NEURAL NETWORKS WITH PIECEWISE CONSTANT ARGUMENT AND IMPACT ACTIVATION

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ENES YILMAZ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
SCIENTIFIC COMPUTING

JUNE 2011

Approval of the thesis:

NEURAL NETWORKS WITH PIECEWISE CONSTANT ARGUMENT AND IMPACT ACTIVATION

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

NEURAL NETWORKS WITH PIECEWISE CONSTANT ARGUMENT AND IMPACT ACTIVATION

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June 2011, 137 pages

This dissertation addresses the new models in mathematical neuroscience: artificial neural networks, which have many similarities with the structure of human brain and the functions of cells by electronic circuits. The networks have been investigated due to their extensive applications in classification of patterns, associative memories, image processing, artificial intelligence, signal processing and optimization problems. These applications depend crucially on the dynamical behaviors of the networks. In this thesis the dynamics are presented by differential equations with discontinuities: differential equations with piecewise constant argument of generalized type, and both impulses at fixed moments and piecewise constant argument. A discussion of the models, which are appropriate for the proposed applications, are also provided.

Qualitative analysis of existence and uniqueness of solutions, global asymptotic stability, uniform asymptotic stability and global exponential stability of equilibria, existence of periodic solutions and their global asymptotic stability for these networks are obtained. Examples with numerical simulations are given to validate the theoretical results.

All the properties are rigorously approved by using methods for differential equations with discontinuities: existence and uniqueness theorems; stability analysis through the Second Lyapunov method and linearization. It is the first time that the problem of stability with the method of Lyapunov functions for differential equations with piecewise constant argument of generalized type is investigated. Despite the fact that these equations are with deviating argument, stability criteria are merely found in terms of Lyapunov functions.

Keywords: Neural Networks, Piecewise Constant Argument, Impulses, Periodic Solutions, Stability

ÖZ

PARÇALI SABİT ARGUMANLI VE ÇARPMA AKTİVASYONLU SİNİR AĞLARI

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Tez Yöneticisi : Prof. Dr. Marat Akhmet

Haziran 2011, 137 sayfa

Bu tez, matematiksel sinir bilimindeki yeni modellerden: hücre fonksiyonları ve insan beyninin yapısı ile birçok benzerlik gösteren yapay sinir ağlarından ve elektronik devreler yardımıyla hücrelerin fonksiyonlarından bahsetmektedir. Bu ağlar örüntülerin sınıflandırılması, çağrışımlı bellekler, görüntü işleme, yapay zeka, sinyal işleme ve optimizasyon problemlerindeki geniş uygulamalarından dolayı incelenmektedir. Bu uygulamalar önemli bir şekilde ağların dinamik davranışlarına bağlıdır. Bu tezde dinamikler süreksiz diferensiyel denklemler: genel tipteki parçalı sabit argumanlı diferensiyel denklemler, ve hem sabit zamanlı itmeler ve parçalı sabit arguman, ile gösterilmiştir. Ayrıca, sözkonusu olan uygulamalara örnek teşkil eden modellerin tartışması yapılmıştır.

Bu ağlar için çözümlerin varlık ve tekliği, denge noktalarının global asimtotik kararlılığı, düzgün asimtotik kararlılığı ve global üstel kararlılığı, periyodik çözümlerin varlığı ve bunların global asimtotik kararlılığının niteliksel analizi elde edilmiştir. Teorik sonuçları doğrulamak amacıyla nümerik simülasyon örnekleri verilmiştir.

Tüm özellikler: varlık ve teklik teoremleri; ikinci Lyapunov metodu ve lineerizasyon ile kararlılık analizi, süreksiz diferensiyel denklemler için olan metotlar kullanılarak kesin olarak

onaylanmıştır. Genel tipteki parçalı sabit argumanlı diferensiyel denklemler için Lyapunov fonksiyonlar metodu ile kararlılık problemi ilk defa incelenmiştir. Bu denklemler sapma argumanlı olmasına rağmen kararlılık kriterleri sadece Lyapunov fonksiyonları cinsinden bulunmuştur.

Anahtar Kelimeler: Sinir Ağları, Parçalı Sabit Arguman, İtmeler, Periyodik Çözümler, Kararlılık

To my precious wife and sons

ACKNOWLEDGMENTS

I would like to record my gratitude to Prof. Marat Akhmet for his supervision, advice, and precious guidance from the very early stage of this research as well as giving me extraordinary experiences throughout this research. Above all and the most needed, he provided me unflinching encouragement and continuous support in various ways. His truly scientist intuition has made him as a wide variety of ideas and passions in science, which exceptionally inspire and enrich my growth as a student, a researcher and a scientist who I want to be. I am indebted to him more than he knows and it has been a great honor for me to work with him.

I would like to extend my thanks to the members of the examining committee, Prof. Dr. Gerhard Wilhelm Weber, Assoc. Prof. Dr. Songül Kaya Merdan, Assist. Prof. Dr. Mehmet Turan and Assist. Prof. Dr. Konstantin Zheltukhin for their valuable comments and feedback on the manuscript of my thesis. I deeply thank to all my instructors and friends in the Institute of Applied Mathematics and Department of Mathematics at METU for their friendship, understanding and help. The members of the Applied Dynamics Group at METU also deserve to be mentioned here. It has been a great pleasure of working with them for the fruitful discussions during our meetings. Many thanks go to all my colleagues and office mates at METU. I also wish to thank all my close friends who were important to the successful realization of thesis, as well as expressing my apology that I could not mention personally one by one.

My devoted parents, Bekir and Fadime Yılmaz, deserve the greatest appreciation and respect for the unconditional love, generous support, trust and everything they have done for me throughout my whole life. I owe my loving thanks to my beloved sister Ümmühan and my dear brother M. Emin for their encouragements during the research.

Last, but not the least, words fail me to express my appreciation to my wonderful and beautiful wife Sümeyra whose dedication, love and persistent confidence in me, has taken the load off my shoulder. I would like to give my heartfelt thanks to her for enormous support, great patience and belief in me during this long process. With her endless understanding and generous encouragement, I have been able to continue my career in a sweet and peaceful environment.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

The dynamics of artificial neural networks is one of the most applicable and attractive objects for the mathematical foundations of neuroscience. In the last decades, Recurrent Neural Networks (RNNs), Cohen-Grossberg Neural Networks (Hopfield neural networks as a special version) and Cellular Neural Networks (CNNs) have been deeply investigated by using various types of difference and differential equations due to their extensive applications in classification of patterns, associative memories, image processing, artificial intelligence, signal processing, optimization problems, and other areas [27, 28, 30, 31, 32, 35, 36, 37, 38, 39, 117, 119, 120, 121, 122, 123, 124, 131, 134, 135, 136, 137, 138, 139, 148, 149, 150, 153, 154]. One of the ways to extend considered equations is to involve discontinuities of various kinds. The first one is to assume that functions on the right hand side are discontinuous. Also, we can use the independent argument as a piecewise constant function. In both cases one has a discontinuity of the velocity of the network process. Another way to obtain the dynamics with discontinuities is to consider it when the space variables, that is, the electrical characteristics themselves are discontinuous. Besides continuous activations, discontinuous/singular activations started to be used to develop for these applications. This phenomena immediately brings a great interest to the theory of networks with different types of discontinuity. An exceptional practical interest is connected with discontinuities, which appear at prescribed moments of time. Moreover, as it is well known, nonautonomous phenomena often occur in many realistic systems. Particularly, when we consider a long-term dynamical behavior of a system, the parameters of the system usually will change along with time. Thus, the research on nonautonomous neural networks is of prime significance. These problems provide very difficult theoretical and mathematical challenges, which will be analyzed in this thesis. The results of Akhmet's studies [1, 2, 3, 4, 5, 6, 7, 9, 19, 20, 21, 22, 23, 24] give a solid theoretical background for the proposed investigation on differential equations with discontinuity of types mentioned above: systems with piecewise constant argument and impulsive differential equations.

The main purpose of this thesis is the mathematical analysis of RNNs. It is well known that these applications mentioned above depend crucially on the dynamical behavior of the networks. In these applications, stability and convergence of neural networks are prerequisites. However, in the design of neural networks one is not only interested in the global asymptotic stability but also in the global exponential stability, which guarantees a neural network to converge fast enough in order to achieve fast response. In addition, in the analysis of dynamical neural networks for parallel computation and optimization, to increase the rate of convergence to the equilibrium point of the networks and to reduce the neural computing time, it is necessary to ensure a desired exponential convergence rate of the networks trajectories, starting from arbitrary initial states to the equilibrium point which corresponds to the optimal solution. Thus, from the mathematical and engineering points of view, it is required that the neural networks have a unique equilibrium point which is globally exponentially stable. Moreover, for example, if a neural network is employed to solve some optimization problems, it is highly desirable for the neural network to have a unique globally stable equilibrium point [140, 141, 142, 111, 54, 61, 64, 66]. Therefore, the problem of stability analysis of RNNs has received great attention and many results on this topic have been reported in the literature; see, e.g., [43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 62, 63, 65, 66], and the references therein.

Further, RNNs have been developed by implementing impulses and delays [11, 13, 15, 32, 33, 35, 37, 39, 40, 41, 42, 43, 72, 99, 117, 120] issuing from different reasons: In implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise. This leads to the model of RNNs with impulses. Due to the finite switching speed of amplifiers and transmission of signals in electronic networks or finite speed for signal propagation in biological networks, time delays exist.

In numerical simulations and practical implementations of neural networks, it is essential to formulate a discrete-time system, an analogue of the continuous-time system. Hence, stability for discrete-time neural networks has also received considerable attention from many researchers [134, 135, 136, 137, 138, 139]. As we know, the reduction of differential equations with piecewise constant argument to discrete equations has been the main and possibly a unique way of stability analysis for these equations [69, 77]. As a consequence of the existing method, initial value problems are considered only for the case when initial moments are integers or their multiples. In addition, one can not study stability in the complete form as only integers or their multiples are allowed to be discussed for initial moments. Hence, the concept of differential equations with piecewise constant argument of generalized type [3, 4, 5, 6, 7, 10, 17, 18] will be applied to RNNs by considering arbitrary piecewise constant functions as arguments.

It is well known that the studies on neural dynamical systems not only involve stability and periodicity, but also involve other dynamic behaviors such as synchronization, bifurcation and chaos et al. Nevertheless, in this thesis, we aim to consider the following two mathematical problems for neural networks with piecewise constant argument and impact activation:

- Sufficient conditions for the global existence-uniqueness of solutions and global asymptotic stability of equilibria.
- Existence of periodic solutions and their global asymptotic stability.

This dissertation is organized as follows: In this chapter, a brief review of neural networks that provides a clearer understanding for the modeling of RNNs is given. Also a mathematical background for the theory of differential equations with piecewise constant argument of generalized type, and the theory of impulsive differential equations with their qualitative properties are discussed.

In Chapter 2, we consider neural networks systems as well as impulsive neural networks systems with piecewise constant argument of generalized type. For the first system, we obtain sufficient conditions for the existence of a unique equilibrium and a periodic solution and investigate the stability of these solutions. For the second one, we introduce two different types of impulsive neural networks; $(\theta, \theta)-$ type neural networks and $(\theta, \tau)-$ type neural networks. For these types, sufficient conditions for the existence of the unique equilibrium are obtained, the existence and uniqueness of the solutions and the equivalence lemma for such systems are established, the stability criterion for the equilibrium based on linear approximation is proposed, some sufficient conditions for the existence and stability of periodic solutions are derived and examples with numerical simulations are presented to illustrate the results.

Chapter 3 deals with the problem of stability for differential equations with piecewise constant argument of generalized type through the method of Lyapunov functions. Besides this theoretical results, we analyze the stability for neural networks models with piecewise constant argument based on the Second Lyapunov method. That is to say, we use the method of Lyapunov functions and Lyapunov-Razumikhin technique for the stability of RNNs and CNNs, respectively. Examples with numerical simulations are given to illustrate the theoretical results.

Finally, in Chapter 4, some concluding remarks and future works are discussed.

The main parts of this thesis come from the following papers:

- M. U. Akhmet, E. Yılmaz, Neural networks with non-smooth and impact activations, (Revised version is submitted to Physica D: Nonlinear Phenomena).

- M. U. Akhmet, D. Aruğaslan, E. Yılmaz, Method of Lyapunov functions for differential equations with piecewise constant delay, J. Comput. Appl. Math., 235, pp. 4554-4560, 2011.

- M. U. Akhmet, E. Yılmaz, Impulsive Hopfield-type neural network system with piecewise constant argument, Nonlinear Anal: Real World Applications, 11, pp. 2584-2593, 2010.

- M. U. Akhmet, D. Aruğaslan, E. Yılmaz, Stability in cellular neural networks with a piecewise constant argument, J. Comput. Appl. Math., 233, pp. 2365-2373, 2010.

- M. U. Akhmet, D. Aruğaslan, E. Yılmaz, Stability analysis of recurrent neural networks with piecewise constant argument of generalized type, Neural Networks, 23, pp. 305-311, 2010.

- M. U. Akhmet, E. Yılmaz, Global attractivity in impulsive neural networks with piecewise constant delay. Proceedings of Neural, Parallel, and Scientific Computations, Dynamic Publishers, Inc, USA, pp. 11-18, 2010.

- M. U. Akhmet, E. Yılmaz, Hopfield-type neural networks systems equations with piecewise constant argument. International Journal of Qualitative Theory of Differential Equations and Applications, 3, no: 1-2, pp. 8-14, 2009.

1.1 A Brief Review of Neural Networks

The Artificial neural networks, commonly referred to as "neural networks", have been motivated by the fact that the human brain computes in an entirely different way from the conventional computer. Conventional computer has a single processor implementing a sequence of arithmetic and logical operations, now at speed about 10^9 operations per second [148, 152]. However, these devices have not an ability to adapt their structure and to learn in a way that a human being does.

What todays computers can not do? We know that there is a large number of tasks for which it is impossible to make an algorithm or sequence of arithmetic and/or logical operations. For example, in spite of many attempts, a machine has not yet been produced which can automatically read handwritten characters, or recognize words spoken by any speaker let alone can translate from one language to another, or identify objects in visual scenes, or drive a car, or walk and run as human does [151].

Neither the processing speed of the computers nor their processing ability makes such a difference. Today's computers have a speed 10^6 times faster than the main and basic processing elements of the brain called "neuron" [148]. If one compares the abilities, the neurons are much simpler. The main difference comes from the structural and operational trend. Although, the brain is a massively parallel interconnection of relatively simple and slow processing elements, in a conventional computer the instructions are executed sequentially in a complicated and fast processor.

Simon Haykin in his book [148] gives a definition of a neural network viewed as an adaptive machine:

Definition 1.1.1 *A neural network is a massively parallel distributed processor made up of simple processing units that has a natural prospensity for storing experiential knowledge and making it available for use. It resembles the brain in two respects:*

- *Knowledge is accuried by the network from its environment through a learning process;*
- *Interneuron connection strengths, known as synaptic weights are used to store the acquired knowledge.*

It is clear that a neural network gets its power from, first, its massively parallel distributed structure and, second, its ability to learn.

1.1.1 From Biological to Artificial Neuron

The human nervous system may have three-stage system [152](See Fig. 1.1). The main part of this system is the brain denoted by *neural net*, which continualy receives information, perceives it, and makes suitable decisions. In this figure we have two sets of arrows; from left to right indicates the *forward* transmission of information and the arrows from right to left shown in red represents the *feedback* in the system. The *receptors* change stimuli from the human body or the external environment into electrical impulses that transmit information to the neural net. The *effectors* transform electrical impulses produced by the neural net into discernible responses as systems outputs.

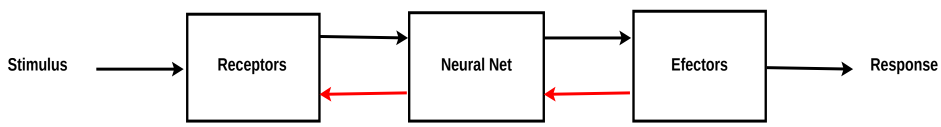

Figure 1.1: Representation of nervous system

It is declared that the human central nervous system consists of $1,3 \times 10^{10}$ neurons and that 1×10^{10} of them takes place in the brain [148]. Some of these neurons are firing and their power decreases because of this electrical activity is assumed to be in the order of 10 watts.

A neuron has a roughly spherical body called soma (Fig. 1.2). The signals produced in soma are converted to the other neurons through an extension on the cell body called *axon* or *nerve fibres*. Another kind of extensions on the cell body like bushy tree is the *dendrites*, which are responsible from receiving the incoming signals generated by other neurons. The axon is divided into several branches, at the very end of which the axon enlarges and forms terminal *buttons*. These buttons are placed in special structures called *synapses*. Synapses are the junctions transmitting signals (electrical or chemical signals) from one neuron to another. A neuron typically drive 10^3 to 10^4 synaptic junctions [148, 149, 153, 154].

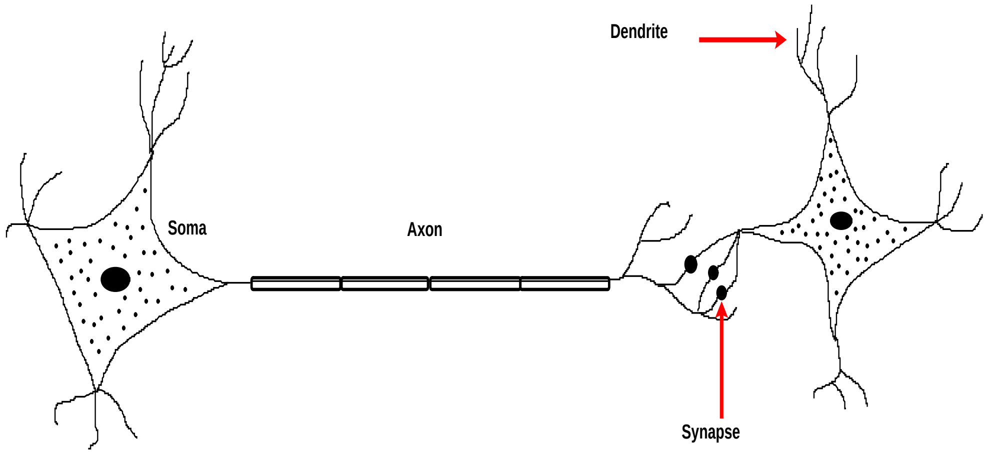

Figure 1.2: Typical biological neuron [161]

As it is mentioned before, the transmision of a signal from one neoron to another via synapses is a complex chemical process. So, we need potentials to transmit signal from one neuron to another. The effect is to raise or lower the electrical potential inside the body of the receiving cell. If this potential reaches a threshold, the neuron fires. By describing the characteristics of neuron, in 1943, McCulloch and Pittz published their famous paper [156], "A Logical Calculus of the Ideas Immanent in Nervous Activity." In this paper, it was the first time that they proposed an artificial neuron model which was widely used in artificial neural networks with some minor modifications. This model is shown in Fig. 1.3. The representation of this model can also be found in many books [148, 153, 154].

This model has N inputs, denoted by $x_1, \ldots, x_N$. Each line connecting these inputs to the neuron represents a weight, denoted as $w_1, \ldots, w_N$, respectively. Weights in this model mean the synaptic connections in the biological neurons. The activation function of the model is a threshold function and represented by θ and the activation corresponding to the potential is given by

$$u = \sum_{j=1}^{N} x_j w_j + \theta.$$

The inputs and the weights are real values. A negative value for a weight shows an inhibitory connection, while a positive value indicates an excitatory connection. In biological neurons, θ has a negative value, but in artificial neuron models it may be assigned as a positive value. For

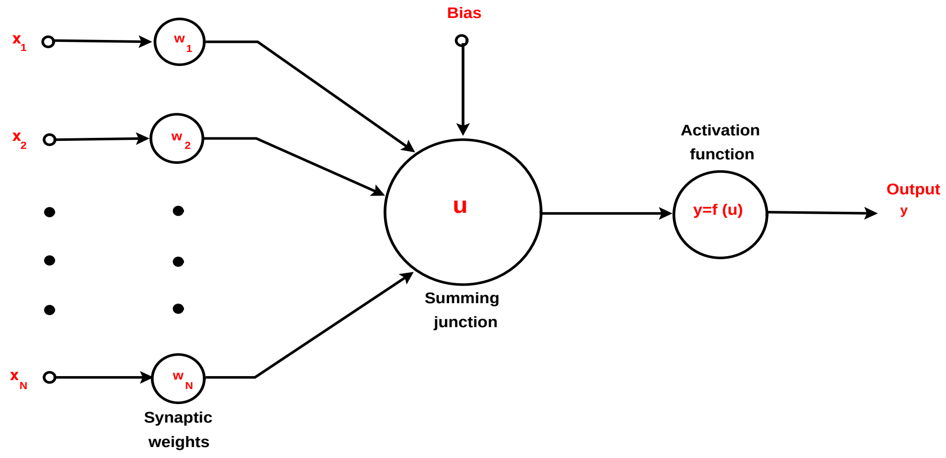

Figure 1.3: Artificial neuron model

this reason, it is usually referred as *bias* if θ is positive. Here, for convenience we take (+) sign in the activation formula. Orginally, McCulloch and Pittz proposed the threshold function for the neuron activation function in the artificial model, however there are also different types of activation functions (or output functions). For example linear, ramp and sigmoid functions are also widely used as output functions. In what follows, we identify the following basic types of activation functions described in Fig. 1.4:

- (a) Threshold Function:

$$f(v) = \begin{cases} 1, & if\ v \geq 0 \\ 0, & if\ v < 0 \end{cases}$$

- (b) Linear Function:

$$f(v) = \kappa v$$

- (c) Ramp Function:

$$f(v) = \begin{cases} 0, & if\ v \leq 0 \\ \frac{v}{\kappa}, & if\ 0 < v < \kappa \\ 1, & if\ \kappa < v \end{cases}$$

- (d) Sigmoid Function:

$$f(v) = \frac{1}{1 + e^{-\kappa v}}$$

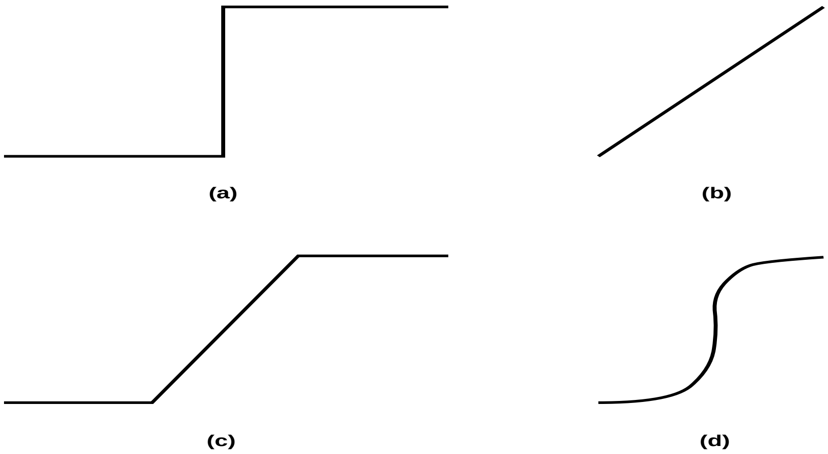

Figure 1.4: Some neuron activation functions

1.1.2 Basics of Electrical Circuits

Electrical circuits are so important for understanding the activity of our bodies . For example, most neurons especially the ones in the brain are electrical. In order to activate neurons we need brief voltage pulses. These pulses are known as action potentials or spikes which are used for communication among neurons [150]. The reason for this phenomenon is based on the physics of axons described in previous section. Therefore, the axon may be modeled as resistance-capacitance (RC)-circuit. A simple electric circuit made up of a voltage and resistor is shown in Fig.1.5.

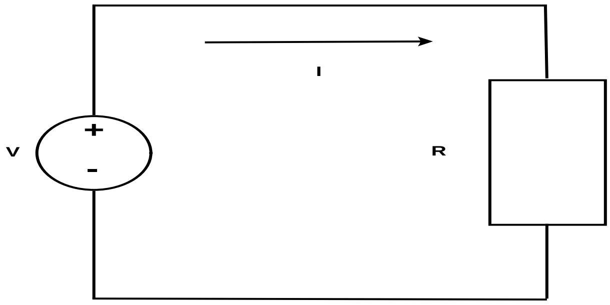

Figure 1.5: A simple electric circuit

We define electrical cicuits in terms of the physical quantities of voltage (V) and the current (I). We know that these quantities are solutions to the mathematical models. These models can be derived from Maxwell's equations by an advanced work or in this dissertation can be derived from Kirchoff's laws for elementary circuits. It is well known that circuits are combinations of physical devices like resistors and capacitors. Now, let us give some useful electrical circuits to understand our neural networks model, which will be described in detail in the next section.

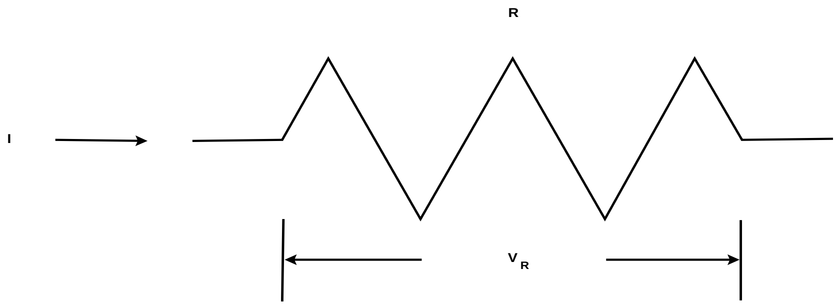

Figure 1.6: A resistor

The resistors are devices that limit or regulate the flow of electrical current in an electrical circuit. The relationship between V_R, I, and resistance R (as shown in Fig. 1.6) through an object is given by a simple equation known as *Ohm's law*:

$$V_R = IR,$$

where V is the voltage across the object in volts, I is the current through the object in amperes, and R is the resistance in ohms.

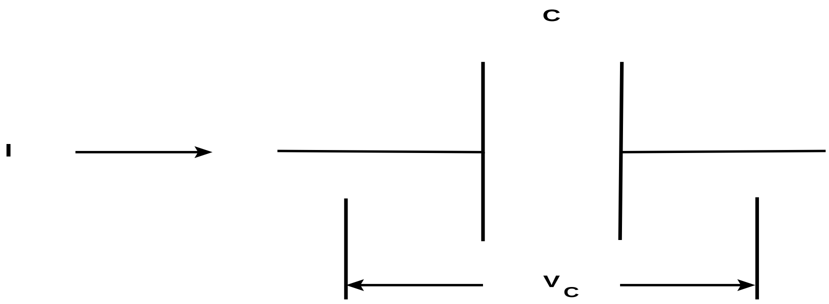

Figure 1.7: A capacitor

As seen in Fig.1.7, a capacitor is a device that stores charge on the plates at a rate proportional to I , and a voltage change on a capacitor can be shown by

$$V_C' = I/C$$

or, equivalently by,

$$V_C = \frac{1}{C}\int_0^t I dt.$$

The constant C is called the *capacitance* in units of farads.

As mentioned before, the circuit models are obtained from Kirchhoff's laws. Kirchhoff's laws state that:

- Kirchhoff's current law: The sum of all currents entering a node is equal to the sum of all currents leaving the node.
- Kirchhoff's voltage law: The directed sum of the voltage differences around any closed loop must be zero.

1.1.3 Models of Recurrent Neural Networks

The structures, which allow the connections to the neurons of the same layer or to the previous layers, are called *recurrent neural networks* [148, 155]. That is, a recurrent network may consist of a single layer of neurons with each neuron feeding its output signal back to the inputs of all other neurons. Moreover, there is no self-feedback loops in the network; self-feedback means a situation where the output of a neuron is fed back into its own input [148]. This is illustrated in Fig. 1.8.

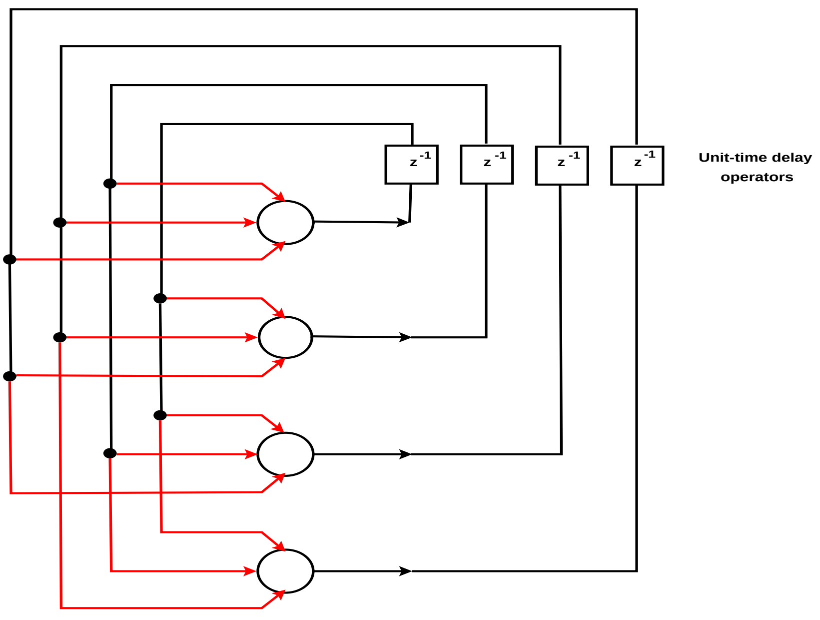

Figure 1.8: Recurrent neural networks with no hidden neurons

We are now ready to consider some of the important RNNs model involved in the literature and needed for our investigations.

1.1.3.1 Additive Model

The neurodynamic model of a neuron is illustrated in Fig. 1.9 where the conductances are denoted by synaptic weights $w_{j1}, w_{j2}, \ldots, w_{jN}$ and the potentials by relevant inputs $x_1, x_2, \ldots, x_N$.

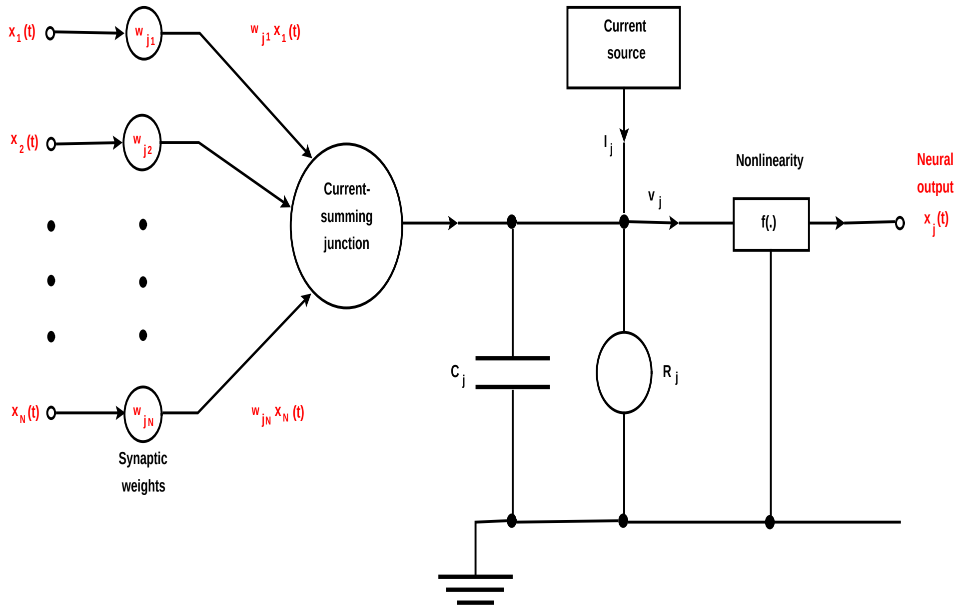

Figure 1.9: Additive model

The total current entering a node in Fig. 1.9 is

$$\sum_{i=1}^{N} w_{ji}x_i(t) + I_j,$$

where the current source I_j representing an externally applied bias. Let $v_j(t)$ be the induced local field at the input of the nonlinear activation function $f(\cdot)$. Then, the total current sum leaving the node is

$$\frac{v_j(t)}{R_j} + C_j\frac{dv_j(t)}{dt}.$$

By Kirchhoff's current law, the following nonlinear differential equations can be obtained:

$$\frac{v_j(t)}{R_j} + C_j\frac{dv_j(t)}{dt} = \sum_{i=1}^{N} w_{ji}x_i(t) + I_j, \tag{1.1}$$

where $x_j(t) = f(v_j(t))$. Thus,the model described by the equation (1.1) is called *additive model*. It is assumed that the activation function $f(\cdot)$ is a continuously differential function with respect to t. Here, the activation function is the logistic function

$$f(v_j) = \frac{1}{1 + e^{-v_j}} \qquad j = 1, 2, \ldots, N.$$

1.1.3.2 Hopfield (Additive) Model

In 1984, Hopfield [28] proposed the continuous deterministic model which is based on continuous variables and responses. The model contains a set of neurons and corresponding set of unit-time delays shown in Fig. 1.10. Each neuron has same architectural graph as shown in Fig. 1.9. Therefore, the neurons are modeled as amplifiers in conjuction with feedback circuits made up of wires, resistors and capacitors.

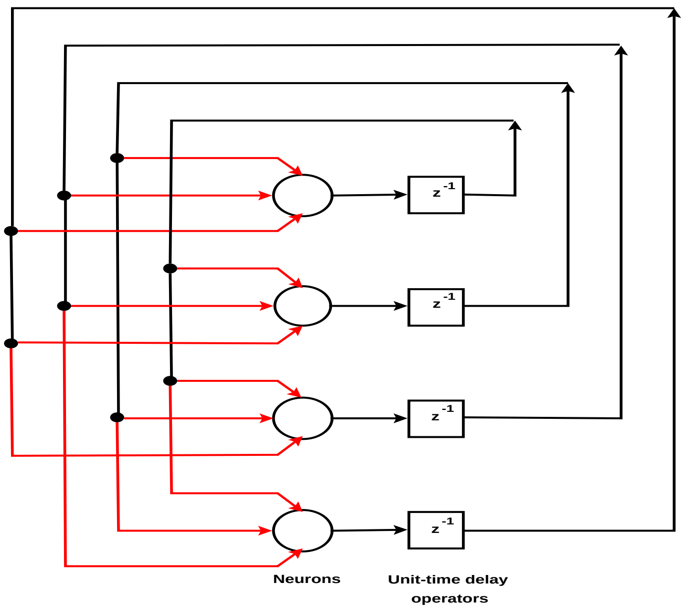

Figure 1.10: Hopfield network with $N = 4$ neurons

The Hopfield network can be considered as a nonlinear associative memory or content-addressable memory (CAM). We know priori the fixed points of the network so that they correspond to the patterns to be stored. However, the synaptic weights of the network that produce the desired fixed points are unknown. Thus, the problem is how to determine them. In the application of the Hopfield network, an important property of a CAM is the ability to retrieve a stored pattern, given a reasonable subset of the information content of that pattern [29]. The essence of a CAM is mapping a fundamental memory A onto a fixed (stable) point B of a dynamical system (see Fig. 1.11). The arrow from left to right describes the encoding mapping, whereas the arrow from right to left describes the decoding mapping.

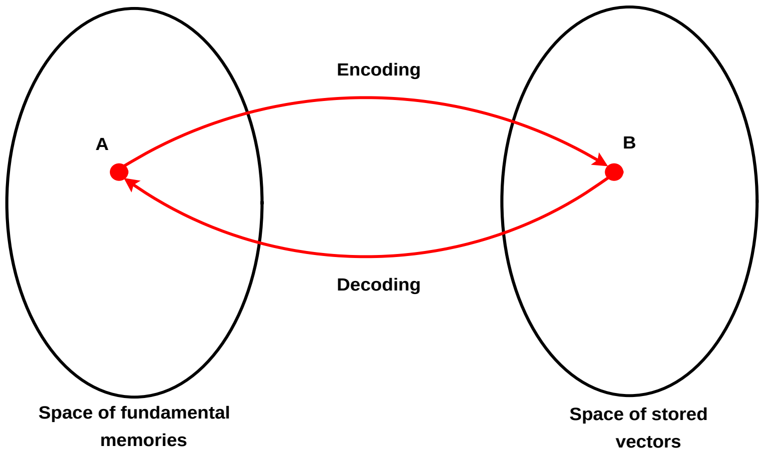

Figure 1.11: Encoding-decoding performed by a network

The stable points of the phase space of the network are the fundamental memories, or prototype states of the network. For example, when a network has a pattern containing partial but sufficient information about one of the fundamental memories, we may represent it as a starting point in the phase space. Provided that the starting point is close to the stable point representing the memory being retrieved, finally, the system converges onto the memory state itself. Consequently, it can be said that Hopfield network is a dynamical system whose phase space contains a set of fixed (stable) points representing the fundamental memories of the system [148, 155].

Let us consider the dynamics of the Hopfield network which is based on the additive model of a neuron described in the previous section:

$$C_j \frac{dv_j(t)}{dt} = -\frac{v_j(t)}{R_j} + \sum_{i=1}^{N} w_{ji} f_i(v_i(t)) + I_j, \quad j = 1, 2, \ldots, N \tag{1.2}$$

where

$$x_j = f_i(v_j)$$

and

$$\frac{1}{R_j} = \frac{1}{r_i} + \sum_{i=1}^{N} |w_{ji}|.$$

Here, r_i denotes the resistance representing the cell membrane impedance. The equation (1.2) has a neurobiological background as explained below:

- C_j is the total input capacitance of the amplifier representing the capacitance of cell membrane of neuron j.
- w_{ji} is the value of the conductance of the connection from the output of the jth amplifier to the input of the ith amplifier, representing strengths of the synaptic connection strengths among the neurons.
- $v_j(t)$ is the voltage of the amplifier of the jth neuron at time t representing the soma potential of neuron j.
- I_j is a constant external input current to the jth neuron representing the threshold for activation of neuron.
- f_i is the activation function representing the response of the ith neuron to its membrane potential.

The activation function or the input-output relation of the ith amplifier is given by

$$f_i(v) = \tanh(\rho_i v),$$

where ρ_i is a constant gain parameter. It can be seen that this function is differentiable and increasing (see Fig.1.12). Specifically, its derivative at origin gives us the the constant gain parameter ρ_i.

