RISK MEASUREMENT, MANAGEMENT AND OPTION PRICING VIA A NEW LOG-NORMAL SUM APPROXIMATION METHOD

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ABSTRACT

RISK MEASUREMENT, MANAGEMENT AND
OPTION PRICING VIA A NEW LOG-NORMAL SUM APPROXIMATION METHOD

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In this thesis we mainly focused on the usage of the Conditional Value-at-Risk (CVaR) in risk management and on the pricing of the arithmetic average basket and Asian options in the Black-Scholes framework via a new log-normal sum approximation method. Firstly, we worked on the linearization procedure of the CVaR proposed by Rockafellar and Uryasev. We constructed an optimization problem with the objective of maximizing the expected return under a CVaR constraint. Due to possible intermediate payments we assumed, we had to deal with a re-investment problem which turned the originally one-period problem into a multi-period one. For solving this multi-period problem, we used the linearization procedure of CVaR and developed an iterative scheme based on linear optimization. Our numerical results obtained from the solution of this problem uncovered some surprising weaknesses of the use of Value-at-Risk (VaR) and CVaR as a risk measure.

In the next step, we extended the problem by including the liabilities and the quantile hedging to obtain a reasonable problem construction for managing the liquidity risk. In this problem construction the objective of the investor was assumed to be the maximization of the probability of liquid assets minus liabilities bigger than a threshold level, which is a type of quantile hedging. Since the quantile hedging is not a perfect hedge, a non-zero probability of having a liability value higher than the asset value exists. To control the amount of the probable deficient amount we used a CVaR constraint. In the Black-Scholes framework, the solution of this problem necessitates to deal with the sum of the log-normal distributions. It is known that sum of the log-normal distributions has no closed-form representation. We introduced a new, simple and highly efficient method to approximate the sum of the log-normal distributions using shifted log-normal distributions. The method is based on a limiting approximation of the arithmetic mean by the geometric mean. Using our new approximation method we reduced the quantile hedging problem to a simpler optimization problem.

Our new log-normal sum approximation method could also be used to price some options in the Black-Scholes model. With the help of our approximation method we derived closed-form approximation formulas for the prices of the basket and Asian options based on the arithmetic averages. Using our approximation methodology combined with the new analytical pricing formulas for the arithmetic average options, we obtained a very efficient performance for Monte Carlo pricing in a control variate setting. Our numerical results show that our control variate method outperforms the well-known methods from the literature in some cases.

Keywords: Risk measures, linearization of conditional value-at-risk, quntile hedging, pricing options based on arithmetic averages, variance reduction with control variates

ÖZ

RİSK ÖLÇÜMÜ, YÖNETİMİ VE LOG-NORMAL DAĞILIMLARIN TOPLAMINA YENİ BİR YAKLAŞIM METODU İLE OPSİYON FİYATLAMA

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Bu tezde temel olarak Koşullu Riske Maruz Değer (CVaR)'in risk yönetiminde kullanımı ile geometrik ortalama sepet ve Asya tipi opsiyonların log-normal dağılımların toplamına yeni bir yaklaşım metodu ile fiyatlanması üzerine odaklandık. Öncelikli olarak, Rockafeller ve Uryasev tarafından ortaya atılan CVaR'ın doğrusallaştırılması yöntemi üzerinde çalıştık. Amaç fonksiyonu beklenen getiriyi maksimize etmek olan ve CVaR kısıtına sahip bir optimizasyon problemi kurduk. Olası ara ödemelerden dolayı, orjinal hali tek dönem olan problemi çok dönemli probleme dönüştüren bir yeniden yatırım problemi ile uğraşmamız gerekti. Çok dönemli problemi çözmek için CVaR'ın doğrusallaştırılma yöntemini kullandık ve lineer optimizasyona dayanan bir iteratif plan geliştirdik. Sayısal sonuçlarımız Riske Maruz Değer (VaR) ve CVaR'ın risk ölçüm aracı olarak kullanılmasının bazı şaşırtıcı zayıflıklarını ortaya çıkardı.

Bir sonraki adımda, likidite riskini kontrol etmemize yardımcı olacak bir problem yapısı elde etme amacıyla probleme pasifi (borçları) ve quantile hedging'i ekledik. Bu problem yapısında amaç fonksiyonunu quantile hedging'in bir türü olan, likit varlıkların değeri ile borçların değeri arasındaki farkın bir eşik seviyesinden daha büyük olma olasılığının maksimize edilmesi olarak kabul ettik. Quantile hedging tam koruma (perfect hedge) sağlamadığı için borçların değerinin varlıkların değerinden büyük olmasının sıfırdan farklı bir olasılığı bulunmaktadır. Olası açıkların miktarını kontrol etmek için bir CVaR kısıtı kullandık. Bu problemin Black-Scholes modelinde çözümü log-normal dağılımların toplamını ele almayı gerektirmektedir. Log-normal dağılımların toplamının kapalı-formda bir gösteriminin olmadığı bilinmektedir. Kaydırılmış (shifted) log-normal dağılımları kullanarak log-normal dağılımların toplamı için yeni, basit ve çok etkili bir metot geliştirdik. Metot aritmetik ortalamanın geometrik ortalamaya limit ile yaklaşımına dayanmaktadır. Yaklaşım metodumuzu kullanarak problemimizi daha basit bir optimizasyon problemine indirgedik.

Log-normal dağılımların toplamına yaklaşım metodumuz Black-Scholes modelinde bazı opsiyonların fiyatlanmasında da kullanılabilmektedir. Yaklaşım metodumuzu kullanarak aritmetik ortalama sepet ve Asya tipi opsiyonları için kapalı-form yaklaşım formülleri elde ettik. Yaklalaşım metodolojimizi aritmetik ortalamaya dayalı opsiyonların fiyat formülleri ile birlikte kullanarak, kontrol değişkenli Monte Carlo metodunda çok etkili bir performans elde ettik. Sayısal sonuçlarımız kontrol değişken metodumuzun bazı durumlarda literatürdeki iyi bilinen metotlardan daha iyi sonuç verdiğini göstermektedir.

Anahtar Kelimeler: Risk ölçümü, koşullu riske maruz değerin doğrusallaştırılması, quantile hedging, aritmetik ortalamaya dayalı opsiyonların fiyatlandırılması, kontrol değişkeni ile varyans azaltma

In memory of my father Mahmut Zeytun.

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CHAPTER 1

INTRODUCTION

In the process of measuring the risk, the question of which risk measure should be taken into account has a critical importance. Several risk measures have been considered in the literature, however none of them has superiority in all aspects. The use of the variance as the measure of risk has been popular since the introduction of Markowitz's classical mean-variance model [40]. Variance is a measure of variability which takes both the upward and downward price movements into consideration. This feature is the main drawback of the use of variance as a risk measure since the upward movements are desired by investors. Contrary to the variance, Value-at-Risk (VaR) is a risk measure which takes only the lower quantile of the return distribution into account. VaR has a big popularity among banks and other financial institutions. It is the amount of money that expresses the maximum expected loss from an investment over a specific investment horizon for a given confidence level. Although VAR is commonly used by practitioner it has some drawbacks such as lack of coherency and convexity. In this thesis we will focus on the Conditional Value-at-Risk (CVaR) which is a coherent and convex risk measure. More importantly, its optimization problem can be reduced to a linear optimization problem. Due to these features, in the last years CVaR has gained interest by the researchers.

In this thesis, firstly we will focus on the linearization procedure of the CVaR proposed by Rockafeller and Uryasev [48]. We will construct an optimization problem aiming the maximization of the expected return under a CVaR constraint. Our problem construction will be a dynamic version of the problem used in Martinelli et al. [42]. To solve this problem we will use the linearization procedure of the CVaR and propose an iterative scheme based on linear optimization. We will also compare the performance of VaR, CVaR and the variance as a risk measure.

In the next step, we will include liabilities to the problem, which turns the problem into a type of asset-liability management problem. In construction of an asset-liability management problem hedging plays a significant role. Since perfect (or super-) hedging eliminates the opportunity of getting a profit higher than the risk-free investment together with the risk of a loss, the quantile hedging could be reasonable for some investors. Therefore, we will also include the quantile hedging into our problem. In our new problem, the objective of the investor will be to maximize the probability of the (liquid) assets minus the (current) liabilities bigger than a threshold level. We will only consider the liquid assets since we aim to construct a strategy strengthening the liquidity of the investor, therefore helps to reduce the liquidity risk of the investor. Since the quantile hedging is not a perfect hedge, a non-negative probability exist to have liability value higher than the asset value. We will control the probable deficient amount by a CVaR constraint. This problem will be solved in a Black-Scholes framework where the assets and the liabilities are log-normally distributed. To calculate the probability in our objective function we have to deal with the problem of the summation of the log-normal distributions. It is well-know that sum of the log-normal distributions has no closed-form representation. Some log-normal sum approximation methods are proposed in the literature. We will introduce a new, simple and highly efficient method to approximate the sum of the log-normal distributions using shifted log-normal distributions. The method is based on a limiting approximation of the arithmetic mean by the geometric mean. Using our approximation method we will reduce our problem to a simpler optimization problem.

Using an approximation for the sum of the log-normal distributions is a strategy commonly utilized to price some types of options. In the Black-Scholes model we cannot find an exact closed form price formula for the options based on the arithmetic averages since there is no closed form distribution representing the sum of the log-normal distributions. In the literature, these kind of options are generally priced by Monte-Carlo methods or by the use of an approximation for the sum of the log-normal distributions. We will derive some closed-form approximation formulas in the Black-Scholes framework for the prices of the arithmetic average basket and Asian options by using our method. We will also show how to use the extrapolation methods to accelerate the convergence of our method. Another important feature of our log-normal sum approximation method is that it is a good candidate for the Monte-Carlo control variate approach. We will describe the use of our closed form formulas and the methodology as a control variate, and conduct some numerical examples to assess its efficiency.

The outline of this thesis is as follows: In Chapter 2, we will describe some important concepts from the risk measurement theory and define some important risk measures. The linearization procedure of the CVaR proposed by Rockafeller and Uryasev will be introduced in Chapter 3. In Chapter 3, we will also provide information about the possibility of different problem constructions containing CVaR, which is proved by Krokhmal et al. [37]. Our problem construction which is a dynamic version of the problem used in Martinelli et al. [42] as specified above will be introduced and solved in Chapter 4. Chapter 4 will mainly be based on Korn and Zeytun [35]. In Chapter 5, our quantile hedging problem and its reduction to a simpler problem will be given. Our now log-normal sum approximation and its usage in option pricing will be described in Chapter 6, which will be based on Korn and Zeytun [34]. Finally, in Chapter 7, we will summarize our works, give the conclusions and outlook to future studies.

CHAPTER 2

RISK MEASURES

In investment and risk management processes, risk plays a crucial role. Investors generally construct their portfolios by taking their risk perception and risk appetite into the consideration. A risk measure is a mapping from the set of random variables to the real numbers. The mathematical definition of a risk measure can be given as follows.

Definition 2.0.1. ***Risk Measure:*** *Let* Ω *be the set of all possible states of nature (world), and* $\mathcal{G}$ *be the set of all real-valued functions (random variables) on* Ω*. Then, a risk measure* ρ *is a mapping from the set of real-valued functions into the set of real numbers, i.e.,*

$$\rho : \mathcal{G} \to \mathbb{R}.$$

By using a risk measure we assign a single number to the risk of a portfolio and this number is generally used in the investment decisions.

There are different types of risk measures which are used in finance. In the process of measuring the risk, the question of which risk measure should be taken into account has a critical importance. In Markowitz's modern portfolio theory (see Markowitz [40]), the investor's goal is to optimally allocate its investments between different assets by maximizing the expected value of the portfolio subject to a selected level of risk. In this theory, Markowitz uses the variance as the measure of the risk. Although the risk of an investor is actually to face with a large negative return (loss) realization, variance also takes into account the upward return realizations which are desired by the investors.

Although there are several risk measures proposed so far in the literature, none of them has superiority in all aspects. In recent years, particular stress is laid on the definition and use of the more sophisticated risk measures that have some desired properties instead of the use of standard risk measures such as variance and expected absolute deviation. In this chapter firstly we will define coherent and convex risk measures and then provide information about two well-known risk measures: Value-at-Risk and Conditional Value-at-Risk.

2.1 Coherent Risk Measures

The concept of coherent risk measure is first introduced in Artzner et al. [2] and then extended by the same authors in [3]. In these works, Artzner et al. assumed finite probability spaces without complete market assumption and defined 4 axioms for the coherency of the risk measures. Later, Delbaen [16] extended the definition of Artzner et al. to general probability spaces. Here we will give the definition of Artzner et al. and provide the economic interpretation of the axioms.

Definition 2.1.1. ***Coherent Risk Measure:*** *Let Ω be the set of all possible states of nature and $\mathcal{G}$ be the set of all real valued functions on Ω. Then, a risk measure ρ is called coherent if it satisfies the following axioms:*

- ***Translation invariance:*** *For all $\alpha \in \mathbb{R}$ and all $X \in \mathcal{G}$,* $\quad \rho(X + \alpha r) = \rho(X) - \alpha$ *where r is the return of a reference instrument (e.g. risk-free rate).*
- ***Subadditivity:*** *For all $X, Y \in \mathcal{G}$,* $\quad \rho(X + Y) \leq \rho(X) + \rho(Y)$.
- ***Positive homogeneity:*** *For all $\alpha \geq 0$ and all $X \in \mathcal{G}$,* $\quad \rho(\alpha X) = \alpha\rho(X)$.
- ***Monotonicity:*** *For all $X, Y \in \mathcal{G}$ with $X \geq Y$,* $\quad \rho(X) \leq \rho(Y)$.

Translation invariance is also called "cash invariance" and it implies that if we add a risk-free investment with an initial amount α to the portfolio then the risk of the portfolio decreases by α. *Subadditivity* means that the risk of a portfolio is always less than or equal to the sum of the risks of the individual components. This axiom is in line with the common economic intuition that diversification decreases the risk. *Positive homogeneity* says that there is a positive linear relationship between the size of the portfolio position and its risk. Positive homogeneity axiom assumes liquidity in the market and it may not be reasonable in an illiquid market since in such a market the risk of the portfolio might increase in a non-linear way with the size of the position. *Monotonicity* axiom means that, among two portfolios if a portfolio has higher returns for all possible states of nature then this portfolio has a lower risk.

2.2 Convex Risk Measures

Convex risk measure is an extension of the coherent risk measure and its notion was introduced by Föllmer et al. [22]. As stated in the previous section, although positive homogeneity axiom of coherent risk measures implies a liner relationship between the size of the position and its risk, in some situations the relationship might be in a nonlinear way. For example, when the size of the position multiplied by a large factor then an additional liquidity risk may arise. Due to this fact, Föllmer et al. suggested to relax the positive homogeneity and subadditivity axioms of the coherent risk measures by a weaker property of convexity, and called the new risk measure as convex risk measure.

Definition 2.2.1. ***Convex Risk Measure:*** *Let Ω be the set of all possible states of nature and $\mathcal{G}$ be the set of all real valued functions on Ω. Then, a risk measure ρ is called* ***convex*** *if it satisfies the following axioms:*

- ***Convexity:*** *For all $X, Y \in \mathcal{G}$ and any $\lambda \in [0, 1]$,* $\quad \rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.

- ***Monotonicity:*** *For all $X, Y \in \mathcal{G}$ with $X \geq Y$,* $\quad \rho(X) \leq \rho(Y)$.

- ***Translation invariance:*** *If $m \in \mathbb{R}$ then for all $X \in \mathcal{G}$,* $\quad \rho(X + m) = \rho(X) - m$.

Convexity implies that the risk of a diversified position $\lambda X + (1 - \lambda)Y$ is less or equal to the weighted average of the individual risks, or, in other words, diversification does not increase the risk. Notice that, the convexity axiom and the subadditivity axiom which is defined for the coherent risk measures have the same intuition. Actually, if a risk measure satisfies positive homogeneity then convexity implies subadditivity when $\lambda = 1/2$. Therefore, a convex risk measure is coherent if it satisfies positive homogeneity.

Convexity is an important feature in portfolio optimization problems since convex functions or risk measures have a unique global minimum and therefore easy to optimize. However, when a risk measure is non-convex with respect to the portfolio position then it may has many local minima and therefore it is difficult to optimize.

2.3 Value-at-Risk

Contrary to the risk measures like variance and expected absolute deviation which use both the lower and upper quantiles of the return distribution to calculate the risk of a position, Value-at-Risk (VaR) is a risk measure which takes only the lower quantile of the return distribution into the account. VaR is a risk measure that aims to find an answer to the question “what is the most an investor can lose on a specific investment?”. More expressly, it is the amount of money that expresses the maximum expected loss from an investment over a specific investment horizon for a given confidence level. In mathematical terms, VaR can be defined as follows.

Definition 2.3.1. ***Value-at-Risk (VaR):*** *Let L^w be the loss of an investor using the portfolio vector w, let $\beta \in [0, 1]$. The probability of L^w not exceeding a threshold α is denoted by*

$$\psi(w, \alpha) = \mathbb{P}\left(L^w \leq \alpha\right).$$

Then, the ***Value-at-Risk*** *$VaR\left(L^w, \beta\right)$ of the loss with a confidence level of β can be defined via*

$$VaR\left(L^w, \beta\right) = \min\{\alpha \in \mathbb{R} : \psi(w, \alpha) \geq \beta\}.$$

VaR has a big popularity among banks and other financial institutions due to its simplicity to understand and the approval of Basel Committee on Banking Supervision for the usage of VaR in calculations of capital requirements for banks. Although VaR is a very popular risk measure, it has some undesirable characteristics:

VaR is generally not a coherent risk measure since it does not satisfy the subadditivity property (see, for example, Artzner at al. [3], Föllmer and Schied [23]). Therefore, when we use VaR as our risk-measure diversification may increase the risk of the portfolio. VaR is coherent only when underlying risk factors are normally distributed [48].

Another undesirable property of VaR is that it is generally difficult to optimize. When the underlying risk factors are normally distributed then VaR can be efficiently optimized. However, when the underlying risk factors are not normally distributed, for example when we have discrete distributions or when we use scenarios in calculations, VaR is non-convex (since it does not satisfy the subadditivity property), non-smooth as a function of positions and it is difficult to optimize since it has multiple local extremum points [56].

2.4 Conditional Value-at-Risk

As explained in the previous section, VaR has some undesirable features such as lack of sub-additivity and convexity. Beside these, VaR has the shortcoming that it does not handle/give any information about the loses that might be suffered beyond the VaR value. An alternative measure which handle the losses that might be encountered beyond VaR is the Conditional Value-at-Risk (CVaR). For continuous loss distributions, the CVaR at a given confidence level is the expected loss given that the loss is greater than (or equal to) the VaR at that level [49]. In this sense, CVaR can be defined in mathematical terms as follows.

Definition 2.4.1. ***Conditional Value-at-Risk (CVaR):*** *In addition to the assumptions of the 2.3.1, let L^w have a finite expected value. Then the* ***Conditional Value-at-Risk*** $CVaR\,(L^w, \beta)$ *of the loss funcion L^w with a confidence level of β is given as*

$$CVaR\,(L^w, \beta) = \mathbb{E}\,(L^w | L^w \geq VaR\,(L^w, \beta))\,.$$

For general loss distributions, Rockafellar and Uryasev [49] defined the upper and lower CVaR (CVaR$^+$ and CVaR$^-$, respectively) as

$$CVaR^+\,(L^w, \beta) = \mathbb{E}\,(L^w | L^w > VaR\,(L^w, \beta))$$

and

$$CVaR^-\,(L^w, \beta) = \mathbb{E}\,(L^w | L^w \geq VaR\,(L^w, \beta))\,.$$

CVaR$^+$ is sometimes called "mean shortfall" or "expected shortfall", while CVaR$^-$ is also called "tail VaR" [49]. Generally CVaR$^-$ $\leq$ CVaR $\leq$ CVaR$^+$, and equality holds for continuous loss distributions. For general loss distributions, Rockafellar and Uryasev defined CVaR as the weighted average of VaR and CVaR$^+$ as

$$CVaR\,(L^w, \beta) = \lambda VaR\,(L^w, \beta) + (1 - \lambda) CVaR^+\,(L^w, \beta)\,,$$

where

$$\lambda = [\psi\,(w, VaR(L^w, \beta)) - \beta] \,/\, [1 - \beta]\,.$$

Since CVaR is the expected value of the VaR and the losses beyond it, VaR never exceeds CVaR. When the return-loss distribution is normal, these two measures provide the same optimal portfolio. However, for very skewed distributions, the optimal portfolios provided by CVaR and VaR may be quite different [37].

CVaR is a coherent risk measure, and its coherency is first proved by Pflug [46] (see also, for example Rockafellar and Uryasev [49], Acerbi and Tasche [1]). In addition to be coherent, CVaR is a convex risk measure (see, Rockafellar and Uryasev [48]), therefore it is easier to optimize. Furthermore, it is possible to reduce the problem of optimizing CVaR to a linear optimization problem. The linearization procedure of CVaR is proposed by Rockafellar and Uryasev [48], and it will be introduced in-depth in the next chapter.

For more detailed information about the risk measures described above and the other type of risk measures we refer the readers for example to Down [18], Ekşi [19] and Yıldırım [59].

