STOCHASTIC CREDIT DEFAULT SWAP PRICING

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ABSTRACT

STOCHASTIC CREDIT DEFAULT SWAP PRICING

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Credit risk measurement and management has great importance in credit market. Credit derivative products are the major hedging instruments in this market and credit default swap contracts (CDSs) are the most common type of these instruments. As observed in credit crunch (credit crisis) that has started from the United States and expanded all over the world, especially crisis of Iceland, CDS premiums (prices) are better indicative of credit risk than credit ratings. Therefore, CDSs are important indicators for credit risk of an obligor and thus these products should be understood by market participants well enough. In this thesis, initially, advanced credit risk models firsts, the structural (firm value) models, Merton Model and Black-Cox constant barrier model, and the intensity-based (reduced-form) models, Jarrow-Turnbull and Cox models, are studied. For each credit risk model studied, survival probabilities are calculated. After explaining the basic structure of a single name CDS contract, by the help of the general pricing formula of CDS that result from the equality of in and out cash flows of these contracts, CDS price for each structural models (Merton model and Black-Cox constant barrier model) and CDS price for general type of intensity based models are obtained. Before the conclusion, default intensities are approximately es-

timated from the distribution functions of default under two basic structural models; Merton and Black-Cox constant barrier. Finally, we conclude our work with some inferences and proposals.

Keywords: Credit risk, credit derivatives, single name credit default swap, credit crunch, structural model, intensity-based model, Merton model, Black-Cox constant barrier model, Jarrow-Turnbull model, Cox model, default intensity, survival probability, probability of default.

ÖZ

KREDİ TEMERRÜT TAKASI SÖZLEŞMELERİNİN STOKASTİK FİYATLANMASI

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Kredi piyasalarında kredi riskinin ölçülmesi ve yönetilmesi büyük önem arz etmektedir. Kredi türevleri, kredi riskinin azaltılmasında kullanılan önemli enstrümanlardır. Amerika Birleşik Devletleri'nden başlayarak tüm dünyaya yayılan kredi krizinde, özellikle de İzlanda krizinde gözlendiği gibi kredi temerrüt takası sözleşmelerinin fiyatları yani CDS primleri, krizi ön görmede kredi derecelerinden daha iyi birer gösterge olmuştur. Dolayısı ile bu enstrümanların piyasa katılımcıları tarafından iyi bir şekilde anlaşılmaları, kredi riskinin yönetilmesi ve olası risklerin önceden tahmin edilmesi noktasında değerlidir. Bu tezde, öncelikle gelişmiş kredi riski modellerinin ilkleri olan ve firma değerine dayalı modellerden Merton modeli ve Black-Cox sabit bariyer modeli ile yoğunluk değerine dayalı modellerden olan Jarrow-Turnbull ve Cox modelleri calışılmıştır. Çalışılan her model için ayrıca batmama olasılıkları hesaplanmıştır. Tek isimli CDS kontratlarının genel yapısı açıklandıktan sonra, koruma satın alan tarafça ödenen primler ile batma durumunda elde edilecek koruma tutarının eşitliğine dayanan genel CDS fiyatı formüllerinden faydalanılarak, ayrıca daha önceden hesaplanan batmama ihtimalleri kullanılarak firma değerine dayalı modeller, Merton modeli ve Black-Cox

sabit değer modeli için ayrı ayrı ve ayrıca yoğunluk değerine dayalı modeller için ise genel bir CDS fiyatı hesaplanmıştır. Daha sonra firma değerine dayalı modellerin olasılık fonksiyonları kullanılarak yoğunluk değerlerine dayalı modellerde kullanılabilecek yoğunluk değerleri ortalama olarak hesaplanmıştır. Son olarak sonuç kısmında çeşitli çıkarsamalar ve öneriler yapılmıştır.

Anahtar Kelimeler: Kredi riski, kredi türevleri, tek isimli kredi temerrüt takası, kredi krizi, firma değerine dayalı modeller, yoğunluk değerine dayalı modeller, Merton modeli, Black-Cox sabit bariyer modeli, Jarrow-Turnbull modeli, Cox modeli, yoğunluk değeri, batmama olasılığı, batma olasılığı.

To my daughter Nil and my wife Pınar

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PREFACE

Credit risk is the most important risk of financial institutions regarding its relative amount and contagion effects among the other risks of these institutions. Therefore, credit risk deserves better understanding. Especially, after US credit crisis, every agent in the financial market has understood the importance of credit risk again. Academicians have started to study credit risk and related product more and more after that crises. After credit crunch, hedging instruments of credit risk, credit derivatives were criticized because of their complex structure and speculative use. However, we believe that, complexity of derivative products is not the reason for the credit crunch. Political intervention and lack of understanding of supervisors about the credit risk and its hedging products are the main causes of the credit crisis. To contribute better understanding of credit risk and most commonly used credit derivative, Credit Default Swap (CDS), this work is written on these subjects.

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CHAPTER 1

INTRODUCTION

According to the Basel II rules, there are three major risks. These risks that any firm or especially banks can face are; market, credit and operational risks. Market risk is the risk of the losses that take place in trading portfolio of a firm resulting from changes of general market conditions. According to the Basel Amendment [2], operational risk is the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. Credit risk, for a specified time horizon T, is the risk of a credit obligor or a reference entity that does not fulfil its credit obligation [11]. Credit risk is the most important risk among these market, credit and operational risk because of its larger scale and higher complexity. According to April 2012 monthly bulletin data of Banking Supervision and Regulation of Turkey (BRSA), credit risk constitutes 86% of all risk of Turkish banking sector. The Basel Committee permits to banks a choice between two broad methodologies in order to calculate their capital requirements for credit risk. The first choice is the Standardised Approach (SA) that is supported by external credit assessments and the alternative one is the Internal Ratings-based Approach (IRB), which is subject to the explicit approval of the bank's supervisor. In other words, supervisors would allow banks to use their internal rating systems for estimate credit risk. The IRB approach is based on the measures of unexpected losses (UL) and expected losses (EL). For many of the asset classes, the Committee has made two broad approaches available: a foundation and an advanced. Typically, a bank uses the internal estimates of the default probabilities ($P_{\mathcal{D}}$) but uses external sources for other model inputs such as loss given default (LGD). Supervisors often provide later information. However, generally bank generates all model inputs, exposure at default (EAD), $P_{\mathcal{D}}$ and LGD, under IRB approach.

As stated above, credit risk measurement has three main components for individual entity, $P_{\mathcal{D}}$, LGD and EAD, whereas a portfolio has two more components: default and credit quality correlation, risk contribution and concentration.

Banks generally keep capital for financial risks that they can face. Expected loss is the loss that mean of the loss distribution of a firm that predictable earlier. Unexpected loss is the dispersion of losses from expected loss. Expected loss EL and unexpected loss UL for given probability of default $P_{\mathcal{D}}$, loss given default LGD and exposure at default EAD are given by following formulas, respectively [30]:

$$EL = P_{\mathcal{D}} \times LGD \times EAD, \tag{1.1}$$

$$UL = EAD \times \sqrt{P_{\mathcal{D}} \times \sigma^2_{LGD} + LGD \times \sigma^2_{P_{\mathcal{D}}}} \tag{1.2}$$

where σ^2_{LGD} is the volatility of LGD and $\sigma^2_{P_{\mathcal{D}}}$ is the volatility of $P_{\mathcal{D}}$.

Figure 1.1 illustrates how variation in realised losses over time leads to a distribution of losses for a bank.

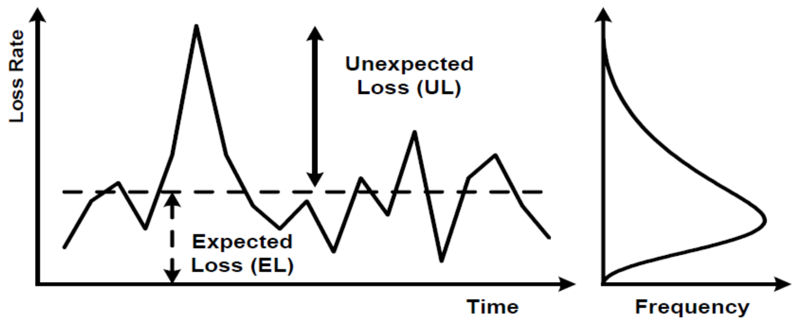

Figure 1.1: Expected and unexpected Loss[1].

As it can be realized, both EL and UL are basically functions of the same parameters, $P_{\mathcal{D}}$, LGD and EAD, with additional second order terms, $\sigma^2_{P_{\mathcal{D}}}$ and σ^2_{LGD}, in the UL expression. It can be also observed that, EL increases linearly with $P_{\mathcal{D}}$ but UL is a non-linear function of same parameter $P_{\mathcal{D}}$ and uniformly larger than EL for non-zero $P_{\mathcal{D}}$ as it is displayed in Figure 1.2 [30]. Adjusted exposure (AE) is the adjusted value of the EAD regarding the portion of credit that has been already used and portion of credit that expected to be used by obligor in case of his/her financial distress.

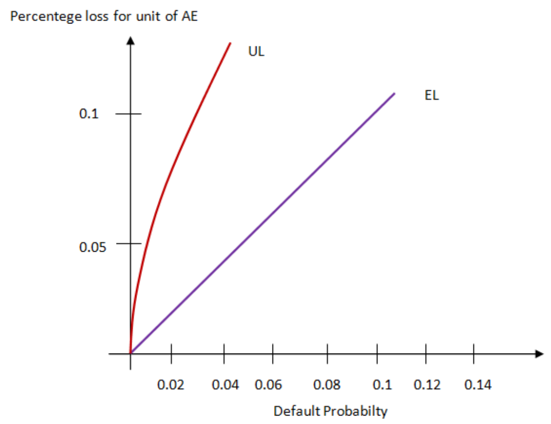

Figure 1.2: EL and UL for $LGD = 0.5$.

Therefore, in order to estimate the credit quality of a firm or a bank, one should determine these three credit risk factors. These factors are generally in the area of interest of the IRB models.

In general, there are two main IRB credit risk models categories: structural (firm value based) and intensity based (reduced-form) models. The structural models are based on the article "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates" written by Robert C. Merton [29]. These models use the information embedded in the equity prices in order to solve the default probabilities. The intensity based models use bond and other security prices in order to calculate the default probabilities. These models were originally introduced by Jarrow and Turnbull (1992) [22] and they are constructed on counting process.

Structural models are based on Black-Scoles option pricing formula, they take the asset values of a firm as underlying asset and the value of the liabilities as strike price then value firm equities as option premium written on firm assets. The first utilization of Black-Scholes formula was applied by Robert C. Merton in order to value a firm equity. After Merton's contribution, there have been many new models which improve Merton's study by changing some of its assumptions.

On the other hand, intensity based models focus directly on modelling the default probability. The basic idea lies in it is at any instant there exists a possibility of default for an obligor and this possibility depends on obligor's overall health. Default is defined at the first jump of a counting process $N = \{N_t;\ 0 \leq t \leq T\}$ with intensity $\lambda = \{\lambda_t;\ 0 \leq t \leq T\}$, which thus determines the price of credit risk.

Being very crucial risk of financial and non-financial global firms, it is important for creditors to hedge their credit risk. For that purpose, credit derivatives have become popular instruments and they have traded on the every side of the world after 1996. A credit derivative contract is a credit transaction that credit protection buyer makes periodic payments to the credit protection seller in exchange for right to have some compensation when a default event occurs on underlying asset or name [13]. Credit derivatives generally traded on the over the counter (OTC) market. The credit default swap (CDS), total return swap (TRS), credit linked note (CLN), portfolio protection products, collateralized debt obligations (CDOs) are the credit derivatives that are mainly traded on the market. Credit default swaps are more common and relatively complex type of the credit derivatives. A CDS is similar to a typical swap in that one party makes payments to another party. There are two counterpart in CDS transactions, one is the protection seller and the other one is the protection buyer. The protection buyer of the CDS seeks credit protection and makes fixed payments, CDS premium to the seller of the CDS for the life of the swap, or until credit event occurs [35]. Figure 1.3 shows the basic mechanism of a single name CDS. The CDS contract must specify the underlying reference name, a specific issuer or obligor of the underlying asset in advance, so both parties agree when a credit event occurs. Credit event specified by International Swaps and Derivatives Association, Inc. (ISDA) can be bankruptcy, failure to pay, restructuring, repudiation, moratorium, obligation acceleration and obligation default.

It is the responsibility of the protection seller to compensate the protection buyer for a credit event. Cash settlement or physical settlement are two standard settlement methods. Under cash settlement, the protection seller makes a one time cash payment to the protection buyer equal to par value of reference assets minus market value of that assets. On the other hand, if contract specified physical settlement, the protection buyer delivers the underlying reference to the protection seller and receives cash payment in amount of the par value [13].

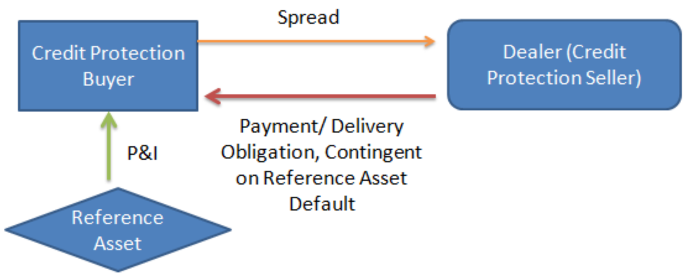

Figure 1.3: Single name CDS.

Ownership, recovery rights and liquidity concerns are the issues that may arise after a credit event. The main distinction between a CDS and a credit insurance is that the credit protection buyer in a CDS need not to have the reference asset.

The expected recovery of on the asset is reflected in the current market price of a security following its default. In a cash-settled CDS, the credit protection buyer receives a payment equal to the par value of the security minus the expected recovery, but if protection buyer owns the asset, it can try to improve on the actual recovery relative to the expectation reflected in the security price. On the contrary, if CDS contract is physically settled, the bond and the recovery right are given to the swap dealer in exchange for a cash payment equal to the par value of the security. If the protection buyer owns the asset and believes that it can improve on the recovery rate priced into the security, the protection buyer is clearly better off using the cash settled CDS [13].

The CDS protection can be sold on one single reference or on a portfolio containing more than one asset or name. Portfolio protection products entitle their buyer to a payment following one or more defaults in a reference portfolio consisting of multiple names and/ or assets. The CDS written on basket of assets is called basket CDS. There are different types of basket CDSs:

1. Nth to Default Swap,

2. Senior and Subordinated Basket CDSs,

3. Credit Indexes.

An nth default CDS pays off when the nth default occurs in the reference asset portfolio. For example, consider a reference portfolio that consists of the public bonds issued by 100 different companies or reference names. A first to default CDS will pay off when the first default occurs in the reference portfolio. A second to default will pay off when the second default occurs in the reference portfolio. This CDS does not pay anything for the first default, and terminates the following the payout that associated with the second default [13].