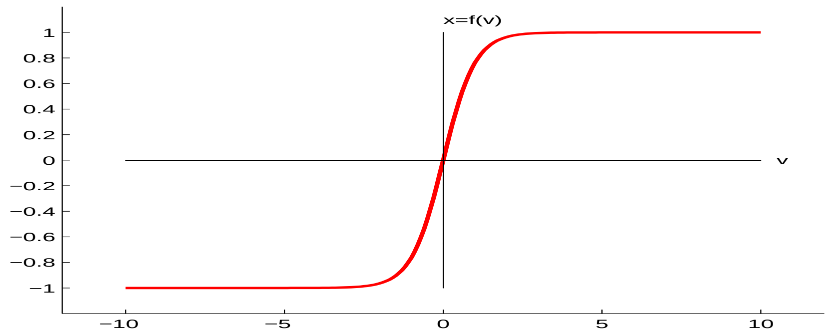

Figure 1.12: The graph of $\tanh(v)$

In this model, our aim is to find stable fixed points to read or to understand the fundamental memory. We have nonlinear dynamical system. So, we analyze the stability by using Lyapunov functions. It is also well known that the use of Lyapunov functions makes it possible to decide the stability of equilibrium points without solving the state-space equation of the system (1.2). In what follows, we need the following basic but useful definition and theorem:

Definition 1.1.2 *[106] A continuous function $L(x)$ with a continuous derivative $L'(x)$ is a definite Lyapunov function if it satisfies:*

(i) *$L(x)$ is bounded;*

(ii) *$L'(x)$ is negative definite, that is: $L'(x) < 0$ for $x \neq x^*$ and $L'(x) = 0$ for $x = x^*$.*

If the condition (ii) is in the form $L'(x) \leq 0$ for $x \neq x^*$ the Lyapunov function is called semidefinite.

Theorem 1.1.3 *[106] The equilibrium state x^* is stable (asymptotically stable), if there exists a semidefinite (definite) Lyapunov function in a small neighborhood of x^*.*

To study the stability of the system (1.2), we need three assumptions:

(i) The matrix of synaptic weights is symmetric: $w_{ji} = w_{ij}$ for all i and j;

(ii) Each neuron has a nonlinear activation function of its own;

(iii) The inverse of the nonlinear activation function exists.

Particularly, the inverse of the function f_i illustrated in Fig. 1.13 is

$$f_i^{-1}(x) = -\ln \frac{1-x}{1+x}.$$

The energy function of the Hopfield network in Fig. 1.10 is defined by

$$E = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}x_i x_j + \sum_{j=1}^{N}\frac{1}{R_j}\int_0^{x_j} f_j^{-1}(x)dx - \sum_{j=1}^{N} I_j x_j$$

and Hopfield used this function E as a Lyapunov function. That is, the energy function E is bounded since the integral of the inverse of the function $\tanh(v)$ is bounded when $-1 < x_j < 1$

and $\frac{dE}{dt} < 0$ except at a fixed point. Then, by the Definition 1.1.2 and Theorem 1.1.3, the Hopfield network is globally asymptotically stable in the Lyapunov sense [28]. That is to say, whatever the initial state of the network is, it will converge to one of the equilibrium states.

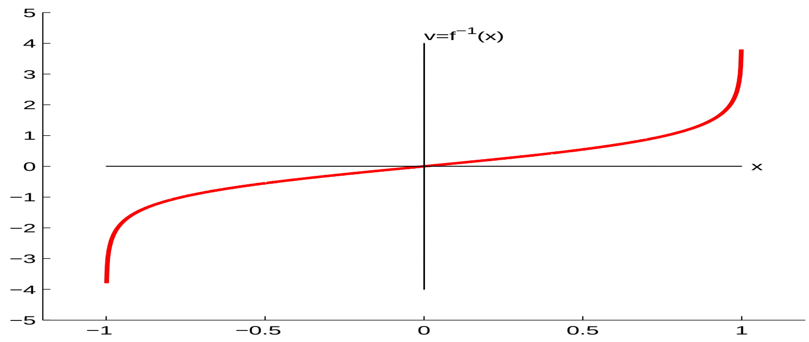

Figure 1.13: The graph of $\tanh^{-1}(x)$

1.1.3.3 Cohen-Grossberg Theorem

In paper [27], Cohen-Grossberg states a very useful theorem in deciding the stability of a certain class of neural networks.

Theorem 1.1.4 *[27] Given a neural network with N processing elements having bounded output signals $\varphi_i(u_i)$ and transfer functions of the form*

$$\frac{du_j}{dt} = a_j(u_j)\Big[b_j(u_j) - \sum_{i=1}^{N} c_{ji}\varphi_i(u_i)\Big], \quad j = 1, \dots, N \tag{1.3}$$

satisfying the conditions:

(i) *Symmetry:* $c_{ji} = c_{ij}$;

(ii) *Nonnegativity:* $a_j(u_j) \geq 0$;

(iii) *Monotonicity:* $\varphi_j'(u_j) = \frac{d\varphi_j(u_j)}{du_j} \geq 0$.

Then the network will converge to some stable point and there will be at most a countable number of such stable points.

In equation (1.3), u_j denotes the state variable associated with the ith neuron, a_j represents an amplification function, b_j is an appropriately behaved function, c_{ji} represents the connection strengths between neurons, and φ_i means the activation function which shows how neurons respond to each other.

Obviously, the equation of this model (1.3) reduces to the equation for Hopfield model (1.2). Thus, one can easily see that the Hopfield model is a special case of the system defined in the Theorem 1.1.4. The relations between the general system of equation the (1.3) and the system of (1.2) are summarized as follows in Table 1.1:

Table 1.1: Relations between the Cohen-Grossberg Theorem and the Hopfield Model

Cohen-Grossberg Theorem	Hopfield Model
u_j	$C_j v_j$
$a_j(u_j)$	1
$b_j(u_j)$	$-(v_j/R_j) + I_j$
c_{ji}	$-w_{ji}$
$\varphi_i(u_i)$	$f_i(v_i)$

In order to prove the stability of the equation (1.3), Cohen-Grossberg use energy function E, defined as

$$E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ji}\varphi_i(u_i)\varphi_j(u_j) - \sum_{j=1}^{N}\int_0^{u_j} b_j(s)\varphi_j^{'}(s)ds.$$

Then, under the certain conditions in Theorem 1.1.4, one can show that the energy function E of the system (1.3) is a Lyapunov function satisfying $\frac{dE}{dt} < 0$ for $u_j \neq u_j^*$ and therefore the Hopfield network is globally asymptotically stable [27, 148].

In the light of above discussions, the qualitative analysis of model equations (1.3) and (1.2) have been attracted by many scientists and have been investigated through different types of difference and differential equations. Now we will give some examples considering different types of model equations from the literature. Of course, there are many papers dealing with discrete-time and continuous-time neural networks; see, e.g., [33, 35, 39, 42, 117, 119, 120, 121, 122, 123, 124, 131, 134, 135, 136, 137, 138, 139] and the references cited therein. Nevertheless, here we will review a few of them:

Firstly, in paper [135], the authors consider the following discrete-time recurrent neural networks with time-varying delays and discuss the analysis of exponential stability for a class of discrete-time recurrent neural networks with time delays:

$$u_i(k+1) = a_i u_i(k) + \sum_{j=1}^{n} b_{ij} f_j(u_j(k)) + \sum_{j=1}^{n} d_{ij} g_j(u_j(k-\tau(k))) + J_j,$$

where $\tau(k)$ denotes the time-varying delay satisfying $\tau_m \leq \tau(k) \leq \tau_M,\ k \in \mathbb{N}$ with positive unknown integers $\tau_m,\ \tau_M$. The delay is of the time-varying nature, and the activation functions are assumed to be neither differentiable nor strict monotonic.

Secondly, in paper [138], the global exponential stability of a discrete-time recurrent neural network with impulses is discussed. The equation of the model is given as

$$\begin{cases} x(n+1) = Dx(n) + A\sigma(Bx(n) + I) \\ x(n_0) = x_0 \in \mathbb{R}^n \\ I_i(x_i(n_k)) = x_i(n_k+1) - x_i(n_k),\ i = 1, \ldots, m,\ k = 1, 2, \ldots \\ n_0 < n_1 < n_2 < \ldots < n_k \to \infty \ as \ k \to \infty, \end{cases}$$

where $\sigma_i(x_i) = \frac{1}{2}\Big(|x_i + 1| - |x_i - 1|\Big)$ and the impulsive functions $I_k : \mathbb{R} \to \mathbb{R}$ are assumed to be discrete.

Then, in paper [42], the delayed CNNs model described by differential equations with delays is considered:

$$\begin{aligned} x_i'(t) &= -c_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t-\tau_j)) + I_i, \\ &\quad a_i > 0,\ i = 1, 2, \ldots, m, \end{aligned}$$

where τ_j corresponds to the transmission delay along the axon of the jth unit and is non-negative constant. In this paper, a set of criteria ensuring the global asymptotic stability of delayed CNNs is derived.

Next, in paper [118], Akça et al. investigate the following Hopfield-type model of neural network with impulses:

$$\begin{aligned} & x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + c_i,\ t > 0,\ t \neq t_k \\ & \Delta x_i(t_k) \mid_{t=t_k} = I_k(x_i(t_k)), \quad i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, \end{aligned}$$

where $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0)$ are the impulses at moments t_k and $t_1 < t_2 < \ldots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$. They investigate the global stability characteristics of a system of equations modelling the dynamics of additive Hopfield-type neural networks with impulses in the continuous-time case.

Finally, in paper [123], by using Lyapunov functions and analysis technique, Zhang and Sun get a result for the uniform stability of the equilibrium point of the following impulsive Hopfield-type neural networks systems with time delays:

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{m} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} b_{ij} g_j(x_j(t-\tau_j)) + I_i,\ t > 0,\ t \neq\ t_k,$$
$$x_i(t_k) = d_k(x_i(t_k^-)), \quad i = 1, 2, \ldots, m, \quad k \in \mathbb{N},$$

where τ_j are the time delays, and satisfy $\tau_j > 0$.

We end this section by getting attention to the important novelties of this thesis: From the mathematical and engineering points of view, the modeling process for a real-world problem is generally given by the Fig.1.14.

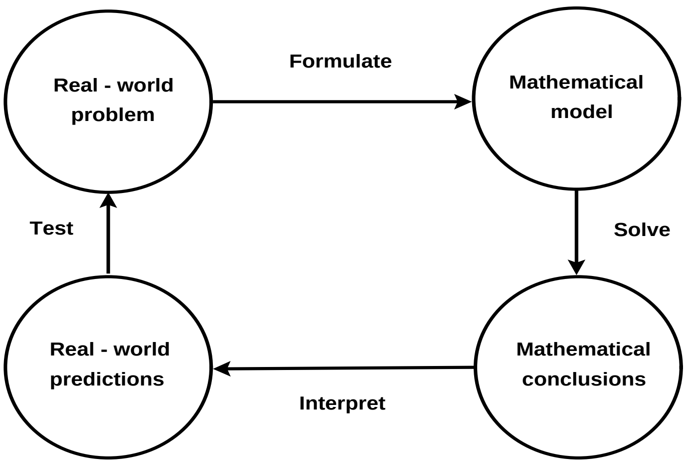

Figure 1.14: The modeling process

However, in order to be realistic, the modeling process in our investigations is considered only from a mathematical point of view and is illustrated in Fig. 1.15. From the perspective of this illustration, our approaches developed in this thesis can be regarded as an extension of the many conventional techniques which have been investigated by engineers in the mathematical neuroscience.

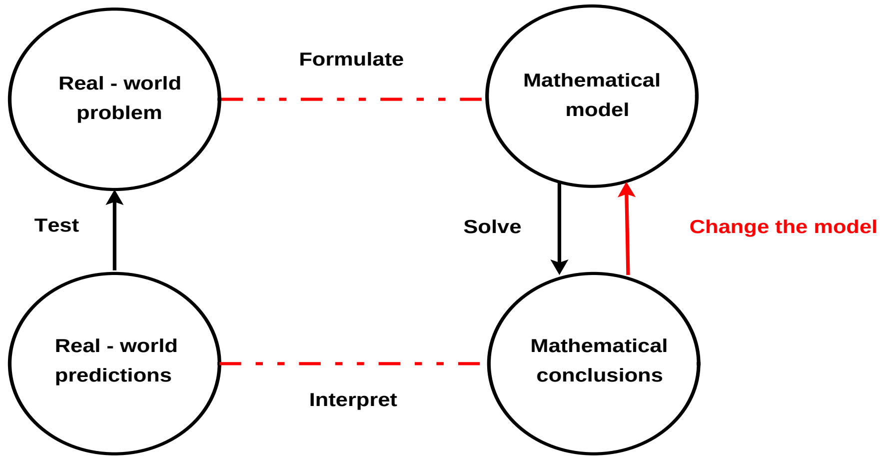

Figure 1.15: The modeling process in our investigations

1.2 A Brief Historical Review of Piecewise Constant Argument

The theory of differential equations with piecewise constant argument (EPCA) was initiated in [69, 70, 77]. These equations have been under intensive investigation of researchers in mathematics, biology, engineering and other fields for the last twenty years. The studies of such equations were motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both the differential and difference equations. The first mathematical model including piecewise constant argument was given by Busenberg and Cooke [68] whose work was on a biomedical problem. In their work, first order differential equations with piecewise constant argument was developed based on the investigation of vertically transmitted diseases. Since then, several papers were published by Shah, Cooke, Aftabizadeh and Wiener [69, 70, 78]. A typical EPCA studied by them is in the following form:

$$y'(t) = a_0 y(t) + a_1 y([t]) + a_2 y([t] \pm a_3),$$

where a_0, a_1, a_2 and a_3 are constants, $y(t)$ represents an unknown function, and $[t]$ denotes the greatest integer function. The initial value problems so defined have the structure of a continuous dynamical system within each of the intervals of unit length. There are many mathematical models involving a piecewise constant argument such as Froude pendulum, Workpiece-Cutter system, Geneva wheel, electrodynamic shaker, damped loading system, undamped systems, vibration systems and so on. For a brief description of the models, we can refer to the book by Dai [71]. These differential equations are closely related to delay differential equations (DDE) which provide a mathematical model for a physical, mechanical or biological system in which the rate of change of a system depends upon its past history as they contain arguments of delayed or advanced type [71, 72, 73]. Examples of numerrous applications can be found from literature [100, 101, 102].

The theory of differential equations with piecewise constant argument of the form

$$x'(t) = f(t, x(t), x(h(t))), \tag{1.4}$$

where the argument $h(t)$ has interval of constancy. For example, equations with $h(t) = [t],\ [t-n],\ t - n[t]$ were investigated in [69], where n is a positive integer and [.] denotes the greatest integer function.

An equation (1.4) in which $x'(t)$ is given by a function x evaluated at t and at arguments $[t], \dots, [t - n]$, where n is a non-negative integer, is called of retarded or delay type. If the arguments are t and $[t + 1], \dots, [t + n]$, then the equation is of advanced type. If both these types of arguments appear in the equation, it is called of mixed type.

The literature shows a general progress of an extensive interest in the properties of solutions to the governing differential equations with piecewise constant arguments. The system with retarded type and advanced type was investigated in [69, 70, 78] and the references therewith. Existence and uniqueness of the solution of this system and the asymptotic stability of some of its solutions, the oscillatory properties of its solution and many qualitative results were formulated and analyzed by researchers in the field of differential equations. A brief summary of the theory can be found in [67, 71, 77] and the references cited therein.

It is not surprising to expect that the investigations of EPCA are continuously attracting the attention from the scientists for the behaviors of piecewise constant systems, as can be found from the current literature. Examples of such researchs are on the existence of almost periodic solutions of retarded EPCA by Yuan [79], quasiperiodic solutions of EPCA by Küpper and Yuan [80], existence of periodic solutions of retarded EPCA by Wang [81], Green's function and comparison principles for first-order periodic EPCA by Cabada, Ferreiro and Nieto [82], existence, uniqueness and asymptotic behavior of EPCA by Papaschinopoulos [83]. Remarks on the development of the theory of EPCA can be also found in thesis [143, 144].

In the light of above discussions, most of the results for differential equations with piecewise constant argument are obtained by the method of reducing them into discrete equations. The method of reduction to discrete equations has been the main and possibly a unique way of stability analysis for these equations [69, 77]. Hence, qualitative properties of solutions which start at non-integer values can not be achieved. Particularly, one can not investigate the problem of stability completely, as only elements of a countable set are allowed to be discussed for initial moments. Consequently, we need a different kind of investigation.

By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with piecewise constant argument has been generalized in papers [3, 4, 5, 6, 7, 10, 17, 18]. It has been assumed that there is no restriction on the distance between the switching moments of the argument. Only equations which are linear with respect to the values of solutions at non-deviated moments of time have been investigated. That narrowed significantly the class of systems. All of these equations are reduced to equivalent integral equations such that one can investigate many problems, which have not been solved properly by using discrete equations, i.e., existence and uniqueness of solutions, stability and existence of periodic solutions. Since we do not need additional assumptions on the reduced discrete equations, the new method requires more easily verifiable conditions, similar to those for ordinary differential equations.

In papers [3, 4, 5, 6, 7], the theory of differential equations with piecewise constant argument has been generalized by Akhmet. Later, Akhmet gathered all results for differential equation with piecewise constant argument of generalized type in the book [2]. There, it has been proposed to investigate differential equations of the form

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{1.5}$$

where $\beta(t) = \theta_k$ (see Fig. 1.16) if $\theta_k \leq t < \theta_{k+1}$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$, is an identification function, $\theta_k, k \in \mathbb{Z}$, is a strictly increasing sequence of real numbers, $|\theta_k| \to \infty$ as $|k| \to \infty$. Clearly, the greatest integer function $[t]$ is a particular case of the function $\beta(t)$. That is, if we choose $\theta_k = k$, $k \in \mathbb{Z}$, then $\beta(t) = [t]$. System (1.5) is called a differential equation with piecewise constant argument of generalized type. That is to say, equation (1.5) is of delayed type.

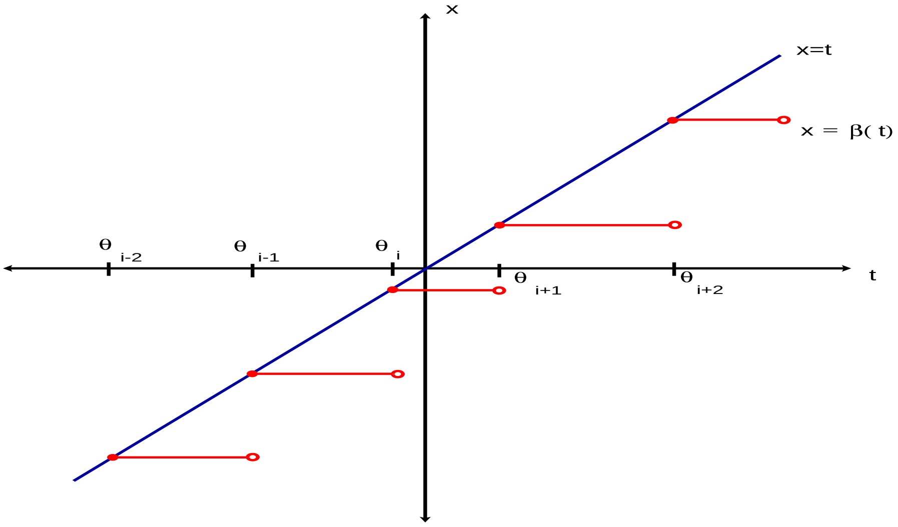

Figure 1.16: The graph of the argument $\beta(t)$.

Another generalization of differential equations with piecewise constant argument of type

$$x'(t) = f(t, x(t), x(\gamma(t))), \tag{1.6}$$

where $\gamma(t) = \zeta_k$ (see Fig.1.17) if $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$, are piecewise constant functions, ζ_k are strictly increasing sequence of real numbers, unbounded on the left and on the right such that $\theta_k \le \zeta_k \le \theta_{k+1}$ for all k.

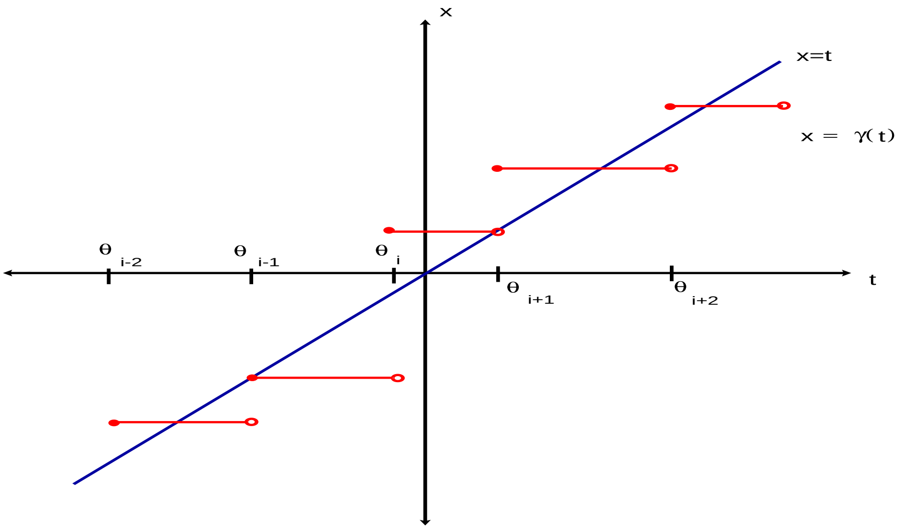

Figure 1.17: The graph of the argument $\gamma(t)$.

Let us clarify why the system (1.6) is of mixed type [90], that is, the argument can changed its deviation character during the motion. The argument is deviated if it is advanced or delayed.

Fix $k \in \mathbb{N}$, and consider the system on the interval $[\theta_k, \theta_{k+1})$. Then, the identification function $\gamma(t)$ is equal to ζ_k. If the argument t satisfies $\theta_k \le t < \zeta_k$, then $\gamma(t) > t$ and (1.6) is an equation with advanced argument. Similarly, if $\zeta_k < t < \theta_{k+1}$, then $\gamma(t) < t$ and (1.6) is an equation with delayed argument. Consequently, the equation (1.6) changes the type of deviation of the argument during the process. In other words, the system is of the mixed type.

1.3 Differential Equations with Piecewise Constant Argument of Generalized Type

In this section we shall give some useful definitions, lemmas and fundamental theorems for differential equations with piecewise constant argument of generalized type proposed by Akhmet [2, 4, 6, 8].

1.3.1 Description of Systems

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers and integers, respectively. Denote by $\|\cdot\|$ the Euclidean norm for vectors in $\mathbb{R}^m$, $m \in \mathbb{N}$, and denote the uniform norm by $\|C\| = \sup\{\|C x\| \mid \|x\| = 1\}$ for $m \times m$ matrices.

1.3.2 Existence and Uniqueness Theorems

We now consider the existence and uniqueness theorems for differential equations with piecewise constant argument of generalized type due to Akhmet [4, 6, 8] based on the construction of an equivalent integral equation.

1.3.2.1 Equations with Delayed Argument

In this part we consider the following quasilinear system with delayed argument

$$y' = Ay + f(t, y(t), y(\beta(t))), \tag{1.7}$$

where $y \in \mathbb{R}^m$, $t \in \mathbb{R}$, A is a constant $n \times n$ real valued matrix, $f \in C(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m)$ is a real valued $n \times 1$ function, $\beta(t) = \theta_i$ if $\theta_i \le t < \theta_{i+1}$, $i \in \mathbb{Z}$, is an identification function, θ_i, $i \in \mathbb{Z}$, is a strictly ordered sequence of real numbers, $|\theta_i| \to \infty$ as $|i| \to \infty$.

The following assumptions will be needed throughout this section:

(H1) $f(t,\ x,\ z)$ is continuous in the first argument, $f(t,\ 0,\ 0) = 0,\ t \in \mathbb{R}$, and f is Lipschitzian such that $\|f(t,\ y_1,\ w_1) - f(t,\ y_2,\ w_2)\| \leq \ell(\|y_1 - y_2\| + \|w_1 - w_2\|)$;

(H2) there exists real number $\overline{\theta} > 0$ such that $\theta_{i+1} - \theta_i \leq \overline{\theta},\ i \in \mathbb{Z}$;

Denote by $X(t, s) = e^{A(t-s)},\ t, s \in \mathbb{R}$ the fundamental matrix of the following linear homogeneous system

$$y'(t) = Ay$$

associated with (1.7). It is known that there exists a constant $\mu > 0$ such that $\|e^{A(t-s)}\| \leq e^{\mu|t-s|},\ t,\ s \in \mathbb{R}$. One can also show that $\|e^{A(t-s)}\| \geq e^{-\mu|t-s|}\ t,\ s \in \mathbb{R}$. Thus, there exist positive numbers N and n such that $n \leq \|e^{A(t-s)}\| \leq N$, where $N = e^{\mu\theta},\ n = e^{-\mu\theta}$ if $t, s \in [\theta_i, \theta_{i+1}]$ for all $i \in \mathbb{Z}$.

Definition 1.3.1 *A solution* $y(t) = y(t, \theta_i, y_0), y(\theta_i) = y_0,\ i \in \mathbb{Z}$, *of* (1.7) *on* $[\theta_i, \infty)$ *is a continuous function such that*

(i) *the derivative* $y'(t)$ *exists at each point* $t \in [\theta_i,\ \infty)$, *with the possible exception of the points* $\theta_j,\ j \geq i$, *where one-sided derivatives exist;*

(ii) *equation* (1) *is satisfied by* $y(t)$ *at each interval* $[\theta_j,\ \theta_{j+1}),\ j \geq i$.

Definition 1.3.1 is a new version of [69] adapted to our genaral case.

Theorem 1.3.2 *Suppose conditions* (H1) – (H2) *are fulfilled. Then for every* $y_0 \in \mathbb{R}^m$ *and* $i \in \mathbb{Z}$, *there exists a unique solution* $y(t)$ *of* (1.7) *in the sense of Definition* 1.3.1.

Assume additionally that:

(H3) $2N\ell\overline{\theta} < 1$;

(H4) $N\ell\overline{\theta}[1 + N(1 + \ell\overline{\theta})e^{N\ell\overline{\theta}}] < n$.

We continue with the following assertion which provides the conditions of existence and uniqueness of solutions for arbitrary initial moment t_0.

Lemma 1.3.3 *Assume that conditions* (H1) – (H4) *are fulfilled. Then, for every* $y_0 \in \mathbb{R}^m$, $t_0 \in \mathbb{R}$, $\theta_i < t_0 \leq \theta_{i+1}$, $i \in \mathbb{Z}$, *there exists a unique solution* $\overline{y}(t) = y(t, \theta_i, \overline{y}_0)$ *of* (1.7) *in sense of Definition* 1.3.1 *such that* $\overline{y}(t_0) = y_0$.

Let us introduce the following definition from [84], modified for our genaral case.

Definition 1.3.4 *A function* $y(t)$ *is a solution of* (1.7) *on* $\mathbb{R}$ *if:*

(i) *$y(t)$ is continuous on $\mathbb{R}$;*

(ii) *the derivative $y'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points θ_i, $i \in \mathbb{Z}$, where one-sided derivatives exist;*

(iii) *equation* (1.7) *is satisfied on each interval* $[\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

Now, we give the following equivalence lemma which is major importance of our investigations throughout the thesis. The proof of the assertion is very similar to that of Lemma 3.1 in [4].

Lemma 1.3.5 *A function* $y(t) = y(t, t_0, y_0)$, $y(t_0) = y_0$, *where* t_0 *is a fixed real number, is a solution of* (1.7) *in the sense of Definition 1.3.4 if and only if it is a solution of the following integral equation:*

$$y(t) = e^{A(t-t_0)}y_0 + \int_{t_0}^{t} e^{A(t-s)} f(s, y(s), y(\beta(s)))ds.$$

In the following theorem the conditions for the existence and uniqueness of solutions on $\mathbb{R}$ are established. The proof of the assertion is similar to that of Theorem 2.3 in [4].

Theorem 1.3.6 *Suppose that conditions* (H1) – (H4) *are fulfilled. Then, for every* $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^m$, *there exists a unique solution* $y(t) = y(t, t_0, y_0)$ *of* (1.7) *in sense of Definition* 1.3.4 *such that* $y(t_0) = y_0$.

The last theorem arranges the correspondence between points $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^m$ and all solutions of (1.7), and there is not a solution of the equation out of the correspondence. Using the assertion we can say that definition of the initial value problem for differential equations with piecewise constant argument of generalized type is similar to the problem for an ordinary differential equation, although the equation is of delayed type.

1.3.2.2 Equations with both Delayed and Advanced Arguments

Let us fix two real-valued sequences $\theta_i, \zeta_i, i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}, \theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{Z}, |\theta_i| \to \infty$ as $|i| \to \infty$.

We shall consider the following system of differential equations:

$$z'(t) = Cz(t) + f(t, z(t), z(\gamma(t))), \tag{1.8}$$

where $z \in \mathbb{R}^m, t \in \mathbb{R}, \gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1}), i \in \mathbb{Z}$.

The following assumptions will be needed:

(H1′) $C \in C(\mathbb{R})$ is a constant $m \times m$ real valued matrix;

(H2′) $f(t, x, y) \in C(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m)$ is an $m \times 1$ real valued function;

(H3′) $f(t, x, y)$ satisfies the condition

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq \ell_0(\|x_1 - x_2\| + \|y_1 - y_2\|), \tag{1.9}$$

where $\ell_0 > 0$ is a constant, and the condition

$$f(t, 0, 0) = 0, \; t \in \mathbb{R};$$

(H4′) there exists real number $\overline{\theta} > 0$ such that $\theta_{i+1} - \theta_i \leq \overline{\theta}, i \in \mathbb{Z}$.

One can see that system (1.8) on $t \in [\theta_i, \theta_{i+1}), i \in \mathbb{Z}$ have the form of a special functional-differential equations

$$z'(t) = Cz(t) + f(t, z(t), z(\zeta_i)). \tag{1.10}$$

We introduce the following definition, which is a version of a definition from [84], modified for the general case.

Definition 1.3.7 *A continuous function* $z(t)$ *is a solution of* (1.8) *on* $\mathbb{R}$ *if:*

(i) *the derivative* $z'(t)$ *exists at each point* $t \in \mathbb{R}$ *with the possible exception of the points* $\theta_i,\ i \in \mathbb{Z}$*, where the one-sided derivatives exist;*

(ii) *the equation is satisfied for* $z(t)$ *on each interval* $(\theta_i,\ \theta_{i+1}),\ i \in \mathbb{Z}$*, and it holds for the right derivative of* $z(t)$ *at the points* $\theta_i,\ i \in \mathbb{Z}$*.*

We now give some useful investigations for the fundamental matrix of solutions.

Denote by $Z(t,s) = e^{C(t-s)},\ t,s \in \mathbb{R}$ the fundamental matrix of the following linear homogeneous system

$$z'(t) = Cz$$

associated with (1.8). There exist positive numbers N and n such that $n \le \|e^{C(t-s)}\| \le N$ if $t,s \in [\theta_i, \theta_{i+1}]$ for all $i \in \mathbb{Z}$.

In the following lemma a correspondence between points $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^m$ and the solutions of (1.8) in the sense of Definition 1.3.7 is established. Using this result we can say that the definition of the IVP for our system is similar to that for ordinary differential equations, although it is an equation with a deviating argument. The proof of the assertion is very similar to that of Lemma 2.1 in [5].

Lemma 1.3.8 *A function* $z(t) = z(t, t_0, z_0),\ z(t_0) = z_0$*, where* t_0 *is a fixed real number, is a solution of* (1.8) *in the sense of Definition 1.3.7 if and only if it is a solution of the following integral equation:*

$$z(t) = e^{C(t-t_0)}z_0 + \int_{t_0}^{t} e^{C(t-s)} f(s, z(s), z(\gamma(s)))ds.$$

From now on we need the assumption

$$\text{(H5}')\ \ N\ell_0\bar{\theta}e^{N\ell_0\bar{\theta}} < 1,\ 2N\ell_0\bar{\theta} < 1,\ N^2\ell_0\bar{\theta}\{\tfrac{N\ell_0\bar{\theta}e^{N\ell_0\bar{\theta}}+1}{1-N\ell_0\bar{\theta}e^{N\ell_0\theta}} + N\ell_0\bar{\theta}e^{N\ell_0\bar{\theta}}\} < n.$$

Now, we continue with the following lemma which provides the conditions of existence and uniqueness of solutions for arbitrary initial moment ξ on $[\theta_i,\ \theta_{i+1}]$.

Lemma 1.3.9 *Assume that conditions* (H1′) − (H5′) *are fulfilled, and fix* $i \in \mathbb{Z}$*. Then, for every* $(\xi, z_0) \in [\theta_i, \theta_{i+1}] \times \mathbb{R}^m$*, there exists a unique solution* $z(t) = z(t, \xi, z_0)$ *of (1.10) on* $[\theta_i, \theta_{i+1}]$.

Theorem 1.3.10 *Assume that conditions* (H1′) − (H5′) *are fulfilled. Then, for every* $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^m$ *there exists a unique solution* $z(t) = z(t, t_0, z_0)$ *of (1.8) in the sense of Definition 1.3.7 such that* $z(t_0) = z_0$.

The last two assertions can be verified in exactly the same way as Lemma 1.1 and Theorem 1.1 from [5].

1.3.3 Basics of Lyapunov-Razumikhin Technique

In this section, the following results due to the paper [17] obtained by applying Razumikhin technique for differential equations with piecewise constant argument of generalized type [4, 5]. We will give sufficient conditions for stability, uniform stability and uniform asymptotic stability of the trivial solution of such equations.

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the set of natural numbers and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, 3, ...\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the m-dimensional real space by $\mathbb{R}^m$, $m \in \mathbb{N}$, and the Euclidean norm in $\mathbb{R}^m$ by $\|\,.\,\|$. Fix a real-valued sequence θ_i such that $0 = \theta_0 < \theta_1 < ... < \theta_i < ...$ with $\theta_i \to \infty$ as $i \to \infty$.

Let us introduce special notations:

$\mathcal{K} = \{\, a \in C(\mathbb{R}^+, \mathbb{R}^+) :$ strictly increasing and $a(0) = 0\}$,

$\Lambda = \{\, b \in C(\mathbb{R}^+, \mathbb{R}^+) : \; b(0) = 0, \; b(s) > 0$ for $s > 0\}$.

Here, we consider the following differential equation:

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{1.11}$$

where $x \in S(\rho)$, $S(\rho) = \{x \in \mathbb{R}^m : \|x\| < \rho\}$, $t \in \mathbb{R}^+$, $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$.

The following assumptions are needed:

(C1) $f(t, y, z) \in C(\mathbb{R}^+ \times S(\rho) \times S(\rho))$ is an $m \times 1$ real valued function;

(C2) $f(t,0,0) = 0$ for all $t \geq 0$;

(C3) $f(t,y,z)$ satisfies the condition

$$\|f(t,y_1,z_1) - f(t,y_2,z_2)\| \leq \ell(\|y_1 - y_2\| + \|z_1 - z_2\|)$$

for all $t \in \mathbb{R}^+$ and $y_1,\ y_2,\ z_1,\ z_2 \in S(\rho)$, where $\ell > 0$ is a Lipschitz constant;

(C4) there exists a positive number θ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}$;

(C5) $\ell\theta[1 + (1 + \ell\theta)e^{\ell\theta}] < 1$;

(C6) $3\ell\theta e^{\ell\theta} < 1$.

We give now some definitions and preliminary results which enable us to investigate stability of the trivial solution of (1.11).

Definition 1.3.11 *[4] A function* $x(t)$ *is a solution of* (1.11) *on* $\mathbb{R}^+$ *if:*

(i) $x(t)$ *is continuous on* $\mathbb{R}^+$*;*

(ii) *the derivative* $x'(t)$ *exists for* $t \in \mathbb{R}^+$ *with the possible exception of the points* θ_i, $i \in \mathbb{N}$, *where one-sided derivatives exist;*

(iii) *equation* (1.11) *is satisfied by* $x(t)$ *on each interval* (θ_i, θ_{i+1}), $i \in \mathbb{N}$, *and it holds for the right derivative of* $x(t)$ *at the points* θ_i, $i \in \mathbb{N}$.

For simplicity of notation in the sequel, let us denote

$$K(\ell) = \frac{1}{1 - \ell\theta[1 + (1 + \ell\theta)e^{\ell\theta}]}.$$

The following lemma is an important auxiliary result of that paper.

Lemma 1.3.12 *Let* (C1) − (C5) *be fulfilled. Then the following inequality*

$$\|x(\beta(t))\| \leq K(\ell)\,\|x(t)\|$$

holds for all $t \geq 0$.

We give the following assertion which establishes the existence and uniqueness of solutions of (1.11).

Theorem 1.3.13 *Assume that conditions* (C1) *and* (C3) – (C6) *are satisfied. Then for every* $(t_0, x_0) \in \mathbb{R}^+ \times S(\rho)$ *there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of* (1.11) *on* $\mathbb{R}^+$ *in the sense of Definition* 1.3.11 *such that* $x(t_0) = x_0$.

Definition 1.3.14 *Let* $V : \mathbb{R}^+ \times S(\rho) \to \mathbb{R}^+$. *Then, V is said to belong to the class* ϑ *if:*

(i) *V is continuous on $\mathbb{R}^+ \times S(\rho)$ and $V(t,0) \equiv 0$ for all $t \in \mathbb{R}^+$;*

(ii) *$V(t,x)$ is continuously differentiable on $(\theta_i, \theta_{i+1}) \times S(\rho)$ and for each $x \in S(\rho)$, right derivative exists at $t = \theta_i$, $i \in \mathbb{N}$.*

Let the derivative of V with respect to system (1.11) be defined by

$$V'(t,x,y) = \frac{\partial V(t,x)}{\partial t} + grad_x^T V(t,x) f(t,x,y)$$

for all $t \neq \theta_i$ in $\mathbb{R}^+$ and $x, y \in S(\rho)$ if a function $V \in \vartheta$.

Definitions of Lyapunov stability for the solutions of discussed systems can be given in the same way as for ordinary differential equations. Let us give the followings.

Definition 1.3.15 *[6] The zero solution of* (1.11) *is said to be*

(i) *stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;*

(ii) *uniformly stable if δ is independent of t_0.*

Definition 1.3.16 *[6] The zero solution of* (1.11) *is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x_0\| < \delta_0$.*

Now we give the formulation for the stability of the zero solution of (1.11) based on the Lyapunov-Razumikhin method. In the next theorems, we assume that conditions (C1)-(C6) are satisfied.

Theorem 1.3.17 *Assume that there exists a function $V \in \vartheta$ such that*

(i) $u(\|x\|) \leq V(t, x)$ *on* $\mathbb{R}^+ \times S(\rho)$*, where* $u \in \mathcal{K}$*;*

(ii) $V'(t, x, y) \leq 0$ *for all* $t \neq \theta_i$ *in* $\mathbb{R}^+$ *and* $x,\ y \in S(\rho)$ *such that*

$V(\beta(t), y) \leq V(t, x)$.

Then the zero solution of (1.11) *is stable.*

Theorem 1.3.18 *Assume that there exists a function* $V \in \vartheta$ *such that*

(i) $u(\|x\|) \leq V(t, x) \leq v(\|x\|)$ *on* $\mathbb{R}^+ \times S(\rho)$*, where* $u, v \in \mathcal{K}$*;*

(ii) $V'(t, x, y) \leq 0$ *for all* $t \neq \theta_i$ *in* $\mathbb{R}^+$ *and* $x,\ y \in S(\rho)$ *such that*

$V(\beta(t), y) \leq V(t, x)$.