CHAPTER 3

OPTIMIZATION OF CVaR

In this chapter we will introduce the linearization procedure that can be used for the optimization (minimization) of Conditional Value-at-Risk (CVaR), which was proposed by Rockafellar and Uryasev. Rockafellar and Uryasev in [48] introduced this procedure for continuous loss distributions and, then, in [49], they extended their study to assume general loss distributions. In this part we will describe the methodology for the case of continuous loss distributions. Krokhmal et al. [37] extended the CVaR minimization approach of Rockafellar and Uryasev to other classes of problems with CVaR functions. Krokhmal et al. showed that the approach of Rockafellar and Uryasev can be used for the maximization of reward functions (e.g., expected returns) under CVaR constraints and for the minimization of CVaR subject to a constraint on a reward function.

Here, firstly we will describe the approach of Rockafellar and Uryasev for continuous loss distributions and then show how this approach can be used for other classes of problems with CVaR functions. In this part our main resources will be [37] and [48].

Let $L(w, y)$ be the loss of an investor (it is a random variable) using the portfolio vector w and $y \in \mathbb{R}^m$ is the set of uncertainties which determine the loss function. As in [48], we will assume that the probability distribution of y have a density denoted by $p(y)$ (Rockafellar and Uryasev indicated that an analytical expression of $p(y)$ is not needed and it is sufficient to have an algorithm which generates random samples from $p(y)$).

We denote the probability of $L(w, y)$ not exceeding a threshold α by

$$\psi(w, \alpha) = \mathbb{P}\left(L(w, y) \leq \alpha\right).$$

In general, $\psi(w, \alpha)$ is nondecreasing with respect to α and continuous from the right, but not necessarily from the left because of the possibility of jumps. However, as in [48], we assume that $\psi(w, \alpha)$ is everywhere continuous with respect to α, that means there is no jumps.

In order to avoid confusion in terms of the appearance, inside the parenthesis of VaR and CVaR we will use the notation L^w instead of $L(w, y)$.

For $\beta \in [0, 1]$, the value-at-risk $VaR\,(L^w,\beta)$ of the loss with a confidence level of β can be defined via

$$VaR\,(L^w,\beta) = \min\{\alpha \in \mathbb{R} : \psi(w, \alpha) \geq \beta\}. \tag{3.1}$$

In addition, let $L(w, y)$ (that is, L^w) have a finite expected value. Then its Conditional Value-at-Risk $CVaR\,(L^w,\beta)$ with a confidence level of β is given as

$$CVaR\,(L^w,\beta) = \mathbb{E}\left(L^w | L^w \geq VaR\,(L^w,\beta)\right). \tag{3.2}$$

Rockafellar and Uryasev characterize the $VaR\,(L^w,\beta)$ and $CVaR\,(L^w,\beta)$ in terms of a function F_β on $W \times \mathbb{R}$ where W is the set of available portfolios. They defined F_β as

$$F_\beta(w, \alpha) = \alpha + (1 - \beta)^{-1} \int_{y \in \mathbb{R}^m} [L(w, y) - \alpha]^+ p(y)dy,$$

where $[t]^+$ is equal to t for $t > 0$, and 0 otherwise.

Theorem 3.0.2. *([48]) The function $F_\beta(w, \alpha)$ is convex and continuously differentiable as a function of α. The $\beta-$CVaR associated with the portfolio vector $w \in W$ can be determined form the formula*

$$CVaR\,(L^w,\beta) = \min_{\alpha \in \mathbb{R}} F_\beta(w, \alpha).$$

Here, the set consisting of the values of α for which the minimum is attained, namely

$$A_\beta(w) = \arg\min_{\alpha \in \mathbb{R}} F_\beta(w, \alpha)$$

is a nonempty closed bounded interval (perhaps reducing to a single point), and the $\beta-$VaR of the loss is given by

$$VaR\,(L^w,\beta) = \textit{left endpoint of } A_\beta(w).$$

In particular, we have

$$VaR\,(L^w,\beta) \in \arg\min_{\alpha \in \mathbb{R}} F_\beta(w, \alpha) \quad \textit{and} \quad CVaR\,(L^w,\beta) = F_\beta(w, VaR\,(L^w,\beta)).$$

The proof of Theorem 3.0.2 is given in [48]. The convexity and continuously differentiability of $F_\beta(w, \alpha)$ is based on Shapiro and Wardi [51]. The expression of VaR and CVaR in terms of $F_\beta(w, \alpha)$ can be obtained by taking the derivative of $F_\beta(w, \alpha)$ with respect to α and equating it to zero, and rearranging the integral expression placed in $F_\beta(w, \alpha)$.

To calculate $\beta-$CVaR by using its definition (i.e., Equation (3.2)) first we need to calculate $\beta-$VaR. This could be complicated because of the non-convexity of VaR. Instead, by using Theorem 3.0.2, one can calculate $\beta-$CVaR by minimizing $F_\beta(w, \alpha)$ over α. This would be easier since the function $F_\beta(w, \alpha)$ is convex and continuously differentiable as a function of α. Furthermore, in this method there is no need to calculate VaR value.

Another important advantage of this methodology is that the $\beta-$CVaR can be minimized over all portfolio weights w using the following theorem:

Theorem 3.0.3. *([48]) The minimization of $\beta - CVaR$ of the loss over all possible portfolio vectors $w \in W$ is equivalent to the minimization of $F_\beta(w, \alpha)$ over all $(w, \alpha) \in W \times \mathbb{R}$, in the sense that*

$$\min_{w \in W} CVaR\,(L^w, \beta) = \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w, \alpha). \tag{3.3}$$

Here, (w^*, α^*) is a solution of the right-hand-side minimization problem if and only if w^* is a solution of the left-hand-side minimization problem and $\alpha^* \in A_\beta(w^*)$, where $A_\beta(w^*)$ is defines as in Theorem 3.0.2. When the set $A_\beta(w^*)$ reduces to a single point then the minimization of $F_\beta(w, \alpha)$ produces a pair (w^*, α^*), not necessarily unique, such that w^* minimizes the CVaR and α^* gives the corresponding VaR with the confidence level of β.

Moreover, when the loss function $L(w, y)$ is convex with respect to w then $F_\beta(w, \alpha)$ is convex with respect to (w, α), and CVaR is convex with respect to w. In addition to the convexity of $L(w, y)$, if the constraints are such that W is a convex set then the joint minimization is an instance of convex programming.

The proof of Theorem 3.0.3 is provided in [48]. The equality of minimums in (3.3) can be obtained by using the expression of $CVaR\,(L^w, \beta)$ given in Theorem 3.0.2, and carrying out the minimization of $F_\beta(w, \alpha)$ with respect to (w, α) by first minimizing over $\alpha \in \mathbb{R}$ for fixed w and then minimizing the result over $w \in W$. For the justification of the convexity claim we refer the interested readers to [48].

Theorem 3.0.3 says that, to find the optimum w values which minimizes the CVaR, there is no need to directly work with the Equation (3.2) which may be hard to do since it is defined in terms of the VaR value. Instead, we can work with $F_\beta(w,\alpha)$ which is convex with respect to w, and even commonly with respect to (w,α) [48].

The integral in the definition of $F_\beta(w,\alpha)$ can be approximated in different ways. For example, a sample from the historical data, or a sample from the distribution of the uncertainty vector y can be used. In this case, an approximation of the form

$$\widetilde{F}_\beta(w,\alpha) = \alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^{N} [L(w,y_i) - \alpha]^+,$$

can be used, where N is the size of the sample.

The approximation $\widetilde{F}_\beta(w,\alpha)$ is convex and piecewise linear with respect to α when we have a linear loss function $L(w,y)$ with respect to w [48]. The function $\widetilde{F}_\beta(w,\alpha)$ is not differentiable with respect to α but it can be minimized by reducing the minimization problem of $\widetilde{F}_\beta(w,\alpha)$ to a linear optimization problem. The minimization of $\widetilde{F}_\beta(w,\alpha)$ over $X \times \mathbb{R}$ is equivalent to the minimization of the linear expression

$$\alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^{N} z_k \tag{3.4}$$

subject to linear constraints

$$z_i \geq 0 \quad \text{and} \quad L(w,y_i) - \alpha \leq z_i \quad \text{for} \quad i = 1, ..., N$$

where z_i $(i = 1, ..., N)$ are dummy variables [48].

Note that, the quality of the approximation given above may depends on different factors, such as the number of Monte-Carlo simulations, types of the random numbers used for simulations and types of the descretization methods used for processes.

Although the above theorems are given for the case of continuous loss distribution, the reduction to linear programming does not depend on the distribution of y and it can be applied for different distributions.

The procedure of Rockafeller and Uryasev which is described above deals with the minimization of CVaR. In [48], the authors required a minimum expected return, therefore they admitted only the portfolios that can be expected to return at least that minimum return. By considering different levels of the expected return in the setting of Rockafeller and Uryasev the efficient frontier can be generated. Krokhmal et. al. [37] assumed different types of problem constructions containing CVaR and they showed how the procedure of Rockafeller and Uryasev can be used for these types of optimization problems.

In the next theorem Krokhmal et. al. [37] show the equivalence of three optimization problems in the sense that they produce the same efficient frontier.

Theorem 3.0.4. *[37] Consider the functions $R(w)$ and $\phi(w)$ depending on decision vector w, and the following optimization problems:*

$$\min_{w \in W} \phi(w) - \mu R(w) \quad \textit{subject to} \quad \mu \geq 0, \tag{3.5}$$

$$\min_{w \in W} \phi(w) \quad \textit{subject to} \quad R(w) \geq \rho, \tag{3.6}$$

$$\min_{w \in W} -R(w) \quad \textit{subject to} \quad \phi(w) \leq \varepsilon. \tag{3.7}$$

If $\phi(w)$ is convex, $R(w)$ is concave and the set W is convex, then the above three optimization problems generate the same efficient frontier provided that the constraints $R(w) \geq \rho$ and $\phi(w) \leq \varepsilon$ have internal points.

When the loss function $L(w, y)$ is linear with respect to w then Theorem 3.0.3 implies that the CVaR risk function which is given by Equation (3.2) is convex with respect to w. Furthermore, when the reward function $R(x)$ is linear and the constraints are linear then the conditions of the above theorem are satisfied for the CVaR risk function and the reward function R(x). In this case, the function $\phi(x)$ in Theorem 3.0.4 can be replaced by the CVaR function. Therefore, minimization of CVaR under a constraint on a concave reward function and maximization of a concave reward function under a CVaR constraint generate the same efficient frontier.

Remember that, in Theorem 3.0.3 Rockafeller and Uryasev showed that in the problem (3.6) the function $F_\beta(w, \alpha)$ can be used instead of $CVaR\,(L^w, \beta)$. In the following theorems, Krokhmal et. al. showed that the usage of $F_\beta(w, \alpha)$ instead of $CVaR\,(L^w, \beta)$ is also possible for the problems (3.5) and (3.7):

Theorem 3.0.5. *[37] The objective functions of the optimization problems*

$$\min_{w \in W} -R(w) \quad \textit{subject to} \quad CVaR(L^w, \beta) \leq \varepsilon \tag{3.8}$$

and

$$\min_{(\alpha, w) \in \mathbb{R} \times W} -R(w) \quad \textit{subject to} \quad F_\beta(w, \alpha) \leq \varepsilon \tag{3.9}$$

achieve the same minimum value as the solution. If the CVaR constraint in (3.8) is active then a pair (w^*, α^*) *minimizes (3.9) if and only if* w^* *minimizes (3.8) and* $\alpha^* \in A_\beta(w^*)$*. Furthermore, when the interval* $A_\beta(w^*)$ *reduces to a single point then the minimization of* $-R(x)$ *produces a pair* (w^*, α^*) *such that* w^* *minimizes the return and* α^* *gives the corresponding VaR value.*

Theorem 3.0.6. *[37] The objective functions of the optimization problems*

$$\min_{w \in W} CVaR(L^w, \beta) - \mu R(w) \quad \textit{subject to} \quad \mu \geq 0, \tag{3.10}$$

and

$$\min_{(\alpha, w) \in \mathbb{R} \times W} F_\beta(w, \alpha) - \mu R(w) \quad \textit{subject to} \quad \mu \geq 0, \tag{3.11}$$

achieve the same minimum value as the solution. A pair (w^*, α^*) *minimizes (3.11) if and only if* w^* *minimizes (3.10) and* $\alpha^* \in A_\beta(w^*)$*. Furthermore, when the interval* $A_\beta(w^*)$ *reduces to a single point then the minimization of* $F_\beta(w, \alpha) - \mu R(w)$ *produces a pair* (w^*, α^*) *such that* w^* *minimizes* $CVaR(L^w, \beta) - \mu R(w)$ *and* α^* *gives the corresponding VaR value.*

The proofs of Theorems 3.0.4, 3.0.5 and 3.0.6 are based on the Kuhn-Tucker Theorem and the detailed proofs can be found in [37].

CHAPTER 4

SOLVING OPTIMAL INVESTMENT PROBLEMS WITH STRUCTURED PRODUCTS UNDER CVAR CONSTRAINTS

In this chapter we will use the linearization procedure of Rockafellar and Uryasev [48] for CVaR (as described in Chapter 3) to solve a problem which is a variant of the one used by Martinelli et al. [42], and compare the performance of VaR, CVaR and the variance as a risk measure.

4.1 Introduction

As a consequence of the Solvency II regulations buying structured products offered by various banks might be a reasonable strategy for insurance companies to hedge their liabilities. Structured products (or, structured investment products) are pre-packed investment strategies designed to meet investors' financial needs depending on the risk tolerance. Structured products can have very sophisticated forms (see, for example, Blundell-Wignall [9]) such as various types of cliquet structures, interest rate derivatives linked to the equity market or even instruments to hedge mortality and interest rate risk at the same time. However, they can also be simpler options such as standard calls or hindsight options (see, for example, Martinelli et al. [42]).

There are various questions institutional investors such as insurance companies have to deal with if they are considering the use of structured products in their investment portfolio. Among them are:

- How to decide about which type of structured product to buy?

- How to judge the advantage of the use of a structured product?
- How to measure the risk of the investment into a portfolio containing structured products?
- Is the structured product worth its price?

While the last question can only be judged in connection with a pricing routine for the structured product and while the first question is closely related to the structure of the firm's liabilities, the remaining two points will be addressed in this part. We will therefore take up the approach given in Martinelli et al. [42], refine the problem and analyze the properties of the corresponding optimal solutions. In particular, we will consider the (highly relevant) aspect of using a structured product that does not exactly match the investor's needs as it has a maturity that lies before the investor's investment horizon. The same situation would turn out if the structured product features intermediate payments. Then, the refined one-period Martinelli et al. problem gets a dynamic aspect, the problem of optimally reinvesting the payments resulting from the structured product.

In recent years, particular stress is laid on the use of the so-called coherent risk measures (see Chapter 2) as a risk measure instead of the use of the variance as in the standard Markowitz mean-variance model. We will examine the use of the Conditional Value-at-Risk (which is a coherent risk measure) as is done in Martinelli et al. We will assume a financial market with investment possibility in a bond, a stock and a structured product, and impose a constraint for the conditional value at risk as is done in Martinelli et al. However, the use of option-type securities in the portfolio will result in a peculiar behaviour of the mean-conditional-value-at-risk optimal portfolio: Using the option with the higher strike leads to a higher expected return while keeping the risk constant.

We will set up our investment problem in the next section which will also review the necessary theoretical background. The remaining sectons will be devoted to the numerical solution of some concrete problems and the interpretation of the special forms of the solutions.

4.2 Optimal investment with structured products

In Martinelli et al. [42] the authors considered an investment problem where the investor can choose a buy and hold strategy in a riskless bond, a stock and an option on the stock. In particular, the option is assumed to mature exactly at the investor's investment horizon. The utility criterion considered is the minimization of a convex combination of the portfolio CVaR and the negative of the expected portfolio return. The authors can then make use of the property of Conditional Value-at-Risk that allows to solve this problem via a combination of linear optimization and Monte-Carlo simulation, a fact that has been pointed out in Rockafellar and Uryasev [48] as described in the Chapter 3.

We will use the same notations with the previous chapters. We denote the loss of an investor using the portfolio vector w by L^w and the probability of L^w not exceeding a threshold α by

$$\psi(w, \alpha) = \mathbb{P}\left(L^w \leq \alpha\right).$$

Then, we define the value-at-risk $VaR\left(L^w, \beta\right)$ of the loss with a confidence level of $\beta \in [0, 1]$ via

$$VaR\left(L^w, \beta\right) = \min\{\alpha \in \mathbb{R} : \psi(w, \alpha) \geq \beta\}.$$

f we assume that L^w have a finite expected value then its Conditional Value-at-Risk $CVaR\left(L^w, \beta\right)$ with a confidence level of β is given a

$$CVaR\left(L^w, \beta\right) = \mathbb{E}\left(L^w | L^w \geq VaR\left(L^w, \beta\right)\right).$$

Note that $L^w = -R^w$, where R^w is the return associated with the portfolio vector w, and we define the return as

$$R^w = \frac{\text{final wealth}}{\text{initial wealth}} - 1.$$

The use of CVaR in portfolio optimization problems as a measure of allowed risk is particularly attractive because, as formulated thus, the optimization problem can be reduced to a linear optimization problem with linear constraints. The resulting problem can then be solved by the standard simplex method (for the simplex method see, for example, Maros [41], Burke [11]).

We will illustrate this by looking at a one-period problem closely related to the one in Martinelli et al. [42]. We assume that we can invest into a stock with return R^S, a bond with return R^B, and an option (or a structured product) with maturity T and return R^O. Choosing the investment portfolio $w = (w_S, w_B, w_O)$, where w_S, w_B and w_O represent the weights of the stock, the bond and the option in the portfolio, respectively, then leads to a portfolio return of

$$R_T^w = w_S R_T^S + w_B R_T^B + w_O R_T^O.$$

To take the risk under the control, we aim the CVaR of the loss to be bounded by a constant C. We are trying to maximize the expected return over all portfolios w with a loss that has a CVaR that does not exceed an upper bound C. We therefore consider the

Problem (RCVaR):

$$\max_{w \in \mathbb{R}^3} \mathbb{E}\left(R_T^w\right),$$

such that

$$\begin{aligned} CVaR\left(-R_T^w, \beta\right) &\leq C, \\ R_T^w &= w_S R_T^S + w_B R_T^B + w_O R_T^O, \\ w_S + w_B + w_O &= 1, \\ w_S, w_B, w_O &\geq 0. \end{aligned}$$

Besides the CVaR-constraint this is a linear optimization problem in w. However, the CVaR-constraint seems to be highly non-linear in w. Fortunately, the linearization procedure for CVaR which is poposed by Rockafellar and Uryasev [48], can be used to overcome this difficulty. As we described in the Chapter 3, Rockafellar and Uryasev showed that using the function $F_\beta(w, \alpha)$ defined by

$$F_\beta(w, \alpha) = \alpha + (1-\beta)^{-1} \int_{y \in \mathbb{R}^m} [L(w, y) - \alpha]^+ p(y) dy$$

one can minimize the CVaR over all possible portfolio vectors $w \in W$ by

$$\min_{w \in W} CVaR\,(L^w, \beta) = \min_{(w,\alpha) \in W \times \mathbb{R}} F_\beta(w, \alpha).$$

Furthermore, Rockafellar and Uryasev proposed to approximate the integral appearing in $F_\beta(w, \alpha)$ by using a sample from the distribution of the uncertainty vector y. Then, the integral can be replaced by a summation, and in this case minimization of $F_\beta(w, \alpha)$ is equivalent to the minimization of the linear expression

$$\alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^{N} z_k$$

subject to linear constraints

$$z_i \geq 0 \quad \text{and} \quad L(w, y_i) - \alpha \leq z_i \quad \text{for} \quad i = 1, ..., N$$

where z_i $(i = 1, ..., N)$ are dummy variables and N is the size of the sample.

Later, Krokhmal et al. [37] showed the equivalence of the optimization problems

$$\min_{w \in W} \phi(w) \quad \text{subject to} \quad R(w) \geq \rho$$

and

$$\min_{w \in W} -R(w) \quad \text{subject to} \quad \phi(w) \leq \varepsilon$$

in the sense that they produce the same efficient frontier for two functions $R(w)$ and $\phi(w)$ depending on decision vector w, if $\phi(w)$ is convex, $R(w)$ is concave and set W is convex, provided that the constraints have internal points.

In our problem, $CVaR(L^w, \beta)$ is convex since the loss function L^w is linear with respevt to w (for the proof, see Rockafellar and Uryasev [48]), the reward function (that is, the expected return) is linear, therefore concave, and the constraints are assumed to be linear. Therefore, the minimization of CVaR under an expected return constraint can be replaced by the maximization of the expected return under a CVaR constraint.

Remember that, Krokhmal et al. [37] also showed the possibility of the usage of $F_\beta(w, \alpha)$ instead of $CVaR\,(L^w, \beta)$ in the optimization problems given above.

With the help of the above information, our problem ”Problem (RCVaR)” can be converted to a linear optimization problem. The new problem mainly consists of

- **Step 1:** Simulate N paths of the market prices of the stock, bond and the option.

- **Step 2:** Set up a suitable linear problem on those simulated paths that can be solved by the well-known simplex method.

More precisely, we can consider the

Problem (LRCVaR):

$$\max_{w \in \mathbb{R}^3} \frac{1}{N} \sum_{i=1}^{N} R^w_{T,i},$$

such that:

$$
\begin{aligned}
R^w_{T,i} &= w_S R^S_{T,i} + w_B R^B_{T,i} + w_O R^O_{T,i} \;\; (i = 1, ..., N), \\
w_S + w_B + w_O &= 1, \quad w_S, w_B, w_O \;\geq\; 0, \\
R^w_{T,i} + \alpha + z_i &\geq 0, \;\; i = 1, ..., N, \\
\alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^{N} z_i &\leq C, \\
z_i &\geq 0, \;\; i = 1, ..., N.
\end{aligned}
$$

Here, β is the given confidence level for the CVaR and α is a free parameter which gives the Value-at-Risk in the optimum solution of our problem (see Krokhmal et al. [37]). The index i corresponds to the values that occur in simulation run number i. Note also that the dimension of the problem is of the order of the number of simulated path N. In our computations, stability of the solutions typically is obtained for $N = 20000$ which means that the linear problem is of a quite big size. However, this also shows that the number of simulation runs determines the size of the problem as considering more investment opportunities would only slightly increase the dimension of the problem (in fact, each further security leads to just one more variable, the corresponding component of the portfolio vector).

