PORTFOLIO INSURANCE STRATEGIES

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ABSTRACT

PORTFOLIO INSURANCE STRATEGIES

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The selection of investment strategies and managing investment funds via employing portfolio insurance methods play an important role in asset liability management. Insurance strategies are designed to limit downside risk of portfolio while allowing some participation in potential gain of upside markets. In this thesis, we provide an extensive overview and investigation, particularly on the two most prominent portfolio insurance strategies: the Constant Proportion Portfolio Insurance (CPPI) and the Option-Based Portfolio Insurance (OBPI).

The aim of the thesis is to examine, analyze and compare the portfolio insurance strategies in terms of their performances at maturity, via some of their statistical and dynamical properties, and of their optimality over the maximization of expected utility criterion.

This thesis presents the financial market model in continuous-time containing no arbitrage opportunities, the CPPI and OBPI strategies with definitions and properties, and the analysis of these strategies in terms of comparing their performances at maturity, of their statistical properties and of their dynamical behaviour and sensitivities to the key parameters during the investment period as well as at the terminal date, with both formulations and simulations. Therefore, we investigate and compare optimal portfolio strategies which maximize the expected utility criterion. As a contribution on the optimality results existing in the literature, an extended study is provided by proving the existence and uniqueness of the appropriate number of shares invested in the unconstrained allocation in a wider interval.

Keywords: Portfolio insurance strategies: CPPI and OBPI, portfolio optimization, Merton optimization problem, financial market model in continuous-time, CRRA utility function.

ÖZ

PORTFÖY SİGORTA STRATEJİLERİ

Güleroğlu, Çiğdem
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Portföy sigorta metodları kullanılarak yatırım stratejilerinin seçimi ve yatırım sermayelerinin yönetimi, aktif-pasif yönetimi alanında önemli bir rol oynamaktadır. Portföy sigorta stratejileri, düşen piyasalarda portföyün zarar riskini sınırlarken, yükselen piyasalardaki kazanç potansiyelinden de yararlanmaya olanak sağlayacak biçimde dizayn edilmiştir. Bu tezde, portföy sigorta stratejilerine, özellikle de en önemli ve göze çarpan iki yöntem olan Sabit Oranlı Portföy Sigortası (SOPS) ve Opsiyon Tabanlı Portföy Sigortası (OTPS) stratejilerine kapsamlı bir genel bakış ve inceleme sunulmaktadır.

Bu tezin amacı, portföy sigorta stratejilerini, bazı istatistiksel ve dinamik özellikleri yolu ile vade sonundaki performansları açısından, ve beklenen fayda kriteri üzerinden optimallikleri açısından incelemek, analiz etmek ve karşılaştırmaktır.

Bu tezde sürekli zamanda arbitraj olanağı içermeyen finansal market modeli, tanım ve özellikleri ile SOPS ve OTPS stratejileri, ve hem vade sonunda hem de vade boyunca bu stratejilerin performanslarının, istatistiksel özellik ve dinamik davranışlarının, ve kilit parametrelerinin değişimine olan duyarlılıklarının formülasyon ve simülasyon yoluyla incelenmesi ve karşılaştırılması sunulmaktadır. Ayrıca, beklenen fayda kriteri maksimizasyonu yolu ile optimal portföy stratejileri de incelenmiş ve karşılaştırılmıştır. Optimallik sonuçları konusunda var olan literature katkı olarak, kısıtsız (sigortalanmamış) tahsisata yatırılan uygun sayıda payın daha geniş bir aralıkta varlık ve tekliğinin ispatlanmasını içeren genişletilmiş bir çalışma sunulmuştur.

Anahtar Kelimeler: Portföy sigorta stratejileri: SOPS ve OTPS, portföy optimizasyonu, Merton optimizasyon problemi, sürekli zamanda finansal market modeli, sabit nispi riskten kaçınma fayda fonksiyonu.

To My Father

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Asset liability management (ALM) is one of the most prominent branches of financial mathematics which provides the decision making techniques to achieve a coordination between assets and liabilities. The selection of investment strategies and managing investment funds via employing portfolio insurance methods play an important role in ALM.

Among various investment strategies, the portfolio insurance strategies are designed to give the investor the ability to limit downside risk by insuring a predefined floor; at the same time, they allow to get some benefit from upside potential in case of rising markets. The portfolio can make a heavy loss if the market experiences sharp downturns. To avoid such a loss, a portfolio manager may need to insure a specified value for the portfolio, which results in the sacrifice of gains, since the guarantee component causes a reduction in the expected utility. Particularly, in falling markets, portfolio insurance methods enable to recover a given percentage of initial capital at the end of the investment period.

Portfolio insurance strategies are preferred by the investors who desire for insurance against not only the sudden falls in markets, but also general down trends. In recent history of financial markets, especially in early 2000s, the examples of these kinds of movements are common, such as the default of Lehman Brother's and dot.com bubble collapse (see [31], [49] and [50] for details). Consequently, portfolio insurance strategies have an increasing popularity among the investors who wants to ensure some predefined amount (or percentage) of their initial investment capital at the end of the investment period, in case of an unexpected downside market behaviour. There are many other quite different reasons to explain the demand for the insurance strategies such as regulatory sanctions, bank and insurance policies and legal constraints which require a guaranteed minimum performance at the end of the investment horizon.

The variety of portfolio insurance models is wide. However, this study focuses on the two most prominents and effective strategies, the *Constant Proportion Portfolio Insurance (CPPI)* and the *Option Based Portfolio Insurance (OBPI).*

1.2 Related Literature

The CPPI approach has first been introduced by Perold (1986) [67] and improved by Perold and Sharpe (1988) [66] for fixed-income instruments, and by Black and Jones (1987) [16] for equity instruments. The CPPI method uses a simplified strategy to allocate assets dynamically over the investment period. The investor starts by setting a *floor* equal to the lowest acceptable value of the portfolio. Then, the *cushion* is computed as the difference between the portfolio value and the floor. The amount allocated to the risky asset is determined by multiplying the cushion by a predetermined *multiple*, and is called as the *exposure*. The remaining funds are invested in the risk-free asset. The floor and the multiple are functions of the investor's risk tolerance and are exogenous to the model.

The OBPI approach, which has been introduced by Leland and Rubinstein (1976) [52], consists of a portfolio invested in a risky asset S covered by a put option written on it. Thus, the portfolio value will always be greater than the strike K of the put, so that the method guarantees a fixed amount K at the terminal date.

The comparison between two popular and important portfolio insurance strategeis, CPPI and OBPI, has been performed by several authors in the literature, since the question of which insurance strategy is more preferable to the other one is of interest for both practitioners and researchers. Black and Rouhani (1989) [18] compared CPPI with OBPI in terms of their payoffs and of both expected and actual volatilities when the put option has to be synthesized. Zhu and Kavee (1988) [81] performed a Monte Carlo simulation to compare some statistical properties of the payoffs of two methods CPPI and OBPI. Furthermore, Bertrand and Prigent (2005) [13], and Zagst and Kraus (2011) [79] compared OBPI and CPPI in terms of their stochastic dominance. Bertrand and Prigent (2005) [13] concludes that there is no strong or weak stochastic dominance between two strategies in case of modelling stock prices with geometric Brownian motion, but there may be dominance between two strategies in a mean-variance sense, depending on the CPPI multiplier. The extension of their study is done by Zagst and Kraus (2011) [79] with consideration of second-order and third-order stochastic dominance.

Related to the optimization of the portfolio insurance strategies, the literature provides an extensive research. Generally, practitioners should have a good understanding of the separation principle. In modern portfolio theory, the two fund separation princible was introduced by Markowitz (1959) [56] and extended by Merton (1971, 1973) [58, 59], which implies that the optimal asset allocation is obtained by using two basic funds: a combination of risky assets and one risky-free asset. The CPPI strategy is an application of this two fund separation princible which allocates the assets held in portfolio by rebalancing them between risky assets and risk-free asset. On the other hand, further studies have extended this two fund separation princible to three fund by adding a third fund which is a derivative written on the portfolio of risky assets. The OBPI strategy can be considered as an example of this three fund separation princible. El Karoui, Jeanblanc and Lacoste (2005) [34] investigated the optimal OBPI strategy when the optimal unconstrained portfolio is covered by a put option that achieves a predetermined guarantee. Balder and Mahayni (2009) [54] cite the following works that study CPPI and OBPI as the optimal solution of a utility maximization problem: Cox and Huang (1989) [28], Brennan and Schwartz (1989) [22], Grossman and Villa (1989) [39], Black and Perold (1992) [17], Grossman and Zou (1993, 1996) [40, 41], Cvitanic and Karatzas (1996, 1999) [43, 44], Browne (1999) [23], Basak (1995, 2002) [9, 10] and El Karoui, Jeanblanc and Lacoste (2005) [34]. Among these, we will be reviewing El Karoui, Jeanblanc and Lacoste (2005) [34] and Balder and Mahayni (2009) [54] in Section 4.2 and 4.3, and extend one of the results in El Karoui, Jeanblanc and Lacoste (2005) [34], which is explained further in the next section.

On the other hand, if the risky asset price follows a geometric Brownian motion, theoretically, the portfolio value can never reach or go under the floor value. However, in reality, jumps may occur in market prices which causes the risk that the portfolio value goes under the floor at the terminal date. Bertrand and Prigent (2002) [14] and Cont and Tankov (2009) [76] study on the effects of jumps in asset prices on the insured portfolio. Moreover, Cont and Tankov (2009) [76] examine gap risk, the risk of violating the floor, and determine its loss distribution in the context of a jump diffusion model which allows to measure and manage the gap risk.

1.3 Contribution and Outline

This thesis brings together comparisons of CPPI and OBPI conducted by the following papers: Bertrand and Prigent (2005) [13], El Karoui, Jeanblanc and Lacoste (2005) [34] and Balder and Mahayni (2009) [54]. Except for the contributions we have listed below, the mathematical results and notations in this thesis related to CPPI and OBPI are contained in these works.

One of the contributions of this thesis is our extension of Proposition 2.1 of El Karoui, Jeanblanc and Lacoste (2005) [34]. This extension consists of the following: The existence and uniqueness of the appropriate number of shares invested in the risky asset is proven in El Karoui, Jeanblanc and Lacoste (2005) [34] via assuming that the initial values of the asset price S_0 and of the initial investment V_0 are normalized to 1. Here, in this study, we extend their work to any values of S_0 and V_0, as a contribution. This extension is given as Proposition 4.1 of Section 4.2 of this thesis. Our second contribution is as follows: Bertrand and Prigent (2005) [13] compares CPPI and OBPI by considering the OBPI method based on a call option and by determining the CPPI multiple under the assumption of equality of portfolio returns, whereas the analysis of Section 3.5 considers the OBPI strategy with a put option and CPPI strategy with various multiplier values.

This study is organized as follows:

In *Chapter 2*, we review the financial market model in a Black-Scholes model which is already presented by Karatzas and Shreve (1998) [46], Dana and Jeanblanc (2007) [29], Rockafellar (1970) [69], Merton (1971) [58], and Touzi and Tankov [78]. Throughout this chapter, we carefully examine the model setup and matematically analyse market properties by providing some further steps and details of the arguments and ideas used in the referred sources. We represent the portfolio and wealth processes in terms of this market model and explain the concept of self-financing portfolios, admissible portfolios and no-arbitrage condition in a continuous-time framework. Therefore, we carefully review that the market described in this chapter contains no arbitrage opportunities. Furthermore, we reintroduce the portfolio optimization problem that maximizes the expected utility with no constraints. This unconstrained problem is first presented by Merton (1971) [58], and known as the *Merton Portfolio Opti-*

mization Problem. We review the optimal solution of this problem by martingale approach by supplying some further steps of the existing arguments in the literature. Moreover, we perform the solution of the problem by considering the utiliy function as a *Constant Relative Risk Aversion (CRRA)* utility function, as an example.

In *Chapter 3*, we describe, analyze and compare two standard portfolio insurance methods CPPI and OBPI. Both strategies ensure a predefined level of insurances, while the CPPI strategy is dynamically rebalancing allocations between a risky asset and a risk free asset, the OBPI strategy does the same with a put option written on the risky asset. Within the concept of this chapter, we aim to make comparisons between CPPI and OBPI methods in terms of their statistical and dynamic properties. We examine the performances of two strategies at the terminal date and determine the important parameters of both strategies. Furthermore, we numerically compute and compare their expected portfolio values and variances at the terminal date, and examine their behaviour due to the changes in their key parameters. Moreover, the dynamic properties of two strategies are also examined in the Black-Scholes framework. Thoughout this chapter, we provide a study by focusing on several articles existing in the related literature. For the market model, definitions of two strategies and derivations of portfolio values, expectations and variances, the outline is complied from Black and Jones (1987) [16], Leland and Rubinstein (1976) [52], Bertrand and Prigent (2005) [13] and Zagst and Kraus [79].

Chapter 4 includes a careful and detailed study of two important articles El Karoui, Jeanblanc and Lacoste (2005) [34], and Balder and Mahayni (2009) [54]. First, we focus on the optimality of the OBPI strategy specified by a particular underlying and a particular put option on this underlying. The optimality of OBPI strategy has already been studied and proven by the work of El Karoui, Jeanblanc and Lacoste (2005) [34] by considering an OBPI strategy written on the optimal solution of the Merton problem, which is presented in Chapter 2. In other words, their work considers the unconstrained allocation of OBPI strategy as the solution of the portfolio optimization problem with no constraints. For the optimality of such an OBPI strategy, first, the existence and uniqueness of the appropriate number of shares invested in the unconstrained allocation must be proven. It is shown by El Karoui, Jeanblanc and Lacoste (2005) [34] via assuming the initial values of the asset price S_0 and of the initial investment V_0 are equal to 1. Here, in this study, we extend their work to any values of S_0 and V_0 as a contribution. This is the only contribution that occurs in Proposition 4.1. After that, we present the optimality results and their proofs, which are already published in El Karoui, Jeanblanc and Lacoste (2005) [34], by providing further details of the ideas and arguments used in this paper. Second, we review the derivations of the optimal solutions of the CPPI and OBPI strategies by using a benchmark strategy, the constant mix (CM) strategy, in order to compare their optimal solutions. This idea is presented by Balder and Mahayni (2009) [54], and the comparison of the optimal solutions is performed. In this study, we provide a detailed exposition on their derivation and comparison. Finally, we perform a numerical example on the optimal solutions of three strategies CM, CPPI and OBPI, and we get the same graph, to replicate their results.

In *Chapter 5*, we briefly review the concept of utility loss of portfolio insurance strategies that is caused by the guarantee component and represent the comparison of the loss rates which is made in Balder and Mahayni (2009) [54].

Finally, we conclude the thesis with *Chapter 6*, including a short outlook to possible future studies.

CHAPTER 2

FINANCIAL MARKETS, PORTFOLIO PROCESSES AND PORTFOLIO OPTIMIZATION

2.1 Introduction

This chapter reviews in details the Black-Scholes model presented in Karatzas and Shreve (1998) [46], Dana and Jeanblanc (2007) [29], Rockafellar (1970) [69], Merton (1971) [58], and Touzi and Tankov [78]. Furthermore, we review the portfolio optimization problem that maximizes the expected utility with no constraints. This unconstrained problem is first presented by Merton (1971) [58], and known as the *Merton Portfolio Optimization Problem.* We review again in detail, the optimal solution of this problem by martingale approach. Moreover, we perform the solution of the problem by considering the utiliy function as a *Constant Relative Risk Aversion (CRRA)* utility function, as an example.

Throughout this chapter, we use some basic tools from probability theory and stochastic calculus (such as stochastic processes, continuous-time martingales, Brownian motion, Itô formula, etc.). The definitions and properties of these concepts can be found in Karatzas and Shreve (1991) [45], Shreve (2004) [74], Lamberton and Lapeyre (1996) [48]. Unless otherwise noted, the notation of this chapter is adapted from the works cited above.

2.2 Financial Markets in Continuous Time

In this section, we review the financial market in a continuous time basis and the evolution of assets included in this market. All the information given in this section is taken from Dana and Jeanblanc (2007) [29] and Karatzas and Shreve (1998) [46].

Let us consider an investor who wants to optimize his investment in a finite horizon T. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space providing a standard Brownian motion $(W_t)_{0 \le t \le T} \in \mathbb{R}^N$ and let $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ denotes the completed canonical (natural) filtration of W_t. One considers a market with N risky assets S^i and one risk free asset B. The value of the risky asset i evolves according to

$$dS^i_t = S^i_t(\mu_i dt + \sigma^i dW_t) \quad \forall t \in [0, T] \quad (i = 1, 2, ..., N), \tag{2.1}$$

while the value of risk free asset evolves according to

$$dB_t = rB_t dt, \quad B_0 = 1,$$

where $r \in \mathbb{R}^+$ is the deterministic interest rate function, $\mu \in \mathbb{R}^N$ is the drift of the risky asset price, and $(\sigma^i)_{1 \le i \le N} = (\sigma^{i,j})_{1 \le i,j \le N}$ is the vector in $\mathbb{R}^N$, where σ is the volatility of the risky asset which is a $N \times N$ matrix supposed by us to be invertable.

Related to the coefficients of the above equations, we assume that

$$\int_0^T r(u) du < \infty,$$

and

$$\int_0^T (|\mu(t)|^2 + |\sigma(t)|^2) dt < \infty,$$

in order for the coefficients and the stochastic integrals to be well defined.

Let us introduce θ as the vector of market prices of risk, it is also called the *risk premium*, for each asset is given by

$$\theta = \sigma^{-1}(\mu - r\mathbf{1}_N), \tag{2.2}$$

where $\mathbf{1}_N := (1, 1, ..., 1)^T \in \mathbb{R}^N$. Then, by using the risk premium in (2.2), the dynamics of the risky assets in (2.1) becomes

$$dS^i_t = S^i_t(rdt + \sigma^i(\theta dt + dW_t)) \quad \forall t \in [0, T] \quad (i = 1, 2, ..., N). \tag{2.3}$$

It follows from *Cameron Martin Change of Measure Theorem* (see [78, Section 6.4, page 100]) that

$$W_t^{\mathbb{Q}} = W_t + \theta t \quad (0 \leq t \leq T)$$

is a Brownian motion under the equivalent martingale measure $\mathbb{Q} = Z_T \mathbb{P}$ on $\mathcal{F}_T$, where

$$Z_T = e^{-\theta^\top W_T - \frac{\|\theta\|_2^2 T}{2}} \tag{2.4}$$

is the density of the martingale measure. Here, $\top$ represents the vector transpose and $\|\cdot\|_2$ represents the Euclidean norm.

