Cox approach stated in [10], and other approaches are also left for future investigations.

Finally, for future research, application of the quantile hedging methodology stated in [24] to the hedging problem would be another interesting study.

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A1. Workshop on Asset Liability Models for Life Insurances, April 22, 2009, Ankara, Turkey given by Prof. Dr. Angelika May in Workshop on Recent Developments in Applied Probability and Statistics,

A2. Workshop on Simulation Toolbox for Financial Engineer, April 23–24, 2008, Ankara, Turkey given by Prof. Dr. Ralf Korn and Dr. Elke Korn in Workshop on Recent Developments in Financial Mathematics and Stochastic Calculus,

A3. Workshop On Risk Analysis & Management, April 23–25, 2006, Antalya, Tukey given by Prof.Dr. Monique Jeanblanc, Prof. Dr. Ralf Korn and Prof. Dr. Uwe Schmock in First Conference of Advanced Mathematical Methods for Finance,

A4. Dependable Adaptive Systems and Mathematical Modeling, July 31 – August 13, 2006, Kaiserlautern, Germany, 1st Summer School of the Rheinland-Pfalz Cluster of Excellence