The basket CDS products are generally all about default correlation risk inside the reference portfolio. If defaults are uncorrelated across names, for example, an nth to default CDS with a one year tenor is unlikely to pay off for n at or above three. In other words, especially for portfolios of investment grade credits, more than three uncorrelated defaults in a year would be considered highly unusual. But if instead defaults on the reference names are perfectly correlated, nth to default CDS is no different at all from the first to default CDS [13].

The standard market model for pricing an nth-to-default CDS is one-factor Gaussian copula model for the time to default. Copulas provide an alternative measure of the dependence between random variables. A copula is defined as a function that joins a multivariate distribution function to a collection of univariate marginal distribution functions [18].

Although they are hedging instruments, credit derivatives are also vulnerable against credit risk, and this credit risk that a corporation can faces because of its derivative positions is called *counter-party credit risk*. For example, an option with default risk is called a *vulnerable option* [35]. To manage the counter-party credit risk, there are some methods; *netting agreements*, *collateralization* and also *downgrade triggers*.

The organization of this thesis is shortly as follows: After a general short explanatory introduction about credit risk models, credit risk management and credit derivatives in Chapter 1, details of credit risk models, structural and intensity based models, are explained in Chapter 2. Also in Chapter 2, probability of default functions for each model is calculated. In Chapter 3, CDS market is examined in details and then default functions found in Chapter 2, are used for calculating CDS prices for each model examined. In Chapter 4, hazard rate of intensity models is predicted from density functions of structural models, Merton and Black-Cox. Finally, Chapter 5 is a short summary of this work.

CHAPTER 2

CREDIT RISK MODELS

As we have mentioned before, there are two main approaches are in use for modelling the credit risk of a single entity: the reduced-form (intensity-based) and firm-value based (structural)approaches. According to intensity-based models, the timing of default depends on exogenous stochastic process, and these models assume that the default event is not interrelated with any observable characteristics of the issuer. However, the structural models that contracted on Black-Scohles [8] and Robert C. Merton's [29] option pricing methodology, and they are directly rely on the asset quality and debt servicing ability of the firm. Under absolute priority rules, stockholders have residual claims on the asset of the firm in case of default [9]. That is, equity shareholders, have long position of European call option which written on assets of the firm with exercise price equal to value of liabilities of the same firm. In the same manner, debt holders have position that can be replicated with long default-free bond plus short position on a put option on the assets of the firm.

According to the structural models, default occurs when the asset values of the firm is less than the liabilities of the firm. Whereas, their assumptions regarding to the time of default differ. For example, Merton model assumes the firm have just one liability and so it can be default only at maturity of the bond if its asset values are not sufficient to pay face value of the debt. The models that rely on the merton assumptions are the examples of the *default-at-maturity* models. In the other structural models, default occurs when ever asset values of the firm falls below the pre-determined barrier level, L, at some *default time* τ. These models are called *first passage time models.* Black-Cox [7] model is the pioneer of the first-passage time models. Kim, Ramaswamy, and

Sundaresan [25], Longstaff and Schwartz [28] and Saá-Requejo and Santa Clara [31] are other works that concentrated on the *first passage time problem.*

Contrary to structural models, reduced-form models do not relate the default to the features of the firm. The works studying these intensity-based models are generally depend on the market data. These models basically started with Jarrow and Turnbull (1992)[22], and subsequently studied by Jarrow and Turnbull (1995) [23], Duffie and Singleton (1999) [14] among others [21].

Rating agencies, like Moody's, S&P and Fitch, are the rating providers and these ratings represent the credit quality of the corporation. Determination of rating is the process of evaluating default probability for a specific entity. For example, the rating with reduced-form models firstly needs the valuation of the corporate bond which is issued by the rated firm. After this valuation we can find *spread* between price of that corporate bond and risk-free asset. Then, by using recovery rate data which is estimated from the market data, we can evaluate the default probability for the issuer. This default probability is the basic indicator of the firm rating. The Moody's rating system is given in Table 2.1.

As we have mentioned, measurement of credit risk needs especially two inputs, recovery rate (R) and probability of default ($P_{\mathcal{D}}$). As we mentioned before, loss given default (LGD) is just equal to $1 - R$.

In finance literature, there are two different types of probability used, which are historical probability and risk natural probabilities. Historical probabilities are the real world probabilities of events occur in the real world. For example, if we have 100 firms and 20 of these firms default during last year we can conclude that yearly $P_{\mathcal{D}}$ is 0.20 for last year. The risk-neutral probability is an artificial measure that used to value derivative contracts depending on the event[10]. According to Jhon Hull [18], risk-neutral default probabilities calculate the present value of expected future cash flows, so they should be used when estimating the impact of default risk on the pricing of debt instruments. On the other hand, real-world default probabilities should be used when carrying out scenario analysis in order to calculate the potential losses from defaults.

Table 2.1: Moody's Rating System.

Moody's Rating	Definitions
Aaa	Highest rating. Capacity to repay principal and interest is high.
Aa	Very strong. Only slightly less secure than highest rating.
A	Determined to be slightly more susceptible to adverse economic conditions.
Baa	Adequate capacity to repay principal and interest. Slightly speculative.
Ba	Speculative and significant chance that issuer could miss an interest payment.
B	Speculative and subject to high credit risk. Issuer has missed one or more interest or principal payments.
Caa	Poor standing and are subject to very high credit risk. No interest is being paid.
Ca	Highly speculative and are likely in, or very near, default, with some prospect of recovery of principal and interest.
D	Issuer in default, with little prospect for recovery of principal or interest.

In the event of default of a firm it is expected that creditors may files a claim against firm's assets to recover partially their receivables. The recovery rate, as a percent of par value, for a bond is its market value immediately following default. Figure 2.2 shows the recovery rates for varying classes of bonds [18]. Historical data implies negative correlation between recovery rates and default rates when they are compared, which means high default rate will likely implies low recovery rate in a year. The combination of these two effects constitute a more adverse outcome for investors [18].

Table 2.2: Moody's recovery rates 1982-2003 [18].

Class Average	Recovery Rate
Senior secured	54.44
Senior unsecured	38.39
Senior subordinated	32.85
Subordinated	31.61
Junior subordinated	24.47

As it is known, there are many credit risk models beginning from primitive expert systems to advanced Lévy jump diffusion models. In this work, we will only study advanced models: intensity-based models constructed on counting processes and structural models constructed on option pricing methodology.

2.1 Intensity Based Models

Intensity based models, generally called reduced form models, answer the question "Why firm default" with the "Because of the market"[3]. Default is defined as an unexpected event whose likelihood is measured by a default-intensity process in these models. The time of default is just the time of first jump of counting process. The basic idea is that at any instant time there is a possibility that an obligor default. Default is defined as the first jump of counting process (Poisson Process) $N = \{N_t; 0 \leq t \leq T\}$ with intensity $\lambda = \{\lambda_t; 0 \leq t \leq T\}$ which determines the price of credit risk for a fixed time horizon T [11]. Let us first define risk free asset, its price process evaluation and the default inference by the help of price differences between defaultable and risk free bonds. Afterwords, we will examine the counting process in detail.

Definition 2.1.1. The price process of a risk free bond $B = \{B_t;\ 0 \leq t \leq T\}$ follows the following dynamics

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \tag{2.1}$$

where $r = \{r_t, 0 \leq t \leq T\}$

Let denote the following ratio with $D(t,T)$

$$D(t,T) = \mathbb{E}\left(\frac{B_t}{B_T}\right), \tag{2.2}$$

where $\mathbb{E}$ denote the expectation operator. When short interest rate process $r = \{r_t, 0 \leq t \leq T\}$ is stochastic then discount factor becomes;

$$D(t,T) = \mathbb{E}\left\{\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\right\}, \tag{2.3}$$

but, if we assume that r is deterministic, then we can write the discount factor as follows:

$$D(t,T] = \exp\left(-\int_t^T r_s\,\mathrm{d}s\right). \tag{2.4}$$

Moreover, if r is constant then we have following equation

$$D(t,T) = \exp\left(-r(T-t)\right). \tag{2.5}$$

The likelihood of default by a obligor on its debt obligation can be most simply predicted from prices of bonds in reduced form models. A corporate zero coupon bond with face value $100 will have less price compared to the risk-free zero coupon bond having same maturity because of credit risk enclosed by corporate bond [18].

Let us define risk notation for risk neutral probabilities and yields of risk-free and corporate bonds as follows:
$P_{\mathcal{S}}(t)$: probability of survive for a obligor in a time interval $[0, t]$,
$P_{\mathcal{D}}(t)$: probability of default for a obligor in a time interval $[0, t]$,
y_T : yield on corporate zero coupon bond with maturity T,
y_T^* :yield on risk-free zero coupon bond with maturity T,
$B_0(T)$: present value of one dollar at maturity T discounted with yield of corporate bond,
$B_0^*(T)$: present value of one dollar at maturity T discounted with risk free rate.

The present value of corporate zero coupon and zero coupon risk free bonds respectively can be defined as follows [11]. Assume both assets have $100 face value:

$$100 \times \exp\left(-y(T)T\right) = 100B_0(T),$$
$$100 \times \exp\left(-y^*(T)T\right) = 100B_0^*(T).$$

To calculate risk natural default probabilities from these bond prices, we assume that present value of the cost of default equals to the excess of the price of risk free bond over the price of corporate bond:

$$100B_0^*(T) - 100B_0(T) = 100\left[\exp\left(-y^*(T)T\right) - \exp\left(-y(T)T\right)\right];$$

if there is no recovery, then

$$100B_0(T) = B_0^*(T)[P_{\mathcal{D}}(T)0 + (1 - P_{\mathcal{D}}(T))100].$$

Therefore,

$$100B_0(T) = 100B_0^*(T)\left[1 - P_{\mathcal{D}}(T)\right],$$
$$P_{\mathcal{D}}(T) = \frac{B_0^*(T) - B_0(T)}{B_0^*(T)} = 1 - \exp\left(-[y(T) - y^*(T)]T\right). \tag{2.6}$$

If there is a recovery and $0 \leq R \leq 1$ is the recovery rate, then

$$100B_0(T) = B_0^*(T)[P_{\mathcal{D}}(T)R + (1 - P_{\mathcal{D}}(T))100],$$

$$P_{\mathcal{D}}(T) = \frac{B_0^*(T) - B_0(T)}{(1-R)B_0^*(T)} = \frac{1 - \exp\left(-[y(T) - y^*(T)]T\right)}{(1-R)}. \tag{2.7}$$

Simply we can say that probability of default can be calculated form the following formula:

$$P_{\mathcal{D}}(t) = \frac{CS}{1-R},$$

where CS is the credit spread of corporate bond yield over the yield of risk-free bond, that is $CS = y - y^*$ and R is recovery rate.

Example 2.1.2. If a corporate bond has 150 bp yield more than risk free yield, and in the event of default have recovery rate 30% then it has default probability per year given no earlier default $P_{\mathcal{D}} = \frac{0.015}{1-0.3} = 2.14\%$. This conditional default per year is also called intensity or hazard rate [18].

As we mentioned before, under reduced form credit risk models, the default probability and the default time defined by counting processes and the process $\lambda = \{\lambda_t; 0 \leq t \leq T\}$ shows default possibility in a small time interval.

Let us first start with the definition of Poisson process. Poisson process is the most simple pure jump Lévy process. In order to construct Poisson process, we should define exponential distribution.

Definition 2.1.3. The random variable τ is called *exponentially distributed* or it has *exponential distribution* if it has following be density

$$f(t) = \begin{cases} \lambda \exp(-\lambda t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases} \tag{2.8}$$

where λ is a positive constant.

The expectation and variance of τ can be found by using partial integration as follows:

$$\mathbb{E}(\tau) = \int_0^\infty t f(t) \mathrm{d}t = \frac{1}{\lambda},$$

$$
\begin{aligned}
Var(\tau) = \mathbb{E}(\tau^2) - [\mathbb{E}(\tau)]^2 & \\
&= \int_0^\infty t^2 f(t)\mathrm{d}t - \left(\frac{1}{\lambda}\right)^2 \\
&= \int_0^\infty \lambda\exp(\lambda t)t^2\mathrm{d}t - \left(\frac{1}{\lambda}\right)^2 \\
&= -t^2\exp(-\lambda t)|_0^\infty + \int_0^\infty 2t\exp(\lambda t)\mathrm{d}t - \left(\frac{1}{\lambda}\right)^2 \\
&= 0 + \frac{2}{\lambda}\mathbb{E}(\tau) - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.
\end{aligned}
$$

To compute cumulative distribution function of τ we have to compute $\mathbb{P}(\tau \leq t)$ and

$$\mathbb{P}(\tau \leq t) = \int_0^t \lambda\exp(-\lambda s)\mathrm{d}s = 1 - \exp(-\lambda t),\ t \geq 0,$$

and since $\mathbb{P}(\tau > t) = 1 - \mathbb{P}(\tau \geq t)$, therefore

$$\mathbb{P}(\tau > t) = \exp(-\lambda t),\ t \geq 0. \tag{2.9}$$

The most interesting feature of exponential distribution is its memoryless property.

Definition 2.1.4. The random variable τ has the memoryless property, if for positive $s, t \in [0, T]$ its distribution satisfies the following equality:

$$\mathbb{P}(\tau > t + s|\tau > s) = \mathbb{P}(\tau > t). \tag{2.10}$$

The Equation (2.10) implies that, if we are at time s and wait additional t unit time, it is not different from starting at time zero and waiting until time t. In other words, probability of survive is just related to how much time is passed, not the starting point or history.

Proposition 2.1.5. *If random variable τ has exponential distribution, then it has memoryless property.*

Proof. Let t and s be two positive real numbers, then

$$
\begin{aligned}
\mathbb{P}(\tau > t + s|\tau > s) &= \frac{\mathbb{P}(\tau > t + s, \tau > s)}{\mathbb{P}(\tau > s)} \\
&= \frac{\mathbb{P}(\tau > t + s)}{\mathbb{P}(\tau > s)} = \frac{\exp(-\lambda(t + s))}{\exp(-\lambda s)} \\
&= \exp(-\lambda t) = \mathbb{P}(\tau > t).
\end{aligned}
$$

$\square$

Definition 2.1.6. A stochastic process $N = \{N_t; t \geq 0\}$ with positive intensity parameter λ is called a *Poisson process*, if it satisfies the following conditions:

- $N_0 = 0$.
- The process has independent and stationary increments.
- The density function of process has form of $\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} \exp(-\lambda t)$.
- If $0 < s < t$, then the random increment $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t-s)$ and
$$\mathbb{P}(N_t - N_s) = \frac{\lambda^n (t-s)^n}{n!} \exp(-\lambda(t-s)).$$

The Poisson process has constant intensity, but we can also define default intensity as a process.