Then the dynamics of the risky assets in (2.3) becomes

$$dS_t^i = S_t^i (rdt + \sigma^i dW_t^{\mathbb{Q}}) \quad \forall t \in [0, T] \quad (i = 1, 2, ..., N). \tag{2.5}$$

In the following two sections, we reintroduce and investigate the concepts of portfolio processes, self-financing portfolios, wealth processes, admissible portfolios and no-arbitrage condition. The content reviews the definitions and explanations given in Karatzas and Shreve (1998) [46], Dana and Jeanblanc (2007) [29] and Touzi and Tankov [78], and provides some further steps of the arguments.

2.3 Portfolio and Wealth Processes

A *portfolio strategy* is an $\mathbb{F}$-adapted process $\pi = \{\pi_t, 0 \leq t \leq T\}$ with values in $\mathbb{R}^N$.

While considering the portfolio processes, we shall assume that the strategies are self-financing. A strategy is *self financing* if there is no money injection or withdraw during the investment period $(0, T)$. In other words, the new portfolio must be financed only by selling the assets that are already held in the portfolio.

Let π be the portfolio strategy such that $(\pi_t)_{0 \leq t \leq T} = (H_t^0, H_t)_{0 \leq t \leq T}$, where $(H_t^0)_{0 \leq t \leq T}$ and $(H_t)_{0 \leq t \leq T}$ are the processes of the quantities of risk-free asset and risky assets, respectively, that are held in the portfolio at time t in a frictionless market with trading in continuous time. Then, the value of portfolio is

$$V_t(\pi) = H_t^0 B_t + H_t S_t.$$

The portfolio strategy is self financing, if

$$dV_t(\pi) = H_t^0 dB_t + H_t dS_t.$$

Therefore, π_t^i represents the amount invested in the risky asset S_t^i at time t with $1 \leq i \leq N$, and $(X_t)_{0 \leq t \leq T}$ denotes the *wealth process* for the portfolio strategy π. Since, the amount invested in the risky asset is π_t^i, the amount invested in the risk free asset is $X_t - \sum_{i=1}^N \pi_t^i$. Then, the wealth process X_t is given as

$$X_t = \sum_{i=1}^{N} \frac{\pi_t^i}{S_t^i} S_t^i + \frac{X_t - \sum_{i=1}^N \pi_t^i}{B_t} B_t.$$

Under the self-financing condition, the dynamics of the wealth process is given by

$$dX_t = \sum_{i=1}^{N} \frac{\pi_t^i}{S_t^i} dS_t^i + \frac{X_t - \sum_{i=1}^N \pi_t^i}{B_t} dB_t.$$

By inserting the dynamics of the risky assets in terms of the equivalent martingale measure and of the risk free asset, it becomes

$$dX_t = \sum_{i=1}^{N} \pi_t^i (r dt + \sigma^i dW_t^{\mathbb{Q}}) + (X_t - \sum_{i=1}^{N} \pi_t^i) r dt.$$

Let $(\tilde{X}_t)_{0 \leq t \leq T}$ be the discounted wealth process

$$\tilde{X}_t = \frac{X_t}{B_t}.$$

Then, by applying *Itô formula* to the discounted wealth process, we get

$$
\begin{aligned}
d(X_t e^{-rt}) &= X_t d(e^{-rt}) + e^{-rt} dX_t \\
&= -re^{-rt} X_t dt + e^{-rt} \sum_{i=1}^{N} \pi_t^i (rdt + \sigma^i dW_t^{\mathbb{Q}}) + re^{-rt} X_t dt - e^{-rt} \sum_{i=1}^{N} \pi_t^i rdt \\
&= e^{-rt} \sum_{i=1}^{N} \pi_t^i \sigma^i dW_t^{\mathbb{Q}}.
\end{aligned}
$$

Thus, the dynamics of the discounted wealth process can be found as

$$
d\tilde{X}_t = \sum_{i=1}^{N} \tilde{\pi}_t^i \sigma^i dW_t^{\mathbb{Q}},
$$

where $\tilde{\pi}$ is the discounted portfolio strategy.

Consequently, we can see that the discounted wealth is a local martingale under the equivalent martingale measure $\mathbb{Q}$, since the drift term disappears (for example, see, Rutrowski and Musiela (2011) [71] Martingale Methods in Financial Modelling).

Let $X_t^{x,\pi}$ denotes the value of the portfolio at time t, for the pair (x, π), where x is the initial wealth and π is the portfolio strategy.

However, we still need a further technical condition on π. The following result from stochastic integration is taken from Touzi and Tankov [78] and is stated for the measurability of the portfolio strategy π.

Theorem 2.1 [78, Theorem 7.18, page 110] Let W be a standard one dimensional Brownian motion. For $T > 0$, and $\xi \in \mathbb{L}^0(T)$, there exists a progressively measurable process π satisfying

$$
\xi = \int_0^T \pi_t dW_t \quad \text{and} \quad \int_0^T |\pi_t|^2 dt < \infty \quad \mathbb{P} - a.s..
$$

2.4 Admissible Portfolios and No-arbitrage

Let $\mathcal{A}$ the set of all admissible strategies. A portfolio strategy is said to be *admissible* if the corresponding stochastic integral exists and if there exist a constant C such that

$$X_t^{x,\pi} \geq C \quad (t \in [0, T]).$$

Now, we reintroduce the *no-arbitrage condition* in the continuous-time framework according to the definition in Touzi and Tankov [78]. We say that the financial market contains no arbitrage opportunities if for any admissible portfolio process $\pi \in \mathcal{A}$,

$$X_T^{0,\pi} \geq 0 \quad \mathbb{P} - \text{a.s.} \quad \text{implies} \quad X_T^{0,\pi} = 0 \quad \mathbb{P} - \text{a.s.}.$$

The purpose of this section is to show that the financial market described above contains no arbitrage opportunities. Our first observation is that $\mathbb{Q}$ is a risk-neutral measure, or an equivalent martingale measure, for the price process S. In other words, the discounted price process $\{\tilde{S}_t = S_t/B_t, 0 \leq t \leq T\}$ is a $\mathbb{Q}$-martingale. We next observe that $\tilde{X}_t^{x,\pi}$, the discounted wealth process with initial wealth x and the portfolio strategy π, is a $\mathbb{Q}$-local martingale for every $(x, \pi) \in \mathbb{R} \times \mathcal{A}$. Then, we notice that it is bounded from below by a constant, since the corresponding portfolio strategy is admissible. Now, we review the following lemma and its proof which are given Touzi and Tankov [78], by supplying some further steps.

Lemma 2.1 [78, Lemma 7.8, 111] If a local martingale $M = (M_t)_{0 \leq t \leq T}$ is bounded from below by some constant m, i.e., $M_t \geq m$, for all t, then M is a supermartingale, i.e., $E(M_t|\mathcal{F}_s) \leq M_s$ $\forall \quad 0 \leq s \leq t \leq T$.

Proof: We know from definition of the local martingale, an $\mathbb{F}$-adapted process $M = (M_t)_{t>0}$ is a local martingale, if there exists a sequence of stopping times $(T_n)_{n \geq 0}$, which is called a localizing sequence, such that $T_n \to \infty$ a.s. as $n \to \infty$, and the stopped process $M^{T_n} = (M_{t \wedge T_n})_{t>0}$ is a martingale for every $n \geq 0$.

Let $(T_n)_{n \in \mathbb{N}_0}$ be a localizing sequence of stopping times for the local martingale M, then

$T_n \to \infty$ a.s., and $(M_{t\wedge T_n})_{0\leq t\leq T}$ is a martingale for every $n \geq 0$. Then,

$$E(M_{t\wedge T_n}|\mathcal{F}_u) = M_{u\wedge T_n} \quad (0 \leq u \leq t \leq T),$$

for every fixed n. As n goes to infinity, by the lower bound on M, we can use *Fatou's lemma*, and we deduce that

$$E(M_t|\mathcal{F}_u) \leq M_u \quad (0 \leq u \leq t \leq T),$$

which is the required inequality.

Hence, it follows from Lemma 3.1 that the discounted value of the portfolio $\tilde{X}_t^{x,\pi}$ is a $\mathbb{Q}$-supermartingale for every $(x, \pi) \in \mathbb{R} \times \mathcal{A}$.

The supermartingale property plays an important role in both no-arbitrage condition and solution to the optimization problem with martingale approach which are stated in later sections (see Theorem 2.2 and Section 2.5). The first consequence of the supermartingale property concerns the no-arbitrage condition in the continuous-time financial market. We give the following theorem from Touzi and Tankov [78].

Theorem 2.2 [78, Theorem 7.19, 112] The continuous-time financial market described above contains no arbitrage opportunities.

Proof: For $\pi \in \mathcal{A}$, the corresponding wealth process is $X_t^{0,\pi}$ is a $\mathbb{Q}$-supermartingale and, equivalently, the corresponding discounted wealth process $\tilde{X}_t^{0,\pi}$ is a $\mathbb{Q}$-supermartingale.

For $\pi \in \mathcal{A}$, the corresponding discounted wealth process $\tilde{X}_t^{0,\pi}$ is a $\mathbb{Q}$-supermartingale. Then, $E^{\mathbb{Q}}(\tilde{X}_T^{0,\pi}) \leq 0$. We recall that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and B_0 is strictly positive. Then, this inequality shows that, whenever $X_T^{0,\pi} \geq 0$ $\mathbb{P}$-a.s. or, equivalently, $\tilde{X}_T^{0,\pi} \geq 0$ $\mathbb{Q}$-a.s., we have $\tilde{X}_T^{0,\pi} = 0$ $\mathbb{Q}$-a.s. and, therefore, $X_T^{0,\pi} = 0$ $\mathbb{P}$-a.s., which corresponds to the no-arbitrage condition.

The second consequence of the supermartingale property leads us to describe the budget constraint for admissible portfolios and, consequently, it helps us to solve the portfolio optimization problem with no constraints. This result is described in details in the next section.

2.5 The Merton Portfolio Optimization Problem

In this section, we review the problem of an investor who has access to the financial market with one risk free and N risky assets, which is described earlier in this chapter, and wants to optimize his consumption and investment strategy. This well-known problem is first introduced by Merton (1971) [58] and has been used as an important mathematical tool in portfolio theory. Here, in this section, we briefly review this problem by using the related literature, particularly Karatzas and Shreve [46].

We represent the portfolio optimization problem without constraints, where the investor wants to maximize his expected utility over all admissible portfolios (consequently, the integrals exist and the wealth remains nonnegative):

$$\underset{\pi \in \mathcal{A}}{\text{maximize}} \quad E[u(X_T^{x,\pi})], \tag{2.6}$$

where u is the utility function of the investor. A utility function is a concave, increasing function $u : [0, \infty) \to \mathbb{R}$ which measures the preferences of the investor by means of investing in some securities. According to the explanations given in Karatzas and Shreve [46], investor prefers the portfolio X at the terminal date to the portfolio Y if and only if $E(u(X)) \geq E(u(Y))$. The increasing property of the utility function represents that X is always preferred to Y if $X \geq Y$, while concavity implies that $E(X)$ is always preferred to X via *Jensen's inequality*. More precisely, in this study, the utility function u also needs to be bijective to provide the existence of its inverse function.

Let $H_t = (H_t^0, H_t^1, ..., H_t^N)^\top$ denote the quantities of the assets that are held in the portfolio and that the investor is allowed to consume continuously at the risk free rate c_t, so that under the self-financing condition, the value of the portfolio X_t satisfies

$$X_t = \sum_{i=0}^{N} H_t^i S_t^i = x + \sum_{i=0}^{N} \int_0^t H_s^i dS_s^i - \int_0^t c_s ds.$$

We shall also impose the positivity constraint on the portfolio, $X_t \geq 0 \; \forall t \in [0, T]$. Then the optimization problem that is given in (2.6) can be stated as

$$\underset{H,r}{\text{maximize}} \quad E\left[\int_0^T e^{-\rho t}u(r_t)dt + e^{-\rho T}u(X_T)\right],$$

where $e^{-\rho t}$ is the discount factor. The first part of the functional, $\int_0^T e^{-\rho t}u(c_t)dt$, represents the expected utility from consumption and the second part, $e^{-\rho T}u(X_T)$, represents the expected utility from the terminal wealth. Throughout this study, we focus on the maximization of the expected utility without considering consumption.

This stochastic optimization (stochastic control) problem has been known as the *Merton Portfolio Optimization Problem* (the *Merton Consumption / Investment Problem*) since it was introduced in Merton's seminar paper [58]. There are several approaches for the solution of this problem. In this study, we focus on the solution of the Merton problem by the so-called *martingale approach*. In the following two section, we first review the solution method with martingale approach. The notation and calculations are taken from Karatzas and Shreve (1998) [46], Merton (1971, 1973) [58, 59], and Davis and Norman (1990) [30] by providing further details of arguments. Second, we perform the solution of the problem by considering the utiliy function as a *Constant Relative Risk Aversion (CRRA)* utility function, as an example.

2.5.1 The Martingale Approach

Let us consider the optimization problem given in (2.6), where u is the utility function of the investor, $X_T^{x,\pi}$ is the discounted wealth process with initial wealth x and the portfolio strategy π, and $\mathcal{A}$ is the set of all strategies corresponding to admissible portfolios.

We remind from the previous section that the discounted value of the portfolio $\tilde{X}_t^{x,\pi}$ is a $\mathbb{Q}$-supermartingale for every $(x, \pi) \in \mathbb{R} \times \mathcal{A}$, which means that the terminal discounted value of the portfolio satisfies

$$E^{\mathbb{Q}}(\tilde{X}_T^{x,\pi}) \leq x \quad \forall \pi \in \mathcal{A}.$$

This is equivalent to

$$E(e^{-rT}Z_T X_T^{x,\pi}) \leq x \quad \forall \pi \in \mathcal{A}.$$

The above inequalities imply that every admissible portfolio satisfies the budget constraint. This is another important consequence of the supermartingale property of the discounted wealth process which is stated before. The budget constraint permits the interpretation that the expected discounted terminal value of the portfolio under the risk neutral probability can not exceed the initial invested capital.

We are looking for the optimal solution of the problem (2.6). Let X_T be an $\mathcal{F}_T$ measurable random variable that satisfies the initial budget constraint. Thus, we can say that

$$\{X_T^{x,\pi}, \pi \in \mathcal{A}\} \subseteq \{X_T, E^{\mathbb{Q}}(X_T) \leq x\}. \tag{2.7}$$

We would like to show that these two sets are equal. Let $\tilde{X}_T$ be the discounted terminal wealth such that $\tilde{X}_T = X_T/B_T$. Therefore, since the discounted terminal value of the portfolio $X_T^{x,\pi}$ for initial wealth x and portfolio strategy π is a $\mathbb{Q}$-martingale, then by *Martingale Representation Theorem* (see [48, Theorem 4.2.4, page 91]), there exists a portfolio strategy φ such that

$$\tilde{X}_T^{x,\pi} = E^{\mathbb{Q}}(\tilde{X}_T) + \int_0^t \varphi_s dW_s^{\mathbb{Q}}.$$

Therefore, since X_T bounds the budget constraint, then $E^{\mathbb{Q}}(\tilde{X}_T) = x$. This means,

$$\tilde{X}_T^{x,\pi} = x + \int_0^t \varphi_s dW_s^{\mathbb{Q}}.$$

By setting $\varphi = \pi\sigma$, i.e., by putting the portfolio process π to $\varphi\sigma^{-1}$, we can say that the sets in (2.7) are equal.

Consequently, we can rewrite the optimization problem (2.6) in an equivalent form as

$$
\begin{aligned}
&\text{maximize} \quad E[u(X_T)] \\
&\text{subject to} \quad E(e^{-rT} Z_T X_T) \leq x.
\end{aligned}
\tag{2.8}
$$

The problem above can be solved by the *Lagrangian Method* on *Karush-Kuhn-Tucker Necessary Optimality Conditions* which provide a strategy for finding the local maxima, local minima and saddle points of an objective function subject to some equality and inequality constraints, under constraint qualifications (see, for example, Nash, Sofer and Griva [63]). The following computation is taken from Rockafellar (1970) [69]. The Lagrangian of this problem is

$$E[u(X_T) - y(e^{-rT} Z_T X_T - x)].$$

The *Fenchel transform* of the utility function u, which is a strictly concave and increasing function (see Section 3.5) is given as

$$F(y) = \sup_{x \in \mathbb{R}} (u(x) - xy),$$

with the maximizer of $F(y)$ being given by

$$x^* = (u')^{-1}(y) := I(y). \tag{2.9}$$

The strict concavity of u implies that the optimizer is unique.

For information and details on Fenchel transformation, we refer to Rockafellar (1970) [69] and Fenchel (1953) [38].

Then, the optimal solution of the problem (2.8), consequently of (2.6), corresponding to initial wealth x and the portfolio strategy π, is

$$X_T^* = I(y e^{-rT} Z_T), \tag{2.10}$$

where the Lagrangian multiplier y is found from the condition

$$E[e^{-rT}Z_T I(ye^{-rT}Z_T)] = x,$$

since the optimal solution of the problem X_T^* bounds the budget constraint.

We notice that X_T^*, given in (2.10), is the optimal solution to the problem (2.6) for a general class of utility functions. Now, we give an example by the CRRA utility function for the solution of the maximization problem without constraints.

2.5.2 Optimal Solution to the Merton Problem for CRRA Utility Function

The CRRA utility function is defined as

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

for all $x \in \mathbb{R}^+$, with $\gamma \in (0,1)$, where $u \in C^2$ and it is strictly increasing and concave, i.e., $u' > 0$ and $u'' < 0$.