Having this problem and also the linearization method of Rockafellar and Uryasev in mind, we now turn to our problem variant. Therefore, suppose that we have again three different investment opportunities on the financial market and suppose further that our desired investment horizon is T, however, some of the financial instruments contain payments at time $T_1 < T$. A simple example of such an instrument would be an option which expires before time T or a coupon bond with a coupon payment before T. The presence of such intermediate payments is the main extension to the Martinelli et al. [42] problem. More precisely, when we receive these intermediate payments we are facing the problem of (re-)investing them in the remaining investment opportunities at the intermediate time. As a consequence, the one-period problem has turned into a special multi-period one. This however destroys the linearity in the corresponding Rockafellar-Uryasev version. To cope with this fact, we first choose a fixed re-investment portfolio v^i (with v^i being a vector of non-negative components adding up to one) for the payments received at time T_1 from security i and then add the payments received from v^i at time T to security i. Thus, we can identify this *new security i* as a structured product. With this interpretation, we can apply the Rockafellar-Uryasev linearization method to the new problem of the type (LRCVaR) and find the optimal (initial) portfolio w (given the fixed choice of v). An outer optimization loop for the best choice of the re-investment strategy v completes our method.

If, for instance, we use a call option with maturity $\frac{T}{2}$ as the (very simple) structured product then its return $R_T^{O,v}$, given a fixed re-investment strategy $v = (v_S, v_B)$ where v_S and v_B represent the weights of the stock and the bond, respectively, satisfies

$$R_T^{O,v} = (1 + \Pi^O)[v_S(1 + r^S) + v_B(1 + r^B)] - 1,$$

where r^S and r^B denote the return of the stock and the bond on the interval $\left[\frac{T}{2}, T\right]$, respectively, Π^O denotes the call option's return at maturity $\frac{T}{2}$, i.e.,

$$\Pi^O = \frac{(K - S_{\frac{T}{2}})^+}{C(S_0, K, \frac{T}{2})} - 1$$

with

$$(K - S_{\frac{T}{2}})^+ = \max(0, K - S_{\frac{T}{2}}),$$

and $C(S_0, K, \frac{T}{2})$ is the price of the call option with initial stock price S_0, strike price K and maturity time $\frac{T}{2}$. Our problem to solve then reads as

Problem (RCVaR-Mult(v)):

$$\max_{w \in \mathbb{R}^3} \mathbb{E}\left(R_T^{w,v}\right),$$

such that

$$\begin{aligned} R_T^{w,v} &= w_S R_T^S + w_B R_T^B + w_O R_T^{O,v}, \\ w_S + w_B + w_O &= 1, \quad w_S, w_B, w_O \geq 0, \\ R_T^{O,v} &= (1 + \Pi^O)[v_S(1 + r^S) + v_B(1 + r^B)] - 1, \\ CVaR\left(-R_T^{w,v}, \beta\right) &\leq C. \end{aligned}$$

Its corresponding linearized version then consists of

Problem (LRCVaR-Mult(v)):

$$\max_{w \in \mathbb{R}^3} \frac{1}{N} \sum_{i=1}^{N} R_{T,i}^{w,v},$$

such that

$$\begin{aligned}
R^{w,v}_{T,i} &= w_S R^S_{T,i} + w_B R^B_{T,i} + w_O R^{O,v}_{T,i} \ \ (i = 1, ..., N),\\
R^{O,v}_{T,i} &= (1 + \Pi^O_i)[v_S(1 + r^S_i) + v_B(1 + r^B_i)] - 1,\\
w_S + w_B + w_O &= 1, \quad w_S, w_B, w_O \ \geq \ 0,\\
R^{w,v}_{T,i} + \alpha + z_i &\geq 0 \ \ (i = 1, ..., N),\\
\alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^{N} z_i &\leq C,\\
z_i &\geq 0 \ \ (i = 1, ..., N).
\end{aligned}$$

Here again, the subscript i indicates the value of the indexed variable corresponding to simulation run number i.

Remark 4.2.1. *1. The choice of the optimal re-investment strategy v mostly depends on the option (or, in general, the available structured product) that is the alternative to the standard investment possibilities bond and stock. In our example above, optimal v can be determined by a combination of a simple line search on* $[0, 1]$ *and the solution of a sequence of corresponding problems (LRCVaR-Mult(v)). To see this, note that due to $v_S + v_B = 1$, v is indeed determined by its first component.*
2. We can also benefit from the linear optimization theory if we want to decide a priori about the usefulness of including an option (or a specific structured product) into our portfolio. Suppose we have an optimization problem which does not contain the investment opportunity in options and has the form of

$$\max \quad C'X \quad s.t. \quad AX \leq b,$$

where C is the coefficient vector of the objective function, A is the constraint matrix and b is the right-hand-side vector of the constraints. Then, using the relationship between the primal and dual problems in the simplex method and the Strong Duality Theorem (see, for example, Burke [11]), including an option in our portfolio improves the quality of our portfolio (i.e., leads to a better risk-return trade-off) if vector $Y = (y_1, y_2, ..., y_{N+2})^T \in \mathbb{R}^{N+2}$ of dual prices corresponding to the above problem satisfies

$$\begin{aligned}
y_1 + \sum_{i=1}^{N} \{(1 + \Pi^O_i)[v_S(1 + r^S_i) + v_B(1 + r^B_i)] - 1\} y_{i+1}\\
< \frac{1}{N} \sum_{i=1}^{N} (1 + \Pi^O_i)[v_S(1 + r^S_i) + v_B(1 + r^B_i)] - 1
\end{aligned} \tag{4.1}$$

where

$$\Pi_i^O = \frac{(K - S_{\frac{T}{2},i})^+}{C(S_0, K, \frac{T}{2})} - 1$$

and, $C(S_0, K, \frac{T}{2})$ *is the price of the call option at time* 0 *(initial time) with strike price* K *and maturity* $\frac{T}{2}$*, and* $S_{\frac{T}{2},i}$ *is the stock price at time* $\frac{T}{2}$ *under scenario i. Thus, by the inequality (4.1) we can decide whether using an option with strike price K and re-investment weights* v_S*,* v_B *for the payoff of the option will be beneficial or not. Inequality (4.1) can be used to find a sample of options with different strike prices and different re-investment weights which improve the quality of the portfolio. However, this inequality is not enough to find the best initial investment weights and re-investment weights for the payoff of the option.*

We will illustrate both our method and the particular consequences of using options in our portfolio in the next sections.

4.3 Results for the optimization problem with a call option

In the following, to consider a realistic financial market model, we assume stock prices following a Heston [26] type process

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sqrt{V_t} dW_t^S, \\ dV_t &= \kappa(\theta - V_t)\, dt + \sigma_v \sqrt{V_t} dW_t^V, \end{aligned}$$

and interest rates are assumed to follow a Vasicek [57] process

$$dr_t = a(b - r_t)\, dt + \sigma_B dW_t^B$$

with non-negative constants κ, θ, a, b, σ_S, σ_V, σ_B, a volatility $\sqrt{V_t}$ (where V_t is the variance) of the stock price return and a three-dimensional Brownian motion $W = \left(W^S, W^V, W^B\right)$. As in [42], we assume that W^B and W^V are independent Brownian motions while W^S is correlated with the others . μ_t is a time-varying expected return which can be derived by using market prices of risk and arbitrage-free market assumption, and given by

$$\mu_t + r_t = \sqrt{V_t}\left(\lambda_S - \frac{1 - e^{-at}}{a}\sigma_B \rho \lambda_B\right),$$

where λ_B and λ_S are the riskpremiums associated with interest rate risk and stock price risk, respectively, and ρ is the correlation between the drift term and stock return (see Martinelli et al. [42]).

We now look at an investment problem with a 4-year investment horizon. Consider a European call option which expires in 2 years and has a strike of K. If the option ends up in-the-money in 2 years then we allocate this payoff to the stock and bond with pre-specified weights v.

We performed our simulations by discretising the stock price and interest rate processes by the Euler method. In the discretization, a step size of 0.004 year is used for time. We simulated 5000 paths for the stock price and the interest rate. Then, applying the simplex method to these simulated scenarios, by changing the re-investment weights v successively (indeed, we performed a simple line search by changing v_S), we optimized our problem for the initial weights of the stock, bond, option and re-investment weights for the payoff of the option.

The results for different strike prices of the call option, with initial stock price equal to 100, an upper bound of $C = 10$ percent loss for the CVaR with a confidence level of $\beta = 0.95$ (we used similar parameters with [42] and all the parameter values used in the optimization problem are given in Table 4.3), are given in Table 4.1.

K	CVaR	w_S	w_B	w_O	v_S	v_B	$\mathbb{E}(R)$	VaR
60	10	42.52	52.04	5.44	100	0	35.34	4.39
70	10	36.94	54.98	8.08	100	0	35.50	4.63
80	10	19.92	64.12	15.96	100	0	36.11	6.73
90	10	0	75.25	24.75	100	0	38.25	10
100	10	0	75.25	24.75	100	0	41.19	10
120	10	0	75.25	24.75	100	0	47.87	10
150	10	0	75.25	24.75	100	0	57.16	10
200	10	0	75.25	24.75	100	0	63.82	10

Table 4.1: *Optimization results (in percentages) for different strike prices of the call option.*

CVaR	w_S	w_B	$\mathbb{E}(R)$	VaR
10	51.02	48.98	35.29	4.41

Table 4.2: *Optimization results (in percentages) for the problem without option.*

The results given in Table 4.1 have some peculiar consequences. First, the optimization results show that by using a call option with high strike price we can increase the expected return of our portfolio substantially. When the strike price of the option is increased then in the optimal portfolio the weight of the stock in the initial investment decreases and the weights of the

Investment horizon (T)	4 years
Size of time steps in the discretization	0.004 year
Number of scenarios (N)	5000
Upper bound for CVaR (C)	10 percent (loss)
Confidence level for CVaR (β)	95 percent
Initial stock price	100
Initial variance of the stock price	0.045
Speed of mean-reversion of the variance (κ)	5
Long-run mean of the variance (θ)	0.045
Volatility of the variance (σ_V)	0.48
Initial interest rate	0.04
Speed of mean-reversion of the interest rate (a)	0.15
Long-run mean of the interest rate (b)	0.04
Volatility of the interest rate (σ_B)	0.015
Correlation between W^S and W^V	-0.77
Correlation between W^S and W^B	-0.25
Correlation between W^V and W^B	0
Market price of risk of the stock (λ_S)	0.343
Market price of risk of the bond (λ_B)	-0.207

Table 4.3: *Parameter values that are used in the optimization problem.*

bond and option increase. Over a specific level of the strike price, the optimal weight of the stock always takes the value 0 and the weights of the bond and option remain constant. Also, over this level of the strike price, CVaR and VaR attain the same value. The only difference is seen in the expected returns. When we use a call option with a higher strike price then our expected return is also higher.

In the first instance, these results look very surprising. In particular, by using different strike prices for the call option we can obtain different portfolios with the same risk level (if we take CVaR or VaR as our risk measure) but different expected returns. This counterintuitive result calls for an explanation. Indeed, when a certain level of the strike price of the call is exceeded then losses from call investment which are relevant for the VaR or CVaR calculation are always caused when the option ends up out of the money, i.e., when the call investment leads to the total loss. This explains the fact that when initially the only risky investment is the call investment (which is the case in our example in Table 4.1 if the strike K of the call is at least 90) VaR and CVaR coincide. Further, the relative return of the call increases with increasing strike (in the Heston setting we can only show this numerically, in the Black-Scholes case this can even be proved). As, however, in our example the risk measured in terms

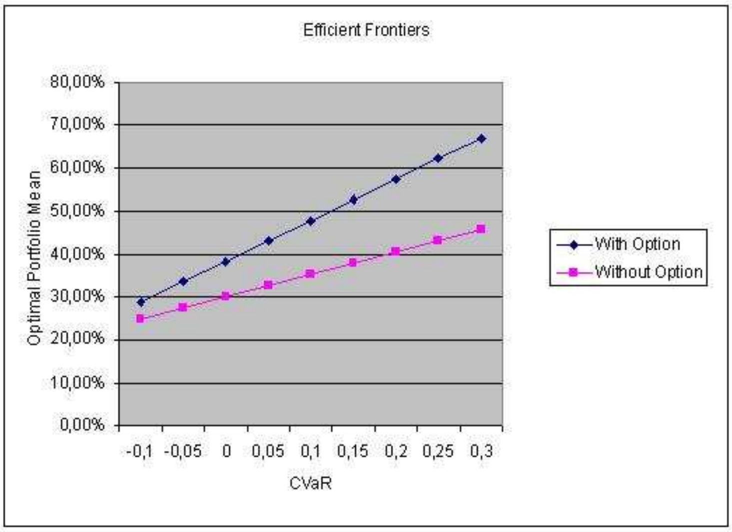


Figure 4.1: Efficient frontiers with and without call option based structured product where $K = 120$.

of CVaR stays constant above (a value slightly smaller than) $K = 90$, there is an increase in the relative return of the total portfolio although the CVaR of the return remains constant.

Thus, this behaviour of the risk-return characteristics does not explain the subjective fact that investing in calls with a higher strike price is more risky. If, however, instead of CVaR we use the traditional mean-variance description, then the optimal portfolios of Table 4.1 are judged more and more risky with increasing call strike (which indeed is what we expected). As a consequence, if we use the classical Markowitz mean-variance method, then for a constant variance bound S (say, S=2342 which corresponds to the mean-CVaR-portfolio in the case of $K = 90$, see Table 4.4) optimal portfolios have to involve initial stock investment for strikes $K > 90$.

K	60	70	80	90	100	120	150	200
$\mathbb{E}(R)$	35.34	35.50	36.11	38.25	41.19	47.87	57.16	63.82
CVaR	10	10	10	10	10	10	10	10
VaR	4.39	4.63	6.73	10	10	10	10	10
Variance	979	1032	1306	2342	3762	10185	42705	336789

Table 4.4: *The values (in percentages) of the three risk measures for the portfolios with different strike price of the call option.*

Another view on this remarkable result can be obtained via Figure 4.2. Here, we present the return of a fixed amount of money invested in options (in our case, w_O for one unit of money) with different strikes. Obviously, the higher the strike, the cheaper the option price. Consequently, for the same amount of money, more options with higher strike can be bought compared to one with a lower strike which results in the different forms of the final payoffs.

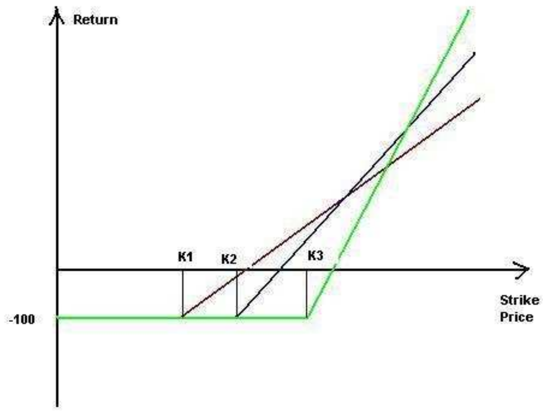


Figure 4.2: Percentage return of an investment in call options with different strike prices.

Here, for all considered values of the strike price of the call option we will in the worst case lose all the invested money, so all graphs start from the same point. Depending on the strike price of the call option, the return starts to increase at different levels of the stock price. An investment in a call option with a higher strike price has a higher slope of return, because the price of the call is lower with a higher strike price and this implies the possibility of buying more options. Suppose the probability of obtaining a final stock price under K_1 is equal to $1 - \beta$. Then, with this probability, the options will expire out-of-money and we will lose all the invested money. This explains why in our results we obtained identical values for both VaR and CVaR above a certain level of the strike price. Therefore, with a strike price above K_1, all investments have the same level of VaR and CVaR (and, thus, are assigned the same risk when VaR or CVaR are taken as our risk measures), but different levels of expected return since the slopes of the increase in total return are different for different strike prices. However, if we take the variance as our risk measure then all investments have different levels of risk since the variations (so, the variances) are different.

Another result, which in the first instance looks surprising, is the re-investment weights for the payoff of the option. Optimization results always favor full re-investment in the stock. This can easily be explained in the cases where VaR and CVaR coincide. Here, the only risk that is measured is the one of the call ending out of the money, i.e. the loss of all the money. Thus, using the stock after the option has ended up in the money adds no risk that enters the CVaR computation. However, reinvesting everything in the stock is the re-investment strategy that yields the highest expected return. In the case where VaR and CVaR still differ, the explanation is not as easy but can still be given: Since the option is a call, we receive a payoff if the stock price at time $\frac{T}{2}$ is above the strike price. In this case, we have got rid of the risk of losing all the invested money in the option. Moreover, the conditional probability of losing more than a pre-specified level from the stock investment has decreased substantially as the stock has already done well until $\frac{T}{2}$. Thus, as the risk of losing from call investment is highly correlated with the risk of losing from stock investment, the full stock re-investment strategy does not add to the risk already taken, but it increases the return. Therefore, we always optimally reinvest all call payments in the stock.

We can also obtain the results of the optimization problem when we replace the call option with a put option. In this case the optimal re-investment strategy always consists of investing the whole option payoff into the bond. This can be explained by the negative relationship between the return of a stock and a put option written on that stock.

The two extreme re-investment strategies in the cases of a call option (full re-investment in the stock) and a put option (full re-investment in the bond) which are mentioned above make it worth asking what would happen if we used a combination of a call option and a put option as the structured product.

4.4 Results for the optimization problem when a combined call-plus-put option is traded

Assume we have the opportunity of investing into a call and a put option with the same strike price (the assumption of the same strike price can be relaxed). Applying the same methodology and using the same parameter values as in the case of the call option above, we get the results outlined in Table 4.5 for the case of this call-plus-put option.

The results show that the optimal re-investment weights of the option payoffs change with

K	CVaR	w_S	w_B	w_O	v_S	v_B	$\mathbb{E}(R)$	VaR
80	10	26.64	57.85	15.51	100	0	36.72	6.19
100	10	41.99	48.49	9.53	100	0	36.79	5.17
105	10	45.58	46.46	7.96	100	0	36.57	4.96
105.7	10	46.65	45.63	7.72	86	14	36.54	4.96
106	10	49.77	43.08	7.15	23	77	36.53	5.04
110	10	52.25	40.82	6.93	0	100	36.55	4.80
130	10	57.10	36.41	6.49	0	100	36.39	4.80

Table 4.5: *Optimization results for different strike prices of the call-plus-put option.*

different strike prices.

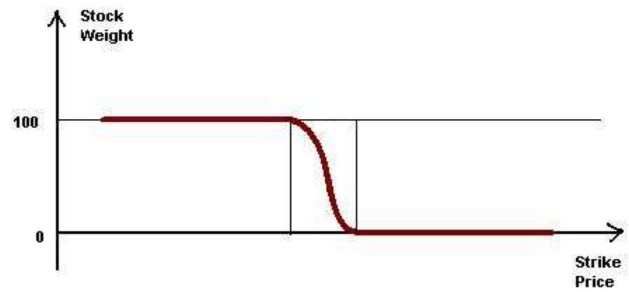


Figure 4.3: Weight of the stock in the re-investment of the call-plus-put option payoff for different strike prices.

Figure 4.3 shows the change of the re-investment weight in the stock for different strike prices. Up to a specific level of the strike price we obtain full re-investment in the stock. After this specific level of the strike price, when we increase the strike price, the weight of the stock starts to decrease and above a certain level of the strike, a full bond re-investment will be optimal. We can explain this result by combining the explanations for the call option and put option we have given above. When the strike price of the call-plus-put option is sufficiently low the put option will be cheap relative to the call option. If we buy the combined call-plus-put option with this strike price, most of the premium we have to pay will be paid for the call option part of the strategy. Also, for the call option the probability of expiring in the money is higher than the put option in the case of low strike price. The call option will effect the total return much more than the put option does since the weight of the call option in the investment and the probability of getting a payoff from the call option are higher. Thus, the investment of the call-plus-put option will behave like a call option. Therefore, in this case we get a pure stock re-investment, as in the case of the call option example above. Likewise, if the strike price of the call-plus-put option is sufficiently high, the call-plus-put option will behave like a put option. Therefore, we get a pure bond re-investment, as in the put option example above. In between these two values of the strike price none of the options dominate the other sufficiently, and therefore we end up with a mixed re-investment strategy with weights depending on the level of the strike price.

4.5 Summary and Concluding Remarks

In this chapter we have looked at a particular investment problem where -besides stocks and bonds- the investor can also include options (or more complicated, structured products) into a portfolio. Compared to the Martinelli et al. [42] approach, we allow for intermediate payments of the securities and are thus faced with a re-investment problem which turns the originally one-period model into a (special kind of a) multi-period problem. We developed a method to deal with this problem by solving a series of those one-period problems.

Our numerical results uncovered some surprising weaknesses of the use of VaR and CVaR as a risk measure. In the presence of the opportunity to invest into options with relatively high strikes, using the option with the higher strike leads to a higher expected return while keeping the risk constant. However, our subjective feeling of an increasing risk is much better matched by the use of the variance, although this is a non-coherent risk measure.