Definition 2.1.7. Let τ be default time, then the intensity of default $\lambda = \{\lambda_t, 0 \leq t \leq T\}$ is defined as
$$\lambda_t = \lim_{h \to 0} \frac{\mathbb{P}(t < \tau < t+h) | \tau > t)}{h}. \tag{2.11}$$

This equation tells us that, roughly for a small time interval $\Delta t > 0$

$$\mathbb{P}[\tau \leq t + \Delta t | \tau > t] \approx \lambda_t \Delta t \tag{2.12}$$

Definition 2.1.8. The τ be a default time is a arbitrary positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with cumulative distribution function F, where
$$F(t) = \int_0^t f(s) \mathrm{d}s$$
for a probability density function f of τ

From the above definition of default time, we can define a cumulative probability distribution of survive Γ as $\Gamma(t) = \mathbb{P}(\tau > t) = 1 - \mathbb{P}(\tau \leq t)$. Then we have $\Gamma(t + \Delta t) = \mathbb{P}(\tau > t + \Delta t) = 1 - \mathbb{P}(\tau \leq t + \Delta t)$. We can calculate the probability of small time interval by the difference of cumulative distribution functions:

$$\Gamma(t) - \Gamma(t + \Delta t) = \mathbb{P}(\tau \leq t + \Delta t) - \mathbb{P}(\tau \leq t),$$

and from equation (2.11) we know that

$$\lambda_t \Delta t \approx \mathbb{P}\left(t < \tau \leq t + \Delta t / \tau > t\right) = \frac{\mathbb{P}(t < \tau \leq t + \Delta t)}{\mathbb{P}(\tau > \tau)}).$$

Therefore,

$$\lambda_t \Delta t \approx \frac{\Gamma(t) - \Gamma(t + \Delta t)}{\Gamma(t)};$$

then, we have

$$\frac{\Gamma(t + \Delta t) - \Gamma(t)}{\Gamma(t)} \approx -\lambda_t \Delta t.$$

This implies that

$$\frac{\mathrm{d}\Gamma(t)}{\Gamma(t)} = -\lambda_t \mathrm{d}t.$$

Finally, we get the following result for probability of survive:

$$\Gamma(t) = P_{\mathcal{S}}(t) = \exp\left(-\int_0^t \lambda_s \mathrm{d}s\right). \tag{2.13}$$

Taking $\bar{\lambda}_t$ as the average hazard rate between time 0 and t we can write

$$P_{\mathcal{S}}(t) = \exp(-t\bar{\lambda}_t).$$

We have already showed that $\mathbb{P}(\tau > t) = \exp(-\lambda t)$ for constant λ. Therefore, $P_s(t) = \mathbb{P}(\tau > t) = \exp(-\lambda t)$ for constant λ. Since $P_{\mathcal{D}} = 1 - P_{\mathcal{S}}$ then, $P_{\mathcal{D}} = 1 - \exp(\lambda t)$. As we mention in next topic, this result known as *homogeneous* case of Jarrow-Turnbull Model.

2.1.1 Jarrow-Turnbull Model

Jarrow-Turnbull model has two case: the *homogeneous* and the *inhomogeneous*.

Homogeneous Case: Poisson process with constant intensity λ that we have mentioned before is the one of the standard example of *homogeneous* case. Probability of survive under this model is defined as

$$P_{\mathcal{S}}(t) = \exp(-\lambda t), \tag{2.14}$$

whose corresponding expected time of default is $\tau = \frac{1}{\lambda}$ [11]. As we explained before, the survival probability is the probability that the counting process N_t is equal to 0.

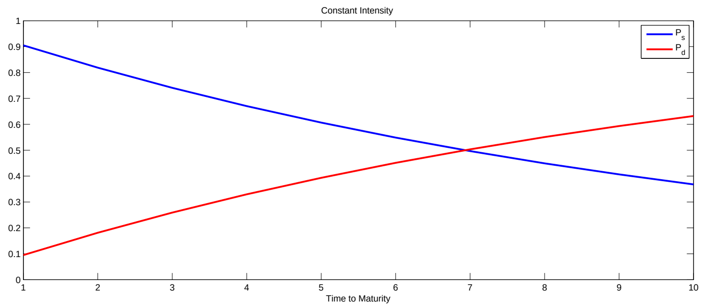

Figure 2.1: Constant intensity.

Figure 2.1 shows that $P_{\mathcal{D}}$ increases when ever the time to maturity increases for $\lambda = 0.1$.

Inhomogeneous: In this case, Jarrow and Turnbull modelled λ as a process of deterministic function of time $\lambda = \{\lambda_t : 0 \leq t \leq T\}$. Since λ is deterministic function of time, as we stated before, the probability of default is equal to equation (2.13) here.

We can define λ_t as a stepwise function [11]:

$$\lambda_t = K_i,\, T_{i-1} \leq t \leq T_i,\, i = 1, 2, 3, 4. \tag{2.15}$$

In this case we can model survival probability as follows:

$$P_{\mathcal{S}}(t) = \begin{cases} \exp(-K_1 t), & \text{if } 0 \leq t < T_1, \\ \exp(-K_1 T_1 - K_2(t - T_2)), & \text{if } T_1 \leq t < T_2, \\ \exp(-K_1 T_1 - K_2(T_2 - T_1) - K_3(t - T_2)), & \text{if } T_2 \leq t < T_3, \\ \exp(-K_1 T_1 - K_2(T_2 - T_1) - K_3(T_3 - T_2) - K_4(t - T_3)), & \text{if } T_3 \leq t < T_4. \end{cases}$$

Figure 2.2 shows evaluation of probability of default taking following T and K values:

$T_1 = 1$; year $\quad K_1 = 0.02$,

$T_2 = 3$ years $\quad K_2 = 0.05$,

$T_3 = 5$ years $\quad K_3 = 0.07$,

$T_4 = 4$ years $\quad K_4 = 0.1$.

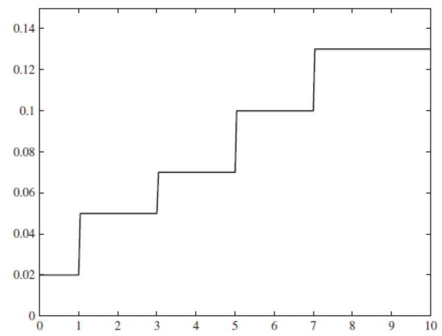

Figure 2.2: Probability of default with stepwise intensity[11].

2.1.2 Cox Model

Deterministic intensity implies that default risk related to information that arrives over time is the only fact of survival date. However, different information about credit worthiness of the reference entity constitutes different states which will accessible as time passes. That is, the intensity would change randomly as new additional information arrive. For example, credit ratings, distance to default and equity price related new information will change default intensity randomly [15].

Given all current information, in general, this model approaches arrival intensity as a random process. That is, the approach is conditional on current information. Therefore, to model different state of information with time-varying intensity $\lambda = \{\lambda_t; 0 \leq t \leq T\}$, the survival probability

$$P_{\mathcal{S}}(t) = \mathbb{E}\left\{\mathbb{P}(\tau > t)|\lambda_s : 0 \leq s \leq t\right\} \tag{2.16}$$

which can also be written as

$$P_{\mathcal{S}}(t) = \mathbb{E}\left\{\exp\left(-\int_0^t \lambda_s \mathrm{d}s\right)\right\}. \tag{2.17}$$

Given all current available information, the conditional probability of survive at time t, of survival to a future time v, is given by

$$P_{\mathcal{S}}(t, v) = \mathbb{E}\left\{\exp\left(-\int_t^v \lambda_u \mathrm{d}u\right)\right\}. \tag{2.18}$$

There are various models that are used to model default intensity and one of the fundamental is Cox, Ingersoll, Ross (CIR) short-interest-rate model [12]:

$$\mathrm{d}\lambda_t = \kappa(\theta - \lambda_t)\mathrm{d}t + \sigma_\lambda \sqrt{\lambda_t}\mathrm{d}W_t \tag{2.19}$$

where W is a standard Brownian motion and κ is the mean rate of reversion to the long run mean, σ_λ is a volatility coefficient and θ is the long run mean of λ.

By taking λ as CIR process, the conditional survival probability is given by

$$P_{\mathcal{S}}(t) = \exp(\alpha(v - t) - \beta(v - t)\lambda_t), \tag{2.20}$$

where α and β are time-dependent coefficients. For details of the proof and later information see [15].

2.2 Structural Models (Firm Value Based Models)

Structural models are models based on Black-Scholes-Merton option pricing theory build up by Merton with his article [29]. Merton modelled firm equities as European call option written on asset values with strike price equal to the value of only single zero coupon bond which is the liability of a levered firm and can be paid at some certain maturity T. Other structural models are just extensions of Merton model. The aim of these models is to improve this elegant but naive idea to answer the real economy problems. The article of Black and Cox [7] is one of the earliest extensions of Merton model, just adding early default possibility to the model. They modelled the default time as first passage time, that is the default time was modelled as the first time that value of asset V break down barrier L.

After the nineties, as globalization accelerated, the interaction of financial system has also increased. Therefore, especially, world wide investment banks such as J.P. Morgan Chase build up new credit risk models, CreditMatrics in 1997, to deal with complicated credit risk. KMV model is an another structural model developed by rating agency Moody's and it is also based on Merton model. It includes new idea of liquidity that a firm can have more than one liabilities with different maturities, classified as short term and long term. KMV model is built on Merton model and adjusts this model considering some of its shortcomings, most notably (1) that all debt matures at the same maturity and (2) that the value of firm follows a log-normal diffusion process [33].

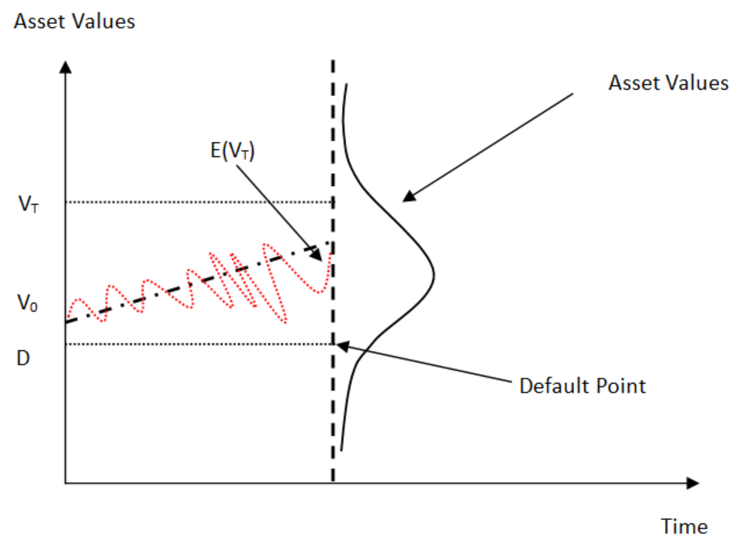

Figure 2.3: Distance to default.

For the Merton model as we will mention later, distance to default (D) as in Figure 2.3 is a crucial concept. Later we will formulate it as d_2. As it will be explained later, Merton modelled the probability of default of a firm as $\Phi(-D)$, KMV modelled the expected default frequencies just assign the real defaults to $\Phi(-D)$. In other words, Merton compute P_D by computing normal cumulative distribution of $-D$, but KMV firstly compute D just as in Merton Model and assign it to the value of the real default frequencies.

Since Merton model and Black-Cox model are the mathematical base of firm value models, we will just explain these two models in details.

2.2.1 Merton Model

The Merton model suggests that a company's equity value can be estimated by modelling a European call option on the asset of the company [18]. In order to use Black-Scholes option pricing methodology to price firm equity, Merton assumes the following assumptions [29].

1. There are no transaction costs, taxes and assets traded on market can be infinity divisible.

2. There is a sufficient number of investors with a comparable level of wealth participate in market, and there is not any restriction on quantity of asset which can be bought or sold at the market price by investors.

3. There exists an exchange market for borrowing and lending at the same rate of interest.

4. There is no restriction on short sale, with full use of proceeds.

5. Assets are continuously traded in time.

6. The value of the firm is invariant to its capital structure obtains as stated in the Modigliani-Miller Theorem.

7. The Term-Structure is flat and known with certainty. That is, the price of risk-free discount bond which pays one dollar at time t in future is $D(0,t) = \exp(-rt)$, where r is the risk-free rate of interest, the same for all time.

8. Firm asset values, V, follow a diffusion type stochastic process with stochastic differential equation

$$\mathrm{d}V_t = \mu V_t \mathrm{d}t + \sigma V_t \mathrm{d}W_t, \tag{2.21}$$

for $t \in [0,T]$ and $V_0 = v \in \mathbb{R}$.

Let us assign the following notation to value of company's assets, debt and equity respectively:

V_t: Firm value at time t for $t \in [0,T]$

Z_t^T: Value of a single zero coupon bond at time t with maturity T and face value L.

E_T: Value of equity at time T.

The asset values of the firm behaves as a geometric Brownian motion, as shown in Figure 2.4. Furthermore, since in Merton model the firm has just one liability with

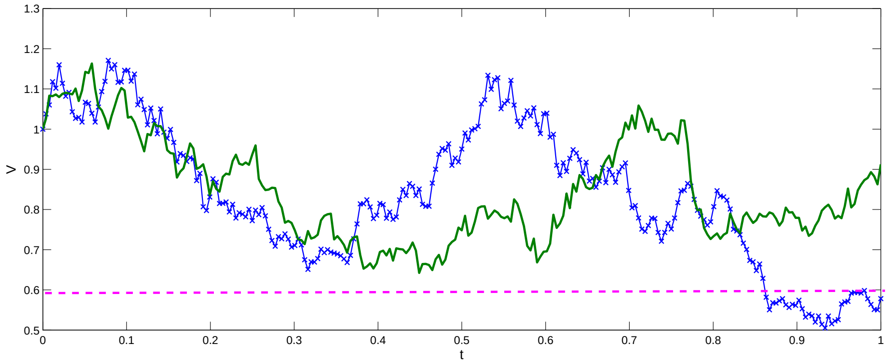

Figure 2.4: Asset values under Merton model.

maturity T, we can decide whether there is default or not just by observing these asset values at maturity T. For maturity $T = 1$ year, we can say if asset values follow green path there is no default, but if asset values evolve as blue path there is default since asset values become less than L at maturity.