In the case of CRRA utility,

$$x^* = (u')^{-1}(y) := I(y) = y^{-\frac{1}{\gamma}},$$

and the Fenchel transform of u is

$$F(y) = \sup_x (u(x) - xy) = \frac{\gamma}{1-\gamma} y^{\frac{\gamma-1}{\gamma}}.$$

Thus, the candidate optimal solution is given by

$$X_T^* = I(ye^{-rT}Z_T) = (ye^{-rT}Z_T)^{-\frac{1}{\gamma}}. \tag{2.11}$$

Since the optimal solution in (2.11) bounds the budget constraint $E(e^{-rT} Z_T X_T^*) = x$, it follows that

$$E(e^{-rT} Z_T (y e^{-rT} Z_T)^{-\frac{1}{\gamma}}) = x. \tag{2.12}$$

Then, by rearranging (2.12), we have

$$E(e^{rT(\frac{1-\gamma}{\gamma})} y^{-\frac{1}{\gamma}} Z_T^{\frac{\gamma-1}{\gamma}}) = x,$$

which also implies

$$y^{-\frac{1}{\gamma}} E(Z_T^{\frac{\gamma-1}{\gamma}}) = x e^{rT(\frac{\gamma-1}{\gamma})}. \tag{2.13}$$

Finally, from (2.13), we get

$$y^{-\frac{1}{\gamma}} = \frac{x e^{rT(\frac{\gamma-1}{\gamma})}}{E(Z_T^{\frac{\gamma-1}{\gamma}})}. \tag{2.14}$$

By inserting (2.14) into the optimal solution X_T^* in (2.10), it becomes

$$X_T^* = x e^{rT} \frac{Z_T^{-\frac{1}{\gamma}}}{E(Z_T^{\frac{\gamma-1}{\gamma}})}.$$

From the density of the martingale measure Z_T, is given in (2.4), we can calculate the components $Z_T^{-\frac{1}{\gamma}}$ and $E(Z_T^{\frac{\gamma-1}{\gamma}})$ of the optimal solution X_T^*.

Finally, the optimal solution of the maximization problem in the case of CRRA utility becomes

$$X_T^* = x e^{rT} \left(e^{\frac{\theta^\top W_T}{\gamma} + \frac{2\gamma-1}{2\gamma^2} ||\theta||_2^2 T} \right).$$

Thus, it is clear that the optimal solution X_T^* of the maximization problem without constraints is linear with respect to the initial wealth x for CRRA utility case. Hence, it can be referred as to

$$X_T^* = x\mathcal{S}_T,$$

where the process $\mathcal{S}$ represents an arbitrary allocation of the risky assets that is available in the market.

CHAPTER 3

PORTFOLIO INSURANCE STRATEGIES

3.1 Introduction

Among various investment strategies, the portfolio insurance strategies are designed to give the investor the ability to limit downside risk by insuring a given floor while allowing participation in upside potential in the market. The portfolio can make a heavy loss if the market experiences sharp downturns. To avoid such a loss, a portfolio manager may need to insure a predetermined value for the portfolio, which results in the sacrifice of potential gains, since the guarantee component causes a reduction in the expected utility. Particularly, in falling markets, portfolio insurance methods allow investors to recover a given percentage of their initial investment at maturity.

In this chapter, we review several articles Black and Jones (1987) [16], Leland and Rubinstein (1976), Bertrand and Prigent (2005) [13] [52] and Zagst and Kraus [79] to bring together explanation and comparison of CPPI and OBPI, and compute the various statistics of two methods. We particularly review the study of Bertrand and Prigent (2005) [13], in which the comparison of CPPI and OBPI has been performed in terms of their statistical and dynamic properties, by considering the OBPI method based on a call option and by determining the CPPI multiple under the assumption of equality of portfolio returns. We make a similar comparison by considering the OBPI strategy with a put option and CPPI strategy with various multiplier values. We examine the performances of two strategies at the terminal date and investigate the important parameters of both strategies. Furthermore, we numerically compute and compare their expected portfolio values and variances at the terminal date, and examine their behaviour due to the changes in their key parameters. Moreover, the dynamic properties of two strategies are also analyzed in the Black-Scholes framework.

Thoughout this chapter, for the development of our study, we use some standard tools from finance theory (such as financial derivatives, bullish and bearish market trends, Black-Scholes option pricing formula, etc.) and stochastic calculus (such as Brownian motion, stochastic differential equations, etc.). For the definitions and properties of these well-known concepts, we refer to Chance and Brooks (2007) [26], and Shreve (2004) [74], Lamberton and Lapeyre (1996) [48], respectively.

We note that, in this chapter, we consider the basic structure of OBPI strategy which consists of one risky asset and one put option, in order to make the comparisons possible and the similations easier, which will be improved and discussed in the later chapter. Both European and American options are available for this concept, however, we restrict ourselves to European options because of the same reasons, which provide a guarantee only at the terminal date.

3.2 Financial Market Model

To define and compare the two portfolio strategies, we reintroduce the financial market conditions as a natural first step.

Let us consider a classic Black-Scholes model where the portfolio manager is assumed to invest in two basic assets which are traded continuously in the time during the investment period $[0, T]$. The first of the two assets is a risk-free asset, like a money market account or zero coupon bond, and is denoted by B. The value of the riskless asset B evolves according to

$$dB_t = rB_t dt, \tag{3.1}$$

where r is the deterministic interest rate and the initial value $B_0 > 0$. The second asset, denoted by S, is a risky asset, such as a stock or a stock portfolio. The dynamics of the market value of the risky asset is given by the classic diffusion process

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{3.2}$$

where $(W_t)_{0 \leq t \leq T}$ is a standard Brownian Motion process and the initial value $S_0 > 0$. Therefore, $\mu > r > 0$ and $\sigma > 0$ are constants that represent the drift and volatility of the asset price S, respectively.

We note that the assumption $\mu > r$ is understandable by observing that a typical risk averse investor is characterized by a strictly concave and strictly increasing utility function u. Hence, if $\mu < r$, there would be no reason for a rational investor to invest in stocks rather than a risk free asset, since at any time $t \in [0, T]$ it holds

$$E[u(S_0 e^{rt})] = u(S_0 e^{rt}) \geq u(S_0 e^{\mu t}) \geq u(E[S_t]) \geq E[u(S_t)].$$

Then, by applying *Itô formula* to risky asset dynamics, the solution can be found as

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \quad (t \in [0, T]). \tag{3.3}$$

We remark that Equation (3.3) is a well known solution of the stochastic differential equation (3.2) under Itô interpretation. We refer to Shreve (2004) [74], and Lamberton and Lapeyre (1996) [48] for information on stock price modelling with Geometric Brownian Motion.

Thus, the log returns of the risky asset are normally distributed according to

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right). \tag{3.4}$$

Within the context of this study, we assume the market to be complete, arbitrage free and frictionless. We also assume that the investment strategies are self-financing. We recall that a strategy is self financing if there is no money injection or withdraw during the invesment period $(0, T)$. Therefore, all dividends and coupons are assumed to be reinvested in such a way that the portfolio remains self-financing.

Furthermore, following Black and Scholes (1973) [19], we assume *ideal conditions* in the market for stocks and options. When the options are considered, we restrict ourselves to European options that can only be exercised on a predetermined date.

In the following, we give the definitions and basic properties of two portfolio strategies the CPPI and the OBPI, respectively. The content of the following two sections are complied from Black and Jones (1987) [16], Leland and Rubinstein (1976) [52], Bertrand and Prigent (2005) [13] and Zagst and Kraus [79]. We give the definitions and do calculations according to these sources by supplying further steps and explanations of the arguments.

3.3 Constant Proportion Portfolio Insurance (CPPI)

The CPPI strategy is introduced by Perold (1986) [67] and improved by Perold and Sharpe (1988) [66] for fixed-income instruments, and by Black and Jones (1987) [16] for equity instruments. The CPPI method uses a simplified strategy to allocate assets dynamically over the investment period. The investor starts by setting a *floor* equal to the lowest acceptable value of the portfolio. Then, the *cushion* is computed as the difference between the portfolio value and the floor. The amount allocated to the risky asset is determined by multiplying the cushion by a predetermined *multiple*, and is called as the *exposure*. The remaining funds are invested in the risk free asset. The floor and the multiple are functions of the investor's risk tolerance and are exogenous to the model.

More precisely, the CPPI method consists of managing a dynamical portfolio so that its terminal value V_T^{CPPI} at the end of the investment horizon T lies almost surely above a predetermined floor value F_T.

The *floor* value, i.e., the guarantee, is determined by the investor at the beginning of the investment period, given as a percentage $\alpha_T \geq 0$ of the initial investment V_0^{CPPI}. The terminal value of the guarantee, the so-called *floor*, is given by

$$F_T = \alpha_T V_0^{CPPI}.$$

We note that, in the absence of any arbitrage opportunities, it is impossible to find an investment that returns more than the risk-free rate of return r with no risk and, thus, the maximum guaranteed portfolio value at time T is limited by

$$\alpha_T \leq e^{rT}.$$

Let $(F_t)_{0\leq t\leq T}$ denote the time t value of the floor with respect to a given guarantee F_T, in order to define its dynamics over the investment period $[0, T]$. The value of the floor gives the dynamical insured amount and it is assumed to evolve according to

$$dF_t = F_t r dt, \quad d\alpha_t = \alpha_t r dt,$$

where

$$F_t = \alpha_t V_0^{CPPI} \quad \text{and} \quad \alpha_t = \alpha_T e^{-r(T-t)}. \tag{3.5}$$

Obviously, the initial floor F_0 is less than the initial portfolio value V_0^{CPPI}.

The *cushion* at time zero is defined by the difference between the portfolio value, and the given guarantee $V_0^{CPPI} - F_0$ is denoted by C_0. Its value C_t at any time t in $[0, T]$ is given by

$$C_t = V_t^{CPPI} - F_t. \tag{3.6}$$

The *exposure*, which is denoted by E_t, is the total amount invested in the risky assets. The standard CPPI method consists of letting

$$E_t = mC_t, \tag{3.7}$$

where m is a constant called the *multiple*. Now, the basic idea of the CPPI approach consists of investing a constant proportion m of the cushion C_t in the risky assets. This is the reason that the strategy is named as *Constant Proportion Portfolio Insurance*.

The remaining part of the portfolio $V_t^{CPPI} - E_t$ is invested in the riskless asset.

Thus, the value of the self financing CPPI portfolio at time $t \in [0, T]$ is given by

$$dV_t^{CPPI} = (V_t^{CPPI} - E_t)\frac{dB_t}{B_t} + E_t\frac{dS_t}{S_t}. \tag{3.8}$$

Then by the formula of exposure, given in (3.7), and by inserting the dynamics of the risk-free asset (3.1) and the risky asset (3.2), Equation (3.8) becomes

$$dV_t^{CPPI} = (V_t^{CPPI} - mC_t)rdt + mC_t(\mu dt + \sigma dW_t).$$

By the formulas of the dynamic floor (3.5) and of the cushion (3.6), we have

$$\begin{aligned} dV_t^{CPPI} &= (V_t^{CPPI} - m(V_t^{CPPI} - \alpha_t V_0^{CPPI}))rdt + mC_t(\mu dt + \sigma dW_t) \\ &= (V_t^{CPPI}(1-m) + m\alpha_t V_0^{CPPI})rdt + mC_t(\mu dt + \sigma dW_t). \end{aligned} \tag{3.9}$$

Hence, by (3.6), the stochastic dynamics of the cushion C_t satisfies

$$dC_t = d(V_t^{CPPI} - \alpha_t V_0^{CPPI}) = dV_t^{CPPI} - V_0^{CPPI} d\alpha_t. \tag{3.10}$$

Now, we insert the dynamics of the portfolio value which is found in (3.9) into the stochastic dynamics of the cushion C_t in (3.10). Then we get

$$\begin{aligned} dC_t &= (V_t^{CPPI}(1-m) + m\alpha_t V_0^{CPPI})rdt + mC_t(\mu dt + \sigma dW_t) - V_0^{CPPI}\alpha_t rdt \\ &= ((V_t^{CPPI} - V_0^{CPPI}\alpha_t)(1-m))rdt + mC_t(\mu dt + \sigma dW_t) \\ &= C_t(1-m)rdt + mC_t(\mu dt + \sigma dW_t). \end{aligned}$$

This leads to

$$\frac{dC_t}{C_t} = [m\mu + r(1-m)]dt + m\sigma dW_t.$$

By applying *Itô Formula*, it can be deduced that

$$\ln C_t - \ln C_0 = m(\ln S_t - \ln S_0) + (1-m)(r + m\frac{\sigma^2}{2})t. \tag{3.11}$$

Thus,

$$C_t = C_0 \left(\frac{S_t}{S_0}\right)^m e^{(1-m)(r+m\frac{\sigma^2}{2})t}.$$

Substituting the lognormal distribution for the risky asset S_t, given in (3.4), we can deduce for the form (3.11) that the cushion is lognormally distributed with

$$\ln\left(\frac{C_t}{C_0}\right) \sim N\left((r + m(\mu - r) - \frac{1}{2}m^2\sigma^2)t, m^2\sigma^2 t\right). \tag{3.12}$$

Finally, by inserting the value of the floor and the value of cushion into $V_t^{CPPI} = F_t + C_t$, we find the value of the CPPI portfolio at time $t \in [0, T]$ as

$$\begin{aligned} V_t^{CPPI} &= \alpha_T e^{-r(T-t)} V_0^{CPPI} + C_0 \left(\frac{S_t}{S_0}\right)^m e^{(1-m)(r+m\frac{\sigma^2}{2})t} \\ &= \alpha_T e^{-r(T-t)} V_0^{CPPI} + \beta_t S_t^m \\ &= F_t + \beta_t S_t^m, \end{aligned}$$

where

$$\beta_t = \frac{C_0}{S_0^m} e^{(1-m)(r+m\frac{\sigma^2}{2})t}, \tag{3.13}$$

and

$$C_t = \beta_t S_t^m.$$

Formula (3.13) was first proved by Black and Perold (1992) [17], then extended by different authors for different cases. With respect to the derivation of the formula, we refer to Bertrand and Prigent (2005) [13] and Zagst and Kraus [79].

Hence, it is clear that the CPPI method is parameterized by the level of insurance α_T and the multiplier m. Both of these parameters are initially determined according to investor's risk preferences. Since the exposure to risky assets is $E_t = mC_t$, m represents the level of risk taken by the investor. If $0 < m < 1$, the *payoff function* of CPPI, which is the terminal value of the CPPI portfolio, is concave with respect to the risky asset values, which means that the investor is risk averse. Also in the case that $m \geq 1$, the payoff function is convex.

Before we continue with the OBPI strategy, we conclude this section with the determination of the expected value as well as the variance of the value of the CPPI portfolio V_T^{CPPI} at the end of the investment horizon T.

For the derivation of the expected value (mean) and the variance, we especially need the probability distribution of the cushion C_t which is given in (3.12). We also recall that the mean and variance of a lognormally distributed random variable X, with $\ln X \sim N(\mu, \sigma^2)$, are given by

$$\begin{aligned} E(X) &= e^{\mu+\frac{\sigma^2}{2}}, \\ Var(X) &= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1). \end{aligned} \tag{3.14}$$

On the other hand, the terminal value of the CPPI portfolio is given by

$$V_T^{CPPI} = \alpha_T V_0^{CPPI} + C_0 \left(\frac{S_T}{S_0}\right)^m e^{(1-m)(r+m\frac{\sigma^2}{2})T} = \alpha_T V_0^{CPPI} + C_T.$$

Thus, the expected terminal value of the CPPI portfolio can be obtained as

$$E(V_T^{CPPI}) = E(\alpha_T V_0^{CPPI} + C_T) = \alpha_T V_0^{CPPI} + E(C_T).$$

Then, by (3.12) and (3.14), we have

$$\begin{aligned} E(V_T^{CPPI}) &= \alpha_T V_0^{CPPI} + e^{\ln C_0 + (r+m(\mu-r)-\frac{1}{2}m^2\sigma^2)T + \frac{m^2\sigma^2 T}{2}} \\ &= \alpha_T V_0^{CPPI} + C_0 e^{(r+m(\mu-r))T}. \end{aligned}$$

On the other hand, the variance of the terminal value of the CPPI portfolio can be found as

$$\begin{aligned} Var(V_T^{CPPI}) &= Var(\alpha_T V_0^{CPPI} + C_T) \\ &= Var(C_T). \end{aligned}$$

Again, by using (3.12) and (3.14), we obtain

$$
\begin{aligned}
Var(V_T^{CPPI}) &= e^{2(\ln C_0+(r+m(\mu-r)-\frac{1}{2}m^2\sigma^2)T+\frac{m^2\sigma^2T}{2})}(e^{m^2\sigma^2T}-1) \\
&= C_0^2 e^{2((r+m(\mu-r))T)}(e^{m^2\sigma^2T}-1),
\end{aligned}
$$

where

$$
C_0 = V_0^{CPPI}(1-\alpha_T e^{-rT}).
$$

At this point, we leave the comments and implementations on the expectation and variance to the later section where the comparisons between the two portfolio insurance strategies CPPI and OBPI are made. Now, we go on with the description of OBPI strategy.

3.4 Option-Based Portfolio Insurance (OBPI)

The OBPI strategy, which was introduced by Leland and Rubinstein (1976) [52], consists of a portfolio invested in a risky asset S covered by a put option written on it. Thus, the portfolio value will always be greater than the strike K of the put, so that the method guarantees a fixed amount K at the terminal date. In other words, the guaranteed amount F_T is equal to the strike price K of the put.

In contrast to the CPPI strategy, the OBPI strategy is a static investment strategy, i.e., no rebalancing of the portfolio occurs during the investment period $(0, T)$.

In this section, we consider the OBPI strategy as a put-based portfolio insurance strategy which guarantees a minimum terminal portfolio value of

$$
V_T^{OBPI} = \alpha_T V_0^{OBPI},
$$

for a portfolio consisting of one shares of the risky asset S, by purchasing one shares of European put option written on S with maturity T and strike K. We assume that the put option is financed at the risk-free interest rate r at $t = 0$.