Then the zero solution of (1.11) *is uniformly stable.*

Theorem 1.3.19 *Assume that all of the conditions in Theorem* 1.3.18 *are valid and there exist a continuous nondecreasing function* ψ *such that* $\psi(s) > s$ *for* $s > 0$ *and a function* $w \in \Lambda$*. If condition (ii) is replaced by*

(iii) $V'(t, x, y) \leq -w(\|x\|)$ *for all* $t \neq \theta_i$ *in* $\mathbb{R}^+$ *and* $x,\ y \in S(\rho)$ *such that*

$V(\beta(t), y) < \psi(V(t, x))$,

then the zero solution of (1.11) *is uniformly asymptotically stable.*

1.4 The Theory of Impulsive Differential Equations

The theory of impulsive differential equations play its own modest role in attracting the attention of researchers to the symbiosis of continuity and discontinuity for the definition of a motion. It is well known that impulsive differential equation [1, 112, 113] is one of the basic instruments so the role of discontinuity has been understood better for the real world problems. In real world, many evolutionary processes are characterized by abrupt changes at certain time. These changes are called to be impulsive phenomena, which are included in many fields such as biology involving thresholds, bursting rhythm models, physics, chemistry, population dynamics, models in economics, optimal control, neural networks, etc. For example, when an oscillating string is struck by a hammer, it experiences a sudden change of velocity; a pendulum of a clock undergoes a rapid change of momentum when it crosses its equilibrium position; harvesting and epidemics lead to a significant decrease in the population density of a species, etc. To explain such processes mathematically, it becomes necessary to study impulsive differential equations, also called differential equations with discontinuous trajectories.

A well known example of such phenomena is the mathematical model of clock [145, 146, 147]. Although, general novelties of impulsive differential equations were introduced by Pavlidis [128, 129, 130]. The book of Samoilenko and Perestyuk [112] is a fundamental work in the area as it contains many qualitative theoretical problems such as the existence and uniqueness of solutions, stability, periodic and almost periodic solutions, integral sets, optimum control, etc. Later, Akhmet [1] gathered all previous results and introduced new approaches in applied mathematics. In this book, main purpose is to present the theory of differential equations with solutions that have discontinuities either at the moments when the integral curves reach certain surfaces in the extended phase space (t, x), as time increases (decreases), or at the moments when the trajectories enter certain sets in the phase space x. That is to say, the moments when solutions have discontinuities are not prescribed.

There are two different kinds of impulsive differential equations: with impulses at fixed times; and with impulsive action at variable times. The first one has the form [1]

$$\begin{aligned} x'(t) &= f(t, x) \\ \Delta x \,|_{t=\tau_k} &= I_k(x), \end{aligned} \tag{1.12}$$

where $x \in \mathbb{R}^m,\ m \in \mathbb{N},\ t \in \mathbb{R},\ \{\tau_k\}$, is a given sequence of times indexed by a finite or an infinite set J, f and I_k are $m-$ dimensional vector-valued, continuous functions. A phase point of (1.12) moves along one of the trajectories of the differential equation $x'(t) = f(t, x)$ for all $t \neq \tau_k$. When $t = \tau_k$, the point has a jump $\Delta x \,|_{t=\tau_k} = x(\tau_k^+) - x(\tau_k^-) = I_k(x(\tau_k^-))$. Thus, a solution $x(t)$ of (1.12) is a piecewise continuous function that has discontinuities of the first kind at $t = \tau_k$.

In the latter one, impulse action occurs when the phase point of a system intersects the prescribed surfaces in the phase space. It is known that systems with impulses at variable times generate more difficult theoretical challenges if one compares that systems with impulses at fixed moments. Systems with impulses at variable (non-fixed) times is of the form

$$\begin{aligned} x'(t) &= f(t,x) \\ \Delta x \mid_{t=\tau_k(x)} &= I_k(x), \end{aligned} \tag{1.13}$$

where $\tau_k(x),\ k \in J$ defines for the surfaces of discontinuities. Despite the systems (1.12), the solutions of equations (1.13) are nevertheless piecewise continuous but the points of discontinuity depend on the solutions. For this reason, this makes the investigations of such systems more difficult.

We note that it is the first time that the differential equation with impulses is given in the form (1.12) in [1]. Generally, the system has been used in the following form [112, 113]

$$\begin{aligned} x'(t) &= f(t,x), \quad t \neq \tau_k, \\ \Delta x \mid_{t=\tau_k} &= I_k(x). \end{aligned} \tag{1.14}$$

The system (1.12) is more convenient for the motion with discontinuities than (1.14), since the existence of the left derivative at disontinuity points is not disregarded in system (1.12).

In this thesis, we focus on the systems with impulses at fixed times. Let us give some definitions and theoretical results for systems with fixed moments of impulses due to the books proposed by Akhmet, Samoilenko and Perestyuk [1, 112].

1.4.1 Description of the System

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers and integers, respectively. Denote by $\tau = \{\tau_k\}$ a strictly increasing sequaence of real numbers such that the set $\mathcal{A}$ of indexes k is an interval in $\mathbb{Z}$.

Definition 1.4.1 *τ is a $B-$ sequence, if one of the following alternatives is valid:*

(i) *$\tau = \emptyset$;*

(ii) *τ is a nonempty and finite set;*

(iii) *τ is an infinite set such that $|\tau_k| \to \infty$ as $|k| \to \infty$.*

Definition 1.4.2 *A function $\varphi : T \to \mathbb{R}^m$ is from the set $PC(T,\ \tau)$ if:*

(i) *φ is left continuous;*

(ii) *φ is continuous, except, possibly, points from τ, where it has discontinuities of the first kind.*

Definition 1.4.3 *A function $\varphi : T \to \mathbb{R}^m$ belongs to the set $PC^1(T,\ \tau)$ if:*

(i) *$\varphi \in PC(T,\ \tau)$;*

(ii) *$\varphi'(t) \in PC(T,\ \tau)$, where the derivative at points τ is assumed to be the left derivative.*

In this part of the section, we consider the following impulsive differential equation which has maximal correspondence to our investigations throughout the thesis:

$$\begin{aligned} x'(t) &= Ax + f(t,x) \\ \Delta x \mid_{t=\tau_k} &= I_k(x), \end{aligned} \tag{1.15}$$

where $A = diag(-a_1, \ldots, -a_m)$ with $a_i > 0,\ i = 1, \ldots, m$ is a constant diagonal matrix.

We understand a solution of (1.15) as a function from $PC^1(T, \tau), T \subset \mathbb{R}^+$, which satisfies the differential equation and the impulsive condition of (1.15). The differential equation is satisfied for all $t \in T$, except possibly at the moments of discontinuity τ, where the left side derivative exists and it satisfies the differential equation as well.

Denote by $X(t,s) = e^{A(t-s)},\ t, s \in \mathbb{R}^+$ the fundamental matrix of the following linear homogeneous system

$$x'(t) = Ax \tag{1.16}$$

associated with (1.15). One can easily see that $\|X(t,s)\| \le e^{-\sigma(t-s)}$, where $\sigma = \min\limits_{1 \le i \le m} a_i$.

We now give equivalent integral equations for the initial value problem (1.15).

Lemma 1.4.4 *A function $\varphi \in PC^1(T,\ \tau),\ \varphi(t_0) = x_0$, is a solution of (1.15) if and only if it is a solution of the following integral equation:*

$$\varphi(t) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s) f(s, \varphi(s)) ds + \sum_{t_0 \le \tau_k < t} X(t, \tau_k^+) I_k(\varphi(\tau_k)),\ t \ge t_0.$$

Lemma 1.4.5 *A function $\varphi \in PC^1(T,\ \tau),\ \varphi(t_0) = x_0$, is a solution of (1.15) if and only if it is a solution of the following integral equation:*

$$\varphi(t) = x_0 + \int_{t_0}^{t} \Big(A\varphi(s) + f(s, \varphi(s))\Big) ds + \sum_{t_0 \le \tau_k < t} I_k(\varphi(\tau_k)),\ t \ge t_0.$$

Let us give the Gronwall-Bellman Lemma for piecewise continuous functions, which is one of the simplest and most useful integral inequalities.

Lemma 1.4.6 *Let $u, v \in PC(J,\ \tau),\ u(t) \ge 0,\ v(t) > 0,\ t \in J,\ \beta_k \ge 0,\ k \in \mathcal{A},\ t_0 \in J$, and $c \in \mathbb{R}$ is a nonnegative constant. If $u(t)$ satisfies the inequality*

$$u(t) \le c + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0 \le \tau_k < t} \beta_k u(\tau_k),\ t \ge t_0,$$

then the following estimates holds for the function $u(t)$,

$$u(t) \le ce^{\int_{t_0}^{t} v(s)ds} \prod_{t_0 \le \tau_k < t} (1 + \beta_k),\ t \ge t_0.$$

1.4.2 Existence and Uniqueness Theorems

Let us denote by $J \subseteq \mathbb{R}^+, \tau$ and $G \subseteq \mathbb{R}^m, m \in \mathbb{N}$ as an open interval, a nonempty $B-$ sequence with set of indexes $\mathcal{A}$ and an open connected set, respectively. Consider a continuous function $f : J \times G \to \mathbb{R}^m$ and a map $I : \mathcal{A} \times G \to \mathbb{R}^m$. The domain of the equation (1.15) is the set $\Omega = J \times \mathcal{A} \times G$.

Theorem 1.4.7 *[1](Local existence theorem) Suppose $f(t, x)$ is continuous on $J \times G$ and $\prod_k G \subseteq G,\ k \in \mathcal{A}$. Then for any $(t_0, x_0) \in J \times G$ there is $\alpha > 0$ such that a solution $x(t, t_0, x_0)$ of (1.15) exists on $(t_0 - \alpha, t_0 + \alpha)$.*

Theorem 1.4.8 *[1](Uniqueness theorem) Assume that $f(t, x)$ satisfies a local Lipschitz condition, and every solution $x = v + I_k(v),\ k \in \mathcal{A}, x \in G$, has at most one solution with respect to v. Then any solution $x(t, t_0, x_0),\ (t_0, x_0) \in J \times G$, of (1.15) is unique. That is, if $y(t, t_0, x_0)$ is another solution of (1.15), and the two solutions are defined at some $t \in J$, then $x(t, t_0, x_0) = y(t, t_0, x_0)$.*

Now, let us consider existence and uniqueness theorems of (1.15) on Ω.

Fix $(t_0, x_0) \in J \times G$, and take

$$J_0 = [t_0 - h, t_0 + h],\ G_0 = \{x \in \mathbb{R}^m : \|x - x_0\| < H\},$$

with some fixed positive numbers H and h. Suppose that $J_0 \times G_0 \subset J \times G$ with small numbers. Let $p_+ = i([t_0, t_0 + h]),\ p_- = i([t_0 - h, t_0]),\ \mathcal{A}_0 = \{k \in \mathcal{A} : \tau_k \in J_0\}$ and $\tau^0 = \{\tau_k\},\ k \in \mathcal{A}_0$.

We need the following assumptions:

(C1) A is a continuous $m \times m$ diagonal matrix and $\|A\| \le N < \infty$;

(C2) there exists Lipschitz constant $\ell_f > 0$ such that $\|f(t, x) - f(t, y)\| \le \ell_f \|x - y\|$ for arbitrary $x, y \in G$, uniformly in all $(t, k) \in J \times \mathcal{A}$;

(C3) I_k satisfies $\|I_k(x) - I_k(y)\| \le \ell_I \|x - y\|$ for arbitrary $x, y \in G$, uniformly in all $(t, k) \in J \times \mathcal{A}$, where ℓ_I is a positive Lipschitz constant;

(C4) $\sup\limits_{I \times \mathcal{G}} \|f(t, x)\| + \sup\limits_{\mathcal{A} \times \mathcal{G}} \|I_k(x)\| = M < \infty$;

(C5) $(N + M)h + \max(p_+, p_-)M < H$;

(C6) $(N + \ell_f)h + \ell_I \max(p_+, p_-) < 1$.

Theorem 1.4.9 *Assume that* (C1) – (C6) *are valid. Then the initial value problem (1.15) and* $x(t_0) = x_0$ *admit a unique solution on* J_0.

1.4.3 Stability Based on Linear Approximation

In this section, we will give sufficient conditions for the global asymptotic stability of the zero solution of (1.15) based on linearization [112]. Here, we assume that $\varphi(t)$ is a solution of (1.15) such that $\varphi : J = [0, \infty) \to G$ and $\tau_k \to \infty$ as $k \to \infty$.

Definition 1.4.10 *[1] The solution* $\varphi(t)$ *is stable if for any* $\epsilon > 0$ *and* $t_0 \in J$ *there corresponds* $\delta(t_0, \epsilon) > 0$ *such that for any other solution* $\psi(t)$ *of (1.15) with* $\|\varphi(t_0) - \psi(t_0)\| < \delta(t_0, \epsilon)$ *we have* $\|\varphi(t) - \psi(t)\| < \epsilon$ *for* $t \ge t_0$.

Definition 1.4.11 *[1] The solution $\varphi(t)$ is asymptotically stable if it is stable in the sense of Definition 1.4.10 and there exists a positive number $\kappa(t_0)$ such that if $\psi(t)$ is any other solution of (1.15) with $\|\varphi(t_0) - \psi(t_0)\| < \kappa(t_0)$, then $\|\varphi(t) - \psi(t)\| \to 0$ as $t \to \infty$.*

From now on we make the following assumptions:

(C7) there exists positive number $\underline{\tau}$ such that $\underline{\tau} \leq \tau_{k+1} - \tau_k,\ k \in \mathbb{N}$;

(C8) $\sigma - \ell_f - \frac{\ln(1+\ell_I)}{\underline{\tau}} > 0$.

Theorem 1.4.12 *Assume that* (C1) – (C8) *are fulfilled.Then, the zero solution of (1.15) is globally asymptotically stable.*

1.4.4 Existence of Periodic Solutions

We shall need the following additional conditions of periodicty:

(C9) $f(t+\omega, x) = f(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times G$;

(C10) the sequence τ_k satisfies $\tau_{k+p} = \theta_k + \omega, k \in \mathbb{N}$ and $I_{k+p} = I_k$ for a fixed positive real period ω and some positive integer p.

A function $\varphi \in PC(R^+, \tau)$ is a piecewise continuous $\omega-$ periodic function if $\varphi(t+\omega) = \varphi(t)$, for all $t \in \mathbb{R}^+$.

Theorem 1.4.13 *(Poincare' criterion) Assume that* (C1), (C2) *and* (C4) – (C10) *are fulfilled. Then, a solution $\varphi \in PC(R^+, \tau)$ of (1.15) is $\omega-$ periodic if and only if $\varphi(0) = \varphi(\omega)$.*

In what follows, we introduce a Green's function

$$\mathcal{G}(t,s) = (1 - e^{A\omega})^{-1} \begin{cases} e^{A(t-s)}, & 0 \leq s \leq t \leq \omega, \\ e^{A(\omega+t-s)}, & 0 \leq t < s \leq \omega, \end{cases} \tag{1.17}$$

so that the unique $\omega-$ periodic solution of the system (1.15) can be written as

$$x^*(t) \quad = \quad \int_0^{\omega} \mathcal{G}(t,s) f(s,\varphi(s))ds + \sum_{k=1}^{p} \mathcal{G}(t,\tau_k^+) I_k(\varphi(\theta_k)).$$

For $\tau_k,\ k \in \mathbb{N}$, let $[0,\omega] \cap \{\tau_k\}_{k\in\mathbb{N}} = \{\tau_1, \ldots, \tau_p\}$ and $\max\limits_{t,s\in[0,\omega]} \|\mathcal{G}(t,s)\| = \lambda.$

Theorem 1.4.14 *Assume that* (C1) – (C10) *are valid. Moreover, the linear homogeneous $\omega-$ periodic system (1.16) does not have nontrivial $\omega-$ periodic solutions and the inequality* $\lambda(\ell_f \omega + \ell_I p) < 1$ *holds, then (1.15) has a unique $\omega-$ periodic solution.*

CHAPTER 2

STABILITY OF PERIODIC MOTIONS AND EQUILIBRIA

In this chapter we consider neural networks systems as well as impulsive neural networks systems with piecewise constant argument of generalized type. Sufficient conditions for the existence of a unique equilibrium and a periodic solution are obtained. The stability of these solutions is investigated. Examples with numerical simulations are presented to illustrate the results.

2.1 Equilibria of Neural Networks with Piecewise Constant Argument

In this section we obtain sufficient conditions for the existence of the unique equilibrium. Existence and uniqueness of the solutions are established. We get a criteria for the global asymptotic stability of the Hopfield-type neural networks with piecewise constant arguments of generalized type by using linearization.

2.1.1 Introduction and Preliminaries

In recent years, dynamics of delayed neural networks have been studied and developed by many authors and many applications have been found in different areas such as associative memory, image processing, signal processing, pattern recognition and optimization (see [42, 94, 111, 46] and references cited therein). As is well known, such applications depend on the existence of an equilibrium point and its stability.

One of the most crucial idea of the present section is that we assume Hopfield-type neural networks may "memorize" values of the phase variable at certain moments of time to utilize the values during middle process till the next moment. Thus, we arrive to differential equations with piecewise constant delay. Obviously, the distances between the moments may be very variable. Consequently, the concept of generalized type of piecewise constant argument may be fruitful for the theory of neural networks.

Let us denote the set of all real numbers, natural numbers and integers by $\mathbb{R}, \mathbb{N}, \mathbb{Z}$, respectively, and a norm on $\mathbb{R}^m$ by $\|\cdot\|$ where $\|u\| = \sum_{j=1}^{m} |u_i|$.

In the present section we shall consider the following Hopfield-type neural networks system with piecewise constant argument

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i, \\ & \quad a_i > 0, \ i = 1, 2, \ldots, m. \end{aligned} \tag{2.1}$$

where $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$, is an identification function, $\theta_k, k \in \mathbb{Z}$, is a strictly increasing sequence of real numbers, $|\theta_k| \to \infty$ as $|k| \to \infty$, and there exists a positive real number $\bar{\theta}$ such that $\theta_{k+1} - \theta_k \le \bar{\theta}, k \in \mathbb{Z}$. Moreover, m denotes the number of neurons in the network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\beta(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and $\theta_k, k \in \mathbb{Z}$; b_{ij}, c_{ij}, d_i are real constants; b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t, c_{ij} denotes the synaptic connection weight of the unit j on the unit i at time θ_k, d_i is the input from outside the network to the unit i.

The following assumptions will be needed throughout the section:

(C1) The activation functions $f_j, g_j \in C(\mathbb{R}^m)$ with $f_j(0) = 0, \ g_j(0) = 0$ satisfy

$$|f_j(u) - f_j(v)| \le L_j |u - v|,$$

$$|g_j(u) - g_j(v)| \le \bar{L}_j |u - v|$$

for all $u, v \in \mathbb{R}$, where $L_j, \bar{L}_j > 0$ are Lipschitz constants, for $j = 1, 2, \ldots, m$;

(C2) $\bar{\theta}[\alpha_3 + \alpha_2] < 1$;

(C3) $\bar{\theta}\left[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\right] < 1$,

where

$$\alpha_1 = \sum_{i=1}^{m}\sum_{j=1}^{m} |b_{ji}| L_i, \ \alpha_2 = \sum_{i=1}^{m}\sum_{j=1}^{m} |c_{ji}| \bar{L}_i, \ \alpha_3 = \sum_{i=1}^{m} a_i + \alpha_1.$$

Theorem 2.1.1 *Suppose* (C1) *holds. If the neural parameters* a_i, b_{ij}, c_{ij} *satisfy*

$$a_i > L_i \sum_{j=1}^{m} |b_{ji}| + \bar{L}_i \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \ldots, m.$$

Then, (2.1) has a unique equilibrium $x^* = (x_1^*, \ldots, x_m^*)^T$.

The proof of the theorem is almost identical to the verification in [46] with slight changes which are caused by the piecewise constant argument.

We understand a solution $x(t) = (x_1, \ldots, x_m)^T$ of (2.1) as a continuous function on $\mathbb{R}$ such that the derivative $x'(t)$ exists at each point $t \in \mathbb{R}$, with the possible exception of the points $\theta_k, k \in \mathbb{Z}$, where one-sided derivative exists and the differential equation (2.1) is satisfied by $x(t)$ on each interval (θ_k, θ_{k+1}) as well.

In the following theorem the conditions for the existence and uniqueness of solutions on $\mathbb{R}$ are established. The proof of the assertion is similar to that of Theorem 2.3 in [3]. Nevertheless, detailed proof will be given in Lemma 3.3.2 and Theorem 3.3.3 of Section 3.3.

Theorem 2.1.2 *Suppose that conditions* (C1) − (C3) *are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R} \times \mathbb{R}^m$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T,\ t \in \mathbb{R}$, *of (2.1), such that* $x(t_0) = x^0$.

Now, let us give the following two equivalence lemmas of (2.1). The proofs are omitted here, since they are similar to that of Lemma 3.1 in [3].

Lemma 2.1.3 *A function* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$, *where* t_0 *is a fixed real number, is a solution of (2.1) on* $\mathbb{R}$ *if and only if it is a solution of the following integral equation on* $\mathbb{R}$: *For* $i = 1, \ldots, m$,

$$\begin{aligned} x_i(t) \quad = \quad & e^{-a_i(t-t_0)} x_i^0 + \int_{t_0}^{t} e^{-a_i(t-s)} \left[\sum_{j=1}^{m} b_{ij} f_j(x_j(s)) \right. \\ & \left. + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i \right] ds. \end{aligned}$$

Lemma 2.1.4 *A function* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$, *where* t_0 *is a fixed real number, is a solution of (2.1) on* $\mathbb{R}$ *if and only if it is a solution of the following integral equation on* $\mathbb{R}$:

For $i = 1, \dots, m$,

$$x_i(t) = x_i^0 + \int_{t_0}^{t} \Bigg[-a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i \Bigg] ds.$$

2.1.2 Stability of Equilibrium

In this section, we will give sufficient conditions for the global asymptotic stability of the equilibrium x^*. The system (2.1) can be reduced as follows. Let $y_i = x_i - x_i^*$, for each $i = 1, \dots, m$. Then,

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} \phi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} \psi_j(y_j(\beta(t))), \\ & a_i > 0, \ i = 1, 2, \dots, m, \end{aligned} \tag{2.2}$$

where $\phi_i(y_i) = f_i(y_i + x_i^*) - f_i(x_i^*)$ and $\psi_i(y_i) = g_i(y_i + x_i^*) - g_i(x_i^*)$. For each $j = 1, \dots, m$, $\phi_j(\cdot)$, $\psi_j(\cdot)$, are Lipschitzian since $f_j(\cdot)$, $g_j(\cdot)$ are Lipschitzian with $L_j, \bar{L}_j$ respectively, and $\phi_j(0) = 0$, $\psi_j(0) = 0$.

Definition 2.1.5 *The equilibrium* $x = x^*$ *of* (2.1) *is said to be globally asymptotically stable if there exist positive constants* α_1 *and* α_2 *such that the estimation of the inequality* $\|x(t) - x^*\| < \alpha_1 \|x(t_0) - x^*\| e^{-\alpha_2 (t - t_0)}$ *is valid for all* $t \geq t_0$.

For simplicity of notation in the sequel, let us denote

$$\zeta = \left\{ 1 - \bar{\theta} \left[\alpha_2 + \alpha_3 \left(1 + \bar{\theta} \alpha_2 \right) e^{\bar{\theta} \alpha_3} \right] \right\}^{-1}.$$

The following lemma, which plays an important role in the proofs of further theorems has been considered in [17]. But, for convenience of the reader we place the full proof of the assertion.

Lemma 2.1.6 *Let* $y(t) = (y_1(t), \dots, y_m(t))^T$ *be a solution of (2.2) and* (C1) − (C3) *be satisfied. Then, the following inequality*

$$\|y(\beta(t))\| \leq \zeta \|y(t)\| \tag{2.3}$$

holds for all $t \in \mathbb{R}$.

Proof. For a fixed $t \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $t \in [\theta_k, \theta_{k+1})$. Then, from Lemma 2.1.4, we have

$$\begin{aligned}
\|y(t)\| &= \sum_{i=1}^{m} |y_i(t)| \\
&\le \|y(\theta_k)\| + \sum_{i=1}^{m} \left\{ \int_{\theta_k}^{t} \left[a_i |y_i(s)| + \sum_{j=1}^{m} |b_{ji}| L_i |y_i(s)| \right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} |c_{ji}| \bar{L}_i |y_i(\theta_k)| \right] ds \right\} \\
&\le (1 + \bar{\theta}\alpha_2)\|y(\theta_k)\| + \int_{\theta_k}^{t} \alpha_3 \|y(s)\| ds.
\end{aligned}$$

The Gronwall-Bellman Lemma yields that

$$\|y(t)\| \le \left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3} \|y(\theta_k)\|. \tag{2.4}$$

Furhermore, for $t \in [\theta_k, \theta_{k+1})$ we have

$$\begin{aligned}
\|y(\theta_k)\| &\le \|y(t)\| + \sum_{i=1}^{m} \left\{ \int_{\theta_k}^{t} \left[a_i |y_i(s)| + \sum_{j=1}^{m} |b_{ji}| L_i |y_i(s)| \right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} |c_{ji}| \bar{L}_i |y_i(\theta_k)| \right] ds \right\} \\
&\le \|y(t)\| + \bar{\theta}\alpha_2 \|y(\theta_k)\| + \int_{\theta_k}^{t} \alpha_3 \|y(s)\| ds.
\end{aligned}$$

The last inequality and (2.4) imply that

$$\|y(\theta_k))\| \le \|y(t)\| + \bar{\theta}\alpha_2 \|y(\theta_k)\| + \bar{\theta}\alpha_3 \left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3} \|y(\theta_k)\|.$$

Thus, it follows from condition (C3) that

$$\|y(\theta_k)\| \le \zeta \|y(t)\|, \qquad t \in [\theta_k, \theta_{k+1}).$$

Accordingly, (2.3) holds for all $t \in \mathbb{R}$, which is the desired conclusion. □

From now on we need the following assumption:

(C4) $\gamma - \alpha_1 - \zeta\alpha_2 > 0$, where $\gamma = \min_{1 \le i \le m} a_i$ is positive.

Theorem 2.1.7 *Assume that* (C1) − (C4) *are fulfilled. Then, the zero solution of (2.2) is globally asymptotically stable.*

Proof. Let $y(t) = (y_1(t), \ldots, y_m(t))^T$ be an arbitrary solution of (2.2). From Lemma 2.1.3, we have

$$\begin{aligned} \|y(t)\| &\leq e^{-\gamma(t-t_0)}\|y_0\| + \sum_{i=1}^{m}\left\{\int_{t_0}^{t} e^{-\gamma(t-s)}\left[\sum_{j=1}^{m}|b_{ji}|L_i|y_i(s)|\right.\right. \\ &\quad \left.\left. + \sum_{j=1}^{m}|c_{ji}|\bar{L}_i|y_i(\beta(s))|\right]ds\right\} \\ &\leq e^{-\gamma(t-t_0)}\|y_0\| + (\alpha_1 + \zeta\alpha_2)\int_{t_0}^{t} e^{-\gamma(t-s)}\|y(s)\|ds. \end{aligned}$$

It follows that

$$e^{\gamma(t-t_0)}\|y(t)\| \leq \|y_0\| + (\alpha_1 + \zeta\alpha_2)\int_{t_0}^{t} e^{\gamma(s-t_0)}\|y(s)\|ds.$$

By virtue of Gronwall-Bellman inequality, we obtain that

$$\|y(t)\| \leq e^{-(\gamma-\alpha_1-\zeta\alpha_2)(t-t_0)}\|y_0\|.$$

The last inequality, in conjunction with (C4), deduces that the zero solution of system (2.2) is globally asymptotically stable. □

2.2 Periodic Motions of Neural Networks with Piecewise Constant Argument

In this section we derive some sufficient conditions for the existence and stability of periodic solutions.

2.2.1 Existence and Stability of Periodic Solutions

In this part, we study the existence and global asymptotic stability of the periodic solution of (2.1). The following conditions are to be assumed:

(C5) there exists a positive integer p such that $\theta_{k+p} = \theta_k + \omega,\ k \in \mathbb{Z}$ with a fixed positive real period ω;

(C6) $\kappa\left[\omega\left(\alpha_1 + \zeta\alpha_2\right)\right] < 1, \quad$ where $\kappa = \frac{1}{1-e^{-\gamma\omega}}$.

Theorem 2.2.1 *Assume that conditions* (C1) – (C3) *and* (C5) – (C6) *are valid. Then, the system (2.1) has a unique $\omega-$ periodic solution.*

We omit the proof of this assertion, since it can be proved in the same way as existence of the periodic solution for the quasilinear system of ordinary differential equations in noncritical case [110].

Theorem 2.2.2 *Assume that conditions* (C1) − (C6) *are valid. Then, the periodic solution of (2.1) is globally asymptotically stable.*

Proof. By Theorem 2.2.1, we know that (2.1) has an $\omega-$periodic solution $x^*(t) = (x_1^*, \ldots, x_m^*)^T$. Suppose that $x(t) = (x_1, \ldots, x_m)^T$ is an arbitrary solution of (2.1) and let $z(t) = x(t) - x^*(t) = (x_1 - x_1^*, \ldots, x_m - x_m^*)^T$. Then, from Lemma 1.1, we have

$$
\begin{aligned}
\|z(t)\| \quad &\leq \quad e^{-\gamma(t-t_0)}\|z_0\| + \sum_{i=1}^{m}\left\{\int_{t_0}^{t} e^{-\gamma(t-s)}\left[\sum_{j=1}^{m}|b_{ji}|L_i|z_i(s)|\right.\right. \\
&\qquad \left.\left. + \sum_{j=1}^{m}|c_{ji}|\bar{L}_i|z_i(\beta(s))|\right]ds\right\} \\
&\leq \quad e^{-\gamma(t-t_0)}\|z_0\| + (\alpha_1 + \zeta\alpha_2)\int_{t_0}^{t} e^{-\gamma(t-s)}\|z(s)\|ds.
\end{aligned}
$$

Also, the previous inequality can be written as,

$$
e^{\gamma(t-t_0)}\|z(t)\| \quad \leq \quad \|z_0\| + (\alpha_1 + \zeta\alpha_2)\int_{t_0}^{t} e^{\gamma(s-t_0)}\|z(s)\|ds.
$$

By applying Gronwall-Bellman inequality, we obtain that

$$
\|z(t)\| \leq e^{-(\gamma-\alpha_1-\zeta\alpha_2)(t-t_0)}\|z_0\|.
$$

Thus, using (C4), the periodic solution of system (2.1) is globally asymptotically stable. □

2.3 Equilibria of Neural Networks with Impact Activation

In this section we introduce following two different types of impulsive neural networks system with piecewise constant argument of generalized type. For these types, sufficient conditions for the existence of the unique equilibrium are obtained, existence and uniqueness of solutions and the equivalence lemma for such systems are established and stability criterion for the equilibrium based on linear approximation is proposed.

2.3.1 $(\theta,\theta)-$ Type Neural Networks

In this type, switching moments of constancy of arguments $\theta_k,\ k \in \mathbb{N}$ and the moments of discontinuity are same for impulsive Hopfield-type neural networks system with piecewise constant arguments.

2.3.1.1 Introduction and Preliminaries

Scientists often are interested in systems, which are either continuous-time or discrete-time. They are widely studied in the theory of neural networks, but there is a somewhat new category of dynamical system, which is neither continuous-time nor purely discrete-time; among them are dynamical systems with impulses, and systems with piecewise constant arguments [72, 117, 118, 119, 120]. It is obvious that processes of 'integrate-and-fire' type in neural networks [125, 126, 127, 128] request the systems as a mathematical modeling instrument. Significant parts of pioneer results for impulsive differential equations (IDE) and differential equations with piecewise constant argument (EPCA) can be found in [1, 2, 72, 77, 112].

In recent years, dynamics of Hopfield-type neural networks have been studied and developed by many authors by using IDE [117, 120, 121, 122, 123, 124] and EPCA [114]. To the best of our knowledge, there have been no results on the dynamical behavior of impulsive Hopfield-type neural networks with piecewise constant arguments. Our investigation contains an attempt to fill the gap by considering differential equations with piecewise constant arguments of generalized type [4, 5, 6].

Denote by $\mathbb{N}$ and $\mathbb{R}^+ = [0,\infty)$ the sets of natural and nonnegative real numbers, respectively, and denote a norm on $\mathbb{R}^n$ by $\|\cdot\|$, where $\|u\| = \sum_{j=1}^{m} |u_i|$. The main subject under investigation in this section is the following impulsive Hopfield-type neural networks system with piecewise constant argument

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i,\ t \neq \theta_k \\ \Delta x_i \mid_{t=\theta_k} &= I_k(x_i(\theta_k^-)), \quad i = 1,2,\ldots,m, \quad k \in \mathbb{N}, \end{aligned} \tag{2.5}$$

where $\beta(t) = \theta_{k-1}$ if $\theta_{k-1} \le t < \theta_k,\ k \in \mathbb{N},\ t \in \mathbb{R}^+$, is an identification function, $\theta_k > 0, k \in \mathbb{N}$, is a sequence of real numbers such that there exist two positive real numbers $\underline{\theta}$, $\overline{\theta}$ such that $\underline{\theta} \le \theta_{k+1} - \theta_k < \overline{\theta}, k \in \mathbb{N}$, $\Delta x_i(\theta_k)$ denotes $x_i(\theta_k) - x_i(\theta_k^-)$, where $x_i(\theta_k^-) = \lim_{h\to 0^-} x_i(\theta_k + h)$.

Moreover, $a_i > 0, i = 1, 2, \ldots, m$ are constants, m denotes the number of neurons in the network, $x_i(t)$ corresponds to the state of the i th unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\beta(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time $t \in [\theta_{k-1}, \theta_k), k = 1, 2, \ldots,$ and θ_{k-1}; b_{ij}, c_{ij}, d_i are constants; b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t, c_{ij} denotes the synaptic connection weight of the unit j on the unit i at time θ_{k-1}, d_i is the input from outside the network to the unit i.

We denote $PC(J, \mathbb{R}^m)$, where $J \subset \mathbb{R}^+$ is an interval, the set of all right continuous functions $\varphi : J \to \mathbb{R}^m$ with possible points of discontinuity of the first kind at $\theta_k \in J, k \in \mathbb{N}$.

Moreover, we introduce a set $PC^{(1)}(J, \mathbb{R}^m)$ of functions $\varphi : J \to \mathbb{R}^m$ such that $\varphi, \varphi' \in PC(J, \mathbb{R}^m)$, where the derivative at points θ_k is assumed to be the right derivative.

Throughout this section, we assume that the functions $I_k : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, the parameters b_{ij}, c_{ij}, d_i are real, the activation functions $f_j, g_j \in C(\mathbb{R}^m)$ with $f_j(0) = 0, g_j(0) = 0$, and they satisfy the following conditions:

(C1) there exist Lipschitz constants $L_j, \bar{L}_j > 0$ such that

$$|f_j(u) - f_j(v)| \leq L_j|u - v|,$$

$$|g_j(u) - g_j(v)| \leq \bar{L}_j|u - v|$$

for all $u, v \in \mathbb{R}^m$, $j = 1, 2, \ldots, m$;

(C2) the impulsive operator I_i satisfies

$$|I_i(u) - I_i(v)| \leq \ell|u - v|$$

for all $u, v \in \mathbb{R}^m$, $i = 1, 2, \ldots, m$, where l is a positive Lipschitz constant.

For the sake of convenience, we adopt the following notations:

$$\alpha_1 = \sum_{i=1}^{m} \sum_{j=1}^{m} |b_{ji}| L_i, \ \alpha_2 = \sum_{i=1}^{m} \sum_{j=1}^{m} |c_{ji}| \bar{L}_i, \ \alpha_3 = \sum_{i=1}^{m} a_i + \alpha_1.$$

Furthermore, the following assumptions will be needed throughout the section:

(C3) $\bar{\theta}[\alpha_3 + \alpha_2] < 1$;

(C4) $\bar{\theta}\left[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\right] < 1.$

Taking into account the definition of solutions for differential equations with piecewise constant arguments of generalized type [2] and IDE [113], we understand a solution of (2.5) as a function from $PC^{(1)}(J, \mathbb{R}^m)$, $J \subset \mathbb{R}^+$, which satisfies the differential equation and the impulsive condition of (2.5). The differential equation is satisfied for all $t \in J$, except possibly at the moments of discontinuity θ_k, where the right side derivative exists and it satisfies the differential equation as well.

Let us denote an equilibrium solution for the differential equation of (2.5) as the constant vector $x^* = (x_1^*, \ldots, x_m^*)^T \in \mathbb{R}^m$, where each x_i^* satisfies

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*)) + d_i.$$

The proof of following lemma is almost identical to the verification of Lemma 2.2 in [117] with slight changes which are caused by the piecewise constant argument.

Lemma 2.3.1 *Assume that the neural parameters a_i, b_{ij}, c_{ij} and Lipschitz constants $L_j, \bar{L_j}$ satisfy*

$$a_i > L_i \sum_{j=1}^{m} |b_{ji}| + \bar{L_i} \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \ldots, m.$$

Then, the differential equation of (2.5) has a unique equilibrium.

Theorem 2.3.2 *If the equilibrium $x^* = (x_1^*, \ldots, x_m^*)^T \in \mathbb{R}^m$ of the differential equation of (2.5) satisfies $I_k(x_i^*) = 0$ for all $i = 1, \ldots, m,\ k \in \mathbb{N}$. Then, x^* is an equilibrium point of (2.5).*

Particularly, if $c_{ij} = 0$, the system (2.5) reduces to the system in [117].

In the theory of differential equations with piecewise constant arguments of generalized type [2], we take the function $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}$, that is, $\beta(t)$ is right continuous. However, as it is usually done in the theory of IDE, at the points of discontinuity θ_k of the solution, solutions are left continuous. One should say that for the following investigation the right continuity is more convenient assumption if one considers equations with piecewise constant arguments.

The rest of this section is organized as follows: In the next section, we obtain sufficient conditions for the existence and uniqueness of the solutions and the equivalence lemma for (2.5). We get a criteria for the global asymptotic stability of the impulsive Hopfield-type neural networks with piecewise constant arguments of generalized type by using linearization.

2.3.1.2 Existence and Uniqueness of Solutions

Consider the following system

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\theta_{r-1})) + d_i, \\ & \quad a_i > 0,\ i = 1, 2, \ldots, m \end{aligned} \tag{2.6}$$

for $\theta_{r-1} \leq t \leq \theta_r$.

Now, we continue with the following lemma which provides the conditions of existence and uniqueness of solutions for arbitrary initial moment ξ.