Our investment decision problem can also be solved when we have more securities than above. They can also have multiple internal payments. One can think of coupon bonds or exotic options. In particular, we can also deal with more than just two periods in our optimization problem. However, here the outer optimization loop(s) for obtaining the optimal re-investment strategy gets more complicated. Each additional time period will add one more outer loop, consequently finding the solution of the optimization problem will take longer.

CHAPTER 5

QUANTILE HEDGING
IN THE BLACK-SCHOLES FRAMEWORK

In the previous chapter we worked on an asset management problem with a CVaR constraint. In this part, we will improve our problem by including liabilities to our problem. Furthermore, we will consider a quantile hedging problem to obtain a reasonable problem construction from the point of view of the *Basel Committee on Banking Supervision*'s proposal on liquidity risk management.

5.1 Introduction

In a complete market every contingent claim can be hedged. In such a market there always exists a self-financing strategy that replicates the contingent claim (this could also be referred as "perfect hedge"). In other words, for every contingent claim H, we could find a self-financing strategy φ such that the value of this strategy at maturity time T is equal to the contingent claim, that is,

$$V_T(\varphi) = H$$

for each state of the world, where V_T denotes the value at time T. The cost of the replicating strategy defines the price of the claim. This price can be computed as the expected value of the claim under a unique risk neutral measure (or, equivalent martingale measure).

In an incomplete market every contingent claim can not be hedged. In this market the risk neutral measure is not unique, therefore we can find different prices for a contingent claim by using different risk neutral measures. Although in an incomplete market a perfect hedge is not possible, the investors can stay on the safe side by using a super-hedging strategy. For a contingent claim H, a super-hedging strategy is a self-financing strategy φ that satisfies

$$\mathbb{P}\left(V_T(\varphi) = V_0 + \int_0^T \varphi dS \geq H\right) = 1$$

where S is the set of the available instruments in the market. The cost of the cheapest super-hedging strategy is called the cost of the super-hedging and it is given by

$$\Pi(H) = inf\left\{V_0; \exists\varphi \ \ s.t. \ \mathbb{P}\left(V_T(\varphi) = V_0 + \int_0^T \varphi dS \geq H\right) = 1, \right\}$$

where inf denotes the infimum [14]. The cost of a super-hedging strategy is generally too high from a practical point of view.

Although the investors have the opportunity to stay on the safe side by using a hedging or super-hedging strategy, they are generally unwilling to put up the initial amount of capital required for a hedge or a super-hedge. Furthermore, the investors may be unwilling to use a hedging or super-hedging strategy since these strategies take away the opportunity of making a profit together with the risk of a loss. Therefore, the investors may prefer the use of quantile hedging instead of a perfect or a super hedge. Quantile hedging is a partial hedge which can be achieved with a smaller amount of capital. This type of hedge generally aims to maximize the probability of success of hedge under a given initial capital. In mathematical terms, its aim is to find an admissible strategy (V_0, φ) such that $\mathbb{P}\left(V_T(\varphi) = V_0 + \int_0^T \varphi dS \geq H\right)$ is maximum under the constraint $V_0 \leq \widetilde{V}_0$, for a given initial $\widetilde{V}_0$. This type of problem was used in many studies, and its solution can be found, for example, in Spivak and Cvitanic [54] in the context of the classical Black-Scholes model (by using a duality approach familiar from utility maximization literature), in Föllmer and Leukert [20] for the general complete and incomplete cases (with the help of the Neyman-Pearson lemma), in Klusik and Palmowski [31] and in the references provided there for the problem adapted to the insurance setting.

Another commonly used problem construction for the quantile hedging is the minimization of the cost of the hedging strategy for a given level of shortfall probability, i.e., finding the minimum value of V_0 such that there exists an admissible strategy (V_0, φ) with

$$\mathbb{P}\left(V_T(\varphi) = V_0 + \int_0^T \varphi dS \geq H\right) \geq 1 - \epsilon,$$

where $\epsilon \in (0, 1)$ is a pre-specified shortfall probability. The solution of this kind of problems can be found, for example, in Föllmer and Leukert [20].

In the literature, some works construct the quantile hedging problem by using the "success ratio" for the hedge, where the success ration can be defined as

$$\xi(V_0, H) = \mathbf{1}_{\{V_T \geq H\}} + \frac{V_T}{H}\mathbf{1}_{\{V_T < H\}},$$

and **1** is the indicator function. This kind of problem construction and its solution can be found, for example, in Föllmer and Leukert [20, 21] and in Klusik and Palmowski [31].

Quantile hedging does not take into the account the size of the shortfall just as the value-at-risk. Here, we will use a problem construction which also aims to control the shortfall.

5.2 A Quantile Hedging Problem in the Black-Scholes Framework

5.2.1 Description of the Problem

In classical approaches, default occur when the value of a firm (or, assets) is less than the value of its liabilities. For example, see Merton [43], where the default is assumed to occur only at time of liability payment, or Black and Cox [7], where the intermediate default (prior to the liability payment) is also allowed. Here, we will not exactly assume the default, instead assume the ability of the firm to pay its liabilities using its liquid assets. Assuming the liquid assets is important in the following sense: At the maturity time of a liability, if the firm has no enough cash to pay the liability then it will borrow money from the market, or, if the borrowing is not desirable, it will sell some liquid assets. Selling un-liquid assets will not be desirable for the firm since it is hard to sell these assets in a short time with their intrinsic value.

More importantly, the liquidity has been emphasized by the *Basel Committee on Banking Supervision*. In 2008, the Committee published the document *Principles for Sound Liquidity Risk Management and Supervision* [4] as the foundation of its liquidity framework. This document provide detailed guidance on the risk management and supervision of funding liquidity risk. Later, in the documents *Basel III: International framework for liquidity risk measurement, standards and monitoring* [5] and *Basel III: A global regulatory framework for more resilient banks and banking systems* [6] the Committee proposed new standards for the liquidity. In these documents, the Committee developed the Liquidity Coverage Ratio to achieve the objective of promoting short-term resilience of a bank's liquidity risk profile by ensuring

that it has sufficient high-quality liquid assets to survive a significant stress scenario lasting for one month [5]. The Committee described the importance of the liquidity in [6] by the following words:
"During the early liquidity phase of the financial crisis, many banks despite adequate capital levels still experienced difficulties because they did not manage their liquidity in a prudent manner. The crisis again drove home the importance of liquidity to the proper functioning of financial markets and the banking sector."

Assume an investor having assets and liabilities. At time t, let the value of the investor's liquid assets be $X(t)$, and let the value of the liabilities of the investor be $L(t)$. Due to the facts which we mentioned above, the investor may aims to have a liquid asset value larger than the liability value at any time of liability payment. However, since the financial markets generally are not complete a perfect hedge against the liability payment may not be possible, and/or a super-hedging strategy may not be desirable due to the reasons outlined above. In this case, trying to maximize the probability $P(X(t) \geq L(t))$ seems to be reasonable for the investor. Therefore, for a pre-specified cost, the investor may aim to construct a strategy with the objective

$$\max \quad \mathbb{P}(X(T) \geq L(T)), \tag{5.1}$$

where T is the time of the liability payment.

Remember that, this quantile hedging strategy does not take into account the size of the shortfall (that is, $L(T) - X(T)$) since it only deals with the probability of success. However, the amount of the shortfall might be important since, at maturity time, if $X(T) < L(T)$ then the investor will barrow money from the market or sell some un-liquid assets to pay his liabilities. If the amount of the shortfall is large then in the case of borrowing the cost (the yield) could be higher, and in the case of selling an un-liquid asset the loss due to un-liquidity could be high. Since, both of these cases are undesirable for the investor, the investor could aim to take the level of the shortfall amount under the control. In this case, a constraint of Conditional Value-at-Risk

$$\text{CVaR}(L(T) - X(T), \beta) \leq c$$

could be appropriate for the investor, where β is the confidence level. By using such a CVaR constraint, the investor could get ride of the drawback of the quantile hedging, which is related to the shortfall amount.

If the investor also aims to control the expected value of the surplus (i.e., $X(T) - L(T)$), then he could have a constraint of the type

$$\mathbb{E}(X(T) - L(T)) \geq \varepsilon$$

where ε is a constant (it may also be taken as depended on $X(T)$ or $L(T)$).

Under the above conditions, the problem of the investor can be formalized as follows:

$$\max \quad \mathbb{P}(X(T) \geq L(T))$$

subject to

$$\begin{aligned} \mathbb{E}(X(T) - L(T)) &\geq \varepsilon, \\ \mathrm{CVaR}(L(T) - X(T)) &\leq c. \end{aligned}$$

By using the linearization procedure of Rackafeller and Uryasev described in Chapter 3, the above problem can be approximated by the following problem:

$$\max \quad \mathbb{P}(X(T) \geq L(T))$$

such that

$$\begin{aligned} \frac{1}{N}\sum_{j=1}^{N}(X^j(T) - L^j(T)) &\geq \varepsilon, \\ X^j(T) - L^j(T) + \alpha + z_j &\geq 0 \ \ (j = 1, ..., N), \\ \alpha + \frac{1}{N(1-\beta)}\sum_{j=1}^{N} z_j &\leq c, \\ z_j &\geq 0 \ \ (j = 1, ..., N). \end{aligned}$$

Note that, this problem construction is important since it both eliminates the drawback of the quantile hedging as mentioned above and may help to achieve the proposal of the Basel Committee on liquidity.

5.2.2 Case of Geometric Brownian Motion for Asset and Liability Processes

In this part we assume a Black-Scholes type market for asset and liability processes. Assume we have n assets $S_1(t)$, $S_2(t)$, ..., $S_n(t)$, and their prices evolve according to geometric

Brownian motions

$$dS_i(t) = S_i(t)\{\mu_i dt + \sigma_i dW_i(t)\}, \quad S_i(0) = s_i, \quad i = 1, ..., n$$

where the constants μ_i and σ_i $(i = 1, ..., n)$ are the drift and volatility terms, respectively. For simplicity, the Brownian motions W_i $(i = 1, ..., n)$ are assumed to be uncorrelated, however this assumption can be relaxed to include the correlation structure to the problem and in this case the methodology that will be used in Section A.2 can be applied here.

We also assume that liability process is also follows the geometric Brownian motion

$$dL(t) = L(t)\{bdt + \eta dB(t)\}, \quad L(0) = l_0,$$

where the constant b is drift term, η is the volatility term and B is a Brownian motion.

Modeling asset processes as a geometric Brownian motion is very common in the finance literature while it is not common in modeling of liabilities. However, there are studies in the literature that are modeling liabilities using geometric Brownian motion (see, for example, Chiu and Li[13], Josa-Fombellida and Rincon-Zapatero [28], Gerber and Shiu [24]).

Since the asset processes follow geometric Brownian motion then stock price equations have analytic solutions as

$$S_i(t) = s_i exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t\right) \quad (i = 1, ..., n).$$

where s_i is the initial value of the process. In this case, stock prices log-normally distributed with expected value

$$\mathbb{E}(S_i(t)) = s_i e^{\mu_i t} \tag{5.2}$$

and variance

$$Var(S_i(t)) = s_i^2 e^{2\mu_i t}\left(e^{\sigma_i^2 t} - 1\right) \tag{5.3}$$

for $i = 1, ..., n$.

If the investor construct his investment portfolio by using the assets S_i $(i = 1, ..., n)$, then the value of the investment at time t, which will be denoted here by X_t, can be written as

$$X(t) = \sum_{i=1}^{n} w_i S_i(t)$$

where w_i $(i = 1, ..., n)$ are the investment weights of the stocks and they satisfy $\sum_{i=1}^{n} w_i = 1$.

Since we assumed log-normal distributions for the stock prices then $X(t)$ can be interpreted as the sum of log-normal distributions. The distribution of sum of log-normal distributions is not known. In the literature, the distribution of the sum of log-normal distributions is generally approximated by a log-normal distribution having same moments as the sum, or the multiplication of the log-normal distributions, which is again a log-normal distribution, is used to find a lower bound for the sum. To calculate the probability included in our objective function we will use our log-normal sum approximation method introduced in Chapter 6. For the details of our log-normal sum approximation we refer the reader to the Chapter 6 of this thesis.

Our aim is to replace the arithmetic mean with the geometric mean by using the information given in Theorem 6.2.1. To get an approximation between the arithmetic mean and the geometric mean of the price processes, firstly we shift each S_i by a sufficiently large positive constant C. In this case,

$$\tilde{S}_i := S_i + C \quad (i = 1, ..., n),$$

has a shifted log-normal distribution with expectation $\mathbb{E}(S_i) + C$ and variance $Var(S_i)$, where $\mathbb{E}(S_i)$ and $Var(S_i)$ are given by the Equations (5.2) and (5.3), respectively. Now, if we approximate each shifted log-normal distribution $\tilde{S}_i$ by a log-normal distribution $\hat{S}_i$ having same expected value and variance with the shifted log-normal distribution, then the corresponding parameters of the $\hat{S}_i$ are

$$\begin{aligned} \hat{\mu}_i &= \log\left(\frac{(\mathbb{E}(S_i) + C)^2}{\sqrt{(\mathbb{E}(S_i) + C)^2 + Var(S_i)}}\right), \\ (\hat{\sigma}_i)^2 &= \log\left(1 + \frac{Var(S_i)}{(\mathbb{E}(S_i) + C)^2}\right). \end{aligned}$$

Denote

$$S_i^* = w_i \hat{S}_i \quad (i = 1, ..., n).$$

Then, each S_i^* has a log-normal distribution with parameters

$$\begin{aligned} \mu_i^* &= \log(w_i) + \hat{\mu}_i, \\ (\sigma_i^*)^2 &= (\hat{\sigma}_i)^2. \end{aligned}$$

From Theorem 6.2.1 we know that the arithmetic mean and the geometric mean of S_i^* for $i = 1, ..., n$ are close to each other. Using this fact and knowing that the multiplication of log-normal distribution has a known distribution (which is again a log-normal distribution) it will be beneficial to work with the geometric mean instead of to arithmetic mean. Therefore, in our calculations of the probability contained in the objective function, we will replace the arithmetic mean by the geometric mean to find an approximation to the arithmetic mean of stock prices.

Since the corresponding log-normal distribution for S_i^* has parameters μ_i^* and $(\sigma_i^*)^2$ for $i = 1, ..., n$, and since the product of the log-normal distributions has a log-normal distribution, the product of the S_i^*'s can be approximated with a log-normal distribution having parameters as

$$\prod_{i=1}^{n} S_i^* \sim \text{LogN}\left(\sum_{i=1}^{n} \mu_i^*, \quad \sum_{i=1}^{n} (\sigma_i^*)^2\right).$$

Using the properties of the log-normal distributions it can be easily concluded that

$$n\left(\prod_{i=1}^{n} S_i^*\right)^{1/n} \sim \text{LogN}\left(\log(\text{n}) + \frac{1}{n}\sum_{i=1}^{n} \mu_i^*, \quad \frac{1}{n^2}\sum_{i=1}^{n} (\sigma_i^*)^2\right).$$

We also know that, for a sufficiently large positive constant C, the arithmetic mean and the geometric mean of the price processes S_i^* are approximately equal to each other, that is $\frac{1}{n}\sum_{i=1}^{n} S_i^* \cong \left(\prod_{i=1}^{n} S_i^*\right)^{1/n}$. Therefore, for the summation of the S_i^* for $i = 1, ..., n$ we find the approximation

$$\sum_{i=1}^{n} S_i^* \sim \text{LogN}\left(\log(n) + \frac{1}{n}\sum_{i=1}^{n} \mu_i^*, \quad \frac{1}{n^2}\sum_{i=1}^{n} (\sigma_i^*)^2\right).$$

Then, the distribution of the log-returns for the approximation of the asset portfolio

$$X^*(t) := \sum_{i=1}^{n} S_i^*(t) = \sum_{i=1}^{n} w_i \hat{S}_i(t) \cong \sum_{i=1}^{n} w_i(S_i + C) = X(t) + C$$

is a normal distribution with

$$R_{X^*}(t) := \log\left(\frac{X^*(t)}{X^*(0)}\right) \sim N\left(\log\left(\frac{n}{X^*(0)}\right) + \frac{1}{n}\sum_{i=1}^{n} \mu_i^*, \quad \frac{1}{n^2}\sum_{i=1}^{n} (\sigma_i^*)^2\right).$$

where $X^*(0) = X(0) + C$.

We will use the same methodology for the liability process. If we denote $L^*(t) := L(t) + C$, then

$$\mathbb{E}(L^*(t)) = l_0 e^{bt} + C, \quad Var(L^*(t)) = l_0^2 e^{2bt}\left(e^{\eta^2 t} - 1\right).$$

L^* has a shifted log-normal distribution and we will approximate it by a log-normal distribution having same expected value and variance. The parameters of the corresponding log-normal distribution with expected value $\mathbb{E}(L^*(t))$ and $Var(L^*(t))$ are

$$\begin{aligned} b^* &= \log\left(\frac{(\mathbb{E}(L^*(t)))^2}{\sqrt{(\mathbb{E}(L^*(t)))^2 + Var(L^*(t))}}\right), \\ (\eta^*)^2 &= \log\left(1 + \frac{Var(L^*(t))}{(\mathbb{E}(L^*(t)))^2}\right). \end{aligned}$$

If we denote $R_{L^*}(t) := \log\left(\frac{L^*(t)}{L^*(0)}\right)$, then it has a normal distribution with

$$R_{L^*}(t) \sim N\left(b^* - \log(L^*(0)), \quad (\eta^*)^2\right),$$

where $L^*(0) = L(0) + C$.

Therefore, it can be concluded that

$$R_{L^*}(t) - R_{X^*}(t) \sim N\left(b^* - \log\left(n\frac{L^*(0)}{X^*(0)}\right) - \frac{1}{n}\sum_{i=1}^{n}\mu_i^*, \quad (\eta^*)^2 + \frac{1}{n^2}\sum_{i=1}^{n}(\sigma_i^*)^2\right). \tag{5.4}$$

Note that, we can write our objective as

$$\max \quad \mathbb{P}(L(t) - X(t) \leq 0).$$

The probability $\mathbb{P}(L(t) - X(t) \leq 0)$ can be written in the form of (log-)returns of $X(t)$ and $L(t)$ when the initial values are known. Here we will assume the initial values $X(0)$ and $L(0)$ are equal, that means, the investor construct a hedge portfolio with a cost equal to the initial value of liabilities (this assumption is not mandatory since we can write the objective function in terms of the returns for each values of $X(0)$ and $L(0)$). In this case, the objective of

$$\max \quad \mathbb{P}(L(t) - X(t) \leq 0)$$

is equivalent to

$$\max \quad \mathbb{P}(R_L(t) - R_X(t) \leq 0).$$

Since $X(0)$ and $L(0)$ are equal then $X^*(0)$ and $L^*(0)$ are equal where $X^*(0) = X(0) + C$ and $L^*(0) = L(0) + C$. Then, it can be concluded that we can use the objective

$$\max \quad \mathbb{P}(R_{L^*}(t) - R_{X^*}(t) \leq 0) \tag{5.5}$$

instead of the objective

$$\max \quad \mathbb{P}(R_L(t) - R_X(t) \leq 0).$$

Therefore, the problem (5.5) can be used (as an approximation) to find an optimum solution for our original problem (5.1). Since $X^*(0) = L^*(0))$ then by using (5.4) we get

$$\mathbb{P}(R_{L^*}(t) - R_{X^*}(t) \leq 0) = \frac{1}{2}\left\{1 + erf\left(\frac{-b^* + \log(n) + \frac{1}{n}\sum_{i=1}^{n}\mu_i^*}{\sqrt{2\left[(\eta^*)^2 + \frac{1}{n^2}\sum_{i=1}^{n}(\sigma_i^*)^2\right]}}\right)\right\}$$

where $erf(u)$ represent the error function of u and it is given by

$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt.$$

Since the error function is a strictly increasing function then the problem of

$$\max \quad \mathbb{P}(R_{L^*}(t) - R_{X^*}(t) \leq 0)$$

is equivalent to the problem of

$$\min \quad \frac{b^* - \log(n) - \frac{1}{n}\sum_{i=1}^n \mu_i^*}{\sqrt{(\eta^*)^2 + \frac{1}{n^2}\sum_{i=1}^n (\sigma_i^*)^2}}.$$

Therefore, our problem can be written in the form of

$$\min_{w,\alpha,z} \quad \frac{b^* - \log(n) - \frac{1}{n}\sum_{i=1}^n \mu_i^*}{\sqrt{(\eta^*)^2 + \frac{1}{n^2}\sum_{i=1}^n (\sigma_i^*)^2}}$$

such that

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}(R_X^j(t) - R_L^j(t)) &\geq \varepsilon, \\
R_X^j(t) - R_L^j(t) + \alpha + z_j &\geq 0 \; (j = 1, ..., N), \\
\alpha + \frac{1}{N(1-\beta)}\sum_{j=1}^{N} z_j &\leq c, \\
z_j &\geq 0 \; (j = 1, ..., N).
\end{aligned}$$

As we mentioned before, the above problem construction could be used to manage the liquidity risk and therefore, it is reasonable from the point of view of the *Basel Committee on Banking Supervision*'s proposal on liquidity risk management. Moreover, one could obtain more conservative or speculative problem constructions by using a higher or lower value for the ratio $X(T)/L(T)$, or, by allowing a higher or lower CVaR value on the probable shortfall amount.