Thus, the portfolio value of the put-based strategy at terminal date, V_T^{OBPI}, is given by

$$
V_T^{OBPI} = S_T + (K - S_T)^+,
$$

which is also equal to $K + (S_T - K)^+$ due to the *put/call parity*. This relation shows that the insured amount at maturity is the exercise price K. It also implies that the initial value of the investment is equal to

$$V_0^{OBPI} = S_0 + P(0, S, K).$$

We note that, normally, it is more accurate to consider the OBPI portfolio as a combination of λ shares of the initial investment in the risky asset and a put written on this underlying. Here, we have assumed that λ is equal to one. In fact, it is not a natural way to determine the initial investment V_0 for a fixed value $\lambda = 1$. It is more consistent to search for an appropriate λ satisfying the no-arbitrage condition, for a fixed value of the initial investment V_0. Here, we choose to consider the OBPI strategy in a basic framework without loss of generality, for simplification of simulations and comparisons. However, we refer to the next section where the structure of the OBPI strategy is discussed in details.

One can also consider the value of the OBPI portfolio at any time t in the period $[0, T]$ as

$$V_t^{OBPI} = S_t + P(t, S_t, K),$$

which is also equal to $Ke^{-r(T-t)} + C(t, S_t, K)$ due to the put/call parity; here, $P(t, S_t, K)$ and $C(t, S_t, K)$ are the Black-Scholes values of the European put and call given by the formulas

$$\begin{aligned} C(t, S_t, K) &= N(d_1)S_t - N(d_2)Ke^{-r(T-t)}, \\ P(t, S_t, K) &= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t, \end{aligned} \tag{3.15}$$

where

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$
$$d_2 = d_1 - \sqrt{T - t},$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. We note that, for all dates t before T, the portfolio value is always above the deterministic level

$Ke^{-r(T-t)}$.

Consequently, the strike price K of the put option, which represents the guaranteed level at the terminal date, and which consequently depends on the desired level of insurance α_T and V_0, represents the only parameter of the OBPI method.

Similar to the CPPI portfolio, we finally try to determine the expectation and variance of the terminal value of the OBPI portfolio V_T^{OBPI}. First, we obtain the terminal value of the OBPI strategy as

$$V_T^{OBPI} = S_T + P(T, S_T, K).$$

For the derivation of the expectation and variance of the OBPI strategy, we directly reintroduce the findings in the article of Zagst and Kraus [79] without any contribution. In their study, they first recall the following definition which is used to obtain the expectation and variance of the OBPI strategy.

Definition 3.1 [79, Definition 1, page 8] Given the benchmark X and a randon variable Y, the *Lower Partial Moment* LPM_z and the *Upper Partial Moment* UPM_z of Y with respect to X and some $z \in \mathbb{N}_0$ are defined as

$$LPM_z(Y, X) = E(\max\{X - Y, 0\}^z),$$

$$UPM_z(Y, X) = E(\max\{Y - X, 0\}^z).$$

If we consider the random variable Y as an asset price S, and the corresponding benchmark X, the lower partial moment LPM_0 denotes the shortfall probability and LPM_1 stands for the expected value of the loss, in the case that the asset price falls below the benchmark. (The shortfall probability is basically the probability that the return lies below a target amount at the end of the investment period.) Vice versa, UPM_0 represents the probability of outperformance and UPM_1 the expected value of the profit when the asset price beats the benchmark X.

Based on this definition, the mean and variance of the OBPI strategy at maturity T is determined by Zagst and Kraus [79] as follows:

$$E(V_T^{OBPI}) = UPM_1(S_T, K) + \alpha_T V_0^{OBPI},$$
$$Var(V_T^{OBPI}) = UPM_2(S_T, K) - UPM_1(S_T, K)^2;$$

We refer to the related article for the proof.

Now, after defining two different portfolio strategies, namely, the CPPI and the OBPI, we continue with the comparison of these two methods in terms of their statistical and dynamical properties.

3.5 Comparison Between Standard CPPI and OBPI

The comparison between two prominent portfolio insurance strategeis, CPPI and OBPI, has been performed by several authors in the literature, since the question of which insurance strategy is more preferable to the other one is of interest for both practitioners and researchers. Thoughout this section, while comparing two methods, we particularly examine the study Bertrand and Prigent (2005) [13] in which they investigate and compare two methods by their statistical and dynamical properties. In their study, they consider the OBPI strategy consists of a risky asset and a call option, and compare two methods by means of the equality of expected returns, quantiles, and hedge parameters. Here, in this thesis, we consider the OBPI strategy with a risky asset and a put option written on the same risky asset. We analyze and compare two methods in terms of their performances at maturity, sensitivities to important parameters and dynamical behaviour.

As it has been previously mentioned while the CPPI method is parameterized by the proportion of the initial amount α_T (or by the guarantee F_T itself) and by the multiple m, the OBPI method has just one parameter, the strike K.

According to the method used in Bertrand and Prigent (2005) [13], to compare two methods, first, one should consider the same initial investment for both methods. Thus, the initial portfolio values V_0^{CPPI} and V_0^{OBPI} are assumed to be equal to

$$V_0 := V_0^{CPPI} = V_0^{OBPI} = S_0 + P(0, S_0, K).$$

Secondly, the two strategies are supposed to provide the same guarantee α_T, at the end of the investment period T, expressed as proportion of the initial investment V_0:

$$F_t = \alpha_t V_0 \quad \text{and} \quad F_T = K,$$

where $F_t = e^{-r(T-t)} F_T$, ($t \in [0, T]$). Thus,

$$K = \alpha_t e^{r(T-t)} V_0.$$

We note that these two conditions do not impose any constraint on the multiplier m. Hence, this leads us to consider the CPPI strategies for various values of the multiplier m.

To compare the two strategies, we examine their performances at maturity. The analysis of the payoff functions, i.e., the values of the two portfolios at the terminal date as a function of risky asset values, gives a first insight.

3.5.1 Comparison of the Payoff Functions

As stated before, the value of the CPPI portfolio strategy at the terminal date, the so-called payoff function of the CPPI, is given by

$$V_T^{CPPI} = \alpha_T V_0 + C_0 \left(\frac{S_T}{S_0} \right)^m e^{(1-m)(r+m\frac{\sigma^2}{2})T},$$

while the payoff function of the OBPI strategy, i.e., the value of the OBPI portfolio at maturity, is given by

$$V_T^{OBPI} = S_T + P(T, S_T, K).$$

Let us start with looking at the behaviour of the payoff functions of both strategies. However, one must take account of the probabilities of the market behaviour, for example bullish or bearish market, in order to compare the methods. The terms *bull market* and *bear market* describe upward and downward market trends, respectively. As a mathematical definition, the market is said to be *bullish* when $S_T \geq E(S_T)$. Vice versa, the market is *bearish* when the prices at time T are less than or equal to their expected values, $E(S_T) \geq S_T$.

Example 3.1 A simple numerical example is illustrated with the following values of the financial market: μ=7.5%, σ=8% and r=3.5% ($\mu > r > 0$). In this market, the two portfolio strategies are set up assuming $T = 5$ years, S_0=80, K=80, α_T=0.90 and $V_0 = S_0 + P(0, S_0, K) = 81.15$. The value of CPPI strategy is calculated for different values of multiplier $m = 2, 4, 6, 8$. Under no-arbitrage assumption, the upper bound of the level of insurance $\alpha_T \leq e^{rT}$ is equal to 1.1912 for the given values of this example.

Figure 3.1 illustrates the payoff functions, i.e., the terminal values of CPPI and OBPI portfolio strategies as functions of S in the Black-Scholes framework.

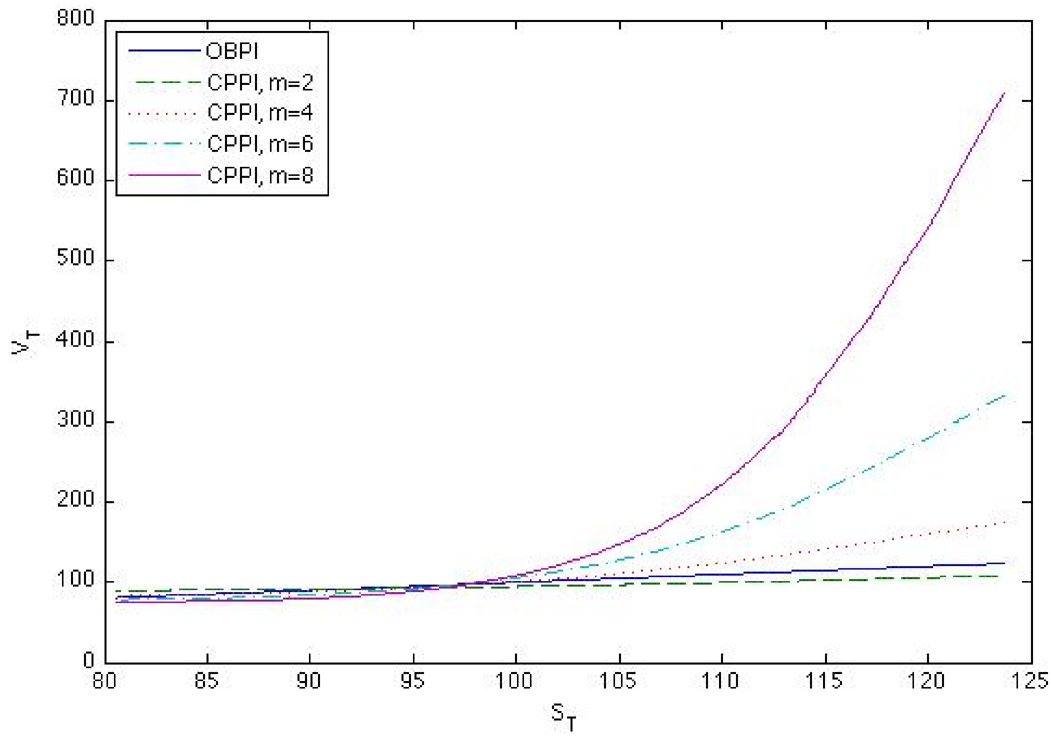

Figure 3.1: CPPI and OBPI portfolio values at maturity.

We also illustrate Figure 3.2 to be able to see the graph with closer details for small terminal risky asset price S_T.

We notice that as m increases, for the CPPI method, the portfolio value function becomes

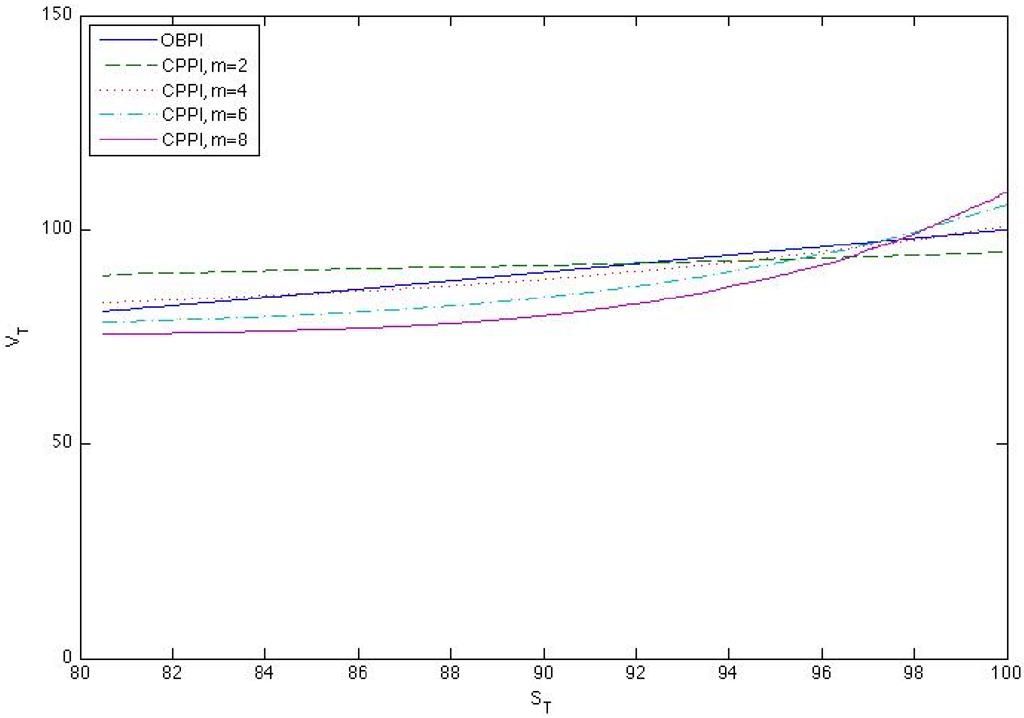

Figure 3.2: CPPI and OBPI portfolio values at maturity.

more convex. Since m represents the level of exposure to risky asset, as m increases, the investor's tolerance to risk also increases, which makes the payoff function more convex, too.

On the other hand, for each value of the multiplier m, the payoffs of the CPPI and the OBPI strategy intersect at least once. We can only compare the terminal portfolio values according to the expected terminal risky asset price $E(S_T)$ which is equal to 100.27. For example, when $S_T \geq E(S_T)$, i.e., the market is bullish, we can see from Figure 3.1 and Figure 3.2 that the OBPI strategy dominates the CPPI strategy with multiplier $m = 2$, but it is dominated by the CPPI strategy with multipliers $m = 4, 6, 8$. The converse comparisons can be made for the bearish market as well, however, we see that neither of the two payoff functions is greater than the other for all terminal values of the risky asset. Thus, in terms of the payoff functions, it does not seem possible to make a general comparison between two methods. One can prefer one to another only under some specified parameter and variable considerations.

3.5.2 Comparison of the Expectations and Variances

In Bertrand and Prigent (2005) [13], the comparison of the expectatiton, variance, skewness and kurtosis of the terminal portfolio values of CPPI and OBPI strategies, is done by obtaining a multiplier value m^* under equality of expected returns. In this subsection, we perform the comparison of expectations and variances at maturity (without considering the value of m^*) for different values of the multiplier m and level of insurance α_T.

In the following numerical example, we examine the two portfolio strategies CPPI and OBPI in terms of their statistical properties, for example, their expectations and variances.

Example 3.2 We calculate the expectations and the variances of our two strategies CPPI and OBPI at the terminal date T due to the parameter set given in Example 3.1. We again consider the CPPI strategy for different values of multiplier, $m = 2, 4, 6, 8$. Therefore, we perform the calculation for two different values of insurance level, $\alpha_T = 0.90$ and $\alpha_T = 1.050$, in order to understand the effect of the change of desired level of insurance on the expected values and variances of our two portfolio strategies. Table 3.1 shows the numerical results for the desired level of insurance $\alpha_T = 0.90$.

Table 3.1: Expectations and variances for the desired level of insurance $\alpha_T = 0.90$.

	Expectation	Variance
OBPI	126.5785	37.4250
CPPI, m=2	138.7102	24.6376
CPPI, m=4	162.4223	217.9744
CPPI, m=6	185.7115	853.9417
CPPI, m=8	202.4067	2080.1003

We observe, as the multiple m increases, the expected terminal value of the CPPI strategy $E(V_T^{CPPI})$ increases. However, its variance $Var(V_T^{CPPI})$ increases dramatically at the same time. This is because the multiple m represents the level of exposure to risky assets; in other words, it represents the investor's risk tolerance. While m increases, investment becomes more risky.

Furthermore, these numerical results also justify the formulations (which are given in Sections 3.3 and 3.4) of the expected returns and the variances of the terminal values of two portfolio strategies.

Table 3.2: Expectations and variances for the desired level of insurance $\alpha_T = 1.050$.

	Expectation	Variance
OBPI	124.7979	35.4607
CPPI, m=2	128.7872	6.0335
CPPI, m=4	140.6223	53.8142
CPPI, m=6	152.4274	212.5394
CPPI, m=8	161.1215	521.8999

Table 3.2 shows the results for an increased level of insurance up to $\alpha_T = 1.050$. For CPPI strategy, the expected terminal value again increases with the increasing values of the multiple m, while the variance of the terminal value increases. Also, we notice that an increase in the level of insurance α_T significantly reduces the investment risk, i.e., the variance of the portfolio value at the terminal date $Var(V_T^{CPPI})$. However, the expected portfolio value $E(V_T^{CPPI})$ is decreasing at the same time. Infact, increasing the guaranteed amount means investing less in the risky assets and more in the risk free asset, causes a decrease both in the expected terminal value and the variance. Therefore, in the case of OBPI strategy, an increase in the level of insurance α_T or, correspondingly, in the strike K, results in a lower expected terminal value of the OBPI strategy $E(V_T^{OBPI})$. At the same time, the variance of the terminal value of the OBPI strategy $Var(V_T^{OBPI})$ decreases as the guarantee increases.

3.5.3 Comparison of the Greeks

Now, we aim to analyze the hedge parameters, in other words, the *greeks*, of both portfolio strategies: the CPPI and the OBPI. This kind of analysis is also performed in Bertrand and Prigent (2005) [13] by considering the OBPI method with a call option. In our study, we obtain the hedge parameters of OBPI strategy with a put option. In terms of portfolio insurance, the greeks are the quantities representing the sensitivities of the value of the portfolio to a change of the parameter on which the value of the portfolio depends.

Before defining and examining the greeks of two portfolios CPPI and OBPI, we recall the values of these portfolios at any time t during the investment period $[0, T]$, as in follows:

The time t value of the CPPI portfolio is given as

$$V_t^{CPPI} = \alpha_T e^{-r(T-t)} V_0^{CPPI} + \beta_t S_t^m, \tag{3.16}$$

where β_t is defined in (3.13).

Therefore, the value of the OBPI portfolio at time t is defined as

$$V_t^{OBPI} = S_t + P(t, S_t, K), \tag{3.17}$$

where $P(t, S_t, K)$ is the Black-Scholes put option price given in (3.15).

Thus, we can evaluate the greeks of the portfolios with the derivatives of Equation (3.16) and Equation (3.17) with respect to the corresponding parameters.