Lemma 2.3.3 *Let* (C1), (C3), (C4) *be satisfied. Then for each* $x^0 \in \mathbb{R}^m$, *and* ξ, $\theta_{r-1} \leq \xi < \theta_r, r \in \mathbb{N}$, *there exists a unique solution* $x(t) = x(t, \xi, x^0) = (x_1(t), \ldots, x_m(t))^T$, *of (2.6),* $\theta_{r-1} \leq t \leq \theta_r$, *such that* $x(\xi) = x^0 = (x_1^0, \ldots, x_m^0)^T$.

Proof. *Existence* : It is enough to show that the equivalent integral equation

$$z_i(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i z_i(s) + \sum_{j=1}^{m} b_{ij} f_j(z_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(z_j(\theta_{r-1})) + d_i \right] ds$$

has a unique solution $z(t) = (z_1(t), \ldots, z_m(t))^T$.

Define a norm $\|z(t)\|_0 = \max_{[\theta_{r-1}, \theta_r]} \|z(t)\|$ and construct the following sequences $z_i^n(t)$, $z_i^0(t) \equiv x_i^0$, $i = 1, \ldots, m$, $n \geq 0$ such that

$$z_i^{n+1}(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i z_i^n(s) + \sum_{j=1}^{m} b_{ij} f_j(z_j^n(s)) + \sum_{j=1}^{m} c_{ij} g_j(z_j^n(\theta_{r-1})) + d_i \right] ds.$$

One can find that

$$\|z^{n+1}(t) - z^n(t)\|_0 = \max_{[\theta_{r-1}, \theta_r]} \|z^{n+1}(t) - z^n(t)\| \leq \left[\overline{\theta} \left[\alpha_3 + \alpha_2 \right] \right]^n \kappa,$$

where

$$\kappa = \bar{\theta}\left[(\alpha_3 + \alpha_2)\,\|x^0\| + \sum_{i=1}^{m} d_i\right].$$

Hence, the sequences $z_i^n(t)$ are convergent and their limits satisfy the integral equation on $[\theta_{r-1}, \theta_r]$. The existence is proved.

Uniqueness : It is sufficient to check that for each $t \in [\theta_{r-1}, \theta_r]$, and $x^2 = (x_1^2, \ldots, x_m^2)^T$, $x^1 = (x_1^1, \ldots, x_m^1)^T \in \mathbb{R}^m$, $x^2 \neq x^1$, the condition $x(t, \theta_{r-1}, x^1) \neq x(t, \theta_{r-1}, x^2)$ is valid. Let us denote the solutions of (2.5) by $x^1(t) = x(t, \theta_{r-1}, x^1)$, $x^2(t) = x(t, \theta_{r-1}, x^2)$. Assume on the contrary that there exists $t^* \in [\theta_{r-1}, \theta_r]$ such that $x^1(t^*) = x^2(t^*)$. Then, we have

$$\begin{aligned} x_i^1 - x_i^2 &= \int_{\theta_{r-1}}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s) \right) + \sum_{j=1}^{m} b_{ij} [f_j(x_j^2(s)) - f_j(x_j^1(s))] \right. \\ &+ \left. \sum_{j=1}^{m} c_{ij} [g_j(x_j^2(\theta_{r-1})) - g_j(x_j^1(\theta_{r-1}))] \right] ds, \quad i = 1, \ldots, m. \end{aligned} \tag{2.7}$$

Taking the absolute value of both sides for each $i = 1, \ldots, m$ and adding all equalities, we obtain

$$\begin{aligned} \|x^2 - x^1\| &= \sum_{i=1}^{m} \left| \int_{\theta_{r-1}}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s) \right) + \sum_{j=1}^{m} b_{ij} \left[f_j(x_j^2(s)) - f_j(x_j^1(s)) \right] \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{m} c_{ij} \left[g_j(x_j^2(\theta_{r-1})) - g_j(x_j^1(\theta_{r-1})) \right] \right] ds \right| \\ &\leq \sum_{i=1}^{m} \left\{ \int_{\theta_{r-1}}^{t^*} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} L_i |b_{ji}| |x_i^2(s) - x_i^1(s)| \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{m} \bar{L}_i |c_{ji}| |x_i^2 - x_i^1| \right] ds \right\} \\ &\leq \int_{\theta_{r-1}}^{t^*} \alpha_3 \|x^1(s) - x^2(s)\| ds + \bar{\theta}\alpha_2 \|x^1 - x^2\|. \end{aligned} \tag{2.8}$$

Furthermore, for $t \in [\theta_{r-1}, \theta_r]$, the following is valid:

$$\begin{aligned} \|x^1(t) - x^2(t)\| &\leq \|x^1 - x^2\| + \sum_{i=1}^{m} \left\{ \int_{\theta_{r-1}}^{t} \left[a_i |x_i^2(s) - x_i^1(s)| \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{m} L_i |b_{ji}| |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} \bar{L}_i |c_{ji}| |x_i^2 - x_i^1| \right] ds \right\} \\ &\leq \left(1 + \bar{\theta}\alpha_2 \right) \|x^1 - x^2\| + \int_{\theta_{r-1}}^{t} \alpha_3 \|x^1(s) - x^2(s)\| ds. \end{aligned}$$

Using Gronwall-Bellman inequality, it follows that

$$\|x^1(t) - x^2(t)\| \leq \left(1 + \bar{\theta}\alpha_2 \right) e^{\bar{\theta}\alpha_3} \|x^1 - x^2\|. \tag{2.9}$$

Consequently, substituting (2.9) in (2.8), we obtain

$$\|x^1 - x^2\| \le \left[\bar{\theta}\alpha_3\left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3} + \bar{\theta}\alpha_2\right]\|x^1 - x^2\|. \tag{2.10}$$

Thus, one can see that (C4) contradicts with (2.10). The lemma is proved. □

Theorem 2.3.4 *Assume that conditions* (C1), (C3), (C4) *are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^m$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^T,\ t \ge t_0$, *of (2.5), such that* $x(t_0) = x^0$.

Proof. Fix $t_0 \in \mathbb{R}^+$. There exists $r \in \mathbb{N}$ such that $t_0 \in [\theta_{r-1}, \theta_r)$. Use Lemma 2.3.3 with $\xi = t_0$ to obtain the unique solution $x(t, t_0, x^0)$ on $[\xi, \theta_r]$. Then apply the impulse condition to evaluate uniquely $x(\theta_r, t_0, x^0) = x(\theta_r^-, t_0, x^0) + I(x(\theta_r^-, t_0, x^0))$. Next, on the interval $[\theta_r, \theta_{r+1})$ the solution satisfies the ordinary differential equation

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(y_j(\theta_r)) + d_i. \\ & \quad a_i > 0,\ i = 1, 2, \ldots, m. \end{aligned}$$

The system has a unique solution $y(t, \theta_r, x(\theta_r, t_0, x^0))$. By definition of the solution of (2.5), $x(t, t_0, x^0) = y(t, \theta_r, x(\theta_r, t_0, x^0))$ on $[\theta_r, \theta_{r+1}]$. The mathematical induction completes the proof. □

Let us introduce the following two lemmas. We will prove just the second one, the proof for the first one is similar.

Lemma 2.3.5 *A function* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^T$, *where* t_0 *is a fixed real number, is a solution of (2.5) on* $\mathbb{R}^+$ *if and only if it is a solution, on* $\mathbb{R}^+$, *of the following integral equation:*

$$\begin{aligned} x_i(t) &= e^{-a_i(t-t_0)}x_i^0 + \int_{t_0}^{t} e^{-a_i(t-s)}\left[\sum_{j=1}^{m} b_{ij} f_j(x_j(s))\right. \\ &\quad \left. + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i\right] ds + \sum_{t_0 \le \theta_k < t} e^{-a_i(t-\theta_k)} I_k(x_i(\theta_k^-)), \end{aligned}$$

for $i = 1, \ldots, m,\ t \ge t_0$.

Lemma 2.3.6 *A function $x(t) = x(t, t_0, x^0) = (x_1(t), \dots, x_m(t))^T$, where t_0 is a fixed real number, is a solution of (2.5) on $\mathbb{R}^+$ if and only if it is a solution, on $\mathbb{R}^+$, of the following integral equation:*

$$\begin{aligned} x_i(t) \quad &= \quad x_i^0 + \int_{t_0}^{t} \Bigg[-a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i \Bigg] ds + \sum_{t_0 \le \theta_k < t} I_k(x_i(\theta_k^-)), \end{aligned}$$

for $i = 1, \dots, m,\ t \ge t_0$.

Proof. Sufficient part of this lemma can be easily proved. Therefore, we only prove the necessity of this lemma. Fix $i = 1, \dots, m$. Assume that $x(t) = (x_1(t), \dots, x_m(t))^T$ is a solution of (2.5) on $\mathbb{R}^+$. Denote

$$\begin{aligned} \varphi_i(t) \quad &= \quad x_i^0 + \int_{t_0}^{t} \Bigg[-a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i \Bigg] ds + \sum_{t_0 \le \theta_r < t} I_r(x_i(\theta_r^-)). \end{aligned} \tag{2.11}$$

It is clear that the expression in the right side exists for all t.

Assume that $t_0 \in (\theta_{r-1}, \theta_r)$, then differentiating the last expression, we get

$$\varphi_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i.$$

We also have

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i.$$

Hence, for $t \ne \theta_r,\ r \in \mathbb{N}$, we obtain

$$[\varphi_i(t) - x_i(t)]' = 0. \tag{2.12}$$

Moreover, it follows from equation (2.11) that

$$\Delta\varphi_i(\theta_r) = \varphi_i(\theta_r) - \varphi_i(\theta_r^-) = I_r(\varphi_i(\theta_r^-)). \tag{2.13}$$

One can see that $\varphi_i(t_0) = x_i^0$. Then, by (2.12), we have that $\varphi_i(t) = x_i(t)$ on $[t_0, \theta_r)$, which implies $\varphi_i(\theta_r^-) = x_i(\theta_r^-)$. Next, using equation (2.13) and the last equation, we obtain

$$\varphi_i(\theta_r) = \varphi_i(\theta_r^-) + I_r(\varphi_i(\theta_r^-)) = x_i(\theta_r^-) + I_r(x_i(\theta_r^-)) = x_i(\theta_r).$$

Therefore, one can conclude that $\varphi_i(\theta_r) = x_i(\theta_r)$ for $t \in [t_0, \theta_r)$. Similarly, in the light of above discussion, one can also obtain that $\varphi_i(t) = x_i(t)$ on $[\theta_r, \theta_{r+1})$. We can complete the proof by using mathematical induction. $\square$

2.3.1.3 Stability of Equilibrium

In this section, we will give sufficient conditions for the global asymptotic stability of the equilibrium, x^*, of (2.5) based on linearization [112]. The system (2.5) can be simplified as follows. Let $y_i = x_i - x_i^*$, for each $i = 1, \ldots, m$. Then,

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}\phi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}\psi_j(y_j(\beta(t))),\ t \neq \theta_k \\ \Delta y_i \mid_{t=\theta_k} &= \overline{I_k}(y_i(\theta_k^-)), \quad i = 1, 2, \ldots, m, \quad k \in \mathbb{N}, \end{aligned} \tag{2.14}$$

where $\phi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$, $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*)$ and $\overline{I_k}(y_j(\theta_k^-)) = I_k(y_j(\theta_k^-) + x_j^*) - I_k(y_j(x_j^*))$. For each $j = 1, \ldots, m$, and $k \in \mathbb{N}$, $\phi_j(\cdot), \psi_j(\cdot)$, and $\overline{I_k}$ are Lipschitzian since $f_j(\cdot), g_j(\cdot)$ and I_k are Lipschitzian with $L_j, \overline{L_j}$ and l respectively, and $\phi_j(0) = 0, \psi_j(0) = 0$; furthermore, $\overline{I_k}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $\overline{I_k}(0) = 0$.

It is clear that the stability of the zero solution of (2.14) is equivalent to the stability of the equilibrium x^* of (2.5). Therefore, in what follows, we discuss the stability of the zero solution of (2.5).

Let us denote

$$\overline{B} = \left\{1 - \overline{\theta}\left[\alpha_2 + \alpha_3\left(1 + \overline{\theta}\alpha_2\right)e^{\overline{\theta}\alpha_3}\right]\right\}^{-1}.$$

The following lemma is an important auxiliary result of the section (see, also, [17]).

Lemma 2.3.7 *Let $y(t) = (y_1(t), \ldots, y_m(t))^T$ be a solution of (2.14) and* (C1), (C3), (C4) *be satisfied. Then, the following inequality*

$$\|y(\beta(t))\| \leq \overline{B}\|y(t)\| \tag{2.15}$$

holds for all $t \in \mathbb{R}^+$.

Proof. Fix $t \in \mathbb{R}^+$, there exists $k \in \mathbb{N}$ such that $t \in [\theta_{k-1}, \theta_k)$. Then, from Lemma 2.3.6, we have

$$
\begin{aligned}
\|y(t)\| &= \sum_{i=1}^{m} |y_i(t)| \\
&\le \|y(\theta_{k-1})\| + \sum_{i=1}^{m}\left\{\int_{\theta_{k-1}}^{t}\left[a_i|y_i(s)| + \sum_{j=1}^{m} L_i|b_{ji}||y_i(s)|\right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} \bar{L}_i|c_{ji}||y_i(\theta_{k-1})|\right] ds\right\} \\
&\le (1+\bar{\theta}\alpha_2)\|y(\theta_{k-1})\| + \int_{\theta_{k-1}}^{t} \alpha_3\|y(s)\|ds
\end{aligned}
$$

By using Gronwall-Belmann Lemma, we get

$$
\|y(t)\| \le \left(1+\bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\|y(\theta_{k-1})\|. \tag{2.16}
$$

Moreover, for $t \in [\theta_{k-1}, \theta_k)$, we have

$$
\begin{aligned}
\|y(\theta_{k-1})\| &= \|y(t)\| + \sum_{i=1}^{m}\left\{\int_{\theta_{k-1}}^{t}\left[a_i|y_i(s)| + \sum_{j=1}^{m} L_i|b_{ji}||y_i(s)|\right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} \bar{L}_i|c_{ji}||y_i(\theta_{k-1})|\right] ds\right\} \\
&\le \|y(t)\| + \bar{\theta}\alpha_2\|y(\theta_{k-1})\| + \int_{\theta_{k-1}}^{t} \alpha_3\|y(s)\|ds.
\end{aligned}
$$

It follows from (2.16) that

$$
\|y(\theta_{k-1}))\| \le \|y(t)\| + \bar{\theta}\alpha_2\|y(\theta_{k-1})\| + \bar{\theta}\alpha_3\left(1+\bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\|y(\theta_{k-1})\|.
$$

Then, we have from condition (C4) that

$$
\|y(\theta_{k-1})\| \le \bar{B}\|y(t)\|, \quad t \in [\theta_{k-1}, \theta_k).
$$

Thus, (2.15) holds for all $t \in \mathbb{R}^+$. This proves the lemma. □

Now, we are ready to give sufficient conditions for the global asymptotic stability of (2.5). Let us denote the solution of linear homogeneous system of (2.14) as $\bar{y} = diag(y_1, \ldots, y_m)$.

From now on we need the following assumption:

(C5) $\gamma - \alpha_1 - \bar{B}\alpha_2 - \frac{ln(1+\ell)}{\underline{\theta}} > 0$, where $\gamma = \min\limits_{1\le i\le m} a_i$ is positive.

The following theorem is a modified version of the theorem in [112], for our system.

Theorem 2.3.8 *Assume that* (C1) − (C5) *are fulfilled.Then, the zero solution of (2.14) is globally asymptotically stable.*

Proof. Let $y(t) = (y_1(t), \ldots, y_m(t))^T$ be an arbitrary solution of (2.14). From Lemma 2.3.5, we have

$$
\begin{aligned}
\|y(t)\| \quad \le \quad & e^{-\gamma(t-t_0)}\|y_0\| + \sum_{i=1}^{m}\Bigg\{\int_{t_0}^{t} e^{-\gamma(t-s)}\Bigg[\sum_{j=1}^{m} L_i|b_{ji}||y_i(s)| \\
& + \sum_{j=1}^{m} \bar{L}_i|c_{ji}||y_i(\beta(s))|\Bigg]ds + \ell \sum_{t_0\le\theta_k<t} e^{-\gamma(t-\theta_k)}|y_i(\theta_k^-)|\Bigg\} \\
\le \quad & e^{-\gamma(t-t_0)}\|y_0\| + \left(\alpha_1 + \bar{B}\alpha_2\right)\int_{t_0}^{t} e^{-\gamma(t-s)}\|y(s)\|ds \\
& +\ell \sum_{t_0\le\theta_k<t} e^{-\gamma(t-\theta_k)}\|y(\theta_k^-)\|.
\end{aligned}
$$

Also, previous inequality can be written as,

$$
\begin{aligned}
e^{\gamma(t-t_0)}\|y(t)\| \quad \le \quad & \|y_0\| + \left(\alpha_1 + \bar{B}\alpha_2\right)\int_{t_0}^{t} e^{\gamma(s-t_0)}\|y(s)\|ds \\
& +\ell \sum_{t_0\le\theta_k<t} e^{\gamma(\theta_k-t_0)}\|y(\theta_k^-)\|.
\end{aligned}
$$

By applying Gronwall-Bellman inequality [112], we obtain

$$
e^{\gamma(t-t_0)}\|y(t)\| \le e^{\left(\alpha_1+\bar{B}\alpha_2\right)(t-t_0)}[1+\ell]^{i(t_0,t)}\|y_0\|,
$$

where $i(t_0, t)$ is the number of points θ_k in $[t_0, t)$. Then, we have that

$$
\|y(t)\| \le e^{-(\gamma-\alpha_1-\bar{B}\alpha_2-\frac{ln(1+\ell)}{\underline{\theta}})(t-t_0)}\|y_0\|.
$$

So, using (C5), we see that $\|y(t)\| \to 0$ as $t \to \infty$. That is, the zero solution of system (2.14) is globally asmptotically stable. □

2.3.2 $(\theta, \tau)-$ Type Neural Networks

In this section we investigate same problems for the second type of an impulsive neural networks system with piecewise constant argument of generalized type. That is, the sequence of moments $\theta_k,\ k \in \mathbb{N}$, where the constancy of the argument changes, and the sequence of impulsive moments, τ_k, are different.

2.3.2.1 Introduction

Recurrent neural networks and impulsive recurrent neural networks have been investigated due to their extensive applications in classification of patterns, associative memories, image processing, optimization problems, and other areas [28, 31, 32, 35, 37, 39, 131, 117, 119, 120, 121, 122, 123, 124]. It is well known that these applications depend crucially on the dynamical behavior of the networks. For example, if a neural network is employed to solve some optimization problems, it is highly desirable for the neural network to have a unique globally stable equilibrium [140, 141, 142, 111, 54, 61, 64, 66]. Therefore, stability analysis of neural networks has received much attention and various stability conditions have been obtained over the past years.

In this section, we develop the model of recurrent neural networks to differential equations with both impulses and piecewise constant argument of generalized type. In the literature, recurrent neural networks have been developed by implementing impulses and piecewise constant delay [13, 14, 117, 119, 120, 121, 122, 123, 124] issuing from different reasons: In implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants which may be caused by switching phenomenon, frequency change or other sudden noise. Furthermore, the dynamics of quasi-active dendrites with active spines is described by a system of point hot-spots (with an integrate-and-fire process), see [132, 133] for more details. This leads to the model of recurrent neural network with impulses. It is important to say that the neighbor moments of impulses may depend on each other. For example, the successive impulse moment may depend upon its predecessor. The reason for this phenomenon is the interior design of a neural network. On the other hand, due to the finite switching speed of amplifiers and transmission of signals in electronic networks or finite speed for signal propagation in neural networks, time delays exist [32, 35, 39, 131]. Moreover, the idea of involving delayed arguments in the recurrent neural networks can be explained by the fact that we assume neural networks may *"memorize"* values of the phase variable at certain moments of time to utilize the values during middle process till the next moment. Thus, we arrive to differential equations with piecewise constant delay. Obviously, the distances between the *"memorized"* moments may be very variative. Consequently, the concept of generalized type of piecewise constant argument is fruitful for recurrent neural networks [13, 14]. Therefore, it is possible to apply differential equations with both impulses and piecewise constant delay to neural networks theory.

The intrinsic idea of this section is that our model is not only from the applications point of view, but also from a new system of differential equations. That is, we develop differential equations with piecewise constant argument of generalized type to a new class of systems; impulsive differential equations with piecewise constant delay and apply them to recurrent neural networks [3, 5, 6, 17, 13, 14]. Another novelty is that the sequence of moments $\theta_k,\ k \in \mathbb{N}$, where the constancy of the argument changes, and the sequence of impulsive moments, τ_k, are different. More precisely, each moment $\tau_i,\ i \in \mathbb{N}$, is an interior point of an interval (θ_k, θ_{k+1}). This gives to our investigations more biological sense, as well as provides new theoretical opportunities.

2.3.2.2 Model Formulation and Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{R}^+ = [0, \infty)$ be the sets of natural and nonnegative real numbers, respectively, and denote a norm on $\mathbb{R}^m$ by $\| \cdot \|$ where $\|u\| = \sum_{i=1}^{m} |u_i|$. Fix two real valued sequences $\theta = \{\theta_k\}, \tau = \{\tau_k\},\ k \in \mathbb{N},\ \tau \cap \theta = \phi$ such that $\theta_k < \theta_{k+1}$ with $\theta_k \to \infty$ as $k \to \infty$ and $\tau_k < \tau_{k+1}$ with $\tau_k \to \infty$ as $k \to \infty$, and there exist two positive numbers $\overline{\theta}, \underline{\tau}$ such that $\theta_{k+1} - \theta_k \leq \overline{\theta}$ and $\underline{\tau} \leq \tau_{k+1} - \tau_k,\ k \in \mathbb{N}$. The condition of the empty intersection is caused by the investigation reasons. Otherwise, the proof of auxiliary results needs several additional assumptions.

The main subject under investigation in this section is the following impulsive recurrent neural networks with piecewise constant delay

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i,\ t \neq \tau_k \\ \Delta x_i \mid_{t=\tau_k} &= I_k(x_i(\tau_k^-)), \quad a_i > 0,\ i = 1, 2, \ldots, m, \quad k \in \mathbb{N}, \end{aligned} \tag{2.17}$$

where $\beta(t) = \theta_k$ if $\theta_k \leq t < \theta_{k+1}$, $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, is an identification function, $\Delta x_i(\tau_k)$ denotes $x_i(\tau_k) - x_i(\tau_k^-)$, where $x_i(\tau_k^-) = \lim_{h \to 0^-} x_i(\tau_k + h)$. Moreover, m corresponds to the number of units in a neural network, $x_i(t)$ stands for the state vector of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\beta(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and $\beta(t)$, b_{ij}, c_{ij}, d_i are real constants, b_{ij} means the strength of the jth unit on the ith unit at time t, c_{ij} infers the strength of the jth unit on the ith unit at time $\beta(t)$, d_i

signifies the external bias on the ith unit and a_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs.

In the theory of differential equations with piecewise constant argument [3, 5, 6], we take the function $\beta(t) = \theta_k$ if $\theta_k \leq t < \theta_{k+1}$, that is, $\beta(t)$ is right continuous. However, as it is usually done in the theory of impulsive differential equations, at the points of discontinuity τ_k of the solution, solutions are left continuous. Thus, the right continuity is more convenient assumption if one considers equations with piecewise constant arguments, and we shall assume the continuity for both, impulsive moments and moments of the switching of constancy of the argument.

We say that the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^m$ is from the set $PC_\tau(\mathbb{R}^+, \mathbb{R}^m)$ if:

(i) φ is right continuous on $\mathbb{R}^+$;

(ii) it is continuous everywhere except possibly moments τ where it has discontinuities of the first kind.

Moreover, we introduce a set of functions $PC_{\tau\cup\theta}(\mathbb{R}^+, \mathbb{R}^m)$ if we replace τ by $\tau \cup \theta$ in the last definition. In our investigation, we understand that $\varphi : \mathbb{R}^+ \to \mathbb{R}^m$ is a solution of (2.17) if $\varphi \in PC_\tau(\mathbb{R}^+, \mathbb{R}^m)$ and $\varphi' \in PC_{\tau\cup\theta}(\mathbb{R}^+, \mathbb{R}^m)$.

Throughout this section, we assume the following hypotheses:

(H1) there exist Lipschitz constants $L_j^f, L_j^g > 0$ such that

$$|f_j(u) - f_j(v)| \leq L_j^f |u - v|,$$

$$|g_j(u) - g_j(v)| \leq L_j^g |u - v|$$

for all $u, v \in \mathbb{R}^m,\ j = 1, 2, \ldots, m$;

(H2) the impulsive operator $I_i : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$|I_i(u) - I_i(v)| \leq \ell |u - v|$$

for all $u, v \in \mathbb{R}^m,\ i = 1, 2, \ldots, m$, where ℓ is a positive Lipschitz constant.

For the sake of convenience, we adopt the following notations in the sequel:

$$\begin{aligned} k_1 &= \max_{1\le i\le m}\Big(a_i + L_i^f \sum_{j=1}^{m} |b_{ji}|\Big),\ k_2 = \max_{1\le i\le m}\Big(L_i^g \sum_{j=1}^{m} |c_{ji}|\Big),\ k_3 = \max_{k\ge 1}\Big(|I_k(0)|\Big),\\ k_4 &= \max_{1\le i\le m}\Big(\sum_{j=1}^{m}\Big(|b_{ji}||f_i(0)| + |c_{ji}||g_i(0)|\Big)\Big). \end{aligned}$$

Denote by p_k the number of points τ_i in the interval $(\theta_k, \theta_{k+1}),\ k \in \mathbb{N}$. We assume that $p = \max_{k\in\mathbb{N}} p_k < \infty$.

Assume, additionally, that

(H3) $\Big[(k_1 + 2k_2)\bar{\theta} + \ell p\Big]\Big(1 + \ell\Big)^p e^{k_1\bar{\theta}} < 1;$

(H4) $k_2\bar{\theta} + (k_1\bar{\theta} + \ell p)(1 + k_2\bar{\theta})(1 + \ell)^p e^{k_1\bar{\theta}} < 1,$

We denote an equilibrium state for the differential equation of (2.17) by the constant vector $x^* = (x_1^*, \ldots, x_m^*)^T \in \mathbb{R}^m$, where the components x_i^* are governed by the algebraic system

$$0 = -a_i x_i^* + \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*)) + d_i.$$

The proof of following lemma is very similar to that of Lemma 2.2 in [117] and therefore we omit it here.

Lemma 2.3.9 *Assume* (H1) *holds. If the condition*

$$a_i > L_i^f \sum_{j=1}^{m} |b_{ji}| + L_i^g \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \ldots, m.$$

is satisfied, then the differential equation of (2.17) has a unique equilibrium state x^*.

Theorem 2.3.10 *If the equilibrium* $x^* = (x_1^*, \ldots, x_m^*)^T \in \mathbb{R}^m$ *of the differential equation of (2.17) satisfies* $I_k(x_i^*) = 0$ *for all* $i = 1, \ldots, m,\ k \in \mathbb{N}$. *Then,* x^* *is an equilibrium point of (2.17).*

Now we need the following equivalence lemmas which will be used in the proof of next assertions. The proofs are omitted here, since it is similar in [3, 5, 6, 13, 112].

Lemma 2.3.11 *A function* $x(t) = x(t, t_0, x^0) = (x_1(t), \dots, x_m(t))^T$, *where* t_0 *is a fixed real number, is a solution of (2.17) on* $\mathbb{R}^+$ *if and only if it is a solution, on* $\mathbb{R}^+$, *of the following integral equation:*

$$\begin{aligned} x_i(t) \quad &= \quad e^{-a_i(t-t_0)}x_i^0 + \int_{t_0}^{t} e^{-a_i(t-s)}\left[\sum_{j=1}^{m} b_{ij}f_j(x_j(s))\right. \\ &\qquad \left. + \sum_{j=1}^{m} c_{ij}g_j(x_j(\beta(s))) + d_i\right] ds + \sum_{t_0 \le \tau_k < t} e^{-a_i(t-\tau_k)} I_k(x_i(\tau_k^-)), \end{aligned}$$

for $i = 1, \dots, m,\ t \ge t_0$.

Lemma 2.3.12 *A function* $x(t) = x(t, t_0, x^0) = (x_1(t), \dots, x_m(t))^T$, *where* t_0 *is a fixed real number, is a solution of (2.17) on* $\mathbb{R}^+$ *if and only if it is a solution, on* $\mathbb{R}^+$, *of the following integral equation:*

$$\begin{aligned} x_i(t) \quad &= \quad x_i^0 + \int_{t_0}^{t}\left[-a_ix_i(s) + \sum_{j=1}^{m} b_{ij}f_j(x_j(s))\right. \\ &\qquad \left. + \sum_{j=1}^{m} c_{ij}g_j(x_j(\beta(s))) + d_i\right] ds + \sum_{t_0 \le \tau_k < t} I_k(x_i(\tau_k^-)), \end{aligned}$$

for $i = 1, \dots, m,\ t \ge t_0$.

Consider the following system

$$\begin{aligned} & x_i'(t) = -a_ix_i(t) + \sum_{j=1}^{m} b_{ij}f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij}g_j(x_j(\theta_r)) + d_i,\ t \ne \tau_r \\ & \Delta x_i \mid_{t=\tau_r} = I_r(x_i(\tau_r^-)), \quad i = 1, 2, \dots, m, \quad k \in \mathbb{N}. \end{aligned} \tag{2.18}$$

In the next lemma the conditions of existence and uniqueness of solutions are established for arbitrary initial moment ξ.

Lemma 2.3.13 *Assume that conditions* (H1) − (H3) *are fulfilled, and fix* $r \in \mathbb{N}$. *Then for every* $(\xi, x^0) \in [\theta_r, \theta_{r+1}] \times \mathbb{R}^m$ *there exists a unique solution* $x(t) = x(t, \xi, x^0) = (x_1(t), \dots, x_m(t))^T$ *of (2.18) on* $[\theta_r, \theta_{r+1}]$ *with* $x(\xi) = x^0$.

Proof. *Existence* : Denote $\vartheta(t) = x(t, \xi, x^0), \vartheta(t) = (\vartheta_1(t), \dots, \vartheta_m(t))^T$. From Lemma 2.3.12, we have

$$\vartheta_i(t) = x_i^0 + \int_\xi^t \left[-a_i\vartheta_i(s) + \sum_{j=1}^m b_{ij} f_j(\vartheta_j(s)) + \sum_{j=1}^m c_{ij} g_j(\vartheta_j(\theta_r)) + d_i \right] ds + \sum_{\xi \le \tau_r < t} I_r(\vartheta_i(\tau_r^-)). \tag{2.19}$$

Define a norm $\|\vartheta(t)\|_0 = \max\limits_{[\theta_r,\ \theta_{r+1}]} \|\vartheta(t)\|$ and construct the following sequences $\vartheta_i^n(t),\ \vartheta_i^0(t) \equiv x_i^0,\ i = 1, \ldots, m,\ n \ge 0$ such that

$$\vartheta_i^{n+1}(t) = x_i^0 + \int_\xi^t \left[-a_i\vartheta_i^n(s) + \sum_{j=1}^m b_{ij} f_j(\vartheta_j^n(s)) + \sum_{j=1}^m c_{ij} g_j(\vartheta_j^n(\theta_r)) + d_i \right] ds + \sum_{\xi \le \tau_r < t} I_r(\vartheta_i^n(\tau_r^-)).$$

One can find that

$$\|\vartheta^{n+1}(t) - \vartheta^n(t)\|_0 \le \left(\left[(k_1 + k_2)\overline{\theta} + \ell p \right] \right)^n \kappa,$$

where

$$\kappa = \left(\left[(k_1 + k_2)\overline{\theta} + \ell p \right] \|\vartheta^0\| + \overline{\theta}\Big(\sum_{i=1}^m d_i \Big) + \overline{\theta} m k_4 + m p k_3 \right).$$

Since the condition (H3) implies $\left[(k_1 + k_2)\overline{\theta} + \ell p \right] < 1$, then the sequences $\vartheta_i^n(t)$ are convergent and their limits satisfy (2.19) on $[\theta_r,\ \theta_{r+1}]$. The existence is proved.

Uniqueness : Let us denote the solutions of (2.17) by $x^1(t) = x(t, \xi, x^1)$, $x^2(t) = x(t, \xi, x^2)$, where $\theta_r \le \xi \le \theta_{r+1}$. It is sufficient to check that for each interval $t \in [\theta_r, \theta_{r+1}]$, and $x^2 = (x_1^2, \ldots, x_m^2)^T, x^1 = (x_1^1, \ldots, x_m^1)^T \in \mathbb{R}^m, x^2 \ne x^1$, the condition $x^1(t) \ne x^2(t)$. Then, we have

$$\begin{aligned} \|x^1(t) - x^2(t)\| &\le \|x^1 - x^2\| + \sum_{i=1}^m \Bigg\{ \int_\xi^t \Bigg[\Big(a_i + \sum_{j=1}^m L_i^f |b_{ji}| \Big) \left| x_i^2(s) - x_i^1(s) \right| \\ &\quad + \sum_{j=1}^m L_i^g |c_{ji}| \left| x_i^2(\theta_r) - x_i^1(\theta_r) \right| \Bigg] ds + \ell \sum_{\xi \le \tau_r < t} \left| x_i^2(\tau_r^-) - x_i^1(\tau_r^-) \right| \Bigg\} \\ &\le \|x^1 - x^2\| + k_2\overline{\theta} \|x^1(\theta_r) - x^2(\theta_r)\| + k_1 \int_\xi^t \|x^1(s) - x^2(s)\| ds \\ &\quad + \ell \sum_{\xi \le \tau_r < t} \|x^1(\tau_r^-) - x^2(\tau_r^-)\|. \end{aligned}$$

Using Gronwall-Bellman Lemma for piecewise continuous functions [112, 113], one can obtain that

$$\|x^1(t) - x^2(t)\| \le \Big(\|x^1 - x^2\| + k_2\overline{\theta} \|x^1(\theta_r) - x^2(\theta_r)\| \Big) (1 + \ell)^p e^{k_1 \overline{\theta}}.$$

Particularly,

$$\|x^1(\theta_r) - x^2(\theta_r)\| \le \Big(\|x^1 - x^2\| + k_2\bar{\theta}\|x^1(\theta_r) - x^2(\theta_r)\|\Big)(1+\ell)^p e^{k_1\bar{\theta}}.$$

Hence,

$$\|x^1(t) - x^2(t)\| \le \left[\frac{(1+\ell)^p e^{k_1\bar{\theta}}}{1 - k_2\bar{\theta}(1+\ell)^p e^{k_1\bar{\theta}}}\right]\|x^1 - x^2\|. \tag{2.20}$$

Also, we peculiarly have

$$\|x^1(\tau_r^-) - x^2(\tau_r^-)\| \le \left[\frac{(1+\ell)^p e^{k_1\bar{\theta}}}{1 - k_2\bar{\theta}(1+\ell)^p e^{k_1\bar{\theta}}}\right]\|x^1 - x^2\|. \tag{2.21}$$

On the other hand, assume on the contrary that there exists $t \in [\theta_r, \theta_{r+1}]$ such that $x^1(t) = x^2(t)$. Then

$$\begin{aligned}
\|x^1 - x^2\| \quad &\le \quad \sum_{i=1}^{m}\Bigg\{\int_{\xi}^{t}\Bigg[\Big(a_i + \sum_{j=1}^{m} L_i^f|b_{ji}|\Big)\left|x_i^2(s) - x_i^1(s)\right| \\
&\quad + \sum_{j=1}^{m} L_i^g|c_{ji}|\left|x_i^2(\theta_r) - x_i^1(\theta_r)\right|\Bigg]ds + \ell\sum_{\xi\le\tau_r<t}\left|x_i^2(\tau_r^-) - x_i^1(\tau_r^-)\right|\Bigg\} \\
&\le \quad k_1\int_{\xi}^{t}\|x^1(s) - x^2(s)\|ds + k_2\bar{\theta}\|x^1(\theta_r) - x^2(\theta_r)\| \\
&\quad + \ell p\|x^1(\tau_r^-) - x^2(\tau_r^-)\|.
\end{aligned} \tag{2.22}$$

Consequently, substituting (2.20) and (2.21) in (2.22), we obtain

$$\|x^1 - x^2\| \le \Big[(k_1 + 2k_2)\bar{\theta} + \ell p\Big]\Big(1+\ell\Big)^p e^{k_1\bar{\theta}}\|x^1 - x^2\|. \tag{2.23}$$

Thus, one can see that (H3) contradicts with (2.23). The lemma is proved. □

Theorem 2.3.14 *Assume that conditions* (H1) − (H3) *are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^m$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^T,\ t \ge t_0$, *of (2.17), such that* $x(t_0) = x^0$.

Proof. Fix $t_0 \in \mathbb{R}^+$. It is clear that there exists $r \in \mathbb{N}$ such that $t_0 \in [\theta_r, \theta_{r+1})$. Using previous lemma for $\xi = t_0$, one can obtain that there exists a unique solution $x(t) = x(t, t_0, x^0)$ on $[\xi, \theta_{r+1}]$. Next, we again apply the last lemma to obtain the unique solution on interval $[\theta_{r+1}, \theta_{r+2})$. The mathematical induction completes the proof. □

2.3.2.3 Global Asymptotic Stability

In this section, we will focus our attention on giving sufficient conditions for the global asymptotic stability of the equilibrium, x^*, of (2.17) based on linearization [6, 112].

The system (2.17) can be simplified as follows. Substituting $y(t) = x(t) - x^*$ into (2.17) leads to

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}\phi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}\psi_j(y_j(\beta(t))),\ t \neq \tau_k \\ \Delta y_i \mid_{t=\tau_k} &= W_k(y_i(\tau_k^-)), \quad i = 1, 2, \ldots, m, \quad k \in \mathbb{N}, \end{aligned} \tag{2.24}$$

where $\phi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$, $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_i^*)$ and $W_k(y_i(\tau_k^-)) = I_k(y_i(\tau_k^-) + x_i^*) - I_k(x_i^*)$. From hypotheses (H1) and (H2), we have the following inequalities: $|\phi_j(\cdot)| \leq L_j^f|(\cdot)|$, $|\psi_j(\cdot)| \leq L_j^g|(\cdot)|$ and $|W_k(\cdot)| \leq \ell|(\cdot)|$.

It is clear that the stability of the zero solution of (2.24) is equivalent to the stability of the equilibrium x^* of (2.17). Therefore, in what follows, we discuss the stability of the zero solution of (2.24).

First of all, we give the lemma below which is one of the most important results of the present section. One can see that this lemma is generalized version of the lemmas in [3, 5, 6, 17, 13, 14].