Although our problem reduction with the help of our log-normal sum approximation method and the linearization procedure of CVaR leads to a simpler problem with linear constraints, the solution of the problem is still not so easy since the reduced objective function is not linear. Furthermore, we may need to use more developed programming languages since the dimension of the problem is about the number of the simulated paths, which should be high to obtain good approximation values. Nevertheless, numerical examples show that our log-normal sum approximation method performs good in the above problem when we are working around the mean (or median) of the shortfall amount, while the approximation is getting worse in the tails. Actually, the results are in line with the ones we obtain in the next chapter, where we use our log-normal sum approximation method to price options on geometric averages. However, in option pricing we would have the opportunity to use our methodology and formulas as a control variate to obtain excellent approximation values.

CHAPTER 6

PRICING OPTIONS BASED ON ARITHMETIC AVERAGES BY USING A NEW LOG-NORMAL SUM APPROXIMATION

6.1 Introduction

The problem of pricing options on an arithmetic mean (average) of stock prices cannot be solved analytically, neither for basket options (where the final payoff is based on the arithmetic mean over a basket of prices from different stocks) nor for Asian options (where the mean is built over the evolution in time of one stock), just to name the two most popular such options.

The main reason for this difficulty is that exponentials of families of random variables which are stable under convolution are typically not stable under convolution themselves. This can easily be seen in the case of a basket option in the multi-asset Black-Scholes model: While the arithmetic mean of the exponents of the different stock prices is again normally distributed, the arithmetic mean of the stock prices has no simple distribution. It is in particular not log-normally distributed. This fact prevents us to find a closed-form solution for the price of the options based on the arithmetic averages.

In the literature, there are numerous studies aiming to price such options and different types of methods are proposed by the authors. One way to price such options is to use the Monte-Carlo methods. Especially when the variance reduction methods, such as control variate methods and antithetic variate methods (for theoretical background and other variance reduction methods see, for example, Uğur [55]), are used one can obtain very effective results from the Monte-Carlo method. However, this method is time consuming and in risk management it is difficult to work with since, for example, some risk measures are non-convex when we use scenarios, the sensitivity analysis is more difficult when we use Monte-Carlo, etc.. Therefore, the focus in the literature is on finding closed-form approximation formulas and bounds for the option price.

The product of log-normally distributed stock prices is however log-normally distributed. Thus, as a consequence, the geometric mean of asset prices in the Black-Scholes model is again log-normally distributed. This allows a Black-Scholes type closed analytical pricing formula for a basket option which is based on the geometric mean. On top of this, the geometric mean of a set of numbers is always bounded by the arithmetic mean of those numbers. Kemna and Vorst [30] used this fact for two different types of approximation:

- By replacing the arithmetic mean in a basket (or an Asian) option by a geometric one, one can obtain a lower bound for the actual price of a basket (resp. an Asian) option.
- As the approximation of the arithmetic mean by the geometric mean is not really good if the numbers entering the means are not close to each other a mean correction is used. In the case of a basket call this amounts to using a modified strike price $\tilde{K}$ such that we have

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} S_i(T) - K\right) = \mathbb{E}\left(\left(\prod_{i=1}^{n} S_i(T)\right)^{1/n} - \tilde{K}\right),$$

where $\mathbb{E}$ denotes the expectation, K is the original strike price, $S_i(T)$ is the price of $i-$th stock at time T and n is the number of stocks deriving the payoff of the option. In both cases the approximation price can be computed immediately via a suitable variant of the Black-Scholes formula. However, in the first case the lower bound might be a poor one, while in the second case the resulting approximating price is not necessarily a lower bound at all.

Other analytical approximation methods are mainly based on the approximation of the sum of the log-normal distributions with a simple distribution by matching some moments. For instance, Levy [38] approximates the sum of the log-normal distributions by a log-normal distribution, Milevsky and Posner [44] approximate by a reciprocal gamma distribution, Posner and Milevsky [47] approximate by a shifted log-normal distribution and Zhou and Wang [60] approximate by some log-extended-skew-normal distribution. Besides finding closed-form approximation formulas, finding analytical lower and upper bounds is also a popular way to price arithmetic average based options. Such kind of methods can be found, for example, in Curran [15], Rogers and Shi [50], Kaas et al. [29] and Deelstra et al. [17]. Another class of the pricing methods are the model-free approaches which are based on the observed option prices on the individual stocks. Such methods can be found, for instance, in Chen et al. [12], Hobson et al. [27] and Linders et al. [39].

The readers interested in other pricing techniques of options based on arithmetic averages, such as partial differential equation approaches, binomial trees and lattice techniques, are referred to the list of references given in Zhou and Wang [60] and Milevsky and Posner [45].

In this part we describe a new methodology to price the basket and Asian options based on the arithmetic averages in the Black-Scholes model. We will use the simple fact that arithmetic and geometric mean coincide if the numbers entering them are equal to derive an asymptotic relation between the two means. Followed by an approximation of a shifted log-normal distribution by a log-normal one, we obtain an analytic Black-Scholes type approximation formula for basket and Asian options which is very accurate for low volatilities of the underlying stock price(s). For medium to high volatilities this approximation serves as the basis for a Monte-Carlo control variate approach which performs well.

6.2 Approximating the arithmetic by a geometric mean

It is well-known that for a set of non-negative real numbers (and, thus, also for realizations of non-negative random variables) $S_1, ..., S_n$ we have the following relation between their arithmetic and their geometric mean:

$$\frac{1}{n}\sum_{i=1}^{n} S_i \geq \left(\prod_{i=1}^{n} S_i\right)^{1/n} .$$

We only have equality if all the S_i coincide. Further, the relative difference between the two means is the smaller, the smaller the relative variation inside the set $S_1, ..., S_n$. The relative variation inside this set vanishes if we add a sufficiently big number C to each S_i. Then the relative differences

$$\frac{(S_i + C) - (S_j + C)}{S_j + C} = \frac{S_i - S_j}{S_j + C}$$

vanish asymptotically with growing C leading to the following result.

Theorem 6.2.1. *Let $S_1, ..., S_n$ be a set of non-negative numbers. Denote $Y_i = S_i + C$ for $i = 1, ..., n$, where C is a positive constant. Then, the geometric mean converges asymptotically* *to the arithmetic mean of the sequence Y_i as C tends to infinity, that is,*

$$\lim_{C\to\infty} \frac{\frac{1}{n}\sum_{i=1}^{n}(S_i + C)}{\left[\prod_{i=1}^{n}(S_i + C)\right]^{\frac{1}{n}}} = 1.$$

Proof. Equivalent to our aim, we will show that

$$\lim_{C\to\infty} \frac{\left[\frac{1}{n}\sum_{i=1}^{n}(S_i + C)\right]^n}{\prod_{i=1}^{n}(S_i + C)} = 1.$$

For suitable functions $f_1, ..., f_n$ and $g_1, ..., g_n$ depending on $S_1, ..., S_n$, and by the use of binomial expansion we get

$$\begin{aligned}
\frac{\left[\frac{1}{n}\sum_{i=1}^{n}(S_i + C)\right]^n}{\prod_{i=1}^{n}(S_i + C)} &= \lim_{C\to\infty} \frac{\left[C + \frac{1}{n}\sum_{i=1}^{n} S_i\right]^n}{\prod_{i=1}^{n}(S_i + C)} \\
&= \lim_{C\to\infty} \frac{C^n + f_1(S_1, ..., S_n)C^{n-1} + ... + f_n(S_1, ..., S_n)C^0}{C^n + g_1(S_1, ..., S_n)C^{n-1} + ... + g_n(S_1, ..., S_n)C^0} \\
&= \lim_{C\to\infty} \frac{1 + \frac{1}{C} f_1(S_1, ..., S_n) + ... + \frac{1}{C^n} f_n(S_1, ..., S_n)}{1 + \frac{1}{C} g_1(S_1, ..., S_n) + ... + \frac{1}{C^n} g_n(S_1, ..., S_n)}.
\end{aligned}$$

By taking the limit $C \to \infty$ in both sides of the equality, one directly ends up with the equality that we would like to obtain. ■

The rate of convergence of the geometric mean to the arithmetic mean in Theorem 6.2.1 is furnished in the following theorem:

Theorem 6.2.2. *Again, let $S_1, ..., S_n$ be a set of non-negative numbers, and denote $Y_i = S_i + C$ for $i = 1, ..., n$ where C is a positive constant. Then, the convergence rate of the geometric mean to the arithmetic mean of the sequence Y_i is given by*

$$\frac{\left(\frac{1}{n}\sum_{i=1}^{n}(S_i + C)\right)^n}{\prod_{i=1}^{n}(S_i + C)} = 1 + \frac{1}{C}\sum_{i=1}^{n} S_i + O\left(\frac{1}{C^2}\right) \quad \textit{for } C \to \infty.$$

Proof. From the proof of Theorem 6.2.1 we have

$$
\begin{aligned}
\frac{\left[\frac{1}{n}\sum_{i=1}^{n}(S_i + C)\right]^n}{\prod_{i=1}^{n}(S_i + C)} &= \frac{1 + \frac{1}{C} f_1(S_1, ..., S_n) + ... + \frac{1}{C^n} f_n(S_1, ..., S_n)}{1 + \frac{1}{C} g_1(S_1, ..., S_n) + ... + \frac{1}{C^n} g_n(S_1, ..., S_n)} \\
&= \frac{1}{1 + \frac{1}{C} g_1(S_1, ..., S_n) + ... + \frac{1}{C^n} g_n(S_1, ..., S_n)} \\
&+ \frac{1}{C} \frac{f_1(S_1, ..., S_n)}{1 + \frac{1}{C} g_1(S_1, ..., S_n) + ... + \frac{1}{C^n} g_n(S_1, ..., S_n)} \\
&+ \frac{1}{C^2} \frac{f_2(S_1, ..., S_n)}{1 + \frac{1}{C} g_1(S_1, ..., S_n) + ... + \frac{1}{C^n} g_n(S_1, ..., S_n)} \\
&+ ... \\
&+ \frac{1}{C^n} \frac{f_n(S_1, ..., S_n)}{1 + \frac{1}{C} g_1(S_1, ..., S_n) + ... + \frac{1}{C^n} g_n(S_1, ..., S_n)}.
\end{aligned}
$$

Explicit calculation of numerator gives

$$
f_1(S_1, ..., S_n) = n\frac{1}{n}\sum_{i=1}^{n} S_i = \sum_{i=1}^{n} S_i.
$$

Then, by using the properties of limit one can easily obtain

$$
\frac{\left(\frac{1}{n}\sum_{i=1}^{n}(S_i + C)\right)^n}{\prod_{i=1}^{n}(S_i + C)} = 1 + \frac{1}{C}\sum_{i=1}^{n} S_i + O\left(\frac{1}{C^2}\right).
$$

■

Note that, the coefficient of the term $1/C$ may change depending on the assumptions we made on limit calculations.

6.3 Approximate pricing of options on the arithmetic mean

We now use the approximation result of Theorem 6.2.1 such that

- the arithmetic mean will not be changed at all,
- the geometric mean will be modified via adding a large number C to all components entering the geometric mean.

To make this more precise, we consider the situation of an arithmetic average basket option where we have

$$B_{basket,ar} = \left(\frac{1}{n}\sum_{i=1}^{n} S_i - K\right)^+ = \left(\frac{1}{n}\sum_{i=1}^{n} (S_i + C) - (K + C)\right)^+$$
$$\geq \left(\left(\prod_{i=1}^{n} (S_i + C)\right)^{1/n} - (K + C)\right)^+ =: B_{basket,geo}(C), \quad (6.1)$$

with $x^+ = \max\{x, 0\}$. So, in the spirit of Theorem 6.2.1 we have convergence of the option prices for the payments $B_{basket,geo}(C)$ towards the original basket option price. In particular, the necessary exchange of the C-limit with the expectation is ensured by inequality (6.1) by the Dominated Convergence Theorem (see, for example, Körezlioğlu and Hayfavi [36], Shreve [52]). Note that this result is model independent which gives the approach a taste of robustness.

Numerical examples showing the fast convergence of such an approximation of the arithmetic mean-based basket option by a sequence of geometric mean-based ones are given in Table 1, where both the prices and the confidence intervals (inside the parenthesis) are given. There, we assumed three independent stock prices that follow geometric Brownian motion under the risk-neutral measure with drift $r = 0.06$, volatilities $\sigma = (0.3, 0.2, 0.3)$ and initial stock prices $S(t_0) = (40, 60, 80)$. We used two different strikes of K and assumed that the option matures after 6 months. The prices were obtained by using the Monte-Carlo method with 10^5 simulations for each stock price process. The prices $P_{basket,ar}$ and $P_{basket,geo}(C)$ are correspond to the present values of $\mathbb{E}(B_{basket,geo})$ and $\mathbb{E}(B_{basket,geo}(C))$, respectively, where $B_{basket,geo}$ and $B_{basket,geo}(C)$ are as given in (6.1) and $\mathbb{E}$ denotes the expected value. Note that, the case of $C = 0$ for $P_{basket,geo}$ corresponds to the price based on the geometric average of the original stock price realizations.

Although the results given in Table 1 are really impressive, there is a particular drawback of this method: Simulating the geometric mean is at least as slow as simulating the arithmetic one. Even more, as the factors in the newly obtained geometric mean are no longer log-normally distributed for $C \neq 0$, we can also not use the well-known closed pricing formula for the call option based on the geometric mean (see Korn et al. [33]):

Theorem 6.3.1. *[Price of geometric average basket option] The price of a geometric average*

K	C	$P_{basket,ar}$	$P_{basket,geo}(C)$
55	0	7.1214 ([7.0827, 7.1601])	4.6311 ([4.5999, 4.6623])
55	10^2	7.1214 ([7.0827, 7.1601])	6.1113 ([6.0762, 6.1464])
55	10^4	7.1214 ([7.0827, 7.1601])	7.1043 ([7.0656, 7.1429])
55	10^6	7.1214 ([7.0827, 7.1601])	7.1212 ([7.0825, 7.1600])
55	10^7	7.1214 ([7.0827, 7.1601])	7.1214 ([7.0826, 7.1601])
65	0	1.5692 ([1.5482, 1.5902])	0.6373 ([0.6250, 0.6497])
65	10^2	1.5692 ([1.5482, 1.5902])	1.0947 ([1.0780, 1.1114])
65	10^4	1.5692 ([1.5482, 1.5902])	1.5600 ([1.5391, 1.5809])
65	10^6	1.5692 ([1.5482, 1.5902])	1.5691 ([1.5481, 1.5901])
65	10^7	1.5692 ([1.5482, 1.5902])	1.5692 ([1.5482, 1.5902])

Table 6.1: *Convergence of the geometric mean-based basket option price to the arithmetic mean-based one when we use our method.*

basket call with weights $w_i = 1/n$ in the Black- Scholes model is given by

$$P_{basket,geo} = e^{-rT}\left(\tilde{s}e^{\tilde{m}}\Phi\left(\tilde{d}_1\right) - K\Phi\left(\tilde{d}_2\right)\right),$$

$$\nu = \frac{1}{n}\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\sigma_{ij}^2\right)^2}, \quad m = rT - \frac{1}{2n}\sum_{i,j=1}^{n}\sigma_{ij}^2 T,$$

$$\tilde{m} = m + \frac{1}{2}\nu^2, \quad \tilde{s} = \left(\prod_{i=1}^{n} s_i\right)^{1/n}, \quad \tilde{d}_1 = \frac{\log(\tilde{s}/K) + m + \nu^2}{\nu}, \quad \tilde{d}_2 = \tilde{d}_1 - \nu,$$

where r is the risk-free rate, n is the number of stocks entering into the basket, K is the strike price, T is the maturity time, s_i is the initial price of the $i-th$ stock, σ_{ij} are the enteries of the variance-covariance matrics of the stock returns and Φ denotes the cumulative standard normal distribution.

To be able to use closed pricing formulae for geometric mean-based options (such as the one in Theorem 6.3.1), we approximate the shifted log-normally distributed random variables in the geometric mean by a log-normal distribution having the same expected value and variance. The resulting log-normal sum approximation method and the complete methodology to price options based on arithmetic averages can be summarized as follows:

- **Step 1:** Shift each log-normally distributed random variable S_i by a sufficiently large positive constant C. Then, $Y_i := S_i + C$ has a shifted log-normal distribution with expected value $\mathbb{E}(Y_i) = \mathbb{E}(S_i) + C$ and variance $Var(Y_i) = Var(S_i)$ for $i = 1, ..., n$.
- **Step 2:** Approximate each shifted log-normal distribution Y_i by a log-normal distribution Y_i^* with the same expected value and variance. This yields the parameters μ_i and

σ_i for the approximating log-normal distribution as

$$\mu_i = \log\left(\frac{[\mathbb{E}(Y_i)]^2}{\sqrt{[\mathbb{E}(Y_i)]^2 + Var(Y_i)}}\right), \quad \sigma_i^2 = \log\left(1 + \frac{Var(Y_i)}{[\mathbb{E}(Y_i)]^2}\right).$$

- **Step 3:** Now, from Theorem 6.2.1, the arithmetic mean and the geometric mean of Y_i^*s are very close to each other, therefore the geometric mean can be used (with some small error) instead of the arithmetic mean.

- **Step 4:** Finally, since shifting the random variables by C increases the original mean, we need to subtract C from the geometric mean that we obtain (or, instead, when we are pricing an option we can shift the strike price by C).

We are going to illustrate the application and performance of this approximation methodology in Sections 6.4 and 6.5 when applied to the pricing of arithmetic average basket call option and arithmetic average Asian call option in the Black-Scholes framework. However, first we give some information about the accuracy of our log-normal sum approximation method.

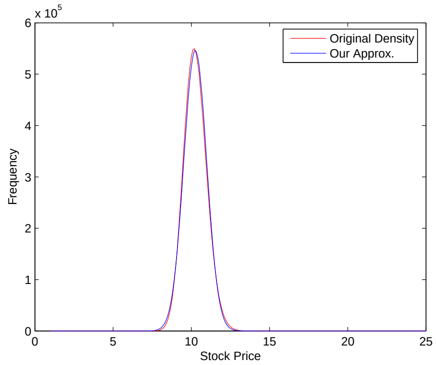
6.3.1 Some facts about our log-normal sum approximation method

In our log-normal sum approximation method each log-normal distribution is shifted by a positive real number C and then the resulting shifted log-normal distribution is approximated by a log-normal distribution. The quality of our log-normal sum approximation is closely related to the quality of the approximation of the shifted log-normal distributions by a log-normal distribution. Numerical examples show that the shifted log-normal distributions can be approximated very well by a log-normal distribution when the original log-normal distribution has a small volatility parameter σ. The reason is as follows: The volatility parameter directly effects the skewness of the log-normal distribution. Remember that, we approximate the shifted log-normal distribution by a log-normal distribution having the same first and second moments, which yields the following volatility parameter for the final log-normal distribution:

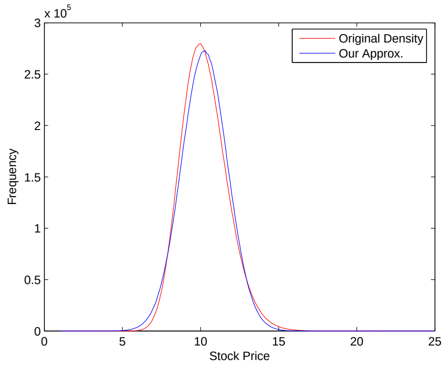
$$(\sigma^*)^2 = \log\left(1 + \frac{Var(S)}{[\mathbb{E}(S) + C]^2}\right).$$

In our approximation method C should be taken sufficiently large to get good approximation values. Since a large value of C causes a value of σ^* close to zero, the skewness of the new log-normal distribution will be small. If the skewness of the original log-normal distribution

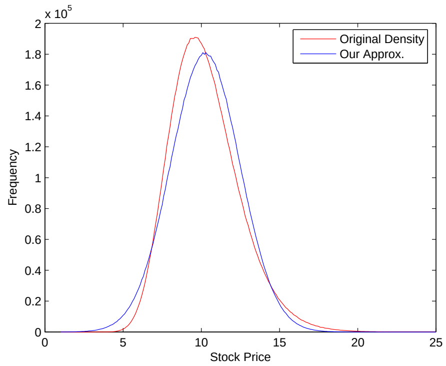
S is small (which means if the volatility parameter of S is small) then the original log-normal distribution and the approximating one will be close to each other. Therefore, in our approximation method smaller volatility parameters imply better approximation values. This situation is illustrated in Figure 6.1.



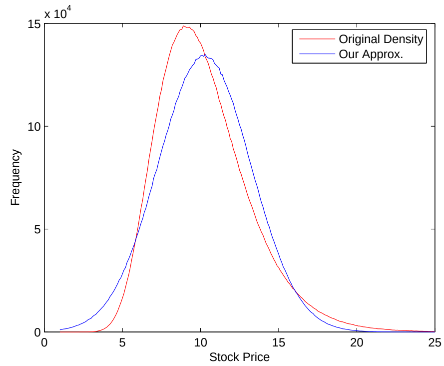
(a) $\sigma = 0.1$



(b) $\sigma = 0.2$



(c) $\sigma = 0.3$



(d) $\sigma = 0.4$

Figure 6.1: Comparison of our approximation with the original distribution.

Figure 6.1 contains the densities of the stock price realizations at maturity time for the original stock price process and the artificial stock price obtained from our approximation method (with density named "Our Approx."). Here, we used $r = 0.06$, $T = 0.5$ and varying values of σ. Note that for $\sigma = 0.1$ our approximation and the original density function nearly coincide. When we increase the σ parameter then the approximation gets worse. Therefore, our approximation method can be used effectively especially for small values of σ parameter or when we have short time to maturity as both situations imply small variance for stock price realizations.