3.5.3.1 The Delta

The *delta* measures the change in portfolio value over the small changes in asset price. For the OBPI strategy, the delta of the OBPI portfolio is given as

$$\Delta_t^{OBPI} = \frac{\partial V_t^{OBPI}}{\partial S_t} = N(d_1).$$

The delta of the CPPI strategy is defined by

$$\Delta_t^{CPPI} = \frac{\partial V_t^{CPPI}}{\partial S_t} = \beta_t m S_t^{m-1},$$

where β_t is given in Equation (3.13).

Figure 3.3 shows the deltas of two insurance strategies as a function of the risky asset value.

We can observe that the delta of CPPI becomes more convex as the multiple m increases and it can be greater than one. An increase in multiple m causes a significant increase in the sensitivity of the CPPI method to the changes in risky asset prices. On the other hand, the delta of OBPI also increases, but always stays below one. Moreover, for small values of the

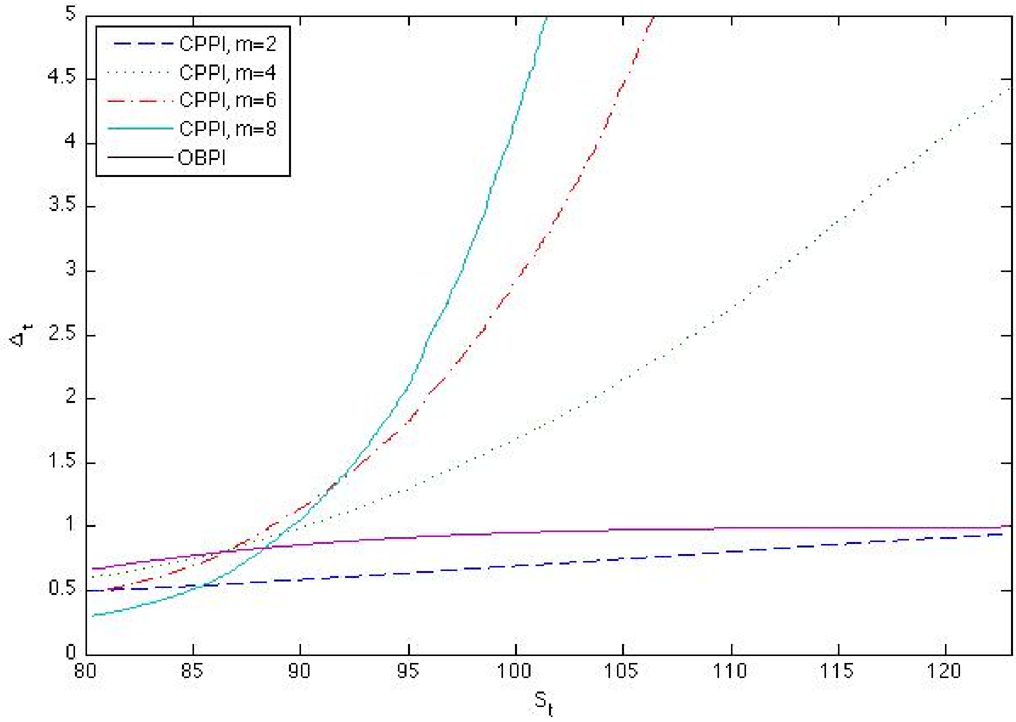

Figure 3.3: CPPI and OBPI deltas.

risky asset, the delta of the OBPI is greater than that of the CPPI. However, as the asset prices increase, the delta of the OBPI stays below the deltas of the CPPI with $m = 4, 6, 8$.

3.5.3.2 The Gamma

The *gamma* measures the change in the portfolio value over larger changes in the asset price. For larger changes, the delta does not accurately reflect the change in the portfolio value. For this reason, this risk is measured by the gamma.

For the OBPI strategy, the gamma of the OBPI portfolio is given by

$$\Gamma_t^{OBPI} = \frac{\partial^2 V_t^{OBPI}}{\partial S_t^2} = \frac{N(d_1)}{S_t \sigma \sqrt{T-t}}.$$

The gamma of the CPPI portfolio is

$$\Gamma_t^{CPPI} = \frac{\partial^2 V_t^{CPPI}}{\partial S_t^2} = \beta_t m(m-1) S_t^{m-2}.$$

Figure 3.4 shows the gammas of the two strategies, the CPPI gamma with various m values and the OBPI gamma, as functions of the risky asset value.

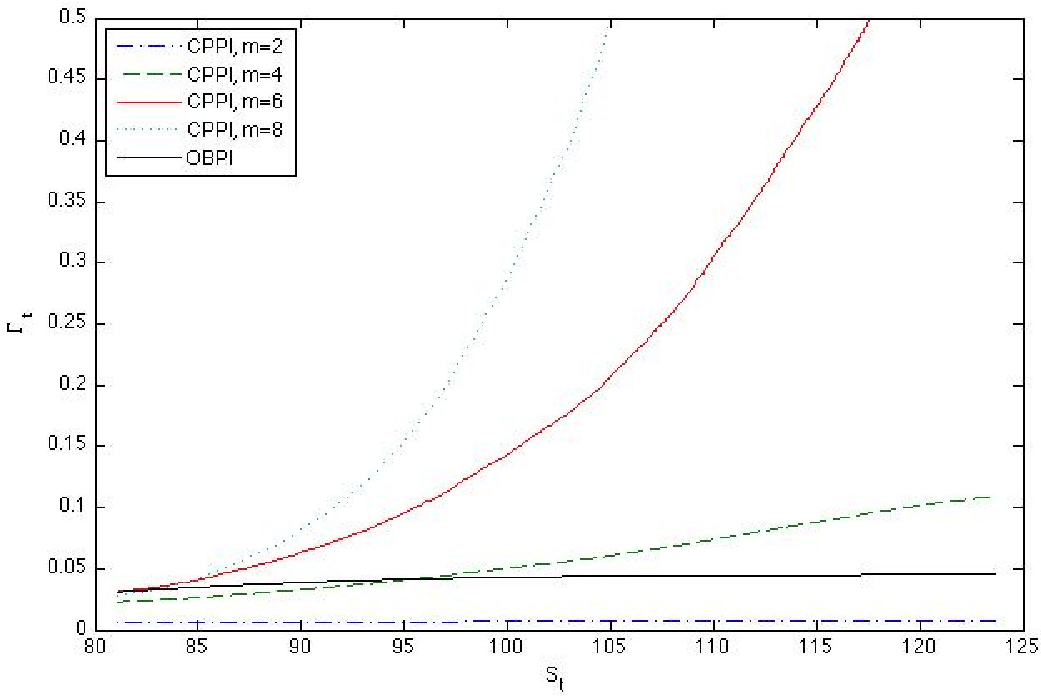

Figure 3.4: CPPI and OBPI gammas.

We remark that, not surprisingly, the gamma of the CPPI again becomes more convex as the multiple m increases. The OBPI gamma is greater than the CPPI gamma for smaller stock prices, but this is not always true for greater stock prices.

3.5.3.3 The Vega

The *vega* measures the changes in portfolio value over changes in volatility, i.e., it measures the risk of gain or loss resulting from changes in volatility. It is known that the volatility is the most critical variable in the Black-Scholes framework, because the values of portfolio components are very sensitive to volatility.

The vega of the OBPI portfolio is the vega of the put which is given by

$$Vega_t^{OBPI} = \frac{\partial V_t^{OBPI}}{\partial \sigma} = S_t N(d_2)\sqrt{T-t}.$$

For the CPPI portfolio, the vega is defined as

$$Vega_t^{CPPI} = \frac{\partial V_t^{CPPI}}{\partial \sigma} = ((m - m^2)\sigma t)V_t^{CPPI}.$$

Thus, the sensitivity of the CPPI value with respect to the volatility is negative as $m > 1$.

Figure 3.5 shows the vegas of CPPI and OBPI strategies.

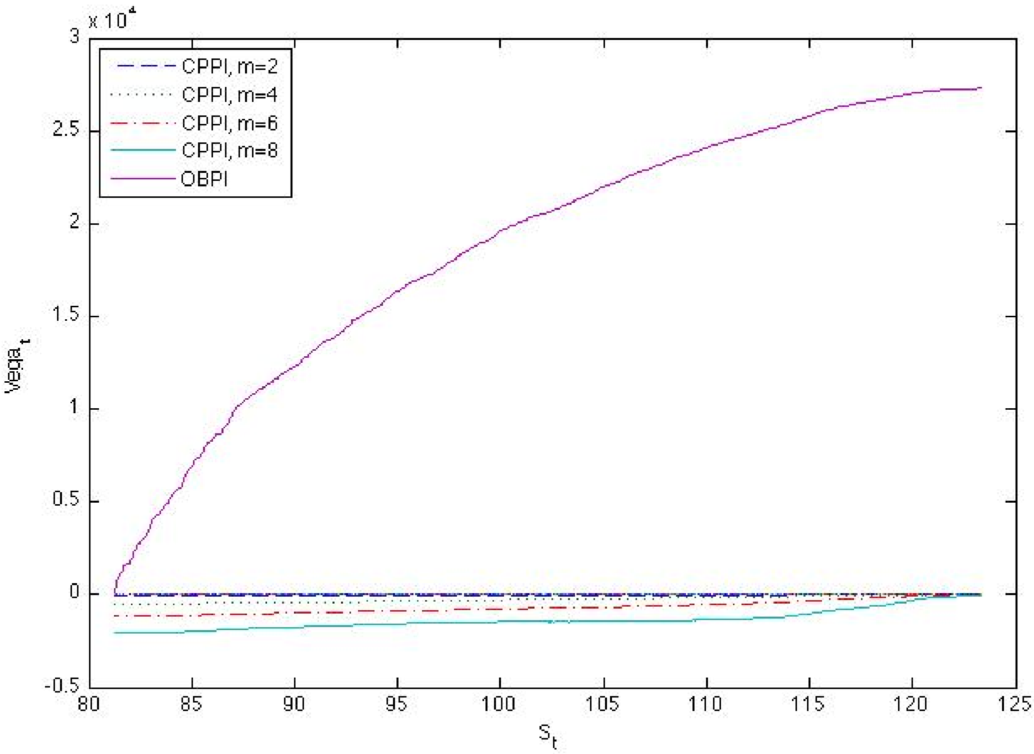

Figure 3.5: CPPI and OBPI vegas.

We can see from the graph of Figure 3.5 that the behaviour of the vegas of two strategies are very different. For CPPI portfolio, the vega is negative and it decreases as the multiple m increases. However, the OBPI vega is always above zero.

CHAPTER 4

OPTIMIZATION OF THE PORTFOLIO INSURANCE STRATEGIES AND OPTIMAL PORTFOLIO SELECTION

4.1 Introduction

The Portfolio Optimization framework attempts to calculate the optimal asset allocations for a portfolio that gives the highest return for the least risk.

In modern portfolio theory, the two fund separation princible was introduced by Markowitz (1959) [56] and extended by Merton (1971, 1973) [58, 59], which implies that the optimal asset allocation is obtained by using two basic funds: the first one is a combination of risky assets and the second one is the risky-free asset. The CPPI strategy is an application of this two fund separation princible which allocates the assets held in portfolio by rebalancing them between risky assets and risk-free asset. On the other hand, further studies have extended this two fund separation princible to three fund by adding a third fund which is a derivative written on the portfolio of risky assets. The OBPI strategy can be considered as an example of this three fund separation princible. As stated before in the previous chapter, we consider the OBPI strategy with two funds, one is the risky assets and the other is an option, to make the simulations and the comparisons easier. The first fund, the risky assets, can be considered as an allocation of assets including both risky assets and a risk-free asset. Now, we focus on an advanced OBPI method with three funds which is more challenging and commonly used in practice.

To manage a portfolio insurance strategy, the investor should first choose the *unconstrained (uninsured) allocation* based on his risk preferences without considering his desire of the guarantee, then he should insure his portfolio by specifying a strategy which is called the insured allocation.

This chapter includes a careful and detailed study of two important articles El Karoui, Jeanblanc and Lacoste (2005) [34], and Balder and Mahayni (2009) [54].

First, we focus on the optimality of the OBPI strategy specified by a particular underlying and a particular put option on this underlying. The optimality of OBPI strategy has already been studied and proven by the work of El Karoui, Jeanblanc and Lacoste (2005) [34] by considering an OBPI strategy written on the optimal solution of the Merton problem, which is presented in Chapter 2. In other words, their work considers the unconstrained allocation of OBPI strategy as the solution of the portfolio optimization problem with no constraints. For the optimality of such an OBPI strategy, first, the existence and uniqueness of the appropriate number of shares invested in the unconstrained allocation must be proven. It is shown by El Karoui, Jeanblanc and Lacoste (2005) [34] via assuming the initial values of the asset price S_0 and of the initial investment V_0 are equal to 1. Here, in this study, we extend their work to any values of S_0 and V_0 as a contribution. This is the only contribution that occurs in Proposition 4.1. After that, we present the optimality results and their proofs, which are already published in El Karoui, Jeanblanc and Lacoste (2005) [34], by providing further details of the ideas and arguments used in this paper.

Second, we review the derivations of the optimal solutions of the CPPI and OBPI strategies by using a benchmark strategy, the constant mix (CM) strategy, in order to compare their optimal solutions. This idea is presented by Balder and Mahayni (2009) [54], and the comparison of the optimal solutions is performed. In this study, we provide a detailed exposition on their derivation and comparison. Finally, we perform a numerical example on the optimal solutions of three strategies CM, CPPI and OBPI, and we get the same graphs, to replicate their results.

4.2 Optimal OBPI Strategy

The content of this section which is used to obtain the optimal OBPI strategy is taken from El Karoui, Jeanblanc and Lacoste (2005) [34], and it includes a revision of their setup and results with some minor contributions in Proposition 4.1 by proving the existence and uniqueness of the appropriate number of shares without assuming the initial asset price S_0 and the initial investment V_0 are normalized at 1. We also use their notation to be consistent with the prior study. Therefore, in Proposition 4.2 and Proposition 4.3, we provide further details of the arguments and ideas used in this paper while proving the optimality of OBPI strategy.

The OBPI strategy is designed as a portfolio insurance strategy to protect the portfolio value at maturity by using a put option written on the same portfolio (the unconstrained allocation). One considers the OBPI strategy as a combination of investing the λ shares of the initial investment in the portfolio and investing the remaining part in a put option written on the same portfolio.

Let $\mathcal{S}_t$ denote the time t value of the unconstrained optimal allocation where $\mathcal{S}_0 \geq 0$. (We notice that by $\mathcal{S}$ denotes the unconstrained allocation and by S the risky asset itself in the market). Consequently, a number of λ shares (as a fraction of the initial investment) is invested in the unconstrained allocation at time 0 evolves in the future following $(\lambda\mathcal{S}_t)_{t\geq 0}$. In order to be precise, the market is assumed to be complete, arbitrage free and frictionless and $(\mathcal{S}_t)_{t\geq 0}$ follows a continous diffusion process. All dividends and coupons are assumed to be reinvested in such a way that the allocation remains to be self financing.

For the optimization of the OBPI portfolio, we focus on the optimization problem in which the investor wants to maximize his expected utility over all self-financing portfolios with the values satisfying the European constraint.

Now, consider the OBPI strategy with a put option written on the optimal solution of the unconstrained problem. In other words, as a first step, one needs to determine the unconstrained (uninsured) allocation $\mathcal{S}$ of the OBPI strategy as the optimal solution of the unconstrained problem described in the previous section; then, as a second step, one defines the insured allocation to achieve the guarantee over the unconstrained portfolio. To define the insured allocation, we use the European put option with underlying being the optimal unconstrained portfolio with the strike being the guarantee itself. When it is mentioned that the protection is European, which means that the guarantee holds only for the terminal date T, the European option is used that the investor cannot exercise it before the terminal date.

We construct the OBPI portfolio as follows. We assume that λ shares of the unconstrained allocation $\lambda\mathcal{S}_0$ are invested at date 0, with the value at the terminal date as $\lambda\mathcal{S}_T$, and the remaining part of the initial wealth, $V_0 - \lambda\mathcal{S}_0$, is used to purchase a put option written on the same underlying λS_t with strike K. We note that the strike price K is the guaranteed amount at the terminal date, i.e., $K = F_T$. We also state that under no-arbitrage condition, the initial value of the portfolio V_0 must be greater than or equal to the time 0 value of the guarantee $Ke^{-rT} = F_0$. If $V_0 = Ke^{-rT}$, the investor has to invest all his initial wealth in the risk-free asset. Since we do not find this case necessary to investigate in the concept of this chapter, we assume that $V_0 > Ke^{-rT}$.

The payoff of the put option at maturity is $\Psi_T = \max\{K - \lambda\mathcal{S}_T, 0\}$. Thus, the payoff of the OBPI strategy at maturity is given by

$$V_T(\lambda) = \lambda\mathcal{S}_T + (K - \lambda\mathcal{S}_T)^+ = \sup\{\lambda\mathcal{S}_T, K\}.$$

It is important to highlight that the value λ, the so-called *leverage* or *gearing* parameter of the fund, must be determined at the initial date and has to satisfy the initial budget $V_0(\lambda) = \lambda\mathcal{S}_0 + P_{\lambda\mathcal{S}}(0, T, K)$, where $P_{\lambda\mathcal{S}}(0, T, K)$ is the time 0 value of the European put option written on λ units of the unconstrained allocation with maturity T and strike K. Thus, for a fixed value of V_0, an appropriate λ value needs to be determined in a way that it satisfies the initial budget condition. The existence and uniqueness of such a λ in this context has been shown by El Karoui, Jeanblanc and Lacoste (2005) [34] via assuming that the initial values of the asset price S_0 and of the initial investment V_0 are normalized to 1, in [34, Proposition 2.1, page 453]. Here, in this study, we extend their work to any values of S_0 and V_0, as a contribution. The proof remains the same except for some minor differences. We also provide all the steps of the proof that appear in [34, Proposition 2.1, page 453]. For this purpose, we give the following proposition.

Proposition 4.1 There exists a unique λ with $0 < \lambda < (V_0/\mathcal{S}_0)$, such that

$$V_0(\lambda) = \lambda\mathcal{S}_0 + P_{\lambda\mathcal{S}}(0, T, K).$$

Proof: The terminal value of the fund is $V_T(\lambda) = \sup\{\lambda\mathcal{S}_T, K\}$ is a non-decreasing function with respect to $\lambda \in \mathbb{R}_+$ and valued in [K,+∞). Thus, for any $\lambda' < \lambda$, it holds

$$V_T(\lambda') \le V_T(\lambda) \Rightarrow 0 \le V_T(\lambda) - V_T(\lambda').$$

The following equalities and inequalities are to be understood in the sense of being almost surely.