For simplicity of notation, we denote

$$\lambda = \left(1 - \left(k_2\bar{\theta} + (k_1\bar{\theta} + \ell p)(1 + k_2\bar{\theta})(1 + \ell)^p e^{k_1\bar{\theta}}\right)\right)^{-1}.$$

Lemma 2.3.15 *Let $y(t) = (y_1(t), \ldots, y_m(t))^T$ be a solution of (2.24) and* (H1) − (H4) *be satisfied. Then, the following inequality*

$$\|y(\beta(t))\| \leq \lambda\|y(t)\| \tag{2.25}$$

holds for all $t \in \mathbb{R}^+$.

Proof. Fix $t \in \mathbb{R}^+$, there exists $k \in \mathbb{N}$ such that $t \in [\theta_k, \theta_{k+1})$. Then, from Lemma 2.3.12, we have

$$
\begin{aligned}
\|y(t)\| &= \sum_{i=1}^{m} |y_i(t)| \\
&\leq \|y(\theta_k)\| + \sum_{i=1}^{m} \left\{ \int_{\theta_k}^{t} \left[\Big(a_i + \sum_{j=1}^{m} L_i^f |b_{ji}| \Big) |y_i(s)| \right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} L_i^g |c_{ji}| \, |y_i(\theta_k)| \right] ds + \ell \sum_{t_0 \leq \tau_k < t} \Big| y_i(\tau_k^-) \Big| \right\} \\
&\leq (1 + k_2\overline{\theta})\|y(\theta_k)\| + k_1 \int_{\theta_k}^{t} \|y(s)\| ds + \ell \sum_{t_0 \leq \tau_k < t} \|y(\tau_k^-)\|.
\end{aligned}
$$

Applying the analogue of Gronwall-Bellman Lemma [112, 113], we obtain

$$
\|y(t)\| \leq (1 + k_2\overline{\theta})(1 + \ell)^p e^{k_1\overline{\theta}} \|y(\theta_k)\|. \tag{2.26}
$$

Particularly,

$$
\|y(\tau_k^-)\| \leq (1 + k_2\overline{\theta})(1 + \ell)^p e^{k_1\overline{\theta}} \|y(\theta_k)\|. \tag{2.27}
$$

Moreover, for $t \in [\theta_k, \theta_{k+1})$, we also have

$$
\begin{aligned}
\|y(\theta_k)\| &\leq \|y(t)\| + k_2\overline{\theta}\|y(\theta_k)\| + k_1 \int_{\theta_k}^{t} \|y(s)\| ds \\
&\quad + \ell \sum_{t_0 \leq \tau_k < t} \|y(\tau_k^-)\|.
\end{aligned}
$$

The last inequality together with (2.26) and (2.27) imply

$$
\|y(\theta_k))\| \leq \|y(t)\| + \Big[k_2\overline{\theta} + (k_1\overline{\theta} + \ell p)(1 + k_2\overline{\theta})(1 + \ell)^p e^{k_1\overline{\theta}} \Big] \|y(\theta_k)\|
$$

Thus, we have from condition (H4) that

$$
\|y(\theta_k)\| \leq \lambda \|y(t)\|, \qquad t \in [\theta_k, \theta_{k+1}).
$$

Therefore, (2.25) holds for all $t \in \mathbb{R}^+$. This completes the proof of lemma. □

Now, we are ready to give sufficient conditions for the global asymptotic stability of (2.17). For convenience, we adopt the notation given below in the sequel:

$$
\mu = \max_{1 \leq i \leq m} \Big(L_i^f \sum_{j=1}^{n} |b_{ji}| \Big).
$$

From now on we need the following assumption:

(H5) $\gamma - \mu - \lambda k_2 - \frac{ln(1+\ell)}{\underline{\tau}} > 0, \quad \gamma = \min\limits_{1\leq i\leq m} a_i.$

The next theorem is a modified version of the theorem in [112], for our system.

Theorem 2.3.16 *Assume that* (H1) – (H5) *are fulfilled.Then, the zero solution of (2.24) is globally asymptotically stable.*

Proof. Let $y(t) = (y_1(t), \ldots, y_m(t))^T$ be an arbitrary solution of (2.24). From Lemma 2.3.11, we have

$$\begin{aligned} \|y(t)\| &\leq e^{-\gamma(t-t_0)}\|y_0\| + \sum_{i=1}^{m}\Bigg\{\int_{t_0}^{t} e^{-\gamma(t-s)}\Bigg[\sum_{j=1}^{m} L_i^f |b_{ji}||y_i(s)| \\ &\quad + \sum_{j=1}^{m} L_i^g |c_{ji}||y_i(\beta(s))|\Bigg]ds + \ell\sum_{t_0\leq\tau_k<t} e^{-\gamma(t-\tau_k)}|y_i(\tau_k^-)|\Bigg\} \\ &\leq e^{-\gamma(t-t_0)}\|y_0\| + (\mu + \lambda k_2)\int_{t_0}^{t} e^{-\gamma(t-s)}\|y(s)\|ds \\ &\quad + \ell\sum_{t_0\leq\tau_k<t} e^{-\gamma(t-\tau_k)}\|y(\tau_k^-)\|. \end{aligned}$$

Then, we can write the last inequality as,

$$\begin{aligned} e^{\gamma(t-t_0)}\|y(t)\| &\leq \|y_0\| + (\mu + \lambda k_2)\int_{t_0}^{t} e^{\gamma(s-t_0)}\|y(s)\|ds \\ &\quad + \ell\sum_{t_0\leq\tau_k<t} e^{\gamma(\tau_k-t_0)}\|y(\tau_k^-)\|. \end{aligned}$$

By virtue of Gronwall-Bellman Lemma [112], we obtain

$$e^{\gamma(t-t_0)}\|y(t)\| \leq e^{(\mu+\lambda k_2)(t-t_0)}[1+\ell]^{i(t_0,t)}\|y_0\|,$$

where $i(t_0, t)$ is the number of points τ_k in $[t_0, t)$. Then, we have

$$\|y(t)\| \leq e^{-(\gamma-\mu-\lambda k_2-\frac{ln(1+\ell)}{\underline{\tau}})(t-t_0)}\|y_0\|.$$

Hence, using the condition (H5), we see that the zero solution of system (2.24) is globally asymptotically stable. □

2.4 Periodic Motions of Neural Networks with Impact Activation

In this section we derive some sufficient conditions for the existence and stability of periodic solutions for each $(\theta, \theta)-$ type neural networks and $(\theta, \tau)-$ type neural networks, respectively. Examples with numerical simulations are given to illustrate our results.

2.4.1 $(\theta, \theta)-$ Type Neural Networks

Here, we investigate some sufficient conditions for the existence and stability of periodic solutions for $(\theta, \theta)-$ type neural networks discussed in Section 2.3.1.

2.4.1.1 Existence and Stability of Periodic Solutions

In this part, we will establish some sufficient conditions for existence of periodic solutions of (2.5). Then, we will study the stability of these solutions. Firstly, we shall need the folowing assumptions:

(C6) the sequence θ_k satisfies $\theta_{k+p} = \theta_k + w, k \in \mathbb{N}$ and $I_{k+p} = I_k$ for a fixed positive real period w and some positive integer p.

(C7) $\beta^* = \mathcal{K}\left[\omega\left(\alpha_1 + \bar{B}\alpha_2\right) + \ell p\right] < 1, \quad$ where $\mathcal{K} = \frac{1}{1-e^{-\gamma\omega}}$.

From now on, let us denote $\alpha_4 = \max\limits_{k \geq 1}\left(|I_k(0)|\right)$ and for $\theta_k,\ k \in \mathbb{N}$, let $[0, \omega] \cap \{\theta_k\}_{k\in\mathbb{N}} = \{\theta_1, \ldots, \theta_p\}$.

Here, we will give the following version of the Poincare' criterion for system (2.5). One can easily prove the following lemma (see, also, [112]).

Lemma 2.4.1 *Suppose that conditions* (C1), (C3), (C4), (C6) *are valid. Then, solution* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$ *of (2.5) with* $x(t_0) = x^0$ *is* $\omega-$*periodic if and only if* $x(\omega) = x(0)$.

Theorem 2.4.2 *Assume that conditions* (C1) – (C4), (C6), (C7) *are valid. Then system (2.5) has a unique* $\omega-$*periodic solution.*

Proof. Let $PC_\omega = \{\varphi \in PC^{(1)}(\mathbb{R}^+, \mathbb{R}^m) \mid \varphi(t+\omega) = \varphi(t),\ t \geq 0\}$ be a Banach space of periodic functions with the norm $\|\varphi\|_0 = \max\limits_{0\leq t\leq\omega} \|\varphi(t)\|$.

Let $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))^T \in PC_\omega$. Using Lemma 2.3.5, similarly to the proof in [112], one can show that if $\varphi \in PC_\omega$ then the system

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(\varphi_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(t))) + d_i,\ t \neq \theta_k,$$
$$\Delta x_i \mid_{t=\theta_k} = I_k(\varphi_i(\theta_k^-)),\ i = 1, \ldots, m, \quad k = 1, 2, \ldots, p$$

has the unique $\omega-$ periodic solution

$$\begin{aligned} x_i^*(t) &= \int_0^{\omega} \mathcal{H}_i(t,s)\left[\sum_{j=1}^{m} b_{ij} f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(s))) + d_i\right] ds \\ &\quad + \sum_{k=1}^{p} \mathcal{H}_i(t,\theta_k) I_k(\varphi_i(\theta_k^-)), \end{aligned}$$

where

$$\mathcal{H}_i(t,s) = (1-e^{-a_i\omega})^{-1} \begin{cases} e^{-a_i(t-s)}, & 0 \le s \le t \le \omega \\ e^{-a_i(\omega+t-s)}, & 0 \le t < s \le \omega \end{cases}$$

The function $\{\mathcal{H}_i(t,s)\}_{i=1,\ldots,m}$ is a Green's function. One can find that

$$\max_{t,s\in\mathcal{D}} \left|\{\mathcal{H}_i(t,s)\}_{i=1,\ldots,m}\right| = \frac{1}{1-e^{-a_i\omega}},$$

where $\mathcal{D} = [0,\omega]\times[0,\omega]$.

Define the operator $\mathcal{E}$ in PC_ω by

$$\mathcal{E} : PC_\omega \to PC_\omega$$

such that if $\varphi \in PC_\omega$, then

$$\begin{aligned} (\mathcal{E}\varphi)_i(t) &= \int_0^{\omega} \mathcal{H}_i(t,s)\left[\sum_{j=1}^{m} b_{ij} f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(s))) + d_i\right] ds \\ &\quad + \sum_{k=1}^{p} \mathcal{H}_i(t,\theta_k) I_k(\varphi_i(\theta_k^-)), \quad i = 1,\ldots,m. \end{aligned}$$

Let $PC_\omega^* = \{\varphi \mid \varphi \in PC_\omega, \|\varphi-\varphi_0\|_0 \le \frac{\beta^* C}{1-\beta^*}\}$, where $C = \mathcal{K}\omega \sum_{i=1}^{m} d_i$ and $(\varphi_0)_i(t) = \int_0^{\omega} \mathcal{H}_i(t,s) d_i ds,\ i = 1,\ldots,m$. Then it is easy to see that PC_ω^* is a closed convex subset of PC_ω. According to the definition of the norm of Banach space PC_ω, we have

$$\|\varphi_0(t)\| = \sum_{i=1}^{m} \left|\int_0^{\omega} \mathcal{H}_i(t,s) d_i ds\right| \le \frac{1}{1-e^{-a_i\omega}} \sum_{i=1}^{m} \left[\int_0^{\omega} d_i ds\right] \le C < \infty.$$

So, $\|\varphi_0\|_0 \le C$.

Then, for an arbitrary $\varphi \in PC_\omega^*$, we have

$$\|\varphi\|_0 \le \|\varphi-\varphi_0\|_0 + \|\varphi_0\|_0 \le \frac{\beta^* C}{1-\beta^*} + C = \frac{C}{1-\beta^*}.$$

Now, we need to prove that $\mathcal{E}$ maps PC_ω^* into itself. That is, we shall show that $\mathcal{E}\varphi \in PC_\omega^*$ for any $\varphi \in PC_\omega^*$. One can easily verify that $(\mathcal{E}\varphi)(t) = ((\mathcal{E}\varphi)_1,\ldots,(\mathcal{E}\varphi)_m)^T$ is $\omega-$periodic function. Now, if $\varphi \in PC_\omega^*$, then

$$
\begin{aligned}
\|\mathcal{E}\varphi - \varphi_0\| &= \sum_{i=1}^{m} \left| \int_0^\omega \mathcal{H}_i(t,s) \left[\sum_{j=1}^{m} b_{ij} f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(s))) \right] ds \right. \\
&\quad \left. + \sum_{k=1}^{p} \mathcal{H}_i(t,\theta_k) I_k(\varphi_i(\theta_k^-)) \right| \\
&\le \sum_{i=1}^{m} \frac{1}{1-e^{-a_i\omega}} \left\{ \int_0^\omega \left[\sum_{j=1}^{m} L_j |b_{ij}| |\varphi_j(s)| + \sum_{j=1}^{m} \overline{L}_j |c_{ij}| |\varphi_j(\beta(s))| \right] ds \right. \\
&\quad \left. + \sum_{k=1}^{p} |I_k(\varphi_i(\theta_k^-))| \right\} \\
&\le \mathcal{K} \sum_{i=1}^{m} \left\{ \int_0^\omega \left[\sum_{j=1}^{m} L_i |b_{ji}| |\varphi_i(s)| + \sum_{j=1}^{m} \overline{L}_i |c_{ji}| |\varphi_i(\beta(s))| \right] ds \right. \\
&\quad \left. + \ell \sum_{k=1}^{p} |\varphi_i(\theta_k^-)| + \sum_{k=1}^{p} |I_k(0)| \right\} \\
&\le \mathcal{K} \left(\int_0^\omega \Big[\alpha_1 \|\varphi(s)\| + \alpha_2 \|\varphi(\beta(s))\| \Big] ds + \ell \sum_{k=1}^{p} \|\varphi(\theta_k^-)\| + mp\alpha_4 \right).
\end{aligned}
$$

In this periodic case, we take $\alpha_4 = \max\limits_{1\le k\le p} \Big(|I_k(0)|\Big)$. Thus, it follows that

$$
\begin{aligned}
\|\mathcal{E}\varphi - \varphi_0\|_0 &\le \mathcal{K} \left(\left(\omega \left(\alpha_1 + \overline{B}\alpha_2 \right) + \ell p \right) \|\varphi\|_0 + mp\alpha_4 \right) \\
&\le \beta^* \frac{C}{1-\beta^*} + \mathcal{A} = \frac{\beta^* C}{1-\beta^*} + \mathcal{A},
\end{aligned}
$$

where $\mathcal{A} = \mathcal{K} mp\alpha_4$. Choose $\mathcal{A} \le C$ so that in the view of (C7), $\mathcal{E}\varphi \in PC_\omega^*$.

Finally, we shall show that $\mathcal{E}$ is a contraction mapping. If $\varphi^1, \varphi^2 \in PC^*_\omega$, then

$$\begin{aligned}
\|\mathcal{E}\varphi^1(t) - \mathcal{E}\varphi^2(t)\| &= \sum_{i=1}^{m} \left|(\mathcal{E}\varphi^1)_i(t) - (\mathcal{E}\varphi^2)_i(t)\right| \\
&\leq \sum_{i=1}^{m} \Bigg\{ \int_0^\omega |\mathcal{H}_i(t,s)| \Bigg[\sum_{j=1}^{m} L_j |b_{ij}||\varphi_j^1(s) - \varphi_j^2(s)| \\
&\quad + \bar{B} \sum_{j=1}^{m} \bar{L}_j |c_{ij}||\varphi_j^1(s) - \varphi_j^2(s)| \Bigg] ds \\
&\quad + \ell \sum_{k=1}^{p} |\mathcal{H}_i(t,\theta_k)||\varphi_i^1(\theta_k^-) - \varphi_i^2(\theta_k^-)| \Bigg\} \\
&\leq \mathcal{K} \sum_{i=1}^{m} \Bigg\{ \int_0^\omega \Bigg[\sum_{j=1}^{m} L_i |b_{ji}||\varphi_i^1(s) - \varphi_i^2(s)| \\
&\quad + \sum_{j=1}^{m} \bar{L}_i |c_{ji}||\varphi_i^1(\beta(s)) - \varphi_i^2(\beta(s))| \Bigg] ds + \ell \sum_{k=1}^{p} |\varphi_i^1(\theta_k^-) - \varphi_i^2(\theta_k^-)| \Bigg\} \\
&\leq \mathcal{K} \Bigg(\int_0^\omega \Big[\alpha_1 \|\varphi^1(s) - \varphi^2(s)\| + \alpha_2 \|\varphi^1(\beta(s)) - \varphi^2(\beta(s))\| \Big] ds \\
&\quad + \ell \sum_{k=1}^{p} \|\varphi^1(\theta_k^-) - \varphi^2(\theta_k^-)\| \Bigg).
\end{aligned}$$

Hence,

$$\|\mathcal{E}\varphi^1 - \mathcal{E}\varphi^2\|_0 \leq \mathcal{K}\left(\omega\left(\alpha_1 + \bar{B}\alpha_2\right) + \ell p\right)\|\varphi^1 - \varphi^2\|_0.$$

Noting (C7), it can be seen that $\mathcal{E}$ is a contraction mapping in PC^*_ω. Consequently, by using Banach fixed point theorem, $\mathcal{E}$ has a unique fixed point $\varphi^* \in PC^*_\omega$, such that $\mathcal{E}\varphi^* = \varphi^*$, which implies that (2.5) has a unique $\omega-$ periodic solution. □

We are now in a position to give and prove the stability of the periodic solution of (2.5).

Theorem 2.4.3 *Assume that conditions* (C1) – (C7) *are valid. Then the periodic solution of (2.5) is globally asymptotically stable.*

Proof. By Theorem 2.4.2, we know that (2.5) has an $\omega-$periodic solution $x^*(t) = (x_1^*, \ldots, x_m^*)^T$. Suppose that $x(t) = (x_1, \ldots, x_m)^T$ is an arbitrary solution of (2.5) and let $y(t) = x(t) - x^*(t) = (x_1 - x_1^*, \ldots, x_m - x_m^*)^T$. Then, similar to the proof of Theorem 2.3.8, one can show that

$$\begin{aligned}
\|y(t)\| &\leq e^{-\gamma(t-t_0)}\|y_0\| + \sum_{i=1}^{m} \Bigg\{ \int_{t_0}^{t} e^{-\gamma(t-s)} \Bigg[\sum_{j=1}^{m} L_i |b_{ji}||y_i(s)| \\
&\quad + \sum_{j=1}^{m} \bar{L}_i |c_{ji}||y_i(\beta(t))| \Bigg] ds + \ell \sum_{t_0 \leq \theta_k < t} e^{-\gamma(t-\theta_k)}|y_i(\theta_k^-)| \Bigg\}
\end{aligned}$$

and hence

$$\|y(t)\| \le e^{-(\gamma-\alpha_1-\bar{B}\alpha_2-\frac{ln(1+\ell)}{\underline{\theta}})(t-t_0)}\|y_0\|.$$

Thus, the periodic solution of (2.5) is globally asymptotically stable. □

2.4.1.2 An Illustrative Example

Consider the following impulsive Hopfield-type neural networks system with piecewise constant argument

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{2} c_{ij} g_j(x_j(\beta(t))) + d_i, \ t \neq \theta_k \\ \Delta x_i \mid_{t=\theta_k} &= I_k(x_i(\theta_k^-)), \quad i = 1,2, \quad k = 1,2,\ldots, \end{aligned} \tag{2.28}$$

where $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}, k \in \mathbb{N}, \theta_k = k + (-1)^k/12$. The distance $\theta_{k+1} - \theta_k, k \in \mathbb{N}$, is equal to either $\underline{\theta} = 5/6$, or $\bar{\theta} = 7/6$. The output functions are $f_i(x) = tanh(x/2), g_i(x) = (|x+1| - |x-1|)/8$. Obviously, $L_i = 1/2$ and $\bar{L}_i = 1/4$. Taking $b_{ij} = c_{ij} = 1/64$ for $i, j = 1, 2, \ I_k = (-1)^k x/32 + 1/12$ with $l = 1/32$, and $d_1 = 1/6, \ d_2 = 1/7, a_1 = 0.18, \ a_2 = 0.19$, we get $p = 2, \ \omega = 2, \ \gamma = 0.18, \ \bar{B} = 4.53370, \ \mathcal{K} = 3.30786$ and $\beta^* = 0.88194 < 1$. It is easily checked that the system (2.28) satisfies Theorem 2.3.1, Theorem 2.3.8, Theorem 2.4.2 and Theorem 2.4.3. Consequently, the system (2.28) has a unique 2-periodic solution which is globally asymptotically stable. Since it is globally asymptotically stable, any other solution is eventually 2-periodic. The fact can be seen by simulation in Figure 2.1 and Figure 2.2.

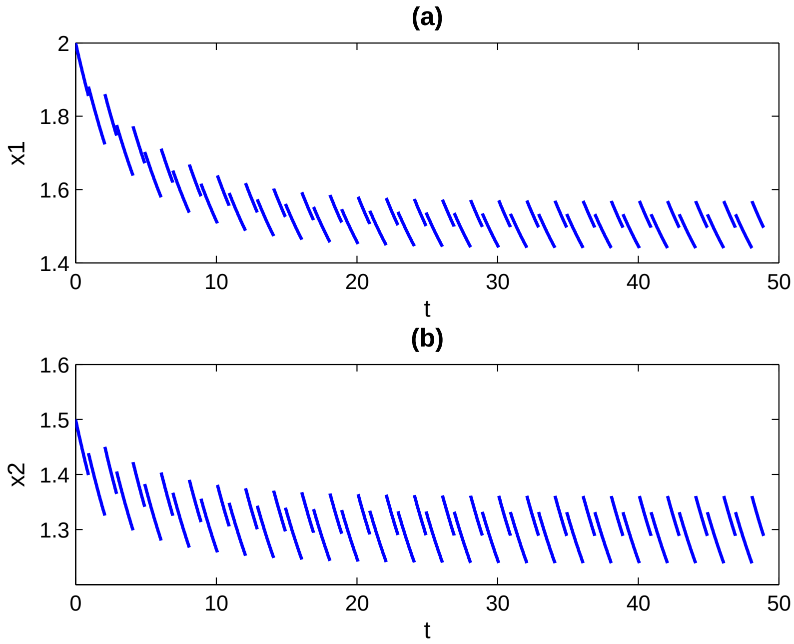

Figure 2.1: (a) 2-periodic solution of $x_1(t)$ of system (2.28) for $t \in [0, 50]$ with the initial value $x_1(t_0) = 2$. (b) 2-periodic solution of $x_2(t)$ of system (2.28) for $t \in [0, 50]$ with the initial value $x_2(t_0) = 1.5$.

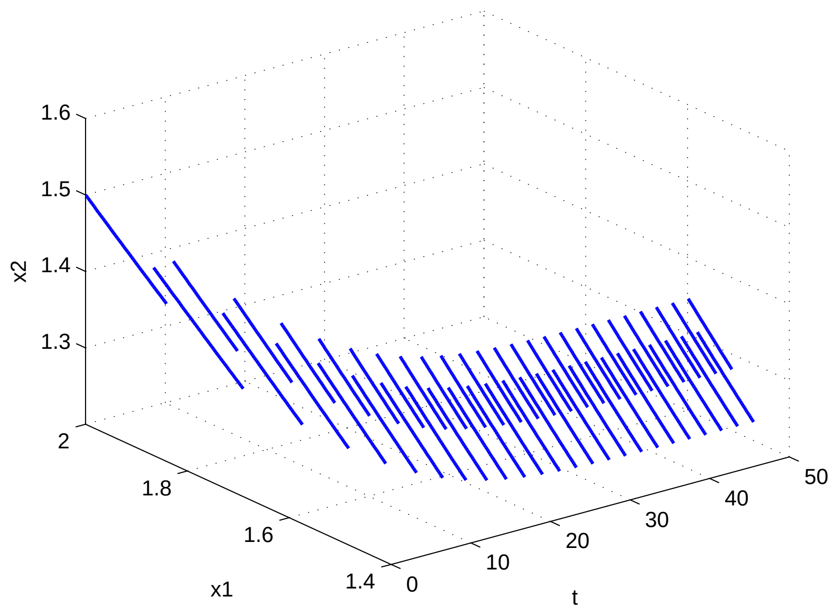

Figure 2.2: Eventually 2-periodic solutions of system (2.28).

2.4.2 $(\theta, \tau)-$ Type Neural Networks

In this part of this section, we continue $(\theta, \tau)-$ type neural networks considered in Section 2.3.2 and obtain some sufficient conditions for the existence and stability of periodic solutions.

2.4.2.1 Existence and Stability of Periodic Solutions

In this section, we shall discuss the existence of periodic solution of (2.17) and its stability. To do so, we need the following assumptions:

(H6) the sequences τ_k and $\theta_k,\ k \in \mathbb{N}$ satisfy (ω, p) and (ω, p_1)-properties; that is, there are positive integers p and p_1 such that the equations $\tau_{k+p} = \tau_k + \omega$ and $\theta_{k+p_1} = \theta_k + \omega$ hold for all $k \in \mathbb{N}$ and $I_{k+p} = I_k$ for a fixed positive real period ω.

(H7) $\alpha_1 = \mathcal{R}\Big(\omega\,(\mu + \lambda k_2) + \ell p\Big) < 1, \quad$ where $\mathcal{R} = \frac{1}{1-e^{-\gamma\omega}}$.

For τ_k and θ_k, let $[0,\omega]\cap\{\tau_k\}_{k\in\mathbb{N}} = \{\tau_1, \ldots, \tau_p\}$ and $[0,\omega]\cap\{\theta_k\}_{k\in\mathbb{N}} = \{\theta_1, \ldots, \theta_{p_1}\}$, respectively.

Here, we will give the following version of the Poincare' criterion for system (2.17) which can be easily proved (see, also, [112]).

Lemma 2.4.4 *Suppose that conditions* (H1) $-$ (H3) *and* (H6) *are valid. Then, solution* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$ *of (2.17) with* $x(t_0) = x^0$ *is* $\omega-$*periodic if and only if* $x(\omega) = x(0)$.

Theorem 2.4.5 *Assume that conditions* (H1) $-$ (H3) *and* (H6) $-$ (H7) *are valid. Then system (2.17) has a unique* $\omega-$*periodic solution.*

Proof. To begin with, let us introduce a Banach space of periodic functions $PC_\omega = \{\varphi \in PC_{\tau\cup\theta}(\mathbb{R}^+, \mathbb{R}^m) \mid \varphi(t+\omega) = \varphi(t),\ t \geq 0\}$ with the norm $\|\varphi\|_0 = \max_{0\leq t\leq\omega} \|\varphi(t)\|$.

Let $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))^T \in PC_\omega$ satisfying the inequality $\|\varphi(t)\|_0 \leq h$. Using Lemma 2.3.11, similarly to the proof in [112], one can show that if $\varphi \in PC_\omega$ then the system

$$
\begin{aligned}
&x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(\varphi_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(t))) + d_i,\ t \neq \tau_k, \\
&\Delta x_i \mid_{t=\tau_k} = I_k(\varphi_i(\tau_k^-)),\ i = 1, \ldots, m, \quad k = 1, 2, \ldots, p
\end{aligned}
$$

has the unique $\omega-$ periodic solution

$$\begin{aligned} x_i^*(t) &= \int_0^{\omega} \mathcal{G}_i(t,s)\left[\sum_{j=1}^{m} b_{ij} f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(s))) + d_i\right] ds \\ &\quad + \sum_{k=1}^{p} \mathcal{G}_i(t,\tau_k) I_k(\varphi_i(\tau_k^-)), \end{aligned}$$

where

$$\mathcal{G}_i(t,s) = (1-e^{-a_i\omega})^{-1} \begin{cases} e^{-a_i(t-s)}, & 0 \le s \le t \le \omega, \\ e^{-a_i(\omega+t-s)}, & 0 \le t < s \le \omega, \end{cases}$$

which is known as *Green's function* [112]. Then, one can easily find that

$$\max_{t,s\in[0,\omega]} \left|\{\mathcal{G}_i(t,s)\}_{i=1,\ldots,m}\right| = \frac{1}{1-e^{-a_i\omega}}.$$

Define the operator $\mathcal{F} : PC_\omega \to PC_\omega$ such that if $\varphi \in PC_\omega$, then

$$\begin{aligned} (\mathcal{F}\varphi)_i(t) &= \int_0^{\omega} \mathcal{G}_i(t,s)\left[\sum_{j=1}^{m} b_{ij} f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(s))) + d_i\right] ds \\ &\quad + \sum_{k=1}^{p} \mathcal{G}_i(t,\tau_k) I_k(\varphi_i(\tau_k^-)), \quad i = 1,\ldots,m. \end{aligned}$$

Now, we need to prove that $\mathcal{F}$ maps PC_ω into itself. That is, we shall show that $\mathcal{F}\varphi \in PC_\omega$ for any $\varphi \in PC_\omega$. It is easy to check that $(\mathcal{F}\varphi)(t) = ((\mathcal{F}\varphi)_1,\ldots,(\mathcal{F}\varphi)_m)^T$ is $\omega-$periodic function. Now, if $\varphi \in PC_\omega$, then

$$\begin{aligned} \|\mathcal{F}\varphi\| &= \sum_{i=1}^{m} \left| \int_0^{\omega} \mathcal{G}_i(t,s)\left[\sum_{j=1}^{m} b_{ij} f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\varphi_j(\beta(s))) + d_i\right] ds \right. \\ &\quad \left. + \sum_{k=1}^{p} \mathcal{G}_i(t,\tau_k) I_k(\varphi_i(\tau_k^-)) \right| \\ &\le \sum_{i=1}^{m} \frac{1}{1-e^{-a_i\omega}} \left\{ \int_0^{\omega} \left[\sum_{j=1}^{m} L_j^f |b_{ij}||\varphi_j(s)| + \sum_{j=1}^{m} L_j^g |c_{ij}||\varphi_j(\beta(s))| \right.\right. \\ &\quad \left.\left. + \sum_{j=1}^{m} |b_{ij}||\varphi_j(0)| + \sum_{j=1}^{m} |c_{ij}||\varphi_j(0)| + d_i \right] ds + \sum_{k=1}^{p} |I_k(\varphi_i(\tau_k^-))| \right\} \\ &\le \mathcal{R} \sum_{i=1}^{m} \left\{ \int_0^{\omega} \left[\sum_{j=1}^{m} L_i^f |b_{ji}||\varphi_i(s)| + \sum_{j=1}^{m} L_i^g |c_{ji}||\varphi_i(\beta(s))| \right.\right. \\ &\quad \left.\left. + \sum_{j=1}^{m} |b_{ji}||\varphi_i(0)| + \sum_{j=1}^{m} |c_{ji}||\varphi_i(0)| + d_i \right] ds + \ell \sum_{k=1}^{p} |\varphi_i(\tau_k^-)| + \sum_{k=1}^{p} |I_k(0)| \right\} \\ &\le \mathcal{R} \left(\int_0^{\omega} \left[\mu\|\varphi(s)\| + k_2\|\varphi(\beta(s))\| \right] ds + \ell \sum_{k=1}^{p} \|\varphi(\tau_k^-)\| + \omega\Big(\sum_{i=1}^{m} d_i\Big) + \omega m k_4 + m p k_3 \right). \end{aligned}$$

In this periodical case, we take $k_3 = \max_{1\le k\le p}\Big(|I_k(0)|\Big)$. Thus, it follows that

$$\begin{aligned}
\|\mathcal{F}\varphi\|_0 &\le \mathcal{R}\Big(\big(\omega\,(\mu+\lambda k_2)+\ell p\big)\|\varphi\|_0 + \omega\Big(\sum_{i=1}^{m} d_i\Big) + \omega m k_4 + m p k_3\Big)\\
&\le \alpha_1 h + \alpha_2.
\end{aligned}$$

Choose h such that $\alpha_2 \le h(1-\alpha_1)$, where $\alpha_2 = \mathcal{R}\Big(\omega\Big(\sum_{i=1}^{m} d_i\Big) + \omega m k_4 + m p k_3\Big)$. Then, we obtain that $\mathcal{F}\varphi \in PC_\omega$.

Next, the proof is completed by showing that $\mathcal{F}$ is a contraction mapping.

If $\varphi^1, \varphi^2 \in PC_\omega$, then

$$\begin{aligned}
\|\mathcal{F}\varphi^1(t) - \mathcal{F}\varphi^2(t)\| &= \sum_{i=1}^{m}\Big|(\mathcal{F}\varphi^1)_i(t) - (\mathcal{F}\varphi^2)_i(t)\Big|\\
&\le \sum_{i=1}^{m}\Bigg\{\int_0^\omega |\mathcal{G}_i(t,s)|\Bigg[\sum_{j=1}^{m} L_j^f |b_{ij}||\varphi_j^1(s) - \varphi_j^2(s)|\\
&\quad + \lambda\sum_{j=1}^{m} L_j^g |c_{ij}||\varphi_j^1(s) - \varphi_j^2(s)|\Bigg]ds\\
&\quad + \ell\sum_{k=1}^{p} |\mathcal{G}_i(t,\tau_k)||\varphi_i^1(\tau_k^-) - \varphi_i^2(\tau_k^-)|\Bigg\}\\
&\le \mathcal{R}\sum_{i=1}^{m}\Bigg\{\int_0^\omega \Bigg[\sum_{j=1}^{m} L_i^f |b_{ji}||\varphi_i^1(s) - \varphi_i^2(s)|\\
&\quad + \sum_{j=1}^{m} L_i^g |c_{ji}||\varphi_i^1(\beta(s)) - \varphi_i^2(\beta(s))|\Bigg]ds + \ell\sum_{k=1}^{p} |\varphi_i^1(\tau_k^-) - \varphi_i^2(\tau_k^-)|\Bigg\}\\
&\le \mathcal{R}\Bigg(\int_0^\omega \Big[\mu\|\varphi^1(s) - \varphi^2(s)\| + k_2\|\varphi^1(\beta(s)) - \varphi^2(\beta(s))\|\Big]ds\\
&\quad + \ell\sum_{k=1}^{p} \|\varphi^1(\tau_k^-) - \varphi^2(\tau_k^-)\|\Bigg).
\end{aligned}$$

Hence,

$$\|\mathcal{F}\varphi^1 - \mathcal{F}\varphi^2\|_0 \le \mathcal{R}\Big(\omega\,(\mu+\lambda k_2)+\ell p\Big)\|\varphi^1 - \varphi^2\|_0.$$

It follows from the condition (H7) that, $\mathcal{F}$ is a contraction mapping in PC_ω. Consequently, by using Banach fixed point theorem, $\mathcal{F}$ has a unique fixed point $\varphi^* \in PC_\omega$, such that $\mathcal{F}\varphi^* = \varphi^*$. This completes the proof. □

Theorem 2.4.6 *Assume that conditions* (H1) – (H7) *are valid. Then the periodic solution of (2.17) is globally asymptotically stable.*

Proof. By Theorem 2.4.5, we know that (2.17) has an $\omega-$periodic solution $z^*(t) = (z_1^*, \dots, z_m^*)^T$. Suppose that $z(t) = (z_1, \dots, z_m)^T$ is an arbitrary solution of (2.17) and let $z(t) = z(t) - z^*(t) = (z_1 - z_1^*, \dots, z_m - z_m^*)^T$. Then, similar to the proof of Theorem 2.3.16, one can show that it is globally asymptotically stable.

2.4.2.2 Numerical Simulations

In this part, we give examples with numerical simulations to illustrate the theoretical results of this section. In what follows, let $\theta_k = k,\ \tau_k = (\theta_k + \theta_{k+1})/2 = (2k+1)/2,\ k \in \mathbb{N}$ be the sequence of the change of constancy for the argument and the sequence of impulsive action, respectively.

Consider the following recurrent neural networks:

$$\begin{cases} \frac{dx(t)}{dt} = -\begin{pmatrix} 5\times10^{-1} & 0 \\ 0 & 5\times10^{-1} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 10^{-4} & 2\times10^{-4} \\ 10^{-4} & 3\times10^{-4} \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(t)}{10}) \\ \tanh(\frac{3x_2(t)}{10}) \end{pmatrix} \\ \qquad + \begin{pmatrix} 2\times10^{-2} & 3\times10^{-3} \\ 3\times10^{-3} & 5\times10^{-3} \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(\beta(t))}{5}) \\ \tanh(\frac{x_2(\beta(t))}{5}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ t \neq \tau_k \\ \Delta x(t) = \begin{pmatrix} I_k(x_1(\tau_k^-)) \\ I_k(x_2(\tau_k^-)) \end{pmatrix} = \begin{pmatrix} \frac{x_1(\tau_k^-)}{40} + \frac{1}{2} \\ \frac{x_2(\tau_k^-)}{40} + \frac{1}{2} \end{pmatrix},\ t = \tau_k, \quad k = 1, 2, \dots, \end{cases} \tag{2.29}$$

By simple calculation, one can see that the corresponding parameters in the conditions of Theorems 2.3.14, 2.3.16, 2.4.5, 2.4.6 are $k_1 = 0.5001,\ k_2 = 0.0046,\ L_1^f = 0.1,\ L_2^f = 0.3,\ L_1^g = L_2^g = 0.2,\ \ell = 0.0250,\ \overline{\theta} = \underline{\tau} = 1,\ p = p_1 = 1,\ \gamma = 0.5,\ \lambda = 9.6421,\ \mu = 0.00015,\ \omega = 1,\ \mathcal{R} = 2.5415,\ \alpha_1 = 0.1766$. For these values, we can check that (H3) $= 0.9032 < 1$, (H4) $= 0.8963 < 1$, (H5) $= 0.4308 > 0$ and $\alpha_1 = 0.1766 < 1$. So, it is easy to verify that (2.29) satisfies the conditions of these theorems. Hence, the system of (2.29) has a 1-periodic solution which is globally asymptotically stable. Specifically, the simulation results with some initial points are shown in Fig. 2.3 and Fig. 2.3. We deduce that the non-smoothness at $\theta_k,\ k \in \mathbb{N}$ is not seen by numerical simulations due to the choosing the parameters small enough to satisfy the theorems. Hence, the *smallness* hides the *non-smoothness*.