6.4 Approximate pricing of arithmetic average basket option

We consider a Black-Scholes type market with n assets where the prices at time t are denoted by $S_1(t)$, $S_2(t)$, ..., $S_n(t)$ and under the risk-neutral measure they follow the following dynamics:

$$dS_i(t) = S_i(t)\{rdt + \sigma_i dW_i(t)\}, \quad S_i(0) = s_i \quad (i = 1, ..., n),$$

Here, r is the risk-free rate and σ_i is the volatility parameter of the $i-$th stock price process. For simplicity we assume the independence between the stock price processes, therefore, the variables W_i are independent Brownian motions. Later, we will also mention about the case of the dependence between stock prices.

By applying our approximation methodology, we would like to find a closed-form approximation formula for the price of a European arithmetic average basket call option with a strike price K and final payoff

$$\left(\frac{1}{n}\sum_{i=1}^{n} S_i(T) - K\right)^+$$

where T is the maturity time.

We start our methodology by shifting each stock price process by a sufficiently big positive constant. If we shift each price process using the maturity time then the process $Y_i(T) := S_i(T) + C$ has a shifted log-normal distribution with expected value $\mathbb{E}(Y_i(T)) = \mathbb{E}(S_i(T)) + C$ and variance $Var(Y_i(T)) = Var(S_i(T))$ where

$$\mathbb{E}(S_i(T)) = s_i e^{rT}, \quad Var(S_i(T)) = s_i^2 e^{2rT}\left(e^{\sigma_i^2 T} - 1\right).$$

In the next step, we use log-normal random variables $Y_i^*(T)$ to replace the shifted log-normal random variables $Y_i(T)$. The corresponding parameters $\mu_i^*, (\sigma_i^*)^2$ of $Y_i^*(T)$ that yield the same mean and variance as $Y_i(T)$ can be easily calculated to obtain

$$\begin{aligned}
\mu_i^* &= \log\left(\frac{[\mathbb{E}(S_i(T)) + C]^2}{\sqrt{[\mathbb{E}(S_i(T)) + C]^2 + Var(S_i(T))}}\right) \\
&= \log\left(\frac{\left[s_i e^{rT} + C\right]^2}{\sqrt{[s_i e^{rT} + C]^2 + s_i^2 e^{2rT}\left(e^{\sigma_i^2 T} - 1\right)}}\right), \qquad (6.2)
\end{aligned}$$

$$(\sigma_i^*)^2 = \log\left(1 + \frac{Var(S_i(T))}{[\mathbb{E}(S_i(T)) + C]^2}\right) = \log\left(1 + \frac{s_i^2 e^{2rT}\left(e^{\sigma_i^2 T} - 1\right)}{[s_i e^{rT} + C]^2}\right). \qquad (6.3)$$

Note that, in accordance with the Theorem 6.2.1, the geometric mean and the arithmetic mean of the variables $Y_i^*(T)$ are very close to each other, therefore we could approximate the arithmetic average by the geometric average. Since the products of the variables $Y_i^*(T)$ has a log-normal distribution with parameters

$$\prod_{i=1}^{n} Y_i^*(T) \sim \text{logN}\left(\sum_{i=1}^{n} \mu_i^*, \quad \sum_{i=1}^{n} (\sigma_i^*)^2\right),$$

we can value the final payoff based on the geometric mean

$$B_{appr,geo} := \left[\left(\prod_{i=1}^{n} Y_i^*(T)\right)^{1/n} - (K + C)\right]^+$$

by using the log-normal valuation formula (see Korn et al. [33]) and obtain the approximate price formula for a European basket option

$$C_{appr,geo}(T, K) = e^{-rT} e^{m+\frac{1}{2}v^2} \Phi(d_1) - e^{-rT}(K + C)\Phi(d_2) \tag{6.4}$$

with

$$\begin{aligned} m &= \frac{1}{n}\sum_{i=1}^{n} \mu_i^*, \quad v = \frac{1}{n}\sqrt{\sum_{i=1}^{n} (\sigma_i^*)^2}, \\ d_1 &= \frac{\log(1/(K + C)) + m + v^2}{v}, \quad d_2 = d_1 - v, \end{aligned}$$

where C is a sufficiently big positive constant and μ_i^*, $(\sigma_i^*)^2$ are given by Equations (6.2) and (6.3).

To assess the efficiency of our closed form pricing formula we give the following example:

Example 6.4.1. Assume an arithmetic average basket call option depending on n independent stock prices and maturing after 6 months. We use the risk-neutral market coefficients of $r = 0.06$,

$$\sigma_i = 0.2 + 0.008i \quad (i = 1, ..., n),$$

$$S_i(0) = 100 - i \quad (i = 1, ..., n).$$

The price of the given option calculated from our approximation formula (6.4) for different values of n and different strikes of K are given in Table 6.2. For comparison purpose, in the Table we also provide the 95% confidence intervals obtained from the crude Monte-Carlo method and the Monte-Carlo method with antithetic variates. In the computations, the value of our shift parameter was taken as $C = 10^5$ and we used 250000 simulations for each stock

n	K	Monte-Carlo Method	Monte-Carlo Method with Antithetic Variates	Approximation Formula (6.4)
20	80	[11.8435, 11.8752]	[11.8627, 11.8674]	11.8645
20	85	[7.0439, 7.0747]	[7.0623, 7.0675]	7.0782
20	90	[2.8803, 2.9047]	[2.8913, 2.9004]	2.9157
40	70	[11.5561, 11.5812]	[11.5664, 11.5713]	11.5662
40	75	[6.7184, 6.7432]	[6.7285, 6.7334]	6.7346
40	80	[2.3941, 2.4137]	[2.3991, 2.4064]	2.4175
60	60	[11.2613, 11.2824]	[11.2708, 11.2757]	11.2695
60	65	[6.4138, 6.4348]	[6.4233, 6.4282]	6.4249
60	70	[2.0133, 2.0298]	[2.0189, 2.0252]	2.0320

Table 6.2: *Price of arithmetic average basket call option obtained from the Monte-Carlo method and our closed form approximation formula.*

price process in the Monte-Carlo calculations. The results show that our closed form approximation formula gives very accurate results with a tendency to slightly overprice the option for higher strikes. This is an excellent basis for a control variate Monte-Carlo method which we present later.

Note that, although in the calculations we assumed independence between the stock prices, our methodology could also be used when there is dependence between the stock prices entering the basket. The only difference is that, in the case of dependence we need to preserve the original covariance structure between the stock prices, as we will do in Subsection A.2 for the Asian options. We will not illustrate the case of the dependence for basket options since we illustrate it for the Asian option case where the idea is completely similar.

Note also that, in all of the calculations given above we assumed equal weights for the stocks entering the basket, however our method could still be used when we have different weights in the basket. If w_i is the weight of the ith stock in the basket then $X_i := w_i S_i$ would have a log-normal distribution, therefore we could apply our methodology to the variables X_i with equal weights, just as we did in Subsection 5.2.2.

6.5 Approximate pricing of arithmetic average Asian options

In this part, we focus on a single stock price process since as per construction Asian options depend on a single path. Let the stock price process follows the dynamics of

$$dS(t) = S(t)\{rdt + \sigma dW(t)\}, \tag{6.5}$$

for a one-dimensional Brownian motion W under the risk-neutral probability measure.

Our aim is to price the discrete fixed-strike arithmetic average Asian option with strike K and payoff

$$B = \left(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K\right)^{+} \text{ with } 0 = t_0 < t_1 < \cdots < t_n = T.$$

We are again in a situation where the sum of (dependent!) log-normal distributions causes the problems. However, there is a well-known theorem that allows to price a corresponding geometric mean via a Black-Scholes type formula (see Korn et al. [33]):

Theorem 6.5.1. *[Price of geometric average Asian option] Under the risk-neutral measure, assume an asset price following*

$$dS(t) = S(t)\{rdt + \sigma dW(t)\},$$

where r and σ are constants and W is a Brownian motion. The price of the discrete fixed-strike geometric average Asian option with payoff

$$B = \left(\left(\prod_{i=1}^{n} S(t_i)\right)^{1/n} - K\right)^{+},\ 0 = t_0 < t_1 < \cdots < t_n = T,$$

is given by

$$p_{GFA} = e^{-rT}\left(S(0)\, e^{m+\frac{1}{2}v^2}\Phi\left(\tfrac{\log(S(0)/K)+m+v^2}{v}\right) - K\Phi\left(\tfrac{\log(S(0)/K)+m}{v}\right)\right),$$

$$m = \left(r - \tfrac{1}{2}\sigma^2\right)\tfrac{1}{n}\sum_{i=1}^{n} t_i, \quad v = \tfrac{\sigma}{n}\sqrt{\sum_{i=1}^{n}(n+1-i)^2(t_i - t_{i-1})},$$

where Φ denotes the cumulative standard normal distribution.

Thus, we can make use of our approximation methodology just presented for the basket option. However, we now have two possible ways to apply our approximation concept. In the first one, we determine the input coefficients for the approximating Black-Scholes type formula only from approximating the shifted log-normal distribution $Y(T) := S(T) + C$ at the terminal time by the appropriate log-normal distribution $Y^*(T)$. In this case we obtain the following approximate pricing formula.

Approximate pricing formula I for a fixed-strike arithmetic average Asian option

$$p_{AFA1} \cong e^{-rT}\left(S(0)+C\right)e^{m+\frac{1}{2}\nu^2}\Phi\left(\frac{\log\left(\frac{S(0)+C}{K+C}\right)+m+\nu^2}{\nu}\right)-e^{-rT}(K+C)\Phi\left(\frac{\log\left(\frac{S(0)+C}{K+C}\right)+m}{\nu}\right), \quad (6.6)$$

with

$$\begin{aligned}
m &= \left(\alpha-\tfrac{1}{2}\gamma^2\right)\tfrac{1}{n}\sum_{i=1}^{n}t_i, \quad \nu=\tfrac{\gamma}{n}\sqrt{\sum_{i=1}^{n}(n+1-i)^2\left(t_i-t_{i-1}\right)},\\
\gamma &= \frac{\tilde{\sigma}}{\sqrt{T}}, \quad \alpha=\frac{\tilde{\mu}-\log\left(S(0)+C\right)}{T}+\frac{\gamma^2}{2},\\
\tilde{\mu} &= \log\left(\frac{\left[S(0)e^{rT}+C\right]^2}{\sqrt{\left[S(0)e^{rT}+C\right]^2+S(0)^2e^{2rT}\left(e^{\sigma^2T}-1\right)}}\right),\\
\tilde{\sigma}^2 &= \log\left(1+\frac{S(0)^2e^{2rT}\left(e^{\sigma^2T}-1\right)}{\left[S(0)e^{rT}+C\right]^2}\right).
\end{aligned}$$

In the second way, we apply our methodology at each time point t_i for $i=1,...,n$. However, in this case, the new processes shall be taken as correlated with each other since the original option is path dependent. This kind of approximation yields the following formula.

Approximate pricing formula II for a fixed-strike arithmetic average Asian option

$$p_{AFA2} \cong e^{-rT}\left(S(0)+C\right)e^{m+\frac{1}{2}\nu^2}\Phi\left(\frac{\log\left(\frac{S(0)+C}{K+C}\right)+m+\nu^2}{\nu}\right)-e^{-rT}(K+C)\Phi\left(\frac{\log\left(\frac{S(0)+C}{K+C}\right)+m}{\nu}\right), \quad (6.7)$$

where

$$\begin{aligned}
m &= \frac{1}{n}\sum_{i=1}^{n}\alpha_i t_i,\\
\nu &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\gamma_i^2 t_i+2\sum_{i\neq j}^{n}\min\{i,j\}\gamma_i\gamma_j(t_i-t_{i-1})^{1/2}(t_j-t_{j-1})^{1/2}},\\
\gamma_i &= \frac{\tilde{\sigma}_i}{\sqrt{t_i}} \quad (i=1,...,n),\\
\alpha_i &= \frac{\tilde{\mu}_i-\log(S(0)+C)}{t_i}+\frac{\gamma_i^2}{2} \quad (i=1,...,n),\\
\tilde{\mu}_i &= \log\left(\frac{\left[S(0)e^{rt_i}+C\right]^2}{\sqrt{\left[S(0)e^{rt_i}+C\right]^2+S(0)^2e^{2rt_i}\left(e^{\sigma^2t_i}-1\right)}}\right) \quad (i=1,...,n),\\
\tilde{\sigma}_i^2 &= \log\left(1+\frac{S(0)^2e^{2rt_i}\left(e^{\sigma^2t_i}-1\right)}{\left[S(0)e^{rt_i}+C\right]^2}\right) \quad (i=1,...,n).
\end{aligned}$$

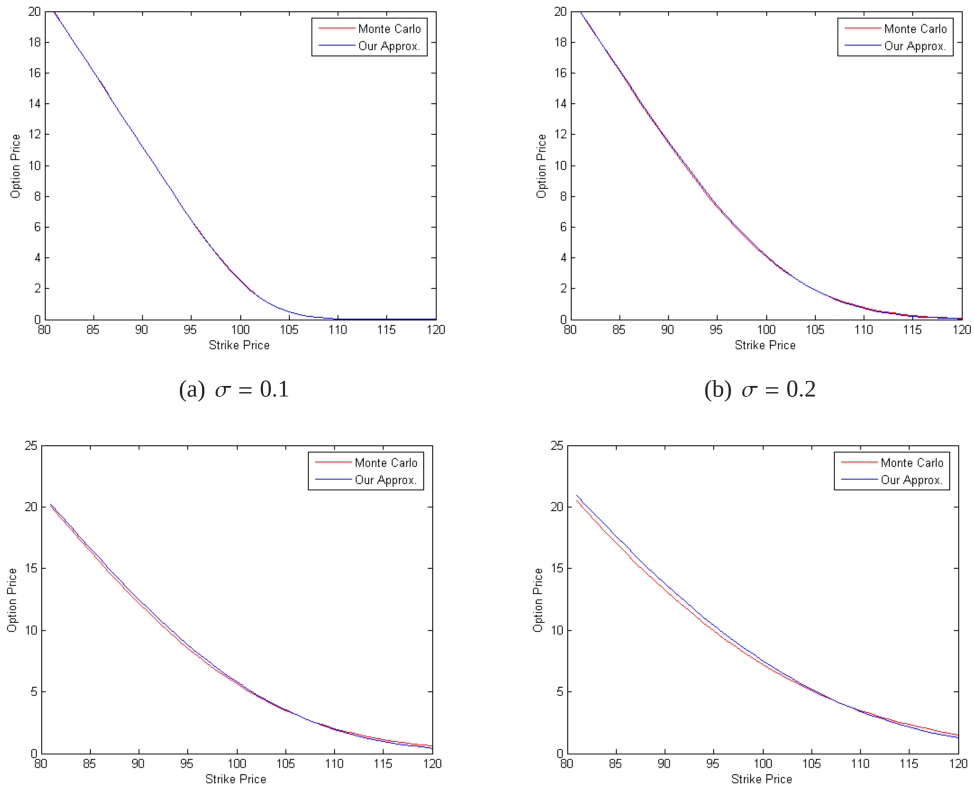
The derivations of the approximate pricing formulas I and II can be found in Appendix A.1 and A.2, respectively.

Example 6.5.2. We consider a discrete fixed-strike arithmetic average Asian call option depending on the stock prices observed at n equidistant time points. We choose $r = 0.06$, $\sigma = 0.2$, $S(0) = 100$, $T = 0.5$. Table 3 contains some approximate option prices obtained from the approximation formulas (6.6) and (6.7) for different values of n and different strikes K. Here, our choice of the shift parameter is $C = 10^6$. For comparison purposes, we also give option prices with 95% confidence interval obtained from the crude Monte-Carlo method and the Monte-Carlo method with antithetic variates based on 250.000 simulations of the stock price paths. From Table 6.3 it is seen that the performance of the approximation formulas is

n	K	Monte-Carlo Method	Monte-Carlo with antithetic Variates	Approx. (6.6)	Approx. (6.7)
25	95	[7.3078, 7.3630]	[7.3387, 7.3554]	7.4863	7.4489
25	100	[4.0759, 4.1202]	[4.0963, 4.1179]	4.1957	4.1499
25	105	[1.9548, 1.9864]	[1.9682, 1.9877]	1.9702	1.9285
50	95	[7.2514, 7.3058]	[7.2773, 7.2937]	7.4266	7.3893
50	100	[4.0102, 4.0538]	[4.0280, 4.0493]	4.1307	4.0847
50	105	[1.8978, 1.9287]	[1.9088, 1.9280]	1.9147	1.8731
75	95	[7.2329, 7.2871]	[7.2593, 7.2756]	7.4068	7.3695
75	100	[3.9925, 4.0359]	[4.0092, 4.0304]	4.1089	4.0629
75	105	[1.8827, 1.9134]	[1.8924, 1.9114]	1.8963	1.8546

Table 6.3: *Price of arithmetic average Asian call option.*

good, but not sufficiently accurate to replace the Monte-Carlo method. On top of that, it is not as good as in the basket option case. In particular, the closed form approximation formulas - and, in particular, Formula (6.6) - perform well when the option is at the money. For low strikes they both tend to underestimate the correct price significantly. For high strikes, the approximation methods tend to overvalue the discrete Asian average option. This is even more pronounced for larger values of the volatility, for small values of σ, the approximations seem to be good approximations. This is also illustrated in the plots given in Figures 6.6 and 6.7. Each figure contains the plots for 4 different values of the volatility parameter σ.



(a) $\sigma = 0.1$ (b) $\sigma = 0.2$

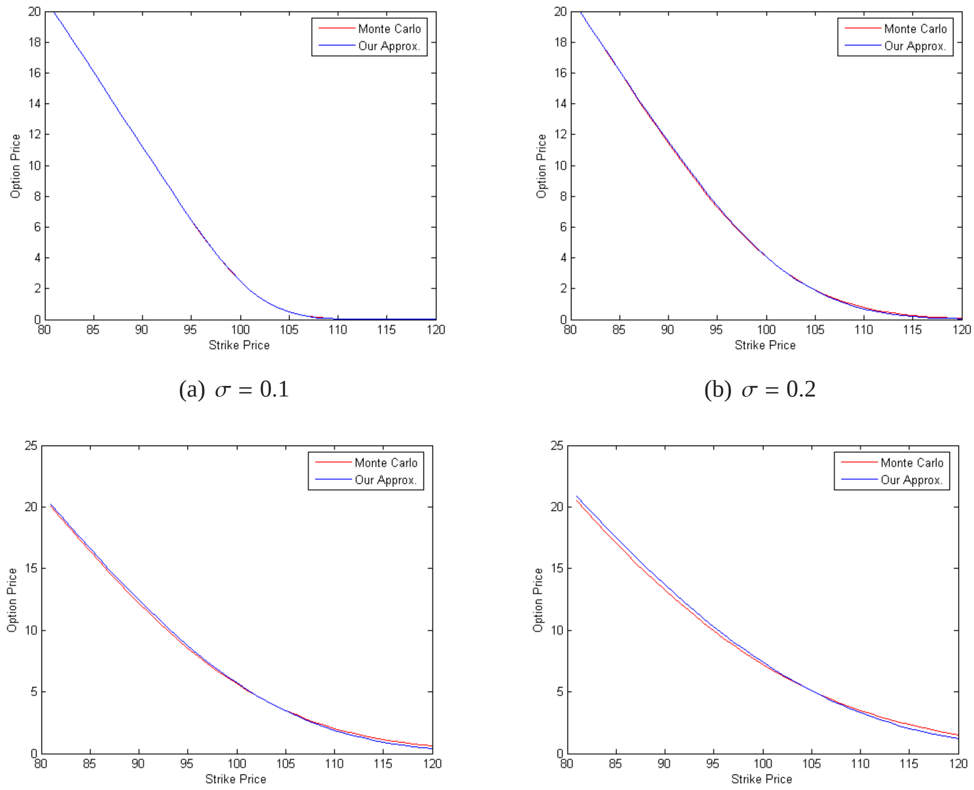
(c) $\sigma = 0.3$ (d) $\sigma = 0.4$

Figure 6.2: Price of the Asian option obtained from the Approximation (6.6).

6.6 A new control variate method for options on arithmetic averages

Numerical results given in the above examples show that our approximation method gives very efficient approximations for the price of the arithmetic average basket option while the price approximations for the arithmetic average Asian options are not as good. In some cases the performance of our approximation formulas is good, but not sufficiently accurate to replace the Monte-Carlo method. However, due to the closed form formula we have derived, the approximating payoffs are candidates to be used as control variates in a Monte-Carlo simulation. For the case of the arithmetic average basket option we could use the Monte-Carlo estimator

$$\overline{X}_N := e^{-rT}\left(\frac{1}{N}\sum_{i=1}^{N}(X_i - M_i) + \mathbb{E}(M)\right) \tag{6.8}$$



(a) $\sigma = 0.1$ (b) $\sigma = 0.2$

(c) $\sigma = 0.3$ (d) $\sigma = 0.4$

Figure 6.3: Price of the Asian option obtained from the Approximation (6.7).

with

$$X_i = \left(\frac{1}{n} \sum_{i=1}^{n} S_i(T) - K \right)^+ , \tag{6.9}$$

$$M_i = \left(\left(\prod_{i=1}^{n} Y_i^* \right)^{1/n} - (K + C) \right)^+ . \tag{6.10}$$

Here, X_i is the payoff (corresponding to the ith scenario) obtained from the Monte-Carlo method when we use the original option, M_i is the payoff (again, corresponding to the ith scenario) obtained from the Monte-Carlo method when we use our methodology and $\mathbb{E}(M)$ is the expected value of the payoff which can be calculated with the help of the price formula (6.4). Y_i^* is the price process having log-normal distribution with the parameters μ_i^* and $(\sigma_i^*)^2$ as defined in (6.2) and (6.3).