If $\lambda\mathcal{S}_T \geq K$ and $\lambda'\mathcal{S}_T \geq K$, then $V_T(\lambda) = \lambda\mathcal{S}_T$ and $V_T(\lambda') = \lambda'\mathcal{S}_T$. Thus, $V_T(\lambda) - V_T(\lambda') = \lambda\mathcal{S}_T - \lambda'\mathcal{S}_T$.

If $\lambda\mathcal{S}_T \leq K$ and $\lambda'\mathcal{S}_T \leq K$, then $V_T(\lambda) = K \geq \lambda\mathcal{S}_T$ and $V_T(\lambda') = K \geq \lambda'\mathcal{S}_T$. Thus, $V_T(\lambda) - V_T(\lambda') = 0$.

If $\lambda\mathcal{S}_T \geq K$ and $\lambda'\mathcal{S}_T \leq K$, then $V_T(\lambda) = \lambda\mathcal{S}_T$ and $V_T(\lambda') = K \geq \lambda'\mathcal{S}_T$. Thus, $V_T(\lambda) - V_T(\lambda') \leq \lambda\mathcal{S}_T - \lambda'\mathcal{S}_T$.

If $\lambda\mathcal{S}_T \leq K$ and $\lambda'\mathcal{S}_T \geq K$, then this contradicts with the initial assumption of $\lambda' < \lambda$.

Thus, in any case, we have

$$0 \leq V_T(\lambda) - V_T(\lambda') \leq \lambda\mathcal{S}_T - \lambda'\mathcal{S}_T = (\lambda - \lambda')\mathcal{S}_T.$$

Then, by discounting, we get

$$\tilde{V}_T(\lambda) - \tilde{V}_T(\lambda') \leq (\lambda - \lambda')\tilde{\mathcal{S}}_T.$$

Since, $\tilde{V}_T$ and $\tilde{S}_T$ are martingales under the equivalent martingale measure, $E^{\mathbb{Q}}(\tilde{V}_T) = V_0$ and $E^{\mathbb{Q}}(\tilde{\mathcal{S}}_T) = \mathcal{S}_0$. Thus, by taking the expectation under $\mathbb{Q}$, we get

$$V_0(\lambda) - V_0(\lambda') \leq (\lambda - \lambda')\mathcal{S}_0.$$

We can see that the initial value of the investment $V_0(\lambda)$ is a non-decreasing and Lipschitzian function with respect to λ with a Lipschitz constant equal to $\mathcal{S}_0$. Since it is Lipschitz continuous, we can use the *Intermediate Value Theorem*. Since we consider V_0 as a function of λ, we need to denote constantly by $V_0 = x$ as the initial value of the wealth.

Let $\lambda \in I = (0, x/\mathcal{S}_0)$ and function V_0 defined on the interval I, then the image set $V_0(I)$ is also an interval which contains either $(V_0(0), V_0(x/\mathcal{S}_0))$ or $(V_0(x/\mathcal{S}_0), V_0(0))$. We know that $V_T(\lambda) = \sup\{\lambda\mathcal{S}_T, K\}$ is valued in $[K, +\infty)$, therefore $V_0(\lambda)$ is valued in $[Ke^{-rT}, +\infty)$. On one hand, we know that $Ke^{-rT} < V_0 = x$, $V_0(0) < x$ since, when $\lambda = 0$, this means that the investor has no investment in the risky asset. On the other hand $V_0(x/\mathcal{S}_0) = x + P_{\lambda\mathcal{S}}(0, T, K) > x$. Hence, there exists a unique $\lambda \in (0, x/\mathcal{S}_0)$ which satisfies the initial budget condition, where x is the initial value of the investment equal to V_0.

Now, for the optimality of the OBPI portfolio strategy with European constraints, we present the following maximization problem and its solution which is given in El Karoui, Jeanblanc and Lacoste (2005) [34].

$$\begin{aligned} \text{maximize} \quad & E[u(V_T^{OBPI})] \\ \text{subject to} \quad & V_T \geq K, \\ & V_0 = x. \end{aligned} \tag{4.1}$$

By $\hat{V}_T$ we denote the optimal terminal strategy. If $\hat{V}_T$ is optimal, for any V_T, it satisfies the first-order condition

$$E(u'(\hat{V}_T)(V_T - \hat{V}_T)) = 0.$$

We give the following proposition for the optimality of the OBPI strategy due to the CRRA utility by providing all the steps of the proof that appear in El Karoui, Jeanblanc and Lacoste (2005) [34].

We recall that, in the case of CRRA utility function, since the optimal terminal value of the unconstrained problem is linear with respect to the initial wealth, it justifies that we construct the OPBI strategy by a put option written on the underlying $\lambda\mathcal{S}_T$. We remind that $u'(\lambda\mathcal{S}_T) = \lambda^{-\gamma}u'(\mathcal{S}_T)$ and $\hat{V}_T = \max\{\lambda\mathcal{S}_T, K\} \geq K$, since these are used in the proof of the below proposition.

Proposition 4.2 [34, Proposition 2.2, page 455] The put-based strategy written on the optimal portfolio with no constraint solves the optimization problem with European constraint for CRRA utility functions.

More precisely, if V_T is the terminal value of a self-financing portfolio with initial wealth V_0

such that $V_T \geq K$ a.s. and $\hat{V}_T$ is the terminal value of the put-based strategy, then

$$E[u(\hat{V}_T)] \geq E[u(V_T)].$$

Proof: We would like to prove that $E[u(V_T) - u(\hat{V}_T)] \leq 0$. Because of the concavity of the utility function u, we can write that

$$u(V_T) - u(\hat{V}_T) \leq u'(\hat{V}_T)(V_T - \hat{V}_T). \tag{4.2}$$

Since $\hat{V}_T = \max\{\lambda \mathcal{S}_T, K\} \geq K$ and $u'(\lambda \mathcal{S}_T) = \lambda^{-\gamma} u'(\mathcal{S}_T)$, we can write $u'(\hat{V}_T)$ as

$$u'(\hat{V}_t) = u'(\lambda \mathcal{S}_T) \wedge u'(K) = [\lambda^{-\gamma} u'(\mathcal{S}_T)] \wedge u'(K).$$

Therefore, since $u'(\hat{V}_T) \geq u'(K)$ is equivalent to $\hat{V}_T = K$ due to the constraint $\hat{V}_T \geq K$, and the decreasing property of u', the right-hand side of Inequality (4.2) becomes

$$\begin{aligned} u'(\hat{V}_T)(V_T - \hat{V}_T) &= [[\lambda^{-\gamma} u'(\mathcal{S}_T)] \wedge u'(K)](V_T - \hat{V}_T) \\ &= \lambda^{-\gamma} u'(\mathcal{S}_T)(V_T - \hat{V}_T) - [\lambda^{-\gamma} u'(\mathcal{S}_T) - u'(K)]^+ (V_T - K). \end{aligned}$$

Then, by taking expectation, we get

$$E[u'(\hat{V}_T)(V_T - \hat{V}_T)] = \lambda^{-\gamma} E[u'(\mathcal{S}_T)(V_T - \hat{V}_T)] - E[[\lambda^{-\gamma} u'(\mathcal{S}_T) - u'(K)]^+ (V_T - K)].$$

On the other hand, we can write the first order condition in the form of

$$E(u'(\mathcal{S}_T)(V_T - \hat{V}_T)) = E[u'(\mathcal{S}_T)(V_T - \hat{V}_T)] + E[u'(\mathcal{S}_T)(\hat{V}_T - \hat{V}_T)] = 0.$$

From the constraint on the terminal wealth $V_T \geq K$, we obtain

$$-E([\lambda^{-\gamma} u'(\mathcal{S}_T) - u'(K)]^+ (V_T - K)) \leq 0.$$

Consequently, we get

$$E[u(V_T) - u(\hat{V}_T)] \le 0.$$

We remark that, the optimality of the OBPI strategy as a put-based strategy is proven where the utility function is considered as CRRA utility. However, this optimality result is extended into the general class of utility functions by El Karoui, Jeanblanc and Lacoste (2005) [34]. In order to do this, one should again consider the OBPI strategy with a put written on the optimal solution of the unconstrained problem, but the linearity property with respect to the initial wealth coming from the CRRA utility function may not be available anymore. Thus, to construct the insured allocation of the OBPI strategy, the unconstrained allocation can be considered as to be parameterized by y, rather than λ, as

$$\mathcal{S}_T(y) = (u')^{-1}(\frac{H_T}{y}),$$

where $H_T = e^{-rt}Z_T$, Z_T is the density of the martingale measure and y is the Lagrangian multiplier which are presented in the previous sections.

Hence, the optimal terminal value of the put-based strategy can be considered as

$$\hat{V}_T = \max\{\mathcal{S}_T(y), K\}.$$

We give the following proposition by providing all the steps of the proof that appear in El Karoui, Jeanblanc and Lacoste (2005) [34].

Proposition 4.3 [34, Proposition 5.1, page 465] The put-based strategy written on the optimal portfolio with no constraint solves the optimization problem with a European constraint for any utility function.

Proof: The proof is similar to CRRA case, but we consider that the optimal solution of the unconstrained problem as $\mathcal{S}_T(y) = (u')^{-1}(H_T/y)$, rather than $\lambda\mathcal{S}_T$, and the optimal terminal value of the put-based strategy as $\hat{V}_T = \max\{\mathcal{S}_T(y), K\} \geq K$, instead of $\hat{V}_T = \max\{\lambda \mathcal{S}_T, K\}$. Then, the first-order condition takes the form of $E(H_T(V_T - \mathcal{S}_T(y)) = 0$.

Again, we would like to prove that $E[u(V_T) - u(\hat{V}_T)] \leq 0$. By concavity of the utility function u,

$$u(V_T) - u(\hat{V}_T) \leq u'(\hat{V}_T)(V_T - \hat{V}_T). \tag{4.3}$$

Since $\mathcal{S}_T(y) = (u')^{-1}(H_T/y)$, we have that $u'(\mathcal{S}_T(y)) = y^{-1}H_T$. Then, we can write $u'(\hat{V}_T)$ as

$$u'(\hat{V}_t) = u'(\mathcal{S}_T(y)) \wedge u'(K) = (y^{-1}H_T) \wedge u'(K).$$

Then, as before, we have

$$u(V_T) - u(\hat{V}_T) \leq u'(\hat{V}_T)(V_T - \hat{V}_T) = [y^{-1}H_T \wedge u'(K)](V_T - \hat{V}_T).$$

The right-hand side of this last equation is equal to

$$y^{-1}H_T(V_T - \hat{V}_T) - [y^{-1}H_T - u'(K)]^+(V_T - K).$$

From the first-order condition, we obtain

$$E[H_T(V_T - \hat{V}_T)] = 0,$$

and from the terminal constraint $V_T \geq K$, we get

$$E[u(V_T) - u(\hat{V}_T)] = -E([u'(\mathcal{S}_T(y)) - u'(K)]^+(V_T - K)) \leq 0.$$

Consequently, we receive the relation $E(u(\hat{V}_T)) \geq E(u(V_T))$.

Hence, we see that the optimal solution of the problem with European constraint can be achieved by the put-based strategy written on the optimal portfolio without any constraint and for any utility function.

We would also like to state that, in this study, we focus on the optimality of the OBPI strategy with European guarantee that the constraint is imposed on a terminal date. The case of the American guarantee, which offers the same guarantee for any terminal date between 0 and T, is also interesting. The optimality of the OBPI portfolio with American guarantee is proven in by El Karoui, Jeanblanc and Lacoste (2005) [34], where the American OBPI strategy is fully described in a Black-Scholes environment as well as in the more general case of complete markets.

4.3 Comparison of the Optimal Portfolio Insurance Strategies

In this section, we focus on another important paper published by Balder and Mahayni (2009) [54], which presents the derivations of the optimal solutions of CPPI and OBPI strategies, in terms of a benchmark strategy, the constant mix (CM) strategy. Their work provides a useful framework in order to compare the optimal solutions of portfolio insurance strategies. As far as we have examined, we can understand that the idea of using CM strategy as a benchmark strategy enables to make comparisons between an uninsured portfolio strategy and an insured portfolio strategy, since the CM strategy contains no guarantee component. On the other hand, it also allows to compare our two portfolio strategies CPPI and OBPI.

Thoughout this section, we review and explain the setup included in Balder and Mahayni (2009) [54], and we briefly introduce the results in this article. The notation and math display are taken from this paper. Finally, we perform a numerical example based on the same setup to replicate their results due to the behaviour of the optimal solutions of the strategies, CM, CPPI and OBPI. Consequently, our graph agrees with their graph.

While constructing the optimization problems, we assume that all stochastic processes are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We consider two assets, a risky asset and a risk-free asset. Therefore, we again assume the standard conventions related to market modelling.

Let $(\phi_t)_{0\leq t\leq T}$ represent the investment strategy for the investment period $[0, T]$. Here, $(\phi_t)_{0\leq t\leq T}$ denotes the proportion invested in the risky asset S, and ϕ denotes the portfolio weight, at time t. Thus, the remaining $(1-\phi)$ is the proportion invested in the riskless asset B.

The value portfolio at time t associated with the strategy ϕ, is usually denoted by $V_t^{\phi}(\omega)$ for any event $\omega \in \Omega$, however, we denote it by $V_t(\phi)$, to be proper and consistent with the terminology of this chapter, which requires to determine the value processes for different strategies. Since we consider that the strategies are self financing, the dynamics of the value of portfolio is given by

$$
\begin{aligned}
dV_t(\phi) &= V_t\left(\phi\frac{dS_t}{S_t} + (1-\phi)\frac{dB_t}{B_t}\right) \\
&= V_t((\phi(\mu - r) + r)dt + \phi\sigma dW_t),
\end{aligned}
$$

where $V_0 = x$.

For the portfolio insurance strategies, we consider the following portfolio optimization problem, in which Φ denotes the set of all self-financing strategies (we remind that it is denoted by $\mathcal{A}$ in the previous section, with an emphasize on being an admissible set).

$$
\begin{aligned}
&\underset{\phi\in\Phi}{\text{maximize}} \quad E[u(V_T(\phi))] \\
&\text{subject to} \quad dV_t(\phi) = V_t((\phi(\mu - r) + r)dt + \phi\sigma dW_t), \\
&\qquad\qquad\quad V_0 = x,
\end{aligned} \tag{4.4}
$$

where u ($u \in C^2$) is the utility function which is strictly increasing and strictly concave.

We now examine the derivations of the optimal solutions of CM, CPPI and OBPI strategies, according to the setup and steps used in Balder and Mahayni (2009) [54]. The following subsections shortly reintroduces the derivations of the optimal solutions and reviews the results, which are already computed and presented in the referred article.

4.3.1 Optimality for the CM Strategy

The CM strategy is a dynamic strategy which maintains an exposure to stocks that is a constant portfolio weight m. Whenever the values of assets change, the investor is required to rebalance the portfolio to return to the desired portfolio weight m. Thus, the portfolio strategy class for the CM strategy can be represented as

$$\Phi^{CM} = \{\phi \in \Phi \mid \phi_t = m, m \geq 0\} = \mathbb{R}^+,$$

while considering the optimization problem (4.4) as

$$\begin{aligned} \underset{\phi \in \Phi^{CM}}{\text{maximize}} \quad & E[u(V_T^{CM}(\phi))] \\ \text{subject to} \quad & dV_t^{CM}(\phi) = V_t^{CM}((m(\mu - r) + r)dt + m\sigma dW_t), \\ & V_0^{CM} = x, \end{aligned}$$

where the utility function is CRRA utility $u(V_T) = V_T^{1-\gamma}/(1-\gamma)$, $\gamma \in (0, 1)$.

Since $\phi \in \Phi^{CM}$, the portfolio value at the terminal date can be obtained as follows:

$$\begin{aligned} dV_t^{CM} &= V_t^{CM}((m(\mu - r) + r)dt + m\sigma dW_t) \\ \Rightarrow V_t^{CM} &= V_0^{CM} e^{(m(\mu - r) + r - \frac{1}{2}m^2\sigma^2)t + m\sigma W_t} \\ \Rightarrow V_T^{CM} &= V_0^{CM} e^{(m(\mu - r) + r - \frac{1}{2}m^2\sigma^2)T + m\sigma W_T}. \end{aligned} \tag{4.5}$$

From the solution of the stochastic differential equation for the risky asset price dynamics, which is given in (3.3), we can derive σW_T as

$$\sigma W_T = \ln \frac{S_T}{S_0} - (\mu - \frac{1}{2}\sigma^2)T. \tag{4.6}$$

By inserting (4.6) into (4.5), it follows that

$$\begin{aligned} V_T^{CM} &= V_0^{CM} e^{(m(\mu - r) + r - \frac{1}{2}m^2\sigma^2)T + m(\mu - \frac{1}{2}\sigma^2)T} \left(\frac{S_T}{S_0}\right)^m \\ &= f(V_0^{CM}, m) S_T^m, \end{aligned}$$

where

$$f(x,y) = x\left(\frac{1}{S_0}\right)^y e^{(1-y)(r+\frac{1}{2}y\sigma^2)T}.$$

Then, the expected utility of the terminal value of the CM strategy is

$$\begin{aligned} E[u(V_T^{CM}] &= E\left[\frac{(f(V_0^{CM},m)S_T^m)^{1-\gamma}}{1-\gamma}\right] \\ &= \frac{f(V_0^{CM},m)^{1-\gamma}}{1-\gamma}E[S_T^{m(1-\gamma)}]. \end{aligned}$$

Since the expectation of a lognormally distributed random variable X, with $\ln X \sim N(\mu, \sigma^2)$ is given by

$$E(X) = e^{\mu+\frac{\sigma^2}{2}},$$

we can compute the expected utility of the terminal value of the CM portfolio, as

$$E[u(V_T^{CM}] = \frac{V_0^{1-\gamma}}{1-\gamma}e^{(1-\gamma)(m(\mu-r)+r-\frac{1}{2}\gamma m^2\sigma^2)T}.$$

Finally, by the strictly monotonical increasing of the exponential function, we can easily find the maximizer of the expected utility of the terminal value as

$$\text{argmax}_m E[u(V_T^{CM}] = \frac{\mu - r}{\gamma\sigma^2} =: m^*.$$

Thus, $\phi^{CM} = m^*$ is the optimal portfolio strategy which gives the optimal solution of CM strategy as

$$\hat{V}_T^{CM} = f(V_0^{CM}, m^*)S_T^{m^*}. \tag{4.7}$$

4.3.2 Optimality for the CPPI Strategy

Now, by following the paths of Balder and Mahayni (2009) [54], we reintroduce the CPPI strategy as a modified CM strategy. As it is already done in their extensive work, the optimal solution of the CPPI strategy can be determined in terms of the optimal solution of the CM strategy, as in follows.