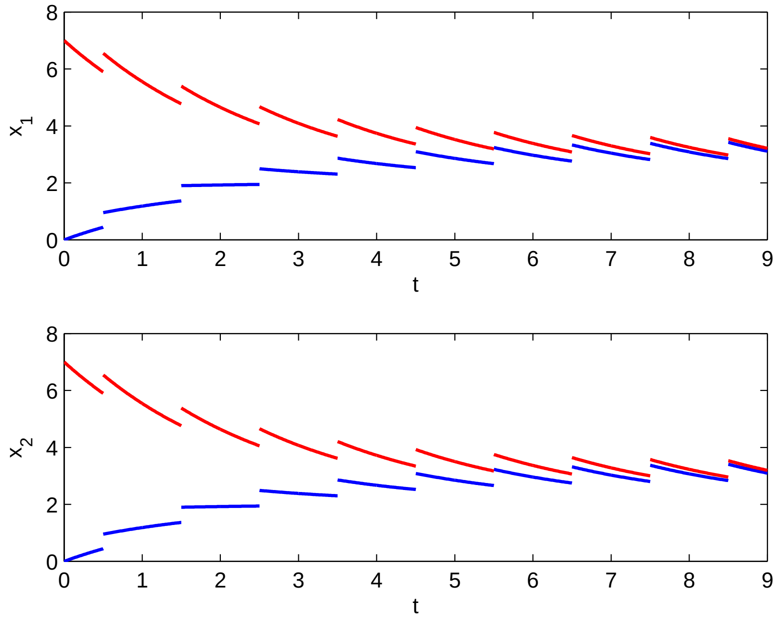

Figure 2.3: Transient behavior of the recurrent neural networks for the system (2.29) with the initial points $[0,0]^T$ and $[7,7]^T$.

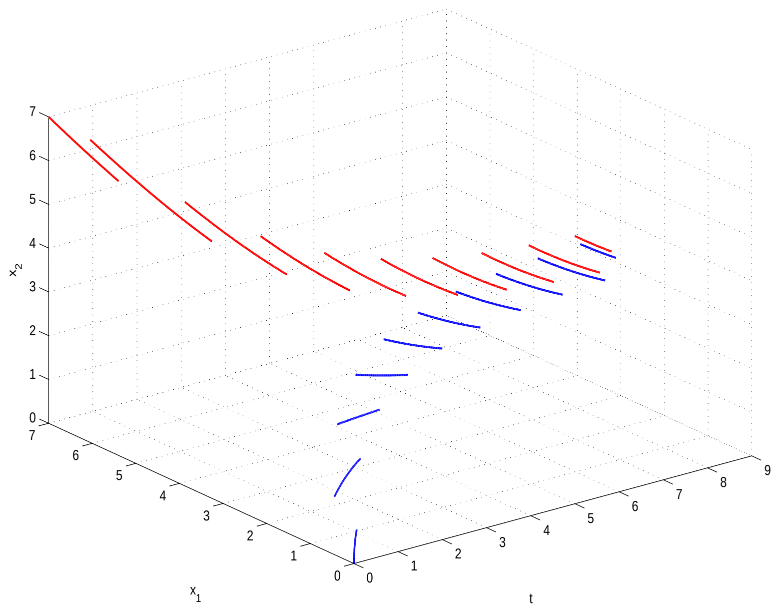

Figure 2.4: Eventually 1-periodic solutions of system (2.29) with the initial points $[0,0]^T$ and $[7,7]^T$.

On the other hand, in the following example, we illustrate a globally stable equilibrium appearance for our system of differential equations:

$$\begin{cases} \frac{dx(t)}{dt} & = -\begin{pmatrix} 5\times10^{-1} & 0 \\ 0 & 5\times10^{-1} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 10^{-4} & 2\times10^{-3} \\ 10^{-4} & 3\times10^{-3} \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(t)}{10}) \\ \tanh(\frac{3x_2(t)}{10}) \end{pmatrix} \\ & \quad + \begin{pmatrix} 2\times10^{-2} & 3\times10^{-2} \\ 3\times10^{-2} & 5\times10^{-2} \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(\beta(t))}{5}) \\ \tanh(\frac{x_2(\beta(t))}{5}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ t \neq \tau_k \\ \Delta x(t) & = \begin{pmatrix} I(x_1(\tau_k^-)) \\ I(x_2(\tau_k^-)) \end{pmatrix} = \begin{pmatrix} \frac{(x_1(\tau_k^-)-x_1^*)^2}{30} \\ \frac{(x_2(\tau_k^-)-x_1^*)^2}{30} \end{pmatrix}, \ t = \tau_k, \quad k = 1, 2, \ldots, \end{cases} \tag{2.30}$$

where $x_1^* = 2.0987$, $x_2^* = 2.1577$. One can check that the point $x^* = (x_1^*, x_2^*)$ satisfies the algebraic system

$$-a_i x_i^* + \sum_{j=1}^{2} b_{ij} f_j(x_j^*) + \sum_{j=1}^{2} c_{ij} g_j(x_j^*)) + d_i = 0, \tag{2.31}$$

approximately. And it is clear that $I(x_i^*) = 0$ for $i = \overline{1,2}$. By simple calculation, we can see that all conditions of Theorem 2.3.1 are satisfied and the point x^* is a solution of (2.31), approximately with the error, which is less than 10^{-11}(evaluated by MATLAB).

The simulation, where the initial value is chosen as $[10, 10]^T$, is shown in Fig. 2.5 and it illustrates that all trajectories converge to x^*.

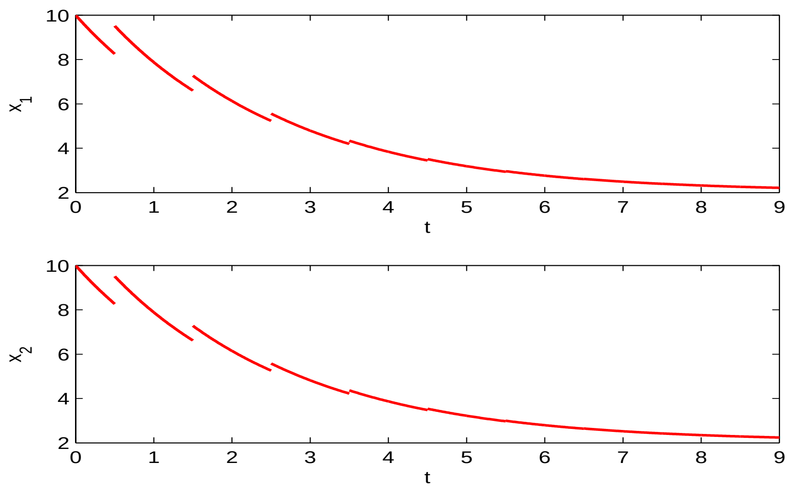

Figure 2.5: The first and the second coordinates of the solution for the the system (2.30) with the initial point $[10, 10]^T$ approaches x_1^* and x_2^*, respectively, as time increases.

Now, let us take the parameters so that the non-smoothness can also be seen. Consider the following recurrent neural networks with non-smooth and impact activations:

$$
\begin{cases}
\frac{dx(t)}{dt} = -\begin{pmatrix} 20 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 10 & 3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{pmatrix} \\
\qquad + \begin{pmatrix} 20 & 1 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} \tanh(x_1(\beta(t))) \\ \tanh(x_2(\beta(t))) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ t \neq \tau_k \\
\Delta x(t) = \begin{pmatrix} I_k(x_1(\tau_k^-)) \\ I_k(x_2(\tau_k^-)) \end{pmatrix} = \begin{pmatrix} \frac{x_1(\tau_k^-)}{3} + \frac{1}{6} \\ \frac{x_2(\tau_k^-)}{3} + \frac{1}{6} \end{pmatrix}, \ t = \tau_k, \quad k = 1, 2, \ldots,
\end{cases} \tag{2.32}
$$

Clearly, one can see that our parameters are big now. Therefore, the system of equations (2.32) does not satisfy the conditions of the theorems. However, we can see the *non-smoothness* of the solution with the initial value $[0, 0]^T$, which is illustrated by simulations in Fig. 2.6 and Fig. 2.7.

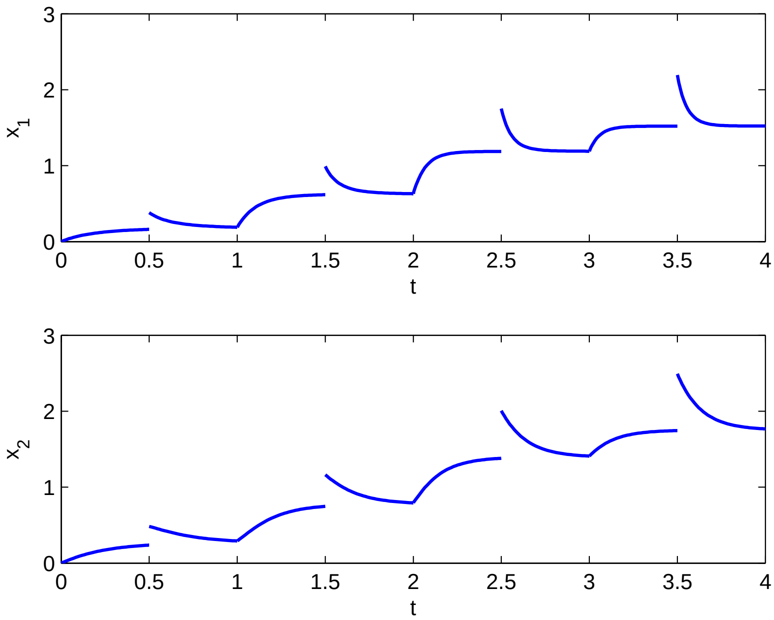

Figure 2.6: The impact and non − smoothness are seen at discontinuity points τ_k : (0.5; 1.5; 2.5; 3.5) and at switching points θ_k : (1; 2; 3), respectively.

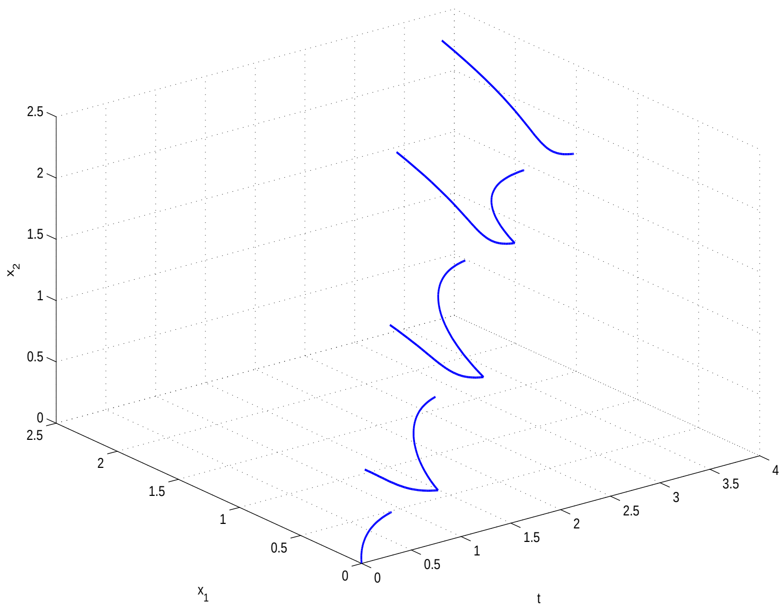

Figure 2.7: Eventually 1− periodic solutions of system (2.32) with the initial point $[0, 0]^T$.

2.4.2.3 Conclusions

This is the first time that global asymptotic stability of periodic solutions for recurrent neural networks with both impulses and piecewise constant delay is considered. Furthermore, our model gives new ideas not only from the implementation point of view, but also from the system of differential equations. In other words, we develop differential equations with piecewise constant argument to a new class of system, so called impulsive differential equations with piecewise constant delay. For applications, we have also nice properties on the system of equations that the moments of discontinuity τ_k and switching moments of constancy of arguments θ_k are not related to each other. That is, our investigations are more applicable to the real world problems like recurrent neural networks. Finally, the results given in this section could be developed for more complex systems [20, 21, 22].

CHAPTER 3

THE SECOND LYAPUNOV METHOD

In this chapter we investigate the problem of stability for differential equations with piecewise constant argument of generalized type based on the method of Lyapunov functions. In addition to this theoretical results, we analyze the stability for neural networks with piecewise constant argument of generalized type through the Second Lyapunov method. That is, we use the method of Lyapunov functions and Lyapunov-Razumikhin technique for the stability of RNNs and CNNs, respectively. Examples with numerical simulations are given to illustrate the theoretical results.

3.1 The Method of Lyapunov Functions: Theoretical Results

In this section, we address differential equations with piecewise constant argument of generalized type [4, 5, 6, 8] and investigate their stability with the second Lyapunov method. Despite the fact that these equations include delay, stability conditions are merely given in terms of Lyapunov functions; that is, no functionals are used. Several examples, one of which considers the logistic equation, are discussed to illustrate the development of the theory.

3.1.1 Introduction

K. L. Cooke, J. Wiener and their co-authors [68, 69, 70, 77] introduced differential equations with piecewise constant argument, which play an important role in applications [4, 5, 6, 8, 9, 12, 13, 14, 17, 68, 69, 70, 72, 77, 85, 87, 100, 101, 102, 115]. By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with piecewise constant argument has been generalized in [4, 5, 6].

We should mention the following novelties of the present section. The main and possibly a unique way of stability analysis for differential equations with piecewise constant argument has been the reduction to discrete equations [72, 77, 80, 86, 88, 89, 101, 115]. Particularly, the problem of exploring stability with Lyapunov functions of continuous time has been remaining open. Moreover, the results of our investigation have been developed through the concept of "total stability" [75, 106], which is stability under persistent perturbations of the right hand side of a differential equation, and they originate from a special theorem by Malkin [76]. Then, one can accept our approach as comparison of stability of equations with piecewise constant argument and ordinary differential equations. Finally, it deserves to emphasize that the direct method for differential equations with deviating argument necessarily utilizes functionals [67, 90, 105], but we use only Lyapunov functions to determine criteria of the stability, and this can be an advantage in applications.

3.1.2 The Subject and Method of Analysis

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the set of natural numbers and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the n-dimensional real space by $\mathbb{R}^n$, $n \in \mathbb{N}$, and the Euclidean norm in $\mathbb{R}^n$ by $\| \,.\, \|$.

Let us introduce a special notation:

$\mathcal{K} = \{\psi : \psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a strictly increasing function and $\psi(0) = 0\}$.

We fix a real-valued sequence θ_i, $i \in \mathbb{N}$, such that $0 = \theta_0 < \theta_1 < ... < \theta_i < ...$ with $\theta_i \to \infty$ as $i \to \infty$, and shall consider the following equation

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{3.1}$$

where $x \in B(h)$, $B(h) = \{x \in \mathbb{R}^n : \|x\| < h\}$, $t \in \mathbb{R}^+$ and $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, is an identification function.

We say that a continuous function $x(t)$ is a solution of equation (3.1) on $\mathbb{R}^+$ if it satisfies (3.1) on the intervals $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$ and the derivative $x'(t)$ exists everywhere with the possible exception of the points θ_i, $i \in \mathbb{N}$, where one-sided derivatives exist.

In the rest of this section, we assume that the following conditions hold:

(C1) $f(t, u, v) \in C(\mathbb{R}^+ \times B(h) \times B(h))$ is an $n \times 1$ real valued function;

(C2) $f(t, 0, 0) = 0$ for all $t \geq 0$;

(C3) f satisfies a Lipschitz condition with constants ℓ_1, ℓ_2 :

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \ell_1 \|u_1 - u_2\| + \ell_2 \|v_1 - v_2\| \tag{3.2}$$

for all $t \in \mathbb{R}^+$ and $u_1,\ u_2,\ v_1,\ v_2 \in B(h)$;

(C4) there exists a constant $\theta > 0$ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}$;

(C5) $\theta[\ell_2 + \ell_1(1 + \ell_2\theta)e^{\ell_1\theta}] < 1$;

(C6) $\theta(\ell_1 + 2\ell_2)e^{\ell_1\theta} < 1$.

We give now some definitions and preliminary results which enable us to investigate stability of the trivial solution of (3.1).

Definition 3.1.1 *[6] The zero solution of* (3.1) *is said to be*

(i) stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;

(ii) uniformly stable if δ is independent of t_0.

Definition 3.1.2 *[6] The zero solution of* (3.1) *is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x_0\| < \delta_0$.*

Next, we shall describe the method, which is in the base of our investigation. Let us rewrite the system (3.1) in the form

$$x'(t) = f(t, x(t), x(t)) + h(t, x(t), x(\beta(t))),$$

where $h(t, x(t), x(\beta(t))) = f(t, x(t), x(\beta(t))) - f(t, x(t), x(t))$. If the constant θ mentioned in $(C4)$ is small, then we can consider $h(t, x(t), x(\beta(t)))$ as a small perturbation. That is to say, system (3.1) is a perturbed system for the following ordinary differential equation,

$$y'(t) = g(t, y(t)), \tag{3.3}$$

where $g(t, y(t)) = f(t, y(t), y(t))$.

Our intention is to consider systems (3.1) and (3.3) involved in the perturbation relation, and then extend these systems to the problem of stability based on the approach of I. G. Malkin [76].

Before applying the method, it is useful to consider a simple example. Let the following linear scalar equation with piecewise constant argument be given:

$$x'(t) = ax(t) + bx(\beta(t)) \tag{3.4}$$

where $\theta_i = ih,\ i \in \mathbb{N}$. The solution of (3.4) if $t \in [ih, (i+1)h)$ is given by [74, 77]

$$x(t) = \{e^{a(t-ih)}(1 + \frac{b}{a}) - \frac{b}{a}\}\{e^{ah}(1 + \frac{b}{a}) - \frac{b}{a}\}^i x_0.$$

Then, one can easily see that the zero solution of (3.4) is asymptotically stable if and only if

$$-\frac{a(e^{ah} + 1)}{e^{ah} - 1} < b < -a. \tag{3.5}$$

On the other side, consider the following ordinary differential equation, which is associated with (3.4), and plays the role of (3.3),

$$y'(t) = ay(t) + by(t) = (a + b)y(t). \tag{3.6}$$

It is seen that the trivial solution of (3.6) is asymptotically stable if and only if

$$b < -a. \tag{3.7}$$

When the insertion of the greatest integer function is regarded as a "perturbation" of the linear equation (3.6), it is seen for (3.4) that the stability condition (3.5) is necessarily stricter than the one given by (3.7) for the corresponding "nonperturbed" equation (3.6). Moreover, it is seen that the condition (3.5) transforms to (3.7) as $h \to 0$.

If we discuss stability of equation (3.1) on the basis of (3.3), we expect that a comparison, similar to the relation of the conditions of (3.5) and (3.7), can be generalized. Furthermore, stability conditions for the ordinary differential equation (3.3) may not be enough for the issue system (3.1). By means of the following theorems, we demonstrate that stability of (3.1) depends on that of the corresponding ordinary differential equation (3.3).

3.1.3 Main Results

The following lemma plays a crucial role in the proofs of stability theorems.

Lemma 3.1.3 *If the conditions* (C1) – (C5) *are fulfilled, then we have the estimation*

$$\|x(\beta(t))\| \leq m\,\|x(t)\| \tag{3.8}$$

for all $t \in \mathbb{R}^+$*, where* $m = \left\{1 - \theta[\ell_2 + \ell_1(1 + \ell_2\theta)e^{\ell_1\theta}]\right\}^{-1}$.

Proof. Fix $t \in \mathbb{R}^+$, then one can find $k \in \mathbb{N}$ such that $t \in I_k = [\theta_k, \theta_{k+1})$. For $t \in I_k$, we have $x(t) = x(\theta_k) + \int_{\theta_k}^{t} f(s, x(s), x(\theta_k))ds$, which yields to

$$\|x(t)\| \leq (1 + \ell_2\theta)\,\|x(\theta_k)\| + \ell_1 \int_{\theta_k}^{t} \|x(s)\|\,ds.$$

By the Gronwall-Bellman Lemma, we obtain $\|x(t)\| \leq (1 + \ell_2\theta)e^{\ell_1\theta}\,\|x(\theta_k)\|$. Moreover,

$$x(\theta_k) = x(t) - \int_{\theta_k}^{t} f(s, x(s), x(\theta_k))ds,\ \ t \in I_k.$$

Thus,

$$\begin{aligned}
\|x(\theta_k)\| &\leq \|x(t)\| + \int_{\theta_k}^{t} (\ell_1\,\|x(s)\| + \ell_2\,\|x(\theta_k)\|)\,ds \\
&\leq \|x(t)\| + \int_{\theta_k}^{t} \ell_1 \left[(1 + \ell_2\theta)e^{\ell_1\theta} + \ell_2\right] \|x(\theta_k)\|\,ds \\
&\leq \|x(t)\| + \theta\left[\ell_1(1 + \ell_2\theta)e^{\ell_1\theta} + \ell_2\right] \|x(\theta_k)\|,
\end{aligned}$$

proves that $\|x(\theta_k)\| \leq m\,\|x(t)\|$ for $t \in I_k$. As the function $x(t)$ is continuous on $\mathbb{R}^+$, (3.8) holds for all $t \geq 0$.

Next, we need the following theorem which provides conditions for the existence and uniqueness of solutions on $\mathbb{R}^+$. Since the proof of the assertion is almost identical to the one given in [4], we omit it here.

Theorem 3.1.4 *Suppose that conditions* (C1) *and* (C3) – (C6) *are fulfilled. Then for every* $(t_0, x_0) \in \mathbb{R}^+ \times B(h)$ *there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of* (3.1) *on* $\mathbb{R}^+$ *with* $x(t_0) = x_0$.

Let the derivative of V with respect to system (3.3) be defined by

$$V'_{(3.3)}(t,y) = \frac{\partial V(t,y)}{\partial t} + \frac{\partial V(t,y)}{\partial y} g(t,y)$$

for all t in $\mathbb{R}^+$ and $y \in B(h)$.

Theorem 3.1.5 *Suppose that* (C1) – (C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$, $V(t,0) = 0$ *for all* $t \in \mathbb{R}^+$, *and a positive constant* α *such that*

(i) $u(\|y\|) \leq V(t,y)$ *on* $\mathbb{R}^+ \times B(h)$, *where* $u \in \mathcal{K}$*;*

(ii) $V'_{(3.3)}(t,y) \leq -\alpha\ell_2(1+m)\|y\|^2$ *for all* $(t,y) \in \mathbb{R}^+ \times B(h)$, *where m is the constant defined in Lemma* 3.1.3*;*

(iii) $\|\dfrac{\partial V(t,y)}{\partial y}\| \leq \alpha \|y\|$.

Then the zero solution of (3.1) *is stable.*

Proof. Let $h_1 \in (0,h)$. Given $\varepsilon \in (0,h_1)$ and $t_0 \in \mathbb{R}^+$, choose $\delta > 0$ sufficiently small that $V(t_0, x(t_0)) < u(\varepsilon)$ if $\|x(t_0)\| < \delta$. If we evaluate the time derivative of V with respect to (3.1), we get for $t \neq \theta_i$

$$\begin{aligned} V'_{(3.1)}(t, x(t), x(\beta(t))) &= \frac{\partial V(t, x(t))}{\partial t} + < \frac{\partial V(t, x(t))}{\partial x}, f(t, x(t), x(\beta(t))) > \\ &= V'_{(3.3)}(t, x(t)) + < \frac{\partial V(t, x(t))}{\partial x}, h(t, x(t), x(\beta(t))) > . \end{aligned}$$

Hence, we have

$$\begin{aligned} V'_{(3.1)}(t, x(t), x(\beta(t))) &\leq -\alpha\ell_2(1+m)\|x(t)\|^2 + \|\frac{\partial V(t, x(t))}{\partial x}\| \|h(t, x(t), x(\beta(t)))\| \\ &\leq -\alpha\ell_2(1+m)\|x(t)\|^2 + \alpha\ell_2(1+m)\|x(t)\|^2 = 0, \end{aligned}$$

which implies that $V(t, x(t)) \leq V(t_0, x(t_0)) < u(\varepsilon)$ for all $t \geq t_0$, proving that $\|x(t)\| < \varepsilon$. □

Theorem 3.1.6 *Suppose that* (C1) – (C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$ *and a constant* $\alpha > 0$ *such that* *(i)* $u(\|y\|) \le V(t,y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u,\ v \in \mathcal{K}$*;*

(ii) $V'_{(3.3)}(t,y) \le -\alpha\ell_2(1+m)\,\|y\|^2$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*

(iii) $\|\dfrac{\partial V(t,y)}{\partial y}\| \le \alpha\,\|y\|$.

Then the zero solution of (3.1) *is uniformly stable.*

Proof. Let $h_1 \in (0,h)$. Fix $\varepsilon > 0$ in the range $0 < \varepsilon < h_1$ and choose $\delta > 0$ such that $v(\delta) < u(\varepsilon)$. If $t_0 \ge 0$ and $\|x(t_0)\| < \delta$, then as a consequence of the condition (i) we have $V(t_0, x(t_0)) < v(\delta) < u(\varepsilon)$. Using the same argument used in the proof of Theorem 3.1.5, one can obtain that $V(t, x(t)) \le V(t_0, x(t_0)) < u(\varepsilon)$ for all $t \ge t_0$. Hence $\|x(t)\| < \varepsilon$ for all $t \ge t_0$.

Theorem 3.1.7 *Suppose that* (C1) – (C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$*, constants* $\alpha > 0$ *and* $\tau > 1$ *such that*

(i) $u(\|y\|) \le V(t,y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u,\ v \in \mathcal{K}$*;*

(ii) $V'_{(3.3)}(t,y) \le -\tau\alpha\ell_2(1+m)\,\|y\|^2$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*

(iii) $\|\dfrac{\partial V(t,y)}{\partial y}\| \le \alpha\,\|y\|$.

Then the zero solution of (3.1) *is uniformly asymptotically stable.*

Proof. In view of the Theorem 3.1.5, the equilibrium $x = 0$ of (3.1) is uniformly stable. We need to show that it is asymptotically stable as well. For $t \ne \theta_i$,

$$\begin{aligned} V'_{(3.1)}(t, x(t), x(\beta(t))) &\le -\tau\alpha\ell_2(1+m)\,\|x(t)\|^2 + \alpha\ell_2(1+m)\,\|x(t)\|^2 \\ &= -(\tau-1)\alpha\ell_2(1+m)\,\|x(t)\|^2 . \end{aligned}$$

Denote $w(\|x\|) = (\tau-1)\alpha\ell_2(1+m)\,\|x\|^2$. Let $h_1 \in (0,h)$. Choose $\delta > 0$ such that $v(\delta) < u(h_1)$. We fix $\varepsilon > 0$ in the range $(0, h_1)$ and pick $\eta \in (0,\delta)$ such that $v(\eta) < u(\varepsilon)$. Let $t_0 \in \mathbb{R}^+$ and $\|x(t_0)\| < \delta$. We define $T = \dfrac{u(h_1)}{w(\eta)}$. We shall show that $\|x(\bar{t})\| < \eta$ for some $\bar{t} \in [t_0, t_0 + T]$. If this were not true, then we would have $\|x(t)\| \ge \eta$ for all $t \in [t_0, t_0 + T]$.

For $t \in [t_0, t_0 + T]$, $t \ne \theta_i$, we have

$$V'_{(3.1)}(t, x(t), x(\beta(t))) \le -w(\|x(t)\|) \le -w(\eta).$$

Since the function $V(t, x(t))$ and the solution $x(t)$ are continuous, we obtain that

$$V(t_0 + T, x(t_0 + T)) \leq V(t_0, x(t_0)) - w(\eta)T < v(\delta) - w(\eta)\frac{u(h_1)}{w(\eta)} < 0,$$

which is a contradiction. Hence, $\bar{t}$ exists. Now for $t \geq \bar{t}$ we have

$$V(t, x(t)) \leq V(\bar{t}, x(\bar{t})) < v(\eta) < u(\varepsilon).$$

In the end, it follows from the hypothesis (i) that $\|x(t)\| < \varepsilon$ for all $t \geq \bar{t}$ and in turn for all $t \geq t_0 + T$. □

Remark 3.1.8 *Theorems* 3.1.5-3.1.7 *provide criteria for stability, which are entirely constructed on the basis of Lyapunov functions. As for the functionals, they appear only in the proofs of theorems. Although the equations include deviating arguments, and functionals are ordinarily used in the stability criteria* [*72*, *90*], *we see that the conditions of our investigations, which guarantee stability, are definitely formulated without functionals.*

Next, we want to compare our present results, which are obtained by the method of Lyapunov functions with the ones proved in [17] by employing the Lyapunov-Razumikhin technique. To this end, let us discuss the following linear equation with piecewise constant argument of generalized type taken from [17],

$$x'(t) = -a_0(t)x(t) - a_1(t)x(\beta(t)), \tag{3.9}$$

where a_0 and a_1 are bounded continuous functions on $\mathbb{R}^+$. We suppose that the sequence $\theta_i,\ i \in \mathbb{N}$, with $\ell_1 = \sup_{t\in\mathbb{R}^+} |a_0(t)|$, $\ell_2 = \sup_{t\in\mathbb{R}^+} |a_1(t)|$, satisfies the conditions (C4)-(C6). One can check easily that conditions (C1)-(C3) are also valid. Under the assumption

$$0 \leq a_0(t) + a_1(t) \leq 2a_0(t), \quad t \geq 0, \tag{3.10}$$

it was obtained via the Lyapunov-Razumikhin method in [17] that the trivial solution of (3.9) is uniformly stable. Let us consider this equation using the results obtained in the present section. We set

$$(1 + m) \sup_{t\in\mathbb{R}^+} |a_1(t)| \leq a_0(t) + a_1(t), \quad t \geq 0, \tag{3.11}$$

In order to apply our results, we need the following equation besides (3.9);

$$y'(t) = -(a_0(t) + a_1(t))y(t). \tag{3.12}$$

Let us define a Lyapunov function $V(y) = \frac{\alpha}{2}y^2,\ y \in B(h), \alpha > 0$. It follows from (3.11) that the derivative of $V(y)$ with respect to equation (3.12) is given by

$$\begin{aligned} V'_{(3.12)}(y(t)) &= -\alpha(a_0(t) + a_1(t))y^2(t) \\ &\leq -\alpha\ell_2(1+m)y^2(t). \end{aligned}$$

Then, by Theorem 3.1.6, the zero solution of (3.9) is uniformly stable.

In addition, taking $(a_0(t) + a_1(t)) \geq \tau\ell_2(1+m)$, $\tau > 1$, one can show that the trivial solution of (3.9) is uniformly asymptotically stable by Theorem 3.1.7.

We can see that theorems obtained by Lyapunov-Razumikhin method provide larger class of equations with respect to (3.9). However, from the perspective of the constructive analysis, the present method may be more preferable, since, for example, from the proof of Theorem 3.1.6, we have $V'_{(3.9)}(t, x(t), x(\beta(t))) \leq 0$, which implies $|x(t)| \leq |x(t_0)|,\ \ t \geq t_0$, for our specific Lyapunov function. Thus, by using the present results, it is possible to evaluate the number δ needed for (uniform) stability in the Definition 3.1.1 as $\delta = \varepsilon$.

Besides Theorems 3.1.5, 3.1.6 and 3.1.7, the following assertions may be useful for analysis of the stability of differential equations with piecewise constant argument. These theorems are important and have their own distinctive values with the newly required properties of the Lyapunov function and can be proved similarly.

Theorem 3.1.9 *Suppose that* (C1) – (C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$ *and a positive constant* M *such that*

(i) $u(\|y\|) \leq V(t, y)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u \in \mathcal{K}$*;*

(ii) $V'_{(3.3)}(t, y) \leq -M\ell_2(1+m)\|y\|$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*

(iii) $\|\dfrac{\partial V(t, y)}{\partial y}\| \leq M$.

Then the zero solution of (3.1) *is stable.*

Theorem 3.1.10 *Suppose that* (C1) – (C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$ *and a positive constant* M *such that* *(i)* $u(\|y\|) \le V(t, y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u, v \in \mathcal{K}$*;*

(ii) $V'_{(3.3)}(t, y) \le -M\ell_2(1 + m)\|y\|$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*

(iii) $\|\dfrac{\partial V(t, y)}{\partial y}\| \le M.$

Then the zero solution of (3.1) *is uniformly stable.*

Theorem 3.1.11 *Suppose that* (C1) − (C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$*, constants* $M > 0$ *and* $\tau > 1$ *such that*

(i) $u(\|y\|) \le V(t, y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u, v \in \mathcal{K}$*;*

(ii) $V'_{(3.3)}(t, y) \le -\tau M\ell_2(1 + m)\|y\|$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*

(iii) $\|\dfrac{\partial V(t, y)}{\partial y}\| \le M.$

Then the zero solution of (3.1) *is uniformly asymptotically stable.*

3.1.4 Applications to The Logistic Equation

In this section, we are interested in the stability of the positive equilibrium $N^* = \dfrac{1}{a + b}$ of the following logistically growing population subjected to a density-dependent harvesting;

$$N'(t) = rN(t)[1 - aN(t) - bN(\beta(t))], \quad t > 0, \tag{3.13}$$

where $N(t)$ denotes the biomass of a single species, and r, a, b are positive parameters. There exists an extensive literature dealing with sufficient conditions for global asymptotic stability of equilibria for the logistic equation with piecewise constant argument (see [115, 86, 87, 89, 101] and the references therein). For example, Gopalsamy and Liu [115] showed that N^* is globally asymptotically stable if $a/b \ge 1$. In these papers, the initial moments are taken as integers owing to the method of investigation: reduction to difference equations. Since our approach makes it possible to take not only integers, but also all values from $\mathbb{R}^+$ as initial moments, we can consider the stability in uniform sense.

Let us also discuss the biological sense of the insertion of the piecewise constant delay [115, 72, 86, 87, 89, 101]. The delay means that the rate of the population depends both on the present size and the memorized values of the population. To illustrate the dependence, one may think populations, which meet at the beginning of a season, e.g., in springtime, with their instinctive evaluations of the population state and environment and implicitly decide which living conditions to prefer and where to go [9] in line with group hierarchy, communications and dynamics and then adapt to those conditions.

By means of the transformation $x = b(N - N^*)$, equation (3.13) can be simplified as

$$x'(t) = -r[x(t) + \frac{1}{1+\gamma}][\gamma x(t) + x(\beta(t))], \tag{3.14}$$

where $\gamma = a/b$. Let us specify for (3.14) general conditions of Theorems 3.1.5, 3.1.6 and 3.1.7. We observe that $f(x, y) := -r[x + \frac{1}{1+\gamma}][\gamma x + y]$ is a continuous function and has continuous partial derivatives for $x, y \in B(h)$. It can be found easily that

$$\ell_1 = r(2\gamma h + h + \frac{\gamma}{1+\gamma}), \quad \ell_2 = r(h + \frac{1}{1+\gamma}).$$

One can see that (C1), (C2) and (C3) hold if r is sufficiently small. Moreover, we assume that (C4), (C5) and (C6) are satisfied.

Consider the following equation associated with (3.14);

$$y'(t) = -r(1+\gamma)y(t)[y(t) + \frac{1}{1+\gamma}]. \tag{3.15}$$

Suppose h is smaller than $\dfrac{1}{1+\gamma}$ and consider a Lyapunov function defined by $V(y) = \frac{\alpha}{2}y^2$, $y \in B(h)$, $\alpha > 0$. Then,

$$\begin{aligned} V'_{(3.15)}(y(t)) &= -\alpha r(1+\gamma)y^2(t)[y(t) + \frac{1}{1+\gamma}] \\ &\leq -\alpha r[1 - h(1+\gamma)]y^2(t). \end{aligned}$$

For sufficiently small h, we assume that

$$\varphi(h, m) \leq \gamma, \tag{3.16}$$

where

$$\varphi(h, m) = \frac{1 - h(3+m) - \sqrt{(h(1+m))^2 - 6h(1+m) + 1}}{2h}.$$

It follows from (3.16) that

$$(h + \frac{1}{1+\gamma})(1+m) \leq 1 - h(1+\gamma),$$

which implies in turn

$$V'_{(3.15)}(y(t)) \leq -\alpha\ell_2(1+m)y^2(t).$$

By Theorem 3.1.6, the zero solution of (3.14) is uniformly stable.

Next, we consider uniform asymptotic stability. Assuming for $\tau > 1$;

$$\psi(h, m, \tau) \leq \gamma, \tag{3.17}$$

where

$$\psi(h, m, \tau) = \frac{1 - h\tau(3+m) - \sqrt{(h\tau(1+m))^2 - 6h\tau(1+m) + 1}}{2h},$$

we obtain that

$$\tau(h + \frac{1}{1+\gamma})(1+m) \leq 1 - h(1+\gamma).$$

One can show easily that $\psi(h, m, \tau) \geq 1$ for small h. Then for $V(y) = \frac{\alpha}{2}y^2$, we have

$$V'_{(3.15)}(y(t)) \leq -\tau\alpha\ell_2(1+m)y^2(t).$$

That is, condition (iii) of Theorem 3.1.7 is satisfied. Thus, the trivial solution $x = 0$ of (3.14) is uniformly asymptotically stable.

In the light of the above reduction, we see that the obtained conditions are valid for the stability of the equilibrium $N = N^*$ of (3.13).

Finally, we see that the condition (3.17) is stronger than the one $\gamma \geq 1$ taken from [115]. However, our results are for all values from $\mathbb{R}^+$ as initial moments, whereas [115] considers only integers. Moreover, the piecewise constant argument is of generalized type.

3.2 The Method of Lyapunov Functions

In this section, we apply the method of Lyapunov functions for differential equations with piecewise constant argument of generalized type to a model of RNNs. The model involves both advanced and delayed arguments. Sufficient conditions are obtained for global exponential stability of the equilibrium point. Examples with numerical simulations are presented to illustrate the results.

3.2.1 Introduction

Lyapunov functions and functionals are among the most popular tools in studying the problem of the stability for RNNs (see, [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 61, 62, 63, 64, 65, 66]). However, it is difficult to construct Lyapunov functions or functionals that satisfy the strong conditions required in the classical stability theory. In this part, we investigate some new stability conditions for RNNs model based on the second Lyapunov method. Although, this model includes both advanced and delayed arguments, it deserves to be mentioned that new stability conditions are given in terms of inequalities, and it is known that for equations with deviating argument, this method necessarily utilizes functionals [67, 69, 77, 90].

To the best of our knowledge, the equations with piecewise constant arguments were not considered as models of RNNs, except possibly [12, 13]. In papers [3, 5, 6, 7, 10, 12, 13, 17, 18] we discuss the stability problems. Unlike these papers, the stability was analyzed by the second Lyapunov method in [10]. Nevertheless, it is the first time that the second method is applied to the equations, whose arguments are not only delayed but also advanced in this thesis. Moreover, one should emphasize that there is an opportunity application of Lyapunov functions technique to estimate domains of attraction which has a particular interest to evaluate the performance of RNNs [54, 60].

The crucial novelty of this part is that the system is of mixed type, in other words; the argument can be advanced during the process. In the literature, biological reasons for argument to be delayed are discussed well [91, 92]. Due to the finite switching speed of amplifiers and transmission of signals in electronic networks or finite speed for signal propagation in neural networks, time delays exist [32, 33, 35, 39]. In the present section, we issue from the fact that delayed as well as advanced argument play a significant role in electromagnetic fields, see, for example, the paper [25], where the *symmetry of the physics laws* were emphasized with respect to *time reversal*. Consequently, one can suppose that analysis of neural networks, which is based on electrodynamics, may bring us to the comprehension of the deviation, especially the advanced one, in the models more clearly. Therefore, in the future analysis of RNNs, the systems introduced in this section can be useful. Furthermore, different types of deviation of the argument may depend on traveling waves emergence in CNNs [26]. Understanding the structure of such traveling waves is important due to their potential applications including image processing (see, for example, [30, 31, 32, 33, 34, 35]). On the other hand, the importance of anticipation for biology, which can be modeled with advanced arguments, is mentioned by some authors. For instance, in paper [93], it is supposed that synchronization of biological oscillators may request anticipation of counterparts behavior.