According to general accounting principles, the total value of the assets of the firm equals to the sum of its debt and equity

$$V_t = Z_t^T + E_t. \tag{2.22}$$

On the other hand, the payoff of the European call option at maturity T is equal to the maximum of the zero and absolute difference of asset value and strike price. Thus, the equity value E_T of the firm as an option price can be given by following equation:

$$E_T = max[V_T - L, 0] = (V_T - L)^+ = \begin{cases} (V_T - L), & \text{if } V_T \geq L, \\ 0, & \text{if } V_T < L. \end{cases}$$

In this chapter, we assume that the value of the firm follows a geometric Brownian motion model. Hence, the value of the firm can be found by applying Ito Lemma to logarithm function $f(x) = \log(x)$:

$$\begin{aligned} f(V_t) = \log(V_t) &= \log(V_0) + \int_0^t \frac{1}{V_s}\,\mathrm{d}V_s - \frac{1}{2}\int_0^t \frac{1}{V_s^2}\sigma^2\,\mathrm{d}s \\ &= \log(V_0) + \int_0^t (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}s + \int_0^t \sigma\,\mathrm{d}W_s \\ &= \log(V_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t, \end{aligned}$$

which implies that for $V_0 > 0$; we have

$$V_t = V_0 \exp\{(\mu - \frac{1}{2}\sigma^2) + \sigma W_t\}; \tag{2.23}$$

therefore,

$$\log(\frac{V_t}{V_0}) \sim \Phi\{(\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\}. \tag{2.24}$$

Asset values under Merton model follow geometric Brownian motion as it can be seen from Figure 2.4, we can conclude that from Equation 2.24 logarithm of asset values is distributed normally with mean $\mu - \frac{1}{2}\sigma^2$ and variance $\sigma^2 t$.

The value of the equity can be calculated just by taking equities as contingent claim $E_T = h$ of stockholder. Therefore, under the assumption of risk-natural world, the value of equity at a time in $[0, T]$ is the expectation of discounted value of that contingent claim:

$$\begin{aligned}
E_t &= \mathbb{E}^* \left[\exp\left(-r(T-t)\right) h | \mathcal{G}_t\right] \\
&= \mathbb{E}^* \left[\exp\left(-r(T-t)\right) f(V_T) | \mathcal{G}_t\right] \\
&= \mathbb{E}^* \left[\exp\left(-r(T-t)\right) f\left(V_t \exp\left((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^* - W_t^*)\right)\right) | \mathcal{G}_t\right].
\end{aligned}$$

Since V_t is $\mathcal{G}_t$ measurable and $W_T^* - W_t^*$ is independent of the filtration $\mathcal{G}_t$ we can write $E_T = F(t, V_t)$ by Proposition given in Appendix A.1.1. Therefore,

$$\begin{aligned}
F(t,x) &= \mathbb{E}^* \left[\exp\left(-r(T-t)\right) f\left(x \exp\left((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^* - W_t^*)\right)\right)\right] \\
&= \int_{-\infty}^{\infty} \exp\left(-rm\right) f\left(x \exp\left((r - \frac{1}{2}\sigma^2)m + \sigma y \sqrt{m}\right)\right) \frac{\exp(\frac{-y^2}{2})}{\sqrt{2\pi}} \,\mathrm{d}y \\
&= \exp\left(-rm\right) \int_{-\infty}^{\infty} \left(x \exp\left((r - \frac{1}{2}\sigma^2)m + \sigma y \sqrt{m}\right) - L\right)^+ \frac{\exp(\frac{-y^2}{2})}{\sqrt{2\pi}} \,\mathrm{d}y
\end{aligned}$$

for $T - t = m$.

Now, we have to find region where $f(V_T)$ is positive:

$$\begin{aligned}
& x \exp\left((r - \frac{1}{2}\sigma^2)m + \sigma y \sqrt{m}\right) - L \geq 0 \\
& \Rightarrow \log(x) + (r - \frac{1}{2}\sigma^2)m + \sigma y \sqrt{m} \geq \log(L) \\
& \Rightarrow \sigma y \sqrt{m} \geq \log(L/x) - \left((r - \frac{1}{2}\sigma^2)m\right) \\
& \Rightarrow -y \leq \frac{\log(x/L) + (r - \frac{1}{2}\sigma^2)m}{\sigma\sqrt{m}}
\end{aligned}$$

If we set

$$d_2 = \frac{\log(x/L) + (r - \frac{1}{2}\sigma^2)m}{\sigma\sqrt{m}}$$

and taking $y = -y$ then, we can write $F(t, x)$ as follows:

$$\begin{aligned}
F(t,x) &= \exp\Big(-rm\Big)\int_{-\infty}^{d_2}\Big(x\exp\Big((r-\frac{1}{2}\sigma^2)m+\sigma y\sqrt{m}\Big)-L\Big)\frac{\exp(\frac{-y^2}{2})}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= \exp\Big(-rm\Big)\int_{-\infty}^{d_2}\Big(x\exp\Big((r-\frac{1}{2}\sigma^2)m+\sigma y\sqrt{m}\Big)\frac{\exp(\frac{-y^2}{2})}{\sqrt{2\pi}}\,\mathrm{d}y \\
&- L\exp\Big(-rm\Big)\int_{-\infty}^{d_2}\frac{\exp(\frac{-y^2}{2})}{\sqrt{2\pi}}\,\mathrm{d}y \\
&= \exp\Big(-rm\Big)\int_{-\infty}^{d_2}\Big(x\exp\Big((r-\frac{1}{2}\sigma^2)m+\sigma y\sqrt{m}\Big)\frac{\exp(\frac{-y^2}{2})}{\sqrt{2\pi}}\,\mathrm{d}y - L\exp\Big(-rm\Big)\Phi(d_2) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\Big(x\exp\Big(-\frac{1}{2}\sigma^2 m+\sigma y\sqrt{m}-\frac{-y^2}{2}\Big)\,\mathrm{d}y - L\exp\Big(-rm\Big)\Phi(d_2).
\end{aligned}$$

We can write $-\frac{1}{2}\sigma^2 m + \sigma y\sqrt{m} - \frac{-y^2}{2}$ as the square of sum:

$$\begin{aligned}
-\frac{1}{2}\sigma^2 m+\sigma y\sqrt{m}-\frac{-y^2}{2} &= -\frac{1}{2}\Big[\sigma^2 m+2\sigma y\sqrt{m}+y^2\Big] \\
&= -\frac{1}{2}\Big[\sigma\sqrt{m}+y\Big]^2
\end{aligned}$$

then,

$$F(t,x) = \int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}x\exp\Big(-\frac{1}{2}(y+\sigma\sqrt{m})^2\Big)\,dy - L\exp\Big(-rm\Big)\Phi(d_2).$$

By taking $y = y + \sigma\sqrt{m}$ we can simplify $F(t, x)$,

$$\begin{aligned}
F(t,x) &= \int_{-\infty}^{d_2+\sigma\sqrt{m}}\frac{1}{\sqrt{2\pi}}x\exp\Big(-\frac{1}{2}y^2\Big)\,dy - L\exp\Big(-rm\Big)\Phi(d_2) \\
&= \int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}x\exp\Big(-\frac{1}{2}y^2\Big)\,dy - L\exp\Big(-rm\Big)\Phi(d_2) \\
&= \frac{1}{\sqrt{2\pi}}x\int_{-\infty}^{d_1}\exp\Big(-\frac{1}{2}y^2\Big)\,dy - L\exp\Big(-rm\Big)\Phi(d_2) \\
&= x\,\Phi(d_1) - \exp\Big(-rm\Big)\,L\Phi\,(d_2).
\end{aligned}$$

That is, the value of equity under Merton model assumption can be written:

$$E_t = V_0\,\Phi(d_1) - \exp\left(-rm\right)\,L\Phi\,(d_2), \tag{2.25}$$

where

$$
\begin{aligned}
d_2 &= \frac{\log(x/L) + (r - \frac{1}{2}\sigma^2)m}{\sigma\sqrt{m}}, \\
d_1 &= \frac{\log(x/L) + (r + \frac{1}{2}\sigma^2)m}{\sigma\sqrt{m}}, \\
d_2 &= d_1 - \sigma\sqrt{m}.
\end{aligned}
$$

In terms of the lender, if (s)he knew certainly that; (s)he would get the principal at the maturity for the zero coupon bond, (s)he will get the face value L and value of debt is equal to $V_T - E_T$. However, if the creditors expect that (s)he would not get par value, that is, if $V_T < L$ she will get V_T. Therefore, the value of debt at maturity T is:

$$
D_T = L - \max(L - V_T, 0).
$$

As it can be understood, the claim of the bondholder can be seen as a long risky bond and a short European put option written on firm's assets with strike price L [35].

Under Merton model, the survival probability for a firm can be calculated by evaluating the asset values of firm and comparing the sum of these values by the value of liabilities at maturity T. If $V > L$, then there is no default. Therefore, we should evaluate $\mathbb{P}[V_T \geq L]$. Since V_t evolves as a geometric Brownian motion, and under risk natural probability $\mathbb{P}^*$, firm value at maturity T is $V_T = V_t \exp\big((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^* - W_t^*)\big)$ as we already shown. Therefore,

$$
\begin{aligned}
P_{\mathcal{S}}(T|\mathcal{G}_t) &= \mathbb{P}[V_T \geq L] \\
&= \mathbb{P}\Big(V_t \exp\big((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^* - W_t^*)\big) \geq L|\mathcal{G}_t\Big) \\
&= \mathbb{P}\Big(x \exp\big((r - \frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}^*\big) \geq L\Big) \\
&= \mathbb{P}\Big(x \exp\big((r - \frac{1}{2}\sigma^2)m + \sigma y\sqrt{m}\big) \geq L\Big) \\
&= \mathbb{P}\Big(\log(x) + (r - \frac{1}{2}\sigma^2)m + \sigma y\sqrt{m} \geq \log(L)\Big) \\
&= \mathbb{P}\Big(y \geq \frac{\log(\frac{L}{x}) - (r - \frac{1}{2})m}{\sigma\sqrt{m}}\Big) \\
&= \mathbb{P}\Big(-y < \frac{\log(\frac{x}{L}) + (r - \frac{1}{2})m}{\sigma\sqrt{m}}\Big).
\end{aligned}
$$

For $V_t = x$ and $y = \frac{W^*_{T-t}}{\sqrt{m}}$, when we take $z = -y$, we get following equation:

$$= \mathbb{P}\Big(z < \frac{\log(\frac{x}{L}) + (r - \frac{1}{2})m}{\sigma\sqrt{m}}\Big)$$

Immediately we get the following formula for the probability of survive according to Merton model

$$P_{\mathcal{S}}(T|\mathcal{G}_t) = \mathbb{P}[V_T \geq L] = \Phi(d_2), \tag{2.26}$$

where

$$d_2 = \frac{\log(x/L) + (r - \frac{1}{2}\sigma^2)m}{\sigma\sqrt{m}}.$$

Since $P_{\mathcal{D}} = 1 - P_{\mathcal{S}}$ then probability of default under Merton model given by

$$P_{\mathcal{D}} = \Phi(-d_2), \tag{2.27}$$

where Φ is the cumulative distribution function of standard normal distribution.

To calculate $\Phi(-d_2)$, we should know the value of the firm at time t, V_t, and the standard deviation of assets denoted by σ. However, neither of them are directly observable in market. But if the company is publicly traded, E_t can be observable. We should use two equations since we have two unknowns. Equation (2.25) can be used as the first equation. For the second equation, we will benefit from the Ito lemma. When we apply Ito formula, we can write the dynamics of equity by the following equation [29]:

$$\mathrm{d}E_t = \mu_e E_t \mathrm{d}t + \sigma_e E_t \mathrm{d}W^e_t \tag{2.28}$$

From Ito lemma, if we take $E = f(V, t)$, then we can write following stochastic equation:

$$\mathrm{d}E_t = \left[\mu V_t \frac{\partial f}{\partial V} + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 f}{\partial V^2} + \frac{\partial f}{\partial t}\right] \mathrm{d}t + \sigma V_t \frac{\partial f}{\partial V} \mathrm{d}W_t \tag{2.29}$$

When we solve both Equations, (2.28) and (2.29) we get following equation that can be used to find values σ and V_t:

$$\sigma_e E_t = \sigma V_t \frac{\partial f}{\partial V}. \tag{2.30}$$

We can find $\frac{\partial f}{\partial V}$ just by differentiating Equation (2.25) and this differential is equal to $\Phi(d_1)$. Just by inserting $\Phi(d_1)$ instead of $\frac{\partial f}{\partial V}$, the second equation becomes the following form:

$$\sigma_e E_t = \sigma V_t \Phi(d_1) \tag{2.31}$$

The above result shows that if a firm stocks are publicly traded, then by observing equity values and volatility of equity, by using equity values as proxy of asset values and equity volatility as proxy of the asset volatility, Merton model can be used on historical data to predict the probability of default for that firm [18].

2.2.2 Black-Cox Constant Barrier Model

As we mentioned before, the first-passage models have been introduced in order to include the possibility of an early default for the reference entity. As opposed to Merton model, first-passage models assume there can be default before T, taking into account that there are more than one liability of a firm with different maturities. The default occurs whenever the value of firm $V = \{V_t, 0 \leq t \leq T\}$ goes below barrier level L. In [7], Black and Cox model L as a time dependent process, but for the sake of simplicity we take it as constant.

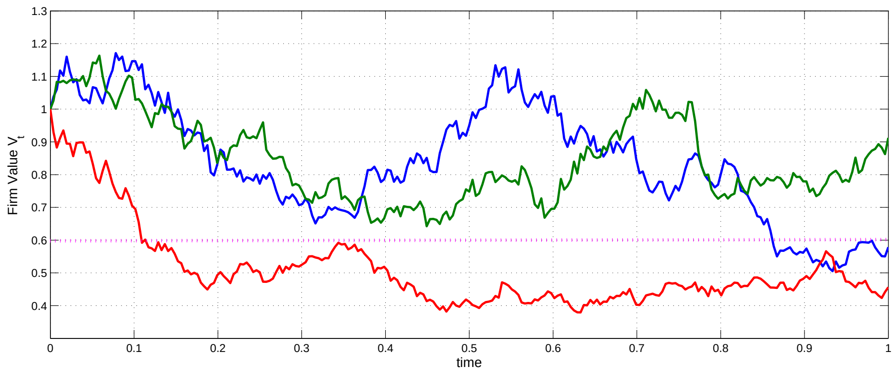

Figure 2.5: Asset values under Black-Cox model.

Figure 2.5 shows the possible firm values. The firm has initial value 1 and maturity one year. If we take barrier $L = 0.6$, then, as it can be observed, if firm value follows red or blue paths then default will occur before maturity. However, if firm value follows green path the firm will survive up to maturity.