Note that, this control variate methodology can also be applied for pricing arithmetic average Asian options.

6.7 Numerical results

In this section we give some numerical examples to illustrate the efficiency of our closed form formulas and the use of our methodology as a control variate method by comparing with some of the well-known methods form the literature. Here we provide the numerical results for the basket option case, however we will also mention about the efficiency of our methodology in Asian option pricing.

We compare our approximation method (the closed-form formula) with the Monte-Carlo price, the confidence interval obtained from Monte-Carlo method, the log-normal approximation of Levy [38], the reciprocal gamma approximation of Milevsky and Posner [44], the approximation of Kemna and Vorst [30], the geometric average basket option price which is in some cases used as an approximation (actually, a lower bound) for the arithmetic average price, and the lower and upper bounds obtained from Deelstra et al. [17]. Besides, we compare the use of our methodology and closed-form formula as a control variate with the use of the geometric average, the log-normal approximation of Levy [38] and the approximation of Kemna and Vorst [30] as a control variate. We do not use the the reciprocal gamma approximation of Milevsky and Posner [44] as a control variate since this approximation uses a distribution different from the log-normal one and therefore yields week results due to the low correlation structure with the original payoff.

We assume a basket option based on the arithmetic average of 4 independent stocks. We also assume that the risk free rate and the maturity time are $r = 0.06$ and $T = 0.5$, respectively. In computations, we used 10^6 simulations for each stock in the Monte-Carlo methods (and the control variate methods) and took the shift parameter as $C = 10^7$ in our method. The results obtained from our method and the methods from the literature are given in Tables 6.4, 6.5, 6.6 and 6.7. The notations MC, MCCI, LB, UB, GA, KV, LN, RG and SLN that we used in the Tables are the Monte-Carlo price, the confidence interval of the Monte-Carlo method, the lower bound of Deelstra et al. (the maximum of the lower bounds obtained by using the conditioning variables $FA1$ and $FA2$ as defined in [17]), the upper bound of Deelstra et al. (the minimum of the partially exact/comonotonic upper bounds obtained by using the conditioning variables $FA1$ and $FA2$), the option price based on the geometric average, Kemna and Vorst's approximation, Levy's log-normal approximation, Milevsky and Posner's Reciprocal Gamma approximation and our shifted log-normal approximation, respectively.

K	MC	MCCI	LB	UB	GA	KV	LN	RG	SLN
				$\sigma_1 = 0.2; \quad \sigma_2 = 0.2; \quad \sigma_3 = 0.2; \quad \sigma_4 = 0.2$					
55	9.1527	[9.1433, 9.1621]	9.1578	9.2548	2.4230	9.1312	9.1606	9.1546	9.1828
60	4.7236	[4.7153, 4.7318]	4.7251	5.3557	0.4583	4.5069	4.7345	4.7179	4.7816
65	1.6678	[1.6623, 1.6733]	1.6648	3.5181	0.0394	1.2847	1.6708	1.6704	1.6653
				$\sigma_1 = 0.5; \quad \sigma_2 = 0.5; \quad \sigma_3 = 0.5; \quad \sigma_4 = 0.5$					
60	7.1455	[7.1267, 7.1642]	7.0786	10.8761	1.7784	6.1027	7.1951	7.0933	7.4087
65	4.6357	[4.6199, 4.6514]	4.5576	10.1025	0.8379	3.4568	4.6653	4.6170	4.6991
70	2.8672	[2.8545, 2.8799]	2.7888	10.2626	0.3660	1.7772	2.8723	2.8834	2.7315
				$\sigma_1 = 0.8; \quad \sigma_2 = 0.8; \quad \sigma_3 = 0.8; \quad \sigma_4 = 0.8$					
65	7.7500	[7.7213, 7.7788]	7.4316	18.2651	0.6246	2.2716	7.8781	7.6444	8.1004
70	5.9464	[5.9206, 5.9723]	5.6183	18.9463	1.0087	3.5026	6.0349	5.8965	5.9495
75	4.5320	[4.5090, 4.5550]	4.2075	20.3043	0.0394	1.2847	4.5782	4.5315	4.2289
				$\sigma_1 = 0.6; \quad \sigma_2 = 1.2; \quad \sigma_3 = 0.3; \quad \sigma_4 = 0.9$					
65	7.8928	[7.8602, 7.9255]	7.5437	17.1976	1.6470	5.3787	8.3693	8.0901	8.6403
70	6.2128	[6.1829, 6.2428]	5.8375	17.9661	1.0480	3.6136	6.5294	6.3500	6.4729
75	4.8941	[4.8668, 4.9214]	4.5070	19.5807	0.6590	2.3721	5.0548	4.9730	4.7116

	Approximate Price by Using Control Variate				Length of the Confidence Interval			
K	GA	LN	KV	SLN	GA	LN	KV	SLN
				$\sigma_1 = 0.2; \quad \sigma_2 = 0.2; \quad \sigma_3 = 0.2; \quad \sigma_4 = 0.2$				
55	9.1556	9.1550	9.1569	9.1584	0.0110	0.0240	0.0081	0.0018
60	4.7239	4.7263	4.7279	4.7288	0.0136	0.0212	0.0073	0.0016
65	1.6673	1.6673	1.6693	1.6704	0.0104	0.0143	0.0055	0.0013
				$\sigma_1 = 0.5; \quad \sigma_2 = 0.5; \quad \sigma_3 = 0.5; \quad \sigma_4 = 0.5$				
60	7.1478	7.1481	7.1524	7.1548	0.0266	0.0486	0.0194	0.0099
65	4.6349	4.6338	4.6393	4.6427	0.0246	0.0412	0.0175	0.0091
70	2.8652	2.8634	2.8695	2.8728	0.0213	0.0335	0.0154	0.0081
				$\sigma_1 = 0.8; \quad \sigma_2 = 0.8; \quad \sigma_3 = 0.8; \quad \sigma_4 = 0.8$				
65	7.7488	7.7451	7.7550	7.7621	0.0450	0.0751	0.0362	0.0251
70	5.9439	5.9398	5.9503	5.9569	0.0422	0.0678	0.0342	0.0237
75	4.5282	4.5234	4.5332	4.5408	0.0391	0.0607	0.0322	0.0223
				$\sigma_1 = 0.6; \quad \sigma_2 = 1.2; \quad \sigma_3 = 0.3; \quad \sigma_4 = 0.9$				
65	7.8928	7.8873	7.8990	7.9059	0.0510	0.0859	0.0421	0.0339
70	6.2111	6.2056	6.2176	6.2241	0.0488	0.0786	0.0403	0.0325
75	4.8922	4.8848	4.8964	4.9035	0.0463	0.0714	0.0386	0.0310

Table 6.4: *Arithmetic average basket call option price when the initial stock prices are* $s_1 = 25, \;\; s_2 = 50, \;\; s_3 = 75, \;\; s_4 = 100.$

Numerical results show that the closed form formulas of RG and LN approximations generally gives better results than the other closed form formulas. Our closed form formula generally slightly under estimates the price when the option is (deep-)out-the-money, and slightly over estimates the price when the option is (deep-)in-the-money, however it generally gives much better results than the KV approximation and the GA price. Furthermore, the performance of our closed form formula is better for some levels of the strike price. This level is slightly higher than the arithmetic average of the initial stock prices when we have low volatilities, and it is higher for higher volatilities.

K	MC	MCCI	LB	UB	GA	KV	LN	RG	SLN
$\sigma_1 = 0.2$; $\sigma_2 = 0.2$; $\sigma_3 = 0.2$; $\sigma_4 = 0.2$									
50	6.5355	[6.5280, 6.5431]	6.5392	6.6987	5.0540	6.5249	6.5412	6.5340	6.5653
55	2.5063	[2.5006, 2.5121]	2.5042	3.5819	0.4583	2.4362	2.5104	2.5010	2.5343
60	0.5041	[0.5014, 0.5068]	0.5005	3.0705	0.2145	0.4439	0.5037	0.5133	0.4719
$\sigma_1 = 0.5$; $\sigma_2 = 0.5$; $\sigma_3 = 0.5$; $\sigma_4 = 0.5$									
55	4.8324	[4.8184, 4.8465]	4.7575	9.1269	2.7725	4.4322	4.8499	4.7920	4.9492
60	2.7402	[2.7292, 2.7512]	2.6592	9.0931	1.3581	2.3371	2.7463	2.7444	2.6729
65	1.4468	[1.4387, 1.4549]	1.3741	9.9795	0.6090	1.1183	1.4413	1.4831	1.2550
$\sigma_1 = 0.8$; $\sigma_2 = 0.8$; $\sigma_3 = 0.8$; $\sigma_4 = 0.8$									
60	5.3401	[5.3186, 5.3616]	5.0103	16.8742	2.1135	4.0846	5.3897	5.2725	5.3819
65	3.8179	[3.7993, 3.8365]	3.4889	18.2588	1.3236	2.6443	3.8418	3.8123	3.5776
70	2.7011	[2.6852, 2.7170]	2.3909	20.3315	0.8153	1.6722	2.7003	2.7430	2.2590
$\sigma_1 = 0.6$; $\sigma_2 = 1.2$; $\sigma_3 = 0.3$; $\sigma_4 = 0.9$									
60	5.5569	[5.5312, 5.5826]	5.2084	16.0580	2.1475	4.1943	5.9128	5.7558	5.9371
65	4.1555	[4.1324, 4.1786]	3.7902	17.7751	1.3628	2.7471	4.3459	4.2836	4.0874
70	3.1196	[3.0989, 3.1403]	2.7517	20.5267	0.8521	1.7609	3.1607	3.1798	2.6941

	Approximate Price by Using Control Variate				Length of the Confidence Interval			
K	GA	LN	KV	SLN	GA	LN	KV	SLN
$\sigma_1 = 0.2$; $\sigma_2 = 0.2$; $\sigma_3 = 0.2$; $\sigma_4 = 0.2$								
50	6.5399	6.5375	6.5399	6.5404	0.0036	0.0172	0.0034	0.0015
55	2.5082	2.5078	2.5092	2.5093	0.0041	0.0135	0.0029	0.0013
60	0.5038	0.5028	0.5044	0.5051	0.0028	0.0067	0.0017	0.0008
$\sigma_1 = 0.5$; $\sigma_2 = 0.5$; $\sigma_3 = 0.5$; $\sigma_4 = 0.5$								
55	4.8355	4.8328	4.8375	4.8385	0.0123	0.0330	0.0098	0.0079
60	2.7413	2.7378	2.7430	2.7445	0.0110	0.0264	0.0085	0.0068
65	1.4457	1.4428	1.4471	1.4495	0.0092	0.0199	0.0071	0.0058
$\sigma_1 = 0.8$; $\sigma_2 = 0.8$; $\sigma_3 = 0.8$; $\sigma_4 = 0.8$								
60	5.3409	5.3355	5.3443	5.3481	0.0265	0.0509	0.0217	0.0199
65	3.8161	3.8113	3.8204	3.8242	0.0245	0.0446	0.0201	0.0183
70	2.6981	2.6933	2.7003	2.7061	0.0223	0.0386	0.0185	0.0167
$\sigma_1 = 0.6$; $\sigma_2 = 1.2$; $\sigma_3 = 0.3$; $\sigma_4 = 0.9$								
60	5.5587	5.5517	5.5622	5.5664	0.0337	0.0645	0.0292	0.0286
65	4.1546	4.1482	4.1587	4.1643	0.0324	0.0580	0.0279	0.0271
70	3.1178	3.1109	3.1200	3.1251	0.0308	0.0518	0.0267	0.0257

Table 6.5: *Arithmetic average basket call option price when the initial stock prices are* $s_1 = 40$, $s_2 = 50$, $s_3 = 60$, $s_4 = 70$.

Numerical results also show that our control variate method performs very well. Especially, when the relative variation inside the initial stock prices high and the volatilities low then our control variate method gives much better results and outperforms the other control variate methods. When the relative variation inside the initial stock prices is smaller then still our method outperforms the other control variate methods, however, in this case and in the case of high volatility environment the difference between our method and the other methods is smaller. When there is no relative variation between the initial stock prices then our method and KV method outperform the other control variate methods.

K	MC	MCCI	LB	UB	GA	KV	LN	RG	SLN
$\sigma_1 = 0.2; \quad \sigma_2 = 0.2; \quad \sigma_3 = 0.2; \quad \sigma_4 = 0.2$									
48	5.9822	[5.9751, 5.9893]	5.9855	6.1555	5.3297	5.9793	5.9870	5.9799	6.0106
52	2.7000	[2.6943, 2.7058]	2.6986	3.5922	2.2148	2.4362	2.7035	2.6934	2.7305
56	0.7722	[0.7689, 0.7755]	0.7682	2.8950	0.5586	0.7431	0.7730	0.7787	0.7518
$\sigma_1 = 0.5; \quad \sigma_2 = 0.5; \quad \sigma_3 = 0.5; \quad \sigma_4 = 0.5$									
50	5.9724	[5.9577, 5.9871]	5.9094	9.1528	4.1446	5.7223	5.9874	5.9127	6.1548
55	3.4087	[3.3970, 3.4204]	3.3284	8.5935	2.1249	3.1189	3.4161	3.3899	3.4215
60	1.7808	[1.7722, 1.7895]	1.7039	9.1467	0.9836	1.5255	1.7803	1.8057	1.6393
$\sigma_1 = 0.8; \quad \sigma_2 = 0.8; \quad \sigma_3 = 0.8; \quad \sigma_4 = 0.8$									
55	5.8777	[5.8562, 5.8991]	5.5589	15.8253	2.7544	4.8301	5.9215	5.7752	6.0396
60	4.1362	[4.1177, 4.1547]	3.8070	16.9183	1.7225	3.1251	4.1605	4.1032	4.0034
65	2.8687	[2.8530, 2.8844]	2.5527	18.7541	1.0554	1.9673	2.8723	2.8930	2.5081
$\sigma_1 = 0.6; \quad \sigma_2 = 1.2; \quad \sigma_3 = 0.3; \quad \sigma_4 = 0.9$									
55	6.0108	[5.9852, 6.0364]	5.6875	14.9213	2.7775	4.9357	6.4436	6.2557	6.6041
60	4.4080	[4.3850, 4.4310]	4.0666	16.2330	1.7586	3.2290	4.6777	4.5851	4.5372
65	3.2438	[3.2233, 3.2643]	2.8972	18.6330	1.0931	2.0596	3.3514	3.3458	2.9729

	Approximate Price by Using Control Variate				Length of the Confidence Interval			
K	GA	LN	KV	SLN	GA	LN	KV	SLN
$\sigma_1 = 0.2; \quad \sigma_2 = 0.2; \quad \sigma_3 = 0.2; \quad \sigma_4 = 0.2$								
48	5.9866	5.9840	5.9865	5.9868	0.0021	0.0152	0.0020	0.0014
52	2.7027	2.7016	2.7031	2.7032	0.0021	0.0126	0.0017	0.0012
56	0.7729	0.7713	0.7733	0.7735	0.0017	0.0076	0.0012	0.0009
$\sigma_1 = 0.5; \quad \sigma_2 = 0.5; \quad \sigma_3 = 0.5; \quad \sigma_4 = 0.5$								
50	5.9771	5.9750	5.9789	5.9796	0.0096	0.0323	0.0081	0.0078
55	3.4114	3.4072	3.4120	3.4133	0.0090	0.0263	0.0071	0.0069
60	1.7807	1.7775	1.7825	1.7837	0.0077	0.0200	0.0059	0.0058
$\sigma_1 = 0.8; \quad \sigma_2 = 0.8; \quad \sigma_3 = 0.8; \quad \sigma_4 = 0.8$								
55	5.8806	5.8740	5.8822	5.8858	0.0235	0.0478	0.0193	0.0194
60	4.1358	4.1305	4.1396	4.1421	0.0219	0.0418	0.0179	0.0178
65	2.8665	2.8616	2.8692	2.8732	0.0200	0.0359	0.0164	0.0162
$\sigma_1 = 0.6; \quad \sigma_2 = 1.2; \quad \sigma_3 = 0.3; \quad \sigma_4 = 0.9$								
55	6.0142	6.0066	6.0166	6.0196	0.0310	0.0626	0.0272	0.0279
60	4.4085	4.4018	4.4123	4.4160	0.0300	0.0565	0.0261	0.0265
65	3.2424	3.2360	3.2454	3.2496	0.0287	0.0504	0.0251	0.0251

Table 6.6: *Arithmetic average basket call option price when the initial stock prices are* $s_1 = 45, \; s_2 = 50, \; s_3 = 55, \; s_4 = 60$.

It is seen that, our control variate method performs better actually when the relative variation inside the expected end prices (that is, expected prices at maturity) is higher. On the top of this, it can be concluded that, our control variate method outperforms all of the other control variate methods when the parameter settings lead to high relative variation inside the expected prices at maturity time, and in the other cases our and KV's control variate methods outperform the other methods.

To compare the computational effort of the different approaches note that

K	MC	MCCI	LB	UB	GA	KV	LN	RG	SLN
$\sigma_1 = 0.2; \quad \sigma_2 = 0.2; \quad \sigma_3 = 0.2; \quad \sigma_4 = 0.2$									
45	6.3618	[6.3550, 6.3686]	6.3651	6.4702	6.0024	6.3635	6.3659	6.3610	6.3831
50	2.2535	[2.2483, 2.2586]	2.2513	3.2410	2.0056	2.2417	2.2561	2.2479	2.2772
55	0.3550	[0.3528, 0.3571]	0.3511	2.8596	0.2863	0.3443	0.3547	0.3637	0.3255
$\sigma_1 = 0.5; \quad \sigma_2 = 0.5; \quad \sigma_3 = 0.5; \quad \sigma_4 = 0.5$									
50	4.3303	[4.3178, 4.3428]	4.2581	8.3077	2.9983	4.0993	4.3388	4.2885	4.4258
55	2.2766	[2.2672, 2.2860]	2.1981	8.3854	1.4187	2.0479	2.2784	2.2828	2.1982
60	1.0915	[1.0849, 1.0981]	1.0247	9.4226	0.6046	0.9152	1.0871	1.1298	0.9131
$\sigma_1 = 0.8; \quad \sigma_2 = 0.8; \quad \sigma_3 = 0.8; \quad \sigma_4 = 0.8$									
55	4.5973	[4.5785, 4.6160]	4.2788	15.5960	2.1080	3.6616	4.6233	4.5330	4.5849
60	3.1407	[3.1249, 3.1566]	2.8264	17.1442	1.2767	2.2850	3.1482	3.1419	2.8703
65	2.1158	[2.1025, 2.1290]	1.8271	19.4057	0.7582	1.3890	2.1065	2.1647	1.6826
$\sigma_1 = 0.6; \quad \sigma_2 = 1.2; \quad \sigma_3 = 0.3; \quad \sigma_4 = 0.9$									
60	3.4530	[3.4324, 3.4735]	3.1366	16.6666	1.3130	2.3788	3.6409	3.6050	3.3613
65	2.5091	[2.4908, 2.5274]	2.1983	19.8250	0.7924	1.4678	2.5448	2.5841	2.0843
70	1.8417	[1.8254, 1.8579]	1.5448	24.3681	0.4720	0.8889	1.7610	1.8525	1.2194

	Approximate Price by Using Control Variate				Length of the Confidence Interval			
K	GA	LN	KV	SLN	GA	LN	KV	SLN
$\sigma_1 = 0.2; \quad \sigma_2 = 0.2; \quad \sigma_3 = 0.2; \quad \sigma_4 = 0.2$								
45	6.3659	6.3635	6.3658	6.3660	0.0012	0.0137	0.0012	0.0013
50	2.2559	2.2548	2.2561	2.2562	0.0012	0.0107	0.0010	0.0011
55	0.3548	0.3538	0.3550	0.3551	0.0008	0.0049	0.0006	0.0007
$\sigma_1 = 0.5; \quad \sigma_2 = 0.5; \quad \sigma_3 = 0.5; \quad \sigma_4 = 0.5$								
50	4.3335	4.3307	4.3350	4.3358	0.0080	0.0259	0.0065	0.0071
55	2.2781	2.2742	2.2792	2.2795	0.0072	0.0201	0.0055	0.0060
60	1.0905	1.0881	1.0913	1.0922	0.0058	0.0146	0.0044	0.0049
$\sigma_1 = 0.8; \quad \sigma_2 = 0.8; \quad \sigma_3 = 0.8; \quad \sigma_4 = 0.8$								
55	4.5985	4.5930	4.6011	4.6027	0.0207	0.0391	0.0168	0.0176
60	3.1391	3.1345	3.1425	3.1452	0.0190	0.0335	0.0154	0.0160
65	2.1129	2.1090	2.1145	2.1174	0.0170	0.0283	0.0140	0.0144
$\sigma_1 = 0.6; \quad \sigma_2 = 1.2; \quad \sigma_3 = 0.3; \quad \sigma_4 = 0.9$								
60	3.4523	3.4461	3.4556	3.4586	0.0277	0.0494	0.0244	0.0246
65	2.5074	2.5013	2.5089	2.5117	0.0265	0.0439	0.0234	0.0233
70	1.8399	1.8348	1.8401	1.8418	0.0251	0.0388	0.0225	0.0222

Table 6.7: *Arithmetic average basket call option price when the initial stock prices are* $s_1 = 50, \; s_2 = 50, \; s_3 = 50, \; s_4 = 50$.