3.2.2 Model Formulation and Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the sets of natural and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the m dimensional real space by $\mathbb{R}^m$, $m \in \mathbb{N}$, and the norm of a vector $x \in \mathbb{R}^m$ by $\|x\| = \sum_{i=1}^{m} |x_i|$. We fix two real valued sequences θ_i, ζ_i, $i \in \mathbb{N}$, such that $\theta_i < \theta_{i+1}$, $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{N}$, $\theta_i \to \infty$ as $i \to \infty$, and shall consider the following RNNs model described by differential equations with piecewise constant argument of generalized type:

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\gamma(t))) + I_i, \\ & \quad a_i > 0, \ i = 1, 2, \ldots, m. \end{aligned} \tag{3.18}$$

where $\gamma(t) = \zeta_k$, if $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, n corresponds to the number of units in a neural network, $x_i(t)$ stands for the state vector of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\gamma(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and $\gamma(t)$, b_{ij}, c_{ij}, I_i are real constants, b_{ij} means the strength of the jth unit on the ith unit at time t, c_{ij} infers the strength of the jth unit on the ith unit at time $\gamma(t)$, I_i signifies the external bias on the ith unit and a_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs.

The following assumptions will be needed throughout this section:

(A1) the activation functions $f_j, g_j \in C(\mathbb{R}^m)$ satisfy $f_j(0) = 0$, $g_j(0) = 0$ for each $j = 1, 2, \ldots, m$;

(A2) there exist Lipschitz constants $L_i^1, L_i^2 > 0$ such that

$$|f_i(u) - f_i(v)| \leq L_i^1 |u - v|,$$

$$|g_i(u) - g_i(v)| \leq L_i^2 |u - v|$$

for all $u, v \in \mathbb{R}^m$, $i = 1, 2, \ldots, m$;

(A3) there exists a positive number θ such that $\theta_{i+1} - \theta_i \le \theta$, $i \in \mathbb{N}$;

(A4) $\theta\,[m_1 + 2m_2]\,e^{m_1\theta} < 1$;

(A5) $\theta\left[m_2 + m_1(1 + m_2\theta)e^{m_1\theta}\right] < 1$,

where

$$m_1 = \max_{1\le i\le m}\left(a_i + L_i^1 \sum_{j=1}^{m} |b_{ji}|\right),\ m_2 = \max_{1\le i\le m}\left(L_i^2 \sum_{j=1}^{m} |c_{ji}|\right).$$

In this part we assume that the solutions of the equation (3.18) are continuous functions. But the deviating argument $\gamma(t)$ is discontinuous. Thus, in general, the right-hand side of (3.18) has discontinuities at moments $\theta_i, i \in \mathbb{N}$. As a result, we consider the solutions of the equations as functions, which are continuous and continuously differentiable within intervals $[\theta_i, \theta_{i+1}), i \in \mathbb{N}$. In other words, by a solution $x(t) = (x_1(t), \ldots, x_m(t))^T$ of (3.18) we mean a continuous function on $\mathbb{R}^+$ such that the derivative $x'(t)$ exists at each point $t \in \mathbb{R}^+$, with the possible exception of the points $\theta_i, i \in \mathbb{N}$, where one-sided derivative exists and the differential equation (3.18) is satisfied by $x(t)$ on each interval (θ_i, θ_{i+1}) as well.

In the following theorem, we obtain sufficient conditions for the existence of a unique equilibrium, $x^* = (x_1^*, \ldots, x_m^*)^T$, of (3.18).

Theorem 3.2.1 *Suppose that* (A2) *holds. If the neural parameters a_i, b_{ij}, c_{ij} satisfy*

$$a_i > L_i^1 \sum_{j=1}^{m} |b_{ji}| + L_i^2 \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \ldots, m,$$

then (3.18) has a unique equilibrium $x^* = (x_1^*, \ldots, x_m^*)^T$.

The proof of the theorem is almost identical to Theorem 2.1 in [46] and thus we omit it here.

The next theorem provides conditions for the existence and uniqueness of solutions on $t \ge t_0$. The proof of the assertion is similar to that of Theorem 1.1 in [5] and Theorem 2.2 in [12]. But, for convenience of the reader we place the full proof of the assertion.

Theorem 3.2.2 *Assume that conditions* (A1) – (A4) *are fulfilled. Then, for every $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^m$, there exists a unique solution $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^T$, $t \ge t_0$, of (3.18), such that $x(t_0) = x^0$.*

Proof. *Existence* : Fix $k \in \mathbb{N}$. We assume without loss of generality that $\theta_k \leq \zeta_k < t_0 \leq \theta_{k+1}$. To begin with, we shall prove that for every $(t_0, x_0) \in [\theta_k, \theta_{k+1}] \times \mathbb{R}^m$, there exists a unique solution $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^T$, of (3.18) such that $x(t_0) = x^0 = (x_1^0, \ldots, x_m^0)^T$.

Let us denote for simplicity $z(t) = x(t, t_0, x^0), z(t) = (z_1, \ldots, z_m)^T$, and consider the equivalent integral equation

$$z_i(t) = x_i^0 + \int_{t_0}^{t} \left[-a_i z_i(s) + \sum_{j=1}^{m} b_{ij} f_j(z_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(z_j(\zeta_k)) + I_i \right] ds.$$

Define a norm $\|z(t)\|_0 = \max_{[\zeta_k, t_0]} \|z(t)\|$ and construct the following sequences $z_i^n(t),\ z_i^0(t) \equiv x_i^0,\ i = 1, \ldots, m,\ n \geq 0$ such that

$$z_i^{n+1}(t) = x_i^0 + \int_{t_0}^{t} \left[-a_i z_i^n(s) + \sum_{j=1}^{m} b_{ij} f_j(z_j^n(s)) + \sum_{j=1}^{m} c_{ij} g_j(z_j^n(\zeta_k)) + I_i \right] ds.$$

One can find that

$$\|z^{n+1}(t) - z^n(t)\|_0 \leq [\theta(m_1 + m_2)]^n \, \tau,$$

where

$$\tau = \theta \left[(m_1 + m_2) \, \|x^0\| + \sum_{i=1}^{m} I_i \right].$$

Thus, there exists a unique solution $z(t) = x(t, t_0, x^0)$ of the integral equation on $[\zeta_k, t_0]$. Then, conditions (A1) and (A2) imply that $x(t)$ can be continued to θ_{k+1}, since it is a solution of ordinary differential equations

$$\begin{aligned} x_i'(t) \quad &= \quad -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\zeta_k)) + I_i, \\ & \quad\ \ a_i > 0,\ i = 1, 2, \ldots, m \end{aligned}$$

on $[\theta_k, \theta_{k+1})$. Next, again, using same argument we can continue $x(t)$ from $t = \theta_{k+1}$ to $t = \zeta_{k+1}$, and then to θ_{k+2}. Hence, the mathematical induction completes the proof.

Uniqueness : Denote by $x^1(t) = x(t, t_0, x^1)$, $x^2(t) = x(t, t_0, x^2)$, the solutions of (3.18), where $\theta_k \leq t_0 \leq \theta_{k+1}$. It is sufficient to check that for every $t \in [\theta_k, \theta_{k+1}]$, $x^2 = (x_1^2, \ldots, x_m^2)^T, x^1 = (x_1^1, \ldots, x_m^1)^T \in \mathbb{R}^m, x^2 \neq x^1$ implies $x^1(t) \neq x^2(t)$. Then, we have that

$$\begin{aligned}\|x^1(t)-x^2(t)\| &\le \|x^1-x^2\| + \sum_{i=1}^{m}\left\{\int_{t_0}^{t}\left[a_i|x_i^2(s)-x_i^1(s)|\right.\right.\\&\quad\left.\left. + \sum_{j=1}^{m} L_i^1|b_{ji}||x_i^2(s)-x_i^1(s)| + \sum_{j=1}^{m} L_i^2|c_{ji}||x_i^2(\zeta_k)-x_i^1(\zeta_k)|\right]ds\right\}\\&\le \left(\|x^1-x^2\| + \theta m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|\right) + \int_{t_0}^{t} m_1\|x^1(s)-x^2(s)\|ds.\end{aligned}$$

The Gronwall-Bellman Lemma yields that

$$\|x^1(t)-x^2(t)\| \le \left(\|x^1-x^2\| + \theta m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|\right)e^{m_1\theta}.$$

Particularly,

$$\|x^1(\zeta_k)-x^2(\zeta_k)\| \le \left(\|x^1-x^2\| + \theta m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|\right)e^{m_1\theta}.$$

Thus,

$$\|x^1(t)-x^2(t)\| \le \Big(\frac{e^{m_1\theta}}{1-m_2\theta e^{m_1\theta}}\Big)\|x^1-x^2\|. \tag{3.19}$$

On the other hand, assume on the contrary that there exists $t\in[\theta_k,\theta_{k+1}]$ such that $x^1(t)=x^2(t)$. Hence,

$$\begin{aligned}\|x^1-x^2\| &= \sum_{i=1}^{m}\left|\int_{t_0}^{t}\left[-a_i\left(x_i^2(s)-x_i^1(s)\right) + \sum_{j=1}^{m} b_{ij}\left[f_j(x_j^2(s))-f_j(x_j^1(s))\right]\right.\right.\\&\quad\left.\left. + \sum_{j=1}^{m} c_{ij}\left[g_j(x_j^2(\zeta_k))-g_j(x_j^1(\zeta_k))\right]\right]ds\right|\\&\le \sum_{i=1}^{m}\left\{\int_{t_0}^{t}\left[a_i|x_i^2(s)-x_i^1(s)| + \sum_{j=1}^{m} L_i^1|b_{ji}||x_i^2(s)-x_i^1(s)|\right.\right.\\&\quad\left.\left. + \sum_{j=1}^{m} L_i^2|c_{ji}||x_i^2(\zeta_k)-x_i^1(\zeta_k)|\right]ds\right\}\\&\le \theta m_2\|x^1(\zeta_k)-x^2(\zeta_k)\| + \int_{t_0}^{t} m_1\|x^1(s)-x^2(s)\|ds.\end{aligned} \tag{3.20}$$

Consequently, substituting (3.19) in (3.20), we obtain

$$\|x^1-x^2\| \le \theta(m_1+2m_2)e^{m_1\theta}\|x^1-x^2\|. \tag{3.21}$$

Thus, one can see that (A4) contradicts with (3.21). The uniqueness is proved for $t\in[\theta_k,\theta_{k+1}]$. The extension of the unique solution on $\mathbb{R}^+$ is obvious. Hence, the theorem is proved. □

Definitions of Lyapunov stability for the solutions of discussed system can be given in the same way as for ordinary differential equations. Let us give only one of them.

Definition 3.2.3 *[6] The equilibrium $x = x^*$ of* (3.18) *is said to be globally exponentially stable if there exist positive constants α_1 and α_2 such that the estimation of the inequality $\|x(t) - x^*\| < \alpha_1 \|x(t_0) - x^*\| e^{-\alpha_2(t-t_0)}$ is valid for all $t \geq t_0$.*

System (3.18) can be simplified as follows. Substituting $y(t) = x(t) - x^*$ into (3.18) leads to

$$y_i'(t) = -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}\varphi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}\psi_j(y_j(\gamma(t))), \tag{3.22}$$

where $\varphi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$ and $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*)$ with $\varphi_j(0) = \psi_j(0) = 0$. From assumption (A2), $\varphi_j(\cdot)$ and $\psi_j(\cdot)$ are also Lipschitzian with L_j^1, L_j^2, respectively.

It is clear that the stability of the zero solution of (3.22) is equivalent to that of the equilibrium x^* of (3.18). Therefore, we restrict our discussion to the stability of the zero solution of (3.22).

First of all, we give the following lemma which is one of the most important auxiliary results of the present section.

Lemma 3.2.4 *Let $y(t) = (y_1(t), \ldots, y_m(t))^T$ be a solution of (3.22) and* (A1) − (A5) *be satisfied. Then, the following inequality*

$$\|y(\gamma(t))\| \leq \lambda \|y(t)\| \tag{3.23}$$

holds for all $t \in \mathbb{R}^+$, where $\lambda = \left\{1 - \theta\left[m_2 + m_1(1 + m_2\theta)e^{m_1\theta}\right]\right\}^{-1}$.

Proof. Fix $k \in \mathbb{N}$. Then for $t \in [\theta_k, \theta_{k+1})$,

$$y_i(t) = y_i(\zeta_k) + \int_{\zeta_k}^{t} \left[-a_i y_i(s) + \sum_{j=1}^{m} b_{ij}\varphi_j(y_j(s)) + \sum_{j=1}^{m} c_{ij}\psi_j(y_j(\zeta_k)) \right] ds,$$

where $\gamma(t) = \zeta_k$, if $t \in [\theta_k, \theta_{k+1})$, $t \in \mathbb{R}^+$. Taking absolute value of both sides for each

$i = 1, 2, \ldots, m$ and adding all equalities, we obtain that

$$\begin{aligned}
\|y(t)\| &\le \|y(\zeta_k)\| + \sum_{i=1}^{m} \left\{ \int_{\zeta_k}^{t} \left[a_i |y_i(s)| + \sum_{j=1}^{m} L_j^1 |b_{ij}| |y_j(s)| \right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} L_j^2 |c_{ij}| |y_j(\zeta_k)| \right] ds \right\} \\
&= \|y(\zeta_k)\| + \int_{\zeta_k}^{t} \left[\sum_{i=1}^{m} \left(a_i + L_i^1 \sum_{j=1}^{m} |b_{ji}| \right) |y_i(s)| \right. \\
&\quad \left. + \sum_{i=1}^{m} \sum_{j=1}^{m} L_i^2 |c_{ji}| |y_i(\zeta_k)| \right] ds \\
&\le (1 + m_2\theta)\|y(\zeta_k)\| + \int_{\zeta_k}^{t} m_1 \|y(s)\| ds.
\end{aligned}$$

The Gronwall-Bellman Lemma yields

$$\|y(t)\| \le (1 + m_2\theta) e^{m_1\theta} \|y(\zeta_k)\|. \tag{3.24}$$

Furthermore, for $t \in [\theta_k, \theta_{k+1})$ we have

$$\begin{aligned}
\|y(\zeta_k)\| &\le \|y(t)\| + \int_{\zeta_k}^{t} \left[\sum_{i=1}^{m} \left(a_i + L_i^1 \sum_{j=1}^{m} |b_{ji}| \right) |y_i(s)| \right. \\
&\quad \left. + \sum_{i=1}^{m} \sum_{j=1}^{m} L_i^2 |c_{ji}| |y_i(\zeta_k)| \right] ds \\
&\le \|y(t)\| + m_2\theta \|y(\zeta_k)\| + \int_{\zeta_k}^{t} m_1 \|y(s)\| ds.
\end{aligned}$$

The last inequality together with (3.24) imply

$$\|y(\zeta_k)\| \le \|y(t)\| + m_2\theta\|y(\zeta_k)\| + m_1\theta(1 + m_2\theta)e^{m_1\theta}\|y(\zeta_k)\|.$$

Thus, it follows from condition (A4) that

$$\|y(\zeta_k)\| \le \lambda \|y(t)\|, \quad t \in [\theta_k, \theta_{k+1}).$$

Hence, (3.23) holds for all $t \in \mathbb{R}^+$. This completes the proof. □

3.2.3 Main Results

In this section we establish several criteria for global exponential stability of (3.22) based on the method of Lyapunov functions.

For convenience, we adopt the following notation in the sequel:

$$m_3 = \frac{1}{m} \min_{1 \le i \le m} \left(a_i - \frac{1}{2} \sum_{j=1}^{m} \left(L_j^1 |b_{ij}| + L_j^2 |c_{ij}| + L_i^1 |b_{ji}| \right) \right).$$

Theorem 3.2.5 *Suppose that* (A1) − (A5) *hold true. Assume, furthermore, that the following inequality is satisfied:*

$$m_3 > \frac{m_2 \lambda^2}{2}. \tag{3.25}$$

Then the system (3.22) is globally exponentially stable.

Proof. We define a Lyapunov function by

$$V(y(t)) = \frac{1}{2} \sum_{i=1}^{m} y_i^2(t).$$

One can easily show that

$$\frac{1}{2m} \|y(t)\|^2 \le V(y(t)) \le \frac{1}{2} \|y(t)\|^2. \tag{3.26}$$

For $t \neq \theta_i, i \in \mathbb{N}$, the time derivative of V with respect to (3.22) is given by

$$\begin{aligned}
V'_{(3.22)}(y(t)) &= \sum_{i=1}^{m} y_i(t) y_i'(t) \\
&= \sum_{i=1}^{m} y_i(t) \left[-a_i y_i(t) + \sum_{j=1}^{m} b_{ij} \varphi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} \psi_j(y_j(\gamma(t))) \right] \\
&\le \sum_{i=1}^{m} \left[-a_i y_i^2(t) + \sum_{j=1}^{m} L_j^1 |b_{ij}| |y_i(t)| |y_j(t)| + \sum_{j=1}^{m} L_j^2 |c_{ij}| |y_i(t)| |y_j(\gamma(t))| \right] \\
&\le \sum_{i=1}^{m} \left[-a_i y_i^2(t) + \frac{1}{2} \sum_{j=1}^{m} L_j^1 |b_{ij}| (y_i^2(t) + y_j^2(t)) + \frac{1}{2} \sum_{j=1}^{m} L_j^2 |c_{ij}| (y_i^2(t) + y_j^2(\gamma(t))) \right] \\
&\le -\sum_{i=1}^{m} \left[\left(a_i - \frac{1}{2} \sum_{j=1}^{m} \left(L_j^1 |b_{ij}| + L_j^2 |c_{ij}| + L_i^1 |b_{ji}| \right) \right) y_i^2(t) \right] \\
&+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} L_i^2 |c_{ji}| y_i^2(\gamma(t)) \\
&\le -\min_{1 \le i \le m} \left(a_i - \frac{1}{2} \sum_{j=1}^{m} \left(L_j^1 |b_{ij}| + L_j^2 |c_{ij}| + L_i^1 |b_{ji}| \right) \right) \sum_{i=1}^{m} y_i^2(t) \\
&+ \frac{1}{2} \max_{1 \le i \le m} \left(L_i^2 \sum_{j=1}^{m} |c_{ji}| \right) \sum_{i=1}^{m} y_i^2(\gamma(t)) \\
&\le -m_3 \|y(t)\|^2 + \frac{m_2}{2} \|y(\gamma(t))\|^2.
\end{aligned}$$

By using Lemma 3.2.4, we obtain

$$\begin{aligned} V'_{(3.22)}(y(t)) &\leq -m_3\|y(t)\|^2 + \frac{m_2\lambda^2}{2}\|y(t)\|^2 \\ &= -(m_3 - \frac{m_2\lambda^2}{2})\|y(t)\|^2. \end{aligned}$$

Now, define β for convenience as follows:

$$\beta = m_3 - \frac{m_2\lambda^2}{2} > 0.$$

Then, we have for $t \neq \theta_i$

$$\begin{aligned} \frac{d}{dt}(e^{2\beta t}V(y(t))) &= e^{2\beta t}(2\beta)V(y(t)) + e^{2\beta t}V'_{(3.22)}(y(t)) \\ &\leq \beta e^{2\beta t}\,\|y(t)\|^2 - \beta e^{2\beta t}\,\|y(t)\|^2 = 0. \end{aligned}$$

From (3.26) and using the continuity of the function V and the solution $y(t)$, we obtain

$$e^{2\beta t}(1/2m)\,\|y(t)\|^2 \leq e^{2\beta t}V(y(t)) \leq e^{2\beta t_0}V(y(t_0)) \leq e^{2\beta t_0}(1/2)\,\|y(t_0)\|^2\,,$$

which implies $\|y(t)\| \leq \sqrt{m}\,\|y(t_0)\|\,e^{-\beta(t-t_0)}$. That is, the system (3.22) is globally exponentially stable. □

In the next theorem, we utilize the same technique, used in previous theorem, to find new stability conditions for RNNs by choosing a different Lyapunov function defined as

$$V(y(t)) = \sum_{i=1}^{m} \alpha_i|y_i(t)|, \quad \alpha_i > 0, \quad i = 1, 2, \ldots, m.$$

For simplicity of notation, let us denote

$$m_4 = \min_{1\leq i\leq m}\left(a_i - L_i^1\sum_{j=1}^{m}|b_{ji}|\right).$$

Theorem 3.2.6 *Suppose that* (A1) − (A5) *hold true. Assume, furthermore, that the following inequality is satisfied:*

$$m_4 > m_2\lambda. \tag{3.27}$$

Then the system (3.22) is globally exponentially stable.

The proof of the assertion is similar to that of Theorem 3.2.5, so we omit it here.

3.2.4 Illustrative Examples

In this section, we give three examples with simulations to illustrate our results. In the sequel, we assume that the identification function $\gamma(t)$ is with the sequences $\theta_k = k/9,\ \zeta_k = (2k+1)/18,\ k \in \mathbb{N}$.

Example 3.2.7 *Consider the following RNNs with the argument function $\gamma(t)$:*

$$\begin{aligned}\frac{dx(t)}{dt} &= -\begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.02 & 0.03 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0.08 & 1 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(\gamma(t))}{7}) \\ \tanh(\frac{x_2(\gamma(t))}{6}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{3.28}$$

It is easy to verify that (3.28) satisfies the conditions of the Theorem 3.2.5 with $L_1^1 = L_2^1 = 1, L_1^2 = 1/7,\ L_2^2 = 1/6,\ m_1 = 2.53,\ m_2 = 0.3333,\ m_3 = 0.6308, m_4 = 0.47,\ \lambda = 1.7337$, Thus, according to this theorem the unique equilibrium $x^* = (0.6011, 1.3654)^T$ of (3.28) is globally exponentially stable. However, the condition (3.27) of Theorem 3.2.6 is not satisfied.

Let us simulate a solution of (3.28) with initial condition $x_1^1(0) = x_1^0,\ x_2^1(0) = x_2^0$. Since the equation (3.28) is of mixed type, the numerical analysis has a specific character and it should be described more carefully. One will see that this algorithm is in full accordance with the approximations made in the proof of Theorem 3.2.2.

We start with the interval $[\theta_0, \theta_1]$, that is; $[0, 1/9]$. On this interval the equation (3.28) has the form

$$\begin{aligned}\frac{dx(t)}{dt} &= -\begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} 0.02 & 0.03 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(x_1(0)) \\ \tanh(x_2(0)) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0.08 & 1 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(1/18)}{7}) \\ \tanh(\frac{x_2(1/18)}{6}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

where $x_i(1/18),\ i = 1, 2$, are still unknown. For this reason, we will arrange approximations in the following way. Consider the sequence of the equations

$$
\begin{aligned}
\frac{dx^{(n+1)}(t)}{dt} &= -\begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x_1^{(n)}(0) \\ x_2^{(n)}(0) \end{pmatrix} + \begin{pmatrix} 0.02 & 0.03 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(x_1^{(n)}(0)) \\ \tanh(x_2^{(n)}(0)) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0.08 & 1 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1^{(n)}(1/18)}{7}) \\ \tanh(\frac{x_2^{(n)}(1/18)}{6}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}
$$

where $n = 0, 1, 2\ldots,$ with $x_1^0(t) \equiv x_1^0,\ x_2^0(t) \equiv x_2^0$. We evaluate the solutions, $x^{(n)}(t)$, by using MATLAB 7.8. and stop the iterations at $(x_1^{(500)}(t), x_2^{500}(t))$. Then, we assign $x_1(t) = x_1^{(500)}(t),\ x_2(t) = x_2^{(500)}(t)$ on the interval $[\theta_0, \theta_1]$. Next, similar operation is done on the interval $[\theta_1, \theta_2]$. That is, we construct the sequence $(x_1^{(n)}, x_2^{(n)})$ of solutions again for the system

$$
\begin{aligned}
\frac{dx^{(n+1)}(t)}{dt} &= -\begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x_1^{(n)}(0) \\ x_2^{(n)}(0) \end{pmatrix} + \begin{pmatrix} 0.02 & 0.03 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(x_1^{(n)}(0)) \\ \tanh(x_2^{(n)}(0)) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0.08 & 1 \\ 0.01 & 1 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1^{(n)}(3/18)}{7}) \\ \tanh(\frac{x_2^{(n)}(3/18)}{6}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

with $x_1^0(t) \equiv x_1^{(500)}(1/9),\ x_2^0(t) \equiv x_2^{(500)}(1/9)$. Then, we reassign $x_1(t) = x_1^{(500)}(t),\ x_2(t) = x_2^{(500)}(t)$ on $[\theta_1, \theta_2]$. Proceeding in this way, one can obtain a simulation which demonstrates the asymptotic property.

Specifically, simulation result with several random initial points is shown in Fig. 3.1. We must explain that the non-smoothness at the switching points $\theta_k,\ k \in \mathbb{N}$ is not seen by simulation. That is why we have to choose the Lipschitz constants and θ small enough to satisfy the conditions of the theorems. So, the smallness "hides" the non-smoothness.

Let us now take the parameters such that the non-smoothness can be seen. Consider the following RNNs:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= -\begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 8 & 0.2 \end{pmatrix}\begin{pmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 20 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \tanh(x_1(\gamma(t))) \\ \tanh(\frac{x_2(\gamma(t))}{2}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned} \tag{3.29}
$$

where $\theta_k = k/2,\ \zeta_k = (2k+1)/4,\ k \in \mathbb{N}$. One can see that θ and the Lipschitz coefficient are large this time. They do not satisfy the conditions of our theorems. It is illustrated in Fig.3.2 that the non-smoothness of the solution with the initial point $[1,2]^T$ can be seen at the switching points $\theta_k,\ k \in \mathbb{N}$. This is important for us to see that the non-smoothness of solutions expected from the equations' nature is seen. Moreover, we can see that the solution converges to the unique equilibrium $x^* = (0.4325, 0.6065)^T$. It shows that the sufficient conditions which are found in our theorems can be elaborated further.

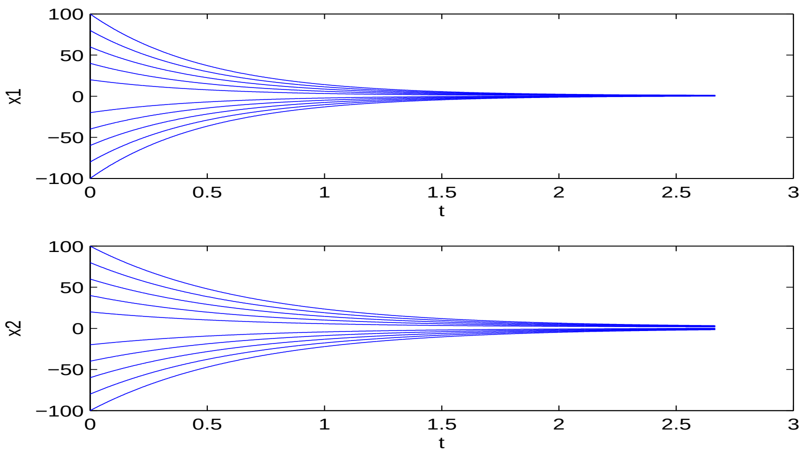

Figure 3.1: Transient behavior of the RNNs in Example 3.2.7.

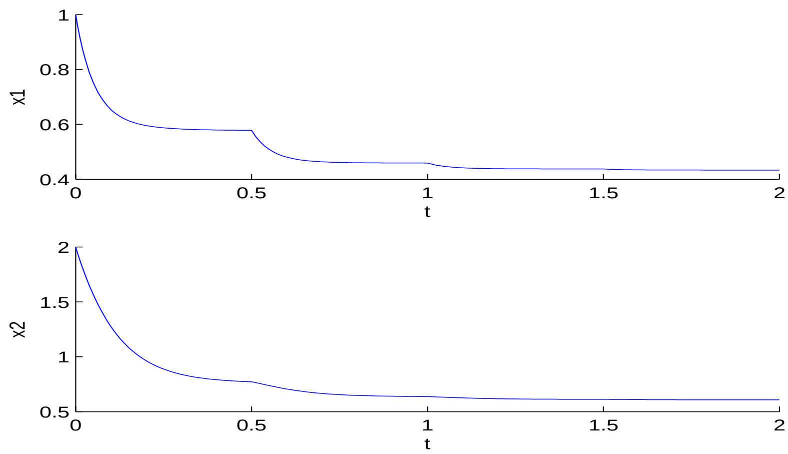

Figure 3.2: The non-smoothness is seen at moments 0.5; 1; 1.5, which are switching points of the function $\gamma(t)$.

Example 3.2.8 *Consider the following RNNs:*

$$\frac{dx(t)}{dt} = -\begin{pmatrix} 2 & 0 \\ 0 & 2.5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 & 0.03 \\ 0.04 & 1 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(t)}{4}) \\ \tanh(x_2(t)) \end{pmatrix}$$
$$+\begin{pmatrix} 1 & 0.04 \\ 0.02 & 0.07 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(\gamma(t))}{4}) \\ \tanh(\frac{x_2(\gamma(t))}{4}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.30}$$

It can be shown easily that (3.30) satisfies the conditions of the Theorem 3.2.6 if $L_1^1 = 1/4, L_2^1 = 1, L_1^2 = 1/4, L_2^2 = 1/4, m_1 = 3.53, m_2 = 0.2550, m_3 = 0.6181, m_4 = 1.47,\ \lambda = 2.6693,$ *whereas the condition (3.25) of Theorem 3.2.5 does not hold. Hence, it follows from Theorem 3.2.6 that the unique equilibrium* $x^* = (0.6737, 0.6265)^T$ *of (3.30) is globally exponentially stable.*

Example 3.2.9 *Consider the following system of differential equations:*

$$\frac{dx(t)}{dt} = -\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.02 & 0.03 \\ 0.04 & 0.25 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(t)}{4}) \\ \tanh(\frac{x_2(t)}{4}) \end{pmatrix}$$
$$+\begin{pmatrix} 0.25 & 0.4 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(\gamma(t))}{4}) \\ \tanh(\frac{x_2(\gamma(t))}{4}) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.31}$$

One can see easily that the conditions of both Theorem 3.2.5 and Theorem 3.2.6 are satisfied with $L_1^1 = 1/4,\ L_2^1 = 1/4, L_1^2 = 1/4,\ L_2^2 = 1/4,\ m_1 = 3.07,\ m_2 = 0.2750,\ m_3 = 1.4081,\ m_4 = 2.93,\ \lambda = 2.1052,\ \tau = 1.1.$ *Thus, according to Theorem 3.2.5 and Theorem 3.2.6 the unique equilibrium* $x^* = (0.4172, 0.4686)^T$ *of (3.31) is globally exponentially stable.*

3.2.5 Conclusion

In this section, it is the first time that the method of Lyapunov functions for differential equations with piecewise constant argument of generalized type is applied to the model of RNNs and this part has provided new sufficient conditions guaranteeing existence, uniqueness, and global exponential stability of the equilibrium point of the RNNs. In addition, our method gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities since the RNNs model equation involves piecewise constant argument of both advanced and delayed types. The obtained results could be useful in the design and applications of RNNs. Furthermore, the method given in this section may be extended to study more complex systems [20]. On the basis of our results, Lyapunov functions give an opportunity to estimate domains of attraction which allows us particular interest to evaluate the performance of RNNs [54, 60].

3.3 Lyapunov-Razumikhin Technique

In this section, by using the concept of differential equations with piecewise constant arguments of generalized type [3, 4, 5, 6], the model of CNNs [31, 32] is developed. Lyapunov-Razumikhin technique is applied to find sufficient conditions for uniform asymptotic stability of equilibria. Global exponential stability is investigated by means of Lyapunov functions. An example with numerical simulations is worked out to illustrate the results.

3.3.1 Introduction

CNNs are introduced by Chua and Yang in 1988. For a brief summary of the theory and applications of CNNs, the reader is referred to the papers [31, 32]. In recent years, dynamical behavior of delayed cellular neural networks (DCNNs) proposed in 1990 by Chua and Roska [33] has been studied and developed by many authors [41, 42, 43, 45, 46, 47, 48, 94, 95, 96, 97, 98] as well as many applications have been found in different areas such as associative memory, image and signal processing, pattern recognition and so on. As is well known, such applications depend on the existence of an equilibrium point and its stability.

In the literature, there are many papers in which Lyapunov-Krasovskii method [67] has been successfully utilized on the stability analysis of CNNs. But, there are few results on the stability of CNNs [103, 44, 104] based on the Lyapunov-Razumikhin technique [90, 105]. Moreover, it deserves to be mentioned that since differential equations with piecewise constant argument are differential equations with deviated argument of delay or advanced type [6, 73], it is reasonable to use this technique.

The intrinsic idea of this section is that we investigate the problem of stability for CNNs with piecewise constant argument through two approaches based on the Lyapunov-Razumikhin method and Lyapunov functions combined with linear matrix inequality technique [98, 107, 108]. In the first one, we apply proper Razumikhin technique with the peculiarity that conditions on derivative are rather vector-like but not functional. For the second one, we utilize Lyapunov functions, not functionals despite the system is a delay differential equation.

In this section, $\mathbb{N}$ and $\mathbb{R}^+$ are the sets of natural and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^m$ denotes the n dimensional real space. The notation $X > 0$ (or $X < 0$) denotes that X is a symmetric and positive definite (or negative definite) matrix. The notations X^T and X^{-1} refer, respectively, the transpose and the inverse of a square matrix X. $\lambda_{max}(X)$ and $\lambda_{min}(X)$ represent the maximal eigenvalue and minimal eigenvalue of X, respectively. The norm $\|\cdot\|$ means either one-norm: $\|x\|_1 = \sum_{i=1}^{m} |x_i|,\ x \in \mathbb{R}^m$ or the induced matrix 2-norm: $\|X\|_2 = \sqrt{\lambda_{max}(X^T X)}$. $*$ refers to the element below the main diagonal of a symmetric block matrix. Let θ_i, $i \in \mathbb{N}$, denote a fixed real-valued sequence such that $0 = \theta_0 < \theta_1 < ... < \theta_i < ...$ with $\theta_i \to \infty$ as $i \to \infty$.

3.3.2 Model Formulation and Preliminaries

In this section, we will focus our attention on some preliminary results which will be used in the stability analysis of CNNs. First, let us give a general description of the mathematical model of cellular neural networks with piecewise constant argument:

$$x'(t) = -Ax(t) + Bf(x(t)) + Cg(x(\beta(t))) + D \tag{3.32}$$

or equivalently,

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i, \\ & \quad a_i > 0,\ i = 1, 2, \ldots, m. \end{aligned} \tag{3.33}$$

where $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, $t \in \mathbb{R}^+$, $x = [x_1, \ldots, x_m]^T \in \mathbb{R}^m$ is the neuron state vector, $f(x(t)) = [f_1(x_1(t)), \ldots, f_m(x_m(t))]^T$, $g(x(\beta(t))) = [g_1(x_1(\beta(t))), \ldots, g_m(x_m(\beta(t)))]^T \in \mathbb{R}^m$ are the activation functions of neurons, $D = [d_1, \ldots, d_m]^T$ is a constant external input vector. Moreover, we have $A = diag(a_1, \ldots, a_m)$, $B = (b_{ij})_{m\times m}$ and $C = (c_{ij})_{m\times m}$, where B and C denote the connection weight and the delayed connection weight matrices, respectively.

The following assumptions will be needed throughout the section:

(H1) The activation functions $f, g \in C(\mathbb{R}^m)$ with $f(0) = 0,\ g(0) = 0$;

(H2) there exist two Lipschitz constants $L = diag(L_1, \ldots, L_m)$, $\overline{L} = diag(\overline{L}_1, \ldots, \overline{L}_m) > 0$ such that

$$|f_i(u) - f_i(v)| \leq L_i|u - v|,$$

$$|g_i(u) - g_i(v)| \leq \overline{L}_i|u - v|$$

for all $u, v \in \mathbb{R}^m,\ i = 1, 2, \ldots, n$;

(H3) there exists a positive number θ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}$;

(H4) $\theta\,[k_3 + k_2] < 1$;

(H5) $\theta\left[k_2 + k_3\,(1 + \theta k_2)\,e^{\theta k_3}\right] < 1$,

where

$$k_1 = \sum_{i=1}^{m}\sum_{j=1}^{m}|b_{ji}|L_i,\ k_2 = \sum_{i=1}^{m}\sum_{j=1}^{m}|c_{ji}|\overline{L}_i \ \text{ and } k_3 = \sum_{i=1}^{m} a_i + k_1.$$

By a solution of equation (3.32) on $\mathbb{R}^+$ we mean a continuous function $x(t)$ satisfying the conditions (i) the derivative $x'(t)$ exists everywhere with the possible exception of the points θ_i, $i \in \mathbb{N}$, where one-sided derivatives exist; (ii) (3.32) is satisfied on each interval $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$.

In the following theorem, we obtain sufficient conditions for the existence of a unique equilibrium, $x^* = (x_1^*, \ldots, x_m^*)^T$, of (3.33).

Theorem 3.3.1 *Suppose that the neural parameters a_i, b_{ij}, c_{ij} and Lipschitz constants $L_j, \overline{L_j}$ satisfy*

$$a_i > L_i \sum_{j=1}^{m}|b_{ji}| + \overline{L_i}\sum_{j=1}^{m}|c_{ji}|, \quad i = 1, \ldots, m.$$

Then, (3.33) has a unique equilibrium.

The proof of the theorem is almost identical to the verification in [46] with slight changes which are caused by the piecewise constant argument.

Now we need the following lemma which provides conditions for the existence and uniqueness of solutions for arbitrary initial moment ξ.

Lemma 3.3.2 *Assume that conditions* (H1) − (H5) *are fulfilled. Then for all* $x^0 \in \mathbb{R}^m, \theta_r \leq \xi < \theta_{r+1}, r \in \mathbb{N}$, *there exists a unique solution* $\overline{x}(t) = x(t, \theta_r, \overline{x}^0) = (x_1(t), \ldots, x_m(t))^T$ *of (3.33),* $\theta_r \leq t < \theta_{r+1}$, *such that* $\overline{x}(\xi) = x^0$.