To model Black-Cox model, firstly we should find the joint distribution of the maximum of Brownian motion and itself[16]. Let define the maximum of Brownian motion with drift zero and arbitrary σ as $M_t = \sup \{X_s, 0 \leq s \leq t\}$ and the maximum of standard Brownian motion as $m_t = \sup\{W_s, 0 \le s \le t\}$ where $X_t = \sigma W_t$ and $W_t \sim N(0,t)$ is a standard Brownian motion. Let us define joint distribution function as

$$F_t(x,y) = \mathbb{P}\{X_t \le x, M_t \le y\}$$

Here, X_t is a Brownian motion with drift zero, so we just need to calculate $F_t(x,y)$ for $x \le y$ since $X_t \le M_t$, $\forall t \in [0,T]$

$$\begin{aligned}F_t(x,y) &= \mathbb{P}\{X_t \le x, M_t \le y\}\\ &= \mathbb{P}\{X_t \le x\} - \mathbb{P}\{X_t \le x, M_t > y\}.\end{aligned}$$

We get this result because

$$\begin{aligned}\mathbb{P}\{X_t \le x\} &= \mathbb{P}\{X_t \le x, M_t \le y\} + \mathbb{P}\{X_t \le x, M_t > y\}\\ &= \mathbb{P}\{\sigma W_t \le x, \sigma m_t \le y\} + \mathbb{P}\{\sigma W_t \le x, \sigma m_t > y\}\\ &= \mathbb{P}\left\{W_t \le \frac{x}{\sigma}, m_t \le \frac{y}{\sigma}\right\} + \mathbb{P}\left\{W_t \le \frac{x}{\sigma}, m_t > \frac{y}{\sigma}\right\}.\end{aligned}$$

Therefore,

$$F_t(x,y) = \Phi(\frac{x}{\sigma\sqrt{t}}) - \mathbb{P}\left\{W_t \le \frac{x}{\sigma}, m_t > \frac{y}{\sigma}\right\}.$$

In order to solve $\mathbb{P}\{W_t \le v, m_t > u\}$ for positive real numbers u and v, we can use *reflection principle* or *reflection property* of Brownian motion. In order to find this probability, let us take $v = \frac{x}{\sigma}$ and $u = \frac{y}{\sigma}$. For every sample path of W that hits the level u before time t but finishes below level of v at time t, there is a another equally probable path (shadow path Figure 2.6) that hits level u before time t and then travel upward at least u-v unit to finish level $u + (u - v) = 2u - v$ at time t. This is because normal distribution has a symmetrical shape around the mean [16].

Thus,

$$\begin{aligned}\mathbb{P}\{W_t \le v, m_t > u\} &= \mathbb{P}\{W_t \ge 2u - v\}\\ &= \mathbb{P}\{W_t < v - 2u\}\\ &= \Phi\left\{\frac{v-2u}{\sqrt{t}}\right\}\\ &= \Phi\left\{\frac{x-2y}{\sigma\sqrt{t}}\right\}.\end{aligned}$$

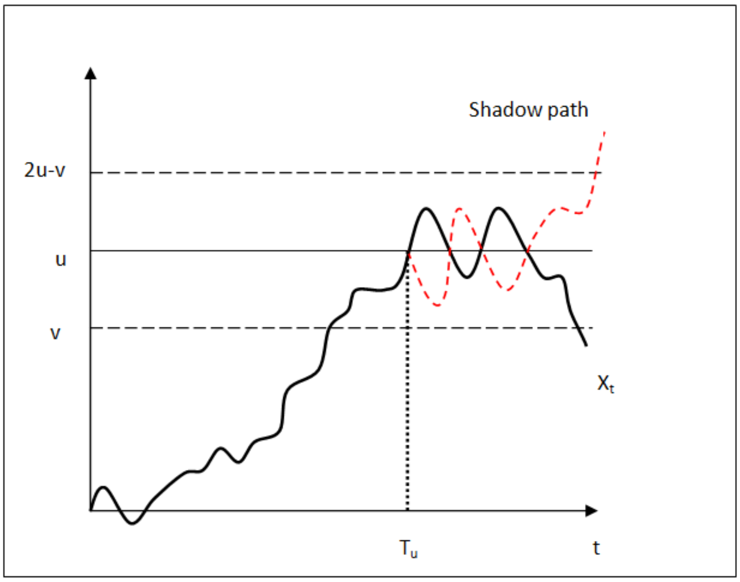

Figure 2.6: Hitting time.

This argument become definite after using the *Strong Markov Property*. Let T be the first time t that at which $W_t = u$ and define $W_t^* = W_{t+T} - W_T$. It follows that

$$\begin{aligned}\mathbb{P}\left\{W_t \leq v, m_t > u\right\} &= \mathbb{P}\left\{T \leq t, W_{t-T}^* \leq v - u\right\} \\ &= \mathbb{P}\left\{T \leq t, W_{t-T}^* \geq v - u\right\}.\end{aligned}$$

(The strong Markov property is needed to justify of these equalities.) By definition, $W_t^* - T = W_t - u$ and thus

$$\mathbb{P}\left\{W_t \leq v, m_t > u\right\} = \Phi\left\{\frac{v - 2u}{\sqrt{t}}\right\},$$

$$\mathbb{P}\left\{W_t \leq \frac{x}{\sigma}, m_t > \frac{y}{\sigma}\right\} = \Phi\left\{\frac{x - 2y}{\sigma\sqrt{t}}\right\};$$

therefore

$$F_t(x, y) = \Phi\left\{\frac{x}{\sigma\sqrt{t}}\right\} - \Phi\left\{\frac{x - 2y}{\sigma\sqrt{t}}\right\}, \tag{2.32}$$

where Φ is the cumulative distribution function of the standard normal distribution.

Proposition 2.2.1. *Let W be a standard Brownian motion, $X_t = \sigma W_t$ and $M_t = \sup\left\{X_s, 0 \leq s \leq t\right\}$. Then we have*

$$\mathbb{P}\left\{X_t \leq x, M_t \leq y\right\} = \Phi\left\{\frac{x}{\sigma\sqrt{t}}\right\} - \Phi\left\{\frac{x - 2y}{\sigma\sqrt{t}}\right\}. \tag{2.33}$$

Proof. The proof is similar to solution of $F_t(x,y)$ given in Equation (2.32). $\square$

Corollary 2.2.2.

$$\mathbb{P}\left\{X_t \in \mathrm{d}x, M_t \le y\right\} = g_t(x,y)\mathrm{dx},$$

where

$$g_t(x,y) = \left[\phi(\frac{x}{\sigma\sqrt{t}}) - \phi\{\frac{x-2y}{\sigma\sqrt{t}}\}\right]\frac{1}{\sigma\sqrt{t}}, \tag{2.34}$$

where

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2}).$$

Proof. The function $g_t(x,y)$ is just derivative of F with respect to x, therefore;

$$\frac{\mathrm{d}F_t(x,y)}{\mathrm{d}x} = \left[\phi(\frac{x}{\sigma\sqrt{t}}) - \phi\{\frac{x-2y}{\sigma\sqrt{t}}\}\right]\frac{1}{\sigma\sqrt{t}}.$$

$\square$

Proposition 2.2.3. *Let* $f_t(x,y) = \mathbb{P}^*\left\{X_t \in \mathrm{d}x, M_t \le y\right\}$ *and let* $X_t \sim N\left\{\mu t, \sigma^2 t\right\}$ *then* $f_t(x,y) = \xi_t g_t(x,y)\mathrm{d}x$*, where* $\xi = \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \exp\left\{\frac{\mu}{\sigma}W_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\right\}$.

Proof. By Girsanov Theorem $W_t^* = W_t + \int_0^t \frac{\mu}{\sigma}\,\mathrm{d}t$ is a standard Brownian Motion under $\mathbb{P}^*$, where θ, the market price of risk is equal to $\frac{\mu - r}{\sigma} = \frac{\mu}{\sigma}$ for $r = 0$. Since $X_t = \mu t + \sigma W_t$ under $\mathbb{P}$, then we have $W_t = \frac{X_t - \mu t}{\sigma}$ and therefore, $W_t^* = \frac{X_t - \mu t}{\sigma} + \frac{\mu t}{\sigma} = \frac{X_t}{\sigma}$. Since $\xi = \exp\left\{\frac{\mu}{\sigma}W_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\right\} = \exp\left\{\frac{\mu}{\sigma^2}X_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\right\}$, then we have

$$\begin{aligned}
\mathbb{P}^*[X_t \le x, M_t \le y] &= \mathbb{E}^*[\mathbf{1}_{\{X_t \le x, M_t \le y\}}] \\
&= \mathbb{E}[\xi_t\,\mathbf{1}_{\{X_t \le x, M_t \le y\}}] \\
&= \mathbb{E}[\exp(\frac{\mu}{\sigma^2}X_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t)\,\mathbf{1}_{\{X_t \le x, M_t \le y\}}] \\
&= \int_{-\infty}^{x} \exp(\frac{\mu}{\sigma^2}X_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t)\,\mathbb{P}\{X_t \in \mathrm{d}z, M_t \le y\} \\
&= \int_{-\infty}^{x} \exp(\frac{\mu}{\sigma^2}X_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t)\,g_t(z,y)\mathrm{d}z.
\end{aligned}$$

Differentiating the last equation with respect to x, we get $f_t(x,y) = \exp\left\{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right\} g_t(x,y)$. $\square$

Corollary 2.2.4. *Let* $F_t(x,y) = \mathbb{P}\{X_t \le x, M_t \le y\}$ *where* $X_t \sim N(\mu t, \sigma^2 t)$ *and,* $M_t = \sup\{X_s, 0 \le s \le t\}$ *then we have*

$$F_t(x,y) = \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - \exp(\frac{2\mu y}{\sigma^2})\,\Phi\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right) \tag{2.35}$$

Proof.

$$\begin{aligned}
F_t(x,y) &= \int_{-\infty}^{x} f_t(z,y)\mathrm{d}z \\
&= \int_{-\infty}^{x} \exp\left\{\frac{\mu z}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right\} \frac{1}{\sigma\sqrt{t}}\Big[\phi\Big(\frac{z}{\sigma\sqrt{t}}\Big) - \phi\Big(\frac{z-2y}{\sigma\sqrt{t}}\Big)\Big]\mathrm{d}z \\
&= \exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\} \int_{-\infty}^{x} \exp\left\{\frac{\mu z}{\sigma^2}\right\} \frac{1}{\sigma\sqrt{t}}\Big[\phi\Big(\frac{z}{\sigma\sqrt{t}}\Big) - \phi\Big(\frac{z-2y}{\sigma\sqrt{t}}\Big)\Big]\mathrm{d}z;
\end{aligned}$$

we take $z' = z - x$, so that

$$\begin{aligned}
F_t(x,y) &= \exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\} \int_{-\infty}^{0} \exp\left\{\frac{\mu(z+x)}{\sigma^2}\right\} \frac{1}{\sigma\sqrt{t}}\Big[\phi\Big(\frac{z+x}{\sigma\sqrt{t}}\Big) - \phi\Big(\frac{z+x-2y}{\sigma\sqrt{t}}\Big)\Big]\mathrm{d}z \\
&= \exp\left\{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right\} \int_{-\infty}^{0} \exp\left\{\frac{\mu z}{\sigma^2}\right\} \frac{1}{\sigma\sqrt{t}}\Big[\phi\Big(\frac{z+x}{\sigma\sqrt{t}}\Big) - \phi\Big(\frac{z+x-2y}{\sigma\sqrt{t}}\Big)\Big]\mathrm{d}z.
\end{aligned}$$

Then, we have

$$F_t(x,y) = \exp\left\{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right\} \Big\{\Psi(x) - \Psi(x-2y)\Big\} \tag{2.36}$$

for

$$\Psi(x) = \int_{-\infty}^{0} \exp\left\{\frac{\mu z}{\sigma^2}\right\} \frac{1}{\sigma\sqrt{t}}\phi\Big(\frac{z+x}{\sigma\sqrt{t}}\Big)\mathrm{d}z$$

Let $h(x,t) = \frac{x-\mu t}{\sigma\sqrt{t}}$ and writhing out $\phi(\cdot)$ we have

$$\begin{aligned}
\Psi(x) &= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{t}} \exp\left\{\frac{\mu z}{\sigma^2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(z+x)^2}{2\sigma^2 t}\right\}\mathrm{d}z \\
&= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{\frac{\mu z}{\sigma^2} - \frac{(z+x)^2}{2\sigma^2 t}\right\}\mathrm{d}z \\
&= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{\frac{1}{\sigma^2}\Big[\mu z - \frac{z^2+2xz+x^2}{2t}\Big]\right\}\mathrm{d}z \\
&= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{\frac{1}{\sigma^2}\left\{\Big[-\frac{z^2+2(x-\mu t)z+(x-\mu t)^2}{2t}\Big] + \Big(\frac{\mu^2 t}{2} - \mu x\Big)\right\}\right\}\mathrm{d}z \\
&= \exp\left\{\frac{\mu^2 t}{2\sigma^2} - \frac{\mu x}{\sigma^2}\right\} \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{1}{2}\Big[\frac{1}{\sigma\sqrt{t}}(z+x-\mu t)\Big]^2\right\}\mathrm{d}z \\
&= \exp\left\{\frac{\mu^2 t}{2\sigma^2} - \frac{\mu x}{\sigma^2}\right\} \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{t}}\phi\Big(z+x-\mu t\Big)\mathrm{d}z \\
&= \exp\left\{\frac{\mu^2 t}{2\sigma^2} - \frac{\mu x}{\sigma^2}\right\} \int_{-\infty}^{h(x,t)} \phi(u)\mathrm{d}z,
\end{aligned}$$

where $u = \frac{z+x-\mu t}{\sigma\sqrt{t}}$ and for $z = 0, u = \frac{x-\mu t}{\sigma\sqrt{t}}$. Therefore

$$\Psi(x) = \exp\left\{\frac{\mu^2 t}{2\sigma^2} - \frac{\mu x}{\sigma^2}\right\} \Phi(\frac{x-\mu t}{\sigma\sqrt{t}}). \tag{2.37}$$

Inserting equation (2.37) into equation (2.36), we get the following result:

$$
\begin{aligned}
F_t(x,y) &= \exp\left\{-\frac{\mu^2 t}{2\sigma^2}+\frac{\mu x}{\sigma^2}\right\}\left[\exp\left\{\frac{\mu^2 t}{2\sigma^2}-\frac{\mu x}{\sigma^2}\right\}\Phi(h(t,x))\right]\\
&= \exp\left\{-\frac{\mu^2 t}{2\sigma^2}+\frac{\mu x}{\sigma^2}\right\}\left[\exp\left\{\frac{\mu^2 t}{2\sigma^2}-\frac{\mu x}{\sigma^2}\right\}\Phi(\frac{x-\mu t}{\sigma\sqrt{t}})-\right.\\
&\left.\exp\left\{\frac{\mu^2 t}{2\sigma^2}-\frac{\mu(x-2y)}{\sigma^2 t}\right\}\Phi(\frac{x-2y-\mu t}{\sigma\sqrt{t}})\right]\\
&= \Phi(\frac{x-\mu t}{\sigma\sqrt{t}})-\exp\left\{\frac{2\mu y}{\sigma^2}\right\}\Phi(\frac{x-2y-\mu t}{\sigma\sqrt{t}}).
\end{aligned}
$$