- All closed-form formulas yield results nearly immediately (even if we have more than hundred stocks in our basket) where the computation times of our method (SLN), GA and KV methods are shorter than those of the RG and LN methods.

- As the closed-form formulas of RG and KV contain nested loops in their calculations their computation times grow much faster than those of SLN, LN and GA if the number of stocks in the basket gets very high.

- In the class of the control variate methods our method is slower than the other ones as it requires nearly twice as much random numbers (for a given number of simulation runs). When we additionally consider the accuracy of the methods via multiplying the computation times with the length of the confidence intervals then our method out performs the other methods, in particular when the relative variation inside the stock prices is high.

Here, we do not provide numerical results for the arithmetic average Asian options. However, in the Asian option case our method is not as suited as for the basket option case as per construction there are high dependencies in the Asian option setting which leads to a low relative variation inside the prices that derive the payoff of the option.

6.8 Use of the extrapolation methods to accelerate the convergence

In general terms, extrapolation is the process of estimating a function value with the help of its other values. With this definition extrapolation may serve for different objectives, however here we will focus on the convergence acceleration.

Finding the limit of an infinite series may be hard when its convergence to the limit value is slow. In this case, extrapolation methods can be applied to accelerate the convergence of the sequence to its limit value. One of the widely used methods for this purpose is the Richardson extrapolation. Richardson extrapolation takes some values of a sequence and process them to obtain better approximation values. Its idea is basically to combine solutions obtained from different terms of the sequence to eleminate some of the error made from the use of them as an approximation. Since we will illustrate the methodology in the next section, here we will not provide more detailed information about the Richardson extrapolation. For a comprehensive information about the Richardson extrapolation method and the other extrapolation methods we refer the interested readers to Sidi [53] and Brezinski and Redivo Zaglia [10].

In this part we will introduce how to use the extrapolation methods to accelerate the convergence of our log-normal sum approximation method. The use of the Richardson and Romberg extrapolation which is based on the successive implementation of the Richardson extrapolation, will be illustrated.

6.8.1 Richardson Extrapolation

To accelerate the convergence of our method by Richardson Extrapolation, we could use both the absolute error and the relative error in our calculations. Here, we will derive the formulas by using the absolute error. The derivation of the formulas for the relative error can be found in Appendix B.

From Theorem 6.2.1 we have the approximation

$$\frac{1}{n}\sum_{i=1}^{n}(S_i + C) \approx \left(\prod_{i=1}^{n}(S_i + C)\right)^{1/n},$$

or, equivalently

$$\frac{1}{n}\sum_{i=1}^{n}S_i \approx \left(\prod_{i=1}^{n}(S_i + C)\right)^{1/n} - C$$

for large values of C. In the remaining part of this chapter and in Appendix B we will denote the arithmetic mean by AM, and the geometric mean obtained from our method with a shift parameter value C by $GM(C)$, that is,

$$AM = \frac{1}{n}\sum_{i=1}^{n}S_i, \tag{6.11}$$

$$GM(C) = \left(\prod_{i=1}^{n}(S_i + C)\right)^{1/n} - C. \tag{6.12}$$

Our extensive numerical applications show that the absolute error of our approximation can be assumed to be of order $1/C$, that is

$$AM = GM(C) + K\frac{1}{C} + O\left(\frac{1}{C^2}\right),$$

where K is a constant and O denotes the big O notation (see, for example, Heath [25]).

For two different values of the shift parameter C, namely, C_1 and C_2, we have

$$AM = GM(C_1) + K\frac{1}{C_1},$$
$$AM = GM(C_2) + K\frac{1}{C_2}.$$

Multiplying the first equation with -1, the second equation with C_2/C_1, and then summing up the equations we get

$$AM(C_1, C_2) = \frac{\frac{C_2}{C_1}GM(C_2) - GM(C_1)}{\frac{C_2}{C_1} - 1}, \tag{6.13}$$

which is the extrapolated value obtained from Richardson Extrapolation with C_1 and C_2.

Remember that, in our method the arithmetic mean and the geometric mean are equal as C goes to infinity. Since in computations we required to use a finite C value it is not possible to find the exact limiting approximation, therefore our calculations contain an error as per the limiting value. If we want to have an approximation error equal to a pre-specified fixed value then the Richardson extrapolation method can be used to find a C value corresponding to this error value. Assume that in our approximation method we want to have an absolute error about ε. To find the C value achieving this absolute error, take two values of C, namely, C_1 and C_2, then we have

$$AM = GM(C_1) + K\frac{1}{C_1},$$
$$AM = GM(C_2) + K\frac{1}{C_2}.$$

Multiplying the second equation by -1 and summing up the equations we get

$$K = (GM(C_2) - GM(C_1))\frac{C_1C_2}{C_2 - C_1}.$$

Note that, in our method we approximate the arithmetic mean by using the Equation (6.12), and we assumed

$$AM = GM(C) + K\frac{1}{C}.$$

Therefore, in our approximation method the absolute error is about $K(1/C)$. Since we would like to have an absolute error about ε, the optimum C value can be calculated from

$$K\frac{1}{C} = \varepsilon,$$

which implies

$$C_{opt} = (GM(C_2) - GM(C_1))\frac{C_1C_2}{C_2 - C_1}\varepsilon^{-1}. \tag{6.14}$$

6.8.2 Romberg Extrapolation

If we use k different C values to find an approximation to the arithmetic mean by the use of our method and denote

$$GM_0^i := GM(C_i) \quad (i = 1, ..., k),$$

then the Romberg extrapolation method implies the following iterative formula that provide a faster convergence to the limiting approximation value:

$$GM_n^j = \frac{\frac{C_{n+j}}{C_j} GM_{n-1}^{j+1} - GM_{n-1}^j}{\frac{C_{n+j}}{C_j} - 1}.$$

To illustrate the above formula, for example use there values of C (namely, C_1, C_2 and C_3), then we have

$$GM_0^1 = GM(C_1), \quad GM_0^2 = GM(C_2), \quad GM_0^3 = GM(C_3),$$

$$GM_1^1 = \frac{\frac{C_2}{C_1} GM_0^2 - GM_0^1}{\frac{C_2}{C_1} - 1}, \quad GM_1^2 = \frac{\frac{C_3}{C_2} GM_0^3 - GM_0^2}{\frac{C_3}{C_2} - 1},$$

$$GM_2^1 = \frac{\frac{C_3}{C_1} GM_1^2 - GM_1^1}{\frac{C_3}{C_1} - 1}.$$

where GM_0^1, GM_0^2, GM_0^3 are the approximation results obtained using C_1, C_2, C_3 in the Equation (6.12), respectively. Furthermore, GM_1^1 is the value of the Richardson extrapolation obtained by using GM_0^1 and GM_0^2; GM_1^2 is the value of the Richardson extrapolation obtained by using GM_0^2 and GM_0^3 and, finally, GM_2^1 is the approximation value obtained from the Romberg extrapolation using C_1, C_2, C_3 (or, equivalently, the value of the Richardson extrapolation obtained by using GM_1^1 and GM_1^2).

Note that, above we illustrated the extrapolation methods to accelerate the convergence of the geometric mean obtained from our methodology to the original arithmetic mean. That means, in the extrapolation methods we did not deal with the approximate option prices. Numerical applications show that the error of our approximation can still be assumed to be of order $1/C$ if we use option prices. Therefore, to use the extrapolation methods for our approximation formulas of option price, we just need to replace AM with the actual option price and $GM(C)$ with the approximate option price obtained from our method.

Note also that, extrapolation methods accelerates the convergence of our approximation method to the limiting value, however that does not mean they accelerate the convergence to the correct option price since our method has a bias in some cases. However, it can be concluded that, for the (deep-) out of money options extrapolation methods speed up the convergance to the right value since in such options our closed-form formula generally underestimate the option price.

Example 6.8.1. Assume an arithmetic average basket call option with payoff depending on the prices of 30 assets in a Black-Scholes market. Under the risk-neutral measure, let the initial stock prices and the volatility parameters of the stock price processes be as follows:

$$S_i(0) = 100 - i \quad (i = 1, ..., n),$$
$$\sigma_i = 0.2 + 0.008i \quad (i = 1, ..., n).$$

In computations, we will take a maturity of 6 months for the option and assume the risk-free rate and the strike price are $r = 0.06$ and $K = 80$.

For different values of shift parameter C we get the following option prices from our approximation formula (6.4):

$p_{GM}(10^1) = 4.6446$; $p_{GM}(10^2) = 5.7207$; $p_{GM}(10^3) = 6.6970$; $p_{GM}(10^4) = 6.8775$;
$p_{GM}(10^5) = 6.8971$; $p_{GM}(10^6) = 6.8990$; $p_{GM}(10^7) = 6.8992$; $p_{GM}(10^8) = 6.8992$.

With four digits there is no difference between the values of $p_{GM}(10^7)$ and $p_{GM}(10^8)$. Therefore, it can be concluded that limiting price of our approximation method is very close to 6.8992.

To test the performance of extrapolation methods we use 4 different values of shift parameter: 10^2, 10^3, 10^4 and 10^5. The results of the Richardson extrapolation for the possible pairwise usage of these C values are given in Table 6.8. From the table it is clearly seen that with increasing C values we get better results. When we use $C = 10^4$ and $C = 10^5$ in the extrapolation method, we obtain the same results with $p_{GM}(10^8)$ in 4 digits.

	$C_1 = 10^2$	$C_2 = 10^3$	$C_3 = 10^4$	$C_4 = 10^5$
$C_1 = 10^2$	-	6.8055	6.8892	6.8982
$C_2 = 10^3$	6.8055	-	6.8975	6.8991
$C_3 = 10^4$	6.8892	6.8975	-	6.8992
$C_4 = 10^5$	6.8982	6.8991	6.8992	-

Table 6.8: *The results of the Richardson extrapolation for different pairwise usage of C values.*

To illustrate the efficiency of the Romberg extrapolation we will use 3 values of C. When $C = 10^2$, $C = 10^3$ and $C = 10^4$ are used we get 6.8985, however when we use $C = 10^2$, $C = 10^3$ and $C = 10^5$ we get 6.8992 which is the result of $p_{GM}(10^8)$.

From the results it is seen that the Richardson and Romberg extrapolation methods accelerates the convergence of our approximation methods, however, there is no big difference between these two methods.

CHAPTER 7

CONCLUSION

In the last decade, CVaR has received considerable attention by the researchers since the introduction of its linearization procedure by Rockafeller and Uryasev [48]. In this thesis, we worked on some types of problems containing CVaR constraints. Firstly, we looked at a particular investment problem where - besides stocks and bonds - the investor can also include options (or more complicated, structured products) into a portfolio. Compared to the Martinelli et al. [42] approach, we allow for intermediate payments of the securities and are thus faced with a re-investment problem which turns the originally one-period model into a (special kind of a) multi-period problem. We used the linearization procedure of Rockafeller and Uryasev for the CVaR constraint and developed a method to deal with our multi-period problem by solving a series of those one-period problems.

Our numerical results obtained from the solution of the problem uncovered some surprising weaknesses of the use of VAR and CVaR as a risk measure. In the presence of the opportunity to invest into options with relatively high strikes, using the option with the higher strike leads to a higher expected return while keeping the risk constant. However, our subjective feeling of an increasing risk is much better matched by the use of the variance, although this is a non-coherent risk measure.

Our investment decision problem can also be solved when we have more securities than the ones assumed above. Each new security increases the number of the unknowns in the problem just by one, therefore it increases the computation time in a negligible level. In particular, we can also deal with more than just two periods in our optimization problem. However, here the outer optimization loop(s) for obtaining the optimal re-investment strategy gets more complicated. Each additional time period will add one more outer loop, consequently finding the solution of the optimization problem will take longer.

In the next step, we focused on a type of quantile hedging problem. Although the investors have the opportunity to stay on the safe side by using a hedging or super-hedging strategy, they are generally unwilling to put up the initial amount of capital required for a hedge or a super-hedge strategy. In this case, the quantile hedging might be reasonable for some investors. We constructed an optimization problem with the objective of maximizing the probability of having a higher value for liquid assets than that for the liabilities at a pre-specified time. In our problem, we only considered the liquid assets since we aimed to construct a strategy strengthening the liquidity of the investor, which is also a convenient problem construction for meeting the Liquidity Coverage Ratio requirement of Basel III Accord. Since the quantile hedging is not a perfect hedge, a non-negative probability for having a liability value higher than the asset value exists. As the shortfall amount between the liability and the asset values effects the cost of financing, we used a CVaR constraint to control the probable deficient amount. Under the assumption of a Black-Scholes market where the assets and the liabilities are log-normally distributed, to calculate the probability placed in our objective function we had to deal with the problem of finding the distribution of summation of the log-normal distributions. It is know that the sum of the log-normal distributions has no specific distribution. To get ride of this problem, some log-normal sum approximation methods are proposed in the literature. We have introduced a new, simple and efficient method to approximate the sum of the log-normal distributions using shifted log-normal distributions. Our method is based on a limiting approximation of the arithmetic mean by the geometric one. In our method, we shift each log-normal distribution by a sufficiently big positive constant and then approximate the resulting shifted log-normal distribution by a log-normal distribution having same moments. This method causes a sharp decrease in the relative variation inside the stock prices and therefore enables us to replace the arithmetic mean by the geometric mean which has a closed form representation. Using our approximation method we reduced our quantile hedging problem to a simpler optimization problem.

Our log-normal sum approximation method could also be used to price some options an the Black-Scholes model. We have derived closed analytical approximation formulas for the prices of the arithmetic average basket and Asian options. Numerical applications show that our approximation method gives very accurate results when we have small variances for the stock prices. If variances of the stock prices are high then we still get reasonable approximate prices but the quality of the approximation is not sufficiently good for the use of the analytical formulas as substitute for Monte-Carlo simulation. However, using the new approximation method in a Monte-Carlo control variate approach results in very accurate and very efficient results. In this case, our control variate method is at lease comparable with the well know control variate methods from the literature and outperforms them when there exists a relative variation inside the expected values of prices at maturity time. Note that, our methodology is not limited to this market model. The same methodology could always be used when options on the approximating geometric averages admit a closed-form option price representation.

Our closed form approximation formulas of the option price are generally exhibit the same characteristic. They generally overestimate the price when we use (deep-) in-the-money options, underestimate the price when we use (deep-) out-of-money options and give better estimates when we use at-the-money options. Therefore, as a future study, it might be possible to define a correction term for our closed form approximation formulas. Furthermore, we are planning to assess the efficiency of our log-normal sum approximation method and the formulas in risk management, by the use of the *Greeks* and by the use of different problem constructions.

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APPENDIX A

Approximate pricing of arithmetic average Asian options

A.1 Approximate pricing by using the parameters of the shifted price process at maturity

Firstly, we shift the random variable $S(T)$ by a sufficiently big positive constant C to obtain the random variable $Y(T) := S(T)+C$ that has a shifted log-normal distribution with expected value $\mathbb{E}(S(T)) + C$ and variance $Var(S(T))$. Then, we approximate this shifted log-normal distribution by a log-normal distribution $Y^*(T)$ having the same mean and variance with $Y(T)$. The parameters of the $Y^*(T)$ are given by

$$\begin{aligned}\tilde{\mu} &= \log\left(\frac{[\mathbb{E}(S(T)) + C]^2}{\sqrt{[\mathbb{E}(S(T)) + C]^2 + Var(S(T))}}\right),\\ \tilde{\sigma}^2 &= \log\left(1 + \frac{Var(S(T))}{[\mathbb{E}(S(T)) + C]^2}\right).\end{aligned}$$

Since $Y^*(T)$ follows log-normal distribution it can be written in the form of a geometric Brownian motion as

$$dY^*(t) = Y^*(t)\{\alpha dt + \gamma dB(t)\},$$

where

$$\gamma = \frac{\tilde{\sigma}}{\sqrt{T}}, \quad \alpha = \frac{\tilde{\mu} - \log(S(0) + C)}{T} + \frac{\gamma^2}{2},$$

and $Y^*(0) = S(0)+C$. We then replace the final payment of the discrete fixed-strike arithmetic average Asian option by that of discrete fixed-strike geometric average Asian option having strike price $K + C$ and payoff

$$B_1 = \left(\left(\prod_{i=1}^{n} Y^*(t_i)\right)^{1/n} - (K + C)\right)^+,$$

and end up with the approximate pricing formula (6.6).

A.2 The price obtained by shifting the price process at each time point

Here, we determine the input parameters for the Black-Scholes type formula by using all stock prices entering the average. For each i, we define a new random variable X_i having a log-normal distribution with mean $\mathbb{E}(S(t_i))$ and variance $Var(S(t_i))$. Then the random variable $Y_i := X_i + C$ has a shifted log-normal distribution and we approximate it by a log-normal distribution Y_i^*, having the same expected value and variance. Then, the corresponding parameters of this log-normal distribution are

$$\begin{aligned} \tilde{\mu}_i &= \log\left(\frac{[\mathbb{E}(S(t_i)) + C]^2}{\sqrt{[\mathbb{E}(S(t_i)) + C]^2 + Var(S(t_i))}}\right), \\ \tilde{\sigma_i}^2 &= \log\left(1 + \frac{Var(S(t_i))}{[\mathbb{E}(S(t_i)) + C]^2}\right). \end{aligned}$$

Each log-normal distribution with parameters $\tilde{\mu}_i$ and $\tilde{\sigma_i}^2$ as defined above corresponds to the geometric Brownian motion

$$dY_i^*(t) = Y_i^*(t)\{\alpha_i dt + \gamma_i dB_i(t)\},$$

with

$$\gamma_i = \frac{\tilde{\sigma}_i}{\sqrt{t_i}}, \quad \alpha_i = \frac{\tilde{\mu}_i - \log(S(0) + C)}{t_i} + \frac{\gamma_i^2}{2},$$

and $Y_i^*(0) = S(0) + C$ for $i = 1, ..., n$. Here, the Brownian motions B_i shall be taken as correlated with each other since the original option is path dependent. More precisely, the price $S(t_i)$ is given as the product of $S(t_{i-1})$ and the randomness between t_{i-1} and t_i. Therefore, the price at time t_i is also affected by the randomness prior to t_{i-1}, or equivalently, by the randomness contained in $S(t_{i-1})$. If we denote the randomness realized in the time interval $(t_{i-1}, t_i]$ by Z_i, then the price $S(t_i)$ will be depended on $Z_1, Z_2, ..., Z_i$ for all i.

We need to transfer the correlation structure to the new stock price process in such a way that the total effect of the correlation structure to the option price shall be preserved in the new construction. We could achieve this by preserving the original covariance structure between the stock prices in our new environment. We thus obtain

$$\begin{aligned} \left(\prod_{i=1}^{n} Y_i^*\right)^{1/n} &= (S(0) + C)\, e^{\frac{1}{n}\sum_{i=1}^{n}\left\{\left(\alpha_i - \frac{\gamma_i^2}{2}\right)t_i + \gamma_i B_i(t_i)\right\}} \\ &= (S(0) + C)\, e^{m + \nu Z}, \end{aligned}$$

where Z has a standard normal distribution and

$$\begin{aligned} m &= \frac{1}{n}\sum_{i=1}^{n}\alpha_i t_i, \\ \nu &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\gamma_i^2 t_i + 2\sum_{i\neq j}^{n}\min\{i,j\}\gamma_i\gamma_j(t_i - t_{i-1})^{1/2}(t_j - t_{j-1})^{1/2}}. \end{aligned}$$

We then replace the final payment of the discrete fixed-strike arithmetic average Asian option by that of a discrete fixed-strike geometric average Asian option having strike price $K+C$ and payoff

$$B_2 = \left(\left(\prod_{i=1}^{n} Y_i^*\right)^{1/n} - (K+C)\right)^+$$

with a sufficiently big constant C, which enables us to end up with the approximate pricing formula (6.7).

APPENDIX B

Richardson extrapolation with the use of relative error

In this part, we will assume that the relative error of our approximation is of order $1/C$, that is

$$\frac{AM - GM(C)}{AM} = K\frac{1}{C} + O\left(\frac{1}{C^2}\right),$$

where K is a constant.

If we take two values of C, namely, C_1 and C_2, then we have

$$\begin{aligned} \frac{AM - GM(C_1)}{AM} &= K\frac{1}{C_1}, \\ \frac{AM - GM(C_2)}{AM} &= K\frac{1}{C_2}. \end{aligned}$$

Multiplying the first equation by -1 and the second equation by C_2/C_1, and then summing up the equations we get

$$AM(C_1, C_2) = \frac{\frac{C_2}{C_1}GM(C_2) - GM(C_1)}{\frac{C_2}{C_1} - 1},$$

which is same as the equation obtained using absolute error.

To find the C value achieving a relative error about ε, again take two values of C, namely C_1 and C_2. Then,

$$\begin{aligned} \frac{AM - GM(C_1)}{AM} &= K\frac{1}{C_1}, \\ \frac{AM - GM(C_2)}{AM} &= K\frac{1}{C_2}. \end{aligned}$$

Multiplying the first equation by $-GM(C_2)/GM(C_1)$ and then summing up the equations we get

$$K = \frac{\frac{GM(C_2)}{GM(C_1)} - 1}{\frac{1}{C_1}\frac{GM(C_2)}{GM(C_1)} - \frac{1}{C_2}}.$$

Since in our approximation method the relative error is about $K(1/C)$ then the C value achieving a relative error about ε can be calculated from

$$K\frac{1}{C} = \varepsilon,$$

which implies

$$C_{opt} = \frac{\frac{GM(C_2)}{GM(C_1)} - 1}{\frac{1}{C_1}\frac{GM(C_2)}{GM(C_1)} - \frac{1}{C_2}}\varepsilon^{-1}.$$

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