We recall that $(F_t)_{0\le t\le T}$ denotes the value of the guarantee, the so-called *floor* of the insurance strategy, and $(C_t)_{0\le t\le T}$ denotes the *cushion* process that is defined as the difference between the portfolio value and the given guarantee $C_t = V_t^{CPPI} - F_t$.

We represent the portfolio strategy class for the CPPI strategy, in terms of the delta hedge of an option payoff, as

$$\Phi^{CPPI} = \{\phi \in \Phi \mid \phi_t = m\frac{V_t - e^{-r(T-t)}F_T}{V_t} = m\frac{C_t}{V_t}, m \ge 0\}.$$

Let us note that for $F_T = 0$, $\Phi^{CPPI} = \Phi^{CM}$. Hence, if we consider that the investor derives utility from the cushion C_t, not from the portfolio value V_t, then the CPPI strategy is equivalent to the CM strategy. In fact, if a constant proportion m is invested in the risky asset, then the dynamics of the investment is given by

$$\begin{aligned} dC_t &= C_t((m(\mu - r) + r)dt + m\sigma dW_t) \\ \Rightarrow C_t &= C_0 e^{(m(\mu-r)+r-\frac{1}{2}m^2\sigma^2)t+m\sigma W_t} \\ \Rightarrow C_T &= C_0 e^{(m(\mu-r)+r-\frac{1}{2}m^2\sigma^2)T+m\sigma W_T}. \end{aligned} \tag{4.8}$$

Again, by inserting σW_T given in (4.6) into (4.8), it follows that

$$\begin{aligned} C_T &= C_0 e^{(m(\mu-r)+r-\frac{1}{2}m^2\sigma^2)T+m(\mu-\frac{1}{2}\sigma^2)T}\left(\frac{S_T}{S_0}\right)^m \\ &= f(C_0, m)S_T^m. \end{aligned}$$

Thus, the optimization problem for CPPI case is

$$\begin{aligned} \underset{\phi\in\Phi^{CPPI}}{\text{maximize}} \quad & E[u(C_T(\phi))] \\ \text{subject to} \quad & dC_t = C_t((m(\mu - r) + r)dt + m\sigma dW_t), \\ & C_0 = x. \end{aligned}$$

As well as the CM strategy, the maximizer of the expected utility of the CPPI cushion $E[u(C_T)]$ turns out to be

$$\text{argmax}_m E[u(C_T] = \frac{\mu - r}{\gamma\sigma^2} =: m^*.$$

Hence, the optimal solution of the problem above is found as

$$\hat{C}_T = f(C_0, m^*) S_T^{m^*}.$$

We remark that this solution is found by assuming that the investor derives utility from the cushion C_t, not from the portfolio value V_t. Thus, by replacing C_t with $V_t^{CPPI} - F_t$, we can find the optimal solution for CPPI strategy as

$$\hat{V}_T^{CPPI} = f(V_0 - e^{-rT} F_T, m^*) S_T^{m^*} + F_T. \tag{4.9}$$

On the other hand, the optimal strategy for CPPI case can be derived as

$$\phi^{CPPI} = \frac{m^* C_t}{V_t} = \frac{m^*(V_t - e^{-r(T-t)} F_T)}{V_t}.$$

Hence, we notice that

$$\phi^{CPPI} \leq m^* = \phi^{CM},$$

which means a reduction in the invested proportion of the CPPI strategy when we compare it with the invested proportion of the CM strategy, and it causes a loss in expected utility coming from the guarantee $F_T > 0$.

4.3.3 Optimality for the OBPI Strategy

As we mentioned previously in this chapter, the optimality of OBPI strategy is proven by El Karoui, Jeanblanc and Lacoste (2005) [34]. According to their optimal solution, the work of Balder and Mahayni (2009) [54] introduces a new derivation of this optimal solution in terms of the optimal solution of CM strategy, by the help of the power call option pricing formula. Here, we only give the final findings of their study in order to review the comparisons. Further explanations and details of formulations can be found in Balder and Mahayni (2009) [54].

We recall that the OBPI strategy with a put option written on the optimal solution $\lambda\mathcal{S}_T$ of the problem with European constraint solves the optimization problem with the European constraint $V_T \geq K$, where $\mathcal{S}_T$ denotes the unconstrained allocation and K is the strike price of the put option which also represents the guaranteed amount F_T at time T. Furthermore, we recall that $\mathcal{S}$ denotes the unconstrained allocation and S stands for the risky asset itself in the market. Thus, we represent the function $h \in C^2$ as a function of the risky asset such that $h(S_T) = \lambda\mathcal{S}_T$ which implies the optimal solution with respect to the unconstrained allocation $\mathcal{S}_T$. Then, we consider the problem (4.4) as

$$\begin{aligned} \underset{\phi\in\Phi^{OBPI}}{\text{maximize}} \quad & E[u(V_T^{OBPI}(\phi))] \\ \text{subject to} \quad & V_T \geq F_T, \\ & V_0 = x, \end{aligned}$$

where the portfolio strategy class Φ^{OBPI} is represented, as

$$\Phi^{OBPI} = \left\{\phi \in \Phi \mid \phi = \frac{\Delta_t S_t}{V_t}, \Delta_t = \frac{\partial}{\partial S_t} E_{\mathbb{Q}}[e^{-r(T-t)}(h(S_T) - F_T)^+ | \mathcal{F}_t], \left(\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}\right)_T = e^{-\frac{1}{2}\theta^2 T - \theta W_T}\right\}. \tag{4.10}$$

Then, the optimal solution $\hat{V}_T^{OBPI}$ can be presented in the form of

$$\hat{V}_T^{OBPI} = \max\{h(S_T), F_T\} = h(S_T) + [F_T - h(S_T)]^+ = F_T + [h(S_T) - F_T]^+.$$

We remind that, for the OBPI strategy, the initial wealth is invested within two parts: the first part in the unconstrained allocation and the second part in the put option written on the same allocation with strike $K = F_T$, which represents the guarantee. If we assume that there was no guarantee, i.e., $F_T = 0$, all initial wealth would be invested in the unconstrained allocation which reduces the constrained optimization problem to the unconstrained one, as in the Merton case. Consequently, the optimal solution of the unconstrained problem can be represented as $h(S_T) = f(\tilde{V}_0, m^*)S_T^{m^*}$, where $\tilde{V}_0$ is the reduced initial investment caused by the guarantee component.

Hence, one can derive the optimal payoff of the OBPI strategy as

$$\hat{V}_T^{OBPI} = F_T + [f(\tilde{V}_0, m^*)S_T^{m^*} - F_T]^+. \tag{4.11}$$

By following the steps taken in Balder and Mahayni (2009) [54], the optimal solution of the OBPI strategy in terms of the optimal solution of CM strategy can be expressed as

$$\hat{V}_T^{OBPI} = \frac{\tilde{V}_0}{V_0}\hat{V}_T^{CM} + \left[F_T - \frac{\tilde{V}_0}{V_0}\hat{V}_T^{CM}\right]^+. \tag{4.12}$$

Furthermore, after calculations and formulations which are done in Balder and Mahayni (2009) [54], the optimal portfolio weight for the OBPI strategy can be obtained as

$$\phi^{OBPI} \le m^* = \phi^{CM}.$$

Similar to the CPPI strategy, the guarantee component, i.e., the terminal constraint $V_T \ge F_T$, of the OBPI strategy gives rise to a reduction of the optimal unconstrained portfolio weight m^*, when it is compared with the CM strategy.

4.3.4 Comparison of the Optimal Solutions

Finally, we summarize and review the optimality results of Balder and Mahayni (2009) [54]. The optimal terminal payoffs of the strategies CM, CPPI and OBPI are found as in follows:

$$\hat{V}_T^{CM} = f(V_0, m^*)S_T^{m^*}, \tag{4.13}$$

$$\begin{aligned}\hat{V}_T^{CPPI} &= F_T + f(V_0 - e^{-rT}F_T, m^*)S_T^{m^*} \\ &= F_T + \frac{V_0 - e^{-rT}F_T}{V_0}\hat{V}_T^{CM},\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
\hat{V}_T^{OBPI} &= F_T + \left[f(\tilde{V}_0, m^*) S_T^{m^*} - F_T\right]^+ \\
&= \frac{\tilde{V}_0}{V_0}\hat{V}_T^{CM} + \left[F_T - \frac{\tilde{V}_0}{V_0}\hat{V}_T^{CM}\right]^+ . \qquad (4.15)
\end{aligned}
$$

In the light of these findings, Balder and Mahayni (2009) [54] first observes the following situations:

> "The optimal terminal payoff of the CM strategy depends on the initial investment and the optimal investment proportion. Therefore, comparing the terminal payoffs shows that both portfolio insurance strategies, CPPI and OBPI, result in payoffs which consist of a fraction of the payoff of a constant mix strategy (which is optimal for the unconstrained CRRA investor) and an additional term due to the guarantee."

Therefore, they conclude with the result which is stated as:

> "The additional term provides an intuitive way to explain the main advantage of the OBPI approach as compared to the CPPI approach. Intiuitively, it is clear that the fraction of wealth which is put into the optimal unconstrained strategy is linlked to the price of the guarantee, i.e., the fraction is less than one. In the case of the CPPI approach, the additional term is simply the guarantee itself, i.e., the payoff of an adequate number of zero bonds. In contrast, the additional term implied by the OBPI is a put option where the underlying is given by the fraction of constant mix strategy and the strike is equal to the guarantee. Obviously, the put is cheaper than the zero bonds. Therefore, an investor who follows the OBPI approach puts a larger fraction of his wealth into the unconstrained optimal portfolio than the investor who follows the CPPI approach."

This means that the cost of the guarantee is cheaper for the OBPI strategy than the cost of the guantee for the CPPI strategy when the investor desires the same level of guarantee.

According to the results of the optimal solutions of the strategies CM, CPPI and OBPI, provided by Balder and Mahayni (2009) [54], we perform a numerical example in order to justify their findings and replicate their results.

Example 4.1 Figure 4.1 illustrates what happens with the optimal payoffs for a numerical example with the parameter constellation as in follows: μ=7.5%, σ=8% and r=3.5% ($\mu > r > 0$). In this market, portfolio strategies are set up assuming $T = 5$ years, S_0=1, K=1, α_T=1 and V_0=1.

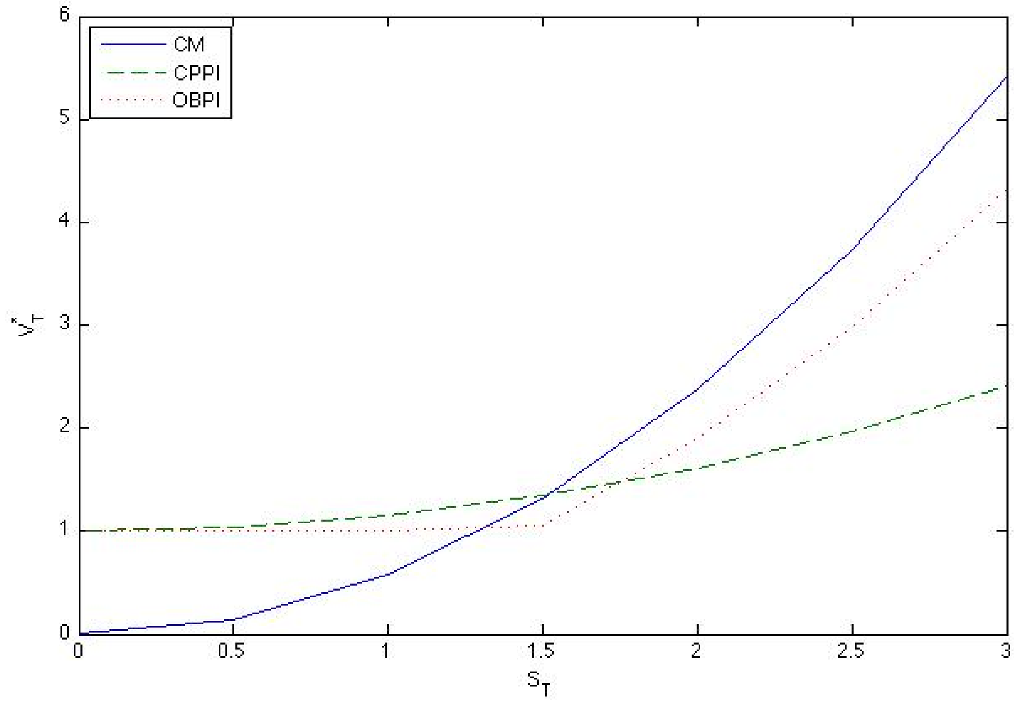

Figure 4.1: Optimal Payoffs of CM, CPPI and OBPI.

In Figure 4.1, we can see that the payoff of the CPPI strategy gives a smooth payoff, while the OBPI payoff provides a nonsmooth one.

Therefore, we can also observe the relation among the intersection points of the optimal terminal payoffs. This also justifies the results in Balder and Mahayni (2009) [54], in which the order of the intersection points that occur among the optimal payoffs of the strategies CM, CPPI and OBPI is given by

$$s_{(CM,OBPI)} \leq s_{(CM,CPPI)} \leq s_{(CPPI,OBPI)},$$

where the terminal asset price S_T is defined by $s_{(i,j)}$ such that $\hat{V}_T^i = \hat{V}_T^j$ for all $i, j \in$ {CM, CPPI,OBPI} with $i \neq j$.

CHAPTER 5

UTILITY LOSS OF THE PORTFOLIO INSURANCE STRATEGIES

5.1 Introduction

In this chapter, we review an important concept of portfolio insurance strategies: the utility loss. The concept of utility loss for insurance strategies depends on the following idea. As we can see from the terminal values of portfolios in the previous chapter, if the investor decides to insure his portfolio, at a fixed amount or a percentage of the initial wealth, against sharp downside movements of the market, the expected utility will be reduced in comparison with the uninsured portfolio. Consequently, limiting downside risk of portfolio to avoid large losses in portfolio value results in the sacrifice of potential upside gains.

The aim of this chapter is to understand and briefly review the utility loss of insurance strategies. For this purpose, throughout this chapter, we review the work of Balder and Mahayni (2009) [54] on the utility loss of CPPI and OBPI strategies without any contribution.

In the following sections, we shortly reintroduce the loss rates of CPPI and OBPI strategies caused by the guarantee component, and the comparison of these loss rates in terms of the CRRA utility function and the CM (constant mix) strategy. All the information included in this part of thesis is compiled from Balder and Mahayni (2009) [54].

5.2 Utility Loss Caused by the Guarantees

The expected utility criterion is an important measure for the performance of the strategies which can also be described by the concept of the certainty equivalent. According to Balder and Mahayni (2009) [54], the *certainty equivalent* is a certain amount that the investor receives, which makes him indifferent between achieving this amount or using the strategy. The notation, definitions and derivations below are taken from Balder and Mahayni (2009) [54] to explain the certainity equivalents, loss rates and utilty losses of portfolio insurance strategies.

Let $CE_T(\phi)$ denote the time T value of the certainty equivalent of the strategy ϕ, and it is defined by

$$u(CE_T(\phi)) = E(u(V_T(\phi))),$$

where E represents the expectation taken under the real-word probability $\mathbb{P}$.

Now, we reintroduce the determination of the certainty equivalents of the aforementioned strategies by considering their payoffs given in the previous chapter. We recall that the payoffs of the CM, CPPI and OBPI strategies are

$$V_T^{CM} = f(V_0^{CM}, m)S_T^m,$$

$$V_T^{CPPI} = F_T + f(V_0 - e^{-rT}F_T, m)S_T^m,$$

and

$$V_T^{OBPI} = F_T + \left[f(\tilde{V}_0, m)S_T^m - F_T\right]^+,$$

respectively. Again we consider the CM strategy as a benchmark strategy, to make comparisons both between insured and uninsured strategies and between the CPPI and the OBPI strategies.

In the following, we represent the certainty equivalent of the CM strategy with invested proportion m in the risky assets, and with respect to the CRRA utility function $u(x) = x^{1-\gamma}/(1-\gamma)$, $\gamma \in (0, 1)$.

Since it holds

$$\frac{(CE_T(\phi^{CM}))^{1-\gamma}}{1-\gamma} = E\Big(\frac{(f(V_0,m)S_T^m)^{1-\gamma}}{1-\gamma}\Big),$$

therefore,

$$CE_T(\phi^{CM}) = E(f(V_0,m)S_T^m).$$

We know that $E(S_T^n) = S_0^n e^{(n\mu-\frac{1}{2}n(1-n)\sigma^2)T}$. Thus, it follows

$$\begin{aligned} CE_T(\phi^{CM}) &= f(V_0,m)S_0^m e^{(m\mu-\frac{1}{2}\gamma\sigma^2 m(1-m(1-\gamma)))T} \\ &= V_0 e^{(r+m(\mu-r)-\frac{1}{2}\gamma\sigma^2 m^2)T}. \end{aligned}$$

Due to the above equation, we can obtain the certainty equivalent of the CPPI strategy, by replacing the payoffs:

$$CE_T(\phi^{CPPI}) = F_T + (V_0 - e^{-rT}F_T)e^{(r+m(\mu-r)-\frac{1}{2}\gamma\sigma^2 m^2)T}.$$

Thus, one can observe that the certainty equivalent of the CPPI strategy is lower than the certainty equivalent of the CM strategy, which means that the certainty equivalent is lower for an investor with insurance than an investor without.