□

Let T_y be the first time t at which $X_t = y$. That is $T_y = \inf\{t : X_t = y\}$ then $T_y > y \Rightarrow M_t < y$ letting $x \nearrow y$ then we have

$$\mathbb{P}(T_y > t) = \mathbb{P}(M_t < y) = F_t(y,y) \Rightarrow$$

$$F_t(y,y) = \Phi(\frac{y-\mu t}{\sigma\sqrt{t}}) - \exp\left\{\frac{2\mu y}{\sigma^2}\right\}\Phi(\frac{-y-\mu t}{\sigma\sqrt{t}}) \tag{2.38}$$

In Black-Cox constant model, our aim is to find probability that asset values of a firm breach a lower barrier. That is, we have to find $\mathbb{P}\{\inf_{s\le t}(X_s) > y\}$. Since $\inf_{s\le t}(X_s) = -\sup_{s\le t}(-X_s)$ and Brownian motion has symmetric property, that is $W_t \sim -W_t$, then we have following result:

$$
\begin{aligned}
\mathbb{P}\left\{\inf_{s\le t}(X_s) > y\right\} &= \mathbb{P}\left\{-\sup_{s\le t}(-X_s) > y\right\}\\
&= \mathbb{P}\left\{\sup_{s\le t}(-X_s) < -y\right\}\\
&= \mathbb{P}\left\{\sup_{s\le t}(-\mu s - \sigma W_s) < -y\right\}\\
&= \mathbb{P}\left\{\sup_{s\le t}(-\mu s + \sigma W_s) < -y\right\}.
\end{aligned}
$$

To compute $\mathbb{P}\left\{\sup_{s\le t}(-\mu s + \sigma W_s) < -y\right\}$, let us define both $Y_t = -\mu t + \sigma W_t$ and its supremum $\mathcal{N}_t = \sup_{s\le t} Y_s$; then we can define the joint distribution function of Y_t and $\mathcal{N}_t$ as

$$G_t(x,y) = \mathbb{P}\{Y_t \le -x, \mathcal{N}_t \le -y\}$$

for $y < x < 0$ or $0 < -x < -y$ and in Corollary 2.2.4 inserting $-x$, $-y$ and $-\mu t$ instead of x, y and μt, we have prove the following results:

Proposition 2.2.5. *The joint distribution of Y_t and its supremum is given by the following equation:*

$$G_t(x,y) = \mathbb{P}\left\{Y_t \le -x, \mathcal{N}_t \le -y\right\} = \Phi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right) - \exp\left\{2\mu\sigma^{-2}y\right\}\Phi\left(\frac{-x+2y+\mu t}{\sigma\sqrt{t}}\right), \tag{2.39}$$

and letting $x \nearrow y$ we have

$$G_t(y,y) = \mathbb{P}\left\{\inf_{s\le t}(X_s) > y\right\} = \Phi\left(\frac{-y+\mu t}{\sigma\sqrt{t}}\right) - \exp\left\{2\mu\sigma^{-2}y\right\}\Phi\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right). \tag{2.40}$$

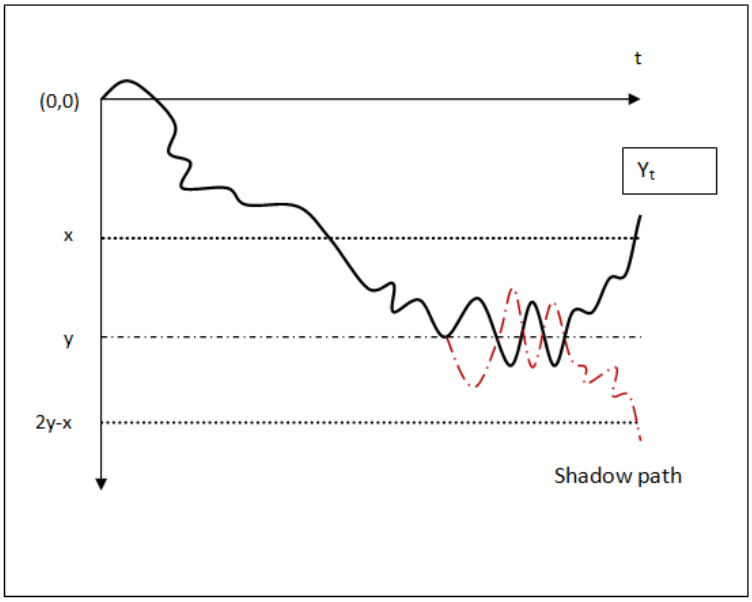

Figure 2.7: Reflection principle of Brownian motion with negative drift.

Corollary 2.2.6. *We have that, for any $t \ge u$ and $u \in (0,T]$, on the event $t < \tau$:*

$$\mathbb{P}(\tau > u|F_t) = \Phi\left(\frac{Z_t + \mu(u-t)}{\sigma\sqrt{u-t}}\right) - \exp(-2\mu\sigma^{-2}Z_t)\Phi\left(\frac{-Z_t+\mu(u-t)}{\sigma\sqrt{u-t}}\right). \tag{2.41}$$

Proof. It is just a consequence of Proposition 2.2.5. □

Theorem 2.2.7. *Let the firm value follows a log normal diffusion process as follows:*

$$\mathrm{d}V_t = V_t\left(r\mathrm{d}t + \sigma\mathrm{d}W_t\right), \tag{2.42}$$

where r is the constant short interest rate and σ is the volatility of the firm value. Then,

$$\mathbb{P}(\tau > u|F_t) = \mathbb{P}(\tau > u|\tau > t).$$

and

$$\mathbb{P}\left(\tau > u | \tau > t\right) = \Phi\left(d_3\right) - \left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1} \Phi\left(d_4\right), \tag{2.43}$$

Proof. The result can be found by taking $Z_t = \log\left(V_t/L\right)$, $\mu = \left(r - \frac{1}{2}\sigma^2\right)$ and using Corollary 2.2.6. $\square$

CHAPTER 3

CREDIT DEFAULT SWAP PRICING

As a financial instrument, a credit derivative payoffs is contingent on credit risk realizations. These payoff depends on the occurrence of a "credit event" for a reference entity. In general, a *credit event* can be one of the following stations [35]:

1. Failure to make a required payment,
2. Restructuring that makes any creditor worse off,
3. Invocation of cross-default clause, and
4. Bankruptcy.

Credit derivatives are the financial instruments designed for hedging the credit risk. For example, as a financial institution, a bank has credit exposure to many obligor. In general, before the use of the credit derivative and the loan sales, financial institutions manage their credit risk through diversification. That approach is not efficient because it enforces a bank to turn down the customers with which it has valuable relationships. A bank can hedge all or a part of its loan exposure to the obligor by using credit derivative. Generally credit derivatives are not traded on exchanges. They are over-the-counter hedging financial products [35].

Swap contracts are the most popular credit derivatives. One of the most popular type is called a *credit default swap* [35]. Credit Default Swaps (CDSs) are the bilateral agreements that include at least two counter parties; the protection buyer and the protection seller. The aim of the protection buyer is deliver credit risk of the reference entity to the protection seller in exchange of CDS price called CDS premium. These premiums are paid yearly up to default of reference entity, occurrence of credit event or maturity of CDS contract [13]. If credit event takes place, the protection seller makes default payment that equal to the notional value of the reference entity. Usually, a credit event requires a final accrual payment by the buyer of CDS [20].

As stated in Chapter 1, there are different types of CDSs. But in this section we will price “Single Name CDS”, that is, the contracts written on single reference entity. The following items have to be explained in order to clarify the mechanism of CDS:

- *Reference entity.* The reference entity is the company upon default of which protection is bought and sold.
- *Credit event.* A credit event is defined event in CDS agreement that would trigger default payment.
- *Reference obligation.* Underlying bond or asset on which protection bought or sold with CDS contract.
- *CDS notional principal.* The face value of the reference obligation sold with a CDS.
- *CDS spread.* The price of CDS yearly paid to protection seller by protection buyer.
- *Default Payment.* Payment that would be paid by protection seller if credit event occurs.

The general mechanism of a single name CDS is given in Figure 3.1.

CDS can be settled by either physical delivery or in cash. If the agreement requires physical delivery, the protection buyer delivers underlying assets in exchange of their par value. In case of cash settlement, after the occurrence of the credit event or default, calculation agent gathers dealers to determine the mid-market price or recovery rate, R. The cash settlement is then equal to $(100 - R)\%$ of the notional principal [20].

In this thesis, we price single name CDS contract. The valuation of a credit default swap necessitates the assess of the risk-neutral probability of default for underling

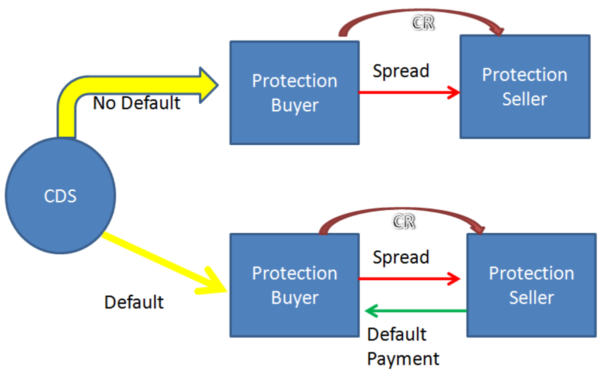

Figure 3.1: Single name CDS mechanism.

reference entity [20]. We have already estimated the risk natural survival probabilities under structural and intensity based models in Chapter 2.

In order to price CDS contract, let us introduce theoretical pricing methodology. Let denote the price (spread) of a single name CDS by c and let denote the notional amount by N. We assume CDS premium paid by the CDS buyer to CDS seller annually and spread payments and default payments are made at discrete times t_i $(i = 1, 2, 3, ..., n)$ with $t_n = T$ and $t_0 = 0$.

For simplicity assume payments made at the and of each period. Let us denote the discount factor for time interval $[0, t_i]$ by $D(0, t_i)$ and define $\Delta t_i = t_i - t_{i-1}$. We can calculate the present value of the CDS premium by the following equation

$$PV_F = cN \sum_{i=1}^{n} D(0, t_i) P_{\mathcal{S}}(t_i) \Delta t_i + A_d, \tag{3.1}$$

where A_d is the accrual payment. Since we assume default can occurs at payment dates or in the middle of the payment dates, average A_d is equal to the following equation:

$$A_d = 0.5cN \sum_{i=1}^{n} D(0, t_i)[P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i)]\Delta t_i \tag{3.2}$$

The present value of the default payment is given by the following formula under the assumption of the recovery rate R:

$$PV_L = (1 - R)N \sum_{i=1}^{n} D(0, t_i)[P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i)]. \tag{3.3}$$

Equation (3.3) comes from the fact that default between the two consequent time t_{i-1} and t_i is equal to $P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i)$, as it can be seen from the following derivation:

$$\begin{aligned}
\mathbb{P}\{t_{i-1} < \tau < t_i\} &= \mathbb{P}\{t_{i-1} < \tau\} + \mathbb{P}\{\tau < t_i\} \\
&= \mathbb{P}\{t_{i-1} < \tau\} - \mathbb{P}\{t_i < \tau\} \\
&= P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i).
\end{aligned}$$

In arbitrage free market, the price of CDS is the price that equate PV_F to PV_L. Therefore, by equating the two Equations (3.1) and (3.3) we have the following result at time $t = 0$:

$$c_0 = \frac{(1-R)\sum_{i=1}^{n} D(0,t_i)[P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i)]}{\sum_{i=1}^{n} D(0,t_i)P_{\mathcal{S}}(t_i)\Delta t_i + (A_d/cN)}. \tag{3.4}$$

By inserting Equation (3.2) in to Equation (3.4), we get the following general formula for the CDS premium under discrete time:

$$c_0 = \frac{(1-R)\sum_{i=1}^{n} D(0,t_i)[P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i)]\Delta t_i}{\sum_{i=1}^{n} D(0,t_i)P_{\mathcal{S}}(t_i)\Delta t_i + 0.5\sum_{i=1}^{n} D(0,t_i)[P_{\mathcal{S}}(t_{i-1}) - P_{\mathcal{S}}(t_i)]}. \tag{3.5}$$

The Equation (3.5) in continuous time resembles the following equation:

$$c_0 = \frac{(1-R)[-\int_0^T D(0,s)\,\mathrm{d}P_{\mathcal{S}}(s)]}{\int_0^T D(0,s)P_{\mathcal{S}}(s)\,\mathrm{d}s - 0.5\int_0^T D(0,s)\,\mathrm{d}P_{\mathcal{S}}(s)]}. \tag{3.6}$$

Before going into CDS price under Merton and other models that we have explained in Chapter 2, the following example taken from Jhon Hull [18] will clarify the pricing process of a single name CDS contract.

Example 3.0.8. For a reference entity, let us assume the probability of default on a year conditional no early default,$\mathbb{P}(\tau = t_i | \tau > t_{i-1})$, for the first year, is equal to 0.02 as given in Table 3.1. This table shows the unconditional default probabilities and the survival probabilities for each year of the five year maturity of CDS. If $P_{\mathcal{D}}$ is 0.02 for the first year, then $P_{\mathcal{S}} = 0.98$. The value of $P_{\mathcal{D}}$ during the second year is $0.02 \times 0.98 = 0.0196$ and the survival probability during the same year, $P_{\mathcal{S}} = 0.98 \times 0.98 = 0.9604$, and other probabilities in Table 3.1 are calculated by the same manner.

We assume risk-free rate is equal to 5% with continuously compounding and a recovery rate is $R = 0.4$. Also we assume that defaults occur in the middle of a year, and CDS premiums are paid once a year. We should calculate the present value of total CDS premium (expected payment), total amount of expected payoff (default payment) and the expected present value of accrual payment to find CDS price.

Table 3.1: Unconditional default probabilities and survival probabilities.

Time(years)	Default probability	Survival probability
1	0.0200	0.9800
2	0.0196	0.9604
3	0.0192	0.9412
4	0.0188	0.9224
5	0.0184	0.9039

Table 3.2 shows the result of the calculation of present value of the expected CDS premium payment c on notional principal \$1. For instance, the possibility of second payment of c is 96.04% for the second year. Therefore, the expected payment is $0.9604c$, and its present value is equal to $0.9604c \times \exp(-0.05 \times 2) = 0.96.04c \times 0.9048 = 0.8690c$. The total present value of CDS price is 4.0704 c.

Table 3.2: Expected payment.