Proof. *Existence* : Consider a solution $v(t) = x(t, \xi, x^0) = (v_1(t), \ldots, v_m(t))^T$ of the equation,

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(\zeta_j) + d_i$$

on $[\theta_r, \xi]$. We need to prove that there exists a vector $\zeta = (\zeta_1, \ldots, \zeta_m)^T \in \mathbb{R}^m$ such that the equation

$$v_i(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i v_i(s) + \sum_{j=1}^{m} b_{ij} f_j(v_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(\zeta_j) + d_i \right] ds \qquad (3.34)$$

has a solution on $[\theta_r, \xi]$, and satisfies $v(\theta_r) = \zeta$. Define a norm $\|v(t)\|_0 = \max_{[\theta_r,\xi]} \|v(t)\|$ and construct the following sequences $v_i^n(t)$,
$i = 1, \ldots, m, \ n \geq 0$.

Take $v_i^0(t) \equiv x_i^0, \ i = 1, \ldots, m$, and sequences

$$v_i^{n+1}(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i v_i^n(s) + \sum_{j=1}^{m} b_{ij} f_j(v_j^n(s)) + \sum_{j=1}^{m} c_{ij} g_j(v_j^n(\theta_r)) + d_i \right] ds.$$

One can find that

$$\|v^{n+1}(t) - v^n(t)\|_0 = \max_{[\theta_r,\xi]} \|v^{m+1}(t) - v^m(t)\| \leq (\theta\,(k_3 + k_2))^n\,\kappa,$$

where

$$\kappa = \theta \max_{[\theta_r,\xi]} \left((k_3 + k_2)\,\|x_0\| + \sum_{i=1}^{m} d_i \right).$$

Hence, the sequences $v_i^n(t)$ are convergent and their limits satisfy (3.34) on $[\theta_r, \xi]$ with $\zeta = v(\theta_r)$. The existence is proved.

Uniqueness : It is sufficient to check that for each $t \in [\theta_r, \theta_{r+1})$, and $x^2 = (x_1^2, \ldots, x_m^2)^T, x^1 = (x_1^1, \ldots, x_m^1)^T \in \mathbb{R}^m, x^2 \neq x^1$, the condition $x(t, \theta_r, x^1) \neq x(t, \theta_r, x^2)$ is valid. Let us denote solutions of (3.33) by $x^1(t) = x(t, \theta_r, x^1)$, $x^2(t) = x(t, \theta_r, x^2), x^1 \neq x^2$. Assume on the contrary that there exists $t^* \in [\theta_r, \theta_{r+1})$ such that $x^1(t^*) = x^2(t^*)$. Then, we have

$$\begin{aligned} x_i^2 - x_i^1 &= \int_{\theta_r}^{t^*} \Bigg[-a_i \left(x_i^2(s) - x_i^1(s) \right) + \sum_{j=1}^{m} b_{ij} [f_j(x_j^2(s)) - f_j(x_j^1(s))] \\ &+ \sum_{j=1}^{m} c_{ij} [g_j(x_j^2(\theta_r) - g_j(x_j^1(\theta_r))] \Bigg] ds, \quad i = 1, \ldots, m. \end{aligned}$$

Taking the absolute value of both sides for each $i = 1, \ldots, m$ and adding all equalities, we obtain that

$$\begin{aligned}
\|x^2 - x^1\| &= \sum_{i=1}^{m} \left| \int_{\theta_r}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s) \right) + \sum_{j=1}^{m} b_{ij} \left[f_j(x_j^2(s)) - f_j(x_j^1(s)) \right] \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{m} c_{ij} \left[g_j(x_j^2(\theta_r) - g_j(x_j^1(\theta_r)) \right] \right] ds \right| \\
&\leq \sum_{i=1}^{m} \left\{ \int_{\theta_r}^{t^*} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} L_i |b_{ji}| |x_i^2(s) - x_i^1(s)| \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{m} \bar{L}_i |c_{ji}| |x_i^2 - x_i^1| \right] ds \right\} \\
&\leq \theta k_2 \|x^1 - x^2\| + \int_{\theta_r}^{t^*} k_3 \|x^1(s) - x^2(s)\| ds. \qquad (3.35)
\end{aligned}$$

Furthermore, for $t \in [\theta_r, \theta_{r+1})$, the following is valid:

$$\begin{aligned}
\|x^1(t) - x^2(t)\| &\leq \|x^1 - x^2\| + \sum_{i=1}^{m} \left\{ \int_{\theta_r}^{t^*} \left[a_i |x_i^2(s) - x_i^1(s)| \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{m} L_i |b_{ji}| |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} \bar{L}_i |c_{ji}| |x_i^2 - x_i^1| \right] ds \right\} \\
&\leq (1 + \theta k_2) \|x^1 - x^2\| + \int_{\theta_r}^{t^*} k_3 \|x^1(s) - x^2(s)\| ds.
\end{aligned}$$

The Gronwall-Bellman lemma yields that

$$\|x^1(t) - x^2(t)\| \leq (1 + \theta k_2)\, e^{\theta k_3} \|x^1 - x^2\|. \qquad (3.36)$$

Consequently, substituting (3.36) in (3.35), we obtain

$$\|x^1 - x^2\| \leq \theta \left[k_2 + k_3 (1 + \theta k_2)\, e^{\theta k_3} \right] \|x^1 - x^2\|. \qquad (3.37)$$

Thus, one can see that (H5) contradicts with (3.37). The lemma is proved. □

Theorem 3.3.3 *Suppose that conditions* (H1) – (H5) *are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^m$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^T,\ t \in \mathbb{R}^+$, *of (3.32), such that* $x(t_0) = x^0$.

Proof. We prove the theorem only for incresing t, but one can easily see that the proof is similiar for decreasing t. It is clear that there exists $r \in \mathbb{N}$ such that $t_0 \in [\theta_r, \theta_{r+1})$. Using Lemma

3.3.2 for $\xi = t_0$, there exists a unique solution $x(t) = x(t, t_0, x^0)$ of (3.33) on $[\theta_r, \theta_{r+1})$. Next, applyig the lemma again, one can obtain a unique solution on interval $[\theta_{r+1}, \theta_{r+2})$. Hence, the mathematical induction completes the proof. □

Consider the equilibrium point, $x^* = (x_1^*, \ldots, x_m^*)^T$, of the system (3.32). Let us give the following definitions, which are adopted for the system (3.32).

Definition 3.3.4 *[6] The equilibrium $x = x^*$ of* (3.32) *is said to be uniformly stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(t_0) - x^*\| < \delta$ implies $\|x(t) - x^*\| < \varepsilon$ for all $t \geq t_0$.*

Definition 3.3.5 *[6] The equilibrium $x = x^*$ of* (3.32) *is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t) - x^*\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x(t_0) - x^*\| < \delta_0$.*

Definition 3.3.6 *[6] The equilibrium $x = x^*$ of* (3.32) *is said to be globally exponentially stable if there exist positive constants α_1 and α_2 such that the estimation of the inequality $\|x(t) - x^*\| < \alpha_1 \|x(t_0) - x^*\| e^{-\alpha_2(t-t_0)}$ is valid for all $t \geq t_0$.*

By means of the transformation $y(t) = x(t) - x^*$, system (3.32) can be simplified as

$$y'(t) = -Ay(t) + B\varphi(y(t)) + C\psi(y(\beta(t))), \tag{3.38}$$

where $\varphi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$ and $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*)$ with $\varphi_j(0) = \psi_j(0) = 0$. From assumption (H2), we have $\varphi_j(\cdot)$ and $\psi_j(\cdot)$ are also Lipschitzian with $L_j, \bar{L}_j$, respectively.

It is obvious that the stability of the zero solution of (3.38) is equivalent to that of the equilibrium x^* of (3.32). Therefore, in what follows, we discuss the stability of the zero solution of (3.38).

To begin with, we introduce the following lemmas which will be used in the proof of the stability of the zero solution for CNNs with piecewise constant argument.

Lemma 3.3.7 *[109] Given any real matrices U, W, Z of appropriate dimensions and a scalar $\epsilon > 0$ such that $0 < W = W^T$, then the following matrix inequality holds:*

$$U^T Z + Z^T U \leq \epsilon U^T W U + \frac{1}{\epsilon} Z^T W^{-1} Z.$$

The following lemma is an important auxiliary result of the section. It can be proved in the same way used for Theorem 2.2 in [4].

Lemma 3.3.8 *Let* $y(t) = (y_1(t), \ldots, y_m(t))^T$ *be a solution of (3.38) and* (H1) − (H5) *be satisfied. Then, the following inequality*

$$\|y(\beta(t))\| \leq l\|y(t)\| \tag{3.39}$$

holds for all $t \in \mathbb{R}^+$, *where* $l = \left\{1 - \theta\left[k_2 + k_3\,(1 + \theta k_2)\, e^{\theta k_3}\right]\right\}^{-1}$.

For convenience, we adopt the following notation in the sequel:

(N) Given $P > 0$, positive diagonal matrices R, S with appropriate dimensions and a real $q > 1$, denote

$$\Omega = PBR^{-1}B^TP + LRL + PCS^{-1}C^TP + qP - AP - PA,$$

or, by Schur complements, it can be rewritten as the following matrix form:

$$-\Omega = \begin{bmatrix} AP + PA - LRL - qP & PB & PC \\ * & R & 0 \\ * & * & S \end{bmatrix},$$

where $L = diag(L_1, \ldots, L_m) > 0$.

We shall consider the quadratic function $V(y) = y^TPy$. The derivative of V with respect to system (3.38) is defined by

$$V'(y, z) = -y^T(AP + PA)y + 2y^TPB\varphi(y) + 2y^TPC\psi(z) \text{ for } y\,, z \in \mathbb{R}^m.$$

3.3.3 Lyapunov-Razumikhin Technique

From now on, we shall need the following assumptions:

(C1) $\Omega < 0$;

(C2) $P > \overline{L}S\,\overline{L}$ where $\overline{L} = diag(\overline{L}_1, \ldots, \overline{L}_m) > 0$.

Lemma 3.3.9 *Assume that conditions* (C1) – (C2) *are fulfilled, and* $y(t) : \mathbb{R}^+ \to \mathbb{R}^m$ *is a solution of (3.38). Then the following conditions hold for* $V(y(t)) = y^T(t)Py(t)$*:*

(1a) $a\,||y(t)||^2 \le V(y(t)) \le b\,||y(t)||^2$, *where* $a = \lambda_{min}(P)$ *and* $b = \lambda_{max}(P)$;

(1b) $V'(y(t), y(\beta(t))) \le -c\,||y(t)||^2$ *for all* $t \ne \theta_i$ *in* $\mathbb{R}^+$ *such that* $V(y(\beta(t))) < qV(y(t))$ *with a constant* $c > 0$.

Proof. It is obvious that $a\,||y(t)||^2 \le V(y(t)) \le b\,||y(t)||^2$, where $a = \lambda_{min}(P)$ and $b = \lambda_{max}(P)$.

For $t \ne \theta_i,\ i \in \mathbb{N}$, the derivative of $V(y(t))$ along the trajectories of system (3.38) is given by

$$\begin{aligned} V'(y(t), y(\beta(t))) &= y'^T(t)Py(t) + y^T(t)Py'(t) \\ &= -y^T(t)(AP + PA)y(t) + 2y^T(t)PB\varphi(y(t)) \\ &\quad +2y^T(t)PC\psi(y(\beta(t))). \end{aligned} \tag{3.40}$$

Let $U = B^T Py(t),\ Z = \varphi(y(t))$. By applying Lemma 3.3.7, we have the following inequality:

$$\begin{aligned} 2y^T(t)PB\varphi(y(t)) &= y^T(t)PB\varphi(y(t)) + \varphi^T(y(t))B^T Py(t) \\ &\le y^T(t)PBR^{-1}B^T Py(t) + \varphi^T(y(t))R\varphi(y(t)) \\ &\le y^T(t)\left(PBR^{-1}B^T P + LRL\right)y(t), \end{aligned} \tag{3.41}$$

since $\varphi^T(y(t))R\varphi(y(t)) \le y^T LRLy(t)$.

Similarly, we have

$$2y^T(t)PC\psi(y(\beta(t))) \le y^T(t)PCS^{-1}C^T Py(t) + y^T(\beta(t))\bar{L}S\bar{L}y(\beta(t)), \tag{3.42}$$

since $\psi^T(y(\beta(t)))S\psi(y(\beta(t))) \le y^T(\beta(t))\bar{L}S\bar{L}y(\beta(t))$.

Substituting (3.41) and (3.42) into (3.40) and using condition (C2), we have

$$\begin{aligned} V'(y(t), y(\beta(t))) \le\ & y^T(t)\left(PBR^{-1}B^T P + LRL + PCS^{-1}C^T P - AP\right. \\ & \left. -PA\right)y(t) + y^T(\beta(t))Py(\beta(t)). \end{aligned}$$

Then, one can conclude that

$$V'(y(t), y(\beta(t))) \le y^T(t)\Omega y(t), \quad t \ne \theta_i \tag{3.43}$$

whenever $y^T(\beta(t))Py(\beta(t)) = V(y(\beta(t))) < qV(y(t)) = y^T(t)qPy(t)$.

It follows from the condition (C1) in terms of Schur complements given in (N) and (3.43) that (1b) is valid. □

From (1a) and (1b) of the last lemma, it implies that V can be taken as a Lyapunov function for system (3.38). Now, we are ready to give sufficient conditions for uniform asymptotic stability of (3.38). To prove the following theorem we shall use the technique which was developed in paper [17].

Theorem 3.3.10 *Suppose that* (H1) – (H5) *and* (C1) – (C2) *hold true, then the equilibrium* x^* *of (3.32) is uniformly asymptotically stable.*

Proof. Fix $h_1 > 0$. Given $\varepsilon > 0$, ($\varepsilon < h_1$), we choose $\delta_1 > 0$ such that $b\delta_1^2 \leq a\varepsilon^2$. Define $\delta = \delta_1/l$ and note that $\delta < \delta_1$ as $l > 1$. We first prove uniform stability when $t_0 = \theta_j$ for some $j \in \mathbb{N}$ and then for $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$, to show that this δ is the needed one in both cases.

If $t_0 = \theta_j$, where $j \in \mathbb{N}$ and $\left\|y(\theta_j)\right\| < \delta$, then $V(y(\theta_j)) < b\delta^2 < b\delta_1^2 \leq a\varepsilon^2$.

We fix $k \in \mathbb{N}$ and consider the interval $[\theta_k, \theta_{k+1})$. Using (1b) in Lemma 3.3.9, we shall show that

$$V(y(t)) \leq V(y(\theta_k)) \ \text{ for } \ t \in [\theta_k, \theta_{k+1}). \tag{3.44}$$

Let us set $v(t) = V(y(t))$. If (3.44) does not hold, then there must exist points η and ρ satisfying $\theta_k \leq \eta < \rho < \theta_{k+1}$ and

$$v(\eta) = v(\theta_k)\ ,\ \ v(t) > v(\theta_k) \ \text{ for } \ t \in (\eta, \rho].$$

Based on the mean value theorem, we can find a $\zeta \in (\eta, \rho)$ satisfying the equation $\dfrac{v(\rho) - v(\eta)}{\rho - \eta} = v'(\zeta) > 0$.

Actually, since $v(\theta_k) < v(\zeta) < qv(\zeta)$, it follows from (1$b$) that $v'(\zeta) < 0$, a contradiction. Hence, (3.44) is true. As the functions V and y are continuous, one can obtain by induction that $V(y(t)) \leq V(y(\theta_j))$ for all $t \geq \theta_j$. Thus, we have $a\left\|y(t)\right\|^2 \leq V(y(t)) \leq V(y(\theta_j)) < a\varepsilon^2$, which implies in turn that $\|y(t)\| < \varepsilon$ for all $t \geq \theta_j$. We see that evaluation of δ does not depend on the choice of $j \in \mathbb{N}$.

Now, consider the case $t_0 \in \mathbb{R}^+$ with $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$. Then there exists $j \in \mathbb{N}$ such that $\theta_j < t_0 < \theta_{j+1}$. For a solution $y(t)$ satisfying $\|y(t_0)\| < \delta$, Lemma 3.3.8 implies that $\left\|y(\theta_j)\right\| < \delta_1$. Using a similar idea used for the case $t_0 = \theta_j$, we conclude that $\|y(t)\| < \varepsilon$ for $t \geq \theta_j$ and hence for all $t \geq t_0$, which completes the proof for the uniform stability. We note that the evaluation is independent of $j \in \mathbb{N}$ and correspondingly it is valid for all $t_0 \in \mathbb{R}^+$. Next we shall prove uniform asymptotic stability.

First, we show "uniform" asymptotic stability with respect to all elements of the sequence θ_i, $i \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. For $t_0 = \theta_j$, we choose $\delta > 0$ such that $b(l\delta)^2 = ah_1^2$ holds. In view of uniform stability, one can obtain that $V(y(t)) < b\delta^2 < b(l\delta)^2$ for all $t \geq \theta_j$ and hence $\|y(t)\| < h_1$ whenever $\left\|y(\theta_j)\right\| < \delta$. In what follows, we present that this δ can be taken as δ_0 in the Definition 3.3.5. That is to say, given $\varepsilon > 0$, $\varepsilon < h_1$, we need to show that there exists a $T = T(\varepsilon) > 0$ such that $\|y(t)\| < \varepsilon$ for $t > \theta_j + T$ if $\left\|y(\theta_j)\right\| < \delta$.

We denote $\gamma = \frac{ac}{b}\varepsilon^2$ and $\delta_1 = l\delta$. We can find a number $\mu > 0$ such that $qs > s + \mu$ for $a\varepsilon^2 \leq s \leq b\delta_1^2$. Let M be the smallest positive integer such that $a\varepsilon^2 + M\mu \geq b\delta_1^2$. Choosing $t_k = k(\dfrac{b\delta_1^2}{\gamma} + \theta) + \theta_j$, $k = 1, 2, ..., M$, we aim to prove that

$$V(y(t)) \leq a\varepsilon^2 + (M - k)\mu \;\text{ for }\; t \geq t_k, \;\; k = 0, 1, 2, ..., M. \tag{3.45}$$

It is easily seen that $V(y(t)) < b\delta_1^2 \leq a\varepsilon^2 + M\mu$ for $t \geq t_0 = \theta_j$. Hence, (3.45) is true for $k = 0$. Now, assuming that (3.45) is true for some $0 \leq k < M$, we will show that $V(y(t)) \leq a\varepsilon^2 + (M - k - 1)\mu$ for $t \geq t_{k+1}$. To prove the last inequality, we first claim that there exists a $t^* \in I_k = [\beta(t_k) + \theta, t_{k+1}]$ such that

$$V(y(t^*)) \leq a\varepsilon^2 + (M - k - 1)\mu. \tag{3.46}$$

Otherwise, $V(y(t)) > a\varepsilon^2 + (M - k - 1)\mu$ for all $t \in I_k$. On the other side, we have $V(y(t)) \leq a\varepsilon^2 + (M - k)\mu$ for $t \geq t_k$, which implies that $V(y(\beta(t))) \leq a\varepsilon^2 + (M - k)\mu$ for $t \geq \beta(t_k) + \theta$. Hence, for $t \in I_k$

$$qV(y(t)) > V(y(t)) + \mu > a\varepsilon^2 + (M - k)\mu \geq V(y(\beta(t))).$$

Since $a\varepsilon^2 \leq V(y(t)) \leq b\,\|y(t)\|^2$ for $t \in I_k$, it follows from (1b) that

$$V'(y(t), y(\beta(t))) \leq -c\,\|y(t)\|^2 \leq -\gamma \text{ for all } t \neq \theta_i \text{ in } I_k.$$

Using the continuity of the function V and the solution $y(t)$, we get

$$\begin{aligned} V(y(t_{k+1})) &\leq V(y(\beta(t_k) + \theta)) - \gamma(t_{k+1} - \beta(t_k) - \theta) \\ &< b\delta_1^2 - \gamma(t_{k+1} - t_k - \theta) = 0, \end{aligned}$$

which is a contradiction. Thus (3.46) holds true. Next, we show that

$$V(y(t)) \leq a\varepsilon^2 + (M - k - 1)\mu \text{ for all } t \in [t^*, \infty). \tag{3.47}$$

If (3.47) does not hold, then there exists a $\bar{t} \in (t^*, \infty)$ such that

$$V(y(\bar{t})) > a\varepsilon^2 + (M - k - 1)\mu \geq V(y(t^*)).$$

Thus, we can find a $\tilde{t} \in (t^*, \bar{t})$ such that $\tilde{t} \neq \theta_i$, $i \in \mathbb{N}$, $V'(y(\tilde{t}), y(\beta(\tilde{t}))) > 0$ and $V(y(\tilde{t})) > a\varepsilon^2 + (M - k - 1)\mu$. However,

$$qV(y(\tilde{t}))) > V(y(\tilde{t})) + \mu > a\varepsilon^2 + (M - k)\mu \geq V(y(\beta(\tilde{t})))$$

implies that $V'(y(\tilde{t}), y(\beta(\tilde{t}))) \leq -\gamma < 0$, a contradiction. Then, we conclude that $V(y(t)) \leq a\varepsilon^2 + (M - k - 1)\mu$ for all $t \geq t^*$ and thus for all $t \geq t_{k+1}$. This completes the induction and shows that (3.45) is valid. For $k = M$, we have

$$V(y(t)) \leq a\varepsilon^2, \; t \geq t_M = M(\frac{b\delta_1^2}{\gamma} + \theta) + t_0.$$

In the end, $\|y(t)\| < \varepsilon$ for $t > \theta_j + T$ where $T = M(\frac{b\delta_1^2}{\gamma} + \theta)$, which proves uniform asymptotic stability for $t_0 = \theta_j$, $j \in \mathbb{N}$.

Take $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$. Then $\theta_j < t_0 < \theta_{j+1}$ for some $j \in \mathbb{N}$. $\|y(t_0)\| < \delta$ implies by Lemma 3.3.8 that $\left\|y(\theta_j)\right\| < \delta_1$. Hence, the argument used for the case $t_0 = \theta_j$ yields that $\|y(t)\| < \varepsilon$ for $t > \theta_j + T$ and so for all $t > t_0 + T$. □

3.3.4 Method of Lyapunov Functions

In this part, Lyapunov-Krasovskii method is used for equation (3.38), which is a delay differential equation, but one must emphasize that Lyapunov functions, not functionals, are used.

In the following condition, the matrices A, B, C, P, R, S, L are described as in (N).

(C3) $\bar{\Omega} = PBR^{-1}B^TP + LRL + PCS^{-1}C^TP + bl^2\kappa P - AP - PA < 0$, where κ is a constant with $\kappa a \geq 1$.

Lemma 3.3.11 *Assume that conditions* (C2) – (C3) *are fulfilled, and* $y(t)$ *is a solution of (3.38). Then the following conditions hold for the quadratic function* $V(y(t)) = y^T(t)Py(t)$*:*

(2a) $a\,\|y(t)\|^2 \le V(y(t)) \le b\,\|y(t)\|^2$, *where* $a = \lambda_{min}(P)$ *and* $b = \lambda_{max}(P)$;

(2b) $V'(y(t), y(\beta(t))) \le -c\,\|y(t)\|^2$ *for all* $t \ne \theta_i$ *in* $\mathbb{R}^+$ *with a constant* $c > 0$.

Proof. It is easily seen that $a\,\|y(t)\|^2 \le V(y(t)) \le b\,\|y(t)\|^2$, where $a = \lambda_{min}(P)$ and $b = \lambda_{max}(P)$.

It follows from Lemma 3.3.8 that $V(y(\beta(t))) \le b\|y(\beta(t))\|^2 \le bl^2\|y(t)\|^2 \le bl^2\kappa a\|y(t)\|^2 \le bl^2\kappa V(y(t))$.

For $t \ne \theta_i$, $i \in \mathbb{N}$, we know from the proof of Lemma 3.3.9 that the derivative of $V(y(t))$ along the trajectories of system (3.38) satisfies

$$\begin{aligned} V'(y(t), y(\beta(t))) \quad &\le \quad y^T(t)\left(PBR^{-1}B^TP + LRL + PCS^{-1}C^TP - AP\right. \\ &\qquad \left.-PA\right)y(t) + y^T(\beta(t))Py(\beta(t)). \end{aligned}$$

Hence, we get

$$V'(y(t), y(\beta(t))) \le y^T(t)\bar{\Omega}y(t), \quad t \ne \theta_i. \tag{3.48}$$

It follows from the condition (C3) and (3.48) that $(2b)$ is valid. □

Theorem 3.3.12 *Suppose that* (H1) – (H5) *and* (C2) – (C3) *hold true, then the equilibrium* x^* *of (3.32) is globally exponentially stable.*

Proof. Using Lemma 3.3.11, we have for $t \ne \theta_i$

$$\begin{aligned} \frac{d}{dt}(e^{(c/b)t}V(y(t))) \quad &= \quad e^{(c/b)t}(c/b)V(y(t)) + e^{(c/b)t}V'(y(t), y(\beta(t))) \\ &\le \quad ce^{(c/b)t}\,\|y(t)\|^2 - ce^{(c/b)t}\,\|y(t)\|^2 = 0. \end{aligned}$$

Using the continuity of the function V and the solution $y(t)$, we obtain

$$e^{(c/b)t}a\,\|y(t)\|^2 \le e^{(c/b)t}V(y(t)) \le e^{(c/b)t_0}V(y(t_0)) \le e^{(c/b)t_0}b\,\|y(t_0)\|^2,$$

which implies that $\|y(t)\| \le \sqrt{\frac{b}{a}}\,\|y(t_0)\|\,e^{-(c/2b)(t-t_0)}$. The theorem is proved. □

3.3.5 An Illustrative Example

Consider the following CNNs with piecewise constant argument:

$$\begin{aligned}\frac{dx(t)}{dt} &= -\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.5 & 0 \\ 0.1 & 0.3 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(t)}{2}) \\ \tanh(\frac{x_2(t)}{2}) \end{pmatrix} \\ &+ \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}\begin{pmatrix} \tanh(\frac{x_1(\beta(t))}{2}) \\ \tanh(\frac{x_2(\beta(t))}{3}) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \qquad (3.49)\end{aligned}$$

Clearly, we obtain

$$L = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \ \bar{L} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Let

$$\begin{aligned} P &= \begin{pmatrix} 1.5 & 1 \\ 1 & 1.5 \end{pmatrix}, \ R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \ S = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \ q = 1.2, \\ \beta(t) &= \theta_i = \frac{i}{10}, \ i \in \mathbb{N}. \end{aligned}$$

By simple calculation, we can check that $k_1 = 0.45,\ k_2 = 0.4333,\ k_3 = 4.45,\ a = \lambda_{min}(P) = 0.5,\ b = \lambda_{max}(P) = 2.5$ and $l = 4.81$. We can choose $\kappa = 2.1$ so that $\kappa a > 1$. It follows from Theorem 3.2.1 that there exists a unique equilibrium such that $x^* = [0.7255, 1.1898]^T$. Then it can be easily verified that

$$\Omega = \begin{pmatrix} -2.9479 & -2.3708 \\ -2.3708 & -3.0604 \end{pmatrix} < 0, \quad P - \bar{L}S\bar{L} = \begin{pmatrix} 0.5 & 1 \\ 1 & 1.0556 \end{pmatrix} > 0.$$

For $\theta = 1/10$, we get $\theta\,[k_3 + k_2] = 0.4883 < 1$ and $\theta\left[k_2 + k_3\,(1 + \theta k_2)\,e^{\theta k_3}\right] = 0.7921 < 1$. So, (H1)-(H5) and (C1)-(C2) hold. Thus, the conditions of the Theorem 3.3.10 for $q = 1.2$ are satisfied. Hence, (3.49) has a uniformly asymptotically stable equilibrium point. However, for the same q we have $q < bl^2\kappa$. Hence, Theorem 3.3.12 is not applicable. That is, using Lyapunov-Razumikhin technique, we may take smaller q values, and that verifies it as more effective in theoretical sense. Nevertheless, the second theorem allows us to obtain exponential evaluation of convergence to the equilibrium, which has a very important peculiarity for applications in practice.

The simulation, where the initial value is chosen as $[1, 1.5]^T$, is shown in Fig. 3.3 and it illustrates that all trajectories uniformly converge to the unique asymptotically stable equilibrium point x^*.

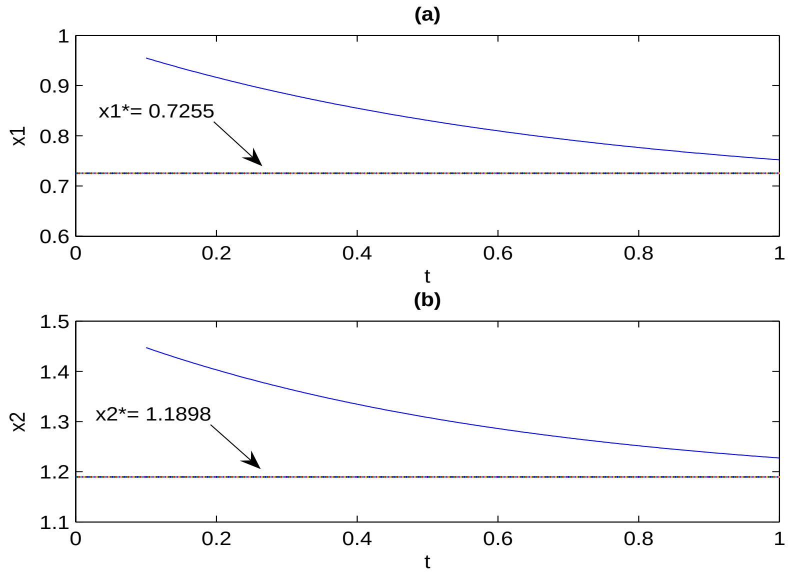

Figure 3.3: Time response of state variables $x_1(t)$ and $x_2(t)$ with piecewise constant arguments in (a) and (b), respectively.

3.3.6 Conclusion

In this section, it is the first time that CNNs with piecewise constant argument of generalized type are investigated. There is not a restriction on the distance between switching neighbors of the argument function and the stability is discussed in the uniform version. The analysis has been available after a new approach was proposed in [4, 5, 6]. It gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities to conjugate with numerical analysis, and take into account the easiness of simulations simplified by the constancy of the argument.

Moreover, comparing two main results of this section, one can see that Theorem 3.3.10 allows to analyze a larger class of equations than Theorem 3.3.12. At the same time, on the basis of Theorem 3.3.12, one can evaluate convergence of solutions to equilibria. Application of Lyapunov functions gives an opportunity to develop further quantitative analysis such as estimation of the domain of attraction, etc.

CHAPTER 4

CONCLUSION AND FUTURE WORKS

This thesis is dedicated to not only the mathematical analysis of RNNs and impulsive RNNs with piecewise constant argument of generalized type but also the problem of stability for differential equations with piecewise constant argument of generalized type through the method of Lyapunov functions. It is the first time that RNNs and impulsive RNNs with piecewise constant argument of generalized type are investigated.

In Chapter 2, we obtain sufficient conditions for the existence of a unique equilibrium and a periodic solution and investigate the stability of RNNs with piecewise constant argument of generalized type. For an impulsive RNNs with piecewise constant argument of generalized type, we introduce two different types of impulsive RNNs; $(\theta, \theta)-$ type neural networks and $(\theta, \tau)-$ type neural networks. For these types, sufficient conditions for the existence of the unique equilibrium are obtained, existence and uniqueness of solutions and the equivalence lemma for such systems are established and stability criterion for the equilibrium based on linear approximation is proposed. In addition to these qualitative analysis, by employing Green's function we derive new result of existence of the periodic solution and the global asymptotic stability of this solution is investigated. Finally, examples with numerical simulations are given to validate our theoretical results.

In Chapter 3, the problem of stability for differential equations with piecewise constant argument of generalized type through the method of Lyapunov functions is investigated. Moreover, Chapter 3 analyzes the problem of stability for neural networks with piecewise constant argument based on the Second Lyapunov method. That is, we use the method of Lyapunov functions and Lyapunov-Razumikhin technique for the stability of RNNs and CNNs, respectively. It is the first time that the method of Lyapunov functions for differential equations with piecewise constant argument of generalized type [10] is applied to the model of RNNs and this part has provided new sufficient conditions guaranteeing existence, uniqueness, and global exponential stability of the equilibrium point of the RNNs. In addition, our method gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities since the RNNs model equation involves piecewise constant argument of both advanced and delayed types. In the last part of Chapter 3, by using the concept of differential equations with piecewise constant arguments of generalized type [2, 3, 4, 5, 6, 17], the model of CNNs is developed. Lyapunov-Razumikhin technique is applied to find sufficient conditions for uniform asymptotic stability of equilibria. Global exponential stability is investigated by means of Lyapunov functions. It gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities to conjugate with numerical analysis, and take into account the easiness of simulations simplified by the constancy of the argument. Application of Lyapunov functions gives an opportunity to develop further quantitative analysis such as estimation of the domain of attraction, etc [54, 60]. Examples with numerical simulations are also given in Chapter 3 to illustrate our theoretical results.

Our approaches developed in papers [10, 11, 12, 13, 14, 15, 16] and based on the methods of analysis for differential equations with discontinuities can be effectively applied to almost all problems concerning neural networks models, including bidirectional associative memory (BAM) neural networks model first introduced by Kosko [162, 163, 164], Cohen-Grossberg neural networks, weakly connected neural networks [149], etc. Exceptionaly, it concerns those problems which relate state-dependent discontinuity [1, 2, 18, 19, 23, 24]. Let us list fields where the activity can be realized, immediately:

- Since these networks have ability to learn, the method under investigation can be applied to learning theory related to an unsupervised Hebbian-type learning mechanism with/without a forgetting term [148, 158, 159, 160] and several learning algorithms modeled by Amari [157] connected to proposal of Hebb [158]. Unsupervised, or self-organized learning means that there is no external teacher to manage the learning process, shown in Fig.4.1.

- It is interesting to study chaos [20, 21, 22, 165, 166, 167, 168, 169, 170] and control of chaos [171, 172, 173] in neural networks models.

- The results in this thesis will be useful for synchronization-desynchronization problems

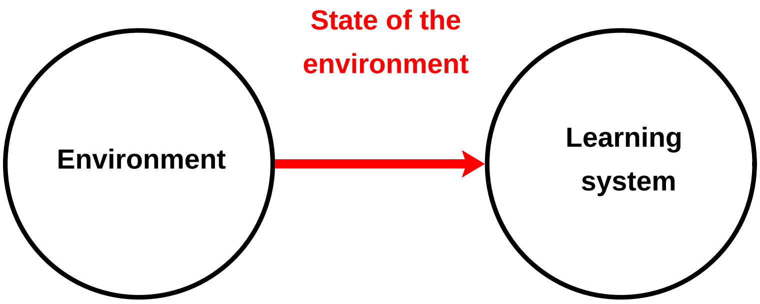

Figure 4.1: Unsupervised Learning

[126, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187] and the references cited therein.

- Neural networks is also widely used in Artificial intelligence [153, 154]. We are sure that the methods established in this thesis will be useful for this subject.

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PUBLICATIONS

A. Papers published in International Journals:

A1. M. U. Akhmet, E. Yılmaz, Neural networks with non-smooth and impact activations, (Revised version is submitted).

A2. M. U. Akhmet, D. Aruğaslan, E. Yılmaz, Method of Lyapunov functions for differential equations with piecewise constant delay, J. Comput. Appl. Math., 235, pp. 4554-4560, 2011.

A3. M. U. Akhmet, D. Aruğaslan, E. Yılmaz, Stability in cellular neural networks with a piecewise constant argument, J. Comput. Appl. Math., 233, pp. 2365-2373, 2010.

A4. M. U. Akhmet, E. Yılmaz, Impulsive Hopfield-type neural network system with piecewise constant argument, Nonlinear Anal: Real World Applications, 11, pp. 2584-2593, 2010.

A5. M. U. Akhmet, D. Aruğaslan, E. Yılmaz, Stability analysis of recurrent neural networks with piecewise constant argument of generalized type, Neural Networks, 23, pp. 305-311, 2010.

A6. M. U. Akhmet, E. Yılmaz, Hopfield-type neural networks systems equations with piecewise constant argument. International Journal of Qualitative Theory of Differential Equations and Applications, 3, no. 1-2, pp. 8-14, 2009.

B. Papers published in International Conference Proceedings:

B1. M. U. Akhmet, E. Yılmaz, Global attractivity in impulsive neural networks with piecewise constant delay. Proceedings of Neural, parallel, and scientific computations, Dynamic Publishers, Inc, USA, pp. 11-18, 2010.

C. Presentations in International Conferences:

C1. E. Yılmaz, Global Attractivity in impulsive neural networks with piecewise constant delay, Invited speaker of The Fourth International Conference on Neural, Parallel and Scientific Computations, Morehouse College, Atlanta, Georgia, August 11-14, 2010.

C2. E. Yılmaz, Stability analysis of recurrent neural networks with deviated argument of mixed type, 14th International Congress on Computational and Applied Mathematics, Antalya, Turkey, September 28- October 2, 2009.

C3. E. Yılmaz, Lyapunov functions for differential equations with piecewise constant delay and its application to neural networks model, International Workshop on Resonance Oscillations and Stability of Nonsmooth Systems, Imperial College, London, UK, June 16-25, 2009.

C4. E. Yılmaz, H. Kalkan, Increasing the Motivation of SMEs for Project Development, Innovative Approaches to University & Small and Medium Enterprises (SMEs) Cooperation, Siauliai, Lithuania, March 19-20, 2009.

C5. H. Kalkan, E. Yılmaz, An SME Oriented Course Design for Information System Development, Innovative Approaches to University & Small and Medium Enterprises (SMEs) Cooperation, Siauliai, Lithuania, March 19-20, 2009.

C6. E. Yılmaz, Impulsive Hopfield- type neural networks system with piecewise constant argument, International Conference on Differential and Difference Equations, Veszprem, Hungary, July 14-17, 2008.

D. Presentations in National Conferences:

D1. E. Yılmaz, Impulsive Hopfield- type neural networks system with piecewise constant argument, Ankara Mathematics Days Symposium, Ankara, Turkey, May 22-23, 2008.

E. Seminars in Applied Dynamics Group:

E1. E. Yılmaz, Analysis of Neural Networks with Piecewise Constant Argument of Generalized Type, Applied Dynamics Group Seminar, Institute of Applied Mathematics, METU, March 19, 2010.

E2. E. Yılmaz, Mathematical Bases of Neural Network Theory, Applied Dynamics Group Seminar, Institute of Applied Mathematics, METU, November 29, 2007.

E3. E. Yılmaz, Modeling of Hopfield Neural Networks, Applied Dynamics Group Seminar, Institute of Applied Mathematics, METU, November 22, 2007.

AWARDS, FELLOWSHIPS AND SCHOLARSHIPS

F. Awards:

F1. Encouragement award for international scientific publications for the paper “Method of Lyapunov functions for differential equations with piecewise constant delay” given by Turkish Academic Network and Information Center (ULAKBIM).

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PROJECTS

G. Projects:

G1. BAP Project, Complex Dynamics of Neural Networks, Project No: BAP-07-05-2008-00-03, December 2007 - December 2010.

G2. CFCU Project, Employment of Young Assistance Health Staff in Ankara, Project No: TR0602.03-02/647, January 2008 - January 2009.