Now, we reintroduce the *loss rate* which allows us to calculate the utility loss of an investment which is done under a suboptimal strategy. Let $l_{T,u}(\phi)$ denote the loss rate of the strategy ϕ and the utility function u at maturity. It is defined by

$$l_{T,u}(\phi) = \frac{1}{T}\ln\Big(\frac{CE_T(\phi^*)}{CE_T(\phi)}\Big), \tag{5.1}$$

where $CE_T(\phi^*)$ denotes the certainty equivalent of the optimal strategy $\phi^* = (\phi^*(t))_{0\le t\le T}$, and $CE_T(\phi)$ denotes one of the suboptimal strategies $\phi = (\phi(t))_{0\le t\le T}$.

Based on Formula (5.1), it is straightforward to calculate the loss rates with respect to the CRRA utility function and with the strategy parameter.

Table 5.1: Loss rates with respect to CRRA utility.

Strategy ϕ	Loss rate $l_{T,u}(\phi)$
$\phi^{CM} = m$	$\frac{1}{2}\gamma\sigma^2(m^* - m)^2$
$\phi^{CPPI} = m\frac{V_t - e^{-r(T-t)}F_T}{V_t}$	$\frac{1}{(1-\gamma)T} \ln\left(\frac{E[(f(V_0,m^*)S_T^{m^*})^{1-\gamma}]}{E[(F_T + f(V_0 - e^{-rT}F_T, m)S_T^m)^{1-\gamma}]}\right)$
$\phi^{OBPI} = \frac{\triangle^{PO}\left(t, S_t; m, \frac{F_T}{f(\tilde{V}_0, m^*)}\right)S_t}{V_t}$	$\frac{1}{(1-\gamma)T} \ln\left(\frac{E[(f(V_0,m^*)S_T^{m^*})^{1-\gamma}]}{E[(F_T + [f(\tilde{V}_0, m)S_T^m - F_T]^+)^{1-\gamma}]}\right)$

The loss rates of the strategies are given in Table 5.1 which is directly compiled from Balder and Mahayni (2009) [54].

According to Table 5.1, one can observe that for $m = m^*$, i.e., if the investor chooses the optimal strategy for his investment, we can order the loss rates as follows:

$$l_{T,u}(\phi^{CM}) = 0 < l_{T,u}(\phi^{OBPI}) < l_{T,u}(\phi^{CPPI}).$$

Consequently, for $m = m^*$, the utility loss caused by the guarantee is less for the OPBI strategy than the CPPI one. Thus, the CPPI investor, investing in the optimal strategy, is exposed to a higher loss caused by the same level of guarantee in comparison with the OBPI investor.

CHAPTER 6

CONCLUSION AND OUTLOOK

This thesis consists of a review of a several papers that explain and compare the portfolio insurance strategies CPPI and OBPI.

First of all, we have reviewed the study of Bertrand and Prigent (2005) [13], in which the comparison of CPPI and OBPI has been performed in terms of their statistical and dynamic properties, by considering the OBPI method based on a call option and by determining the CPPI multiple under the assumption of equality of portfolio returns. We have made a similar comparison by considering the OBPI strategy with a put option and CPPI strategy with various multiplier values. As this paper has found that the comparison with usual criteria such as payoff functions, expectations and variances do not allow one to discriminate clearly between two strategies. But still, one can make comparisons under some parameter specifications such as the multiplier and level of insurance, also under some market assumptions such as bearish or bullish market. Furthermore, we have investigated the effects of the changes in insured amount and multiplier on the portfolio returns and risks, while providing the analyses of the sensitivity parameters, i.e., greeks of two strategies.

Second, we have provided a detailed study of two important articles El Karoui, Jeanblanc and Lacoste (2005) [34], and Balder and Mahayni (2009) [54]. One of the contributions of this thesis is our extension of Proposition 2.1 of El Karoui, Jeanblanc and Lacoste (2005) [34] (which is stated in the introduction). Therefore, we have reviewed Balder and Mahayni (2009) [54], where the derivations of the optimal solutions of the CPPI and OBPI strategies by using a benchmark strategy, the constant mix (CM) strategy, to compare their optimal solutions. We have replicated their conclusions that the guarantee of the OBPI strategy is cheaper than the price of the guarantee of the CPPI strategy. At last, we have performed a numerical example to see the behaviour of the optimal payoffs of CPPI and OBPI strategies to replicate their results.

Finally, to the best of our understanding, a number of open problems could be referred as future work. For example, the optimality of CPPI strategy could be investigated in more details instead of deriving its optimality from the optimal solution of a different strategy. More importantly, we have observed that both strategies CPPI and OBPI seem efficient theoretically, however they have drawbacks that the investors have to face with in practical applications, such as gap risk (the risk of violating the floor protection), caused by the continuous-time trading assumpsions or by the presence of jumps in asset prices. Modelling and managing gap risk is also an inportant concept that we introduce as a further study of this research.

REFERENCES

[1] Ahn, D.H., Boudoukh, J., Richardson, M. and Whitelaw, R., *Optimal Risk Management Using Options*, Journal of Finance, LIV(1), 359-375, 1999.

[2] Akume, D., Luderer, B. and Weber, G.W., *Pricing and Hedging of Swaptions*, Journal of Computational Technologies, 8, 5-14, 2003.

[3] Akume, D., Luderer, B. and Weber, G.W., *Bewertung und Hedging von Swaptions*, Zeitschrift Wirtschafts-Wissenschaftliches Studium (WiSt), 12, 635-640, 2003.

[4] Akume, D., Luderer, B. and Weber, G.W., *Risk-Constrained Dynamic Portfolio Management*, in the Special Issue on Optimization in Finance, of DCDIS-B (Dynamics of Continuous, Discrete and Impulsive Systems), (Series B), 17, 1b, 113-129, 2010.

[5] Avellaneda, M., Lévy, A. and Parâs, A., *Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities*, Applied Mathematical Finance, 2(2), 73-88, 1995.

[6] Arrow K.J., *Aspects of the Theory of Risk-Bearing*, Yrjö Hahnsson Foundation, Helsinki, 1965.

[7] Balder, S. and Mahayni, A., *Cash Lock Comparison of Portfolio Insurance Strategies*, Technical report, 2008.

[8] Balder, S., Brandl, M. and Mahayni, A., *Effectiveness of CPPI Strategies under Discrete-Time Trading*, The Journal of Economic Dynamics and Control, 33, 204-220, 2005.

[9] Basak, S., *A General Equilibrium Model of Portfolio Insurance*, Review of Financial Studies, 8(4), 1059-1090, 1995.

[10] Basak, S., *A Comparative Study of Portfolio Insurance*, The Journal of Economic Dynamics and Control, 26(7-8), 1217-1241, 2002.

[11] Benninga, S. and Blume, M., *On the Optimality of Portfolio Insurance*, Journal of Finance, 40(5), 1341-1352, 1985.

[12] Bergman, Y., Grundy, B. and Wiener, Z., *General Properties of Option Prices*, Journal of Finance, 51, 1573-1610, 1996.

[13] Bertrand, P. and Prigent, J.L., *Portfolio Insurance Strategies: OBPI versus CPPI*, Finance, 26: 5-32, Technical report for GREQAM and Université Montpellier, 2005.

[14] Bertrand, P. and Prigent, J.L., *Portfolio Insurance: The Extreme Value Approach to the CPPI*, Finance, 23: 69-86, 2002.

[15] Bertrand, P. and Prigent, J.L., *Portfolio Insurance Strategies: A Comparison of Standard Methods When the Volatility of the Stock is Stochastic*, International Journal of Business, 8(4), 15-31, 2003.

[16] Black, F. and Jones, R., *Simplifying Portfolio Insurance*, The Journal of Portfolio Management, 14, 48-51, 1987.

[17] Black, F. and Perold, A., *Theory of Constant Proportion Portfolio Insurance*, Journal of Economic Dynamics and Control, 16(3-4), 403-426, 1992.

[18] Black, F. and Rouhani, R., *Constant Proportion Portfolio Insurance and the Synthetic Put Option: A Comparison, in Institutional Investor Focus on Investment Management*, edited by Frank J. Fabozzi, Cambridge, Mass.:Ballinger, 695-708, 1989.

[19] Black F. and Scholes M., *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81: 637-659, 1973.

[20] Bookstaber, R. and Langsam, J., *Portfolio Insurance Trading Rules*, The Journal of Futures Markets, 8, 15-31, 2000.

[21] Brennan, M. and Schwartz, E., *The Pricing of Equity Linked Life Insurance Policies with an Asset Value Guarantee*, Journal of Financial Economics, 3, 195-213, 1976.

[22] Brennan, M. and Schwartz, E., *Portfolio Insurance and Financial Market Equilibrium*, Journal of Business, 62, 455-472, 1989.

[23] Browne, S., *Beating a Moving Target: Optimal Portfolio Strategies for Outperforming a Stochastic Benchmark*, Finance and Stochastics, 3, 275-294, 1999.

[24] Carr, P. and Madan, D., *Option Valuation Using the Fast Fourier Transform*, Math. Finance, 2, pp. 61-73, 1998.

[25] Cesari, R. and Cremonini, D., *Benchmarking, Portfolio Insurance and Technical Analysis: A Monte Carlo Comparison of Dynamic Strategies of Asset Allocation*, Journal of Economic Dynamics and Control, 27, 987-1011, 2003.

[26] Chance, D.M. and Brooks, R., *An Introduction to Derivatives and Risk Management*, 7.th edition, Thomson South-Western, USA, ISBN: 0324321392, 2007.

[27] Cox, C. and Leland, H., *On Dynamic Investment Strategies*, The Journal of Economic Dynamics and Control, 24, 1859-1880, 2000.

[28] Cox, J. and Huang, C.F., *Optimal Consumption and Portfolio Policies When the Asset Price Follows A Diffusion Process*, Journal of Economic Theory, 49, 33-83, 1989.

[29] Dana, R.A. and Jeanblanc, M., *Financial Markets in Continouos Time*, Springer, 2007.

[30] Davis, M.H.A. and Norman, A.R., *Portfolio Selection with Transaction Costs*, Mathematics of Operations Research, 15 (4), pp. 676-713, 1990.

[31] *Dotcom bubble burst: 10 years on*, Business report, BBC News, 9 March 2010, news.bbc.co.uk/hi/business/8558257.stm.

[32] Dupire, B., *Pricing with a Smile*, Risk, 7 (1), 18-20, 1994.

[33] Edwards, R., McGee, J. and Bassetti, W.H.C., *Technical Analysis of Stock Trends*, 9.th ed. Boca Raton, St. Lucie Press, 7 (1), 18-20, 2007.

[34] El Karoui, N., Jeanblanc, M. and Lacoste, V., *Optimal Portfolio Management with American Capital Guarantee*, Journal of Economic Dynamics and Control, 29, 449-468, 2005.

[35] El Karoui, N., Jeanblanc-Picque, M. and Shreve, S., *Robustness of the Black and Scholes Formula*, Mathematical Finance, 8(2), 93-126, 1998.

[36] Esser, A., *General Valuation Principles for Arbitrary Payoffs and Applications to Power Options under Stochastic Volatility Models*, Financial Markets and Portfolio Management, 17, 351-372, 2003.

[37] Fahrmeir, L., Pigeot, I. and Tutz, G., *Statistic der Weg zur Datenanalyse*, Springer, Heidelberg, 2003.

[38] Fenchel, W., *Convex Cones, Sets, and Functions*, Lecture Notes, Princeton University Department of Mathematics, New Jersey, Princeton, 1953.

[39] Grossman, S. and Villa, J., *Portfolio Insurance in Complete Markets: A Note*, Journal of Business, 62, 473-476, 1989.

[40] Grossman, S. and Zhou, J., *Optimal Investment Strategies for Controlling Drawdowns*, Mathematical Finance, 3, 241-276, 1993.

[41] Grossman, S. and Zhou, J., *Equilibrium Analysis of Portfolio Insurance*, Journal of Finance, 51, 1379-1403, 1996.

[42] Hobson, D., *Volatility Misspeciffcation, Option Pricing and Superreplication via Coupling*, Annals of Applied Probability, 8(1), 193-205, 1998.

[43] Karatzas, I. and Cvitanic, J., *On Portfolio Optimization under Drawdown Constraints*, in Davis, M.H., et. al., Mathematical Finance, IMA Volumes in Mathematics and Its Applications, Vol. 65. Springer, Newyork, pp. 35-45, 1996.

[44] Karatzas, I. and Cvitanic, J., *On Dynamic Measures of Risk*, Finance and Stocks, 3, 451-482, 1999.

[45] Karatzas, I. and Shreve, S., *Brownian Motion and Stochastic Calculus*, Springer, New York, 1991.

[46] Karatzas, I. and Shreve, S., *Methods of Mathematical Finance*, Springer, Berlin, 1998.

[47] Kou, S., *A Jump-Diffusion Model for Pricing Options*, Management Science, 48, pp. 1086-1101, 2002.

[48] Lamberton, D. and Lapeyre, B., *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall, 1996.

[49] Lehman 2007 Annual Report, Item 6, Page 29, Ratios, Secinfo.com., 11-11-2011.

[50] *Lehman Folds with Record 613 Billion Dollar Debt,* Marketwatch.com, 09-15-2005., Retrieved 09-15-2008.

[51] Leland, H., *Who Should Buy Portfolio Insurance*, Journal of Finance, 35(2), 581-594, 1980.

[52] Leland, H. and Rubinstein, M., *The Evolution of Potfolio Insurance*, in D.L.Luskin, ed., Portfolio Insurance: A guide to Dynamic Hedging, Wiley, 1976.

[53] Mahayni, A., *Effectiveness of Hedging Strategies Under Model Misspecification and Trading Restrictions*, International Journal of Theoretical and Applied Finance, 6(5), 521-552, 2003.

[54] Mahayni, A. and Balder, S., *How Good Are Portfolio Insurance Strategies*, Mercator School of Management, University of Duisburg-Essen, Lotharstr. 65, 47057 Duisburg, Germany, 2009.

[55] Mahayni, A. and Schlögl, E., *The Risk Management of Minimum Return Guarantees*, Business Research, 1, 55-76, 2008.

[56] Markowitz, H.M., *Portfolio Selection*, Yale University Press, New Haven, CT, 1959.

[57] Merton, R., *Life Time Portfolio Selection under Uncertainity: The Continuous-Time Case*, Rev. Econ. Stat., 51, pp. 247-257, 1969.

[58] Merton, R., *Optimal Consumption and Portfolio Rules in A Continuous Time Model*, Journal of Economic Theory, 3, 373-413, 1971.

[59] Merton, R., *An Intertemporal Capital Asset Pricing Model*, Econometrica, 41, 867-888, 1973.

[60] Merton, R., *Continuous-Time Finance (Macroeconomics and Finance)*, Blackwell Publishing, 1990.

[61] Mosler, K.C., *Entscheidungsregeln bei Risiko*, Multivariate Stochastic Dominance, Springer, Heidelberg, 1982.

[62] Munk, C., *Financial Asset Pricing Theory*, Lecture notes, 2008.

[63] Nash, S.G., Sofer, A. and Griva, I., *Linear and Nonlinear Optimization*, SIAM, 2009.

[64] Nordahl, H. A. and Doskeland, T.M., *Optimal Pension Insurance Design*, Journal of Banking and Finance, 32, 382-392, 2008.

[65] Pafumi, G. and Gerber, H., *Pricing Dynamic Investment Fund Protection*, North American Actuarial Journal, 4, 28-41, 2000.

[66] Perold, A. and Sharpe, W., *Dynamic Strategies For Asset Allocation*, Financial Analyst Journal, January-February, 16-27, 1988.

[67] Perold, A., *Constant Portfolio Insurance*, Harvard Business School, Unpublished manuscript, 1986.

[68] Rincon-Zapatero, J.P. and Josa-Fombellida, R., *Stochastic Pension Funding When the Benefit and the Risky Asset Follow Jump Diffusion Process*, European Journal of Operational Research, Volume 220, Issue 2, 404-413, 2012.

[69] Rockafellar, R.T., *Convex Analysis*, Princeton Universtiy Press, 1970.

[70] Rockafellar, R.T., Ursayev, S. and Kalinchenko, K., *Calibrating Risk Preferences with Generalized CAPM Based on Mixed CVaR Deviation*, Journal of Risk, 2011.

[71] Rutrowski, M. and Musiela, M., *Martingale Methods in Financial Modelling* Springer, 2011.

[72] Sato, K., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK, 1999.

[73] Schlögl, E. and Schlögl, L. and Dudenhausen, A., *Robustness of Gaussian Hedges Under Parameter and Model Misspecification*, Technical report, University of Bonn, Department of Statistics, 1998.

[74] Shreve, S., *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, New York, 2004.

[75] Tankov, P. and Cont, R., *Financial Modelling with Jump Processes*, Chapman and Hall/CRC Financial Mathematics Series, 2007.

[76] Tankov, P. and Cont, R., *Constant Proportion Portfolio Insurance in Presence of Jumps in Asset Prices*, Columbia University Center for Financial Engineering, Financial Engineering Report, No: 2007-10, 2009.

[77] Touzi, N., *Stochastic Control and Finance*, Ecole Polytechnique, Paris, Department of Applied Mathematics Lecture Notes, 2010.

[78] Touzi, N. and Tankov, P., *No-Arbitrage Theory for Derivatives Pricing*, Ecole Polytechnique, Paris, Department of Applied Mathematics Lecture Notes, 2011.

[79] Zagst, R. and Kraus, J., *Stochastic Dominance of Portfolio Insurance Strategies OBPI versus CPPI*, Technical report, 2011.

[80] Zhang, P.G., *Exotic Options: A Guide to Second Generation Options*, World Scientific, 1998.

[81] Zhu, Y. and Kavee, R.C., *Performance of Portfolio Insurance Strategies*, Journal of Portfolio Management, Vol. 14, No: 3, pp. 48-54, 1998.