Time (years)	Survival probability	Expected payment	Discount factor	PV_F
1	0.9800	0.9800 c	0.9512	0.9322 c
2	0.9604	0.9604 c	0.9048	0.8690 c
3	0.9412	0.9412 c	0.8607	0.8101 c
4	0.9224	0.9224 c	0.8187	0.7552 c
5	0.9039	0.9039 c	0.7787	0.7040 c
Total				4.0704 c

The total present value of default payments calculated in Table 3.3 with the assumption of a notional principal of 1\$. Since we assume that default happens in the middle of the year, there is 1.96% possibility for the second year. Default payment for the second year is equal to $0.0196 \times 0.6 \times 1 = 0.0115$ and its present value is $0.0115 \times \exp(-0.05 \times 1.5) = 0.0109$. Totally, we have \$0.0511 as a total present value of expected payoff or default payments.

Finally, Table 3.4 shows the calculations of the accrual payment that would take place conditional on default occurrence. There is a 0.0196 chance of default through the second year. Because of our assumption that default can take place in the middle of a year, the accrual payment is $0.5c$. Thus expected accrual payment for this year is $0.0196 \times 0.5c$ and its present value is equal to $0.0098 \times \exp(-0.05 \times 1.5) = 0.0091c$.

Table 3.3: Expected payoff.

Time(years)	Default prob.	R	Expected payoff	Discount factor	PV_L
0.5	0.0200	0.4	0.0120	0.9753	0.0117
1.5	0.0196	0.4	0.0118	0.9277	0.0109
2.5	0.0192	0.4	0.0115	0.8825	0.0102
3.5	0.0188	0.4	0.0113	0.8395	0.0095
4.5	0.0184	0.4	0.0111	0.7985	0.0088
Total					0.0511

Total present value of accrual payments is $0.0426c$.

Table 3.4: Expected accrual payment.

Time (years)	Default prob.	Expected accrual pay.	Discount factor	PV of $\mathbb{E}(accrual\,pay.)$
0.5	0.0200	0.0100 c	0.9753	0.0097 c
1.5	0.0196	0.0098 c	0.9277	0.0091 c
2.5	0.0192	0.0096 c	0.8825	0.0085 c
3.5	0.0188	0.0094 c	0.8395	0.0079 c
4.5	0.0184	0.0092 c	0.7985	0.0074 c
Total				0.0426 c

From Table 3.2 and Table 3.4 present value of CDS buyer will be $4.0704c + 0.0426c = 4.1130$ and this should be equal to total present value of default payments which is equal to 0.0511. Therefore, CDS price c should be $\frac{0.0511}{4.1130} = 0.0124$, that is 124 basis points(bp). Intuitively this means that a five year CDS contract written on reference entity will has price equal to 0.0124 times the notional amount of underlying reference entity per year.

3.1 CDS Pricing by Using Merton Model

As we stated before, pricing of a CDS requires the estimates of the risk-natural survival probability of an underlying reference entity. The price of the bonds issued by the underlying firm provides the main source of the data for the estimation [20]. However, the bond issuance in the emerging markets like Turkish bond market are not sufficiently developed. For this reason, in this work, we try to fill these bond prices data gap by constructing CDS pricing methodology that relied on the data of the equity prices. In order to carry out this aim, we estimate the CDS price by using the survival probabilities of the structural models. Since, so far, we have studied Merton model and Black-Cox constant barrier model as structural model. First, we will use Merton model to price a single name CDS contract.

Theorem 3.1.1. *Given Equation* (3.6)*, for $t \leq s \leq T$ the price of a single name CDS at time t under Merton Model can be calculated by the following formula:*

$$c_t = \frac{(1-R)[-\int_t^T D(t,s)\,\phi(d_2)\,\frac{log(L/V_t)+(r-\frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}\mathrm{d}s]}{\int_0^T \int_{-\infty}^{d_2} D(t,s)\phi(v)\,\mathrm{d}v\mathrm{d}s - 0.5\int_0^T D(t,s)\,\phi(d_2)\,\frac{log(L/V_t)+(r-\frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}\mathrm{d}s} \tag{3.7}$$

where $D(t,s)$ is the discount factor that equal to present value of one unit of money with maturity $s-t$, R is the recovery rate and ϕ is the density function of standard normal distribution, that is $\phi(v) = (1/\sqrt{2\pi})\exp(-v^2/2)$.

Proof. By inserting Equation (2.26) in to the Equation (3.6) and taking the derivative with respect to s, we will get the following equation:

$$c_t = \frac{(1-R)[-\int_t^T D(t,s)\,\mathrm{d}\Phi(d_2)]}{\int_t^T D(t,s)\Phi(d_2)\,\mathrm{d}s}. \tag{3.8}$$

Since $\Phi(d_2) = \int_{-\infty}^{d_2}\phi(v)\mathrm{d}v$, by Fundamental Theorem of Calculus we have $\mathrm{d}\Phi(d_2) = \phi(d_2)\mathrm{d}d_2$. Therefore, we can rewrite Equation (3.8) as follow:

$$c_t = \frac{(1-R)[-\int_t^T D(t,s)\,\phi(d_2)\mathrm{d}d_2]}{\int_t^T \int_{-\infty}^{d_2} D(t,s)\phi(v)\,\mathrm{d}v\mathrm{d}s}, \tag{3.9}$$

where $\phi(v) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}v^2)$. Note that $\mathrm{d}d_2 \neq \mathrm{d}s$.

To make it clear we have to compute $\frac{\mathrm{d}d_2}{\mathrm{d}s}$. As it was stated before,

$$d_2 = \frac{A + B(s-t)}{\sigma\sqrt{s-t}}$$

for $A = \log(V_t/L)$ and $B = r - (1/2)\sigma^2$

$$\begin{aligned}\frac{\mathrm{d}d_2}{\mathrm{d}s} &= \frac{B\sqrt{s-t} - \frac{\sigma[A+B(s-t)]}{2\sqrt{s-t}}}{\sigma^2(s-t)} \\ &= \frac{2B\sigma(s-t) - \sigma[A+B(s-t)]}{2\sigma^2(s-t)^{\frac{3}{2}}} \\ &= \frac{2B\sigma(s-t) - \sigma A - \sigma B(s-t)}{2\sigma^2(s-t)^{\frac{3}{2}}} \\ &= \frac{-\sigma A + \sigma B(s-t)}{2\sigma^2(s-t)^{\frac{3}{2}}} \\ &= \frac{-A + B(s-t)}{2\sigma(s-t)^{\frac{3}{2}}} \\ &= \frac{\log(\frac{L}{V_t}) + (r - \frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}.\end{aligned}$$

By inserting the last result in Equation (3.9), we get the complete proof. □

One can price single name CDS just by applying Equation (3.7) to the data of firm asset values, liability values, over night (O/N) LIBOR rate, the asset value volatility of the firm and average maturity of firm's liabilities. In Chapter 2, we have explained how one can estimate asset volatility from equity volatility if the return of equity observable or firm stocks are traded at exchanges. These models supply the most naive results considering the firm value models.

3.2 CDS Pricing by Using Black-Cox Model

In order to price CDS under Black-Cox model, we can use same methodology that we have already use for Merton model. The only difference between these two models is the difference of the survival probabilities under each model. Under Black-Cox constant barrier model the survival probability is given by Equation (2.43).

Theorem 3.2.1. *Let denote recovery rate with R, the discount factor with $D(t,s)$, the value of the firm at time t with V_t and magnitude of constant barrier (value of liability) with L, then under Black-Cox constant barrier model price of single name CDS price or premium at time t can be estimated by following equation:*

$$c_t = \frac{(1-R)\left[-\int_t^T D(t,s)\phi(d_3)\mathrm{d}d_3 + \int_t^T D(t,s)\left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1}\phi(d_4)\mathrm{d}d_4\right]}{\int_t^T\int_{-\infty}^{d_3} D(t,s)\phi(v)\mathrm{d}v\mathrm{d}s - \int_t^T\int_{-\infty}^{d_4} D(t,s))\left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1}\phi(v)\mathrm{d}v\mathrm{d}s + X}, \quad (3.10)$$

where

$$X = -0.5\left[\int_t^T D(t,s)\phi(d_3)\mathrm{d}d_3 + \int_t^T D(0,s)\left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1}\phi(d_4)\mathrm{d}d_4\right]$$

OR $X = A_d/cN$ *for accrual payment* A_d *and*

$$\mathrm{d}d_3 = \frac{\log(L/V_t) + (r - \frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}\mathrm{d}s$$

and

$$\mathrm{d}d_4 = \frac{\log(V_t/L) + (r - \frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}\mathrm{d}s$$

Proof. The proof is similar to the proof of Theorem 3.1.1. We just need to calculate $\mathrm{d}d_3$ and $\mathrm{d}d_4$. As it can be realized d_3 is equal to d_2 used in Merton model. Therefore we can write $\mathrm{d}d_3$ immediately as

$$\mathrm{d}d_3 = \frac{\log(L/V_t) + (r - \frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}\mathrm{d}s.$$

Since

$$d_4 = \frac{\ln(L/V_t) + (r - \frac{1}{2}\sigma^2)(s-t)}{\sigma\sqrt{s-t}}$$

we can get $\mathrm{d}d_4$ by just taking $A = \log(L/V_t)$ instead $\log(V_t/L)$ as we took in proof of Theorem 3.1.1. Thus, we get

$$\mathrm{d}d_4 = \frac{\log(V_t/L) + (r - \frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}\mathrm{d}s.$$

□

As we have explained for the Merton model, one can price single name CDS just by applying Equation (3.10) to data of inputs. CDS price with this model probably would give better results than the Merton model because of its more realistic assumptions about the market. But the accurateness of the results obviously needs calibration.

3.3 CDS Pricing by Using Intensity Based Models

One can price CDS contract under intensity based models just by inserting probability survive P_S calculated under these reduced form models into the Equation (3.6). Under different behaviour of λ that leads to different survival probabilities, we get a different CDS premium or price.

Theorem 3.3.1. *Let us denote discount factor with $D(t,s)$ which is equal to* $\exp(-r(s-t))$ *given, then under reduced-form models, the CDS premium can be estimated by using the following formula, for general form of default intensity λ_t:*

$$c_t = \frac{(1-R)\left[-\int\limits_t^T D(t,s)\,\mathrm{d}P_{\mathcal{S}}\right]}{\int_t^T D(t,s)\mathbb{E}\left(\exp(\int_t^s \lambda_u \mathrm{d}u)\right)\mathrm{d}s - \int\limits_t^T D(t,s)\,\mathrm{d}P_{\mathcal{S}}}, \tag{3.11}$$

where

$$P_{\mathcal{S}}(t) = \mathbb{E}\left\{\exp\left(-\int_s^t \lambda_s \mathrm{d}s\right)\right\}$$

as it is given in Equation (2.17).

Proof. The proof of the theorem can be done by just inserting Equation (2.18) into Equation (3.6). □

Following example from J. Cariboni and W. Schoutens [11], shows how we can estimate survival probability for a firm from price of traded CDS contract by using Jarrow-Turnbull model, which is one of the fundamental reduced form model.

Example 3.3.2. Let assume the default time τ is exponentially distributed with parameter λ. Then given survival probability $P_{\mathcal{S}}(t) = e^{-\lambda t}$, we get the following equation:

$$c_0 = \frac{(1-R)[\lambda\int_0^T D(0,s)e^{-\lambda s}\,\mathrm{d}s]}{\int_0^T D(0,s)e^{-\lambda s}\,\mathrm{d}s} = \lambda(1-R)$$

by substituting the survival probability into Equation (3.11).

If we have traded CDS on market with price 90 bps (0.0090) and underlying reference entity has recovery rate $R = 50$, we can estimate probability of default P_d of entity for next five year $\lambda = (c/1-R) = (0.00090/0.5) = 0.18$ and $P_{\mathcal{S}}(5) = e^{-\lambda t} \approx 1 - \lambda t = 1 - 0.18 \times 10 = 0.91$. Since $P_{\mathcal{D}}(5) = 1 - P_{\mathcal{S}}(5)$, we have $P_{\mathcal{D}}(5) = 1 - 0,91 = 0.09$ or 9%.

If we have data about intensity λ and recovery rate R, then we can price a CDS contract by using reduced form models as indicated in Equation (3.11). This means two inputs are enough to price a single name CDS contact under these models. This property as we assume, makes these models more applicable.

CHAPTER 4

INTENSITY ESTIMATION THROUGH THE STRUCTURAL MODELS

So far, we have studied the firm-value models (structural models) which model default by utilizing the process of the issuer asset value process. Default is triggered when assets, or some function thereof, fall below (or hit) some boundary level. And also we have explained intensity-based models which leave aside the question of what exactly triggers the default event, instead we deal with model factors influencing the default event [27].

Intensity-based models are important for two main reasons to be incorporated into study of credit risk. Initially, these models are the most suitable ways of connection between credit scoring models and the models for pricing default risk. In order to incorporate firm's asset values with other relevant indicator of default, we could use default prediction models and ask which variables are relevant for predicting the price. Therefore, we need to understand the variable evaluation of these covariates and we should explain how they influence the default probabilities, and intensity-based models are the natural framework for doing this. Secondly, the entire machinery of default-free term-structure modelling comes into play by the help of the mathematical machinery of intensity models which means that econometric specifications from term-structure modelling and tricks for pricing derivatives can be transferred to defaultable claims. Additionally, some claims, such as basket CDSs, whose equivalent is not readily found in ordinary term-structure modelling, are also conveniently handled in this setting [27].

We have already explained in Chapter 2 what intensity-based models generally are.

Here, we get over the hazard rates (default intensities) once again and we estimate the default intensity approximately from structural models, namely, Merton and Black-Cox models. Let a positive random variable τ have a distribution that can be described in terms of a hazard function h, that is,

$$\mathbb{P}(\tau > t) = \exp\left(- \int_0^t h(s) \mathrm{d}s \right).$$

Then as we mentioned in Chapter 2:

$$\lim_{\delta t \downarrow 0} \mathbb{P}(\tau \leq t + \delta t | \tau > t) = h(t);$$

therefore conditional probability of default in a small time interval containing t is approximately equal to $h(t)\delta t$ [27].

In general model setting literature, there are factors other than time passing that have effects on default probability of a firm which has survived up to time t. As we assume, the firm does not default between time interval $[0, t]$, we have all information available at time t, which can be defined as a filtration $(\mathcal{G}_t)_{t \in \mathbb{R}^+}$, which is a sigma algebra that satisfies property $\mathcal{G}_s \subseteq \mathcal{G}_t$ for time s and t such that $s < t$. Therefore, we can write the conditional default probability in reduced form models as:

$$\mathbb{P}(\tau > t | \mathcal{G}_t) \approx I_{\{\tau > t\}} \lambda(t) \Delta t.$$

As it can be calculated from Table 4.1, the probability of a bond rated Baa bond for the first year is 0.0181 % and it is equal to $1.434 - 0.930 = 0.504\%$ during the fourth year. The survival probability for the firm until the end of fourth year is equal to $100 - 1.434 = 98.566\%$. The probability that it will default during fourth year conditional on no earlier default is equal to $0.00504/0.98566 = 0.0051\%$. That is, $\mathbb{P}(\tau = 4 | \tau > 3) = 0.0051\%$ or $\lambda_4 = 0.0051\%$.

Table 4.1: Average cumulative default rates (%), 1970-2006 [18].

Time/Rates	1	2	3	4	5	7	10	15	20
Aaa	0.000	0.000	0.000	0.026	0.099	0.251	0.521	0.992	1.191
Aa	0.008	0.019	0.042	0.106	0.177	0.343	0.522	1.111	1.929
A	0.021	0.095	0.220	0.344	0.472	0.759	1.287	2.364	4.238
Baa	0.181	0.506	0.930	1.434	1.938	2.959	4.637	8.244	11.362
Ba	1.205	3.219	5.568	7.958	10.215	14.005	19.118	28.380	35.093
B	5.236	11.296	17.043	22.054	26.794	34.771	43.343	52.175	54.421
Caa-C	19.476	30.494	39.717	46.904	52.622	59.938	69.178	70.870	70.870

Definition 4.0.3. Let define hazard function of τ as $\varGamma : \mathbb{R}^+ \to \mathbb{R}^+$, for every $t \in \mathbb{R}^+$,

$$\varGamma(t) = -\log(1 - F(t)),$$

where F is cumulative distribution function that defined in Definition 2.1.8.

Remember that the $\varGamma$ function is also defined in Chapter 2.

Lemma 4.0.4. *If the cumulative distribution function F defined in Definition 2.1.8 is absolutely continuous with respect to the Lebesgue measure, then the hazard function $\varGamma$ is also absolutely continuous. Specifically,*

$$\lambda_t = \frac{f(t)}{1 - F(t)}. \tag{4.1}$$

Proof. The proof is just a consequence of Equation (2.13). $\square$

As we mentioned before, λ_t is called hazard rate or intensity of default time τ. By interpretation of λ_t in Lemma 4.0.4, and definition of hazard function as we have already show in Chapter 2 we can write:

$$\mathbb{P}(\tau > t) = 1 - F(t) = \exp(-\varGamma(t)) = \exp\left(-\int_0^t \lambda_u \mathrm{d}u\right).$$

The hazard rate λ_t can be interpreted by using survival probability functions that we calculated under structural credit risk models, Merton model and Black-Cox constant barrier model.

Theorem 4.0.5. *Default intensity λ_t can be approximated by using cumulative distribution function and probability density function of default probabilities under Merton model for $s, t \in [0, T]$ and $t < s$ as follows:*

$$\lambda_t \approx \frac{\mathrm{d}\Phi(-d_2)}{\Phi(d_2)} \approx \frac{\phi(-d_2)\frac{log(V_t/L)-(r-\frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}}{\Phi(d_2)}, \tag{4.2}$$

where

$$d_2 = \frac{\log(V_t/L) + (r - \frac{1}{2}\sigma^2)(s - t)}{\sigma\sqrt{s - t}}.$$

Proof. As we have already shown, the cumulative distribution function of default under Merton model is equal to $\Phi(-d_2)$ and $1 - \Phi(-d_2) = \Phi(d_2)$. The result will follow after applying Lemma 4.0.4. $\square$

Because of the filtration differences between filtrations generated by λ_t and firm value process V_t, Theorem 4.0.5 is just gives a computational approximation. Also, this is true for the our next Theorem 4. Therefore, in a similar manner, we can write the following theorem to estimate approximately the default intensity by using the distribution function of Black-Cox model.

Theorem 4.0.6. *Under Black-Cox constant barrier model, we can estimate default intensity by the following equation:*

$$\lambda_t \approx \frac{\phi(-d_3)\frac{\log(\frac{V_t}{L})-(r-\frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}} + \left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1}\phi(d_4)\frac{\log(\frac{V_t}{L})+(r-\frac{1}{2}\sigma^2)(s-t)}{2\sigma(s-t)^{\frac{3}{2}}}}{\Phi\left(d_3\right) - \left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(d_4\right)}, \tag{4.3}$$

where Φ is the cumulative distribution function of a standard normal random variable, ϕ is the density function of same random variable, r short interest rate, V_t is the initial value of firm assets and L is the value of the predetermined constant barrier.

Proof. We have estimated the survival and the default probability functions of Black-Cox model in Chapter 2. That is, the survival probability under this model is:

$$P_{\mathcal{S}} = \Phi\left(d_3\right) - \left(\frac{L}{V_t}\right)^{\frac{2r}{\sigma^2}-1}\Phi\left(d_4\right),$$

where

$$d_3 = \frac{\ln(\frac{V_t}{L}) + (r-\frac{1}{2}\sigma^2)(u-t)}{\sigma\sqrt{u-t}},$$

and

$$d_4 = \frac{\ln(\frac{L}{V_t}) + (r-\frac{1}{2}\sigma^2)(u-t)}{\sigma\sqrt{u-t}}.$$

By substituting the probability of survive $P_{\mathcal{S}}$ into Equation 4.1 that we have defined in Lemma 4.0.4, we get the result. □

Because of having more realistic assumptions, we can claim that the intensity estimated from the Black-Cox model discloses real market intensity more than Merton model. However, these intensities estimated by both Merton and Black-Cox model should be calibrated with real data from the market to ensure about the prediction ability of each model. In this thesis, we try to show a way that how default intensities can be estimated from the equity based models, namely, structural models. This is important for the market which is not developed enough regarding the corporate bond issuance. For example, although most of the emerging markets have sufficiently well improved stock market, their bond markets are generally shallow. Therefore, by estimating the default intensities from structural models and using equity prices data, we can calculate cumulative default for a risky corporation or a firm. That is, by this contraction we can rate the firms of the market that have not enough bond prices data.

CHAPTER 5

CONCLUSION

In this thesis, we have examined the need of the credit risk management and the foundation of the advanced credit risk models. After a general introduction, we have started from intensity-based models, that was firstly established by Jarraw and Turnbull [22] and we have shown how to the default intensity can be found from the bond and the similar security prices. We have clarified how to compute the survival probabilities under these reduced-form models. Next, we have explained the firm value or the structural model starting from the Merton model that examined in [29]. In this model, to value the equity of the firm, Black-Scholes option pricing methodology is used. In other words, we have explained the structural models based on Merton model or Black-Scholes option pricing formula and intensity based models (reduced-form) based on counting processes like Poisson process. We have proved the Merton model formula which is used to estimate the equity price of a firm by using expectation under risk natural probability measure. Moreover, we have provided an explanation about how to evaluate volatility of a firm assets that can not be observed in the market by using observed stock return volatility under Merton model. We try to estimate the survival probability for a firm in each model (structural and reduced form). We have given detailed proofs for survival probability formulations under each model. Especially in Black-Cox constant barrier model, we have used the reflection principle of Brownian motion and prove the joint probability of drifted Brownian motion and its maximum and minimum to evaluate default and survival probabilities of a firm for a given maturity which makes basic difference between this model and Merton model. After these issues, we have explained the basic structure of a single name CDS contract and we have shown the basic relation between credit risk of a company and its CDS price. By using general pricing concept that is given in J. Cariboni and W. Schoutens[11], and substituting the survival probabilities into the general CDS pricing formula for each model that we have examined before we have estimated the CDS price for each of these models. Furthermore, we have proposed a CDS pricing formula under the general intensity-based models framework. Finally, we have utilized the hazard function, which is defined by T. Bielecki, M. Jeanblanck, and M. Rutkowski[5], to estimate the default intensities. Because of the *filtration* differences, we have just approximated the default intensities by means of the survival probabilities that we have estimated for each structural credit risk models, Merton and Black-Cox constant barrier.

We can conclude that intensity models and structural models that we have used to price CDS contracts are the fundamental IRB credit risk models. Although they are used by national and international banking and market regulators, other models that based on more realistic assumptions may give better results. However sometimes, naive models can give better result than complex models in relation with market structure. The strength of these models are their simplicity and their intuitive nature. On the other hand, some of unrealistic and insufficient assumptions that these models relaid on, can be considered as the weaknesses of these models.

In short, considering riskiness of the credit market and its destructive effects, as we have already observed in USA credit crisis, understanding credit risk sources and grasp how to manage this risk, is really important. We hope that we could have explained the building blocks of credit risk models and credit risk management in these work.

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APPENDIX A

Preliminaries

A.1 Computations of Conditional Expectations

Proposition A.1.1. *Let $B-$measurable random variable X taking values in (E,ν) and Y, a random variable independent of B with values in $(F,\mathbb{F})$. For any Borel function Ψ non-negative (or bounded)on $(E\times F,\nu\otimes\mathbb{F})$ the function ψ defined by*

$$\forall x\in E\ \psi(x)=\mathbb{E}\left(\Psi(x,Y)\right)$$

is a Borel function on (E,ν) and we have

$$\mathbb{E}\left(\Psi(x,Y)|B\right)=\psi(x)\ \ a.s.$$

.

In other words we can compute $\mathbb{E}\left(\Psi(x,Y)|B\right)$ as if X was a constant. For the proof see [26].

A.2 Hitting Time Distribution

We define the first passage time (hitting time) at which Brownian motion hit the barrier b as:

$$\tau_b=\inf\{t\geq 0,W_t(w)=b\}$$

τ_b. Cumulative distribution of hitting time can be written as follows:

$$\begin{aligned}\mathbb{P}^0[\tau_b<t]&=\mathbb{P}^0[\tau<t,Wt\geq b]+\mathbb{P}^0[\tau_b<t,Wt<b]\\&=\mathbb{P}^0[Wt\geq b]+\mathbb{P}^0[\tau_b<t,Wt<b].\end{aligned}$$

Since by definition of τ_b, we have $\tau_b < t$ implies that $Wt \geq b$ and by reflection property of Brownian motion we can write the following equation:

$$\begin{aligned}\mathbb{P}^0[\tau_b < t, Wt < b] &= \mathbb{P}^0[\tau_b < t, Wt \geq b] \\ &= \mathbb{P}^0[Wt \geq b].\end{aligned}$$

Therefore, we finally get the following result:

$$\mathbb{P}^0[\tau_b < t] = 2\mathbb{P}^0[Wt \geq b]$$

. Since $W_t \sim N(0, t)$, that is, it is normally distributed with mean zero and variance t we can write $\mathbb{P}^0[\tau_b < t]$ as follow:

$$\begin{aligned}\mathbb{P}^0[\tau_b < t] = 2\mathbb{P}^0[Wt \geq b] = 2\mathbb{P}[z > \frac{b-0}{\sqrt{t}}] &= 2\mathbb{P}[z > \frac{b}{\sqrt{t}}] \\ &= \frac{2}{\sqrt{2\pi}} \int_{\frac{b}{\sqrt{t}}}^{\infty} \exp(-\frac{z^2}{2}) \mathrm{d}z.\end{aligned}$$

Thus, τ_b has following cumulative distribution function:

$$F(t) = \sqrt{\frac{2}{\pi}} \int_{\frac{b}{\sqrt{t}}}^{\infty} \exp(-\frac{z^2}{2}) \mathrm{d}z \tag{A.1}$$

$$\begin{aligned}\frac{\partial F(t)}{\partial t} &= \sqrt{\frac{2}{\pi}} \left[0 - \left(- e^{\frac{b^2}{2t}} \frac{b}{2} t^{-\frac{3}{2}} \right) \right] \\ &= \sqrt{\frac{2}{\pi}} \frac{|b|}{2} t^{-\frac{3}{2}} e^{-\frac{b^2}{2t}} \\ &= \frac{|b|}{\sqrt{2\pi t}\, t} \exp(-\frac{b^2}{2t}),\end{aligned}$$

which means hitting time τ_b has following probability density function:

$$f(t) = \frac{|b|}{\sqrt{2\pi t^3}} \exp(-\frac{b^2}{2t}). \tag{A.2}$$

A.3 Mean Value Theorem for Integral

If f is a continuous on interval $[a, b]$ then there exists number $c \in [a, b]$ such that following formula holds:

$$\frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x = f(c). \tag{A.3}$$

A.4 Girsanov Theorem and Risk Natural Pricing

Theorem A.4.1. ***Girsanov Theorem:*** *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space and W_t be a standard Brownian Motion under probability $\mathbb{P}$ and let θ_t , $0 \leq t \leq T$ be an F_t measurable process satisfying $\int_0^T \theta^2 \, \mathrm{d}s < \infty$ a.s. and such that the process*

$$L_t = e^{-\int_0^t \theta_s \, \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2 \, \mathrm{d}s}$$

is a martingale with $E[L_t] = 1$. Then under new probability measure $\mathbb{P}^*$ with density L_T relative to $\mathbb{P}$,the process $(W_t^*)_{0 \leq t \leq T}$ defined by

$$W_t^* = W_t + \int_0^t \theta_s \, \mathrm{d}s,$$

is a standard Brownian Motion.

Following example is important for understanding of *Girsanov Theorem.*

Example A.4.2. Let define the market price of risk as $\theta_t = \frac{\mu - r}{\sigma}$ then by *Girsanov Theorem*, we can obtain the standard Brownian motion under the risk-natural probability as follows:

$$\begin{aligned} W_t^* &= W_t + \int_0^t \frac{\mu - r}{\sigma} \, \mathrm{d}s \\ &= W_t + \frac{\mu - r}{\sigma} t \Longrightarrow \mathrm{d}W_t^* \\ &= \mathrm{d}W_t + \frac{\mu - r}{\sigma} \mathrm{t}. \end{aligned}$$

Therefore, the stochastic differential equation of asset price can be written under the risk natural probability $\mathbb{P}^*$ as follows:

$$\begin{aligned} \mu \mathrm{d}t + \sigma \mathrm{d}W_t &= r \, \mathrm{d}t + \sigma \mathrm{d}W_t^* \\ \Rightarrow \mathrm{d}V_t &= V_t\{\mu \mathrm{d}t + \sigma \mathrm{d}W_t\} \\ &= V_t\{r \mathrm{d}t + \sigma \mathrm{d}W_t^*\}. \end{aligned}$$

And by Ito integral, we get asset price under $\mathbb{P}^*$

$$V_t = V_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t^* \right), \tag{A.4}$$

and if there is a continuous dividend payment q then we have

$$V_t = V_0 \exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t^* \right). \tag{A.5}$$

A.5 Modigliani-Miller Theorem

Theorem A.5.1. *Let* V_U *denotes the value of un-levered firm and and* V_L *denote value levered firm, then* $V_U = V_L$*. That is,there is no any affects of leverage on the value of the firm. In other words, the value of the firm is invariant to its capital structure.*